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X-94-02, *received*: May 1994

ILLC Research Report and Technical Notes Series
Series editor: Dick de Jongh

Technical Notes (X) Series

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Another construction of choiceless ultrapower

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May 20, 1994

Abstract

Another construction of an ultrapower in the case of absense of Choice is proposed. We prove that for any countable transitive model $M \models \mathbf{ZF}$, any $I \in M$ and any $U \in M$, an ultrafilter over I in M, there exists a countable set F of functions defined on I and taking values in M (it is not assumed that $F \subseteq M$, but F includes M^I) such that F modulo U is an elementary extension of M, wellfounded in the case when U is countably additive in M .

This approach is close, or, perhaps, equal to the Spector's extended ultrapowers, see [2]. In particular, it involves forcing, but not the same way as extended ultrapowers do. Roughly it separates the forcing and the ultrapower constituents in the Spector's construction, definitely demonstrating the role of either.

*Moscow Transport Engineering Institute and Moscow State University. Partially supported by the NWO PIONIER Project "Reasoning with Uncertainty" PGS 22 262.

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1 The base of the ultrapower

We fix M, is a countable transitive model of $\mathbf{ZF} + DC$, an *index set* $I \in M$, and an ultrafilter $U \in M$ over I. The variable i is assumed to range over I.

We want to define the ultrapower of M via U. The set $F_0 = M^I \cap M$ of all functions $f \in M$, $f : I \longrightarrow M$ may be not sufficient in the case when M does not satisfy AC or, at least, a rather strong partial form of AC, because of a lack of choice functions necessary to prove the Łoś theorem.

One can, of course, extend U to an ultrafilter over I in the external sense and then define the ultrapower from outside, using *all* functions mapping I into M as the base of the ultrapower. This, however, would be rather useless since the control over the ultrapower from within M should, in general, have been lost. The principal idea of our approach is to use a generically defined set F of functions mapping I into M which includes F_0 and contains functions which may not belong to M, but not *all* of them.

The following definition lists the properties of F essential to guarantee that M still keeps the control over the ultrapower. Take notice that the definition depends on M and I but does not involve U.

Definition 1 A set of functions F is called M, U*-adequate* iff

1. Every $f \in F$ maps I into some $X = X_f \in M$ which depends on f.
2. [M-measurability] Let $f_1, ..., f_n \in F$ and $E \in M$. Then the set $\{i : E(f_1(i), ..., f_n(i))\}$ belongs to M.
3. [Regularity][1] For any $f \in F$ there exists $f' \in M^I \cap M$ such that for any i, if $f(i)$ is an ordinal then $f(i) = f'(i)$.
4. [Choice] Let $f_1, ..., f_n \in F$ and $W \in M$. There exists $f \in F$ such that

$$\forall i \, [\exists x \, W(i, f_1(i), ..., f_n(i), x) \longrightarrow W(i, f_1(i), ..., f_n(i), f(i))].$$

Theorem 2 *There exists an* M, U*-adequate countable set* F.

Proof. F is constructed via forcing. We let P be the set (*class* in the sense of M) of forcing conditions $p \in M$ such that, for some $r = r(p)$:

1. $p = \langle p_1, ..., p_r \rangle$ where each $p_k \in M$ is a function defined on I.

[1]The crucial property to verify wellfoundedness in the countably additive case.

2. Every $p_k(i)$ is nonempty.
3. Every $x \in p_k(i)$ has the form $\langle x_1, ..., x_k \rangle$.
4. if $\langle x_1, ..., x_k, x_{k+1} \rangle \in p_{k+1}(i)$ then $\langle x_1, ..., x_k \rangle \in p_k(i)$. [2]

We say that $p \in P$ is *stronger* than q iff, 1st, $r(q) \leq r(p)$, and 2nd, $p_k(i) \subseteq q_k(i)$ for all $k \leq r(q)$ and all i .

Let $G \subseteq P$ be a P-generic set over M; the genericity here is understood in the sense that G intersects every dense $D \subseteq P$ which is $\in$-definable with parameters in M .

Fact 1 *For any $r \geq 1$, there exists unique function f_r defined on I such that $\langle f_1(i), ..., f_r(i) \rangle \in \bigcap_{p \in G,\, r(p) \geq r} p_r(i)$ for all $i \in I$.*

Proof. For all r and i, the set $\{p \in P : \operatorname{card} p_r(i) = 1\}$ is dense in P. □

We claim that the family of functions $F = \{f_r : r \geq 1\}$ is as required.

Fact 2 *For any $f = f_r \in F$ there exists $X = X_r \in M$ such that* $\operatorname{ran} f_r \subseteq X$.

Proof. For a suitable $p \in G$, therefore $\in M$, we have

$$\forall i\, (\langle f_1(i), ..., f_r(i) \rangle \in p_r(i)).$$

This solves the problem. □

Fact 3 *The family F is M-measurable.*

Proof. Let $E \in M$, $r_1, ..., r_n \geq 1$, $r' = \max\{r_1, ..., r_n\}$. We have to prove that the set

$$A = \{i : E(f_{r_1}(i), ..., f_{r_n}(i))\}$$

belongs to M. To prove this, let $p \in P$ be arbitrary; we show how a stronger condition q which is in some relation to E can be defined.

One can assume w.l.o.g. that $r = r(p) \geq r'$; otherwise expand p by

$$p_k(i) = \{\langle x_1, ..., x_r, \underbrace{\emptyset, \emptyset, ..., \emptyset}_{k-r \text{ emptysets}} \rangle : \langle x_1, ..., x_r \rangle \in p_r(i)\}$$

[2]By this requirement, each p_k contains all the information which all $p_{k'}$, $k' < k$, do contain.

for all k, $r < k \le r'$, and all i.

In this assumption, we put

$$\begin{aligned}
p'_r(i) &= \{\langle x_1, ..., x_r\rangle \in p_r(i) : E(x_{r_1}, ..., x_{r_n})\}; \\
J &= \{i \in I : p'_r(i) \neq \emptyset\}; \\
q_r(i) &= p'_r(i) && \text{provided } i \in J; \\
q_r(i) &= p_r(i) \setminus p'_r(i) && \text{provided } i \notin J; \\
q_k(i) &= p_k(i) && \text{provided } k < r.
\end{aligned}$$

Evidently $q \in P$ is stronger than p. Therefore one may assume w.l.o.g. that a condition q of this type belongs to G. But then $A = J$ belongs to M, as required. □

Fact 4 *The family F satisfies the Choice property.*

Proof. Thus let $f_{r_1}, ..., f_{r_n} \in F$ and $W \in M$. We have to find $f \in F$ such that

$$\forall\, i\, [\exists\, x\, W(i, f_{r_1}(i), ..., f_{r_n}(i), x) \longrightarrow W(i, f_{r_1}(i), ..., f_{r_n}(i), f(i))]. \qquad (1)$$

To prove this, let $p \in P$ be arbitrary; we again show how a stronger condition q which is in some relation to W can be defined. One again may assume that $r = r(p) \ge r' = \max\{r_1, ..., r_n\}$. Let

$$J = \{i : \exists\, x\, W(i, f_{r_1}(i), ..., f_{r_n}(i), x)\}$$

By Replacement, not Choice, and Fact 2 there exists a set $X \in M$ such that

$$\forall\, i \in J\, \exists\, x \in X\, W(i, f_{r_1}(i), ..., f_{r_n}(i), x).$$

We then put $q_k(i) = p_k(i)$ for all $k \le r$ and all i and, separately,

$$\begin{aligned}
q_{r+1}(i) &= \{\langle x_1, ..., x_r, x\rangle : \langle x_1, ..., x_r\rangle \in p_r(i)\ \& \\
&\qquad x \in X\ \&\ W(i, f_{r_1}(i), ..., f_{r_n}(i), x)\} && \text{provided } i \in J \\
q_{r+1}(i) &= \{\langle x_1, ..., x_r, \emptyset\rangle : \langle x_1, ..., x_r\rangle \in p_r(i)\} && \text{provided } i \in I \setminus J
\end{aligned}$$

Then $q \in P$ and $q \ge p$. Moreover, by the construction, (1) holds for $f = f_{r+1}$, as required. □

Fact 5 *The regularity property 1.3 holds for $\mathcal{F}$.*

Proof. Let $f = f_n \in F$; we have to find a function $f' \in M$, $f' : I \longrightarrow M$, such that $f(i) = f'(i)$ whenever $f(i)$ is an ordinal. Let, as above, $p = \langle p_1, ..., p_r \rangle \in P$; $r \geq n$. Let

$$J = \{i : \exists\, x = \langle x_1, ..., x_r \rangle \in p_r(i)\ (x_n \in \text{Ord})\}.$$

For $i \in J$, let $\alpha(i)$ be the least ordinal α equal to x_n for an n-tuple $x = \langle x_1, ..., x_r \rangle \in p_r(i)$, and

$$q_r(i) = \{x = \langle x_1, ..., x_r \rangle \in p_r(i) : x_n = \alpha(i)\}.$$

For $i \notin J$, we set $q_r(i) = p_r(i)$. For $m < r$, let $q_m = p_m$. Thus $q \in P$, $q \geq p$. Let, finally, $f'(i) = \alpha(i)$ for $i \in J$ and (the nonessential case), say, $f'(i) = p_r(i)$ for $i \notin J$.

Then some of the conditions q of such kind belongs to G by genericity. But if $q \in G$ then $f(i) = \alpha(i) = f'(i)$ for $i \in J$, and $f(i)$ is not an ordinal for $i \notin J$, as required. □ □

2 The enlarged ultrapower

We use an M, U-adequate set of functions F as the base for the ultrapower construction. First of all, we use the *generalized quantifier* notation

$$\mathbf{U}i\ \Phi(i) \quad \text{to express} \quad \{i : \Phi(i)\} \in U.$$

Definition 3 Let f, $g \in F$. We define

$$f\ {}^*\!= g \qquad \text{if and only if} \qquad \mathbf{U}i\ (f(i) = g(i))$$

$$f\ {}^*\!\in g \qquad \text{if and only if} \qquad \mathbf{U}i\ (f(i) \in g(i))$$

$\mathcal{F}$ denotes the structure $\langle F; {}^*\!=, {}^*\!\in \rangle$, the *enlarged ultrapower* of M.

Thus we do not factorize the ultrapower via U. Therefore the $\mathcal{F}$-equality ${}^*\!=$ is actually the equivalence on $\mathcal{F}$ which satisfies all logical properties of equality with respect to ${}^*\!\in$. All the following exposition can be easily changed to deal with ${}^*\!=$-classes rather that elements of F themselves.

We let F*-formula* and M*-formula* mean: $\in$-formula having elements of F, or respectively M, as parameters. Let $\Phi(f, g, ...)$ be such a formula, so that $f, g, ... \in F$. We introduce

$$\Phi(f, g, ...)[i] \qquad \text{to denote the formula} \qquad \Phi(f(i), g(i), ...).$$

Thus $\Phi[i]$ is an M-formula.

Proposition 4 *Let* Φ *be an* F *-formula,* $E \in M$. *Then the set* $J = \{i : \Phi^M[i]\}$ *belongs to* M *.*

Proof. This is a reformulation of Property 1.1. □

This is very important: the involved functions may not belong to M, but the related truth sets always belong to M. In particular, for any closed F-formula Φ, either $\mathbf{U}i\ \Phi^M[i]$ or $\mathbf{U}i\ \neg\Phi^M[i]$. (^{M} denotes the relativization to M .)

Theorem 5 [Łoś] *Let* Φ *be a closed* F *-formula. Then*

$$\Phi \text{ is true in } \mathcal{F} \qquad \text{iff} \qquad \mathbf{U}i\ \Phi[i] \text{ is true in } M.$$

Proof. Just follow the canonical patterns (see e.g. Jech [1]). The essential step $\exists$ is carried out using the Choice property 1.4. Assume, indeed, that

$$\mathbf{U}i\ \exists x\ \Phi(f_1, ..., f_n, x)[i] \quad \text{is true in} \quad M; \tag{2}$$

where $f_1, ... f_n \in F$. We have to find $f \in F$ such that

$$\mathbf{U}i\ \Phi^M(f_1, ..., f_n, f)[i]. \tag{3}$$

By Property 1.1, there exist sets $X_1, ..., X_n \in M$ such that

$$\forall i\ [f_1(i) \in X_1\ \&\ ...\ \&\ f_n(i) \in X_n].$$

Therefore, by the **ZFC** Collection in M, there exists a set $X \in M$ satisfying

$$\forall i\ [\exists x\ \Phi^M(f_1, ..., f_n, x)[i] \longrightarrow \exists x \in X\ \Phi^M(f_i, ..., f_n, x)[i]].$$

Let $W = \{\langle i, x_1, ..., x_n, x\rangle \in I \times X_1 \times ... \times X_n \times X : \Phi^M(i, x_1, ..., x_n, x)\}$. By Property 1.4, there exists $f \in F$ such that

$$\forall i\ [\exists x\ W(i, f_1(i), ..., f_n(i), x) \longrightarrow W(i, f_1(i), ..., f_n(i), f(i))].$$

The set $\{i : \exists x\ W(i, f_1(i), ..., f_n(i), x)\}$ belongs to M by Proposition 4. Therefore, using (2), we obtain (3) for f . □

We note that $M^I \cap M \subseteq F$. Indeed, let $g \in M$, $g : I \longrightarrow M$. Applying Property 1.4 to the function $f_1(i) = \{g(i)\}$, we easily obtain the result.

Definition 6 Those elements of F which belong to M are called *representable*.

In particular, if $x \in M$, then the function $^*x$, defined on I by $^*x(i) = x$, all i, belongs to F.

Proposition 7 $x \longmapsto {}^*x$ *is an elementary embedding* M *into* $\mathcal{F}$.

Proof. Use Theorem 5, as usual. □

We treat M as $\langle M; =, \in \rangle$, of course.

Theorem 8 *Let* U *be countably additive in* M. *Then* $\mathcal{F}$ *is wellfounded* [3].

Proof. In other words we have to prove that the class of all $\mathcal{F}$-ordinals is wellordered. By the Regularity property 1.3 and Theorem 5 it suffices to prove that the representable part

$$\{f \in M^I \cap M : \mathcal{F} \models f \text{ is an ordinal}\} = \{f \in M^I \cap M : \mathbf{U}i\ (f(i) \in \mathrm{Ord})\}$$

is wellfounded. But this follows from DC and the assumed countable additivity of U in M. □

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[1] T. Jech, *Set theory*, Academic Press, New York, 1978.

[2] M. Spector, Extended ultrapowers and the Vopenka – Hrbáček theorem without choice. *J. Symbolic Logic*, 1991, 56, no 2, 592 – 607.

[3] In the ground universe. Of course, $\mathcal{F}$ is wellfounded from inside, that is, is a model of the **ZF** Regularity axiom, by Proposition 7.

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