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A note on modal logics for partial belief

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To the memory of Mike Clarke

Abstract

We propose modal logics to reason about degrees of belief. These logics can be used to represent a static state of belief. We also give our picture of dynamics of belief and comment on possibilities to formalize it.

1 Introduction

A modal logic where modalities express partial believing a statement can represent incomplete knowledge more adequately than the standard epistemic logic. The latter cannot express subtleties like 'I believe neither A nor $\neg A$ completely, but I have some evidence which supports A (0.3) and some evidence which supports $\neg A$ (0.2)'. The numbers in brackets may correspond, for example, to degrees of belief in the sense of belief functions theory (cf. Shafer (1976) and Smets & Kennes (1994)). We shall discuss here some logics able of representing degrees of belief.

A major contribution in this field was made by Fagin, Halpern and Megiddo (1988). They constructed two logics, AX' and AX'_{FO}, which are sound and complete both with respect to probabilistic structures (where numbers assigned to statements correspond to the values assigned by an inner measure induced by a probability measure [1]) and to Dempster-Shafer (DS) structures.

Definition 1 *A DS structure is a quadruple* $< W, \Theta, Bel, V >$*, where* W *is a non-empty set (of possible worlds),* Θ *is an algebra of subsets of* W *(such that* $W \in \Theta$*)*[2]*, and* Bel *is a belief function, i.e.* $Bel : \Theta \longrightarrow [0, 1]$ *satisfies*

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[1] **Definition.** Let P be a probability measure on an algebra Θ of subsets of W. Then the *inner probability measure* P_* on 2^W induced by P is defined by

$$P_*(A) = max(P(X) : X \subseteq A, X \in \Theta)$$

[2] In the cited article $\Theta = 2^W$.

B1 $Bel(\emptyset) = 0$;

B2 $Bel(Q_1 \cup \ldots \cup Q_n) \geq \Sigma_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} Bel(\cap_{i \in I} Q_i)$

B3 $Bel(W) = 1$;

and V is a function which assigns propositional variables subsets of W (intuitively, V says in which worlds a propositional variable is true).

The intuitive meaning of belief functions can be explained in different ways. One possibility is to relate them to probability functions. B2 can be understood then by analogy with inclusion-exclusion laws for probabilities. Inequality replacing strict equality which holds for probabilities corresponds to the fact that one may believe in a disjunction, for example, $A \vee \neg A$, without believing in either of the disjuncts. Inner probabilities also have this property. As the work of Fagin, Halpern and Megiddo shows, even a considerably expressive logical language cannot distinguish between belief functions and inner probability measures (at the static level, i.e., without analysing belief revision). However, the contexts where belief functions and probabilities can be meaningfully applied, seem to be very different. This becomes especially clear when conditioning, i.e., changing degrees of belief after learning new information, is considered (cf. section 4).

The language of AX' consists of propositional variables, Boolean connectives, constants denoting integer numbers, functional symbols $+$ and $\times$, a predicate symbol $\geq$, and a functional symbol ω. If φ is a propositional formula, $\omega(\varphi)$ denotes the weight (degree of belief or inner measure) assigned to φ. Propositional formulas are defined in a standard way; a *weight formula* is a Boolean combination of *basic weight formulas* of the form

$$a_1\omega(\varphi_1) + \ldots + a_k\omega(\varphi_k) \geq c,$$

where the a_i and c are integers and φ_i propositional formulas. A conjunction of a propositional formula and a weight formula is not a well formed formula; only weight formulas are evaluated in a model as true or false. A basic weight formula as described above is true in a DS structure if

$$a_1Bel([\varphi_1]) + \ldots + a_kBel([\varphi_k]) \geq c,$$

where $[\varphi_i]$ is the set of worlds assigned to φ_i (the function assigning sets of worlds to propositional formulas is an obvious extension of V).

The language of AX'_{FO} contains in addition variables ranging over real numbers and quantifiers $\forall$ and $\exists$. The class of weight formulas is extended accordingly.

In both logics the reasoning about degrees of belief is axiomatized by the same set of axioms: [3]

W1 $\omega(\varphi) \geq 0$;

W2 $\omega(\top) = 1$;

W3 $\omega(\perp) = 0$;

W4 if $\varphi \equiv \psi$ is a propositional tautology, then $\omega(\varphi) \equiv \omega(\psi)$;

W5 $\omega(\varphi_1 \vee \ldots \vee \varphi_n) \geq \Sigma_{I \subseteq \{1,\ldots,n\}, I \neq \emptyset} (-1)^{|I|+1} \omega(\wedge_{i \in I} \omega(\varphi_i))$.

[3]We have changed the original numbering of the axioms.

The difference between the two logic is in axiomatizing numerical reasoning: AX' contains seven axioms to reason about linear inequalities, and AX'_{FO} - eighteen axioms to reason about real numbers.

Voorbraak [1993] transformed AX'_{FO} into modal logic MO_{FO}. He showed that any weight formula of AX'_{FO} containing $\omega(\varphi_1),\ldots,\omega(\varphi_n)$, is equivalent to a formula of the form

$$\forall x_1 \ldots \forall x_n(\omega(\varphi_1) = x_1 \wedge \ldots \wedge \omega(\varphi_n) = x_n \rightarrow \Psi) \wedge \exists x_1 \ldots x_n(\omega(\varphi_1) = x_1 \wedge \ldots \wedge \omega(\varphi_n) = x_n)$$

where Ψ is a purely arithmetical formula (containing only constants and variables denoting numbers, $+, \times, \geq$ and Boolean connectives). This suggested that AX'_{FO} can be axiomatized in the language containing modal operators P_x (with the meaning 'the degree of belief in ... equals x'), for every variable x. For example, the axioms W1 - W5 above become

MO1 $\forall x(P_x\varphi \rightarrow x \geq 0) \wedge \exists x P_x \varphi$;

MO2 $\forall x(P_x\top \rightarrow x = 1)$;

MO3 $\forall x(P_x\bot \rightarrow x = 0)$;

MO4 $\vdash\ \varphi \equiv \psi \Rightarrow \vdash\ \forall x \forall y(P_x\varphi \wedge P_y\psi \rightarrow x = y)$;

MO5 $\forall x, x_1, \ldots x_m(P_x(\varphi_1 \vee \ldots \vee \varphi_n) \wedge P_{x_1}(\wedge_{i \in I_1}\varphi_i) \wedge \ldots \wedge P_{x_m}(\wedge_{i \in I_m}\varphi_i) \rightarrow x \geq \Sigma_{k=1}^{m}(-1)^{|I_k|+1}x_k)$, where $m = 2^n - 1$, and $\{I_1,\ldots,I_m\} = \mathcal{P}(\{1,\ldots,n\})\backslash\emptyset$.

In addition to these axioms, MO_{FO} contains two more axioms which allow to eliminate nestings of modal operators (the nestings are allowed, but they are always eliminable).

Other axiom schemata of MO_{FO} are the same as those of AX_{FO}.

We propose some other logics with modal operators corresponding to degrees of belief which do not contain explicit reasoning about numbers. Next two sections describe logics for representing a *static* state of belief. The last section deals with *dynamics* of belief.

2 Belief operators

The models for our logics will be the same Dempster - Shafer structures as before, *but without the assumption that the belief in tautology equals 1*, that is, with *Bel* not necessarily satisfying **B3**. To stress the difference we denote the belief function which is required to satisfy only **B1** and **B2** as *bel*. The reason for preferring such structures will become clear in section 4.

Consider a propositional language which contains operators $\Box_r^=$, where $\Box_r^=\varphi$ means 'the degree of belief in φ equals r' and $\Box_r^\geq$, with the meaning 'the degree of belief ... is at least r', for every real r; we shall denote this language $L(\Box_r)$. The definition of a propositional well formed formula (wff) is extended by: if φ is a wff, then $\Box_r^=\varphi$ and $\Box_r^\geq\varphi$ are wff's.

A model $M =< W, \Theta, bel, V >$, makes formulas true or false in accordance with the following definition:

i $M, w \models p \Leftrightarrow w \in V(p)$;

ii $M, w \models \neg\varphi \Leftrightarrow M, w \not\models \varphi$;

iii $M, w \models \varphi \wedge \psi \Leftrightarrow M, w \models \varphi\ \&\ M, w \models \psi$;

iv $M, w \models \Box^{=}_{r}\varphi \Leftrightarrow bel([\varphi]) = r$, where $[\varphi] = \{v \in W : M, v \models \varphi\}$.

v $M, w \models \Box^{\geq}_{r}\varphi \Leftrightarrow bel([\varphi]) \geq r$, where $[\varphi] = \{v \in W : M, v \models \varphi\}$.

Further, we assume that Θ contains all definable subsets of W (i.e. all subsets Q such that there is a formula φ for which $Q = [\varphi]$).

A formula φ is true in a model if it is true in all possible worlds of this model; a formula is valid ($\models \varphi$) if it is true in all models. Finally, a formula is satisfiable if there is a model and a possible world in this model where the formula is true. It is easy to see that φ is valid if, and only if, $\neg\varphi$ is not satisfiable.

Observe that the $\Box^{=}_{r}$ operator is not definable via the $\Box^{\geq}_{r}$ operator (unlike in probabilistic logic). In probabilistic logic, $P^{\leq}_{r}\varphi =_{df} P^{\geq}_{1-r}\neg\varphi$ (which does not hold for belief functions in general), and $P^{=}_{r}\varphi =_{df} P^{\leq}_{r}\varphi \wedge P^{\geq}_{r}\varphi$. It is however possible to define $\Box^{=}_{r}$ via $\Box^{\geq}_{r}$ in an infinitary language (see below).

Another point is that in our language we allow nestings of operators: one can write expressions like $\Box^{=}_{1/2}\Box^{=}_{1/3}p$. However, since the truth of a modal formula in a model does not depend on a possible world, a modal formula is either true in all worlds, or false in all worlds. Namely, if $M, w \models \Box^{=}_{1/3}p$, it means that $bel([p]) = 1/3$. Then all worlds satisfy $\Box^{=}_{1/3}p$, i.e., $[\Box^{=}_{1/3}p] = W$. This implies $bel([\Box^{=}_{1/3}p]) = bel(W)$; $\Box^{=}_{1/2}\Box^{=}_{1/3}p$ is true in M, w if and only if $\Box^{=}_{1/2}\top$ is.

Analogously, if $M, w \not\models \Box^{=}_{1/3}p$, then $bel([p]) \neq 1/3$. The set of worlds satisfying $\Box^{=}_{1/3}p$ is empty, therefore $\Box^{=}_{1/2}\Box^{=}_{1/3}p$ is true in M, w if and only if $\Box^{=}_{1/2}\bot$ is (which never happens, by the way). To summarize,

$$\Box^{=}_{1/2}\Box^{=}_{1/3}p \equiv (\Box^{=}_{1/3}p \wedge \Box^{=}_{1/2}\top) \vee (\neg\Box^{=}_{1/3}p \wedge \Box^{=}_{1/2}\bot).$$

In general, the nestings can be eliminated using the following valid principles ($\Box_r$ stands for $\Box^{\geq}_{r}$ or $\Box^{=}_{r}$):

NF1 $\Box_r(\Box_s\varphi \wedge \psi) \equiv (\Box_s\varphi \wedge \Box_r\psi) \vee (\neg\Box_s\varphi \wedge \Box_r\bot)$

NF2 $\Box_r(\neg\Box_s\varphi \wedge \psi) \equiv (\Box_s\varphi \wedge \Box_r\bot) \vee (\neg\Box_s\varphi \wedge \Box_r\psi)$

analogous to the ones used in Voorbraak (1993).

The nestings of modal operators can be made nontrivial by making *bel* dependent on a possible world (cf. Fattorosi-Barnaba and Amati (1987) for probabilistic operators).

It is unlikely that a finite axiomatization for $L(\Box_r)$ can be found by standard means. The situation resembles the one existing in the probabilistic logic, where the corresponding problem for probability operators has not been solved yet. The following axiom system is obviously sound:

Ax0 propositional tautologies and modus ponens;

Ax1 $\Box^{=}_{0}\bot$

Ax2 $\Box^{\geq}_{1}(\varphi \rightarrow \psi) \rightarrow (\Box^{\geq}_{r}\varphi \rightarrow \Box^{\geq}_{s}\psi)$ if $r \geq s$

Ax3 if $\vdash \varphi \rightarrow \psi$, then $\vdash \Box^{\geq}_{r}\varphi \rightarrow \Box^{\geq}_{s}\psi$ if $r \geq s$;

Ax4 $\Box^{\geq}_{1}\varphi \rightarrow \Box^{=}_{1}\varphi$;

Ax5 $\Box^{=}_{r}\varphi \rightarrow \neg\Box^{=}_{s}\varphi$ if $r \neq s$;

Ax6 $\Box^{=}_{r}\varphi \to \Box^{\geq}_{r}\varphi$;

Ax7 $\Box^{=}_{r}\varphi \to \neg\Box^{\geq}_{s}\varphi$ if $s > r$;

Ax8(2) $\Box^{=}_{r_1}\varphi_1 \wedge \Box^{=}_{r_2}\varphi_2 \wedge \Box^{=}_{s}(\varphi_1 \wedge \varphi_2) \to \Box^{\geq}_{r_1+r_2-s}(\varphi_1 \vee \varphi_2)$;

Ax8(3)

$\vdots$

where $Ax8(n)$ is the analogue of Voorbraak's MO5 for n.

In case where *bel* does not depend on a possible world the normal form principles allowing to eliminate nestings, **NF1** and **NF2**, should be added.

In the class of structures where **B3** holds, $\Box^{=}_{1}\top$ holds (then Ax3 can be replaced by $\vdash \varphi \Rightarrow \vdash \Box^{=}_{1}\varphi$).

However, the resulting systems are probably still incomplete. One can easily show (the argument is analogous to the one used for probability operators in van der Hoek (1992), for example) that it is not compact: there are infinite finitely satisfiable (therefore consistent) sets which are not satisfiable: for example

$$\{\Box^{\geq}_{1-1/n}\varphi : n \in \mathbf{N}\} \cup \{\neg\Box^{=}_{1}\varphi\}.$$

This fact closes the opportunity to prove completeness by the method of canonical models. What we need is to be able to express the fact that for every formula there exists a degree of belief assigned to this formula. This could be expressed in an infinitary language: $\bigvee_{r\in[0,1]} \Box^{=}_{r}\varphi$.

In probabilistic logic where analogous problem appears, there are two approaches which can be both applied here: one (cf. Keisler (1985)) accepts infinite formulas; the other (cf. Fattorosi-Barnaba & Amati (1987)) restricts the range of the probability function to a finite set $F \subseteq [0,1]$.

Assume that the range of belief function is restricted to a set $F = \{0, r_1, \ldots, r_n, 1\}$. The class of models where this holds is axiomatized by the axioms above plus

Ax9 $\bigvee_{r\in F} \Box^{=}_{r}\varphi$.

The proof is analogous to the proof in Fattorosi-Barnaba & Amati (1987).

Completeness for the infinitary language is much more involved; probably it can be proved for the case when infinite disjunctions and conjunctions are taken over *admissible sets* (cf. Keisler (1985)). It is interesting that in the language with arbitrary infinite disjunctions and conjunctions, $\Box^{\geq}_{r}$ and $\Box^{=}_{r}$ are interdefinable:

$$\Box^{\geq}_{r}\varphi =_{df} \bigvee_{s\geq r} \Box^{=}_{s}\varphi$$

$$\Box^{=}_{r}\varphi =_{df} \bigwedge_{s<r} \Box^{\geq}_{s}\varphi \wedge \bigwedge_{s>r} \neg\Box^{\geq}_{s}\varphi.$$

The latter definition could have been written as

$$\Box^{=}_{r}\varphi =_{df} \bigwedge_{s\leq r} \Box^{\geq}_{s}\varphi \wedge \bigwedge_{s>r} \neg\Box^{\geq}_{s}\varphi;$$

but this definition does not imply $\bigvee_{r\in[0,1]} \Box^{=}_{r}\varphi$. In particular, from $\bigwedge_{n\in\mathbf{N}} \Box^{\geq}_{1-1/n}\varphi$ it does not follow that $\Box^{=}_{1}\varphi$.

3 Logic to reason about mass functions

It is well known that any belief function has a unique mass function which corresponds to it. Given a belief function, one can compute the mass function, and vice versa. For some reason, mass functions, although having less intuitive meaning, are more suitable for performing conditioning. Modal operators corresponding to mass functions, as we shall see in this section, allow for an easy completeness result (because the masses assigned to different propositions do not interact, unlike the beliefs: cf. Ax8). Therefore we start with considering *mass structures* instead of DS structures.

Definition 2 *A mass structure* $M = < W, \Theta, m, V >$, *where* W *and* V *are as before and* $m : \Theta \longrightarrow [0,1]$ *satisfies the following condition:*

M1 $\Sigma_{A \in \Theta}\ m(A) = 1$

Further on we consider only *full* mass structures:

Definition 3 *A mass structure is called full if for every possible assignment of values to the propositional variables in the language, there is a possible world in* W *which corresponds to this assignment.*

In a full mass structure, if φ and ψ are not logically equivalent, $[\varphi] \neq [\psi]$.

It is easy to see that defining a mass function on a set of possible worlds in a full model is equivalent to defining it on the Lindenbaum algebra of formulas.

The relation between mass function and belief function is as follows:

$$bel(A) = \Sigma_{\emptyset \neq B \subseteq A} m(B).$$

The condition on mass function corresponding to **B3** for belief functions is

M2 $m(\emptyset) = 0$.

Under this condition the equation above becomes

$$Bel(A) = \Sigma_{B \subseteq A} m(B).$$

The language $L(m)$ of the *basic belief logic* consists of a set of propositional letters $p_1, \ldots, p_n, \ldots$, Boolean connectives and modal operators $m_r^=$ and $m_r^\geq$, for every real number r. The intended meaning of $m_r^= \varphi$ is 'given the available evidence, the amount of belief specifically supporting φ is r', analogously for $m_r^\geq$. The difference with the degree of belief is that m corresponds to support given *exactly* to φ, and not to any proposition which implies φ. That is why m, unlike *bel*, need not be monotone, and may assign 0 to tautologies.

A well formed formula is defined inductively as follows: a propositional variable is a wff; if φ and ψ are wff's, then $\neg\varphi$ and $\varphi \wedge \psi$ are wff's; if φ is a wff which does not contain modal operators, $m_r \varphi$ is a wff, where $m_r \varphi$ is $m_r^= \varphi$ or $m_r^\geq \varphi$, with the following meaning:

vi $M, w \models m_r^= \varphi \Leftrightarrow m([\varphi]) = r$;

vii $M, w \models m_r^\geq \varphi \Leftrightarrow m([\varphi]) \geq r$.

We could have allowed nestings of modal operators, but it makes the story too complicated. Besides, the nestings are again trivial since the truth of a modal formula again does not depend on a world.

Definition 4 *The basic belief logic BBL is the smallest class of formulas closed with respect to the following schemata of axioms and inference rules:*

A0 *all propositional tautologies;*

A1 $m_0^{\geq}\varphi$;

A2 $m_r^{=}\varphi \rightarrow \neg m_s^{\geq}\varphi$ *if* $s > r$;

A3 $m_r^{=}\varphi \rightarrow \neg m_s^{=}\varphi$ *if* $r \neq s$;

A4 $m_r^{=}\varphi \rightarrow m_s^{\geq}\varphi$ *if* $s \leq r$;

A5 $m_r^{\geq}\varphi \rightarrow m_s^{\geq}\varphi$ *if* $s \leq r$;

A6 $\bigwedge_i m_{r_i}^{\geq}\varphi_i \rightarrow \bigwedge_i m_{r_i}^{=}\varphi_i$ *where* φ_i *are pairwise nonequivalent formulas and* $\Sigma_i r_i = 1$;

R1 *if* $\vdash \varphi$ *and* $\vdash \varphi \rightarrow \psi$, *then* $\vdash \psi$;

R2 *if* $\vdash \varphi \equiv \psi$, *and* φ, ψ *are a propositional formulas, then* $\vdash m_r^{=}\varphi \equiv m_r^{=}\psi$

R3 *if* $\vdash \varphi \equiv \psi$, *and* φ, ψ *are a propositional formulas, then* $\vdash m_r^{\geq}\varphi \equiv m_r^{\geq}\psi$

For the structures where $m(\emptyset) = 0$ holds (normalized structures), the following axiom should be added: $m_0^{=}\bot$. This will not influence the following result:

Theorem 1 *BBL is sound, that is, for every formula* φ, $\vdash \varphi$ *implies* $\models \varphi$.

Proof. It suffices to show that all axioms are valid and the inference rules preserve validity. The only nontrivial case is A6. Here we really make use of the fact that the models are full. If not all possible valuations are present, a formula like $m_{1/2}^{\geq}p \wedge m_{1/2}^{\geq}q \wedge \neg m_{1/2}^{=}p \wedge \neg m_{1/2}^{=}q$ is satisfiable: for example, it is true in a model with just one world w which satisfies both p and q, with $m(\{w\}) = 1$. But in full models, nonequivalent formulas define different subsets of W, and the sum of masses assigned to these subsets may not be greater than 1.

□

Theorem 2 *BBL is complete: for every* φ, $\models \varphi$ *implies* $\vdash \varphi$.

Proof. To prove that in $BBL \models \varphi \Rightarrow \vdash \varphi$, we show that $\not\vdash \varphi \Rightarrow \not\models \varphi$; namely, if a formula is not provable, then its negation is satisfiable. This amounts to proving that every consistent formula has a model where it is satisfied in some world.

Let ψ be an arbitrary consistent formula. By propositional reasoning, ψ can be written as a disjunction of conjunctions, where each disjunct is a conjunction of propositional variables, their negations, modal formulas and their negations. If ψ is consistent, then at least one of the disjuncts is consistent. If it is satisfiable, then ψ is also satisfiable. So, it suffices to show that any consistent conjunction as described above is satisfiable.

Assume that ψ is of the form

$$\bar{p_{i1}} \wedge \ldots \wedge \bar{p_{ik}} \wedge m_{r_1}\varphi_1 \wedge \ldots \wedge m_{r_l}\varphi_l \wedge \neg m_{r_{l+1}}\varphi_{l+1} \wedge \ldots \wedge \neg m_{r_m}\varphi_m,$$

where m_r is either $m_r^{=}$ or $m_r^{\geq}$. Due to R2 and R3 we may assume that all formulas φ_i, $1 \leq i \leq m$, are in some normal form (so that there are no two equivalent formulas which are written differently).

For every formula φ_i, $1 \leq i \leq l$, we can leave only one conjunct $m^{\geq}_{r_i}\varphi_i$ with the maximal r_i (other conjuncts follow from this one by A5), or, if $m^{=}_{r_i}\varphi_i$ occurs in ψ, leaving just this conjunct (all conjuncts of the form $m^{\geq}_{s}\varphi_i$, $s \leq r_i$ follow by A4, and there are no conjuncts of the form $m^{\geq}_{s}\varphi_i$ with $s > r_i$, otherwise ψ is inconsistent by A2). Further on we assume that for every φ_i, $1 \leq i \leq l$, there is only one conjunct of the form $m^{=}_{r_i}\varphi_i$ or $m^{\geq}_{r_i}\varphi_i$. We also assume that every formula φ_j, $1 \leq j \leq m$, occurs in ψ with a modal operator positively; we can always add $m^{\geq}_{0}\varphi_j$ by A1.

Analogously, for every formula φ_j, $l+1 \leq j \leq m$, we can assume that there is only one conjunct of the form $\neg m^{\geq}_{r_j}\varphi_j$ (take the smallest r_j and apply A5), and possibly some conjuncts of the form $\neg m^{=}_{s}\varphi_j$, with $s < r_j$.

ψ is satisfiable if there is at least one model such that

(a) at least one world in this models satisfies $\bar{p_{i1}} \wedge \ldots \wedge \bar{p_{ik}}$;

(b) in this model, $m([\varphi_1]) = r_1$ or $m([\varphi_1]) \in [r_1, \infty]$, ..., $m([\varphi_l]) = r_l$ or $m([\varphi_l]) \in [r_l, \infty]$;

(c) $m([\varphi_{l+1}]) \neq (<) r_{l+1}, \ldots, m([\varphi_m]) \neq (<) r_m$

(d) and m is a mass function: for every φ_j $1 \leq j \leq m$, there is only one value $m([\varphi_j])$ which is between 0 an 1, the sum of all masses is 1, and, for normalized structures, the mass of the empty set is 0.

Since ψ is consistent, $\bar{p_{i1}} \wedge \ldots \wedge \bar{p_{ik}}$ is also consistent, therefore in every full model there is at least one world satisfying it.

Conditions (b) and (c) together define an interval to which $m([\varphi_j])$, $1 \leq j \leq m$, belongs. It is a subinterval of $[0, 1]$. This interval is always nonempty: since $m^{\geq}_{r}\varphi_j$ and $\neg m^{\geq}_{s}\varphi_j$, $s \leq r$, can not occur in ψ (by A5). Sometimes it is a point (if $m^{=}_{r_j}\varphi_j$ is in ψ). Sometimes several points are excluded (corresponding to the conjuncts of the form $\neg m^{=}_{s}\varphi_j$). In any case, there is either one possible value, or infinitely many possible values for $m([\varphi_j])$.

A routine check shows that if one of the conditions (d), except for the summing to 1, is violated: for example, if one of the requirements of (b) is $m([\varphi_i]) = r$ and $m([\varphi_i]) = s$, $r \neq s$, then ψ is inconsistent (in the example, by A3).

Now let us show that **M1** can always be satisfied. There are three possibilities for the assignment of values in (b):

- $\Sigma_{i=1}^{i=l} r_i > 1$: then a contradiction is derivable by A6 and A2, therefore this case is excluded;

- $\Sigma_{i+1}^{i=l} r_i = 1$; then instead of $m([\varphi_i]) \in [r_i, \infty]$ we must have $m([\varphi_i]) = r_i$. Suppose this is excluded by one of the conditions in (c): then ψ is contradictory by A6. Since it is not, this case defines a unique mass assignment to the formulas.

- $\Sigma_{i+1}^{i=l} r_i < 1$; we can always assign the rest of the mass to some formula not among φ_j.

□

Operators for belief cannot be defined in $L(m)$. To write a definition we would have to find, for a given formula $\Box^{=}_{r}\varphi$, a formula in the language with mass operators which is true if, and only if, $\Box^{=}_{r}\varphi$ is true. This is not possible, because $bel([\varphi]) = r$ is consistent with infinitely many possible values of masses assigned to the subformulas of φ (except for the trivial case $r = 0$). The mass assigned to φ is determined uniquely only if $bel([\psi])$, for all nonequivalent subformulas ψ of φ, is known.

In a language with finitely many propositional variables and infinite disjunctions $\Box_r^=$ is however definable.

Let the language contain n propositional letters, $p_1, \ldots, p_n$, and all nonequivalent propositional formulas which can be written in this language are $\rho_1, \ldots, \rho_{2^{2^n}}$. Let $S(\varphi)$ be the set of nonequivalent propositional formulas which imply φ (except for $\perp$).

$$\Box_r^= \varphi =_{df} \bigvee_{<r_1,\ldots,r_m>} (m_{r_1}^= \rho_1 \wedge \ldots \wedge m_{r_m}^= \rho_m),$$

where $r_i \in [0,1]$ and $\Sigma r_i = r$ $(1 \leq i \leq m)$, and $\{\rho_1, \ldots, \rho_m\} = S(\varphi)$.

We must acknowledge that the problem can be solved in the languages of Fagin et al (1988) and Voorbraak (1993) much easier: they can write

$$bel(\varphi) = \Sigma m(\rho_i) \qquad \rho_i \in S(\varphi)$$

or

$$\Box_r^= \varphi =_{df} \forall r_1 \ldots \forall r_m (m_{r_1}^= \rho_1 \wedge \ldots \wedge m_{r_m}^= \rho_m \rightarrow r_1 + \ldots + r_m = r).$$

4 Dynamics

4.1 Cognitive changes

The difference between reasoning with inner probabilities and reasoning with belief functions becomes clear when cognitive changes are considered, solutions being different.

Two forms of cognitive change can be described : belief change and world change (cf. Lea Sombe (1994)). In belief change, a rational agent held a belief about which world is the actual world w_0 and the new information puts some further constraint on which worlds can still be the actual world. The most classical information is that some worlds considered as possible candidates for w_0 must be eliminated as candidate.

In world change, evolving worlds are considered. Worlds are transformed into 'new' worlds as information is piling up and new information describes constraint on possible changes.

A typical example of the difference between belief and world changes is given by the banana/apple example (Lea Sombe 1994). The agent knows that a basket contains "a banana or an apple". For belief change, let the information be that a witness looked at the basket and said there is no banana in the basket. So the agent concludes there is an apple in the basket. For world change, let the information be that a banana-eater passed next to the basket and the agent knows that the banana-eater will eat any banana he sees. So the agent updates his knowledge about the content of the basket and he knows now that the basket is either empty or contains an apple.

Within quantified representation of belief, belief change corresponds to the conditioning process and world change to the imaging process (Lewis 1976).

In the context of transferable belief model (TBM) which is an interpretation of the Dempster-Shafer theory (cf. Smets and Kennes (1994)), conditioning is obtained by the application of Dempster's rule of conditioning. Let m/bel be the basic belief mass/belief function that represents the agent initial belief on W. Let the conditioning information be that the actual world does not belong to the complement of A relative to W. Then m/bel are changed into $m(\ |A)/bel(\ |A)$, the conditional basic belief mass/belief function over W given A, with

$$m(X|A) = \Sigma_{B \subseteq \overline{A}} m(X \cup B))$$

$$bel(X|A) = bel(X \cup \overline{A}) - bel(\overline{A})$$

Note that a non-zero mass can be assigned to the contradiction in case a subset of $\overline{A}$ was assigned a non-zero mass before conditioning. That is why we kept the possibility of mass functions with a positive mass on the empty set open. The justification of such rule can be found in Klawonn and Smets (1992), Smets (1993).

For the imaging process, the impact of the updating information $A \subseteq W$ is such that each $w \in W$ is mapped onto a new world in A. Let $\Gamma_A : W \to W$ be such mapping where $\Gamma_A(w) \in A$ and let $\Gamma_A(B) = \cup\{\Gamma_A(w) : w \in B\}$. If an agent has a belief about which world in W is the actual world, represented by the basic belief mass m / belief function bel, after learning that the worlds have been changed so that they must belong to $A \subseteq W$, m/bel are changed into the updated basic belief mass/belief function $m(\ ||A)/bel(\ ||A)$ with:

$$m(X||A) = \Sigma_{Y:Y\subseteq W,\Gamma_A(Y)=X} m(Y) \quad \text{for all } X \subseteq A,$$

and $m(X||A) = 0$ otherwise,

$$bel(X||A) = bel(\{Y : Y \subseteq W, \Gamma_A(Y) \subseteq X, \Gamma_A(Y) \neq \emptyset\}) \quad \text{for all } X \subseteq A.$$

(Assuming $\Gamma_A \neq \emptyset$ for all $w \in W$ seems reasonable but is unnecessary in the presentation as belief functions can be unnormalized).

Belief and world changes can both be described within the framework introduced above. Let the language contain n propositional letters, and ρ_i and $S(\varphi)$ be as before. Introduce a new binary modal operator $m_r^=(\ |\ )$, with $m_r^=(\psi|\varphi)$ meaning "the mass assigned to ψ after learning φ equals r". This operator is not definable via the unary one in a finite language. In the infinite language, the definition would have been

$$m_r^=(\psi|\varphi) =_{df} \bigvee_{r_1,\ldots,r_m:\Sigma r_i = r} (m_{r_1}^= \rho_1 \wedge \ldots \wedge m_{r_m}^= \rho_m),$$

where $\rho_1, \ldots, \rho_m$ are all nonequivalent propositional formulas such that $\vdash \rho_i \wedge \varphi \equiv \psi$. This set of formulas can be denoted as $[\psi]_{\equiv_\varphi}$. One can however easily write the definition of conditional operators in the languages of Fagin et al. (1988) and Voorbraak (1993).

The joint axiomatization of $m_r^=$ and $m_r^=(\ |\ )$ also seem to require infinitary means (as always when nontrivial addition comes into play!).

Conditional operators $m_r^=(\ |\varphi)$ and $m_r^\geq(\ |\varphi)$ alone (without the unconditional ones) can be axiomatized in a finitary language with arbitrary many propositional letters by the axioms A0 – A6 and the rules R1 – R3, replacing the expression "nonequivalent" by "nonequivalent given that φ is true".

The story is essentially the same for the update operators $m_r^=(\ ||\ )$ and $m_r^\geq(\ ||\ )$. In a language with finitely many propositional letters, define on the set of formulas a function which corresponds to Γ. Then it is easy to find a set $[\psi]_{\Gamma_\varphi}$ of all formulas ρ_i such that $\Gamma_\varphi(\rho_i) = \psi$. (In the case of conditioning, $\Gamma_\varphi(\rho_i) = \rho_i \wedge \varphi$.) Again, the definition of $m_r^=(\ ||\ )$ in our language requires an infinite disjunction; in the richer languages, "updated mass" can be defined.

5 Conclusion

We show that reasoning about quantified beliefs in the sense of transferable belief model can be represented in a simple modal logic.

The logic to reason about mass functions has a sound and complete axiomatization. In other cases the systems are sound, but in order to achieve completeness one has either to allow infinite formulas or to restrict the range of belief function to a finite set of values, which we find quite counterintuitive. Extending the language by including reasoning about numbers (cf. Fagin et al (1988), Voorbraak (1993)) can also help to obtain a complete system.

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6 References

FAGIN, R., HALPERN, J.Y., & MEGIDDO, N. (1988). A logic for reasoning about probabilities. *Proceedings 3rd IEEE Symposium on Logic in Computer Science,* 277-291. Also in *Information and Computation* **87** (1990), 78-128.

FATTOROSI-BARNABA, M., & AMATI, G. (1987). Modal operators with probabilistic interpretations, I. *Studia Logica* **4**, 383–393.

HOEK, W. VAN DER (1992). Modalities for reasoning about knowledge and quantities. *PhD Thesis, Free University of Amsterdam.*

KEISLER, H.J. (1985). Probability quantifiers. In J.Barwise & S.Feferman (Eds.), *Model-Theoretic Logics, Springer Verlag, New York.*

F. KLAWONN & P. SMETS (1992). The dynamics of belief in the transferable belief model and specialization-generalization matrices. In D.Dubois, M.P.Wellman, B.d'Ambrosio & P.Smets (Eds.), *Uncertainty in AI 92.* Morgan Kaufmann, San Mateo, CA, USA, 130–137.

D. LEWIS (1976). Probabilities of conditionals and conditional probabilities. *Philosophical Review* 85, 297 - 315.

A. SAFFIOTTI (1991). A belief function logic. *IRIDIA Report 91-25. A shorter version in Proc. AAAI 92.*

G. SHAFER (1976). A mathematical theory of evidence. Princeton Univ. Press, Princeton.

P. SMETS (1993). An axiomatic justification for the use of belief function to quantify beliefs. *IJCAI'93,* San Mateo, CA, 598 - 603.

P. SMETS & R. KENNES (1994). The transferable belief model. *Artificial Intelligence* **66**, 191–234.

LEA SOMBE (1994). A glance at revision and updating in knowledge bases. *Int. J. Intelligent Systems,* **9**(1), 1–28.

VOORBRAAK, F. (1993). As far as I know. Epistemic logic and uncertainty. *PhD Thesis, Utrecht University.*