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Applied Modal Logic: Modal Logics in Information Science

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APPLIED MODAL LOGIC:
MODAL LOGICS IN INFORMATION SCIENCE

doctoral dissertation

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To the memory of my teacher
Professor Helena Rasiowa

PREFACE

The title of the present dissertation is

"APPLIED MODAL LOGIC:MODAL LOGICS IN INFORMATION SCIENCE"

Applied Modal Logic is an intensively developed branch of Mathematical Logic, arising from the applications of logic to some theoretical problems in Information Science, Theoretical Linguistics and Philosophy.

Applied Modal Logic in Bulgaria started at around 1980 by the investigations of Passy, Tinchev, Gargov and Vakarelov in the field of Dynamic Logic ([Pa 84], [P&T 85,91], [Tin 86], [Vak 92c,92d]). Dynamic Logic is one of the first examples of Applied Modal Logic in Information Science, aiming to present some formal theories for reasoning about the behavior of programs.

Another example of Applied Modal Logic, which attracted our attention was an application of modal logic to information systems, found by Orlowska and Pawlak [O&P 84, 84a] at around 1984. The arising new modal logics have to be considered as a formal tool for reasoning about data in information systems. Soon after this new field grew into a more wide area of Rough Sets Theory with many theoretical and practical applications in Information and Decision Science ([Paw 82,84,86,91], [Slo, 91]).

The year 1990 was starting point of a new branch of Applied Modal Logic, called Arrow Logic, developed in different ways in Sofia, Amsterdam and Budapest ([Vak,90,91c,92a,92b,93], [Ar,94], [Ben 92], [Ma 92,92a,95], [MNSM 92], [Ven 91,92], [Mi 92], [Ne 92]). It aims to present formal systems for reasoning about information represented by arrows and their interconnections.

These new fields in Applied Modal Logic have opened many new problems, which needed radically new methods in order to be solved.

The present dissertation is a result of the author's attempt to find such new methods and their applications. Among the new methods are the so called "copying method" and the method of "abstract characterization theorems".

The "copying method" is a technical tool making possible to axiomatize some applied modal logics, whose standard semantics is not modally definable in the corresponding modal language. For the applications of this method see [Vak,87a,88,89,91b,92a,92b,92c, 92d,93], [GPT,88], [G&P,90], [Gor 90], [Pe,88], [Sk&St, 91].

The method of "abstract characterization theorems" consists of characterizing semantical structures based on concretely defined relations, by means of abstract first-order sentences. In some cases this method may be seen as a generalization of the Stone representation theory for distributive lattices and Boolean algebras [Sto 37] to relational structures far away from lattices and Boolean algebras. Applications of this method can be seen in [Vak 87,89,91,91a,91c, 92b,93,94,95,95a]. Let us mention that in many cases the "copying method" and the "method of abstract characterization theorems" have to be applied in a combinations with each other and possibly with some other methods.

The dissertation contains a systematic and uniform study of certain modal logics connected with Information Science by application of the above mentioned new methods and improving some old ones. It includes results in Dynamic Logic, Modal Logics for Information Systems, Approximation Logics based on Rough Sets Theory and Arrow Logics and covers some of the results obtained by the author during the last ten years.

CONTENT

INTRODUCTION

The aim of this introduction is to give the reader some preliminary information about the present dissertation. It is organized as follows.

We started in sec. 1 with a short survey of Modal Logic, containing the main notions and constructions used in some forms in this dissertation. The aim is to help the reader, who is not a specialist in this field, for better understanding of what follows.

Section 2 is devoted to Applied Modal Logic. In order to illustrate the field we give a brief description of four branches of Applied Modal Logic connected with Information Science: Dynamic Logic, Modal Logics for Information Systems, Approximation Logic Based on Rough Sets Theory, Arrow Logic. These branches correspond to the four parts of this dissertation.

Section 3 is about the methods and the results of the dissertation.

1. Modal Logic: a short survey

Modal Logic is a branch of Non-Classical Logic. It arises from the analysis of the modalities of necessity and possibility considered as propositional operators: $\Box A$ - "necessarily A" and $\Diamond A$ - "possibly A". The Golden Age for Modal Logic starts around sixties, when Saul Kripke invented possible world semantics. This semantics makes possible to analyse different fragments of modal thinking, arising from various applied areas in Information Science, Linguistics and Philosophy.

In the next pages we describe shortly the simplest modal Kripke-style theory with a language $\mathcal{L}(\Box,\Diamond)$ containing only the modalities $\Box A$ and $\Diamond A$. In honor of Kripke the smallest logic in this language is denoted by K. We shall use this theory as a representative example in order to illustrate some notions, problems and methods in Modal Logic related to this dissertation. Standard reference books for Modal Logic are the following: Segerberg [Seg 71], Hughes and Cresswell [H&C 84].

1.1. Syntax, semantics and some related notions in Modal Logic

The language $\mathcal{L}(\Box,\Diamond)$ extends the language of the classical propositional logic by two additional one-place connectives: $\Box$ and $\Diamond$. The Kripke semantics of $\mathcal{L}(\Box,\Diamond)$ consists in the following. The language $\mathcal{L}(\Box,\Diamond)$ is interpreted in relational structures called frames, which are of the form $\underline{W}=(W,R)$, where $W\neq\emptyset$ is a set, whose elements are called possible worlds and R is a binary relation between worlds, called accessibility relation. The class of all Kripke frames will be denoted by KRIPKE.

While in the semantics of classical logic each propositional variable takes one of the two truth values - truth and false, now the resulting truth value is a function of the worlds from W. In this way Kripke semantics models the situation that the truth value of a sentence may vary from world to world. One of the formal ways to do this is the following. A valuation of the language $\mathcal{L}(\Box,\Diamond)$ into a Kripke frame $\underline{W}=(W,R)$ is a function v which assigns to each propositional variable p a subset $v(p)\subseteq W$. The triple $M=(W,R,v)$ is called a model. In any model $M=(W,R,v)$ we define a relation, called satisfiability relation denoted by $x \Vdash_v A$ and having the following informal reading: "the formula A is true in the world x at the valuation v". It has the following inductive definition:

$x \Vdash_v p$ iff $x\in v(p)$, where p is a propositional variable,

$x \Vdash_v \neg A$ iff $x \nVdash_v A$ /not $x \Vdash_v A$/,

$x \Vdash_v A\wedge B$ iff $x \Vdash_v A$ and $x \Vdash_v B$,

$x \Vdash_v A\vee B$ iff $x \Vdash_v A$ or $x \Vdash_v B$,

$x \Vdash_v A\Rightarrow B$ iff $x \nVdash_v A$ or $x \Vdash_v B$,

$x \Vdash_v \Box A$ iff $(\forall y\in W)(xRy \longrightarrow y \Vdash_v A)$,

$x \Vdash_v \Diamond A$ iff $(\exists y\in A)(xRy$ and $y \Vdash_v A)$.

According to this definition v(p) contains those and only those worlds from W in which p is true; $\Box A$ is true in x iff A is true in all worlds accessible by R to x; $\Diamond A$ is true in x if it is true in at least one world accessible to x. On the base of this semantics the formulas $\Diamond A$ and $\neg\Box\neg A$ have one and the same truth conditions, which make possible to consider as a primitive only the operation $\Box$ and to introduce $\Diamond$ by definition $\neg\Box\neg$. So, from now on our modal language will be $\mathcal{L}(\Box)$.

Possible worlds and the accessibility relation between them is not the only intuition connected with Kripke frames. There are many possible intuitive interpretations of the elements and the relation R in a frame (W,R). For instance the elements of W may be considered as moments of time and R is the temporal ordering: xRy - "x is before y". In this way we obtain temporal logic and the modalities $\Box A$ and $\Diamond A$ have the meaning "always in the future A" and "sometimes in the future A". In temporal logic we have another pair of modalities $\Box^{-1}$ and $\Diamond^{-1}$ interpreted by R^{-1} with the meaning $\Box^{-1}A$: "Always in the past A" and $\Diamond^{-1}A$: "sometimes in the past A". Assuming different properties of R we obtain different temporal logics. In section 2 we shall give more examples of concrete intuitive interpretations of Kripke semantics.

As we have seen by the example of temporal logic, Kripke semantics can be extended also for modal languages having more than one operation of necessity $\Box$, denoted sometimes by $[\alpha]$, $[\beta]$ and so on, and the corresponding possibility operations - by $\langle\alpha\rangle$, $\langle\beta\rangle$ and so on. In frames, the corresponding relation of the box $[\alpha]$ is sometimes denoted by $R(\alpha)$ or, simply by α. Such modal languages and the logics based on them are called polymodal. Generalizations to modal operations with n-arguments, called poliadic modalities are also possible /[J&T 51]/. Then for the interpretation of such poliadic modalities we use n+1-place relations:

$$x \Vdash_v \Box(A_1,\dots,A_n) \text{ iff } (\forall y_1\dots y_n \in W)(xRy_1\dots y_n \rightarrow y_1 \Vdash_v A_1 \text{ or } \dots \text{ or } y_n \Vdash_v A_n),$$
$$x \Vdash_v \Diamond(A_1,\dots,A_n) \text{ iff } (\exists y\dots y \in W)(xRy\dots y \text{ and } y_1 \Vdash_v A_1 \text{ and } \dots \text{ and } y_n \Vdash_v A_n),$$

We continue with some definitions, considering the simplest modal language $\mathcal{L}(\Box)$, leaving the obvious extensions for arbitrary polymodal languages to the reader.

We say that a formula A is true in a model (W,R,v) if for any $x \in W$ we have $x \Vdash_v A$; A is true in a frame (W,R) if it is true in any model (W,R,v) over (W,R); A is true in a class of frames Σ if it is true in any frame from Σ.

Having in mind the above definitions Kripke semantics can be considered as a translation of modal formulas into second-order logic. Let M=(W,R,v) be a Kripke model. If we write $x \Vdash_v p$ as p(x) for any propositional variable p, and rewrite $x \Vdash_v A$ according to this notation, we obtain a first-order formula $\tau(A)(x)$ of one variable x with the property that

$x \Vdash_v A$ iff $\tau(A)(x)$ is true in the model M.

If $p_1,\dots p_n$ are the propositional variables of A then the second-order translation ST(A) of A is $(\forall x)(\forall p_1)\dots(\forall p_n)\tau(A)(x)$. Then the validity of A in a frame (W,R) is equivalent of the validity of ST(A) in (W,R). Example: $\tau(\Box p \Rightarrow p) = (\forall y)(xRy \Rightarrow p(y)) \Rightarrow p(x)$ and $ST(\Box p \Rightarrow p) = (\forall x)(\forall p)((\forall y)(xRy \Rightarrow p(y)) \Rightarrow p(x))$.

So the Kripke semantics makes possible to consider given modal logics as fragments of second-order logic. Sometimes this a good advantage because the full second-order logic possesses some bad properties, while some of its fragments may have nice properties. This is one of the reasons to use modal logics instead of using the full second-order logic.

Given a class of frames Σ we may consider the set $L(\Sigma)$ of all formulas of $\mathcal{L}(\Box)$ which are true in Σ. In some sense $L(\Sigma)$ may be taken as "the modal logic of Σ" and the formulas of $L(\Sigma)$ are the "logical laws of Σ". Obviously, the larger is Σ, the smaller is the corresponding logic $L(\Sigma)$, i.e. if $\Sigma \subseteq \Sigma'$ then $L(\Sigma') \subseteq L(\Sigma)$.

This is a semantical definition of a modal logic. According to this definition the logic K is defined as the logic of all Kripke frames, i.e. K=L(KRIPKE). Since any class of Kripke frames is included in KRIPKE, K=L(KRIPKE) is the smallest modal logic of the language $\mathcal{L}(\Box)$.

Another way to define a modal logic is the axiomatic one: a formal system F consists of a set of formulas, considered as axioms and a set of rules of inference. Then the modal logic of F is identified with the set L(F) of all theorems of F. This is a syntactic definition of a modal logic.

The following formal system is called a normal axiomatization of the logic K.

Axioms for K:

(Bool) All Boolean tautologies /or any appropriate set of axioms for the classical propositional logic/

(K□) $\Box(A\Rightarrow B)\Rightarrow(\Box A\Rightarrow\Box B)$,

Rules of inference:

(MP) - modus ponens $\dfrac{A,\ A\Rightarrow B}{B}$

(N) - necessitation $\dfrac{A}{\Box A}$

(Sub) - uniform substitution $\dfrac{A}{A[B/p]}$

By a normal formal system in $\mathscr{L}(\Box)$ we will mean formal systems which extend the set of axioms of K by arbitrary additional axioms. If φ is a new axiom then the extension of K with φ will be denoted by $K\varphi$ or $K+\varphi$.

Let F be a formal system in $\mathscr{L}(\Box)$. A formula A is called a theorem of F if there exists a finite sequence of formulas $A_1,\dots,A_n=A$ such that each A_i is either an axiom of F or is obtained from some A_j and A_k for $j,k<i$ by (MP), or is obtained from some A_j for $j<i$ by (N) or (Sub).

The following definitions correlate the semantic and syntactic notions of a modal logic. Let F be a formal modal system and Σ be a class of Kripke frames. We say that F is correct /or sound/ with respect to Σ /or that Σ is a correct semantics for F/ if $L(F)\subseteq L(\Sigma)$, i.e. each theorem of F is true in Σ; we say that F is complete in Σ /or that Σ is a complete semantics for F, or that F is a complete axiomatization of $L(\Sigma)$/ if F is sound in Σ and $L(\Sigma)\subseteq L(F)$, i.e. if $L(F)=L(\Sigma)$. The inclusion $L(\Sigma)\subseteq L(F)$ means that each formula, which is true in Σ is a theorem of F. F is called complete system if there exist Σ such that $L(F)=L(\Sigma)$. Sometimes the later equality is called "completeness theorem of F with respect to Σ". If a logic L is complete in a class of finite frames we say that L is finitely complete. It is possible for a given formal system F to have several complete semantics. There are systems which are not complete, there are complete systems, which are not finitely complete (see [H&C 84]).

The next important notion in Modal Logic is the notion of modal definability. A class of frames Σ is modally definable by a modal formula A from $\mathscr{L}(\Box)$ if the following condition is satisfied:

For any frame $\underline{W}$: $\underline{W}\in\Sigma$ iff the formula A is true in $\underline{W}$.

If Σ is characterized by a first-order condition Φ for R then instead of modal definability of Σ by A we say that Φ is modally definable by A, or that A is first-order definable by ϕ. In this case Φ is called a first-order correspondent of A and A is called a modal correspondent of Φ. In the next table we list some typical examples of such correspondence:

$(\forall x)(xRx)$	$\Box p\Rightarrow p$,	reflexivity,
$(\forall xyz)(xRy \text{ and } yRz \longrightarrow xRz)$	$\Box p\Rightarrow\Box\Box p$,	transitivity,
$(\forall xy)(xRy \longrightarrow yRx)$	$p\Rightarrow\Box\Diamond p$,	symmetry.

Let us note that not all first order conditions are modally definable and that not all modal formulas define a first-order condition. Standard references for Correspondence Theory are [Ben 86] and [Ben 84].

1.2. Some general problems in Modal Logic

The above introduced list of definitions is a formal base of several main problems in modal logic:

- given a modal formal system F find complete semantics for F,
- given a class of frames Σ find a complete axiomatization of $L(\Sigma)$,
- given a logic L prove /or disprove/ its finite completeness,
- Given a class Σ of frames prove /or disprove/ its modal definability.

We shall describe several methods for treating the above problems, used with some modifications in this dissertation: canonical models, filtration, p-morphisms and copying, abstract characterization theorems for frames.

1.3. Canonical models

The main method for proving completeness theorems is the so called method of canonical models, which is consisting in the following /[Seg 71/.

Let L be a given logic with normal axiomatization F. A set of formulas x is called L-inconsistent if for some $A_1,\ldots,A_n \in x$ the formula $\neg(A_1\wedge\ldots\wedge A_n)$ is a theorem of L; x is called maximal consistent set if it is consistent and has no any consistent proper extension. Then we define the canonical frame $\underline{W}_L=(W_L,R_L)$ and the canonical model $M_L=(W_L,R_L,v_L)$ for L as follows: W_L is the set of all maximal L-consistent sets of L, for $x,y\in W_L$ xR_Ly iff $\{A/\Box A\in x\}\subseteq y$ and for any propositional variable p $v_L(p)=\{x\in W_L/p\in x\}$.

The importance of canonical models is in the following

Canonical Model Lemma

For any formula A: A is a theorem of L iff A is true in M_L.

This **lemma** can be used for proving completeness theorems of L. Suppose that L is sound with respect to a class Σ of frames. Then if the canonical frame $\underline{W}_L$ of L is one of the frames **of** Σ then L is complete in Σ. For, suppose that A is true in Σ, i.e. that A is true in all models over frames from Σ. Since the canonical frame is in Σ then A is true in the canonical model M_L of L and by the Canonical Model Lemma A is a theorem of L, which proves the completeness of L with respect to Σ. In this case Σ can be restricted to the set consisting only of the canonical frame and then L is complete in its canonical frame. Let us call such logics canonical. Note that the method of canonical models for proving completeness is not universal: there are logics which are not canonical (see [H&C 84]).

1.4. Filtration

The method of filtration is mainly used to prove completeness with respect to classes of finite frames. The method is based on the following construction /see [Seg 71]/. Suppose we have a model M=(W, R, v) and a finite set of formulas Ψ closed under subformulas. Then the method of filtration gives a way of reconstruction of M into a finite model M'=(W', R', v') depending on Ψ, as follows. For $x,y\in W$ define

$x\equiv_\Psi y$ iff $(\forall A\in\Psi)(x\Vdash_v A \longleftrightarrow y\Vdash_v A)$, $|x|=\{y\in W/x\equiv_\Psi y\}$, $W'=\{|x|/x\in W\}$, and for any propositional variable p set $v'(p)=\{|x|/x\in v(p)\}$. For the relation R' of the model M'=(W',R',v') we do not give a special construction but postulate the following two properties:

(fR1) If xRy then $|x|R'|y|$,

(fR2) If $|x|R'|y|$ then $(\forall\Box A\in\Psi)(x\Vdash_{v}\Box A \rightarrow x\Vdash_{v} A)$.

If we can find R' satisfying (fR1) and (fR2) then the obtained model M'=(W',R',v') is called a filtration of M through Ψ. The importance of this definition is in the following

Filtration lemma

(i) M' is a finite model,

(ii) for any $x\in W$ and $A\in\Psi$: $x\Vdash_{v} A$ iff $|x|\Vdash_{v'} A$.

Note that (fR1) and (fR2) are in some sense the most natural conditions which guarantee (ii).

Filtrations can be used to prove that for some classes of frames Σ we have $L(\Sigma)=L(\Sigma_{fin})$ where Σ_{fin} is the class of finite frames of Σ. This can be done as follows. Obviously we have $L(\Sigma)\subseteq L(\Sigma_{fin})$ and suppose for the sake of contradiction that $L(\Sigma_{fin})\not\subseteq L(\Sigma)$. So there exists $A\in L(\Sigma_{fin})$ but $A\notin L(\Sigma)$. Then for some model M=(W,R,v) over a frame $(W,R)\in\Sigma$ and $x\in W$ we have $x\not\Vdash_{v} A$. Let Ψ be the set of subformulas of A and define the W' and v' as in the definition of filtration. Then if we succeed to define a relation R' in such a way as to satisfy the conditions (R1), (R2) and $(W',R')\in\Sigma$ then by the filtration lemma we will obtain that $|x|\not\Vdash_{v} A$, so A is not true in the finite frame (W',R') and hence $A\notin L(\Sigma_{fin})$, which is a contradiction.

The filtration can be used for proving completeness theorems with respect to a finite frames as follows. Suppose that L(F) is a logic, complete with respect to a class of frames Σ, i.e. we have that $L(F)=L(\Sigma)$. If by filtration we can prove that $L(\Sigma)=L(\Sigma_{fin})$ then obviously we get $L(F)=L(\Sigma_{fin})$, which state that L(F) is complete in Σ_{fin}. If F has finite set of axioms then this fact implies also the decidability of L(F).

Sometimes the method of filtration can be used for a completeness proofs for logics, which are not canonical, applying filtration to the canonical model for the logic in question.

In this dissertation the reader can find many applications of filtration for proving decidability and completeness results for some polymodal logics.

1.5. P-morphisms and copying

The notion of p-morphism /pseudo-epimorphism/ have been introduced by Segerberg [Seg 71]. It is a kind of homomorphism between frames, defined as follows. Let $\underline{W}=(W,R)$ and $\underline{W}'=(W',R')$ be two frames. A mapping $f:W\longrightarrow W'$ from W onto W' is called a p- morphism if it satisfies the following two conditions for any $x,y\in W$ and $y'\in W'$:

(PR1) If xRy then f(x)R'f(y),

(PR2) If f(x)R'y' then $(\exists y\in W)$(f(y)=y' and xRy).

If f is a p-morphism from $\underline{W}$ onto $\underline{W}'$ then $\underline{W}'$ will be denoted by $f(\underline{W})$ and called a p-morphic image of $\underline{W}$ and $\underline{W}$ is called a p-morphic pre- image of $\underline{W}'$. The importance of p-morphisms is in the following

P-morphism lemma

Let f be a p-morphism from $\underline{W}=(W,R)$ onto $\underline{W}'=(W',R')$, v' be a valuation in $\underline{W}'$ and v be the valuation in $\underline{W}$ defined as follows: for any propositional variable p $v(p)=\{x\in W/f(x)\in v'(p)\}$. Then:

(i) for any formula A and $x\in W$: $x\Vdash_{v} A$ iff $f(x)\Vdash_{v'} A$.

(ii) For any formula A: if A is true in $\underline{W}$ then A is true in $f(\underline{W})$, i.e. $L(\{\underline{W}\}) \subseteq L(\{f(\underline{W})\})$.

The p-morphism lemma implies the following

Lemma

If a class of frames Σ is modally definable then it is closed under p-morphisms /hence, if Σ is not closed under p-morfisms then Σ is not modally definable/.

P-morfisms can be used also for completeness proofs as follows. Suppose L is a logic which is complete in a class Σ of frames but the intended semantics is a subclass Σ' of Σ. If for any frame $\underline{W} \in \Sigma$ we can find a frame $\underline{W}' \in \Sigma'$, which is a a p-morphic pre-image of $\underline{W}$, then $L(\Sigma)=L(\Sigma')$ and hence L is complete in Σ'. So, the importance of this construction is in building p-morphic pre-images. There are several methods for constructing p-morphic pre-images: Segerberg's "Bulldozer construction" [Seg 71], Sahlqvist's "unraveling construction" [Sah 75], "Copying construction" (see [Vak 87a, 88, 89, 91b, 92b, 92c, 92d, 93], [Pe 88], [GPT,88], [G&P 90], [Gor 90]. We shall give only the definition of copying for the language $\mathcal{L}(\Box)$.

Let $\underline{W}=(W,R)$ and $\underline{W}'=(W',R')$ be two frames. A nonempty class of mappings I from $\underline{W}$ in $\underline{W}'$ is called a copying from $\underline{W}$ in $\underline{W}'$ if the following conditions are satisfied:

(CI1) $(\forall x' \in W')(\exists x \in W)(\exists f \in I)(f(x)=x')$,

(CI2) $(\forall x,y \in W)(\forall f,g \in I)(f(x)=g(y) \longrightarrow x=y)$,

(CR1) $(\forall x,y \in W)(\forall f \in I)(\exists g \in I)(xRy \longrightarrow f(x)R'g(y))$,

(CR2) $(\forall x,y \in W)(\forall f,g \in I)(f(x)R'f(y) \longrightarrow xRy)$.

Condition (CI1) says that $W' = \bigcup_{f \in I} f(W)$, where $f(W)=\{f(x)/x \in W\}$, called f-th copy of W. From (CI1) and (CI2) it follows that for each $x' \in W'$ there exists unique $x=h(x') \in W$ such that for some $f \in I$ $f(x)=x'$. The mapping $h: W' \longrightarrow W$ is a p-morphism from $\underline{W}'$ onto $\underline{W}$, so $\underline{W}'$ is a p-morphic pre-image of $\underline{W}$. So the copying is another method of constructing p-morphic pre-images. In this dissertation there are many applications of the copying method for proving completeness theorems.

1.6. Abstract characterization theorems

Sometimes the intended semantics (called sometimes "standard semantics") for a given logic L consists of a class Σ of concretely defined frames in which L is sound. In order to prove completeness of L with respect to Σ we may proceed as follows. Find another class of frames Σ' in which L is complete and prove that each frame from Σ' is isomorphic, or can be isomorphically embedded in a frame from Σ. Then L is complete in Σ.

The above described methods for proving completeness theorems may be used in a combination. Sometimes to approach an intended class of "standard" frames for a given logic L, we have to prove that L is complete in some non-intended classes of "non-standard" frames for L and then, using the above described methods, to prove that "standard" and "non-standard" semantics for L are equivalent.

2. Applied Modal Logic

The term "applied modal logic" has been introduced by Segerberg [Seg 80a] for a particular modal system, arising from computer science. Since then the term stands for a name of a field of Modal Logic, studying modal logics arising from some applied area: Information Science (Computer Science, Artificial Intelligence), Cognitive Science, Linguistics, Philosophy, etc. Applied Modal Logic is now a part of a more wide area of Applied Logic. There are several Journals devoted to Applied Logic: The "Journal of Applied Non-Classical Logic", "Logic and Computation", "Logic, Language and Information", "Pure and Applied Logic". Many papers in this area are published in the journals "Artificial Intelligence", "Theoretical Computer Science", "Fundamenta Informaticae", "Information and Computation" etc. There are several regular conferences devoted to Applications of logic in Computer Science, Artificial Intelligence, and Linguistics. Several books on different areas of applied modal logic have been written: Gabbay [Gab 76] "Investigations of Modal and Tense Logic with Applications to Problems of Philosophy and Linguistics", Goldblatt [Go 82] "Axiomatizing the logic of computer programming", Goldblatt [Go 87] "Logics of Time and Computation", Mirkowska and Salwicki [Mi&Sal 87] "Algorithmic Logic".

Normally applied modal logics arise from some classes of Kripke frames, which formalize certain concrete relations between objects in some applied area. The main question here is why we choose modal languages for studying these classes of frames instead of taking some first or higher order classical logical languages. The reasons are several. First, modalities sometimes are more natural for reasoning about the objects and relations in question instead of taking their first-order or second order translations. For instance "Always A" which in temporal logic is formalized by "$\Box A \wedge \Box^{-1} A$" is much more convenient than the equivalent translation: "$(\forall y)(xRy \longrightarrow A(y)) \wedge (\forall y)(yRx \longrightarrow A(y))$". Second, using modal correspondence theory we may express some properties of frames by means of modal formulas, so modal languages allow to talk about frames in an indirect way. And third, as we have mentioned in the previous section, it is more convenient to use modal languages, chosen for some specific purposes and describing some fragments of second-order logic instead of the full second-order logic.

In order to give the reader a preliminary impression of what Applied Modal Logic is like, in the next pages we will give an informal description of the following branches, connected with Information Science: Dynamic Logic, Modal Logics for Information Systems, Approximation Logic based on Rough Sets theory and Arrow Logic. These four branches correspond also to the four parts of the present dissertation.

2.1. Dynamic Logic

Propositional Dynamic Logic PDL is one of the first applied modal logics arising from Information Science, aiming to present some formal theories for reasoning about the behavior of programs.
As a predecessor of PDL we can mention Salwicki's Algorithmic Logic [Mi&Sal 87]. /For the history of PDL see Harel [Ha 84] and also Passy & Tinchev [Pa&Ti 91]/. The main idea in PDL comes from the following intuitive "dynamic" interpretation of Kripke frames. The set W of a frame $(W, R(\alpha))$ is interpreted as a collection of data and the relation $R(\alpha)$ as an input-output relation determined by a program α, acting over the elements of W. Then the relation $xR(\alpha)y$ means that the input x is performed by α in the output y. Now $\Box A$ and $\Diamond A$, denoted by $[\alpha]A$ and $\langle\alpha\rangle A$, have the following intuitive readings:

$[\alpha]A$ - "always after α A",

$\langle\alpha\rangle A$ - "sometimes after α A".

In this "dynamic" interpretation formulas correspond to properties of data and the modal operations $[\alpha]$ and $\langle\alpha\rangle$ as property transformers in the following sense: for a given formula A, considered as a property, $[\alpha]A$ is a new property, which is possessed by the input x of α iff A is possessed by every output y, which is a result of the execution of x by α, and likewise for $\langle\alpha\rangle A$. In this way modal logic can be used for talking about data and programs. For instance the formula $A \Rightarrow [\alpha]B$ express partial correctness of α with respect to **a** precondition A /condition for the input of α/ and **a** postcondition B /condition of the output of α/. In the language of the full PDL we may compose more complex programs using some program constructs and for each program α we have two modalities [α] and <α>.

The main iterative procedure in PDL is the non-deterministic iteration α* - "choose nondeterministically a natural number n and repeat α n-times". The input-output relation R(α*) of α* is the reflexive and transitive closure $R^*(\alpha)$ of the relation R(α). In the axiomatization of PDL given by Segerberg [Seg 82] α* is axiomatized by the following two axioms, which I denote by Seg and Seg_0:

Seg $A\wedge[\alpha^*](A\Rightarrow[\alpha]A)\Rightarrow[\alpha^*]A$,

Seg_0 $[\alpha^*]A\Rightarrow A\wedge[\alpha][\alpha^*]A$.

Seg and Seg_0 are called respectively "induction axiom" and "small induction axiom" in PDL. Seg_0 corresponds to the following first- order condition connected the relations R(α) and $R(\alpha^*)$:

$(\forall x)xR(\alpha^*)x$ & $(\forall xyz)(xR(\alpha)y$ & $yR(\alpha^*)z \rightarrow xR(\alpha^*)z)$.

The conjunction of Seg and Seg_0 modally define the equality $R(\alpha^*)=R^*(\alpha)$. However the semantic condition corresponding to Seg alone has been an open problem.

Part I of this dissertation - Induction Axioms in Dynamic Logic - is devoted to a study of the induction axiom Seg and some of its natural generalizations in some modal logics, regarding as subsystems of PDL ([Vak 92c]). The developed theory is applied to a modal characterization of cyclic repeating of programs ([Vak 92c]). .

2.2. Modal Logics for Information Systems

There are several ways to apply Modal Logic to the theory of information. One approach, based on Dynamic Logic, is given by van Benthem in [Ben 89] /see also De Rijke [Rij 92,93]/. We shall describe here the approach by Orlowska and Pawlak [O&P, 84, 84a], Orlowska [Or 84,85,85a,90] and the author [Vak 87,87a,89,91, 91a,94,95,95a]. It is based mainly on a notion of information system introduced by Pawlak [Paw 81,83,91] and called sometimes knowledge representation system.

By an information system in the sense of Pawlak, we mean any system of the form $S=(Ob_S, At_S, \{Val_S(a)/a\in At\}, f_S)$, where:

- $Ob_S\neq\emptyset$ is a set, whose elements are called objects of S,
- At_S is a set, whose elements are called attributes of S,
- for each $a\in At_S$, $Val_S(a)$ is a set, whose elements are called values of the attribute a,
- f_S is a two-argument total function, called information function, which assigns to each object $x\in Ob_S$ and attribute $a\in At_S$ a subset $f_S(x,a)\subseteq Val_S(a)$, called the information of x according to a.

In [Vak 94,95,95a] Pawlak's information systems are called "Attribute systems" or "A-systems" for short.

An example of an attribute is a="official language", the values of a, Val(a)={E(nglish), G(erman), F(rench), R(ussian)}. If a person x knows Enlish and Russian but not French and German, then f(x)={E,R}. If we have f(x)=∅ then this means that x knows neither of the languages E, G, F and R. Another example of attribute is a="color" and Val(a)={green, red, blue, etc }.

Pawlak's information systems are generalizations of the following more simple information systems, which we call "Property Systems" or P-systems for short /see [Vak 91a,92,94]/. Namely $S=(Ob_S, Pr_S, f_S)$ is a P-system if

- $Ob_S \neq \emptyset$ is a set of elements called objects,
- Pr_S is a set of elements, called properties, and
- f_S is a function, which assigns to each object x a subset $f_S(x) \subseteq Pr_S$ of properties of x.

Let S be a P-system. A relation R_S in the set OB_S is called an informational relation in S if, roughly speaking, R_S can be defined only by using the information contained in S. The following informational relations are some typical examples.

- Positive similarity
  $x \Sigma_S y$ iff $f_S(x) \cap f_S(y) \neq \emptyset$, i.e. x and y have common property,
- Negative similarity
  $x N_S y$ iff $\overline{f}_S(x) \cap \overline{f}_S(y) \neq \emptyset$, where $\overline{f}_S(x)=Pr_S-f_S(x)$, i.e. there is a property in Pr_S which is possessed neither by x nor by y, for instance x and y are not smokers.
- Informational inclusion
  $x \leq_S y$ iff $f_S(x) \subseteq f_S(y)$, i.e. each property of x is possessed by y.
- Indiscernibility
  $x \equiv_S y$ iff $f_S(x)=f_S(y)$.

In Pawlak's information systems informational relations have strong and weak versions:

- Strong positive similarity
  $x \sigma_S y$ iff $(\forall a \in At_S) f_S(x,a) \cap f_S(y,a) \neq \emptyset$,
- Strong negative similarity
  $x \nu_S y$ iff $(\forall a \in At_S) \overline{f}_S(x,a) \cap \overline{f}_S(y,a) \neq \emptyset$,
- Strong informational inclusion
  $x \leq_S y$ iff $(\forall a \in At_S) f_S(x,a) \subseteq f_S(y,a)$,
- Strong indiscernibility
  $x \equiv_S y$ iff $(\forall a \in At_S) f_S(x,a)=f_S(y,a)$,
- Weak positive similarity
  $x \Sigma_S y$ iff $(\exists a \in At_S) f_S(x,a) \cap f_S(y,a) \neq \emptyset$,
- Weak negative similarity
  $x N_S y$ iff $(\exists a \in At_S) \overline{f}_S(x,a) \cap \overline{f}_S(y,a) \neq \emptyset$,
- Weak informational inclusion
  $x <_S y$ iff $(\exists a \in At_S) f_S(x,a) \subseteq f_S(y,a)$,
- Weak indiscernibility
  $x \cong_S y$ iff $(\exists a \in At_S) f_S(x,a)=f_S(y,a)$,
- The set of deterministic elements of S
  $D_S=\{x \in Ob_S / (\forall a \in At_S) Card f_S(x,a) \leq 1\}$.

Information systems in which $D_S=Ob_S$ are called in [Vak 89] deterministic information systems /in [Vak 94] these systems are called single-valued information systems/.

Since the notions of an object, property and attribute are of ontological nature, property systems and attribute systems are information systems of ontological type. Another type of information systems are those of logical type in which the information is represented by a collection of sentences equipped with certain inference mechanism. The first kind of such systems has been introduced by Scott [Sco 82], other kinds can be found in [Vak 92,94], where the connections between logical and ontological systems are studied. The simplest notion of an information system of logical kind is the notion of consequence information system [Vak 94], C-system for short, with the following definition: S=(Sen, $\vdash$) is a C-system if

- Sen is a non-empty set of elements, whose elements are called sentences,
- $\vdash$ is a binary relation between finite subsets of Sen, called Scott consequence relation, satisfying the following axioms, coming from Gentzen sequent calculus:
- (Refl) If $A \cap B \neq \emptyset$ then $A \vdash B$,
- (Mono) If $A \subseteq A'$, $B \subseteq B'$ and $A \vdash B$ then $A' \vdash B'$,
- (Cut) If $A \vdash B \cup \{x\}$ and $\{x\} \cup A \vdash B$ then $A \vdash B$, A,B are finite subsets of Sen.

There is a natural representation theorem of C-systems in P- systems / see [Vak 92,94]/, which implies very unexpected correlation between information relations in P-systems and C- systems. For instance: /$x, y \in Sen$/

- $x \leq y$ iff $\{x\} \vdash \{y\}$,
- $x \Sigma y$ iff $\{x,y\} \not\vdash \emptyset$,
- xNy iff $\emptyset \not\vdash \{x,y\}$.

In [Vak 94] we study a more complex logical information system, called bi-consequence system /B-system/ having two consequence relations: strong $\vdash$ and weak $\succ$. The relation $\vdash$ satisfies the Scott's axioms given above and $\succ$ satisfies some similar conditions:

(Refl $\succ$) If $A \cap B \neq \emptyset$ then $A \succ B$,
(Mono $\succ$) If $A \succ B$, $A \subseteq A'$ and $B \subseteq B'$ then $A' \succ B'$,
(Cut $\succ$1) If $A \vdash \{x\} \cup B$ and $A \cup \{x\} \succ B$ then $A \succ B$,
(Cut $\succ$2) If $A \cup \{x\} \vdash B$ and $A \succ \{x\} \cup B$ then $A \succ B$,
(Incl) If $A \vdash B$ then $A \succ B$.

There is a natural representation theorem of B-systems in A- systems, which gives logical meaning of some information relations in A-systems. For instance: /$x, y \in Sen$/

- $x \Sigma y$ iff $\{x,y\} \not\vdash \emptyset$,
- xNy iff $\emptyset \not\vdash \{x,y\}$,
- $x < y$ iff $\{x\} \succ \{y\}$,
- $x \sigma y$ iff $\{x,y\} \not\succ \emptyset$,
- $x \nu y$ iff $\emptyset \not\succ \{x,y\}$,
- $x \leq y$ iff $\{x\} \vdash \{y\}$.

Let Σ be a class of informational systems of a given sort. A relational system $\underline{W}=(W, \rho)$ is called a standard informational frame of Σ of type ρ if $W=Ob_S$ of some informational system $S \in \Sigma$ and ρ is a set of some informational relations in S. Now in an obvious way we can associate a modal language $\mathcal{L}(\rho)$ to be interpreted in standard informational frames of type ρ and a modal logic of type ρ, corresponding to the language $\mathcal{L}(\rho)$. Such logics are called information modal logics.

The language $\mathcal{L}(\rho)$ has the following "informational" meaning.

Let S be an informational system of a given sort /P- or A- system/, $\underline{W}=(Ob_S$ $\rho)$ be the standard informational frame of type ρ over S and $M=(\underline{W}, v)$ be a model over $\underline{W}$. For any formula A we put $v(A)=\{x \in W / x \Vdash_v A\}$. The set v(A) may have different meanings. One is that it is the set of all objects from $W=Ob_S$ for which A is true (at v). Another meaning is that v(A) may be considered also as a query to S: "give the set of all objects $x \in Ob_S$, for which A is true". This meaning leads to consider interpreted propositional variables in a given model as a simple queries and formulas as compound queries. Them modal formulas will be "modal queries".

Let us consider the following example. Suppose in the above model M that A is a propositional variable such that $v(A)=\{x_0\}$. Then for $v(<\Sigma>A)=\{x\in Ob_S / x \Vdash_v <\Sigma>A\}$ we can compute:

$$v(<\Sigma>A)=\{x\in Ob_S/(\exists y\in Ob_S)(x\Sigma_S y \text{ and } y\in\{x_0\})=\{x\in Ob_S/x\Sigma x_0\}.$$

This is the following query to S: "give all objects of S which are positively similar to x_0".

Boolean connectives correspond to Boolean combinations of queries: $v(A\wedge B)=v(A)\cap v(B)$, $v(A\vee B)=v(A)\cup v(B)$, $v(\neg A)=Ob_S-v(A)$.

Part II of the dissertation - Modal Logics for Information Systems - is devoted to a study of modal logics, arising from Pawlak's information systems and property systems.

2.3. Approximation Logics based on Rough Sets Theory

Approximation Logic is a direction of Applied non-Classical Logic, which studies various kinds of logical systems for reasoning with approximate or incomplete information. First-order systems, based on the Pawlak's Rough Sets approach [Paw82,86,91] have been given by Rasiowa [Ras 86] and Rasiowa and Skowron [R&S 88]. In [Vak 91b] we presents some modal logics based on the Pawlak's idea of rough approximation. This idea consists of the following.

Having an approximating view on an universe W, instead of points we can see classes of points, each class considered as approximation of its elements. Such "big" points should be non-empty, disjoint and covering the universe. In other words, such an approximation defines an equivalence relation R, whose equivalence classes are the approximations of the elements of the universe. This leads to the notion of approximation space - a system S=(W, R), where $W\neq\emptyset$ is a set of objects and R is an equivalence relation in W. Natural examples of approximation spaces are the sets of objects of a given information system with R being an indiscernibility relation in W. Each equivalence class $|x|_R$ in W can be considered as an approximation of x. Then instead of subsets of W we can see unions of eqiuvalence classes, which will be called R-definable sets. Having in mind this for a set $X\subseteq U$ we can define two kinds of approximations:

- lower approximation - $\underline{R}X$ - the biggest R-definable set contained in X, and
- the upper approximation - $\overline{R}X$ - the smallest R-definable set containing X.

In a similar way one can define lower and upper approximation of a relation /see [Paw 86]/.

The title of part three of the dissertation is Approximation Logics Based on Rough Sets Theory. The main aim is to introduce and investigate modal logics, based on the above notion of approximation of a set and relation. As an application a "Rough Boolean Logic - RBL" is constructed, containing rough approximations of the operations of disjunction and conjunction.

2.4. Arrow logic

The year 1990 was starting point of a new branch of Applied Modal Logic, called Arrow Logic, developed in different ways in Sofia, Amsterdam and Budapest ([Vak,90,91c,92a,92b,93], [Ar,94], [Ben 92], [Ma 92,92a,95], [MNSM 92], [Ven 91,92], [Mi 92], [Ne 92]). It aims to present formal systems for reasoning about information represented by arrows and their interconnections.

I would like to say that the idea to investigate arrow structures and modal logics based on them, was suggested to me by Johan van Benthem /[Ben 90]/, during the Kleene Conference, held in June 1990 in Varna. The Johan's advise was that it would be nice to have a simple modal logic, with semantics in two sorted structures, to be called later "arrow structures", having points /"states"/ and arrows /"transitions"/. Such "arrow structures" would have had different models: ordered pairs, directed graphs, categories, vectors, states and transitions, and so on. So, two main problems have arisen: first, the choice of a good enough mathematical structure of arrows, and second, the corresponding choice of appropriate modal language.

I decided to take the following definition of arrow structure, which in Graph theory is called directed multi graph. Namely, arrow structures are systems of the form S=(Ar, Po, 1, 2), where:

- Ar is a nonempty set, whose elements are called arrows,
- Po is a nonempty set, whose elements are called points,
- 1 and 2 are total functions from Ar to Po, called projections. If $x \in Ar$ then 1(x) is called the first end and 2(x) is called the second end of x.

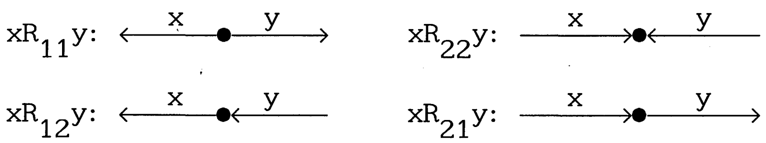
Graphically: **1**(x)● $\xrightarrow{x}$ ●**2**(x)

The next problem was how to associate to arrow structures appropriate modal language and how to interpret this language in such structures. As we know, the standard Kripke semantics requires binary relations in one sorted structure. So one solution of this difficulty is to define in the set of arrows appropriate set of binary relations in such a way, as the new relational system on arrows to contain the information of the whole arrow structure. The following four relations in the set Ar proved to have such property: for $x,y \in Ar$, i, j=1,2

$xR_{ij}y$ iff $i(x)=j(y)$.

The relations R_{ij} , called incidence relations, express the four possible ways for two arrows to have a common end.

Graphically:



The relations R_{ij} satisfy the following simple first-order conditions: for $x,y,z \in Ar$ and i, j, k=1,2:

(ρii) $xR_{ii}x$,

(σij) $xR_{ij}y \rightarrow yR_{ji}x$,

(τijk) $xR_{ij}y \ \& \ yR_{jk}z \rightarrow xR_{ik}z$.

These conditions are characteristic in the following sense: if in a set W we have four relations R_{ij} satisfying the above conditions, then there exists an arrow structure $S=(Ar_S, Po_S, 1, 2)$ such that $Ar_S=W$ and S determines the same relations R_{ij}. So instead of arrow systems, which are two-sorted systems, not convenient for Kripke interpretations, we can use relational systems of the form $\underline{W}=(W,R_{11},R_{12},R_{21},R_{22})$, satisfying the conditions (ρii), (σij) and (τijk). I have called such systems "arrow frames". Arrow frames in this new sense, have two good advantages: first, they are in some sense equivalent to arrow structures, so their abstract elements are real arrows, and second, they have a simple first-order relational definition, suitable for modal purposes.

Now the corresponding modal language for the minimal, or Basic Arrow Logic-BAL - is easy to define. It extends the language of the propositional logic with four unary modalities [ij] i, j=1,2 with standard Kripke interpretation in arrow frames.

The above defined arrow structures can be generalized to the notion of n-dimensional arrow structure S=(Ar, Po, 1,2,...,n) with n projection operations 1,2,...,n. The standard picture of an n-dimensionalarrow is

1(x) 2(x) 3(x)... n(x)
•——•——•–...—>• x

N-dimensional arrow structures are natural generalizations of the notion of a directed multi graph and a graph-theoretic analog of n-ary relations.

The corresponding minimal modal logic is called Basic Arrow Logic of dimension n and denoted by BAL^n.

Another generalization of the notion of arrow structure can be obtained if we consider i(x) not as a single point but as a set of points. In this way we obtain the notion of hyper arrow structure of dimension n.

There is a closed connection between hyper arrow structures of dimension n and attribute systems, studied in part two. Namely, let S be a hyper n-arrow structure. Define an attribute system S' as follows. Put $Ob_{S'}=Ar_S$, $AT_{S'}=(n)$, for $i\in(n)$ define $VALi=\{A\in Po_S/ (\exists x\in Ar_S)\ A\in i.x\}$ and for $x\in Ar_S$ and $i\in(n)$ define $f_{S'}(x,i)=i.x$. Obviously S' is an attribute system. Conversely, let S' be an attribute system with finite number of attributes: $AT_{S'}=\{a_1,\dots,a_n\}$. then S' determines a hyper n-arrow structure S as follows. Put $Ar_S=Ob_{S'}$, $Po_S=\bigcup\{VALa/a\in AT_{S'}\}$, for $i\in(n)$ and $x\in Ob_{S'}$ define $i.x=f(x,a_i)$. Obviously S is a hyper n-arrow structure. This connection between Attribute systems and hyper n-arrow structures suggests many analogies between the theory of attribute systems and the theory of arrow systems.Note that for n=1 the resulting notion is a sort of Property system.

Another example of hyper n-arrow structure is the following. Let $W\neq\emptyset$ be a set and ρ be a nonempty n-place relation in the power set 2^W of W i.e. $\rho\subseteq(2^W)^n$. Put Po=W, Ar=ρ and for $x=(\alpha_1,\dots,\alpha_n)\in\rho$ and $i\in(n)$ define $i.x=\alpha_i$. Then obviously (Po, Ar, (n), .) is a hyper n-arrow structure. This example shows that the theory of hyper n-arrow structures may have some implications to the theory of set-relations /relations in power set/.

In hyper arrow structures of dimension n we can define the following relations: R^S_{ij}, Σ^S_{ij}, N^S_{ij}, $\leq^S_{ij}$, $x,y\in Ar_S$ $i,j\in(n)$

$xR^S_{ij}y$ iff $i.x=j.y$,

$x\Sigma^S_{ij}y$ iff $(i.x)\cap(j.y)\neq\emptyset$,

$xN^S_{ij}y$ iff $\overline{(i.x)}\cap\overline{(j.x)}\neq\emptyset$, where $\overline{(i.x)}=Po-(i.x)$

$x\leq^S_{ij}y$ iff $i.x\subseteq j.y$.

In some sense these relations describe the picture of the of a given hyper arrow structure and also are suitable for a semantics of modal logics, called hyper arrow logics.

"Arrow Logic" is the name of the last part - part VI of the dissertation. It is devoted to a study of different kinds of arrow structures and the corresponding arrow logics. Applications to Algebraic Logic have been done: a natural extension of the logic BAL^n is introduced, which have to be consider as a modal analog of a decidable version of first-order predicate logic. As a consequence of the theory of arrow structures of dimension n we obtain that any first-order theory of one n-place relation can be reduced to a first-order theory of some special binary relations. Another consequence is a reduction of any modal logic based on n-ary modalities to some arrow logic having only unary modalities.

3. On the methods and results of the dissertation

3.1. On the general form of the problems we deal with in the dissertation

The dissertation contains a systematic and uniform study of certain modal logics, naturally arising in some branches of Information Science. It includes results in Dynamic Logic, Modal Logics for Information Systems, Approximation Logics based on Rough Sets Theory and Arrow Logics and covers some of the results obtained by the author during the last ten years.

A typical problem in the field of Applied Modal Logic, and hence in this dissertation, has the following parts:

(1) After an appropriate mathematical analysis of certain domain, connected with a given applied area, to obtain a natural semantical definition of a modal logic L, having its standard semantics in the frames, arising from the analyzing domain.

(2) A mathematical study of the introduced modal logic L. In the dissertation we concentrate ourselves to the following two main mathematical problems concerning L:

(A) Axiomatization of L. This means to find a formal system, which is complete with respect to the intended standard semantics of L and to prove the completeness theorem.

(B) Decidability of L. This means to prove (or disprove) the existence of decision procedure, which recognizes theorems of L.

We will illustrate this by an example from chapter 2.5 of the dissertation. Let S=(Ob, At, {Val(a)/a∈At}, f) be an Attribute system and let (Ob, ≡, ≤, σ, D) be the relational system over S where ≡, ≤, σ and D are the following relations in the set Ob

x≡y iff (∀a∈At)(f(x,a)=f(y,a)) - indiscernibility relation, *x≤y iff (∀a∈AT)(f(x,a)⊆f(y,a)) - informational inclusion,*

xσy iff (∀a∈At)(f(x,a)∩f(y,a)≠∅) - similarity relation,

D={x∈OB/(∀a∈At)(Cardf(x,a)≤1)} - the set of single-valued objects in S.

Such relational systems naturally define a corresponding modal logic, say L. This is part (1) of the problem - the semantical definition of the logic L.

How to solve the problem (A) - the axiomatization of L? First we observe that the class of standard frames for L have concrete definitions in the class of all Attribute systems. However, in Modal Logic there are methods of axiomatization of some classes of frames, satisfying some abstract, for instance, first-order conditions. So, one way to axiomatize L is to obtain an abstract characterization of the frames of L if it is possible. We see that for certain subclass of the frames of L this is possible. These are the frames over the so called separable Attribute systems, satisfying the following condition

(∀A,B∈VALa)((∀x∈OB)(A∈f(x,a) iff B∈f(x,a))⟶ A=B)

We observe that the frames (Ob, ≡, ≤, σ, D) satisfy the following first-order conditions

Lemma

(i) Let S be an Attribute system. Then the following conditions hold:

S1. $x \le x$,

S2. $x \le y$ and $y \le z \longrightarrow x \le z$,

S3. $x\sigma y \longrightarrow y\sigma x$,

S4. $x\sigma y \longrightarrow x\sigma x$,

S5. $x\sigma y$ and $x \le z \longrightarrow z\sigma y$,

S6. $y\in D$ and $x \le y \longrightarrow x\in D$,

S7. $x\in D$ and $x\sigma y \longrightarrow x \le y$,

S8. $x \equiv x$,

S9. $x \equiv y \longrightarrow y \equiv x$,

S10. $x \equiv y$ and $y \equiv z \longrightarrow x \equiv z$,

S11. $x \equiv y \longrightarrow x \le y$,

S12. $x\in D$, $y\in D$, $x\sigma y \longrightarrow x \equiv y$,

Sa. $x \le y$ and $y \le x \longrightarrow x \equiv y$.

(ii) If S is a separable Attribute system then
Sb. $x \notin D \longrightarrow (\exists y \in OB)(x \not\equiv y)$.

Let **U**=(U, ≡, ≤, σ, D) be an abstract relational system with U≠∅, ≡, ≤, σ - binary relations in U, and D⊆U. We say that U is a *D-structure* if it satisfies the conditions S1 - S12, Sa and Sb from lemma 1.2.; **U** is a *generalized D-structure* if it satisfies the conditions S1-S12. Let S=(Ob, At, {Val/a∈AT}, f) be an Attribute system. Then the relational system (Ob, $\equiv_S$, $\leq_S$, σ_S, D_S) is called a *standard D-structure over S*. Standard D-structures constitute the natural semantics of the logic L. Note that they satisfy S1-S12 and Sa and for separable Attribute systems - Sb.

The second step in the axiomatization of the L is the following

Theorem (Abstract Characterization Theorem for D-structures) For any D-structure U=(U, ≡,≤, σ, D) there exists a separable Attribute system S=(Ob, At, {Vala/a∈At}, f) such that Ob=U and for any x,y∈W we have

$x \equiv y$ iff $(\forall a \in At)(f(x)=f(y))$ iff $x \equiv_S y$,

$x \leq y$ iff $(\forall a \in At)(f(x) \subseteq f(y))$ iff $x \leq_S y$,

$x \sigma y$ iff $(\forall a \in At)(f(x) \cap f(y) \neq \emptyset)$ iff $x \sigma_S y$,

$x \in D$ iff $(\forall a \in At)(\mathrm{Card} f(x) \leq 1)$ iff $x \in D_S$.

Let us note that this is not an abstract characterization of the standard frames of the logic L - this is a characterization of the standard frames over separable Attribute systems.

Now we introduce following classes of frames for L: Σ_0 - generalized D-structures, Σ_1 - standard D-structures over arbitrary Attribute systems, Σ_2 - standard D-structures over separable Attribute systems, Σ_3 - D-structures. Models based on structures from Σ_1 and Σ_2 will be called standard and models over structures from Σ_0 will be called non-standard models for L. From the Abstract Characterization Theorem for D-structures we obtain that $\Sigma_2=\Sigma_3$ and hence that for the corresponding logics we have $L(\Sigma_2)=L(\Sigma_3)$. Natural logics are $L(\Sigma_1)$ and $L(\Sigma_2)$ and we intend to define the logic L to be $L(\Sigma_1)$. Our aim is the axiomatization of $L(\Sigma_1)$. We will do this in the following order: first we will axiomatize $L(\Sigma_0)$ and then we will show that $L(\Sigma_0)=L(\Sigma_1)= =L(\Sigma_2)=L(\Sigma_3)$.

The axiomatization of $L(\Sigma_0)$ is easy because all the conditions S1-S12 are canonical and the axiomatization can be done by the canonical method.

The next observation is that we have the following inclusions $\Sigma_3 \subseteq \Sigma_1 \subseteq \Sigma_0$ and hence for the corresponding logics - $L(\Sigma_0) \subseteq L(\Sigma_1) \subseteq L(\Sigma_3)$. If we can prove that $L(\Sigma_3) \subseteq L(\Sigma_0)$ we will obtain the desired equalities $L(\Sigma_0)=L(\Sigma_1)=L(\Sigma_2)=L(\Sigma_3)$. Since we have an axiomatization of $L(\Sigma_0)$ we will obtain that this axiomatization will be also an axiomatization for $L(\Sigma_1)$ - our logic L. The proof of the inclusion $L(\Sigma_3) \subseteq L(\Sigma_0)$ can be done by applying the "copying method", which completes the full solution of the problem A for the logic L (Proposition 5.2). As an additional result we obtain that the logics over separable systems and arbitrary systems coincide.

The solution of the problem (B) for L goes as follows. Let Σ_{0fin} be the class of all finite structures of the class Σ_0. Using the filtration method known from monomodal logic ([Seg 71]) we prove that L possesses finite model property with respect to the class Σ_0 which implies that $L(\Sigma_0)=L(\Sigma_{0fin})$. Since $L=L(\Sigma_0)$ this implies that $L=L(\Sigma_0)$, which implies the decidability of L (Theorem 7.1.). An additional question connected with L is whether L possesses finite model property with respect to its standard semantics - the class Σ_1. It can be proved that this is not true, because it can be shown that $L\neq L(\Sigma_{1fin})$.

This example illustrates how we can proceed with a given semantically defined logic L. It shows also that to solve the standard problems A for L we have to develop a non-trivial mathematical study of the corresponding semantical domain as to obtain the Abstract Characterization Theorem and then the deep combination of this theorem with the copying construction and the canonical construction. To solve the problem B we have to apply a nontrivial version of the filtration technique.

Another kind of typical problem in the dissertation is when the logic L is already defined axiomatically. Then one of the main problems for L is to find an adequate semantics for it, to study its decidability and so on. Examples of such kind of problems are contained in chapter 1.1.

3.2. On the new methods used in the dissertation

The problems which attracted our attention needed new methods for their solution. In some sense this dissertation is a result of finding such new methods and improving some existing ones in order to attack the new problems. Among the new methods I can mention the method of "abstract characterization theorems" and the "copying method".

The method of "abstract characterization theorems" consists of characterizing semantical structures based on concretely defined relations, by means of abstract first-order sentences. The need of this method I saw in 1984, when I have tried to prove some completeness theorems for modal logics based on Pawlak's information systems. These modal logics have its natural semantics in classes of frames with concretely defined relations. First theorem of this kind have been published in [Vak 87]. Then applications of this method can be seen in [Vak 87a,89,91,91a,91c, 92b,93,94,95,95a]. In some nontrivial cases in part two and four of the dissertation this method can be considered as a generalization of the Stone [Sto 37] representation theory for distributive lattices and Boolean algebras to relational systems far away from lattices. The analogy with Stone theory is also seen from the fact that some of the "abstract characterization theorems" are based on a theory of filters and ideals very closed formal analogy with Stone theory of filters and ideals in distributive lattices and Boolean algebras.

The "copying method" is a technical tool making possible to axiomatize some applied modal logics, whose standard semantics is not modally definable in the corresponding modal language. It is a method of constructing p-morphic preimages of frames. Other such methods are the "bulldozer method" of Segerberg [Seg 71] and the "unraveling method" of Sahlqvist [Sah 75]

The history of the "copying method" is the following. At around 1984 Passy and Tinchev found a method of axiomatizing intersection and complement of programs in PDL, called the "method of data constants" (see [Pa 84], [P&T 85,91], [Tin 86]). Since the "method of data constants" requires an additional rule of inference, the need of another method of axiomatization of intersection and complement of modalities have been formulated. In 1987 I saw in [H&C 84] a very simple proof that the logic K is complete in the class of irreflexive frames. The proof have been based on a construction called "duplicating points". I noticed that this construction can be extended to prove completeness theorems for many other cases of modally undefinable conditions, including intersection and complement. I and my colleagues from the Sofia Seminar of Non-Classical Logic decided to name this construction "copying". Gargov, Passy and Tinchev adopted this construction in the "important lemma" in [GPT 88]. Applications of the "copying method" can be seen in [Vak 87a, 88,89,91b,92a,92b,92c,92d,93], [G&P,90], [Gor 90], [Pe,88], [Sk&St, 91]. Let us mention that in many cases the "copying method" and the "method of abstract characterization theorems" have to be applied in a combinations with each other and possibly with some other methods.

Among the methods I have improved is the canonical method for polyadic modalities (chapter 3.1 and [Vak 91b]) based on the notion of co-theory (see [Vak 89a]). For proving finite model property and decidability I have used very intensively the method of filtration. This method for monomodal logics have been invented by Scott and Lemon and improved by Segerberg (see [Seg 71]). In the monomodal case there are several constructions of filtrations for the case of symmetric relation, transitive relation, reflexive and transitive relation and equivalence relation. These constructions, however cannot be used automatically in the polymodal case because the existing of several interacting modalities makes the application of the method very complicated. In this dissertation there are many improvements of these constructions in various nontrivial cases as well as many new ones.

PART I. INDUCTION AXIOMS IN DYNAMIC LOGIC

Propositional Dynamic Logic PDL is one of the first applied modal logics arising from Computer Science. As a predecessor of PDL we can mention Salwicki's Algorithmic Logic [Mi&Sal 87]. /For the history of PDL see Harel [Ha 84] and also Passy & Tinchev [Pa&Ti 91]/. The main idea in PDL comes from the following intuitive "dynamic" interpretation of Kripke frames. The set W of a frame $(W,R(\alpha))$ is interpreted as a collection of data and the relation $R(\alpha)$ as an input-output relation determined by a program α, acting over the elements of W. Then the relation $xR(\alpha)y$ means that the input x is performed by α in the output y. Now $\Box A$ and $\Diamond A$, denoted by $[\alpha]A$ and $<\alpha>A$, have the following intuitive readings:

$[\alpha]A$ - "always after α A",

$<\alpha>A$ - "sometimes after α A".

In this "dynamic" interpretation formulas correspond to properties of data and the modal operations $[\alpha]$ and $<\alpha>$ as property transformers in the following sense: for a given formula A, considered as a property, $[\alpha]A$ is a new property, which is possessed by the input x of α iff A is possessed by every output y, which is a result of the execution of x by α, and likewise for $<\alpha>A$. In this way modal logic can be used for talking about data and programs. For instance the formula $A\Rightarrow[\alpha]B$ express partial correctness of α with respect to the precondition A /condition for the input of α/ and postcondition B /condition of the output of α/. In the language of the full PDL we may compose more complex programs using some program constructs and for each program α we have two modalities $[\alpha]$ and $<\alpha>$.

The main iterative procedure in PDL is the non-deterministic iteration α^* - "choose nondeterministically a natural number n and repeat α n-times". The input-output relation $R(\alpha^*)$ of α^* is the reflexive and transitive closure $R^*(\alpha)$ of the relation $R(\alpha)$. In the axiomatization of PDL given by Segerberg [Seg 82] $\alpha^*$is axiomatized by the following two axioms, which I denote by Seg and Seg_0:

Seg $\quad A\wedge[\alpha^*](A\Rightarrow[\alpha]A)\Rightarrow[\alpha^*]A$,

Seg_0 $\quad [\alpha^*]A\Rightarrow A\wedge[\alpha][\alpha^*]A$.

The axiom Seg is known as "Segerbergs induction axiom". Seg_0 corresponds to the following first-order condition connected the relations $R(\alpha)$ and $R(\alpha^*)$:

$(\forall x)xR(\alpha^*)x$ & $(\forall xyz)(xR(\alpha)y$ & $yR(\alpha^*)z \longrightarrow xR(\alpha^*)z)$.

The conjunction of Seg and Seg_0 modally define the equality $R(\alpha^*)=R^*(\alpha)$. However the semantic condition corresponding to Seg alone has been an open problem.

In this part we study the axiom Seg and some of its natural generalizations. The part is divided into two chapters.

In Chapter 1.1 "On the Segerberg's induction axioms" we study bi-modal logics, called Inductive Modal Lgics, with two modal operations $\Box$ and $\Box^*$, corresponding to $[\alpha]$ and $[\alpha^*]$ and containing the induction axiom Seg. The minimal inductive modal logic is denoted by KInd. The corresponding semantical condition of Seg is found and the completeness theorems and decidability for some natural extensions of KInd are proved. The method which we apply in the completeness theorem is a refined version of the Segerberg's filtration for PDL with combination with copying for one important special case. The results of chapter 1.1 are published in [Vak 92c].

In Chapter 1.2. "A modal characterization of cyclic repeating of programs" we apply the methods, developed in Chapter 1.1 to study a specific PDL-operator **cycle**α called cyclic repeating, which has been introduced by Passy [Pa 84] and studied by Passy and Tinchev [Pa&Ti 91]. To formulate the semantics of **cycle**α we need the following definition. A sequence $C=x_1,x_2,x_3,\ldots,x_n$ is called an α-cycle if $x_1R(\alpha)x_2R(\alpha)x_3..x_nR(\alpha)x_1$. Then we have:

$x \Vdash_V$ **cycle**α iff there exists an α-cycle C such that $x \in C$.

By means of the notion of α-cycle we may define the notion of cyclic iteration α^C with the following semantics:

(C) $xR(\alpha^C)y$ iff there exists an α-cycle C such that $x, y \in C$.

Now **cycle**α can be defined by $\langle\alpha^C\rangle\top$. We introduce the logic MLRC - the Modal Logic for Cyclic Repeating, containing the modalities $[\alpha]$, $[\alpha^*]$ and $[\alpha^C]$. It appears that $[\alpha^C]$ satisfies another induction axiom $[\alpha]A \wedge [\alpha^C](A \Rightarrow [\alpha]A) \Rightarrow [\alpha^C]A$, similar to Seg, which makes possible to apply the methods developed for KInd for the axiomatization of MLCR. The difficulties now are two: non- canonicity of MLCR and the fact that the condition (C) connecting the relations $R(\alpha)$ and $R(\alpha^C)$ is not modally definable. So, in the completeness proof we first apply a suitable filtration to the canonical model followed by a suitable copying construction. The results of chapter 1.2 are published in [Vak 92d].

CHAPTER 1.1 ON THE SEGERBERG'S INDUCTION AXIOMS

Overview. In this chapter we study Segerberg's induction axiom from Dynamic Logic

Seg $A \wedge \Box^*(A \Rightarrow \Box A) \Rightarrow \Box^* A$.

and some of its generalizations. The minimal bi-modal logic containing Seg is called KInd - Inductive Modal Logic. An adequate semantical characterization of Seg is given and completeness theorems and decidability for a number of natural extensions of KInd are proven.

Introduction

One of the main features in PDL /Propositional Dynamic Logic/ is the operation * of non-deterministic iteration of a program α. In PDL each program α is interpreted by a binary relation $R(\alpha)$ and the interpretation $R(\alpha^*)$ of α^* is the transitive and reflexive closure $R^*(\alpha)$ of the relation $R(\alpha)$. The axioms for the corresponding modality $[\alpha^*]A$ in PDL are given by Segerberg [Seg 82a]:

Seg $A\wedge[\alpha^*](A\Rightarrow[\alpha]A)\Rightarrow[\alpha^*]A$

Seg_0 $[\alpha^*]A\Rightarrow A\wedge[\alpha][\alpha^*]A$

The axiom Seg is known as the "Segerberg induction axiom".

The axiom Seg_0 modally define the following first order conditions for the relations $R(\alpha)$ and $R(\alpha^*)$:

(*1) $xR(\alpha^*)x$,

(*2) If $xR(\alpha)y$ and $yR(\alpha^*)z$ then $xR(\alpha^*)z$

The conditions (*1) and (*2) together imply the following inclusion

$R^*(\alpha)\subseteq R(\alpha^*)$

It is known that the conjunction of Seg and Seg_0 define the equality

$R(\alpha^*)=R^*(\alpha)$,

which is the intended interpretation of α^*.

Let us note that the corresponding semantical condition for Seg alone has been an open problem. One of the main aims of this chapter is to present a solution of this problem. We will work in a simpler modal language having two modalities $\Box$ and $\Box^*$ corresponding to the modalities $[\alpha]$ and $[\alpha^*]$ from PDL. The interpretation of this modalities will be in a bi-modal frames (W, R, S), R for the interpretation of $\Box$ and S for the interpretation of $\Box^*$. The minimal normal bi-modal logic containing the modalities $\Box$ and $\Box^*$, extended with the Segerberg's axiom

Seg $A\wedge\Box^*(A\Rightarrow\Box A)\Rightarrow\Box^*A$

will be denoted by KInd and called the minimal inductive logic.

The chapter is divided into four parts.

In part I we give the corresponding semantical condition Ind for the axiom Seg. A frame (U, R, S) satisfying Ind is called an inductive frame. We prove that the logic KInd is complete in the classes of all inductive frames and all finite inductive frames. The later implies the decidability of KInd. The completeness proof generalizes the Segerberg's filtration techniques for PDL from [Seg 82a].

In part II we consider some extensions of KInd with traditional axioms as to illustrate how the filtration techniques for some well known mono-modal logics as KT, KB, K4, S4 and S5 can be generalized for the case of inductive logics.

In part III we consider three special inductive logics - the logic of reflexive and transitive closure LRTC, the logic of cyclic equivalence LCE and their union - LRTC&CE.

The logic of cyclic equivalence is the logic of all frames (U, R, S) where $S=R^c=R^*\cap(R^*)^{-1}$. The name follows from the following fact. Call a non-empty sequence $C=(x_1,x_2,\ldots,x_n)$ an R-cycle if $x_1Rx_2R\ldots Rx_nRx_1$. Then we have xSy iff x=y or there exists an R- cycle C such that $x,y\in C$. Cyclic equivalence is an important notion in graph theory.

The main difference between LRTC and LCE is that the frames for LRTC are modally definable, whereas the frames for LCE are not. We overcome this difficulty by proving first the completeness of the proposed axiomatics with respect to a more general class of frames, which are modally definable, called nonstandard ones, and then by the "copying method", we transform non-standard models into standard ones.

The final part IV is devoted to another Segerberg's induction axiom and its generalization - Seg1 and gSeg:

Seg1 $\Box A\wedge\Box^{+}(A\Rightarrow\Box A)\Rightarrow\Box^{+}A$

gSeg $\Box_{1}A\wedge\Box^{*}(A\Rightarrow\Box A)\Rightarrow\Box^{*}A$

The axiom Seg1 together with the axiom $\Box^{+}A\Rightarrow \Box A\wedge\Box\Box^{+}A$ define exactly that $S=R^{+}$, where R^{+} is the transitive closure of R.

The axiom gSeg generalizes Seg and Seg1 in the following way: if we have $\Box_{1}A=A$ then we obtain Seg and if $\Box_{1}A=\Box A$ then we have Seg1. The normal three-modal logics containing gSeg are called generalized inductive logics. We show briefly a way how the results of this chapter can be modified for generalized inductive logics. An application of these methods are given in the next chapter. The results of this chapter are published in [Vak 92c].

I. ON THE MINIMAL INDUCTIVE LOGIC KInd

1. **Inductive logics and inductive frames**

The bi-modal language $\mathcal{L}(\Box, \Box^{*})$, which we will consider, contains an infinite set VAR of propositional variables, the classical Boolean connectives - $\neg$, $\wedge$ and $\vee$, two modal operations $\Box$, $\Box^{*}$, and parentheses - (,). The definition of the set FOR of all formulas is the usual one.

Abbreviations: $\Diamond$, $\Diamond^{*}$, $\Rightarrow$, $\Leftrightarrow$, $\top$, $\bot$ - are defined as usual.

If A is a formula, then $|A|$ will denote the number of all subformulas of A.

Let L be a set of formulas. We say that L is an inductive logic if it satisfies the following conditions:

(Bool) L contains all Boolean tautologies,

(K$\Box$) $\Box(A\Rightarrow B)\Rightarrow(\Box A\Rightarrow\Box B)\in L$,

(K *) $\Box^{*}(A\Rightarrow B)\Rightarrow(\Box^{*}A\Rightarrow\Box^{*}B)\in L$,

(Seg) $A\wedge\Box^{*}(A\Rightarrow\Box A)\Rightarrow\Box^{*}A\in L$,

(MP) if A, $A\Rightarrow B\in L$ then $B\in L$,

(N$\Box$) if $A\in L$ then $\Box A\in L$,

(N$\Box$*) if $A\in L$ then $\Box^{*}A\in L$,

(Sub) if $A\in L$ then all substitution instances of A are in L. An inductive logic L is called consistent if $L\neq FOR$.

The smallest inductive logic will be denoted by KInd.

The language $\mathcal{L}(\Box, \Box^{*})$ will be interpreted in relational systems (U, R, S), called frames, in which $U\neq\emptyset$ and R, S are binary relations in U. An arbitrary function $v: VAR\rightarrow 2^{U}$ / 2^{U} is the set of all subsets of U/ is called a valuation. The satisfiability relation - $x\Vdash_{v} A$ /the formula A is true in the point $x\in U$ at the valuation v/ is defined inductively as in the usual Kripke semantics:

$x\Vdash_{v} A$ iff $x\in v(A)$ for $A\in VAR$,

$x\Vdash_{v} \neg A$ iff $x\nVdash_{v} A$, ($x\nVdash_{v} A$ means not $x\Vdash_{v} A$),

$x\Vdash_{v} A\wedge B$ iff $x\Vdash_{v} A$ and $x\Vdash_{v} B$,

$x\Vdash_{v} A\vee B$ iff $x\Vdash_{v} A$ or $x\Vdash_{v} B$,

$x\Vdash_{v} \Box A$ iff $(\forall y\in U)(xRy \rightarrow y\Vdash_{v} A)$,

$x\Vdash_{v} \Box^{*}A$ iff $(\forall y\in U)(xSy \rightarrow y\Vdash_{v} A)$.

A frame $\underline{U}=(U, R, S)$ with a valuation v is called a model over $\underline{U}$ and is denoted by (U, R, S, v). A formula A is true in a model (U, R, S, v) if for any $x\in U$ we have $x\Vdash_{v} A$, and A is true in a frame $\underline{U}$ if A is true in any model over $\underline{U}$. A set of formulas L is true in a frame $\underline{U}$, or $\underline{U}$ is a frame for L if all formulas from L are true in $\underline{U}$. A logic L is sound in a class Σ of frames if all frames from Σ are frames for L. A logic L is complete in a class Σ of frames for L if for any formula A: if A is true in all frames from Σ then $A\in L$.

In order to find an adequate semantical condition for Seg we will introduce a new operation on binary relations, which generalizes the operation R^*, of reflexive and transitive closure of R. Let us first remind the inductive definition of R^*. We will use the following denotations:

$id=\{(x,x)/x\in U\}$, $R\circ S=\{(x,z)\in U\times U/(\exists y\in U)(xRy \,\&\, ySz)\}$. Then:

$R^0=id$, $R^{i+1}=R^i\circ R$, $R^*=\bigcup_{i=o}^{\infty} R^i$.

From this definition of R^* we have:

xR^0y iff $x=y$,

for $i>0$ xR^iy iff $\exists\, x_1,x_2,\dots,x_i\in U$ such that $x=x_1Rx_2R\dots Rx_iRy$

xR^*y iff $x=y$ or there exists a nonempty sequence $x_1,x_2,\dots,x_i$ such that $x=x_1Rx_2R\dots Rx_iRy$.

Now we generalize the operation $*$ as follows. Let R and S be binary relations in U. Then we define:

$R_S^0=id$, $R_S^{i+1}=(R_S^i\cap S)\circ R$, $R_S^*=\bigcup_{i=o}^{\infty} R_S^i$

Let us denote by $S(x)=\{y\in U/xSy\}$. Then from the above definitions we obtain:

xR_S^0y iff $x=y$,

for $i>0$ xR_S^iy iff $\exists x_1,x_2,\dots,x_i$ such that $x=x_1Rx_2R\dots Rx_iRy$ and $\{x_1, x_2,\dots,x_i\}\subseteq S(x)$

xR_S^*y iff $x=y$ or there exists a nonempty sequence $x_1,x_2,\dots x_i$ such that $x=x_1Rx_2R\dots Rx_iRy$ and $\{x_1,x_2,\dots,x_i\}\subseteq S(x)$.

The following two properties follow from the above observations:

$R_S^i \subseteq R^i$, $R_S^*\subseteq R^*$

The following fact will be of later use:

Lemma 1.1.
$R^*_{(R_S^*\cap S)}=R_S^*$

Proof. We will show by induction on i that $R^i_{(R_S^*\cap S)}=R_S^i$, which will prove the lemma.

<u>i=0.</u> $R^0_{(R_S^*\cap S)}=id=R^0$.

<u>i=k</u> /i.h/ Let $R^k_{(R_S^*\cap S)}=R_S^k$

<u>i=k+1.</u> $R^{k+1}_{(R_S^*\cap S)}=(R^k_{(R_S^*\cap S)}\cap R_S^*\cap S)\circ R=(R_S^k\cap R_S^*\cap S)\circ R=(R_S^k\cap S)\circ R=R_S^{k+1}$. ■

A frame (U, R, S) is called an inductive frame if it satisfies the following condition:

(Ind) $S\subseteq R_S^*$

Theorem 1.2. */Definability theorem for Seg/*
The formula Seg is true in a frame (U, R, S) iff (U, R, S) is an inductive frame.

Proof. (⟵) Suppose $S\subseteq R_S^*$, $x\in U$ and that for an arbitrary valuation v we have $x\Vdash_v A$, $x\Vdash_v \Box*(A\Rightarrow\Box A)$ and xSy. We have to show that $y\Vdash_v A$.

From xSy and $S\subseteq R_S^*$ we obtain xR_S^*y and hence for some i: xR_S^iy. Now we proceed inductively:

<u>i=0</u>. Then x=y and since $x\Vdash_v A$, so $y\Vdash_v A$.

<u>i=k.</u> (i.h.) Suppose that for some k and for any $z\in U$:

if $xR_S^k z$ then $z \Vdash_v A$.

<u>i=k+1.</u> Then $xR_S^{k+1}y \longleftrightarrow x((R_S^k \cap S)\circ R)y \longleftrightarrow \exists z \in U\ xR_S^k z$, xSz and zRy.

From $xR_S^k z$ by i.h. we have $z \Vdash_v A$.

From xSz, $x \Vdash_v \Box*(A \Rightarrow \Box A)$, $z \Vdash_v A$ and zRy we obtain $y \Vdash_v A$.

($\longrightarrow$) Suppose that not $S \subseteq R_S^*$. Then for some $x, y \in U$ we have xSy and not $xR_S^* y$. We will show that Seg is not true in (U, R, S).

For some fixed $A \in VAR$ define

$v(A)=\{t \in U/\text{not } xSt\} \cup \{t \in U/xR_S^* t\}$ and for $B \neq A$, $B \in VAR$ let $v(B)$ be arbitrary.

We have $y \notin v(A)$ so $y \nVdash_v A$ and by xSy - $x \nVdash_v \Box*A$. Since $xR_S^* x$ we have $x \in v(A)$, so $x \Vdash_v A$. We shall show that $x \Vdash_v \Box*(A \Rightarrow \Box A)$. For that purpose suppose xSt, $t \Vdash_v A$, tRz and proceed to show that $z \Vdash_v A$.

From xSt and $t \Vdash_v A$ we obtain that $xR_S^* t$. Then for some i $xR_S^i t$. But $xR_S^i t$, xSt and tRz give $xR_S^{i+1} z$, so $xR_S^* z$. This shows that $z \in v(A)$ and hence $z \Vdash_v A$.

The obtained conditions $x \Vdash_v A$, $x \Vdash_v \Box*(A \Rightarrow \Box A)$ and $x \nVdash_v \Box*A$ imply that $x \nVdash_v A \wedge \Box*(A \Rightarrow \Box A) \Rightarrow \Box*A$. So Seg is not true in (U, R, S).■

Corollary 1.3.

The logic KInd is sound in the class of all inductive frames.

2. Preliminary facts and definitions

Canonical constructions.

Let L be any consistent inductive logic. By a canonical frame for L ([Seg 71], [H&C 84]) we mean the structure $\underline{U}=(U_L, R_L, S_L)$, where U_L is the set of all maximal consistent sets of formulas of L and the relations R_L and S_L are defined as follows:

$xR_L y$ iff $\{A \in FOR/\ \Box A \in x\} \subseteq y$,

$xS_L y$ iff $\{A \in FOR/\ \Box*A \in x\} \subseteq y$,

If we define

$v_L(A)=\{x \in U_L/A \in x\}$, A is a variable

then $M_L=(U_L, R_L, S_L, v_L)$ will be called canonical model for L.

Fact 2.1.

(i)/Lindenbaum Lemma/ If x is an L-consistent set of formulas then x can be extended to a maximal L-consistent set. In particular if $A \notin L$ then $\{\neg A\}$ is L-consistent set and $A \notin x$ for some $x \in U_L$.

For any $x \in U_L$ and $A, B \in FOR$:

(ii) $\neg A \in x$ *iff* $A \notin x$,

$A \wedge B \in x$ *iff* $A \in x$ *and* $B \in x$,

$A \vee B \in x$ *iff* $A \in x$ *or* $B \in x$,

$\Box A \in x$ *iff* $(\forall y \in x)(xR_L y \longrightarrow A \in y)$,

$\Box*A \in x$ *iff* $(\forall y \in x)(xS_L y \longrightarrow A \in y)$,

$L \subseteq x$, $\top \in x$, $\bot \notin x$.

Filtration. Here we shall list some basic definitions and facts about Segerberg's filtration of canonical frames, adapted for the bi-modal case ([Seg 71], [H&C 84]).

Let Φ be a finite set of formulas closed under subformulas. For $x, y \in U_L$ define $x \sim y$ iff $(\forall A \in \Phi)(A \in x$ iff $A \in y)$, $|x| = \{y \in U_L / x \sim y\}$, $|U_L| = \{|x| / x \in U_L\}$, $v(A) = \{|x| / A \in x\}$.

Let R and S be relations in $|U_L|$. Then the frame $(|U_L|, R, S)$ will be called a filtration of the canonical frame (U_L, R_L, S_L) trough Φ if the following conditions are satisfied:

(FRi) If xR_Ly then $|x|R|y|$.

(FRii) If $|x|R|y|$ then $(\forall \Box A \in \Phi)(\Box A \in x \longrightarrow A \in y)$.

(FSi) If xS_Ly then $|x|S|y|$.

(FSii) If $|x|S|y|$ then $(\forall *A \in \Phi)(*A \in x \longrightarrow A \in y)$.

Fact 2.2.

(i)/Filtration Lemma/ For any formula $A \in \Phi$ and $x \in U_L$ the following equivalence holds: $|x| \Vdash_v A$ iff $A \in x$.

(ii) If $card\Phi = n$ then $card|U_L| \leq 2^n$.

(iii) For any subset $M \subseteq |U_L|$ there exists a formula A which is a Boolean combination of elements of Φ, such that for any $x \in U_L$ we have: $A \in x$ iff $|x| \in M$.

We say that a consistent inductive logic L admits a filtration if:

(i) for any formula A there exists a finite set of formulas Φ, containing A and closed under subformulas, and

(ii) there exists a frame $\underline{U} = (|U_L|, R, S)$ for L, which is a filtration of the canonical frame $\underline{U}_L = (U_L, R_L, S_L)$ through Φ.

If for any finite set of formulas Φ, closed under subformulas, there exists a frame $\underline{U}$ for L, which is a filtration of $\underline{U}_L$ through Φ, then we say that L admits a good filtration.

Fact 2.3.

Let L be a consistent inductive logic which admits a filtration. Then:

(i) L is complete in the class of all finite frames for L; L is complete in the class of all frames for L; L has the finite model property /f.m.p./ and if L is finitely axiomatizable then L is decidable.

(ii) If L admits a good filtration then any non-theorem of L is falsified in a frame (U, R, S) for L such that $cardU \leq 2^{|A|}$. In this case L is decidable.

Proof. (i) Suppose $A \notin L$. Then by the fact 2.1 $A \notin x$ for some $x \in U_L$. Since L admits a filtration, there exist a finite set Φ containing A and closed under subformulas and a filtration $(|U_L|, R, S)$ trough Φ of the canonical frame (U_L, R_L, S_L). Then by the filtration lemma $|x| \not\Vdash_v A$ and hence A is not true in a /finite/ frame for L. This shows that L has the f.m.p. and that it is complete in the class of all finite frames for L and also in the class of all frames for L.

It is a well known fact that f.m.p. and finite axiomatization yields decidability.

(ii) follows from the fact 2.2.ii.■

It will be proved in this chapter that many inductive logics admit a good filtration. In this case we will suppose that fact 2.3 is automatically stated for them without explicit formulation.

3. Filtration for KInd

Theorem 3.1.

The logic KInd admits a good filtration.

Proof. Let L=KInd , A∈FOR and Φ be the set of all subformulas of A. Define:

$|x|R|y|$ iff $\exists x', y' \in U_L$, $x \sim x'$, $y \sim y'$ and $x' R_L y'$,

$|x|S'|y|$ iff $\exists x', y' \in U_L$, $x \sim x'$, $y \sim y'$ and $x' S_L y'$, $S = R^*_{S'} \cap S'$.

It is easy to see that the definitions of R and S' are correct, so the same is for S. The required frame is $\underline{U} = (|U_L|, R, S)$. By the fact 2.2.ii, $|U_L|$ is a finite set such that card$|U_L| \le 2^{|A|}$. It follows from the next lemma and theorem 1.2. that $\underline{U}$ is a frame for L.

Lemma 3.2.

If (U, R, S') is a frame and $S = R^*_{S'} \cap S'$ then (U, R, S) is an inductive frame.

Proof. By lemma 1.1 we have: $S = R^*_{S'} \cap S' = R^*_{(R_{S'} \cap S')} \cap S' = R^*_S \cap S'$, so $S \subseteq R^*_S$ and hence (U, R, S) is an inductive frame.■

For the condition (FRi) of the filtration suppose $xR_L y$, $x \sim x'$, $y \sim y'$, □A∈Φ, □A∈x'and proceed to show that A∈y'. From □A∈x' and $x' \sim x$ we obtain □A∈x and by $xR_L y$ - that A∈y. From □A∈Φ we obtain A∈Φ. Then A∈Φ , A∈y and $y \sim y'$ imply A∈y'.

In the same way we can show that the following is true:

(FS'i) If $xS_L y$ then $|x|S'|y|$.

For the condition (FSi) suppose $xS_L y$. Then by (FS'i) we obtain that $|x|S'|y|$. It remains to show that $|x|R^*_{S'}|y|$ holds. For that purpose let $M = \{|t| \in |U_L| / |x|R^*_{S'}|t|\}$. We will show that $|y| \in M$. Since $M \subseteq |U_L|$ then by the fact 2.2.iii there exists a formula A such that A∈t iff $|t| \in M$. So to prove $|y| \in M$ we have to show that A∈y. This will follow from the following

Assertion:

A∈x and □∗(A⇒□A)∈x.

Indeed, from the assertion we have A∧□∗(A⇒□A)∈x and by Seg - that □∗A∈x. Then from $xS_L y$ we obtain A∈y.

Now let us prove the Assertion.

Since $|x|R^*_{S'}|x|$ we have $|x| \in M$ and hence A∈x.

To show □∗(A⇒□A)∈x we will use fact 2.1. Suppose $xS_L t$, A∈t, $tR_L u$ and proceed to show that A∈u. From A∈t we have $|t| \in M$ and hence $|x|R^*_{S'}|t|$. Then for some i $|x|R^i_{S'}|t|$. From $xS_L t$ and (FS'i) we have $|x|S'|t|$. From $tR_L u$ and (FRi) we have $|t|R|u|$. Then the conditions $|x|R^i_{S'}|t|$, $|x|S'|t|$ and $|t|R|u|$ give $|x|R^{i+1}_{S'}|u|$, so $|x|R^*_{S'}|u|$, which shows that $|u| \in M$ and A∈u.■

To show (FRii) suppose $|x|R|y|$, □A∈Φ, □A∈x and proceed to show that A∈y. From $|x|R|y|$ we have that for some $x', y' \in U_L$: $x' \sim x$, $y' \sim y$ and $x' R_L y'$. □A∈Φ and $x \sim x'$ imply □A∈x'. Then by $x' R_L y'$ and $y' \sim y$ we obtain A∈y.

In a similar way we can prove the following condition:

(FS'ii) If $|x|S'|y|$ then (∀ □∗A∈Φ)(□∗A∈x ⟶ A∈y).

Now to prove the condition (FSii) suppose $|x|S|y|$. Then by the definition of S we have $|x|S'|y|$ and by (FS'ii) we have $(\forall\ \Box^*A\in\Phi)(\ \Box^*A\in x \longrightarrow A\in y)$, which proves (FSii).■

The following proposition will be of later use.

Proposition 3.3.

(i) Let (U, R, S) be an inductive frame and let $x,y\in U$. Then

(i) $xSy \longrightarrow xSx$

(ii) $\Diamond^*\top\Rightarrow(\Box^*A\Rightarrow A)\in KInd$

(iii) Let L be an inductive logic and $x,y\in U_L$. Then $xS_Ly\longrightarrow xS_Lx$.

Proof. (i) Suppose xSy. Then by inductivity we have xR^*_Sy.

<u>Case 1: x=y.</u> Then xSx.

<u>Case 2: x≠y.</u> Then for some $x_1,x_2,\dots,x_i\in U$: $x=x_1Rx_2R\dots Rx_iRy$ and $\{x_1,x_2,\dots,x_i\}\subseteq S(x)$. From the last condition we have xSx_1 and since $x=x_1$ - xSx.

(ii) Applying (i) it can be shown that the formula $\beta=\Diamond^*\top\Rightarrow(\Box^*A\Rightarrow A)$ is true in all inductive frames and since KInd is complete in this class of frames we obtain that $\beta\in$KInd. Of course β can be deduced directly from the axioms of KInd. We left this exercise to the reader.

(iii) Let L be an inductive logic. Then by (ii) $\beta\in L$. Suppose that the assertion is not true. Then for some $x,y\in U_L$ we have xS_Ly but not xS_Lx and hence for some $A\in FOR$ $\Box^*A\in x$ and $A\notin x$. This shows that $\Box^*A\Rightarrow A\notin x$. Since $\beta\in x$ we obtain that $\Diamond^*\top\notin x$, so $\Box^*\neg\top\in x$ and by xS_Ly we have that $\neg\top\in y$, so $\bot\in y$ - a contradiction.■

II. SOME EXTENSIONS OF KInd

4. Extensions of KInd with axioms containing only □

Let L be a monomodal normal modal logic with a modality denoted by □. Then by LInd we will denote the smallest inductive logic containing L. If for example L=K then LInd=KInd. Obviously LInd is axiomatized by adding to the axioms of KInd all theorems of L as new axioms. We will denote this as follows: LInd=KInd+L.

Proposition 4.1.

(i) If (U, R, S) is a frame for LInd then (U,R) is a frame for L.

(ii) If (U, R) is a frame for L then (U, R, S) is a frame for LInd iff $S=R^*_{S'}\cap S'$ for some $S'\subseteq U\times U$.

Proof. (i) Let (U, R, S) be a frame for LInd and let $A\in L$. Since A do not contain □* then A is true in (U, R) so (U, R) is a frame for L.

(ii) Let (U, R) be a frame for L.

(⟶) Then: (U, R, S) is a frame for LInd ⟷ (U, R, S) is a frame for Seg ⟷ $S\subseteq R^*_S$ ⟷ $S=R^*_S\cap S$. So in this case S'=S.

(⟵) Let $S'\subseteq U\times U$ and $S=R^*_{S'}\cap S'$. Then by lemma 3.2 (U, R, S) is an inductive frame. Since (U, R) is a frame for L then for any $A\in L$ we have that A is true in (U, R, S). So (U, R, S) is a frame for LInd.■

Theorem 4.2.

Let L be a normal monomodal logic with modality denoted by □. Then LInd is a conservative extension of L.

Proof. First we define a translation τ of LInd into L in the following way:

$\tau(A)=A$ for $A\in VAR$, $\tau(\neg A)=\neg\tau(A)$, $\tau(A\wedge B)=\tau(A)\wedge\tau(B)$, $\tau(A\vee B)=\tau(A)\vee\tau(B)$, $\tau(\ A)=\ \tau(A)$, $\tau(\ *A)=\tau(A)$.

Let FOR$\Box$ is the subset of all formulas of FOR not containing $\Box*$. Then the following lemma is true.

Lemma 4.3.

(i) If $A\in FOR$ then $\tau(A)=A$.

(ii) If $A\in LInd$ then $\tau(A)\in L$.

Proof. (i) is obvious.

(ii) We shall apply induction on the proof of A in LInd.

1. Suppose A is an axiom of LInd. Since LInd=KInd+L then A can be one of the axioms (K$\Box$), (K$\Box$*), (Seg) or $A\in L$. For the case $A\in L$ we have that $A\in FOR\Box$ and by (i) we have $\tau(A)=A$, so $\tau(A)\in L$.

$\tau(K\Box)=\Box(\tau(A)\Rightarrow\tau(B))\Rightarrow(\Box\tau(A)\Rightarrow\Box\tau(B))$ is again (K$\Box$) and hence belongs to L.

$\tau(K\Box*)=(\tau(A)\Rightarrow\tau(B))\Rightarrow(\tau(A)\Rightarrow\tau(B))$ is a substitution of a Boolean tautology and hence is in L.

$\tau(Seg)=\tau(A)\wedge(\tau(A)\Rightarrow\Box\tau(A))\Rightarrow\tau(A)$ is also a substitution of a Boolean tautology and consequently belongs to L.

2. Suppose B is obtained by (MP) from $A\in LInd$ and $A\Rightarrow B\in LInd$. We have to show that $\tau(B)\in L$. By i.h. we have that $\tau(A)\in L$ and $\tau(A\Rightarrow B)=\tau(A)\Rightarrow\tau(B)\in L$. Then by (MP) for L we obtain that $\tau(B)\in L$

Suppose that $\Box B$ is obtained from $B\in LInd$ by (N$\Box$). We have to show that $\tau(\Box B)=\Box\tau(B)\in L$. By the i.h. from $B\in LInd$ we obtain that $\tau(B)\in L$ and by (N$\Box$) for L we obtain that $\tau(\Box B)\in L$.

Suppose that $\Box*B$ is obtained from $B\in LInd$ by (N$\Box$*). We have to show that $\tau(\Box*B)=\tau(B)\in L$. From $B\in LInd$ we obtain by i.h. that $\tau(B)\in L$.■

To prove the theorem we have to show that if $A\in FOR\Box$ and $A\in LInd$ then $A\in L$. By lemma 4.3 if $A\in LInd$ then $\tau(A)\in L$ and if $A\in FOR\Box$ then $\tau(A)=A$. So if $A\in FOR\Box\cap LInd$ then $A\in L$.■

Theorem 4.4.

Let L be an incomplete normal modal logic with modality denoted by $\Box$. Then the logic LInd is incomplete too.

Proof. The incompleteness of L means that there exists a formula A of L such that $A\notin L$ and A is true in all frames (U,R) for L. By theorem 4.2 LInd is a conservative extension of L, so $A\notin LInd$. Let (U, R, S) be a frame for LInd. Then by proposition 4.1.i (U, R) is a frame for L and hence A is true in (U, R) and consequently in (U, R, S). So all frames for LInd are frames for A but $A\notin LInd$. This shows the incompleteness of LInd.■

Corollary 4.5.

There exist incomplete inductive logics.

Theorem 4.6.

Let L be a normal modal logic with a modality denoted by $\Box$. If L has not the f.m.p. then the logic LInd has not the f.m.p. either.

The proof is similar to that of theorem 4.4.

Corollary 4.7.

There exist inductive logics without f.m.p.

Theorem 4.8.

Let L be an undecidable normal modal logic with a modality denoted by $\Box$. Then LInd is undecidable logic too.

Proof. Suppose that L is an undecidable logic. Since LInd is a conservative extension of L, the same is LInd.■

Corollary 4.9.

There exist finitely axiomatizable undecidable inductive logics.

Problems.

4.1 *If L is a complete modal logic what about LInd?*

4.2 *If L admits a filtration what about LInd?*

4.3 *Is there a complete logic LInd such that L has not f.m.p.?*

We shall show that for some well known normal modal logics L, which admit filtration, the corresponding inductive logics LInd admit filtration too. We will consider the following well known examples for L: KT, KB, K4, S4 and S5 (see [H&C 84]). In the next table we give the axiomatics for the corresponding LInd and the class of inductive frames for it.

$KTInd=KInd+\Box A\Rightarrow A$	xRx
$KBInd=KInd+\Diamond\Box A\Rightarrow A$	$xRy \longrightarrow yRx$
$K4Ind=KInd+\Box A\Rightarrow\Box\Box A$	$xRy\&yRz \longrightarrow xRz$
$S4Ind=KTInd+\Box A\Rightarrow\Box\Box A$	$xRx,\ xRy\&yRz \longrightarrow xRz$
$S5Ind=S4Ind+\Diamond\Box A\Rightarrow A$	$xRx,\ xRy \longrightarrow yRx,\ xRy\&yRz \longrightarrow xRz$

Theorem 4.10.

Let L be any of the logics KT, KB, K4, S4 and S5. Then the logic L'=LInd admits a good filtration.

Proof. For KTInd use the same filtration as for KInd. Then for the canonical relation R_L, we have xR_Lx and by the condition (FRi) we obtain $|x|R|x|$, which says that the frame $(|U_L|, R, S)$ is a frame for KTInd. For all other cases the definition of Φ and the relation S is defined as in the filtration of KInd and for the relation R is used the Lemmon filtration for the corresponding logic L /see for e.g. [H&C 84]/. In the next table we list the corresponding definitions for R.

KBInd:	$\|x\|R\|y\|$ iff $(\forall\Box A\in\Phi)(\Box A\in x\longrightarrow A\in y)\&(\Box A\in y\longrightarrow A\in x)$,
K4Ind:	$\|x\|R\|y\|$ iff $(\forall\Box A\in\Phi)(\Box A\in x\longrightarrow A\wedge\Box A\in y)$,
S4Ind:	$\|x\|R\|y\|$ iff $(\forall\Box A\in\Phi)(\Box A\in x\longrightarrow\Box A\in y)$,
S5Ind:	$\|x\|R\|y\|$ iff $(\forall\Box A\in\Phi)(\Box A\in x\longleftrightarrow\Box A\in y)$.

For all cases the corresponding frame $(|U_{L'}|, R, S)$ is a frame for L', which proves the theorem.∎

5. Extensions of KInd with axioms containing only $\Box *$

Let L^* be a normal modal logic with a modality denoted by $\Box *$. The minimal inductive logic containing L^* will be denoted by L^*Ind. Obviously L^*Ind can be axiomatized by $KInd+L^*$. The completeness problem for the logics of this kind is much more difficult than the corresponding problem for LInd. We will consider only several representative examples for L^*: KT^*, $K4^*$, $S4^*$ and $S5^*$. Here we use $*$ to indicate that the modality in the corresponding logic is denoted by $\Box *$. In the next table we give the axiomatics for the corresponding L^*Ind and their classes of inductive frames.

$KT^*Ind=KInd+\Box *A\Rightarrow A$	xSx
$K4^*Ind=KInd+\Box *A\Rightarrow\Box *\Box *A$	$xSy\&ySz\longrightarrow xSz$
$S4^*Ind=K4^*Ind+\Box *A\Rightarrow A$	$xSx,\ xSy\&ySz\longrightarrow xSz$
$S5^*Ind=S4^*Ind+\Diamond *\Box *A\Rightarrow A$	$xSx,\ xSy\longrightarrow ySx,\ xSy\&ySz\longrightarrow xSz$

Theorem 5.1.

The logic $L=KT^*Ind$ admits a good filtration.

Proof. Use the same filtration as for KInd. For the canonical relation S_L we have xS_Lx for any $x\in U_L$. In the proof of theorem 3.1 we show the following implication:

(FS'i) If $xS_L y$ then $|x|S'|y|$.

Then by (FS'i) from $xS_L x$ we obtain $|x|S'|x|$, so S' is a reflexive relation.

Lemma 5.2.

Let (U, R, S') be a frame in which S' is a reflexive relation and let $S=R^*_{S'}\cap S'$. Then (U, R, S) is an inductive frame and S is a reflexive relation in U.

Proof. By lemma 3.2 the frame (U, R, S) is inductive one. Since $R^*_{S'}$ is a reflexive relation by definition and S' is a such one then obviously $S=R^*_{S'}\cap S'$ is a reflexive relation.■

It follows from lemma 5.2 that $(|U_L|, R, S)$ is an inductive frame with reflexive S, which shows that it is a frame for KT^*Ind.■

Note. The logic KT^*Ind coincides with the logic KD^*Ind, where KD^* is $K+\Diamond*\top$. This fact easily follows from proposition 3.3.iii.

Theorem 5.3.

The logic $K4^*Ind$ admits a good filtration.

Proof. The definitions of Φ, R and S are the same as in the filtration of KInd. The definition of S' is as in the Lemmon's filtration for K4:

$|x|S'|y|$ iff $(\forall\Box*A\in\Phi)(\Box*A\in x\longrightarrow A\wedge\Box*A\in y)$

S' is a transitive relation in $|U_L|$ and satisfies the conditions (FS'i) and (FS'ii) formulated in the proof of theorem 3.1. We need the following

Lemma 5.4.

Let S=(U, R, S') be a frame in which S' is a transitive relation in U and let $S=R^*_{S'}\cap S'$. Then S is a transitive relation in U and (U, R, S) is an inductive frame.

Proof. By lemma 3.2 (U, R, S) is an inductive frame. It remains to show that $R^*\cap S'$ is a transitive relation in U.

We have:

(1) $xR^*_{S'}y$ & $xS'y$ iff ($x=y$ or $\exists m\exists x_1,x_2,\dots,x_m$ such that $x=x_1Rx_2R\dots Rx_mRy$ and $\{x_1,x_2,\dots,x_m\}\subseteq S'(x)$) and $xS'y$

(2) $yR^*_{S'}z$&$yS'z$ iff ($y=z$ or $\exists n\exists y_1,y_2,\dots,y_n$ such that $y=y_1Ry_2R\dots Ry_nRz$ and $\{y_1,y_2,\dots,y_n\}\subseteq S'(y)$) and $yS'z$

We have to show that $xR^*_{S'}z$ & $xS'z$.

By the transitivity of S' we have $xS'z$ and $S'(y)\subseteq S'(x)$. It remains to show that $xR^*_{S'}z$. The cases $x=y$ or $y=z$ are trivial, hence suppose $x\neq y$ and $y\neq z$. Then from $\{x_1,x_2,\dots,x_m\}\subseteq S'(x)$, $\{y_1,y_2,\dots y_n\}\subseteq S'(y)$, $xS'y$, $x=x_1$, $y=y_1$ and $S'(y)\subseteq S'(x)$ we obtain that $\{x_1,x_2,\dots x_m,y_1,y_2,\dots,y_n\}\subseteq S'(x)$. Thus we have:

$x=x_1Rx_2R\dots Rx_mRy_1Ry_2R\dots Ry_nRz$ and $\{x_1,x_2,\dots,x_m,y_1,y_2,\dots,y_n\}\subseteq S'(x)$.

This shows that $xR^*_{S'}z$.■

From lemma 5.4 we have that $(|U_L|, R, S)$ is an inductive frame in which S is a transitive relation in $|U_L|$. This shows that $(|U_L|,R, S)$ is a frame for $K4^*Ind$.■

Theorem 5.5.

The logic $L=S4^{*}Ind$ admits a good filtration.

Proof. Define Φ, R and S as in the filtration of KInd. The definition of S' is as in the Lemmon filtration for S4:

$|x|S'|y|$ iff $(\forall\Box{*}A\in\Phi)(\Box{*}A\in x\rightarrow\Box{*}A\in y)$

Then S' is a reflexive and transitive relation in $|U_L|$ which satisfies the conditions (FS'i) and (FS'ii). The theorem follows from the next

Lemma 5.6.

Let (U, R, S') be a frame with a reflexive and transitive relation S' and let $S=R^{*}_{S'}\cap S'$. Then S is a reflexive and transitive relation in U and (U, R, S) is an inductive frame.

Proof - from lemma 5.2 and lemma 5.4.■

From lemma 5.6 we obtain that $(|U_L|, R, S)$ is an inductive frame in which S is a reflexive and transitive relation in $|U_L|$. This shows that $(|U_L|, R, S)$ is a frame for $S4^{*}Ind$.■

Theorem 5.7.

The logic $L=S5^{*}Ind$ admits a good filtration.

Proof. Define Φ and R as in the filtration of KInd. The definition of S' is as in the Lemmon filtration for S5:

$|x|S'|y|$ iff $(\forall\Box{*}A\in\Phi)(\Box{*}A\in x\leftrightarrow\Box{*}A\in y)$

Then S' is an equivalence relation in $|U_L|$ satisfying the conditions (FS'i) and (FS'ii).

The definition of S is the following: $S=R^{*}_{S'}\cap(R^{*}_{S'})^{-1}\cap S'$.

The theorem follows from the next lemma. To formulate it we need a new notion. Let (U, R, S) be a frame. A nonempty sequence $C=(x_1,x_2,\dots,x_n\}$ of elements of U is called *SR-cycle* if $x_1Rx_2R\dots Rx_nRx_1$ and for any $i,j\in\{1,\dots,n\}$ we have x_iSx_j. If x is a member of C then we will write $x\in C$.

Lemma 5.8.

Let (U, R, S') be a frame in which S' is an equivalence relation in U and let $S=R^{*}_{S'}\cap(R^{*}_{S'})^{-1}\cap S'$. Then:

(i) xSy iff x=y or there exists an S'R-cycle C and $x\in C$.

(ii) (U, R, S) is an inductive frame.

(iii) S is an equivalence relation in U.

Proof. (i) Let $x,y\in U$. Then:

xSy iff $(x=y$ or $\exists m\neq0\exists x_1,x_2,\dots x_m$: $x=x_1Rx_2R\dots Rx_mRy$ and $\{x_1,x_2,\dots x_m\}\subseteq S'(x))$ and $(y=x$ or $\exists n\neq0\exists x_{m+1},x_{m+2},\dots x_{m+n}$: $y=x_{m+1}Rx_{m+2}R\dots Rx_{m+n}Rx$ and $\{x_{m+1},x_{m+2},\dots,x_{m+n}\}\subseteq S'(y))$ and $xS'y$.

Since S' is an equivalence relation, the conditions $\{x_1,x_2,\dots,x_m\}\subseteq S'(x)$, $\{x_{m+1},x_{m+2},\dots,x_{m+n}\}\subseteq S'(y)$, $x=x_1$, $y=y_1$ and $xS'y$ yield that for any $i,j\in\{1,\dots,m+n\}$ we have $x_iS'x_j$. Also we have $x_1Rx_2R\dots Rx_mRx_{m+1}R\dots Rx_{m+n}Rx_1$. This shows that the sequence $(x_1,x_2,\dots x_m,\dots,x_{m+n})$ is an S'R-cycle containing x and y. So we have:

xSy iff x=y or there exists an S'R-cycle C such that $x,y\in C$.

(ii). For $S\subseteq R^{*}_{S}$ suppose xSy and proceed to show that $xR^{*}_{S}y$.

<u>Case 1:x=y.</u> Then we have $xR^{o}_{S}y$ and hence $xR^{*}_{S}y$.

Case 2: $x \neq y$. Then from xSy and (i) we have that there exists an S'R-cycle C such that $x, y \in C$. This implies that for some $x_1, x_2, \dots x_n \in C$ we have $x = x_1 R x_2 R \dots R x_n R y$. Since C is an S'R-cycle then by (i) we have that for any $i \in \{1, \dots, n\}$ xSx_i, so $\{x_1, x_2, \dots, x_n\} \subseteq S(x)$. This shows that we have $xR_S^* y$.

(iii) The fact that S is an equivalence relation in U follows from (i).∎

From lemma 5.8 we obtain that $(|U_L|, R, S)$ is an inductive frame with S an equivalence relation in $|U_L|$, which shows that it is a frame for $S5^*$Ind. It remains to show only that the conditions (FSi) and (FSii) are satisfied.

(FSii) can be proved as the filtration of KInd.

For (FSi) suppose $xS_L y$. By (FS'i) we have $|x|S'|y|$. We have to show that $|x|R_{S'}^*|y|$ and $|y|R_{S'}^*|x|$. For that purpose let

$M_1 = \{|t| / |x|R_{S'}^*|t|\}$ and $M_2 = \{|t| / |y|R_{S'}^*|t|\}$.

Then there exist formulas A_1 and A_2 such that $|t| \in M_1$ iff $A_1 \in t$ and $|t| \in M_2$ iff $A_2 \in t$. In order to prove $|x|R_{S'}^*|y|$ and $|y|R_{S'}^*|x|$ we shall show that $|y| \in M_1$ and $|x| \in M_2$. This will follow from the following

Assertion

(i) $A_1 \in x$ and $\Box*(A_1 \Rightarrow \Box A_1) \in x$,

(ii) $A_2 \in y$ and $\Box*(A_2 \Rightarrow \Box A_2) \in y$.

Indeed, from (i) we have that $A_1 \wedge \Box*(A_1 \Rightarrow \Box A_1) \in x$, so by Seg $\Box*A \in x$. Since we have $xS_L y$ then $A_1 \in y$ and hence $|y| \in M_1$. From (ii) we obtain that $A_2 \wedge \Box*(A_2 \Rightarrow \Box A_2) \in y$ and by Seg $\Box*A_2 \in y$. Note that the canonical relation S_L is symmetric one, so from $xS_L y$ we obtain $yS_L x$. This and $\Box*A_2 \in y$ show that $A_2 \in y$ and that $|y| \in M_2$.

The proof of the Assertion can be given in the same way as for the corresponding assertion in the proof of the filtration of KInd.∎

Problem 5.1.

The logic KB^*Ind=KInd+$\Diamond*\Box*A \Rightarrow A$ is sound in the class of all inductive frames (U, R, S) with S a symmetric relation in U. Is KB^*Ind complete in this class of frames?

6. Extensions of KInd with axioms containing both $\Box$ and $\Box*$

First we will consider the following simple axiom

(R⊆S) $\Box*A \Rightarrow \Box A$

We denote this axiom by (R⊆S) because $\Box*A \Rightarrow \Box A$ is true in a frame (U, R, S) iff R⊆S. This axiom reflects also on the canonical relations: $R_L \subseteq S_L$.

If LInd or L^*Ind is one of the systems we have studied, then the extension of these system with the axiom (R⊆S) will be denoted by L(R⊆S)Ind and L^*(R⊆S)Ind respectively.

A frame (U, R, S) is called an (R⊆S)-frame if R⊆S. Obviously the logic L(R⊆S)Ind /L^*(R⊆S)Ind/ is sound in the class of all inductive (R⊆S)-frames for LInd /for L^*Ind/.

The second type of mixed axioms are combinations of those for KT, KB, K4, S4 and S5 with the axioms for KT^*, $K4^*$, $S4^*$ and $S5^*$. If L is one of the first type logics and L^* is one of the second type logics the combination will be denoted by LL^*Ind. The logic LL^*Ind+(R⊆S) will be denoted by LL^*(R⊆S)Ind. Obviously the logic LL^*Ind /LL^*(R⊆S)Ind/ is sound in the class of all frames

/(R⊆S)-frames/ for LInd and L^*Ind.

Theorem 6.1.

The logic L=K(R⊆S)Ind admits a good filtration.

Proof. Define Φ and S' as in the filtration of KInd. Then define:

$|x|R'|y|$ iff $\exists x',y'\in U_L: x\sim x', y\sim y'$ and $x'R_Ly'$.

Note that R' is defined as R in the filtration of KInd and hence satisfies the following two conditions:

(FR'i) $\quad xR_Ly \longrightarrow |x|R'|y|$,

(FR'ii) $\quad |x|R'|y| \longrightarrow (\forall\Box A\in\Phi)(\Box A\in x \longrightarrow A\in y)$.

Now define:

$|x|R|y|$ iff $|x|R'|y| \& |x|S'|y| \& |x|S'|x|$, $\quad S=R^*_{S'}\cap S'$.

First we shall show that R satisfies the conditions (FRi) and (FRii). For the condition (FRi) suppose xR_Ly. Then by (FR'i) we have $|x|R'|y|$. Since $R_L\subseteq S_L$ we have also xS_Ly and by (FS'i) we have $|x|S'|y|$. By proposition 3.3.iii xS_Ly implies xS_Lx and then by (FS'i) - $|x|S'|x|$. Hence we obtain $|x|R'|y|\&|x|S'|y|\&|x|S'|x|$, so $|x|R|y|$.

For the condition (FR'ii) suppose $|x|R|y|$. Then we have $|x|R'|y|$, which by (FR'ii) implies $(\forall A\in\Phi)(A\in x \longrightarrow A\in y)$.

Now we shall show that R⊆S. Suppose $|x|R|y|$. Then we have $|x|S|y|$ and $|x|S'|x|$. But $|x|S'|x|\&|x|R|y|$ is exactly $|x|R_{S'}|y|$, so $|x|S^*_{S'}|y|$, which with $|x|\ S'|y|$ implies $|x|S|y|$. This shows that the frame $(|U_L|,\ R,\ S)$ is a (R⊆S)-frame.

Then by lemma 3.2 $(|U_L|,\ R,\ S)$ is an inductive frame.■

Theorem 6.2.

Let L be one of the following logics: KT, KB, K4, S4 and S5, and let L'=L(R⊆S)Ind. Then L' admits a good filtration.

Proof. Define Φ, R, S' and S as in theorem 6.1. The definition of R' is as follows:

If L=KT then R' is as in theorem 6.1.

If L=KB, K4, S4, S5 then R' is defined as in the proof of theorem 4.10.■

Theorem 6.3.

Let L^* be one of the following logics: KT^*, $K4^*$, $S4^*$ and $S5^*$, and let $L'=L^*$(R⊆S)Ind. Then L' admits a good filtration.

Proof. Define Φ, R' and R as in theorem 6.1. The definitions of S' and S are as in the proofs of the following theorems:

For $L=KT^*$ - as in theorem 6.1.

For $L=K4^*$ - as in theorem 5.3.

For $L=S4^*$ - as in theorem 5.5.

For $L=S5^*$ - as in theorem 5.7.

The proof is a combination of the proof of theorem 6.1 with the proofs of the cited above theorems.■

Theorem 6.4.

Let L be one of the logics KT, KB, K4, S4 and S5 and L^* be one of the logics KT^*, $K4^*$, $S4^*$ and $S5^*$. Let L' be one of the following logics - LL^*Ind and LL^*(R⊆S)Ind. Then L' admits a good filtration.

Proof. /For LL^*Ind/ Define S' and S as in the filtration of L^*Ind, and R - as in the filtration of LInd.

/For LL^*(R⊆S)Ind/ Define R' and R as in the filtration of L(R⊆S)Ind and S' and S - as in the filtration of L^*Ind.

The proof is a combination of the corresponding proofs for LInd, L^*Ind and K(R⊆S)Ind.■

III. THREE SPECIAL INDUCTIVE LOGICS

7. On the logic of reflexive and transitive closure - LRTC

As it was mentioned in the introduction, the logic of reflexive and transitive closure - LRTC - is the logic of all frames (U, R, S) in which $S=R^*$. This logic is axiomatized by Segerberg [2] in the context of PDL as KInd+Seg_o. LRTC is one of the interesting extensions of KInd. In the following theorem it is stated that LRTC is one of the already introduced systems in the preceding section.

Theorem 7.1.
LRTC=$S4^*$(R⊆S)Ind.
Proof. The proof follows from the following lemma.

Lemma 7.2.
Let (U, R, S) be a frame. Then:
(i) If $S=R^*$ then (U, R, S) is an inductive (R⊆S)-frame with reflexive and transitive S and hence (U, R, S) is a frame for $S4^*$(R⊆S)Ind.
(ii) If (U, R, S) is an inductive R⊆S frame with reflexive and transitive relation S then $S=R^*$ and hence (U, R, S) is a frame for LRTC.
Proof. (i). Suppose $S=R^*$. Then S is a reflexive and transitive relation and R⊆S. The inductivity $S\subseteq R^*_S$ follows from the following equality:

(1) $R^*_{R^*}=R^*$.

This can be proved by induction as follows:
$R^o_{R^*}=id=R^o$.
/i.h./ Suppose that for some i : $R^i_{R^*}=R^i$.
Then: $R^{i+1}_{R^*}=(R^i_{R^*}\cap R^*)\circ R = R^i\circ R = R^{i+1}$.

(ii). Let (U, R, S) be an inductive (R⊆S)-frame with reflexive and transitive relation S. Since $R^*_S\subseteq R^*$ then by the inductivity we have $S\subseteq R^*$. We shall show by induction that for any i $R^i\subseteq S$ and hence $R^*\subseteq S$.
i=0. $R^0=id\subseteq S$ by the reflexivity of S.
i=k./i.h./ Suppose $R^k\subseteq S$. Then by the monotonicity of the composition ∘, condition R⊆S and transitivity of S - S∘S⊆S - we obtain: $R^{k+1}=R^k\circ R\subseteq S\circ S\subseteq S$.■

Notes. Lemma 7.2 has some independent importance. If we call (U, R, R^*) a Segerberg frame, then it follows from (i) that each Segerberg frame is an inductive frame.

Of course, the equality $S4^*$(R⊆S)Ind=LRTC can be proved syntactically. We left this exercise to the reader.

Our filtration for $S4^*$(R⊆S)Ind and hence for LRTC is different from the Segerberg filtration for LRTC. Although our construction is more complicated than the Segerberg construction, it implies a better result: any nontheorem A of LRTC can be refuted in a finite frame (U, R, R^*) with $cardU\le 2^{|A|}$. In the Segerberg construction the inequality is $cardU\le 2^{2|A|}$. This is because the Segerberg's proof holds under the following additional closure condition on Φ: if □∗A∈Φ then A∧□□∗A∈Φ.

8. On the logic of cyclic equivalence LCE

The logic of cyclic equivalence - LCE - is another interesting extension of KInd. It is the logic of all frames (U, R, S) such that $S=R^c=R^*\cap(R^*)^{-1}$. As we have noted in the introduction R^c is an equivalence relation in U, called here the cyclic equivalence determined by R. We will show that LCE=$S5^*$Ind.

Proposition 8.1.
$S5^*$Ind is sound in the class of all frames (U, R, S) such that $S=R^c$.
Proof. The proof follows from the next lemma.

Lemma. 8.2.
Let (U, R, S) be a frame such that $S=R^c$. Then (U, R, S) is an inductive frame in which S is an equivalence relation, and hence (U, R, S) is a frame for $S4^*$Ind.

Proof. Obviously S is an equivalence relation. To prove inductivity, suppose xSy and proceed to show that xR_S^*y. If x=y then this is obvious. Let x≠y. Then there exists an R-cycle C such that x,y∈C. This means that there exists a nonempty sequence $x_1,x_2,\dots,x_n\in C$ such that $x=x_1Rx_2R\dots Rx_nRy$. Since $x=x_1$ then from $x_1, x_2,\dots,x_n\in C$ we obtain that for any $i\in\{1,\dots,n\}$ we have xR^cx_i, so xSx_i which shows that $\{x_1,x_2,\dots,x_n\}\subseteq S(x)$. This proves that we have xR_S^*y.■

The frames of the form (U, R, R^c) will be called standard frames for $S5^*$Ind and the semantics, which they determine - the standard semantics for $S5^*$Ind.

It follows from theorem 5.7 that $S5^*$Ind is complete in the class of all inductive frames (U, R, S) in which S is an equivalence relation. Such frames will be called nonstandard frames for $S5^*$Ind. We adopt this terminology, because standard frames for $S5^*$Ind are indeed standard for LCE. Note also that if (U, R, S) is a nonstandard frame for $S5^*$Ind this does not mean that (U, R, S) is not a standard frame for $S5^*$Ind.

Lemma 8.3.
Let (U, R, S) be a nonstandard frame for $S5^*$Ind. Then: (i) (U, R, S) is a standard frame for $S5^*$Ind iff $R^c\subseteq S$.

(ii) (U, R, S) is a standard frame for $S5^*$Ind iff all R-cycles in U are SR-cycles in U. /The definition of an SR- cycle is given before lemma 5.8./.

Proof. (i) If (U, R, S) is a standard frame for $S5^*$Ind then $S=R^c$ and the inclusion $R^c\subseteq S$ is obvious.

Suppose now that (U, R, S) is a nonstandard frame for $S5^*$Ind and that $R^c\subseteq S$. Since $R_S^*\subseteq R^*$ then by the inductivity we have: $S\subseteq R_S^*\subseteq R^*$. From here we obtain also $S^{-1}\subseteq(R^*)^{-1}$. But $S^{-1}=S$ /S is a symmetric relation/. So $S\subseteq(R^*)^{-1}$ and hence $S\subseteq R^*\cap(R^*)^{-1}=R^c$. Then $S=R^c$ and the frame is a standard one.

(ii) If $S=R^c$ then obviously all R cycles in U are SR-cycles.

Suppose now that (U, R, S) is a nonstandard frame for $S5^*$Ind such that all R-cycles in U are SR-cycles. We shall show that $R^c\subseteq S$, which by (i) will imply that (U, R, S) is a standard frame for $S5^*$Ind.

Suppose xR^cy. If x=y then obviously xSy, so suppose x≠y. Then there exists an R-cycle C such that x,y∈C. But C is by assumption an SR-cycle, so xSy.■

The condition $R^c\subseteq S$ is not modally definable by any class of formulas in our bi-modal language. This will follow from the next theorems.

The following definition is an adaptation of "copying construction" given in the "Introduction" for the bi-modal language of inductive logics.

Let $\underline{U}$=(U, R, S) and $\underline{U}'$=(U',R',S') be two relational structures and $\underline{M}$=($\underline{U}$, v) and $\underline{M}'=(\underline{U}',v')$ be two models over $\underline{U}$ and $\underline{U}'$ respectively. Let $I\neq\emptyset$ be a set of mappings from U into U' and let for any $i\in I$ and $x\in U$ the application of i to x be denoted by x_i. We say that I is a *copying from* $\underline{U}$ *to* $\underline{U}'$ if the following conditions are satisfied for any $x,y\in U$, and $i,j\in I$:

(I1) $U'=\bigcup_{i\in I}U_i$, where $U_i=\{x_i/x\in U\}$,

(I2) if $x_i=y_j$ then $x=y$,

(CR1) if xRy then $(\exists j\in I)(x_iR'y_j)$,

(CR2) if $x_iR'y_j$ then xRy,

(CS1) if xSy then $(\exists j\in I)(x_iS'y_j)$,

(CS2) if $x_iS'y_j$ then xSy.

We say that I is a *copying from the model* $\underline{M}$ *to the model* $\underline{M}'$ if I is a copying from $\underline{U}$ to $\underline{U}'$ and for any propositional variable A, $x\in U$ and $i\in I$ we have:

(Cv) $x\in v(A)$ iff $x_i\in v'(A)$.

The elements of I are called *copying functions* and for each $i\in I$ U_i is called the i-th copy of U.

Lemma 8.4.

(Copying Lemma)(i) Let $\underline{U}=(U,R,S)$ *and* $\underline{U}'=(U',R',S')$ *be two relational structures and I be a copying from* $\underline{U}$ *to* $\underline{U}'$*. Let* $\underline{M}=(\underline{U},v)$ *be a model over* $\underline{U}$ *and let for any propositional variable A* $v'(A)=\{y'\in U'/\exists x\in v(A)$ *and* $\exists i\in I$ *such that* $y'=x_i\}$*. Then I is a copying from* $\underline{M}$ *to the model* $\underline{M}'=\{\underline{U}',v'\}$*.*

(ii) For any formula A, $x\in U$ *and* $i\in I$*:*

$x\Vdash_{v}A$ *iff* $x_i\Vdash_{v'}A$

Proof. (i) ($\rightarrow$) Suppose $x\in v(A)$, then by the definition of v' we have $x_i\in v'(A)$.

($\leftarrow$) Let $x_i\in v'(A)$. Then by the definition of v' there exist $y\in v(A)$ and $j\in I$ such that $x_i=y_j$. This by (I2) implies x=y, so $x\in v(A)$.

(ii) The proof goes by induction on the complexity of the formula A. For $A\in VAR$ the assertion is true by the definition of v'. Boolean combinations of formulas do not present difficulties, so let $A=\Box B$ and suppose by i.h. that the assertion for B holds.

($\rightarrow$) Let $x\Vdash_{v}\Box B$. To show that $x_i\Vdash_{v'}\Box B$ suppose $x_iR'y_j$ and proceed to show that $y_j\Vdash_{v'}B$.

From $x_iR'y_j$, by (CR2) we have xRy and since $x\Vdash_{v}\Box B$ then $y\Vdash_{v}B$. Then by i.h. $y_j\Vdash_{v'}B$.

($\leftarrow$) Let $x_i\Vdash_{v'}\Box B$. To show that $x\Vdash_{v}\Box B$ suppose xRy and proceed to show that $y\Vdash_{v}B$. From xRy, by (CR1) we have that $x_iR'y_j$ for some j and since $x_i\Vdash_{v'}\Box B$ we have $y_j\Vdash_{v'}B$. Then by i.h. $y\Vdash_{v}B$.

The case $A=\Box{*}A$ can be treated in the same way, using (CS1) and (CS2).■

Lemma 8.5.

Let $\underline{U}=(U,R,S)$ *be a nonstandard frame for* $S5^{*}Ind$*. Then there exist a standard frame* $\underline{U}'=(U',R',S')$ *for* $S5^{*}Ind$ *and a copying I from* $\underline{U}$ *to* $\underline{U}'$*.*

Proof. If (U, R, S) is not standard frame for $S5^{*}Ind$ then by lemma 8.3.ii there exists in U an R-cycle C, which is not an SR- cycle. Such an R-cycles will be called defective one. If $C=(x_1,x_2,\dots,x_n)$ is a defective R-cycle then for some $i\in\{1,\dots,n-1\}$ we have not x_iSx_{i+1}. The pair (x_i,x_{i+1}) will be called a defective pair in C. Let $x,y\in U$, the pair (x,y) will be called a defective pair if it is a defective pair in some R-cycle. Now the proof goes as follows.

Let $N=\{0,1,2,\dots\}$. Define $U'=U\times N$, $I=N$ and $x_i=(x,i)$.

$x_iR'y_j$ iff xRy & ($j=i+1$ if (x,y) is a defective pair) &

($j=i$ if (x,y) is not a defective pair),

$x_iS'y_j$ iff xSy & $i=j$,

The required frame is $\underline{U}'=(U', R', S')$. Obviously I is a copying from $\underline{U}$ to $\underline{U}'$ and that S' is an equivalence relation in U'. We shall show that (U', R', S') is an inductive frame. For that purpose suppose $x_iS'y_j$ and proceed to show that $x_iR'^*_{S'}y_j$.

From $x_iS'y_j$ we have xSy and i=j. By the inductivity of the frame (U, R, S), from xSy we obtain xR^*_Sy. If x=y then obviously $xR'^*_{S'}y$. Suppose $x\neq y$. Then there exists a nonempty sequence $x^1,x^2,\dots,x^n\in U$ such that $x=x^1Rx^2R\dots Rx^nRy$ and $\{x^1,x^2,\dots,x^n\}\subseteq S(x)$.

Since S is an equivalence relation in U then for all pairs (x^k,x^{k+1}) $1\leq k<n$ we have x^kSx^{k+1}, which shows that (x^k,x^{k+1}) is not a defective pair. Then by the definition of R' and S' we have:

$x_i=x_i^1R'x_i^2R'\dots R'x_i^nR'y_i$ and $\{x_i^1,x_i^2,\dots,x_i^n\}\subseteq S'(x_i)$.

This shows that we have $x_iR'^*_{S'}y_i$, and by i=j - $x_iR'^*_{S'}y_j$.

Thus, (U', R', S') is an inductive frame in which S' is an equivalence relation. Hence (U',R', S') is a frame for S5*Ind. It remains to show that (U', R', S') is a standard frame for S5*Ind. Suppose that C is an R'-cycle in U'. By the definition of R' it is not possible for the elements of C to be in different copies of U. We shall show that there are no defective pairs (α,β) in U'. Suppose for the contrary that (α,β) is a defective pair in U'. Then (α,β) should be a defective pair of some R'-cycle C' in U'. So we have $\alpha R'\beta$ but not $\alpha S'\beta$. As we have seen C' is contained in some copy U_i, so $C'=(x_i^1,x_i^2,\dots,x_i^n)$ and $C=(x^1,x^2,\dots,x^n)$ is an R cycle in U.

Then we have $\alpha=x_i^k$ and $\beta=x_i^{k+1}$, x^kRx^{k+1} and not x^kSx^{k+1}. Hence (x^k,x^{k+1}) is a defective pair of C. Then by the definition of R' we have not $x_i^kRx_i^{k+1}$, so - not $\alpha R'\beta$, which contradicts $\alpha R'\beta$. This shows that U' has not defective pairs, so U' has not defective R'-cycles. Then all R'-cycles in $\underline{U}'$ are S'R'-cycles and by lemma 8.3.ii (U', R', S') is a standard frame for S5*Ind. ■

Theorem 8.6. */Completeness theorem for S5*Ind/*

For any for mula A the following conditions are equivalent:

*(i) A is a theorem of S5*Ind.*

*(ii) A is true in all nonstandard frames for S5*Ind*

*(iii) A is true in all nonstandard frames (U, R, S) for S5*Ind such that* $cardU\leq 2^{|A|}$.

*(iv) A is true in all standard frames for S5*Ind.*

Proof. The implications (i)⟶(ii)⟶(iii) are obvious. The implication (i)⟶(iv) follows from proposition 8.1. The implication (iii)⟶(i) follows from theorem 5.7. To prove (iv)⟶(ii) suppose that A is not true in some nonstandard frame $\underline{U}=(U, R, S)$ for S5*Ind. Then for some valuation v and $x\in U$ we have $x\Vdash\!\!\!\!/_{v}\ A$. By theorem 8.5 there exist a standard frame $\underline{U}'=(U',R',S')$ and a copying I from $\underline{U}$ to $\underline{U}'$, Define v' as in lemma 8.5.(i). Then by lemma 8.5.(ii) $x_i\Vdash\!\!\!\!/_{v'}A$, so A is not true in the standard frame for S5*Ind. Thus, by contraposition, we have (iv)⟶(ii), which ends the proof of the theorem. ■

Problem 8.1.

Does the logic $S5^*Ind$ posses the finite model property with respect to its standard models?

Corollary 8.7.

(i) $S5^*Ind=LCE$.

(ii) The condition $R^C\subseteq S$ is not modally definable in the bi-modal language $\mathcal{L}(\Box, \Box*)$

9. The logics LRTC and LCE together - the logic of reflexive and transitive closure and cyclic equivalence LRTC&CE.

It is interesting to consider a three-modal logic with modalities $\Box$, $\Box*$ and $\Box^C$ interpreted in the class of all frames (U, R, R^*, R^C), where the interpretations of $\Box$, $\Box*$ and $\Box^C$ is by the relations R, R^* and R^C respectively. The obtained logic is an integration of the logics LRTC and LCE and is denoted by LRTC&CE. It has the following axiomatization.

Axiom schemes and rules for LRTC&CE

(Bool) All /or enough/ Boolean tautologies,

(K$\Box$) $\Box(A\Rightarrow B)\Rightarrow(\Box A\Rightarrow\Box B)$,

(K$\Box*$) $\Box*(A\Rightarrow B)\Rightarrow(\Box*A\Rightarrow\Box*B)$,

(K$\Box^C$) $\Box^C(A\Rightarrow B)\Rightarrow(\Box^C A\Rightarrow\Box^C B)$,

(S4$\Box^*$) $\Box*A\Rightarrow A$, $\Box*A\Rightarrow\Box*\Box*A$, ,

(R$\subseteq$S) $\Box*A\Rightarrow\Box A$,

(Seg $\Box*$) $A\wedge\Box*(A\Rightarrow\Box A)\Rightarrow\Box*A$,

(S5 $\Box^C$) $\Box^C A\Rightarrow A$, $\Box^C A\Rightarrow \Box^C\Box^C A$, $A\vee\Box^C\neg\Box^C A$,

(Seg $\Box^C$) $A\wedge\Box^C(A\Rightarrow\Box A)\Rightarrow\Box^C A$,

(MP) A, $A\Rightarrow B/B$, (N$\Box*$) $A/\Box*A$, (N$\Box^C$) $A/\Box^C A$.

The nonstandard frames for LRTC&CE are all frames (U, R, R^*, S) such that (U, R, S) is a frame for $S5^*Ind$. In other words S is an equivalence relation in U and $S\subseteq R_S$. The frames of the form (U, R, R^*,R^C) will be called standard frames for LRTC&CE. Obviously LRTC&CE is sound with respect to its standard and nonstandard semantics.

Theorem 9.1.

The logic LRTC&CE admits a good filtration with respect to its nonstandard semantics.

Proof. The proof is exactly the "sum" of the proofs of the filtrations for $S4^*(R\subseteq S)Ind$ and $S5^*Ind$.■

In the next theorem we use an obvious extension of the notion of copying and Copying Lemma for frames with three relations.

Theorem 9.2.

Let $\underline{U}=(U, R, R^*, S)$ be a nonstandard frame for LRTC&CE. Then there exists a standard frame $\underline{U}'=(U', R', R'^*, S')$ for LRTC&CE and a copying I from $\underline{U}$ to $\underline{U}'$.

Proof. The definitions of I, U', R', S' are the same as in the proof of theorem 8.5. Then by theorem 8.5 $\underline{U}'=(U', R', R'^*,S')$ is a standard frame for LRTC&CE. It is easy to see that the following two conditions are satisfied:

(CR^*1) If xR^*y then $\forall i\exists j\geq i$ $x_iR'^*y_j$,

(CR^*2) If $x_iR'^*y_j$ then xR^*y.

So I is a copying from $\underline{U}$ to $\underline{U}'$.

Theorem 9.3. */Completeness theorem for LRTC&CE/*

For any formula A of LRTC&CE the following conditions are equivalent

(i) A is a theorem of LRTC&CE,

(ii) A is true in all nonstandard frames for LRTC&CE,

(iii) A is true in all nonstandard frames for LRTC&CE (U, R, R^*, S) *such that* $\mathrm{card}U \le 2^{|A|}$,

(iv) A is true in all standard frames for LRTC&CE.

Proof - from theorem 9.1 and theorem 9.2.■

In connection with the problem 8.1 we have the following

Theorem 9.4.

The logic LRTC&CE do not possess the finite model property with respect to its standard semantics.

Proof. Consider the following formula

Fin $\quad \Box^*(\Box(\Box^c A \Rightarrow \Box A) \Rightarrow A) \Rightarrow A$

Then the theorem follows from the following

Lemma 9.5.

(i) Fin is true in all finite standard frames for LRTC&CE.

(ii) Fin is not a theorem of LRTC&CE.

Proof. (i) Let (U, R, R^*, R^c, v) be a model, $x_0 \in U$ and $x_0 \not\Vdash_v$ Fin. We shall show that U is an infinite set.

From $x_0 \not\Vdash_v$ Fin we obtain $x_0 \not\Vdash_v A$ and

(∗) $\quad x_0 \Vdash_v \Box^*(\Box(\Box^c A \Rightarrow \Box A) \Rightarrow A)$.

Since $x_0 R^* x_0$ we have $x_0 \Vdash_v \Box(\Box^c A \Rightarrow \Box A) \Rightarrow A$, and by $x_0 \not\Vdash_v A$ we obtain $x_0 \not\Vdash_v \Box(\Box^c A \Rightarrow \Box A)$, so there exists x_1 and $x_2 \in U$ such that $x_0 R x_1$, $x_1 \Vdash_v \Box^c A$, $x_1 R x_2$ and $x_2 \not\Vdash_v A$. From $x_0 R x_1$, $x_1 R x_2$ we obtain $x_0 R^* x_1$, $x_1 R^* x_2$ and hence $x_0 R^* x_2$. From $x_1 \Vdash_v \Box^c A$ and $x_0 \not\Vdash_v A$ we obtain not $x_1 R^c x_2$ and by $x_0 R^* x_1$ we get not $x_1 R^* x_0$.

From $x_1 \Vdash_v \Box^c A$ and $x_2 \not\Vdash_v A$ we obtain not $x_1 R^c x_2$ and by $x_1 R^* x_2$ - not $x_2 R^* x_1$.

Define $s<t$ iff sR^*t & not tR^*s.

Since R^* is reflexive and transitive relation then $<$ is an irreflexive and transitive relation in U. So we have obtained:

$x_0 < x_1 < x_2$. From $x_0 R^* x_2$ and (∗) we obtain that $x_2 \Vdash_v \Box(\Box^c A \Rightarrow \Box A) \Rightarrow A$. Since $x_2 \not\Vdash_v A$ we can find as above x_3 and x_4 such that $x_2 < x_3 < x_4$, $x_0 R^* x_4$ and $x_4 \not\Vdash_v A$. So $x_0 < x_1 < x_2 < x_3 < x_4$.

This procedure can be repeated infinitely many times and then we will obtain an infinite sequence of elements of U $x_0 < x_1 < x_2 < x_3 < x_4 < \ldots$ .By the irreflexivity and transitivity of the relation $<$ it follows that all elements of this sequence are different. So U is an infinite set.

(ii) To show that Fin is not a theorem of LRTC&CE we will find a nonstandard model (U, R, R^*, S, v) for LRTC&CE in which fin is not true . Then by theorem 9.3 fin∉LRTC&CE.

Let $U=\{x_0, x_1\}$, $R=\{(x_0, x_1), (x_1, x_0)\}$, $S=id=\{(x_0, x_0), (x_1, x_1)\}$.

Now we shall calculate R_S^*. We have:

$R_S^0 = id$, $R_S^1 = (id \cap S) \circ R = id \circ R = R$, $R_S^2 = (R \cap S) \circ R = \varnothing \circ R = \varnothing$. Then for $i>2$ we obtain $R_S^i = \varnothing$. Hence $R_S^* = id \cup R = S \cup R$. So $S \subseteq R_S^*$ which shows the inductivity. Then (U, R, R^*, S) is a nonstandard frame for LRTC&CE. Define for $A \in VAR$ $v(A)=\{x_1\}$, so $x_1 \Vdash_v A$. We

will show that
$x_0\not\Vdash_v$ Fin. Since $x_0\notin v(A)$ we have $x_0\not\Vdash_v A$. The proof will be completed by showing that $x_0\Vdash_v \Box*(\Box(\Box^c A\Rightarrow\Box A)\Rightarrow A)$. Suppose the contrary. Then for some $y\in U$ we have x_0R^*y, $y\Vdash_v \Box(\Box^c A\Rightarrow\Box A)$ and $y\not\Vdash_v A$. From $y\not\Vdash_v A$ we obtain that $y=x_0$, so $x_0\Vdash_v \Box(\Box^c A\Rightarrow\Box A)$. By x_0Rx_1 we obtain that $x_1\Vdash_v \Box^c A\Rightarrow\Box A$. Since $x_1\Vdash_v A$ and x_1Sx_1 we have $x_1\Vdash_v \Box^c A$. From $x_1\Vdash_v \Box^c A\Rightarrow\Box A$ and $x_1\Vdash_v \Box^c A$ we obtain $x_1\Vdash_v \Box A$ and by x_1Rx_0 - $x_0\Vdash_v A$, which contradicts $x_0\not\Vdash_v A$. ■

Note. Instead of Fin we can use also the following introduced in [Vak 89] formula, called generalized Grzegorczik formula

gGrz $\quad\Box*(\Box*(\Box^c A\Rightarrow\Box*A)\Rightarrow A)\Rightarrow A$

It can be shown that gGrz⇒fin ∈ LRTC&CE.

Problem 9.1.
Axiomatize the logic of all finite frames (U, R, R^*, R^C). Conjecture: LRTC&CE+fin or LRTC&CE+gGrz.

IV. GENERALIZATIONS

10. On the logic of transitive closure LTC. Other inductive axioms

Let (U, R, S) be a frame and $S=R^+=\bigcup_{i=1}^{\infty}R^i$ be the transitive closure of R. The modal logics with modalities $\Box$ and $\Box^+$ /interpreted by R and R^+ respectively/ of all such frames will be called the logic of transitive closure - LTC.

The following formal system axiomatize LTC.

Axiom schemes and rules for LTC.

(Bool), (K$\Box$), (K$\Box^+$),

(R≤S) $\quad\Box^+A\Rightarrow\Box A$,

(K4$\Box^+$) $\quad\Box^+A\Rightarrow\Box^+\Box^+A$,

(Seg1) $\quad\Box A\wedge\Box^+(A\Rightarrow\Box A)\Rightarrow\Box^+A$,

(MP), (N$\Box^+$).

Curiously enough, the logics LTC and LRTC are equivalent in the following sense: let

L_1=LTC + $\Box*A=_{def}A\wedge\Box^+A$ and L_2=LRTC + $\Box^+A=_{def}\Box\Box*A$.

Then $L_1=L_2$. We left to the reader the syntactic proof of this equality, which yields the completeness of the above system with respect to the described semantics.

In the context of this chapter LTC is interesting with the fact that it presents another induction axiom -

Seg_1 $\quad\Box A\wedge\Box^+(A\Rightarrow\Box A)\Rightarrow\Box^+A$

Using Seg_1 we can develop the theory of another type inductive logics. We will not do this because Seg_1 and Seg can be generalized into the following formula denoted by gSeg - generalized Segerberg axiom

gSeg $\quad\Box_1A\wedge\Box*(A\Rightarrow\Box A)\Rightarrow\Box*A$.

When $\Box_1A=A$ we obtain Seg, when $\Box_1A=\Box A$ we obtain Seg_1.

In the next section we will show how can be developed the theory of normal three-modal logics having gSeg as an axiom.

11. Generalized inductive logics

Let $\mathcal{L}(\Box, \Box_1, \Box*)$ be a propositional modal language extending the language of classical propositional logic with three box operations - $\Box$, $\Box_1$ and $\Box*$. This language will be interpreted in a frames (U, R, R_1, S), R - for $\Box$, R_1 - for $\Box_1$ and S - for $\Box*$. To define the adequate condition for gSeg we will introduce a new operation between relations, denoted by $R^*_{(R_1,S)}$, generalizing the operation R^*_S. The definition is as follows:

$$R^0_{(R_1,S)}=R_1,\quad R^{i+1}_{(R_1,S)}=(R^i_{(R_1,S)}\cap S)\circ R,\quad R^*_{(R_1,S)}=\bigcup_{i=0}^{\infty}R^i_{(R_1,S)}$$

Lemma 11.1.

$$R^*_{(R_1,R^*_{(R_1,S)}\cap S)}=R^*_{(R_1,S)}.$$

The proof is similar to that of lemma 1.1.

A frame (U, R, R_1, S) will be called a *generalized inductive frame if* $S\subseteq R^*_{(R_1,S)}$

Theorem 11.2.

The axiom gSeg is true in a frame (U, R, R_1, S) iff $S\subseteq R^*_{(R_1,S)}$.

The proof is similar to those of theorem 1.2.

Let L be a set of formulas in the language $\mathcal{L}(\Box, \Box_1, \Box*)$. The set L will be called a *generalized inductive logic* if it satisfies the following conditions:

(Bool) L contains all Boolean tautologies,

(K$\Box$) $\Box(A\Rightarrow B)\Rightarrow(\Box A\Rightarrow\Box B)\in L$,

(K$\Box_1$) $\Box_1(A\Rightarrow B)\Rightarrow(\Box_1 A\Rightarrow\Box_1 B)\in L$,

(K$\Box*$) $\Box*(A\Rightarrow B)\Rightarrow(\Box*A\Rightarrow\Box*B)\in L$,

(gSeg) $\Box_1 A\wedge\Box*(A\Rightarrow\Box A)\Rightarrow\Box*A\in L$,

(MP) if $A\in L$ and $A\Rightarrow B\in L$ then $B\in L$,

(N) if $A\in L$ then $\Box A\in L$,

(N $_1$) if $A\in L$ then $\Box_1 A\in L$,

(N $*$) if $A\in L$ then $\Box*A\in L$,

(Sub) if $A\in L$ then all substitution instances of A are in L. The smallest generalized inductive logic will be denoted by KgInd.

Lemma 11.3.

KgInd is sound in the class of all generalized inductive frames.

Theorem 11.4.

KgInd admits a good filtration.

Proof. Let ϕ be a finite set of formulas closed under subformulas. The definition of $\sim$, $|U_L|$, R, S' and v are the same as in the filtration of KInd.

Define

$$|x|R_1|y| \text{ iff } \exists x'\sim x,\ y'\sim y \text{ and } x'R_{1L}y',\quad S=R^*_{(R_1,S')}\cap S'.$$

The following lemma assures that $(|U_L|, R_1, R, S)$ is a generalized inductive frame.

Lemma 11.5.

Let (U, R_1, R, S') *be a frame and let* $S=R^*_{(R_1,S')}\cap S'$. *Then* (U, R_1, R, S) *is a generalized inductive frame.*

The proof uses lemma 11.1 and is similar to the proof of lemma 1.5.

Now the rest of the proof of the theorem is similar to the proof of theorem 3.1.

Corollary 11.6.

(i) KgInd is complete in the class of all /finite/ generalized inductive frames.

(ii) Any non-theorem A of KgInd is falsified in a generalized inductive frame (U, R_1, R, S) *with* $\mathrm{card}U\leq 2^{|A|}$.

Let L_1=KgInd + $\Box_1 A\Leftrightarrow A$ and L_2=KgInd + $\Box_1 A\Leftrightarrow\Box A$.

Then L_1 is equivalent to KInd and L_2 is equivalent to the smallest bi-modal logic in the language $\mathcal{L}(\Box, \Box*)$ containing the axiom Seg_1 $\Box A\wedge\Box*(A\Rightarrow\Box A)\Rightarrow\Box*A$.

CHAPTER 1.2

A MODAL CHARACTERIZATION OF CYCLIC REPEATING OF PROGRAMS

Overview. In this chapter we study a polimodal logic MLCR - a Modal Logic for Cyclic Repeating, in which a cyclic repeating of a program α can be expressed. In the axiomatization of MLCR we use the Segerberg's induction axiom $\Box A \wedge \Box^*(A \Rightarrow \Box A) \Rightarrow \Box^* A$, which makes possible to apply the methods from chapter 1.1. A completeness theorem for MLCR with respect to the intended standard semantics is given.. It is proved that MLCR do not posses the finite model property with respect to its standard models but possesses this property with respect to some nonstandard models, which implies its decidability.

Introduction

Propositional dynamic logic with infinite repeating - RPDL - was introduced by Streett [St 82]. It contains an one-argument operator **repeat** which transforms each program α into a formula **repeat**α with the following semantics. Let $W\neq\emptyset$ be a set and $R(\alpha)$ be an interpretation of α as a binary relation in W. An infinite sequence $S=(x_1,x_2,\dots)$ of elements of W is called an *infinite* *α-repeating starting from* x_1 if for any i we have $x_i R(\alpha) x_{i+1}$. Then **repeat**α is true in some point $x\in W$ iff there exists an infinite α-repeating starting from x. Streett proposed an axiomatization of RPDL by the following axioms for **repeat**α:

(Str1) **repeat**$\alpha \Rightarrow$ **<**α**>repeat**α

(Str2) $A\wedge[\alpha^*](A\Rightarrow\langle\alpha\rangle A)\Rightarrow$**repeat**$\alpha$

A completeness proof for RPDL was found by Sakalauskaite and Valiev [S&V 80] and by Gargov and Passy [G&P 88] for some extension of RPDL. A completeness theorem for a fragment of RPDL, containing only the modalities $[\alpha]$, $[\alpha^*]$ and **repeat**α can be derived also from Goldblatt [Go 85].

Let us note that the finite version of α-repeating, namely *finite α-repeating, is clearly definable in PDL by the formula* $\langle\alpha^+\rangle\top$, where $\alpha^+=\alpha\alpha^*$ and $\top=p\vee\neg p$.

A special kind of infinite α-repeating is the notion of cyclic α-repeating. We say that an infinite α-repeating $S=(x_1,x_2,\dots)$ is a *cyclic α-repeating* if there exists $k\geq1$ such that for any i we have $x_{i+k}=x_i$. Then the sequence $(x_1,\dots,x_k)$ is called an α-cycle. Analogous to **repeat**α an operator **cycle**α can be introduced with the following semantics: **cycle**α is true in some point $x\in W$ iff there exists an α-cycle containing x. Cycling is a very natural feature of programs, so it is important to consider extensions of PDL in which **cycle**α can be expressed. Danecki [Da 84] had considered an extension of PDL with an operator **loop**α, called a *strong loop predicate,* with the following semantics: **loop**α is true in a point x iff $xR(\alpha)x$. Then **cycle**α is equivalent to the formula **loop**α^+. Another extension of PDL in which **cycle**α can be expressed is IPDL - PDL with intersection $\alpha\cap\beta$ of programs /see Harel [Ha 83], Danecki [Da 85]/. In IPDL **cycle**α is equivalent to the formula $\langle\alpha^+\cap\top?\rangle\top$. If IPDL contains the operation of converse α^{-1} then another formula, which is equivalent to **cycle**α is $\langle\alpha^c\rangle\top$, where $\alpha^c=\alpha^+\cap(\alpha^{-1})^+$. Unfortunately no completeness results are known for these logics. We will consider an extension of PDL, denoted by PDL^c, in which the above defined operation α^c is taken as a primitive one and called a *cyclic repeating.* The axiomatization of the full PDL^c is also an open problem. However, we found an axiomatization of a very simplified version of PDL^c without program operations and containing only three programs: α, α^* and α^c. This is a kind of a polymodal logic with three modal operations $[\alpha]$, $[\alpha^*]$ and $[\alpha^c]$, which will be denoted by $\Box$, $\Box^*$ and $\Box^c$ respectively. The corresponding modal logic will be called a *modal logic for cyclic repeating* and will be denoted for short by MLCR.

The main result of this chapter is an axiomatization of MLCR and a completeness theorem with respect to the described above standard semantics. The main difficulty of this problem is that the standard semantics for MLCR is not modally definable in the polymodal language of MLCR. So, we found, first, the expected axiom system and then define for it a nonstandard semantics, which is modally definable by the proposed set of axioms. Second, using the *copying method* (see the "Introduction. 1.5") we prove that each nonstandard model of MLCR can be transformed into an equivalent standard one. Third, applying a generalization of the Segerberg's filtration techniques for PDL [Seg 82a] we prove that MLCR is complete with respect to its nonstandard semantics and consequently - complete with respect to its standard semantics. As a side result of this proof we obtain that MLCR has the finite model property with respect to its nonstandard semantics, which implies its decidability. However, MLCR has not the finite model property with respect to its standard semantics.

1. Syntax and semantics of MLCR

The language of MLCR contains an infinite set VAR of propositional variables; $\neg$, $\wedge$, $\vee$ - the Boolean connectives; $\Box$, $\Box^*$, $\Box^c$ - three modal operations, and (,) - parentheses. The set FOR of all formulas is defined in the usual way. Abbreviations: $\top = A\vee\neg A$, $\bot=\neg\top$, $A\Rightarrow B=\neg A\vee B$, $\Diamond=\neg\Box\neg$, $\Diamond^*=\neg\Box^*\neg$, $\Diamond^c=\neg\Diamond^c\neg$, **cycle**$=\Diamond^c 1$. If $A\in$FOR then $|A|$ is the set of all subformulas of A.

Let $\underline{W}=(W, R, Q, S)$, be a relational system with $W\neq\emptyset$, called here a *frame*, and $v: VAR \longrightarrow P(W)$ be a *valuation*, assigning to each propositional variable A a subset $v(A)$ of W. For $x\in W$ and $A\in$FOR define a *satisfiability relation* $x\Vdash_v A$ / A is true in x at v/ according to the usual Kripke semantics:

$x\Vdash_v A$ iff $x\in v(A)$ for $A\in$VAR,

$x\Vdash_v \neg A$ iff $x\not\Vdash_v A$ /non $x\Vdash_v A$/,

$x\Vdash_v A\wedge B$ iff $x\Vdash_v A$ and $x\Vdash_v B$,

$x\Vdash_v A\vee B$ iff $x\Vdash_v A$ or $x\Vdash_v B$,

$x\Vdash_v \Box A$ iff $(\forall y\in W)(xRy \longrightarrow y\Vdash_v A)$,

$x\Vdash_v \Box^* A$ iff $(\forall y\in W)(xQy \longrightarrow y\Vdash_v A)$,

$x\Vdash_v \Box^c A$ iff $(\forall y\in W)(xSy \longrightarrow y\Vdash_v A)$.

If $\underline{W}$ is a frame and v a valuation then the pair $(\underline{W}, v)$ will be called a *model over* $\underline{W}$. A *formula A is true in a model* $(\underline{W}, v)$, or $(\underline{W}, v)$ verifies A, if for any $x\in W$ we have $x\Vdash_v A$. A formula A is *true in a frame* $\underline{W}$ if it is true in any model over $\underline{W}$. A *formula A is true in a class* Γ *of frames* if it is true in each frame from Γ. By $L(\Gamma)$ - *the logic of* Γ- we denote the set of all formulas true in Γ. We say that *two models are equivalent* if they verify one and the same set of formulas.

To define the standard semantics of MLCR we need some notations and facts. Let R and S be two relations in a set $W\neq\emptyset$. Define:

$R^{-1}=\{(x,y)/yRx\}$, $R\circ S=\{(x,z)/(\exists y\in W)xRy \text{ \& } ySz\}$,

$R^0=id_W=\{(x,x)/x\in W\}$, $R^{i+1}=R^i\circ R=R\circ R^i$, $R^*=\bigcup_{i=0}^{\infty}R^i$, $R^+=\bigcup_{i=1}^{\infty}R^i$, $R^c=R^+\cap(R^{-1})^+$.

It is well known fact that R^* is the reflexive and transitive closure of R and that R^+ is the transitive closure of R. To explain the meaning of R^c we need the notion of R-cycle. A finite nonempty sequence $C=(x_1,x_2,\ldots,x_n)$ is called an *R-cycle* if $x_1Rx_2R\ldots x_nRx_1$. By this definition the sequence (x) is an R-cycle if xRx. If x is a member of C we will write $x\in C$. The following equivalence follows directly from the definitions:

xR^cy iff there exists an R-cycle C such that $x,y\in C$.

Obviously R^c is a symmetric and transitive relation in W. Let us say that a relation is a *quasi-equivalence* if it is a symmetric and transitive one. Then, adopting such a terminology we will call R^c the *cyclic quasi-equivalence determined by* R.

We say that a frame $\underline{W}=(W, R, Q, S)$ is a *standard frame* for MLCR if $Q=R^*$ and $S=R^c$. By Σ_s, Σ_s^{fin}, Σ_s^{inf} we denote the classes of all standard frames, all finite standard frames, and all standard infinite frames respectively.

Let $\underline{W}$ be a standard frame. Then for any valuation v we have

$x \Vdash_v$ **cycle** iff $x \in C$ for some R cycle in W.

This shows that the constant **cycle** expresses in MLCR the notion of cyclic repeating of R.

We propose the following axiomatization of MLCR

Axiom schemes and rules for MLCR

(Bool) All or enough Boolean tautologies,
(K□) $\Box(A \Rightarrow B) \Rightarrow (\Box A \Rightarrow \Box B)$,
(K□*) $\Box^*(A \Rightarrow B) \Rightarrow (\Box^* A \Rightarrow \Box^* B)$,
(K□^{c}) $\Box^c(A \Rightarrow B) \Rightarrow (\Box^c A \Rightarrow \Box^c B)$,
(Ref□*) $\Box^* A \Rightarrow A$,
(Tr□*) $\Box^* A \Rightarrow \Box^* \Box^* A$,
(Incl) $\Box^* A \Rightarrow \Box A$,
(Ind□*) $A \wedge \Box^*(A \Rightarrow \Box A) \Rightarrow \Box^* A$,
(Sim□^{c}) $A \vee \Box^c \neg \Box^c A$,
(Tr□^{c}) $\Box^c A \Rightarrow \Box^c \Box^c A$,
(Ind□^{c}) $\Box A \wedge \Box^c(A \Rightarrow \Box A) \Rightarrow \Box^c A$,
(MP) A, A⇒B/B, (N□) A/□A, (N □*) A/□*A, (N□^{c}) A/□^{c}A.

The axioms (Ref□*), (Tr□*) and (Incl) express the fact that the relation Q is a reflexive and transitive relation, containing R. The axiom (Ind□*) is the well-known Segerberg's induction axiom for the iteration in PDL, denoted in chapter 1.1 by Seg. Let us note that instead of (Tr□*), (Ref□*) and (Incl), Segerberg uses a single axiom - (Ind_0) $\Box^* A \Rightarrow A \wedge \Box\Box^* A$, which is equivalent to the ours on the base of (K□) and (Ind□*). The axiom (Ind□^{c}) will be called *the induction axiom for* □^{c}. It was denoted in chapter 1.1 by Seg1.

Lemma 1.1.

The logic MLCR is sound with respect to its standard semantics.

Proof. We shall show only that (Ind□^{c}) is true in all standard frames. Suppose that $\underline{W}=(W, R, R^*, R^c)$ is a standard frame, v is a valuation, $x \in W$, $x \Vdash_v \Box A$, $x \Vdash_v \Box^c(A \Rightarrow \Box A)$ and proceed to show that $x \Vdash_v \Box^c A$. Suppose $xR^c y$, i.e. xR^+y and yR^+x. We have to show that $y \Vdash_v A$.

From xR^+y we have that $xR^i y$ for some $i \ge 1$. Now we apply induction on i.

i=1. /basis/ Then we have xRy and since $x \Vdash_v \Box A$ we obtain that $y \Vdash_v A$.

i=k. /induction hypothesis/ Suppose that for any $z \in W$: if $xR^k z$ and zR^+x then $z \Vdash_v A$.

i=k+1. /induction step/ We have $xR^{k+1}y \longleftrightarrow (\exists z \in W) xR^k z$ & zRy. From zRy and yR^+x we obtain that zR^+x. Then from $xR^k z$ and zR^+x we obtain by i.h. that $z \Vdash_v A$. Also from $xR^k z$ and zR^+x we obtain $xR^c z$. Then, since $x \Vdash_v \Box^c(A \Rightarrow \Box A)$ we get $z \Vdash_v A \Rightarrow \Box A$, and by $z \Vdash_v A$ we obtain $z \Vdash_v \Box A$. From here and zRy we finally obtain $y \Vdash_v A$. This shows that (Ind ^{c}) is true in $\underline{W}$. ■

According to the modality □* a stronger result can be proved: the axioms (Ref□*), (Tr□*), (Incl) and (Ind□*) are true in a frame (W, R, Q, S) iff $Q=R^*$. One can expect that the axioms for □^{c} - (Sim□^{c}), (Tr□^{c}) and (Ind□^{c}) are true in a frame (W, R, Q, S) iff $S=R^c$. But this is not true. More over we shall show later that there is no a set F of formulas such that F is true in a frame (W, R, Q, S) iff $S=R^c$. This means that the notion of a standard frame is not modally definable in the language of MLCR. That is why we shall define a more wide class of frames for MLCR than the standard one, which will be modally definable by the axioms of MLCR.

A frame $\underline{W}=(W, R, Q, S)$ i called a *general frame for MLCR* if $Q=R^{*}$, S is a quasi-equivalence in W and the axiom (Ind ^{c}) is true in $\underline{W}$. General frames sometimes will be called nonstandard frames for MLCR. By Σ_{g}, Σ_{g}^{fin}, Σ_{g}^{inf} we denote the classes of all general frames, all finite general frames, and all infinite general frames respectively. General frames define general or nonstandard semantics for MLCR.

A simple example of a general frame, which is not a standard one can be constructed in the following way. Let $W=\{a\}$, $R=\{(a,a)\}$, $Q=R^{*}$ and $S=\emptyset$. Obviously S is a quasi-equivalence in W. It is easy to verify that (Ind$\Box^{c}$) is true in the frame $\underline{W}=(W, R, Q, S)$, so $\underline{W}$ is a general frame for MLCR. For that frame we have $R^{+}=R$, $R^{-1}=R$ and hence $R^{c}=R$. So $R^{c}\neq S$, which shows that $\underline{W}$ is not a standard frame for MLCR.

To find an adequate semantic condition for (Ind ^{c}) we shall proceed as in chapter 1.1, generalizing the operation of R^{+} in the following way.

Let $W\neq\emptyset$ and R, S be binary relations in W. We define a new relation R_{S}^{+} in W by the following inductive definition:

$R_{S}^{1}=R$, $R_{S}^{i+1}=(R_{S}^{i}\cap S)\circ R$, $R_{S}^{+}=\bigcup_{i=1}^{\infty}R^{i}$.

Let $S(x)=\{y\in W/xSy\}$. Then the following lemma holds.

Lemma 1.2.

(i) $xR_{S}^{1}y$ *iff* xRy,

(ii) for $i\geq 2$: $xR_{S}^{i}y$ *iff* $\exists x_{1},x_{2},\ldots,x_{i}$ *such that* $x=x_{1}Rx_{2}R\ldots x_{i}Ry$ *and* $\{x_{2},\ldots,x_{i}\}\subseteq S(x)$,

(iii) $R_{S}^{i}\subseteq R^{i}$, $R_{S}^{+}\subseteq R^{+}$,

(iv) $R_{S}^{+}=R_{R^{+}\cap S}^{+}$.

Proof. The conditions (i), (ii) and (iii) follow directly from the definitions. For (iv) we shall show by induction that for any i: $R_{S}^{i}=R_{R_{S}^{+}\cap S}^{i}$, which proves the assertion.

i=1. $R_{S}^{1}=R=R_{R_{S}^{+}\cap S}^{1}$.

i=k. /i.h./ Suppose $R_{S}^{k}=R_{R_{S}^{+}\cap S}^{k}$.

i=k+1. $R_{R_{S}^{+}\cap S}^{k+1}=(R_{R_{S}^{+}\cap S}^{k}\cap S)\circ R=$/by i.h./$(R_{S}^{k}\cap S)\circ R=R_{S}^{k+1}$.■

Proposition 1.3

(i) The formula (Ind ^{c}*) is true in a frame* (W, R, R^{*}, S) *iff* $S\subseteq R_{S}^{+}$.

(ii) Let $\underline{W}=(W, R, Q, S)$ *be a frame such that* $Q=R^{*}$ *and* S *be a quasi-equivalence in W. Then* $\underline{W}$ *is a general frame for MLCR iff* $S\subseteq R_{S}^{+}$.

(iii) All standard frames for MLCR are general ones.

Proof. (i)($\Leftarrow$) Let v be a valuation, $x\in W$ and $S\subseteq R_{S}^{+}$. Suppose $x\Vdash_{v}\Box A$ and $x\Vdash_{v}\Box^{c}(A\Rightarrow\Box A)$. We will show that $x\Vdash_{v}\Box^{c}A$. For that purpose suppose xSy and proceed to show that $y\Vdash_{v}A$.

From xSy and $S\subseteq R_{S}^{+}$ we obtain $xR_{S}^{+}y$, so $xR_{S}^{i}y$ for some $i\geq 1$. Now we apply induction on i.

i=1. Then $xR_{S}^{1}y$ implies xRy and since $x\Vdash_{v}\Box A$ we obtain $y\Vdash_{v}A$.

i=k. /i.h./ Suppose that for any $z \in W$: if $xR_S^k z$ then $z \Vdash_v A$.

i=k+1. From $R_S^{k+1}=(R_S^k \cap S) \circ R$ we obtain that $xR_S^{k+1}y$ implies that for some $z \in W$ we have $xR_S^k z$, xSz and zRy. By the i.h. we obtain that $z \Vdash_v A$. From xSz and $x \Vdash_v \Box^c(A \Rightarrow \Box A)$ we get $z \Vdash_v A \Rightarrow \Box A$, and since $z \Vdash_v A$ then $z \Vdash_v \Box A$. From here and zRy we obtain that $y \Vdash_v A$.

($\rightarrow$) Suppose that $S \not\subseteq R_S^+$. Then for some $x,y \in W$ we have xSy but not $xR_S^+ y$. We will show now that the formula $A \wedge \Box^c(A \Rightarrow \Box A) \Rightarrow \Box^c A$, $A \in VAR$, is not true in W. Define a valuation v for A as follows:

$v(A)=\{t \in W / \text{if } xSt \text{ then } xR_S^+ t\}$.

From this definition we have that $y \not\Vdash_v A$ and by xSy we obtain $x \not\Vdash_v \Box^c A$.

We shall show that $x \Vdash_v \Box A$. Suppose xRz. Then we have $xR_S^1 z$ and consequently $xR_S^+ z$. So $z \in v(A)$ and hence $x \Vdash_v \Box A$.

We shall show also that $x \Vdash_v \Box^c(A \Rightarrow \Box A)$. For that purpose suppose xSt, $t \Vdash_v A$, tRz and proceed to show that $z \Vdash_v A$.

From xSt and $t \Vdash_v A$ we obtain $xR_S^+ t$ and then $xR_S^i t$ for some i. From $xR_S^i t$, xSt and tRz we obtain xR_S^{i+1}, so $xR^+ z$ and hence $z \Vdash_v A$.

From $x \not\Vdash_v \Box^c A$, $x \Vdash_v \Box A$ and $x \Vdash_v \Box^c(A \Rightarrow \Box A)$ we obtain that $x \not\Vdash_v \Box A \wedge \Box^c(A \Rightarrow \Box A) \Rightarrow \Box^c A$, which shows that (ind$\Box^c$) is not true in $\underline{W}$.

(ii) is a consequence of (i).

(iii) Suppose $\underline{W}=(W, R, R^*, R^c)$ is a standard frame. The relation R^c is a quasi-equivalence in W and by lemma 1.1 (ind$\Box^c$) is true in $\underline{W}$, so $\underline{W}$ is is a general frame for MLCR.■

The following lemma will be of later use.

Lemma 1.4.

Let $\underline{W}=(W, R, R^*, S)$ be a general frame. Then:

(i) If C is an S-cycle in W then for any $x,y \in C$ we have xSy.

(ii) $\underline{W}=(W, R, R^*, S)$ is a standard frame if $R^c \subseteq S$,

(iii) $\underline{W}$ is a standard frame if all R-cycles in W are S-cycles.

Proof. (i) The assertion follows immediately from the transitivity of S.

(ii) Suppose $R^c \subseteq S$. Since $\underline{W}$ is a general frame then by proposition 1.3.(ii) we have $S \subseteq R_S^+$, and since $R_S^+ \subseteq R^+$ we obtain that $S \subseteq R^+$. From here we obtain also that $S^{-1} \subseteq (R^+)^{-1}=(R^{-1})^+$. But S is a symmetric relation then $S^{-1}=S$ and hence $S \subseteq (R^{-1})^+$. From $S \subseteq R^+$ and $S \subseteq (R^{-1})^+$ we get $S \subseteq R^+ \cap (R^{-1})^+=R^c$, so $S=R^c$.

(iii). Suppose that all R-cycles in W are S-cycles in W. We shall show that $R^c \subseteq S$ and then by (ii) $\underline{W}$ will be a standard frame.

Suppose $xR^c y$. Then there exists an R-cycle C such that $x,y \in C$. But C is also an S-cycle and by (i) we have xSy. So $R^c \subseteq S$.■

2. Equivalence of standard and general semantics for MLCR.

The main aim of this section is to show that $L(\Sigma_s)=L(\Sigma_g)$, i.e. that the logic of general frames coincides with the logic of standard frames. We will do this by means of the *copying* . The general definition, adapted for the case of MLCR is the following.

Let $\underline{W}=(W, R, Q, S)$ and $\underline{W}'=(W', R', Q', S')$ be two frames and $\underline{M}=(\underline{W}, v)$ and

$\underline{M}'=(\underline{W}',v')$ be models over $\underline{W}$ and $\underline{W}'$ respectively. Let $I\neq\emptyset$ be a set of functions from W into W' and let for any $i\in I$ and $x\in W$ the application of i to x be denoted by x_i. We will say that I is a *copying* from $\underline{W}$ to $\underline{W}'$ if the following conditions are satisfied for any $x,y\in W$ and $i\in I$:

(I) $W'=\bigcup_{i\in I} W_i$, where $W_i=\{x_i/x\in W\}$, (CR1) If xRy then $\exists j\in I$ such that $x_iR'y_j$,

(CR2) If $x_iR'y'$ then $\exists y\in W$ and $j\in I$ such that $y'=y_j$ and xRy,

Analogously for Q and S we have the corresponding conditions (CQ1), (CQ2), (CS1) and (CS2).

We say that I is a *copying of the model* $\underline{M}$ *to the model* $\underline{M}'$ if I is a copying of $\underline{W}$ to $\underline{W}'$ and the following condition is satisfied for any $A\in VAR$, $x\in W$ and $i\in I$:

(Cv) $x\in v(A)$ iff $x_i\in v'(A)$.

Lemma 2.1. */Copying Lemma/*

Let $\underline{M}=((W, R, Q, S), v)$ *be a model and* $\underline{M}'$ *be a copying of* $\underline{M}$ *through I. Then:*

(i) For any formula A, $x\in W$ *and* $i\in I$*:*

$x\Vdash_v A$ *iff* $x_i\Vdash_{v'} A$,

(ii) The models $\underline{M}$ *and* $\underline{M}'$ *are equivalent.*

Theorem 2.2. *(Standardization Theorem for general models)*

Let $M=((W, R, R^*,S),v)$ *be a general model. Then there exists an infinite standard model* $M'=((W',R', R'^*,S'),v')$ *equivalent to M.*

Proof. If $\underline{W}=(W, R, R^*, S)$ is a general frame which is not a standard one then by lemma 1.4.(iii) there exists an R-cycle $C=(x_1,\dots,x_n)$ which is not an S-cycle. Such R-cycles will be called defective ones. In each defective R-cycle there exists an index i such that x_iSx_{i+1} is not true (if i=n then i+1 means 1). The pair (x_i,x_{i+1}) is called a defective pair in C. A pair (x,y) is called a defective pair in W if it is a defective pair in some defective R-cycle in W.

Now we shall apply a copying construction. Let $I=N=\{0,1,2,\dots\}$ Define $W'=W\times N$ and let for any $i\in N$ and $x\in W$ $x_i=(x,i)$. Obviously W' is an infinite set and the condition (I) is satisfied. We define the relations R' and S' in W' by the following equivalences:

$x_iR'y_j$ iff xRy and [(j=i+1 and (x,y) is a defective pair in W) or (j=i and (x,y) is not a defective pair in W)],

$x_iS'y_j$ iff xSy and i=j.

The required frame is $\underline{W}'=(W', R', R'^*, S')$. It is easy to see that the conditions (CR1), (CR2), (CQ1), (CQ2), (CS1) and (CS2) from the definition of copying are fulfilled and hence I is a copying of $\underline{W}$ to $\underline{W}'$. It remains to prove that $\underline{W}'$ is a standard frame.

Obviously the relation S' is a quasi-equivalence relation in W'. First we will show that $\underline{W}'$ is a general frame. For that purpose it is sufficient to prove that $S'\subseteq R'^+_{S'}$.

Suppose $x_iS'y_j$ and proceed to show that $x_iR'^+_{S'}y_j$. From $x_iS'y_j$ we have xSy and i=j. Since $\underline{W}$ is a general frame we have that $S\subseteq R^+_S$, so xR^+_Sy and then xR^k_Sy for some k. If k=1 then by lemma 1.2.(i) we have xRy. Since xSy then (x,y) could not be a defective pair in W, so $x_iR'y_i$ and since i=j we have $x_iR'y_j$ and hence $x_iR'^+y_j$. If k≥2 then by lemma 1.2.(ii) there exists a sequence $x^1, x^2, \dots, x^k$ such that $x=x^1Rx^2\dots Rx^kRy$ and $\{x^2,\dots,x^k\}\subseteq S(x)$, so x^1Sx^2, $x^1Sx^3, \dots, x^1Sx^k$. Then, since S is a quasi-equivalence in W, we have x^sSx^{s+1} for any $1\le s<k$. This shows that (x^s, x^{s+1}) could not be a defective pair in W and then, by the definition of R' we obtain

$x_i=x_i^1R'x_i^2R'\dots R'x_i^nR'y_i$ and $\{x_i^2,\dots,x_i^n\}\subseteq S'(x_i)$.

Then, by lemma 1.2.(ii) we obtain $x_iR'^+_{S'}y_i$ and since i=j – $x_iR'^+_{S'}y_j$.

Now we shall show that M' is a standard frame. For that purpose we shall apply lemma 2.2.(iii), namely we shall prove that all R'-cycles in W' are S'-cycles.

Let $C'=\{x_{i_1}^1, x_{i_2}^2, \dots, x_{i_n}^n\}$ be an R'-cycle in W'. So we have $x_{i_1}^1 R'x_{i_2}^2 R'\dots R'x_{i_n}^n R'x_{i_1}^1$. By the definition of R' we have that $i_1\le i_2\le\dots\le i_n\le i_1$ and from here $i_1=i_2=\dots=i_n=i$. So all members of C' are in one and the same copy W_i of W, which says that all pairs (x^k,x^{k+1}), (x^n,x^1), $1\le k<n$, are not defective ones. This implies that x^kSx^{k+1} and x^nSx^1. Then, by the definition of S' we obtain $x_i^1S'x_i^2S'\dots S'x_i^nS'x_i^1$, which shows that C' is an S'-cycle in W'. Hence the frame W̲' is a standard one.

Applying the Copying Lemma we obtain that the model M̲'=(W̲',v') where $v'=\{x_i/x\in v(A), i\in I\}$ is equivalent to the model M̲.■

Corollary 2.3.

(i) $L(\Sigma_g)=L(\Sigma_s)=L(\Sigma_s^{inf})$,

(ii) There is no set F of formulas such that for any general frame W̲=(W, R, R^*, S) : F is true in W̲ iff W̲ is a standard frame, i.e. the class of standard frames is not modally definable in the language of MLCR.

Proof. (i) By proposition 1.3.(iii) $\Sigma_s\subseteq\Sigma_g$, so $\Sigma_s^{inf}\subseteq\Sigma_s\subseteq\Sigma_g$. Consequently $L(\Sigma_g)\subseteq L(\Sigma_s)\subseteq L(\Sigma_s^{inf})$. By theorem 2.2. we have $L(\Sigma_s^{inf})\subseteq L(\Sigma_g)$. Hence $L(\Sigma_g)=L(\Sigma_s)=L(\Sigma_s^{inf})$.

(ii) Suppose that such a set F exists. Then $F\subseteq L(\Sigma_s)$. Let $\underline{W}_0$ be a general frame for MLCR which is not a standard one. Then for some formula A∈F we have that A is not true in $\underline{W}_0$, so $A\notin L(\Sigma_g)$. Since $A\in L(\Sigma_s)$ we obtain that $L(\Sigma_s)\ne L(\Sigma_g)$ which contradicts (i). ■

Let us note that the copying construction in the proof of theorem 2.2 always produces an infinite standard model M̲'. One can think that the infinity of M̲' is a side effect depending on the construction. The following theorem shows that the infinity of M̲' can not be avoided.

Theorem 2.4.

$L(\Sigma_s^{inf})\ne L(\Sigma_s^{fin})$

Proof. Consider the formula

(Fin) $\Diamond^*(\Box\bot\vee$**cycle**$)$

Let $W_0=(N, R, R^*, R^c)$ be a standard frame defined as follows:

N is the set of all natural numbers,

xRy iff y=x+1, then xR^*y iff x≤y, xR^+y iff x<y and $R^c=\emptyset$.

It is easy to see that *Fin* is not true in W_0, so $Fin\notin L(\Sigma_s^{inf})$. However *Fin* is true in the class Σ_s^{fin} of all finite standard frames and hence $Fin \in L(\Sigma_s^{fin})$. To show this let x be a member of some finite standard frame $\underline{W}$. If there exists an R-cycle C, reached from x by R^* then obviously *Fin* is true in x, because **cycle** is true in C. If this is not the case, then, since W is finite, all R-paths from x are finite, and hence in their ends $\bot$ will be true, so, again *Fin* is true in x.■

This theorem shows that MLCR do not posses the finite model property (f.m.p.) with respect to its standard models. However, we shall prove later that MLCR posses f.m.p. with respect to its general models.

3. Preliminary facts about canonical model and filtration

Canonical model. Denote by L the logic MLCR and the set of all theorems of L. The set of all maximal consistent sets of L is denoted by W_L.

The canonical relations in W_L and the canonical valuation are defined as usual:

xR_Ly iff $\{A\in FOR/\Box A\in x\}\subseteq y$,

xQ_Ly iff $\{A\in FOR/\Box^* A\in x\}\subseteq y$,

xS_Ly iff $\{A\in FOR/\Box^c A\in x\}\subseteq y$,

$v_L(A)=\{x\in W_L/A\in x\}$, $A\in VAR$.

The frame $\underline{W}_L=(W_L, R_L, Q_L, S_L)$ is called the canonical frame of L and the model $M_L=(\underline{W}_L, v_L)$ - the canonical model of L.

Fact 3.1.

(i)/Lindenbaum Lemma/ Any consistent set of formulas of L can be extended to a maximal consistent set of L. In particular, if A is not a theorem of L then $\{\neg A\}$ is a consistent set and hence $A\notin x$ for some $x\in W_L$.

For any $x\in W_L$ and $A,B\in FOR$ the following is true:

(ii) If $A\in x$ and $A\Rightarrow B\in L$ then $B\in x$,

(iii) $L\subseteq x$, $\top\in x$, $\bot\notin x$

(iv) $\neg A\in x$ iff $A\notin x$,

(v) $A\wedge B\in x$ iff $A\in x$ and $B\in x$,

(vi) $A\vee B\in x$ iff $A\in x$ or $B\in x$,

(vii) $\Box A\in x$ iff $(\forall y\in W_L)(xR_Ly \longrightarrow A\in y)$,

(viii) $\Box^* A\in x$ iff $(\forall y\in W_L)(xQ_Ly \longrightarrow A\in y)$,

(ix) $\Box^c A\in x$ iff $(\forall y\in W_L)(xS_Ly \longrightarrow A\in y)$.

Fact 3.2.

The following conditions are true for the canonical frame:

(i) Q_L is a reflexive and transitive relation in W_L and $R_L\subseteq Q_L$,

(ii) S_L is a quasi-equivalence in W_L.

Proof. (i) It is a well known fact that the axioms (Ref$\Box^*$) and (Tr$\Box^*$) imply reflexivity and transitivity of the corresponding canonical relation Q_L. The inclusion $R_L\subseteq Q_L$ follows from axiom (Incl).

(ii) The symmetry and transitivity of S_L follows from the axioms (Sim$\Box^c$) and (Tr$\Box^c$) respectively.■

Filtration. Here we shall list some basic definitions and facts about Segerberg's filtration of canonical models, adapted for the logic L.

Let Φ be a finite set of formulas closed under subformulas. For $x,y\in W_L$ define:

$x\sim y$ iff $(\forall A\in\Phi)(A\in x \longleftrightarrow A\in y)$, $|x|=\{y\in W_L/x\sim y\}$, $|W_L|=\{|x|/x\in W_L\}$, $v(A)=\{|x|/A\in x\}$ for $A\in VAR$.

Let R, Q, S be binary relations in $|W_L|$. Then the model $|M_L|=((|W_L|, R, Q, S), v)$ will be called a *filtration of the canonical model* M_L *through* Φ if the following conditions are satisfied:

(FR1) If xR_Ly then $|x|R|y|$,

(FR2) If $|x|R|y|$ then $(\forall\Box A\in\Phi)(\Box A\in x \longrightarrow A\in y)$,

(FQ1) If xQ_Ly then $|x|Q|y|$,

(FQ2) If $|x|Q|y|$ then $(\forall\Box^*A\in\Phi)(\Box^*A\in x \longrightarrow A\in y)$,

(FS1) If xS_Ly then $|x|S|y|$,

(FS2) If $|x|S|y|$ then $(\forall\Box^cA\in\Phi)(\Box^cA\in x \longrightarrow A\in y)$.

Fact 3.3. */Filtration Lemma/*

Let $((|W_L|, R, Q, S), v)$ *be a filtration of the canonical model through* Φ. *Then the following holds:*

(i) *For any* $x\in W_L$ *and* $A\in\Phi$: $|x|\Vdash_{v} A$ *iff* $A\in x$,

(ii) *If* $\mathrm{Card}\Phi=n$ *then* $\mathrm{Card}|W_L|\leq 2^n$

(iii) *For each subset* $M\subseteq|W|$ *there exists a formula A, /the characteristic formula of M/ which is a Boolean combination of elements of* Φ, *such that for any* $x\in W_L$: $A\in x$ *iff* $|x|\in M$.

We say that L *admits a filtration* if for any formula A there exists a finite set of formulas Φ, containing A and closed under subformulas, and a filtration $((|W_L|, R, Q, S),v)$ trough Φ, such that the frame $(|W_L|, R, Q, S)$ is a general frame for L. If Φ consists of all subformulas of A then we say that *L admits a good filtration.*

4. Filtration and completeness theorem for MLCR

Theorem 4.1. */Filtration Theorem for MLCR/*

The logic MLCR admits a good filtration.

Proof. Let Φ be the set of all subformulas of a given formula. Define the following relations in $|W_L|$:

$|x|R'|y|$ iff $(\exists x',y'\in W_L)(x\sim x' \;\&\; y\sim y' \;\&\; x'R_Ly')$,

$|x|Q'|y|$ iff $(\forall\Box^*A\in\Phi)(\Box^*A\in x \longrightarrow \Box^*A\in y)$,

$|x|S'|y|$ iff $(\forall\Box^cA\in\Phi)(\Box^cA\in x \longrightarrow A\wedge\Box^cA\in y)$,

$S''=S'\cap(S')^{-1}$, $R=R'\cap Q'$, $Q=R^*$, $S=R^+_{S''}\cap(R^{-1})^+_{S''}\cap S''$.

We need the following lemmas.

Lemma 4.2.

(i) The relations R', Q' and S' are well defined and satisfy the following conditions:

(FR'1) If xR_Ly *then* $|x|R'|y|$,

(FR'2) If $|x|R'|y|$ *then* $(\forall\Box A\in\Phi)(\ A\in x \longrightarrow A\in y)$,

(FQ'1) If $xQ_L y$ then $|x|Q'|y|$,

(FQ'2) If $|x|Q'|y|$ then $(\forall\Box^*A\in\Phi)(\Box^*A\in x \rightarrow A\in y)$,

(FS'1) If $xS_L y$ then $|x|S'|y|$,

(FS'2) If $|x|S'|y|$ then $(\forall\Box^c A\in\Phi)(\Box^c A\in x \rightarrow A\in y)$.

(ii) 1. Q' is a reflexive and transitive relation in $|W_L|$,

2. S' is a transitive relation in $|W_L|$.

Proof. (i) and (ii) are well known facts in modal logic.Note that R' is known as minimal filtration for R_L, and Q' and S' are known as Lemmon filtrations for the modal logics S4 and K4 respectively.■

Lemma 4.3.

(i) The relations S" and R satisfy the following conditions:

(FS"1) If $xS_L y$ then $|x|S''|y|$,

(FS"2) If $|x|S''|y|$ then $(\forall\Box^c A\in\Phi)(\Box^c A\in x \rightarrow A\in y)$,

(FR1) If $xR_L y$ then $|x|R|y|$,

(FR2) If $|x|R|y|$ then $(\forall\Box A\in\Phi)(\Box A\in x \rightarrow A\in y)$,

(ii) S" is symmetric and transitive relation in $|W_L|$.

Proof. (i) (FS"1). Suppose $xS_L y$. Then by the fact 3.2.(ii) we have $yS_L x$, and by (FS'1) we obtain $|x|S'|y|$ and $|y|S'|x|$. Thus $|x|S''|y|$.

(FS"2). Suppose $|x|S''|y|$. Then we have $|x|S'|y|$ and by (FS'2) we have $(\forall\Box^c A\in\Phi)(\Box^c A\in x \rightarrow A\in y)$.

(FR1). Suppose $xR_L y$. Then by the fact 3.2.(i) we have $xQ_L y$ and by (FR'1) and (FQ'1) we obtain $|x|R'|y|$ and $|x|Q'|y|$, hence $|x|R|y|$.

(FR2). Suppose $|x|R|y|$. Then we have $|x|R'|y|$ and by (FR'2) we obtain $(\forall\Box A\in\Phi)(\Box A\in x \rightarrow A\in y)$.

(ii) By lemma 5.2.(ii) S' is a transitive relation. Then S'^{-1} is a transitive relation too. From here it can be easily obtained that $S''=S'\cap S'^{-1}$ is a symmetric and transitive relation in $|W_L|$.■

Lemma 4.4.

The relation Q satisfies the following conditions:

(FQ1) If $xQ_L y$ then $|x|Q|y|$,

(FQ2) If $|x|Q|y|$ then $(\forall\Box^*A\in\Phi)(\Box^*A\in x \rightarrow A\in y)$,

Proof. (FQ1). We will follow the Segerberg's proof from Segerberg (1982). Suppose $xQ_L y$ and not $|x|Q|y|$ for some $x,y\in W_L$. We will proceed to obtain a contradiction. Define $M=\{|z|/|x|Q|z|\}$. So $|y|\notin M$ and $|x|\in M$, because Q is a reflexive relation. By the fact 3.3.(iii) there exists a formula A such that for any $z\in W_L$: $|z|\in M$ iff $A\in z$. From $|x|\in M$ and $|y|\notin M$ we get $A\in x$ and $A\notin y$. From $xQ_L y$ and $A\notin y$ we obtain that $\Box^*A\notin x$. By axiom (Ind$\Box^*$) $A\wedge\Box^*(A\Rightarrow\Box A)\Rightarrow\Box^*A$, $A\in x$ and $\Box^*A\notin x$ we obtain that $\Box^*(A\Rightarrow\Box A)\notin x$. Then by the fact 3.1.(viii) there exists $t\in W_L$ such that $xQ_L t$ and $A\Rightarrow\Box A\notin t$ and hence $A\in t$ and $\Box A\notin t$. From $A\in t$ we obtain that $|t|\in M$ and hence $|x|Q|t|$. So we have $|x|R^i|t|$ for some i. From $\Box A\notin t$ we obtain by the fact 3.1.(vii) that $tR_L z$ and $A\notin z$ for some $z\in W_L$. Then by lemma 4.3.(i)(FR1) we get $|t|R|z|$. From $|x|R^i|t|$ and $|t|R|z|$ we obtain $|x|R^{i+1}|z|$ and hence $|x|R^*|z|$. So $|x|Q|z|$ and $|z|\in M$, hence $A\in z$ - a contradiction.

(FQ2). Suppose $|x|Q|y|$. Then we have $|x|R^i|y|$ and then for some $x_1,x_2,\dots,x_i$ we have $|x|R|x_1|R|x_2|\dots|x_i|R|y|$. By the definition of R we obtain $|x|Q'|x_1|Q'|x_2|\ldots|x_i|Q'|y|$. Then by lemma 4.2.(ii)1 Q' is a reflexive and transitive relation and hence $|x|Q'|y|$. Then by (FQ'2) we obtain $(\forall\Box^*A\in\Phi)(\Box^*A\in x \longrightarrow A\in y)$, which proves (FQ2).■

Lemma 4.5.

The relation S satisfies the conditions

(FS1) If xS_Ly then $|x|S|y|$,

(FS2) If $|x|S|y|$ then $(\forall\Box^cA\in\Phi)(\Box^cA\in x \longrightarrow A\in y)$.

Proof. (FS1). Suppose xS_Ly and proceed to show that $|x|S|y|$. For that purpose we have to show the following three conditions: (i) $|x|S''|y|$, (ii) $|x|R^+_{S''}|y|$, and (iii) $|y|R^+_{S''}|x|$.

Proof of (i). From xS_Ly we obtain by lemma 4.3.(i)(FS"1) that $|x|S''|y|$.

Proof of (ii). To proof $|x|R^+_{S''}|y|$ we define the set $M=\{|z|/|x|R^+_{S''}|z|\}$ and proceed to show that $|y|\in M$. By the fact 3.3.(iii) there exists a formula A such that for any $z\in W_L$: $A\in z$ iff $|z|\in M$. So we have to show that $A\in y$. This follows from the following

Assertion 1.

(a) $\Box A\in x$,

(b) $\Box^c(A\Rightarrow\Box A)\in x$.

Indeed, from (a) and (b) we obtain that $\Box A\wedge\Box^c(A\Rightarrow\Box A)\in x$. Then by the axiom (Ind$\Box^c$) $\Box A\wedge\Box^c(A\Rightarrow\Box A)\Rightarrow\Box^cA$ we obtain that $\Box^cA\in x$. From here and xS_Ly we get $A\in y$.

To prove (a) suppose xR_Lz and proceed to show that $A\in z$. By (FR1) we have $|x|R|z|$, so $|x|R^+_{S''}|z|$, $|z|\in M$ and hence $A\in z$.

To prove (b) suppose xS_Lt, $A\in t$, tR_Lz and proceed to show that $A\in z$. From $A\in t$ we get $|t|\in M$ and hence $|x|R^+_{S''}|t|$. Then for some $i\geq 1$ we have $|x|R^i_{S''}|t|$. From xS_Lt and tR_Lz, by the conditions (FS"1) and (FR1) from lemma 4.3.(i) we obtain $|x|S''|t|$ and $|t|R|z|$. Then the conditions $|x|R^i_{S''}|t|$, $|x|S''|t|$ and $|t|R|z|$ imply $|x|R^{i+1}_{S''}|z|$ so $|x|R^+_{S''}|z|$, $|z|\in M$ and hence $A\in z$.

Proof of (iii). The proof of $|y|R^+_{S''}|x|$ is similar to that of (ii): define the set $N=\{|z|/|y|R^+_{S''}|z|\}$ and let B be the characteristic formula of N. We have to show that $|x|\in N$. This follows from the following

Assertion 2.

(a) $\Box B\in y$,

(b) $\Box^c(B\Rightarrow\Box B)\in y$.

Indeed, from (a), (b) and axiom (Ind$\Box^c$) we obtain $\Box^cB\in y$. From xS_Ly, by fact 3.2.(ii) we have yS_Lx. Then $\Box^cB\in y$ and by yS_Lx we get $B\in x$ and $|x|\in N$.

The proof of assertion 2 is similar to the proof of the assertion 1.

Proof of (FS2). Suppose $|x|S|y|$. Then we have $|x|S''|y|$ and by (FS"2) we obtain $(\forall\Box^cA\in\Phi)(\Box^cA\in x \longrightarrow A\in y)$.■

Lemma 4.6.

Let $\underline{W}=(W, R, R^*, S'')$ be a frame in which S'' is a quasi-equivalence in W and Let $S=R^+_{S''}\cap(R^{-1})^+_{S''}\cap S''$. Then:

(i) xSy iff there exists an R- and S"-cycle C such that $x,y\in C$,

(ii) $S \subseteq R^{+}_{S}$,

(iii) S is a quasi-equivalence in W,

(iv) (W, R, R^{*}, S) is a general frame for MLCR.

Proof. (i) By the definition of S we have

xSy iff $xR^{+}_{S''}y$ and $yR^{+}_{S''}x$ and $xS''y$.

By lemma 1.2 we have:

$xR^{+}_{S''}y$ iff xRy or $\exists i\geq 2\exists x_1x_2\ldots x_i$: $x=x_1Rx_2R\ldots x_iRy$ and $\{x_2,\ldots,x_i\}\subseteq S''(x)$,

$yR^{+}_{S''}x$ iff yRx or $\exists j\geq 2\exists x_{i+1}x_{i+2}\ldots x_{i+j}$: $y=x_{i+1}Rx_{i+2}R\ldots x_{i+j}Rx$ and $\{x_{i+2},\ldots,x_{i+j}\}\subseteq S''(y)$.

Suppose $xS''y$. We have to consider four cases.

Case 1: xRy and yRx. Since we have $xS''y$ and $yS''x$, obviously the sequence $C=(x,y)$ is an R- and S"-cycle, containing x and y.

Case 2: $\exists i\geq 2\exists x_1x_2\ldots x_i: x=x_1Rx_2R\ldots x_iRy$ and $\{x_2,\ldots,x_i\}\subseteq S''(x)$ and yRx. Then $C=(x,x_2,\ldots,x_i,y)$ is an R-cycle containing x and y. From $xS''y$ and $\{x_2,\ldots,x_i\}\subseteq S''(x)$, by transitivity and symmetry of S", we obtain $xS''x_2S''\ldots x_iS''yS''x$, which shows that C is also an S"-cycle.

Case 3: xRy and $\exists j\geq 2\exists x_{i+1}\ldots x_{i+j}: y=x_{i+1}Rx_{i+2}R\ldots x_{i+j}Rx$. In this case $C=(y,x_{i+2},\ldots,x_{i+j},x)$. The proof that C is an R-and S"-cycle is the same as in the case 2.

Case 4: $\exists i\geq 2\exists x_1x_2\ldots x_i: x=x_1Rx_2R\ldots x_iRy$ and $\{x_2,\ldots,x_i\}\subseteq S''(x)$ and $\exists j\geq 2\exists x_{i+1}\ldots x_{i+j}: y=x_{i+1}Rx_{i+2}R\ldots x_{i+j}Rx$ and $\{x_{i+2},\ldots,x_{i+j}\}\subseteq S''(y)$. In this case $C=(x,x_2,\ldots,x_i,y,x_{i+2},\ldots,x_{i+j})$. Obviously C is an R-cycle, containing x and y. From $\{x_2,\ldots,x_i\}\subseteq S''(x)$, $\{x_{i+2},\ldots,x_{i+j}\}\subseteq S''(y)$ and $xS''y$ we obtain, by transitivity and symmetry of S", that C is an S"-cycle.

To proof (i)($\longrightarrow$) Suppose xSy. Then in the four cases, considered above, we obtain an R- and S"-cycle C, containing both x and y.

To proof (i)($\longleftarrow$) suppose $x,y\in C$ for some R- and S"-cycle C. Since S" is a quasi-equivalence relation, then, applying lemma 2.1. we obtain easily $xS''y$, $xR^{+}_{S''}y$ and $yR^{+}_{S''}x$, and hence xSy.

(ii) Suppose xSy and proceed to show that $xR^{+}_{S}y$.

Case 1: xRy. Then we have $xR^{1}_{S}y$ and hence $xR^{+}_{S}y$.

Case 2: $x\bar{R}y$. By (i) there exists an R- and S-cycle C such that $x,y\in C$. Since $x\bar{R}y$, $\exists i\geq 1\exists x_1\ldots x_i\in C$: $xRx_1R\ldots x_iRy$. By (i) for any $u,v\in C$ we have uSv so $\{x_1,\ldots,x_i\}\subseteq S(x)$. Consequently we obtain $xR^{+}_{S}y$.

(iii). The fact that S is a quasi-equivalence relation in W easily follows from (i).

(iv). From (ii), (iii) and proposition 1.3 we obtain that (W, R, R^{*}, S) is a general frame for MLCR.∎

Now the proof of theorem 4.1 follows from the lemmas 4.3, 4.4, 4.5 and 4.6.∎

Corollary 4.7.

MlCR has the finite model property with respect to its general semantics and is decidable.

Theorem 4.8. */Completeness Theorem for MLCR/*

The following conditions are equivalent for any formula A:

(i) A is a theorem of MLCR,

(ii) A is true in the class Σ_g *of all general frames,*

(iii) A is true in the class Σ_g^{fin} *of all finite general frames with cardinality* $\leq 2^{|A|}$*,*

(iv) A is true in the class Σ_s *of all standard frames,*

(v) A is true in the class Σ_s^{inf} all infinite standard frames.

Proof. The implications (i)$\longrightarrow$(ii)$\longrightarrow$(iii) are obvious and the implications (i)$\longrightarrow$(iv)$\longrightarrow$(v) follow from lemma 1.1. The equivalence (ii)$\longleftrightarrow$(v) is stated in the corollary 2.3.(i). We shall prove the implication (iii)$\longrightarrow$(i), which completes the proof of the theorem.

Suppose A is not a theorem of MLCR. By the Lindenbaum Lemma there exists a maximal consistent set x_0 such that $A \notin x_0$. By the Filtration Theorem MLCR admits a good filtration. Let Φ be the set of all subformulas of A and let $\underline{M}=((|W_L|, R, Q, S), v)$ be a good filtration of the canonical frame through Φ. Then by the Filtration Lemma we have $|x_0| \not\Vdash_v A$. By the fact 3.3.(ii) $Card|W_L| \leq 2^{|A|}$. So A is not true in a general frame with a cardinality $\leq 2^{|A|}$. Then, by contraposition, we have the implication (iii)$\longrightarrow$(i).∎

5. Some open problems

Problem 1. Extend the results of this chapter for PDL^c.

Problem 2. Let $MLCR^-$ be MLCR without $\Box^*$ and the corresponding axioms. Obviously the Standardization Theorem and the Filtration Theorem for $MLCR^-$ are true. The problem is whether $MLCR^-$ is complete in the class of its finite standard models or not. Let us note that the proof that MLCR has not f.m.p. with respect to its standard models does not hold for $MLCR^-$, because the formula (Fin) used in this proof contains $\Box^*$.

Problem 3. Extend the results of this chapter for an extension of MLCR with a propositional constant **repeat** with the following semantics: $x \Vdash_v$ **repeat** iff there exists an infinite R-repeating starting from x. An algebraic study of **repeat**, $\Box$ and $\Box^*$ is given by Goldblatt [Go 85].

PART II. MODAL LOGICS FOR INFORMATION SYSTEMS

This part is devoted to a study of several kind of abstract notions of information systems, information relations in them and modal logics for information systems. This research line have been initiated by Pawlak [Paw 81,82,83,91], Orlowska and Pawlak [O&P 84, 84a] and Orlowska [Or 84,85,85a,87,88,90,93,95] and continued by Arhangelskij [Arh 90], Gargov [Ga 86], Konikowska [Ko 93] and the present author [Vak 87,87a,89,91,91a,92,94,95,95a].

The content of this part is divided into 5 chapters.

Chapter 2.1 is devoted to the notion of information system. There is no a unique mathematical notion of information system, because in the practice several different schemes for representing information work. There are mainly two groups of information systems: systems of logical type in which the information is represented by a collection of sentences, equipped with some inference mechanism, and systems of ontological type, based on the ontological concepts of object, property, attribute and value of attribute. In the chapter we consider information systems of both types: Consequence and Bi-consequence systems - of logical type, and Property systems and Attribute systems - of ontological type.. The main result is a duality like connections between consequence systems and property systems and between bi-consequence systems and attribute systems, based on abstract characterization theorems, generalizing the Stone representation theorems for distributive lattices and Boolean algebras.

In chapter 2.2 we study some information relations between objects in Property systems and Attribute systems and their analogs in Consequence and Bi-consequence systems. The main results here are abstract characterization theorems for some information relations - several kinds of similarity relations, informational ordering and indiscernibility relations.

Chapter 2.3 is devoted to a study of several modal logics for similarity relations in Property systems and Attribute systems: the logic SIM-1 for Property systems, the logic SIM-2 for Attribute systems, and the logic SIM-3 for single-valued attribute systems - the exact definitions are in the text. The main results in the chapter are decidability and completeness theorems for the introduced modal logics. The completeness theorems are based on the abstract characterization theorems of the corresponding informational relations, proved in chapter 2.2. The decidability theorems are obtained by an extension of the filtration method, known from classical modal logic.

In chapter 2.4 we study some modal logics for information systems containing the indiscernibility relation: IND-1 as an extension of SIM-1, IND-2 as an extension of SIM-2 with strong indiscernibility relation, and IND-3, containing both strong and weak indiscernibility relations.

Chapter 2.5 is devoted to a modal logic with special propositional constant for single-valuedness, which makes possible an uniform study of general case and single-valued case (the definition of single-valued Attribute system is given in chapter 2.1).

CHAPTER 2.1

INFORMATION SYSTEMS

Overview. In this chapter we study two types of abstract information systems: ontological and logical. Systems of ontological type are Property systems and Attribute systems in which the information is represented in terms of the ontological concepts of an object, property and attribute. Systems of logical type are Consequence systems and Bi-consequence systems in which the information is represented by a collection of sentences, equipped with some inference mechanism. The main result in this chapter is a kind of duality between logical and ontological information systems

Introduction

In this chapter we introduce four abstract notions of information systems: Property systems, Attribute systems, Consequence systems and Bi-consequence systems .

The information represented in property systems /P-systems/ is in terms of objects and properties which they possess. In attribute systems /A-systems/ the information is in terms of objects, attributes and values of attributes. An example of an attribute is, for instance, "color" and the values of this attribute are concrete colors like "green", "red" etc. The information about some object with respect to some attribute is a subset of the values of this attribute, which this object possesses. The notion of an A-system was introduced for the first time by Pawlak /[Paw 81], [Paw 83]/ under the name of information system. Pawlak's information systems lay down in the so called "Rough sets approach", which now is one of the fundamentals of some directions in AI /[Paw 82], [Paw 91], [Slo 92]/. P-systems was introduced in [Vak 91a] as a simplification of Pawlak's information systems. As can be seen later on, P-systems may help for better understanding the theory of A-systems.

P-systems and A-systems are information systems of non-logical type and since they deal with some basic ontological concepts, such as "object", "property", "attribute" and "value of attribute", it is natural to classify them as "ontological information systems". Another kind of information systems are the systems of logical type. The information in an information system of logical type is represented by a set of sentences equipped with one or more consequence relations. In this chapter we study two information systems of such a kind: consequence systems /C- systems/ and bi-consequence systems /B-systems/.

C-systems are pairs (W, $\vdash$) where W is a nonempty set of sentences and $\vdash$ is a consequence relation between finite subsets of W satisfying the standard Scott's axioms, coming from proof theory. Similar systems were introduced for the first time by Scott [Sco 82]. We found in [Vak 92] a simple duality between P- systems and C-systems. In a sense C-systems are the logical counterpart of P-systems, although they have completely different nature.

B-systems are like C-systems and are equipped with two consequence relations - a strong one $\vdash$ and a weak one $\succ\!\!-$. These systems are introduced as logical counterparts of A-systems.

The chapter is organized as follows. In sec. 1 and 2 we introduce P-systems and A-systems respectively. In sec. 3 we introduce C-systems. The main theorem here is the Characterization theorem for C-systems /theorem 3.3./ which can be considered as a kind of representation theorem of C-systems in P-systems, generalizing the Stone representation theorem for distributive lattices [Sto 37] and Boolean algebras. In sec. 4 we introduce B-systems and prove Characterization theorem for B-systems /theorem 4.3/, which also can be considered as a kind of representation theorem of B-systems in A-systems. The characterization theorems are used in chapter 2.2 for obtaining abstract characterizations of some relations between objects in P-systems and A-systems.

The results of this chapter have been published in the papers [Vak 92, 94, 95, 95a].

1. Property systems

By a property system, P-system for short, we will mean any system of the form S=(Ob, Pr, f) where:

- Ob$\neq\emptyset$ is a set, whose elements are called objects,

- Pr is a set, whose elements are called properties and
- f is a total function, called information function, which assigns to each object $x\in Ob$ a subset $f(x)\subseteq Pr$, called the information of x in S. By $\overline{f}(x)$ we denote the set $Pr\setminus f(x)$.

Very often the components of a given system S will be denoted by Ob_S, Pr_S and f_S. If $Pr_S=\emptyset$ then S is called trivial P-system, otherwise S is called non-trivial P-system. S is called total P- system if for any $x\in Ob_S$ we have $f_S(x)\neq\emptyset$.

The notion of a P-system follows the simple ontological idea that some things are objects, some other things are properties and that objects possess properties. The notions of an object and a property are taken as primitives and the possession of a property is formalized by the function f: "x has /possesses/ the property A" can be expressed by "$A\in f(x)$".

In the natural language the sentence stating that certain object x possesses certain property A is expressed very often by the phrase "x is A", like "x is green". In this example the property is "to be green". There are many other ways, which express the situation of possession of a property. Examples:

- "x speaks English". This means "x has the property to speak English". The property here is "to speak English".
- "x weights 2 kilo". The property here is :"to weight 2 kilo".
- "x drives a car". The property is "to drive a car".
- "x drives a bicycle". The property is "to drive a bicycle".
- "x is born on 18.4.1938". The property here is "to be born on 18.4.1938", etc.

We do not assume in general that in a given P-system S Pr_S is the set of all possible properties of the objects contained in Ob_S. Rather Pr_S is a set of some properties meaningful for the objects from Ob_S , which are interesting from a certain point of view. In trivial systems we have only objects and no properties. Also we do not assume in general that for every $x\in Ob_S$ we have $f_S(x)\neq\emptyset$ /totality/. It is quite possible for some object $x\in Ob_S$ to have $f_S(x)=\emptyset$. This does not say that x has no properties at all, it says only that x does not have properties from the given set Pr_S.

The set-theoretical counterpart of properties are sets. Then $x\in A$ means that x has the property A. This motivates the following set-theoretical construction of P-systems.

Let (W, V) be a pair such that $W\neq\emptyset$ and $V\subseteq\mathbf{P}(W)$ be a set of subsets of W. Define a system S=P(W,V) as follows: $Ob_S=W$, $Pr_S=V$ and $f_S(x)=\{A\in V/x\in A\}$. Then obviously P(W,V) is a P-system, which we will call set-theoretical P-system over the pair (W,V).

Now we shall introduce the notion of a homomorphism between P- systems.

Let S and S' be two P-systems and let h be a function from $Ob_S\cup Pr_S$ into $Ob_{S'}\cup Pr_{S'}$. We say that h is a homomorphism from S into S' if it satisfies the following conditions:

- $x\in Ob_S$ iff $h(x)\in Ob_{S'}$,
- $A\in Pr_S$ iff $h(A)\in Pr_{S'}$,
- if $A\in f_S(x)$ then $h(A)\in f_{S'}(h(x))$.

The function h is called a strong homomorphism if it is a homomorphism, satisfying the condition:

- if $h(A) \in f_{S'}(h(x))$ then $A \in f_S(x)$.

S' is called a homomorphic image of S if there exists a homomorphism h which maps S onto S'.

Let S be a P-system. We shall define a P-system |S|, called set-theoretical P-system associated with S, which will be a strong homomorphic image of S. The construction is as follows: put $W=Ob_S$, for any $A \in Pr_S$ define $|A|=\{x \in Pr_S / A \in f_S(x)\}$, $V=\{|A| / A \in Pr_S\}$ and define |S| to be the set-theoretical P-system P(W,V) over the pair (W,V). The following lemma is true:

Lemma 1.1

(i) Let S be a P-system. Then |S| is a strong homomorphic image of S.

(ii) If S is a set-theoretical P-system then |S|=S.

Proof (i). Define h in the following way: for $x \in Ob_S$ $h(x)=x$, for $A \in Pr_S$ $h(A)=|A|$. Then by an easy calculation one can see that h is a strong homomorphism from S onto |S|.

(ii) If S is a set theoretical P-system then the construction of |S| gives the same system S and h is the identity function.■

The strong homomorphism h, defined in the above proof, is called the natural homomorphism from S onto |S|.

2. Attribute systems

Let S be a concrete P-system and in the set Pr_S we have the following concrete properties: E:"to speak English", F:"to speak french",G:"to speak German" and R:"to speak Russian". But {E, F, G, R} are all official languages. So the above four properties form a general property "to speak an official language". Such a general property will be called an attribute and its concrete cases will be called values of this attribute. Thus the set of values of the attribute a="to speak an official language" /shortly "official language"/ is the set $Val_S(a)=\{E, F, G, R\}$. Let $x \in Ob_S$ be a person and let $f_S(x,a)=f_S(x) \cap Val_S(a)$. Then $f_S(x,a)$ can be considered as the information of x with respect to the attribute a. Let for example $f_S(x,a)=\{E,R\}$. This is an information for x, which says that x speaks English and Russian but not French and German.

Other examples of attributes and their sets of values are the following:

- a="to have a color...", shortly "color". The set Val(a) is the set of all /or some/ concrete colors: red, green, etc.
- a="to weight ... kilo", shortly "weight". Val(a) may be the set /or some subsets/ of non-negative rational numbers.
- a="to drive a ...", Val(a)= {car, bicycle, motorcycle}
- a="to be born on ...", Val(a) = any interval of possible dates.
- a="age", $Val_a=\{1,2,\ldots, 120\}$ or Val(a)={childhood, boyhood, adult}.

These examples show that we can consider attributes as sets of similar properties, which are called values of the corresponding attribute.

Thus we are coming at the following formal definition.

By an attribute system, A-system for short, we mean any system of the form $S=(Ob, At, \{Val(a)/a \in At\}, f)$, where:

- $Ob \neq \varnothing$ is a set, whose elements are called objects,
- $At \neq \varnothing$ is a set, whose elements are called attributes,
- for each $a \in At$, Val(a) is a set, whose elements are called values of the

attribute a,

- f is a two-argument total function, called information function, which assigns to each object $x \in Ob$ and attribute $a \in At$ a subset $f(x,a) \subseteq Val(a)$, called the information of x according to a.

Sometimes the components of a given A-system S will be written with subscript S: Ob_S, At_S, $Val_S(a)$ and f_S. S is called a total A-system if for every $x \in Ob_S$ and $a \in At_S$ $f_S(x,a) \neq \emptyset$. S is called a single-valued A-system if for any $x \in Ob_S$ and $a \in At_S$ the set $f_S(x,a)$ has at most one element, i.e. $Card(f_S(x,a)) \leq 1$. If $f_S(x,a)=\{A\}$ then we will write simply $f_S(x,a)=A$. S is called a finite system if all components of S are finite sets.

The notion of single-valued and total finite A- system is introduced by Pawlak in [Paw 81] under the name of information system. These systems are very simple and can be represented as a table, whose rows are labeled by objects and whose columns are labeled by attributes.

The notion of A-system given above is a slight generalization of the notion of many-valued information system, given by Pawlak in [Paw 83], /Pawlak always assumes that S is total and finite/, /see also [O&P 84], [Vak 87]and [Vak 89], where this notion is used under the name of knowledge representation system, and single-valued systems are under the name of deterministic knowledge representation systems/.

Now we shall define some constructions of P-systems from A- systems.

Let S be an A-system. Then for each $a \in At_S$ we can define a P- system S(a) with the following construction. Put $Ob_{S(a)}=Ob_S$, $Pr_{S(a)}=Val_S(a)$, for $x \in Ob_{S(a)}$ define $f_{S(a)}(x)=f_S(x,a)$. The P-system S(a) is called the restriction of S to the attribute a.

Let S be an A-system. Define a P-system S'=Under(S) - the underlying P-system of S, putting $Ob_{S'}=Ob_S$, $Pr_{S'}=\bigcup\{Val(a)_S/a \in At_S\}$ and $f_{S'}(x)=\bigcup\{f_S(x,a)/a \in At_S\}$.

The following two constructions transform P-systems into A- systems.

Let S be a P-system. We can associate a single-valued A-system S' to S in the following way. Put $Ob_{S'}=Ob_S$, $At_{S'}=Pr_S$, for each $a \in At_{S'}$ define $Val(a)_{S'}=\{a\}$ and for $x \in Ob_{S'}$ and $a \in At_{S'}$ define $f_{S'}(x,a)=f_S(x) \cap \{a\}$. Obviously S' is a single-valued A-system This construction shows that each property A can be considered as an attribute whose set of values is the one-element set {A}.

We can associated with S another A-system S" having only one attribute, with the following construction. Put $Ob_{S''}=Ob_S$, $At_{S''}=\{p\}=\{property\}=Pr_S$, define $Val(p)_{S''}=Pr_S$ and for each $x \in Ob_{S''}$ put $f_{S''}(x,p)=f_S(x)$. Obviously S" is not in general a single valued A-system. This construction shows that each set of properties Pr can be considered as one attribute, namely the attribute p: "to have a property", with the set of values Val(p)=Pr.

Now we shall give a set-theoretical construction of A-systems, following the intuition of set-theoretical P-systems. Since attributes can be considered as sets of properties and the set- theoretical analog of a property is a set of objects, then the set-theoretical analog of an attribute is a set of sets of objects. This leads to the following construction.

Let (W, V) be a pair with $W \neq \emptyset$ and $V \subseteq \mathbf{P}(\mathbf{P}(W))$, i.e. the elements of V are sets of subsets of W. Define an A-system S=A(W, V) as follows: Put $Ob_S=W$, $At_S=V$, for each $a \in At_S$ define $Val_S(a)=a$, and for each $x \in Ob_S$ and $a \in At_S$ put $f_S(x,a)=\{A \in a/x \in A\}$. The system S=A(W,V), will be called the set-theoretical

A-system over the pair (W,V).

Let S and S' be two A-systems. A function h from the union of all components of S to the union of all components of S' will be called a homomorphism from S into S' if it satisfies the following conditions for any $x \in Ob_S$, $a \in At_S$, $A \in Val_S(a)$:

- $x \in Ob_S$ iff $h(x) \in Ob_{S'}$,
- $a \in At_S$ iff $h(a) \in At_{S'}$,
- $A \in Val_S(a)$ iff $h(A) \in Val_{S'}(h(a))$,
- if $A \in f_S(x,a)$ then $h(A) \in f_{S'}(h(x),h(a))$.

h is called a strong homomorphism if in addition it satisfies the condition

- if $h(A) \in f_{S'}(h(x),h(A))$ then $A \in f_S(x,a)$.

S' is a homomorphic image of S if there is a homomorphism h which maps S onto S'.

Let S be an A-system. We shall associate with S a set- theoretical A-system denoted by |S|, which will be a strong homomorphic image of S. The construction is as follows: put $W=Ob_S$, for $a \in At_S$ and $A \in Val_S(a)$ define $g_a(A)=\{x \in W / A \in f_S(x,a)\}$ and $|a|=\{g_a(A)/A \in Val_S(a)\}$, put $V=\{|a|/a \in At_S\}$. Then we let |S| to be the set-theoretical A-system S(W,V) over the pair (W,V). The following lemma is true:

Lemma 2.1.

Let S be an A-system. Then:

(i) |S| is a strong homomorphic image of S.

(ii) If S is a set-theoretical A-system then |S|=S.

(iii) If S is a set-theoretical A-system over some pair (W, V) then S is a single-valued iff the following condition is satisfied:

$(\forall a \in V)(\forall A,B \in a)(A \cap B \neq \emptyset \longrightarrow A=B)$.

Proof. (i). Let h be defined as follows: for $x \in Ob_S$ $h(x)=x$, for $a \in At_S$ $h(a)=|a|$ and for $A \in Val_S(a)$ let $h(A)=g_a(A)$. Then h is the required strong homomorphism from S onto |S|.

The proof of (ii) and (iii) is straightforward. ■

The function h defined in the proof of lemma 2.1 will be called the natural strong homomorphism from S onto |S|.

3. Consequence systems

In this section we will introduce a kind of logical information systems, called consequence systems, which in a sense can be considered as a logical counterpart of property systems. Intuitively, each consequence system contains a set of sentences and certain consequence relation. There are various kinds of consequence relations: intuitionistic, classical, non-monotonic and so on. We will take an abstract version of the so called Scott's consequence relation / [Gab 81], [Seg 82], [Vak 92]/ which is a binary relation $\vdash$ between finite sets of sentences, satisfying some axioms, coming from classical logic. The formal definition is as follows:

By a consequence information system, C-system for short, we will mean any system of the form S=(Sen, $\vdash$), where

- $Sen \neq \emptyset$ is a set, whose elements are called sentences,
- $\vdash$ is a binary relation in the set $\mathbf{P}_{fin}(Sen)$ of all finite subsets of Sen, called Scott's consequence relation, satisfying the following axioms: for

any $A, B, A', B' \in \mathbf{P}_{fin}(Sen)$ and $x \in Sen$:

- (Refl) If $A \cap B \neq \emptyset$ then $A \vdash B$,
- (Mono) If $A \vdash B$, $A \subseteq A'$ and $B \subseteq B'$ then $A' \vdash B'$,
- (Cut) If $A \vdash B \cup \{x\}$ and $\{x\} \cup A \vdash B$ then $A \vdash B$.

S is called non-trivial C-system if $\emptyset \nvdash_S \emptyset$, otherwise S is called trivial.

If S is a C-system then the components of S sometimes will be denoted by Sen_S and $\vdash_S$.

Following the usual practice, instead of $\{a_1, \ldots, a_n\} \vdash \{b_1, \ldots, b_m\}$ and $A \cup \{x_1, \ldots, x_m\} \vdash B \cup \{y_1, \ldots y_n\}$ we will write $a_1, \ldots, a_n \vdash b_1, \ldots, b_m$ and $A, x_1, \ldots, x_m \vdash B, y_1, \ldots y_n$. We assume that for n=0 $\{x_1, \ldots, x_n\} = \emptyset$.

For a set $A \in \mathbf{P}_{fin}(Sen_S)$ and $x, y \in Sen_S$ we say that:

- x implies y in S iff $x \vdash_S y$,
- A is inconsistent /contradictory/ in S iff $A \vdash_S \emptyset$,
- A is consistent in S iff $A \nvdash_S \emptyset$,
- A is complete /tautological/ in S iff $\emptyset \vdash A$
- A is incomplete in S iff $\emptyset \nvdash_S A$.

The axioms (Refl), (Mono) and (Cut) are known under the names of Reflexivity, Monotonicity and Cut.

The following lemma states some easy consequences from the axioms of a C-system.

Lemma 3.1.

Let S be a C-system. Then for any $x, y, z \in Sen_S$, and $A, B, C, D, A', B \in \mathbf{P}_{fin}(Sen_S)$ we have:

- (Ref) $x \vdash_S x$,
- (Trans) If $x \vdash_S y$, $y \vdash_S z$ then $x \vdash_S z$
- (Cut0) If $A \vdash_S B, x$ and $x, A' \vdash_S B'$ then $A, A' \vdash_S B, B'$,
- (Cut1) If $A \vdash_S B$ and $A, x \vdash_S C$ for every $x \in B$ then $A \vdash_S C$.
- (Cut2) If $A \vdash_S B$ and $A, x \vdash_S C$ for every $x \in B$ then $A \vdash_S C$.
- (Cut3) If $A \vdash_S B$ and for every $x \in A$ and $y \in B$ we have $C \vdash_S x, D$ and $C, y \vdash_S D$ then $C \vdash_S D$.

Proof. As an example we shall prove (Cut1). Suppose $A \vdash_S B$ and $A, x \vdash_S C$ for every $x \in B$. Let $B = \{b_1, \ldots, b_n\}$. Then we have $A \vdash_S b_1, \ldots, b_n$. We shall prove by backward induction that for any $0 \leq i \leq n$ $A \vdash_S b_1, \ldots, b_i, C$. Then taking i=0 we get $A \vdash_S C$.

Basic step: i=n. From $A \vdash_S b_1, \ldots, b_n$ by (Mono) we obtain $A \vdash_S b_1, \ldots, b_n, C$.

Induction hypothesis: Suppose that the assertion is true for i=k, $1 \leq k \leq n$. We shall show that the assertion is true for i=k-1.

By assumption we have $A, b_k \vdash_S C$. Then by (Mono) we have $A, b_1, \ldots, b_{k-1}, b_k \vdash_S C$. By i.h. we have $A \vdash_S b_1, \ldots, b_{k-1}, b_k, C$. Then by (Cut) we obtain $A \vdash_S b_1, \ldots, b_{k-1}, C$. ■

Typical example of a C-system is a logical theory L, based on the classical logic. In such an example the elements of Sen are real sentences and the relation $a_1,\dots,a_n \vdash b_1,\dots,b_m$ holds if the implication $(a_1\wedge\dots\wedge a_n)\Rightarrow(b_1\vee\dots\vee b_m)$ is true in L.

Another example, which is connected with the previous one, can be constructed as follows. Let $(D, \leq, \wedge, \vee)$ be a distributive lattice /for the relevant definitions see [R&S 63]/ and for finite subsets $A=\{a_1,\dots,a_m\}$, $B=\{b_1,\dots,b_n\}\subseteq D$ define

$A\vdash B$ iff $a_1\wedge\dots\wedge a_m\leq b_1\vee\dots\vee b_n$.

Then $(D, \vdash)$ is a C-system.

The next example, which is connected with P-systems, will play a fundamental role in this paper.

Let S be a P-system. We shall construct a C-system C(S), called the C-system over S, in the following way. Define $Sen_{C(S)}=Ob_S$ and for the finite subsets $A=\{a_1,\dots,a_m\}$, $B=\{b_1,\dots,b_n\}\subseteq Sen_{C(S)}$ put

$A \vdash_{C(S)} B$ iff $f_S(a_1)\cap\dots\cap f_S(a_m)\subseteq f_S(b_1)\cup\dots\cup f_S(b_n)$

We assume here that for m=0 and n=0 we have $A=\emptyset$, $B=\emptyset$, $f_S(a_1)\cap\dots\cap f_S(a_m)=Pr_S$ and $f(b_1)\cup\dots\cup f_S(b_n)=\emptyset$.

Lemma 3.2.

Let S be a P-system. Then:

(i) The system C(S) is a C-system.

(ii) Let S=P(W,V) be a set-theoretical P-system over the pair (W,V) and $A,B\subseteq\mathbf{P}_{fin}(W)$. Then

$A \vdash_{C(S)} B$ iff $(\forall C\in V)(A\subseteq C\longrightarrow B\cap C\neq\emptyset)$.

Let |S| be the set-theoretical P-system associated with S. Then:

(iii) for any $A,B\subseteq\mathbf{P}_{fin}(Ob_S)$ we have $A \vdash_S B$ iff $A \vdash_{|S|} B$,

(iv) C(|S|)=C(S).

Proof - a long but easy routine check, which is left to the reader.∎

The above example shows that the objects of a P-system, which may be of an arbitrary nature, may be considered as sentences of certain C-system. In an abstract sense this means that P-systems may be considered as a semantic base for C-systems. The next theorem shows that each C-system can be represented as a C-system over some P-system.

Theorem 3.3./Characterization theorem for C-systems/

Let S be a C-system. Then there is a P-system P(S) such that S=C(P(S)).

Proof. The system P(S), we are going to construct, will be of set-theoretical type over some pair (W,V). Since we want to have S=C(P(S)) then W should be equal to Sen_S. So it remains to show how to define the set V as a subset of **P**(W). We shall do this by defining a suitable property of the elements of the set **P**(W), which will be taken as characteristic property for the elements of V. Suppose for a moment that V is defined. Then by lemma 3.2.(ii) we will have the following: for any $A,B\in\mathbf{P}_{fin}(W)$:

$A\vdash_S B$ iff $(\forall x\in V)(A\subseteq x\longrightarrow B\cap x\neq\emptyset)$.

Suppose now that x is an arbitrary element of V. Then we have: for any $A,B\in\mathbf{P}_{fin}(W)$:

(*) If $A \vdash_S B$ and $A\subseteq x$ then $B\cap x\neq\emptyset$.

We take (*) to be the needed characteristic property for the elements of V.

This leads to the following formal definition.

Let S be a C-system. A subset $x \subseteq Sen_S$ is called a prime filter in S, if for any two finite subsets A, B of Sen_S the condition (*) is true. The set of all prime filters of S will be denoted by PrFil(S).

The name "prime filter" comes from the example with distributive lattices, namely, all prime filters in distributive lattices are prime filters according to the definition given above.

Now we define P(S) as follows. Put $W=Sen_S$, V=PrFil(S) and let P(S) be the set-theoretical P-system P(W,V) over the pair (W,V).

We need the following lemma which is a generalization of the separation lemma for distributive lattices /see [R&S 63]/.

Lemma 3.4./Separation Lemma for C-systems/

Let S be a C-system, $A,B \in \mathbf{P}_{fin}(Sen_S)$ and $A \nvdash_S B$. Then there exists a prime filter x in S such that $A \subseteq x$ and $x \cap B = \emptyset$.

Proof. The proof is by an application of the Zorn's Lemma. Let M be the following set:

$M=\{y \subseteq Sen_S / A \subseteq y$ and for any finite $C \subseteq y$: $C \nvdash_S B\}$.

We consider M as a partially-ordered set by the set-inclusion $\subseteq$. As to apply the Zorn's Lemma we have to show the following:

(i) M is not empty,

(ii) Every non-empty chain N in M has an upper bound in M.

To prove (i) we shall show that $A \in M$. We have $A \subseteq A$. Let $C \subseteq A$. Then from $A \nvdash_S B$ and (Mono) we obtain contrapositively that $C \nvdash_S B$.

To prove (ii) suppose that N is a non-empty chain in M. This means that for any two members u, $v \in N$: either $u \subseteq v$ or $v \subseteq u$. This property of the chain can be generalized as follows: for any finite number $u_1, u_2 \ldots u_n$, $n \geq 1$, of elements of N there exists a permutation $i_1 i_2 \ldots i_n$ of the indices $1,2,\ldots,n$ such that $u_{i_1} \subseteq u_{i_2} \subseteq \ldots \subseteq u_{i_n}$. Let $y=\bigcup\{u/u \in N\}$, y is an upper bound of N. We shall show that $y \in M$. Obviously $A \subseteq y$. Let $C=\{c_1, c_2 \ldots, c_n\}$ be a finite subset of y. We shall show that $C \nvdash_S B$.

Case 1: $C=\emptyset$. Then $C \subseteq A$ and by (Mono) and $A \nvdash_S B$ we get $C \nvdash_S B$.

Case 2: $C \neq \emptyset$. Then $n \neq 0$ and there exists $u_1, u_2, \ldots, u_n \in N$ such that $c_1 \in u_1$, $c_2 \in u_2, \ldots, c_n \in u_n$. Now we can find a permutation $i_1, i_2, \ldots, i_n$ such that $u_{i_1} \subseteq u_{i_2} \subseteq \ldots \subseteq u_{i_n} = v$. Obviously $C \subseteq v \in M$, which implies $C \nvdash_S B$. This shows that $y \in M$, which proves the assertion (ii).

Now we can apply the Zorn's Lemma which states that if in a nonempty partially ordered set M, every nonempty chain in M has an upper bound in M, then M has a maximal element, say in our case x. We shall show that x is a prime filter in S.

Suppose, in order to come into a contradiction, that x is not a prime filter in S. Then there exist finite sets P and $Q=\{q_1, \ldots, q_n\}$ such that $P \vdash_S Q$, $P \subseteq x$ and $x \cap Q = \emptyset$.

Case 1: $Q=\emptyset$. Then $Q \subseteq B$ and from $P \vdash_S Q$ we get $P \vdash_S B$. From $P \subseteq x$ and $x \in M$ we get $P \nvdash_S B$ - a contradiction.

Case 2: $Q \neq \emptyset$. Then $n \geq 1$ and for any $i=1,..n$ $q_i \notin x$. Let $x_i = x \cup \{q_i\}$. Then by the maximality of x we have that for any $i=1,..,n$ $x_i \notin M$ and hence there exist finite sets $C_i \subseteq x_i$ such that $C_i \vdash_S B$, $i=1,..,n$.

Case 2.1: $q_i \notin C_i$ for some i. Then we have $C_i \subseteq x$ and thus $C_i \nvdash_S B$ - a contradiction.

Case 2.2: for any $i=1,..,n$ $q_i \in C_i$. Let $C_i \cap x = D_i$. Then $C_i = D_i \cup \{q_i\}$ and $D_i \subseteq x$. From $C_i \vdash_S B$ we get $D_i, q_i \vdash_S B$ for $i=1,...,n$. Let $D = P \cup D_1 \cup ... \cup D_n$. From $P \subseteq x$ and $D_i \subseteq x$ we get $D \subseteq x$ and since $x \in M$ we obtain $D \nvdash_S B$. From $P \vdash_S Q$, $D_i, q_i \vdash_S B$, $P \subseteq D$ and $D_i \subseteq D$ we obtain by (Mono) $D \vdash_S Q$, $D, q_i \vdash_S B$ for any $q_i \in Q$. Then applying (Cut1) from lemma 3.1 we obtain $D \vdash_S B$ - a contradiction.

So we have proved that x is a prime filter in S. Since $x \in M$ we have $A \subseteq x$. It remains to show that $x \cap B = \emptyset$. Suppose, for the sake of contradiction that $x \cap B \neq \emptyset$. Then for some $a \in Sen_S$ $a \in x$ and $a \in B$ and hence $\{a\} \subseteq x$. Consequently $\{a\} \nvdash_S B$, which implies $\{a\} \cap B = \emptyset$. From here get $a \notin B$ - a contradiction.

This completes the proof of the lemma.■

Corollary 3.5.

Let S be a C-system. then for every $A, B \in \mathbf{P}_{fin}(Sen_S)$ we have: $A \vdash_S B$ iff $(\forall x \in PrFil(S))(A \subseteq x \longrightarrow B \cap x \neq \emptyset)$.

Proof. The direction $(\longrightarrow)$ follows from the definition of a prime filter and the direction $(\longleftarrow)$ follows from the separation lemma.■

Now the proof of the theorem 3.3 follows directly from corollary 3.5 and and lemma 3.2.(ii).■

The P-system P(S), constructed in the proof of theorem 3.5. will be called the canonical P-system over S.

Theorem 3.3. may be considered as an abstract sort of "completeness theorem" for C-systems with respect to their "semantics" in the class of P-systems.

Theorem 3.3 shows a very closed interconnection between C- systems and P-systems. We have two constructions - P and C. P performing a C-system S into a P-system P(S), and C performing a P-system S into a C-system C(S). Theorem 3.3 shows that starting from a C-system S and applying alternatively P and C we get a cycle $S \longrightarrow P(S) \longrightarrow C(P(S))=S$. If we start from a P-system S, then the situation is not the same. Namely we have the following

Theorem 3.6.

Let S be a P-system and consider the P-system P(C(S)). Then $Ob_S = Ob_{P(C(S))}$ and there exists a strong homomorphism h from S into P(C(S)), which is the identity in Ob_S.

Proof. The equality $Ob_S = Ob_{P(C(S)}$ is true by definition. Let h be the natural strong homomorphism from S onto |S| /lemma 1.1.(ii)/. In Ob_S h is identity by definition. We shall show that h is a strong homomorphism from S into P(C(S)).

By lemma 3.2.(iv) C(S)=C(|S|), so P(C(S))=P(C(|S|)). We shall show that the identity is a strong homomorphism from |S| into /not onto/ P(C(|S|)), which will prove the theorem. This will follow from the following

Lemma 3.7.

Let S=P(W,V) be a set-theoretical P-system over the pair (W,V) and let C(S) be the C-system over S. Then:

(i) All elements of V are prime filters in C(S).

(ii) $Pr_S \subseteq Pr_{P(C(S))}$.

(iii) The identity function is a strong homomorphism from S into P(C(S)).

Proof. (i) follows directly from the definition of a prime filter and lemma 3.2.(ii), (ii) is the same as (i) and (iii) follows from (ii) and the equality $Ob_S = Ob_{P(C(S))}$. ■

4. Bi-consequence systems

We have seen that C-systems are good logical counterparts of P- systems. Now we will define a new kind of information systems, which may be considered as a logical counterpart of A-systems. There are several possibilities to do this. Since A-systems may be considered as P-systems with a family of property sets, then, following this analogy we may consider C-systems with a family of Scott's consequence relations. However, for the purposes of this paper we will prefer a simpler notion, having two consequence relations, called respectively strong and weak. That is why we will call such systems bi-consequence information systems. First we shall give the formal definition and then we will discuss the intuition connected with this kind of information system.

By a bi-consequence information system, B-system for short, we will mean any system S of the following form S=(Sen, $\vdash$, $\succ$), where:

- (Sen, $\vdash$) is a C-system and $\vdash$ is called here strong consequence relation
- $\succ$ is a binary relation in the set $\mathbf{P}_{fin}$(Sen), called weak consequence relation and satisfying the following axioms for any $A, B, A' B' \in \mathbf{P}_{fin}$(Sen):

(Refl $\succ$) If $A \cap B \neq \emptyset$ then $A \succ B$,
(Mono $\succ$) If $A \succ B$, $A \subseteq A'$ and $B \subseteq B'$ then $A' \succ B'$,
(Cut $\succ$1) If $A \vdash x, B$ and $A, x \succ B$ then $A \succ B$,
(Cut $\succ$2) If $A, x \vdash B$ and $A \succ x, B$ then $A \succ B$,
(Incl) If $A \vdash B$ then $A \succ B$.

Let us note that axiom (Incl) is equivalent on the base of the remaining axioms to the following more simple axiom

(Incl 0) If $\emptyset \vdash \emptyset$ then $\emptyset \succ \emptyset$.

The following example of a B-system will give the main intuition of this notion . Let S_i=(Sen, $\vdash_i$) i∈I be a non-empty class of C-systems with one and the same set of sentences Sen. Define the relations $\vdash$ and $\succ$ in $\mathbf{P}_{fin}$(Sen) as follows:

$A \vdash B$ iff $\forall i \in I\ A \vdash_i B$, $A \succ B$ iff $\exists i \in I\ A \succ_i B$.

Then it is easy to see that the system (Sen, $\vdash$, $\succ$) is a B- system. It is now clear why $\vdash$ and $\succ$ are called respectively strong and weak consequence relations.

Lemma 4.1.

Let S be a B-system. Then the following Cut condition is true in S for any $X, Y, A, B \in \mathbf{P}_{fin}(Sen_S)$

(Cut$\succ$) If $X \succ_S Y$ and for everyy∈Y we have $A \vdash_S x, B$ and $A, y \vdash_S B$ then $A \succ_S B$.

Proof. Let $X=\{x_1, \ldots, x_m\}$ and $Y=\{y_1, \ldots, y_n\}$. Then prove by induction that for any i and j, 0≤i≤m, 0≤j≤m the following is true: $x_1, \ldots, x_i, A \succ_S y, \ldots, y_j, B$. Then the assertion is obtained from i=j=0. ■

The next example of a B-system is connected with A-systems and will be of a great importance.

Let S be an A-system. We shall construct a system B(S), called the B-system over S, in the following way. Put $Sen_{B(S)}=Ob_S$ and for any finite sets $A=\{x_1,\ldots,x_m\}$ and $B=\{y_1,\ldots,y_n\}$ of Ob_S define

$A \vdash_{B(S)} B$ iff $(\forall a\in At_S)\ f_S(x_1,a)\cap\ldots\cap f_S(x_m,a)\subseteq f_S(y_1,a)\cup\ldots\cup f_S(y_n,a)$,

$A \succ_{B(S)} B$ iff $(\exists a\in At_S)\ f_S(x_1,a)\cap\ldots\cap f_S(x_m,a)\subseteq f_S(y_1,a)\cup\ldots\cup f_S(y_n,a)$.

Lemma 4.2.

(i) Let S be an A-system. Then the system B(S) defined as above is a B-system.

(ii) Let S=A(W,V) be set-theoretical A-system over the pair (W,V) and $A,B\subseteq \mathbf{P}_{fin}(W)$. Then:

$A \vdash_{B(S)} B$ iff $(\forall a\in V)(\forall C\in a)(A\subseteq C \longrightarrow B\cap C\neq\emptyset)$.

$A \succ_{B(S)} B$ iff $(\exists a\in V)(\forall C\in a)(A\subseteq C \longrightarrow B\cap C\neq\emptyset)$.

Let |S| be the set-theoretical P-system associated with S. Then:

(iii) for any $A,B\subseteq \mathbf{P}_{fin}(Ob_S)$ we have

$A \vdash_S B$ iff $A \vdash_{|S|} B$, $A \succ_S B$ iff $A \succ_{|S|} B$,

(iv) B(|S|)=B(S).

Proof - straightforward, following the relevant definitions.■

Lemma 4.2. shows that in some abstract sense A-systems constitute a correct "semantics" for the B-systems. The next theorem shows that each B-system can be represented as a B-system over some A-system.

Theorem 4.3. /Characterization theorem for B-systems/

Let S be a B-system. Then there exists an A-system S'=A(S) such that S=B(A(S)).

Proof. The system S', which we are going to define will be a set theoretical A-system A(W,V) over a pair (W, V). Since we want that S=B(A(W,V)) to hold then we have to put $W=Sen_S$. It remains to show how to define the set V /recall that $V\subseteq \mathbf{P}(\mathbf{P}(W))$/. We will do this by finding a characteristic property of the elements of V, which will separate it from the set $\mathbf{P}(\mathbf{P}(W))$. Suppose for the moment that the set V is defined. Then by the equality S=B(A(W,V)) and lemma 4.2.(ii) we will have

$A \vdash_S B$ iff $(\forall a\in V)(\forall C\in a)(A\subseteq C \longrightarrow B\cap C\neq\emptyset)$,

$A \succ_S B$ iff $(\exists a\in V)(\forall C\in a)(A\subseteq C \longrightarrow B\cap C\neq\emptyset)$.

If a is an arbitrary element of V, then it will satisfy the following two conditions for any two sets $A,B\in \mathbf{P}_{fin}(W)$:

(*) $A \vdash_S B \longrightarrow (\forall C\in a)(A\subseteq C \longrightarrow B\cap C\neq\emptyset)$,

(**) $A \not\succ_S B \longrightarrow (\exists C\in a)(A\subseteq C$ and $B\cap C=\emptyset)$.

The required characteristic property for a will be to satisfy (*) and (**).

So we define: a set of subsets of Sen_S is called a good set in S if it satisfies (*) and (**). Note that condition (*) says that the elements of a good set are prime filters in $(Sen_S, \vdash_S)$.

Now we can start the proof of the theorem.

Put $W=Sen_S$ and V be the set of all good sets in S. Since we have $V\subseteq \mathbf{P}(\mathbf{P}(W))$ put S' to be the set-theoretical A-system A(W,V) over the pair (W,V). and denote S' by A(S). A(S) will be called the canonical A-system over S.

To prove the theorem we need some lemmas.

Lemma 4.4.

Let $A,B,X,Y\in \mathbf{P}_{fin}(Sen_S)$. Then:

(i) If $X\nvdash_S Y$, then there exists $C\in PrFil(S)$, denoted by $C(X\nvdash_S Y)$, such that $X\subseteq C$ and $Y\cap C=\emptyset$.

(ii) If $A\not\succ_S B$ then there exists $C\in PrFil(S)$, denoted by $C(A\not\succ_S B)$, such that $A\subseteq C$ and $B\cap C=\emptyset$.

(iii) If $X\succ_S Y$ and $A\not\succ_S B$ then there exists $C\in PrFil(S)$ denoted by $C(X\succ_S Y, A\not\succ_S B)$ such that:

1) ether $X\not\subseteq C$ or $Y\cap C\neq\emptyset$,
2) $A\subseteq C$ and $B\cap C=\emptyset$.

Proof. (i) is exactly the Separation lemma for C-systems /lemma 3.4./.

(ii) Let $A\not\succ_S B$. Then by (Incl) $A\nvdash_S B$ and by (i) there exists a $C\in PrFil(S)$ such that $A\subseteq C$ and $B\cap C$. Put $C(A\not\succ_S B)=C$.

(iii) Suppose $X\succ_S Y$ and $A\not\succ_S B$. Then by lemma 4.1 either $\exists x\in X$ such that $A\nvdash_S x,B$ or $\exists y\in Y$ such that $A,y\nvdash_S B$.

Case 1: $\exists x\in X$ $A\nvdash_S x,B$. Then by (i) there exists $C_1\in PrFil(S)$ such that $A\subseteq X$ and $(\{x\}\cup B)\cap C_1=\emptyset$, hence $x\notin C$, $B\cap C_1=\emptyset$ and $X\not\subseteq C_1$. This yields the conditions 1) and 2) of the assertion. Put in this case $C(X\succ_S Y, A\not\succ_S B)=C_1$.

Case 2: $\exists y\in Y$ $A,y\nvdash_S B$. Then by (i) there exists $C_2\in PrFil(S)$ such that $A\cup\{y\}\subseteq C_2$ and $B\cap C_2=\emptyset$. Then $A\subseteq C_2$, $y\in C_2$ and consequently $Y\cap C_2\neq\emptyset$. This yields the conditions 1) and 2) of the assertion. In this case put $C(X\succ_S Y, A\not\succ_S B)=C_2$. ∎

Lemma 4.5.

For any $X,Y\in \mathbf{P}_{fin}(W)$ the following holds:

(i) $X\vdash_S Y$ iff $(\forall a\in V)(\forall C\in a)(X\subseteq C \longrightarrow Y\cap C\neq\emptyset)$,

(ii) $X\succ_S Y$ iff $(\exists a\in V)(\forall C\in a)(X\subseteq C \longrightarrow Y\cap C\neq\emptyset)$.

Proof. (i) $(\longrightarrow)$ Suppose $X \vdash_S Y$, $a\in V$. Then a satisfies (*) and consequently $(\forall C\in a)(\forall C\in a)(X\subseteq C \longrightarrow Y\cap C\neq\emptyset)$.

$(\longleftarrow)$ Suppose $X\nvdash_S Y$. We shall define a good set $a\in V$ such that $(\exists C\in a)(X\subseteq C$ and $Y\cap C=\emptyset)$. Put $a=\{C(X\nvdash_S Y)\}\cup\{C(A\not\succ_S B)/A,B\in \mathbf{P}_{fin}(W)$ and $A\not\succ_S B\}$ and let $C=C(X\nvdash_S Y)$. We have $C\in a$ and by lemma 4.4.(i) that $X\subseteq C$ and $Y\cap C=\emptyset$. It remains to show that a is a good set. By lemma 4.4. all elements of a are prime filters of S so (*) is fulfilled. To prove (**) suppose $A\not\succ_S B$. Then by the construction of a $C=C(A\not\succ_S B)\in a$ and by lemma 4.4.(ii) $A\subseteq C$ and $B\cap C\neq\emptyset$. So (**) is full filled and hence $a\in V$.

(ii) $(\longrightarrow)$ Suppose $X\succ_S Y$. We shall construct $a\in V$ such that $(\forall C\in a)(X\subseteq C \longrightarrow Y\cap C\neq\emptyset)$.

Put $a=\{C(X\succ_S Y, A\not\succ_S B)/ A,B\in \mathbf{P}_{fin}(W)$ and $A\not\succ_S B\}$.

Lemma 4.4.(iii) guarantees that $a\in V$ and that it satisfies the assertion. This completes the proof of the lemma. ∎

Now the proof of theorem 4.3. follows immediately from lemma 4.5. and lemma 4.2. ∎

In some abstract sense theorem 4.3 may be considered as "completeness" theorem for B-systems with respect to their "semantics" in the class of A-systems.

Theorem 4.3. has also the following meaning. We have defined two operations - A and B, such that for any B-system S A(S) is an A-system, and that for any A-system S B(S) is a B-system. Then theorem 4.3. says that, starting from a given B-system S we may produce first A(S), second B(A(S))=S coming again to the system S. The picture is not the same if we start with some A-system S and then produce B(S) and A(B(S)). The connection between S and A(B(S)) is studied in the next theorem.

Theorem 4.6.

Let S be an A-system and consider the A-system A(B(S)). Then $Ob_S=Ob_{A(B(S))}$ and there exists a strong homomorphism h from S into A(B(S)), which is the identity function in Ob_S.

Proof. The equality $Ob_S=Ob_{A(B(S))}$ is true by definition. Let h be the natural strong homomorphism from S onto |S| /lemma 2.1./.
In Ob_S h is identity by definition. We shall show that h is a strong homomorphism from S into A(B(S)).

By lemma 4.2.(iv) C(S)=C(|S|), so A(B(S))=A(B(|S|)). We shall show that the identity is a strong homomorphism from |S| into /not onto/ A(B(|S|)), which will prove the theorem. This will follow from the following

Lemma 4.7.

Let S=A(W,V) be a set-theoretical A-system over the pair (W,V) and let B(S) be the B-system over S. Then:

(i) Let a be any element of V. Then all elements of a are prime filters in B(S).

(ii) All elements of V are good sets in S.

(iii) $At_S \subseteq At_{A(B(S))}$.

(iv) The identity function is a strong homomorphism from S into A(B(S)).

The proof is similar to the proof of lemma 3.7.

CHAPTER 2.2

SIMILARITY RELATIONS IN INFORMATION SYSTEMS

Overview. In this chapter we give a formal definition of some kinds of similarity relations between objects in information systems. The main results are first-order characterization theorems for similarity relations in P-systems and in A-systems.

Introduction

The main aim of this chapter is to give a formal analysis of sentences of the form "x is similar to y". Information systems are good base to give a precise meaning of various kinds of similarities. In P-systems we formalize the so called positive and negative similarities. Roughly speaking x and y are positively similar if they both possess a common property; x and y are negatively similar if there is a property, which is possessed neither by x nor by y. In A-systems, following Orlowska and Pawlak [O&P 84], [O&P 84a] and Orlowska [Or 84], [Or 85], [Or 85a], we can give a more detailed classification introducing weak and strong versions of similarity relations.

The structure of the chapter is the following.

In sec. 1, following the paper [Vak 91a] we introduce positive similarity, negative similarity and informational inclusion as binary relations between objects in P-systems and their counterparts in C-systems. The main result here is an abstract characterization of these relations by means of a finite set of first-order sentences /theorem 1.7/. This characterization is obtained by a consequence of the duality between P-systems and C- systems, studied in Chapter 2.1.

Similarity relations in P-systems are special cases of the notion of informational relations in P-systems. Informational relations in P-systems are studied in sec. 2. A first-order characterization of all informational relations, as well as a characterization of the two-place informational relations is given.

Sec. 3 is devoted to similarity relations in A-systems and their counterparts in B-systems. The main result is an abstract characterization by means of a finite set of first-order sentences /theorem 3.6./.

In sec. 4 similarity relations in single-valued A-systems are studied. Several characterization theorems for some collections of similarity relations is given.

Section 5 is a short introduction to the more general notion of informational relations in A-systems.

The results of this chapter have been published in the papers [Vak 94,95,95a].

1. Similarity relations in property systems and their counterparts in consequence systems

Very often, when we say that two objects are similar, we mean that they have common property. For instance, the son is similar to his father if, for example, they both have blue eyes. This is a kind of a positive similarity. In this sense we say that two objects x and y are similar if there exists a property A from a given set of properties Pr, such that A is possessed both by x and y. We can say also that the son is similar to his father, because they both are not smokers. This is a kind of a negative similarity. We say for two objects x and y that they are negatively similar if there exists a property A from a given set of properties Pr, such that A is possessed neither by x nor by y. One can say that there is no difference between these two kinds of similarity, interpreting negative similarity "positively" as follows. The argument is that for each property A we can introduce a new property non A, the complement $\overline{A}$ of A, with the following meaning: x possesses $\overline{A}$ iff x do not possess A. For instance if A is "to be a smoker" then $\overline{A}$ is "to be a non-smoker". The difference appears when we are working with limited sets of properties, which are not closed under taking a complement.

Positive and negative similarity relations between objects can be easily formalized in P-systems. The formal definitions are the following:

Let S be a P-system. Positive similarity in S is a binary relation in Ob_S denoted by Σ_S and defined as follows: for any $x,y\in Ob_S$

$x\Sigma_S y$ iff $(\exists A\in Pr_S)(A\in f_S(x)$ and $A\in f_S(y))$ iff $f_S(x)\cap f_S(y)\neq\emptyset$.

Negative similarity in S is a binary relation in Ob_S denoted by N_S and defined as follows: for any $x,y\in Ob_S$

$xN_S y$ iff $(\exists A\in Pr_S)(A\notin f_S(x)$ and $A\notin f_S(y))$ iff $\overline{f}_S(x)\cap\overline{f}_S(y)\neq\emptyset$

The formal theory of these two relations can be better described if we add one more binary relation between objects in P-systems, called informational inclusion.

Let S be a P-system. Informational inclusion in S, or simply inclusion, is a binary relation in Ob_S denoted by $\leq_S$ and defined as follows: for any two objects $x,y\in Ob_S$:

$x\leq_S y$ iff $f_S(x)\subseteq f_S(y)$ iff $f_S(x)\cap\overline{f}_S(y)=\emptyset$.

If for two objects x and y we have $x\leq_S y$ we say that x is (informationally) included in y, or, that y possesses all properties from Pr_S, which are possessed by x. Obviously $\leq_S$ is a quasi ordering relation in Ob_S , i.e. a reflexive and transitive relation in Ob_S.

By means of informational inclusion we can define another important relation between objects, called indiscernibility relation and denoted by $\equiv$ with the following equivalence:

$x\equiv y$ iff $x\leq y$ & $y\leq x$ iff $f(x)=f(y)$.

Lemma 1.1.

(i) Let S be a P-system. Then:

S is a total P-system iff $(\forall x\in Ob_S)$ $x\Sigma_S x$,

S is a nontrivial P-system iff $(\forall x\in Ob_S)$ $x\Sigma_S x$ or $xN_S x$.

(ii) Let $W\neq\emptyset$, $V\subseteq \mathbf{P}(W)$ and let $S=P(W,V)$ be the set-theoretical P-system over the pair (W,V). Then for any $x,y\in W$ we have:

$x\Sigma_S y$ iff $(\exists A\in V)(x\in A$ and $y\in A)$,

$xN_S y$ iff $(\exists A\in V)(x\notin A$ and $y\notin A)$.

$x\le_S y$ iff $(\forall A\in V)(x\in A \longrightarrow y\in A)$,

(iii) Let S and S' be P-systems and let h be a strong homomorphism from S into S'. Then for any $x,y\in Ob_S$:

$x\Sigma_S y$ iff $h(x)\Sigma_{S'}h(y)$,

$xN_S y$ iff $h(x)N_{S'}h(y)$.

$x\le_S y$ iff $h(x)\le_{S'}h(y)$,

(iv) Let S be a P-system and let |S| be the set-theoretical P-system associated with S. Then for any $x,y\in S$ we have:

$x\Sigma_S y$ iff $x\Sigma_{|S|}y$,

$xN_S y$ iff $xN_{|S|}y$.

$x\le_S y$ iff $x\le_{|S|}y$,

Proof - straightforward.■

By lemma 1.1.(i) we see that the relation Σ is not in general reflexive. This means that it is possible for some object x to have $x\overline{\Sigma}x$. In a sense this is contraintuitive, because in this case x is not similar to itself. By the definition of Σ this yields $f(x)=\emptyset$. This says that x does not have any property from Pr.

Lemma 1.2.

Let S be a P-system and let S'=C(S) be the C-system over S. Then the following conditions are true for any $x,y\in Ob_S$:

(i) $x\le_S y$ iff $x \vdash_{S'} y$,

(ii) $x\Sigma_S y$ iff $x,y \nvdash_{S'} \emptyset$,

(iii) $xN_S y$ iff $\emptyset \nvdash_{S'} x,y$.

Proof. Let $x,y\in Ob_S$. Then by the definition of a C(S) we have:

(i) $x\vdash_{S'} y$ iff $f_S(x)\subseteq f_S(y)$ iff $x\le_S y$,

(ii) $x,y\nvdash_{S'} \emptyset$ iff $f_S(x)\cap f_S(y)\neq\emptyset$ iff $x\Sigma_S y$,

(iii) $\emptyset\nvdash_{S'} x,y$ iff $Pr_S\neq f_S(x)\cup f_S(y)$ iff $xN_S y$.■

Lemma 1.3.

Let S' be a C-system and let S=P(S') be the canonical P-system over S. Then for any $x,y\in Sen_S$, we have:

(i) $x\le_S y$ iff $x\vdash_{S'} y$,

(ii) $x\Sigma_S y$ iff $x,y\nvdash_{S'} \emptyset$,

(iii) $xN_S y$ iff $\emptyset\nvdash_{S'} x,y$.

Proof. By the definition of P(S') $Ob_S=W=Sen_{S'}$, $Pr_S=V=PrFil(S')$, and for $x\in Ob_S$, $f_S(x)=\{A\in V)/x\in A\}$. Let $x,y\in Sen_{S'}$. Then:

(i) $x\vdash_{S'} y$ iff /Corollary 3.5 from ch. 2.1/ $(\forall A\in V)(\{x\}\subseteq A \longrightarrow \{y\}\cap A\neq\emptyset)$ iff $(\forall A\in V)(x\in A \longrightarrow y\in A)$ iff /lemma 1.1/ $x\le_S y$.

(ii) $x,y\nvdash_{S'} \emptyset$ iff /Corollary 3.5 from ch.2.1/ $(\exists A\in V)(\{x,y\}\subseteq A$ and $\emptyset\cap A=\emptyset$) iff $(\exists A\in V)(x\in A$ and $y\in A)$ iff /lemma 1.1/ $x\Sigma_S y$.

(iii) $\emptyset \nvdash_S x,y$ iff /Corollary 3.5 from ch. 2.1/ $(\exists A\in V)(\emptyset\subseteq A$ and $\{x,y\}\cap A=\emptyset)$ iff $(\exists A\in V)(x\notin A$ and $y\notin A)$ iff /lemma 1.1./ xN_Sy. ∎

Lemma 1.2. and lemma 1.3 attach the following unexpected new logical meaning to the introduced three relations:

- $x\leq_S y$ iff $x\vdash_{S'} y$ iff "x implies y in S'",
- $x\Sigma_S y$ iff $x,y\nvdash_{S'} \emptyset$, iff "the set $\{x,y\}$ is consistent in S'",
- $xN_S y$ iff $\emptyset\nvdash_{S'} x,y$, iff "the set $\{x,y\}$ is incomplete in S'".

This suggests the following definition. Let S be a C-system. Then we define the relations $\leq$, Σ and N in S as follows:

- $x\leq_S y$ iff $x \vdash_S y$,
- $x\Sigma_S y$ iff $x,y\nvdash_S \emptyset$,
- $xN_S y$ iff $\emptyset\nvdash_S x,y$.

In the next lemma we list some abstract properties of the relations $\leq$, Σ and N.

Lemma 1.4.

Let S be a P-system (C-system). Then the following conditions are true for any $x,y,z\in Ob_S$ $(x,y,z\in Sen_S)$ /the subscript S is omitted/:

S1. $x\leq x$, / $x\vdash x$, (Ref)
S2. $x\leq y$ & $y\leq z \to x\leq z$, / $x\vdash y$ & $y\vdash z \to x\vdash z$, (Cut)
S3. $x\Sigma y \to y\Sigma x$, / $y,x\vdash\emptyset \to x,y\vdash\emptyset$, (Permutation)
S4. $x\Sigma y \to x\Sigma x$, / $x\vdash\emptyset \to x,y\vdash\emptyset$, (Mono)
S5. $x\Sigma y$ & $y\leq z \to x\Sigma z$, / $x,z\vdash\emptyset$ & $y\vdash z$ then $x,y\vdash\emptyset$, (Cut)
S6. $x\Sigma x$ or $x\leq y$, / $x\vdash\emptyset \to x\vdash y$, (Mono)
S7. $xNy \to yNx$, / $\emptyset\vdash y,x \to \emptyset\vdash x,y$, (Permutation)
S8. $xNy \to xNx$, / $\emptyset\vdash x \to \emptyset\vdash x,y$, (Mono)
S9. $x\leq y$ & $yNz \to xNz$, / $x\vdash y$ & $\emptyset \vdash x,z \to \emptyset\vdash y,z$ (Cut)
S10. yNy or $x\leq y$, / $\emptyset\vdash y \to x\vdash y$, (Mono)
S11. $x\Sigma z$ or yNz or $x\leq y$, / $x,z\vdash\emptyset$ & $\emptyset\vdash z,y \to x\vdash y$, (Cut)

Note. On the right side of the above list, the conditions S1 - S11 for the C-system are written in the form to indicate that they are special cases of the conditions (Ref), (Mono) and (Cut).

Proof - straightforward verification. ∎

In order to give some abstract characterization of the relations $\leq$, Σ and N we introduce the following notion.

Let $\underline{W}=(W, \leq, \Sigma, N)$ be a relational system such that $W\neq\emptyset$ and $\leq$, Σ and N are binary relations in W. We will say that $\underline{W}$ is a similarity structure if it satisfies the conditions S1 - S11 from lemma 1.4. If S is a P-system /C-system/ then lemma 1.4 says that the system $Sim(S)=(Ob_S, \leq_S, \Sigma_S, N_S)$ /the system $Sim(S)=(Sen_S, \leq_S, \Sigma_S, N_S)$ / is a similarity structure, called similarity structure over S.

In the following lemma some easy consequences of the axioms S1- S11 are stated.

Lemma 1.5.

Let $\underline{W}=(W, \leq, \Sigma, N)$ be a similarity structure. Then for any $x,y,u,v\in W$ the following is true:

(Σ) If $u\Sigma v$ and $x,y\in\{u,v\}$ then $x\Sigma y$,
(N) If uNv and $x,y\in\{u,v\}$ then xNy,
S5' $x\Sigma y$&$x\leq u$&$y\leq v \to u\Sigma v$ - monotonicity of Σ,
S9' xNy&$u\leq x$&$v\leq y \to uNv$ - antymonotonicity of N,

S11' $(x\bar{\Sigma}z \text{ or } z\bar{\Sigma}x)\&(y\bar{N}z \text{ or } z\bar{N}y) \longrightarrow x\leq y$,

(ΣN1) $u\Sigma v\&u\bar{\Sigma}x\&v\bar{\Sigma}y \longrightarrow xNy$,

(ΣN2) $uNv\&u\bar{N}x\&v\bar{N}y \longrightarrow x\Sigma y$

(Triv). If $\exists a\in W\ a\bar{\Sigma}a\&a\bar{N}a$ then $\forall x,y\in W: x\leq y\ \&\ x\bar{\Sigma}y\ \&\ x\bar{N}y$.

The following lemma is a simple consequence of lemma 1.2. and lemma 1.3.

Lemma 1.6.

(i) Let S be a P-system and C(S) be the C-system over S. Then Sim(S)=Sim(C(S)).

(ii) Let S be a C-system and P(S) be the canonical P-system over S. Then Sim(S)=Sim(P(S)).

The main theorem of this section is the following.

Theorem 1.7. /Characterization theorem for similarity structures/

Let $\underline{W}$=(W, $\leq$, Σ, N) be a similarity structure. Then:

(1) There exists a C-system S such that $\underline{W}$=Sim(S).

(2) There exists a P-system S' such that $\underline{W}$=Sim(S')

Proof. (1). We define S as follows. Put $Sen_S=W$ and for $X,Y\in \mathbf{P}_{fin}(W)$ define:

$X\vdash Y$ iff (I) $\exists x\in X\ \exists y\in Y\ x\leq y$, or

(II) $\exists x,y\in X\ x\bar{\Sigma}y$, or

(III) $\exists x,y\in Y\ x\bar{N}y$, or

(IV) $\exists a\in W\ a\bar{\Sigma}a\&a\bar{N}a$.

Now we have to show that $\vdash$ satisfies the Scott's axioms (Ref), (Mono) and (Cut).

For (Ref) suppose that $X\cap Y\neq\emptyset$. Then $\exists x\in X,Y$. By S1 we have $x\leq x$ and by (I) we obtain $X\vdash Y$.

For (Mono) suppose $X\vdash Y$, $X\subseteq X'$ and $Y\subseteq Y'$. From $X\vdash Y$ we have that one of the cases (I)-(IV) holds. From the inclusions $X\subseteq X'$ and $Y\subseteq Y'$ the same case case will be true for X' and Y', which implies $X'\vdash Y'$.

For (Cut) suppose

(1) $X,a\vdash Y$,

(2) $X\vdash a,Y$

and proceed to show

(3) $X\vdash Y$.

For (1) and (2) we have to consider several cases following the definition of $\vdash$. Then we shall combine each case for (1) with each case for (2).

(1I) $\exists x\in X\cup\{a\}\ \exists y\in Y\ x\leq y$. If $x\in X$ then by (I) we get (3), so we consider in this case that

$x=a$: $a\leq y$, $y\in Y$.

(1II) $\exists x,y\in X\cup\{a\}\ x\bar{\Sigma}y$. If $x,y\in X$ then by (II) we get (3). So we consider here three cases:

(i) $x=a$: $a\bar{\Sigma}y$, $y\in X$,

(ii) $y=a$: $x\bar{\Sigma}a$, $x\in X$.

(iii) $x=y=a$: $a\bar{\Sigma}a$.

(1III) $\exists x,y\in Y\ x\bar{N}y$. Then directly by (III) we obtain (3). So this case will not be combined with the other cases for (2).

(1IV) $\exists a\ a\bar{\Sigma}a\&a\bar{N}a$. By (IV) we get (3) so this case also will not be combined with the other cases for (2).

(2I) $\exists u\in X\ \exists v\in\{a\}\cup Y\ u\leq v$. If $v\in Y$ then by (I) we get (3) so in this case we consider

$v=a$: $u\leq a$ and $u\in X$.

(2II) $\exists u,v\in X\ u\bar{\Sigma}v$. Then by (II) we obtain (3). This case will not be combined with the other cases for (1).

(2III) $\exists u,v\in\{a\}\cup Y\ u\bar{N}v$. If $u,v\in V$ then we obtain (3) by (III), so we will consider here three cases:

(j) $u=a$: $a\bar{N}v$, $v\in Y$,

(jj) $v=a$: $u\overline{N}a$, $u\in Y$,

(jjj) $u=v=a$: $a\overline{N}a$.

(2IV) $\exists a\in W$ $a\overline{\Sigma}a\&a\overline{N}a$. Then by (IV) we get (3), so this case will not be combined with the other cases of (1).

Now we start to combine the possible cases for (1) and (2).

Case (1I)(2I): $a\le y$, $y\in Y$, $u\le a$, $u\in X$. By S1. we get $u\le v$ and by (I) we get (3).

Case (1I)(2III). Here we have three sub-cases:

(j) $a\le y$, $y\in Y$, $a\overline{N}v$, $v\in Y$. By S9' we obtain $y\overline{N}v$ and by (III) we obtain (3).

(jj) $a\le y$, $y\in Y$, $u\overline{N}a$, $u\in Y$ - similar to (j).

(jjj) $a\le y$, $y\in Y$, $a\overline{N}a$. By S9 we obtain $y\overline{N}y$ and by (III) we obtain (3).

Case (1II)(2I) We have to consider three cases.

(i) $a\overline{\Sigma}y$, $y\in X$, $u\le a$, $u\in X$. Then by S5' we get $u\overline{\Sigma}y$ and by (II) we obtain (3).

(ii) $x\Sigma a$, $x\in X$, $u\le a$, $u\in X$. Proceed as in (i).

(iii) $a\overline{\Sigma}a$, $u\le a$, $u\Sigma a$, $u\in X$. By S5' we obtain $u\overline{\Sigma}u$ and by (II) we get (3).

Case (1II)(2III). We have to combine the cases (i)-(iii) with the cases (j)-(jjj).

(i)(j) $a\overline{\Sigma}y$, $x\in X$, $a\overline{N}v$, $v\in Y$. Then by S11' we get $y\le v$ and by (I) we obtain (3). The cases (i)(jj), (ii)(j) and (ii)(jj) can be treated similarly.

(iii)(jjj) $a\overline{\Sigma}a$, $a\overline{N}a$. By (IV) we obtain (3).

This completes the proof that the system $S=(W, \vdash)$ is a C- system. To complete the prove of the first part of the theorem we need the following

Assertion. For any $u,v\in W$ the following is true:

(a) $u\le v$ iff $u\vdash v$,

(b) $u\Sigma v$ iff $u,v\nvdash \emptyset$,

(c) uNv iff $\emptyset\nvdash u,v$.

Proof (a)($\rightarrow$) Suppose $u\le v$. Then by (I) we have $u\vdash v$.

(a)($\leftarrow$) Suppose $u\vdash v$, i.e. $\{u\}\vdash\{v\}$. We have to show $u\le v$. For that purpose we have to consider the possible cases of the definition of $\vdash$.

Case (I): $\exists x\in\{u\}$ $\exists y\in\{v\}$ $x\le y$. Then $x=u$, $y=v$ and hence $u\le v$.

Case (II): $\exists x,y\in\{u\}$ $x\overline{\Sigma}y$. Then $x=y=u$ and $u\overline{\Sigma}u$. By S6 we obtain $u\le v$.

Case (III): $\exists x,y\in\{v\}$ $x\overline{N}y$. Then $x=y=v$ and $v\overline{N}v$. By S10 we obtain $u\le v$.

Case (IV) $\exists a$ $a\overline{\Sigma}a\&a\overline{N}a$. By Lemma 5.5.(Triv) we obtain $u\le v$.

(b)($\rightarrow$) Suppose $u\Sigma v$ and by way of contradiction that $u,v\vdash\emptyset$. There are only two possible cases for $\{u,v\}\vdash\emptyset$ - case (II) and case (IV).

Case (II): $\exists x,y\in\{u,v\}$ $x\overline{\Sigma}y$. Then by (Σ) of lemma 1.5 we get $x\Sigma y$ - a contradiction.

Case (IV): $\exists a\in W$ $a\overline{\Sigma}a\&a\overline{N}a$. By lemma 1.5.(Triv) we obtain $u\overline{\Sigma}v$ - a contradiction.

(b)($\leftarrow$) Suppose $u,v\nvdash\emptyset$ and for the sake of contradiction that $u\overline{\Sigma}v$. Then by (II) we obtain that $u,v\vdash\emptyset$ - a contradiction.

(c)($\rightarrow$) Suppose uNv and by the way of contradiction that $\emptyset\vdash u,v$. There are only two possible cases for $\emptyset\vdash u,v$ - (III) and (IV).

Case (III): $\exists x,y\in\{u,v\}$ $x\overline{N}y$. From uNv, $x,y\in\{u,v\}$ and by lemma 1.5 (N) we obtain xNy - a contradiction.

Case (IV): $\exists a\in W$ $a\overline{\Sigma}a\&a\overline{N}a$. Then by (Triv) we get $u\overline{N}v$ - a contradiction.

(c)($\leftarrow$) Suppose $\emptyset\nvdash u,v$ and for the contrary $u\overline{N}v$. Then by (III) we get $\emptyset\vdash u,v$ - a contradiction.

This completes the proof of the assertion. From the assertion we obtain that $Sim(S)=\underline{W}$, which completes the proof of part (1) of the theorem.

Proof of part (2) of the theorem. We have to find a P-system S' such that $Sim(S')=\underline{W}$. For the constructed C-system S we have by (1) that $Sim(S)=\underline{W}$. Let S' be the canonical P-system over S. Then by lemma 1.6 we have $Sim(S)=Sim(P(S))=Sim(S')$, hence $W=Sim(S')$ This completes the proof of the theorem.■

Let us note that theorem 1.7 (2) can be proved directly without using C-systems and 1.7(1). Such a proof using a relevant theory of filters and ideals in similarity structures is given in [Vak 91a].

The notion of similarity structure is an abstract characterization of the relations $\leq_S$, Σ_S and N_S by means of first- order axioms. It is possible to give similar characterization of any subset of these relations.

We say that (W, ρ) is an ρ-structure if $\rho \subseteq \{\leq, \Sigma, N\}$ and the relations from ρ satisfy those axioms from the list S1-S11, ΣN1 and ΣN2, which contain relations from ρ.

Below we list the relevant axioms and changes in the proof of the Characterization theorem for any concrete ρ.

≤-structure - (W,≤). Axioms: S1 and S2. The definition of $\vdash$ in the proof of the Characterization theorem is the following:

$X \vdash Y$ iff (I) $\exists x \in X\ \exists y \in Y\ x \leq y$, or
(IV') for any u,v: $u \leq v$.

Σ-structure - (W, Σ). Axioms: S3 and S4.

$X \vdash Y$ iff (I') $X \cap Y \neq \emptyset$, or
(II) $\exists x,y \in X\ x\bar{\Sigma}y$, or
(IV") $\forall u,v \in W\ u\bar{\Sigma}v$.

N-structure - (W,N). Axioms: S7 and S8. The definition of $\vdash$:

$X \vdash Y$ iff (I') or (III) $\exists x,y \in Y\ x\bar{N}y$, or (IV"') $\forall u,v \in W\ u\bar{N}v$.

Σ≤-structure - (W, ≤, Σ). Axioms: S1, S2, S3, S4, S5, S6.

$X \vdash Y$ iff (I) or (II) or (IV') or (IV").

N≤-structure - (W, ≤, N). Axioms: S1, S2, S7, S8, S9, S10.

$X \vdash Y$ iff (I) or (III) or (IV') or (IV"').

ΣN-structure - (W, Σ, N). Axioms: S3, S4, S7, S8, ΣN1 and ΣN2 from lemma 1.5.

$X \vdash Y$ iff (I') or (I") $\exists x \in X\ \exists y \in Y\ \exists b \in W\ x\bar{\Sigma}b\ \&\ b\bar{N}y$, or (II), or (III), or (IV) $\exists a \in W\ a\bar{\Sigma}a\ \&\ a\bar{N}a$.

The proof of the characterization theorem for arbitrary ρ-structure can be done in a way similar to the proof of theorem 1.7. The next lemma shows that the characterization theorem for any ρ-structure can can be derived directly from theorem 5.7.

Lemma 1.8.

(i) Let $\underline{W}$=(W, ρ) be an ρ-structure with $\emptyset \neq \rho \neq \{\leq, \Sigma, N\}$. Then we can define the missing relations from $\{\leq, \Sigma, N\} \setminus \rho$ in such a way as to obtain similarity structure.

(ii) Let us add to the axioms of similarity structure the conditions ΣN1 and ΣN2 from lemma 1.5. Then the first-order theory of similarity structures is a conservative extension of the first- order theory of any ρ-structure.

Proof. (i). First we shall give the relevant definitions.

Case 1: $\rho=\{\leq\}$. Define $\Sigma=N=W\times W$.
Case 2: $\rho=\{\Sigma\}$. Define $N=W\times W$ and for $x,y \in W$ $x \leq y$ iff $x\bar{\Sigma}x$ or $x=y$.
Case 3: $\rho=\{N\}$. Define $\Sigma=W\times W$ and for $x,y \in W$ $x \leq y$ iff $y\bar{N}y$ or $x=y$.
Case 4: $\rho=\{\leq, \Sigma\}$. Define $N=W\times W$.
Case 5: $\rho=\{\leq, N\}$. Define $\Sigma=W\times W$.
Case 6: $\rho=\{\Sigma, N\}$. Define for $x,y \in W$:
$x \leq y$ iff $x\bar{\Sigma}x$ or $y\bar{N}y$ or $(\exists z)(x\bar{\Sigma}z\ \&\ z\bar{N}y)$ or $x=y$.

The proof that each of the above systems is a similarity structure is easy.
(ii) - follows from (i). ■

Theorem 1.9. /Characterization theorem for ρ-structures/
Each ρ-structure is a ρ-structure over some P-system.
Proof. Let $\underline{W}=(W,\rho)$ be a ρ-structure. Then by lemma 1.8 we can extend ρ to the set $\{\leq, \Sigma, N\}$ such that $(W, \leq, \Sigma, N)$ to be a similarity structure. Then by theorem 1.7.(ii) we conclude that $(W, \leq, \Sigma, N)$ is a similarity structure over some P-system S. Then obviously $\underline{W}=(W, \rho)$ as a reduct of $(W, \leq, \Sigma, N)$ is a ρ-structure over S. ■

Each $\leq$-structure is a quasi-ordered set, so we have the following

Corollary 1.10.
Each quasi-ordering can be represented as an informational inclusion.

2. Informational relations in Property systems and their counterparts in Consequence systems

Now we shall analyze in more details the way of defining the relations $\leq$, Σ and N /the subscript S is omitted/ in P-systems in order to obtain the more general notion of informational relation. Similar attempt is sketched in Orlowska [Or 93].

In the next table we repeat the definitions of $\leq$, Σ and N and some obvious equivalents:

$x\leq y$ iff $f(x)\subseteq f(y)$ iff $f(x)\cap\overline{f}(y)=\emptyset$,
$x\Sigma y$ iff $\underline{f}(x)\cap\underline{f}(y)\neq\emptyset$,
xNy iff $\overline{f}(x)\cap\overline{f}(y)\neq\emptyset$.

The general form of the above three relations is the following: there exists a Boolean term $B(u,v)$ of two different variables u and v and the relation R is in one of the following two forms:

(1) xRy iff $B(f(x),f(y))\neq\emptyset$,
(2) xRy iff $B(f(x),f(y))=\emptyset$.

Obviously, if R is in in the one of these forms then the complement of R, $\overline{R}$ will be in the other form.

The above observation leads to the following definition.

Let $B=B(u_1,\dots,u_n)$ be a Boolean term of n different variables. Then in each P-system S the n-place relation $R_S(x_1,\dots,x_n)$ is called a primitive informational relation of type B if one of the following conditions holds:

(1) $R_S(x_1,\dots,x_n)$ iff $B(f_S(x_1),\dots,f_S(x_n))=\emptyset$
(2) $R_S(x_1,\dots,x_n)$ iff $B(f_S(x_1),\dots,f_S(x_n))\neq\emptyset$.

Let $A=A(x_1,\dots,x_n)$ be a first-order open formula built from primitive informational relations, considered as predicates, and let $x_1,\dots,x_n$ be a list of individual variables containing all variables of A. Then the standard interpretation of A in any given P-system S will define an n-place relation in S, called informational relation of type A.

Since there are exactly 2^{2^n} non-equivalent Boolean terms of n arguments (including the constant terms 0 and 1), then, there exist exactly 2^{2^n} types of primitive informational relations in the form (1) and 2^{2^n} in the form (2). Since 1 and 0 are in both types then the total sum is $2.2^{2^n}-2$. For n=2 there exist exactly 30 different types of primitive informational relations.

In the next table we list all primitive informational relations of two variables of the form (1). The relations of the form (2) are negations of some relations of the form (1). Instead of $f(x)$ and $f(y)$ we write x and y; instead of $x\cap y$ we write xy; instead of $\emptyset$ we write 0.

1 $x\underline{y}=0$ iff $x\overline{\Sigma}y$,
2 $\underline{x}y=0$ iff $x\le y$,
3 $\underline{x}\underline{y}=0$ iff $y\le x$,
4 $xy=0$ iff $x\overline{N}y$,
$5=1\cup 2=xy\cup\underline{x}y=x=xx=0$ iff $x\overline{\Sigma}y$ and $x\le y$ iff $x\overline{\Sigma}x$,
$6=1\cup 3=xy\cup\underline{x}\underline{y}=y=0$ iff $x\overline{\Sigma}y$ and $\underline{y}\le x$ iff $y\overline{\Sigma}y$,
$7=1\cup 4=x\underline{y}\cup\underline{x}y=0$ iff $x\underline{y}=0$ and $\underline{x}y=0$ iff $x\overline{\Sigma}y$ and $x\overline{N}y$,
$8=2\cup 3=x\underline{y}\cup\underline{x}\underline{y}=\underline{0}$ iff $xy\underline{=}\underline{0}$ and $xy=0$ iff $x\le\underline{y}$ and $y\le\underline{x}$ iff $x\equiv y$,
$9=2\cup 4=\underline{x}y\cup\underline{x}\underline{y}=\underline{y}=0$ iff $\overline{y}y=0$ iff $\underline{x}\le y$ and $x\overline{N}y$ iff $y\overline{N}y$,
$10=3\cup 4=\overline{x}y\cup x\underline{y}=x=0$ iff $y\le x$ and $x\overline{N}y$ iff $x\overline{N}x$,
$11=1\cup 2\cup 3=x\cup\underline{x}\underline{y}=x\cup\underline{y}=0$ iff $x=0$ and $\underline{y}=0$ iff $x\overline{\Sigma}x$ and $y\overline{\Sigma}y$,
$12=1\cup 2\cup 4=x\cup\underline{x}\underline{y}=\underline{x}\cup y=0$ iff $\underline{x}=0$ and $y=0$ iff $\underline{x}\overline{\Sigma}x$ and $y\overline{N}y$,
$13=1\cup 3\cup 4=y\underline{\cup}\underline{x}\underline{y}=\underline{x}\cup\underline{y}=0$ iff $\underline{x}=0$ and $\underline{y}=0$iff $x\overline{N}\underline{x}$ and $y\overline{\Sigma}\underline{y}$,
$14=2\cup 3\cup 4=xy\cup x\underline{=}x\cup y=0$ iff $x=0$ and $y=0$ iff $x\overline{N}x$ and $y\overline{N}y$,
$15=1\cup 2\cup 3\cup 4=x\cup x=1=0$ iff false iff $x\overline{\Sigma}y$ and $x\Sigma y$
16 truth iff $x\overline{\Sigma}y$ or $x\Sigma y$

From the above table we can obtain the proof of the following

Theorem 2.1.

(i) There exist exactly 30 different types of primitive informational relations in P-systems of two variables.

(ii) All primitive informational relations of two variables are first-order definable by the relations $\le$, Σ and N.

(iii) All informational relations of two variables x and y are Boolean combinations of the formulas $x\le y$, $y\le x$, $x\Sigma y$ and xNy.

Theorem 2.2

The first-order theory of all two-place informational relations in P-systems coincides with the first-order theory of similarity structures.

Proof. Direct consequence of theorem 2.1 and Characterization theorem for similarity structures /theorem 1.7/.■

In the above list of two-place informational relations we see only two interesting new relations: the indiscernibility relation No8 and the relation No7: $x\overline{\Sigma}y$ and $x\overline{N}y$. The complement of this relation, denote it by S means: xSy iff $x\Sigma y$ or xNy, x is either positively or negatively similar to y. So this is another kind of similarity relation. The complement $\overline{S}$ of S has the following interesting meaning: $x\overline{S}y$ iff $f(x)\cap f(y)=\emptyset$ and $\overline{f}(x)\cap\overline{f}(y)=\emptyset$ iff $f(x)=\overline{f}(y)$ - $f(x)$ is the complement of $f(y)$. A possible reading of $x\overline{S}y$ is: "x complements y", or with more words, "x possesses those and only those properties /from the given set of properties Pr/, which are not possessed by y". The corresponding relations of $\equiv$ and $\overline{S}$ in C-systems are these: $x\equiv y$ iff $x\vdash y$ and $y\vdash x$ iff "x is equivalent to y"; $x\overline{S}y$ iff $x,y\vdash\emptyset$ and $\emptyset\vdash x,y$ iff $x \vdash \neg y$ and $\neg y\vdash x$ iff "x is equivalent to $\neg$y" iff "x is the negation of y".

For n-place informational relations also there is a basis consisting of n+1 relations, such that all other informational relations are first-order definable by the relations from the basis. One possible basis is in the following table /on the right are the corresponding relations in C-systems/.

$I^n_0(x_1,\ldots,x_n)$ iff $\overline{f}(x_1)\cap\ldots\cap\overline{f}(x_n)=\emptyset$,	$\emptyset\vdash x_1,\ldots,x_n$
$I^n_1(x_1,x_2\ldots,x_n)$ iff $f(x_1)\cap\overline{f}(x_2)\ldots\cap\overline{f}(x_n)=\emptyset$,	$x_1\vdash x_2,\ldots,x_n$
$I^n_2(x_1,x_2,x_3\ldots,x_n)$ iff $f(x_1)\cap f(x_2)\cap\overline{f}(x_3)\cap\ldots\cap\overline{f}(x_n)=\emptyset$,	$x_1,x_2\vdash x_3,\ldots,x_n$
....	....
$I^n_n(x_1,\ldots,x_n)$ iff $f(x_1)\cap\ldots\cap f(x_n)=\emptyset$,	$x_1,\ldots,x_n\vdash\emptyset$

Theorem 2.3.

(i) There exist exactly $2.2^{2^n}-2$ different types of primitive n-place informational relations in P-systems.

(ii) All n-place informational relations in P-systems are first-order definable by the relations $I_0^n,\ldots,I_n^n$.

Proof. The proof of (i) is given above. To prove (ii) Let $f_1=f$ and $f_0=\overline{f}$. Then obviously all relations of the form $R_{i_1\ldots i_n}(x_1,\ldots,x_n)$ iff $f_{i_1}(x_1)\cap\ldots\cap f_{i_n}(x_n)=\emptyset$, $i_1,\ldots i_n\in\{0,1\}$ are definable by the relations $I_0^n,\ldots,I_n^n$ using only permutation of variables. Then all informational relations of type (1) are some conjunctions of the relations $R_{i_1\ldots i_n}$. The relations of the type (2) are negations of the relations of type (1). ■

The relations I_m^n can be used to reformulate the definition of a C-system in order to consider C-systems as first-order structures. Let $I_{mn}=I_m^{m+n}$, $m=0,1,2\ldots$ $n=1,2,\ldots$. Then the following set of axioms is an equivalent definition of a C-system.

(Ref) $I_{12}(x,x)$

(Mono') $I_{mn}(x_1,\ldots,x_m;y_1,\ldots,y_m)\longrightarrow I_{m+1n}(a,x_1,\ldots,x_m;y_1,\ldots,y_n)$,

(Mono") $I_{mn}(x_1,\ldots,x_m;y_1,\ldots,y_m)\longrightarrow I_{mn+1}(x_1,\ldots,x_m;a,y_1,\ldots,y_n)$,

(Perm') $I_{mn}(x_1,\ldots,x_m;y_1,\ldots,y_n)\longrightarrow I_{mn}(x_{i_1},\ldots,x_{i_m};y_{j_1},\ldots,y_{j_n})$, where $(i_1,\ldots,i_m)$ and $(j_1,\ldots,j_n)$ are permutations of $(1,\ldots,m)$ and $(1,\ldots,n)$ respectively.

(Weak') $I_{m+1n}(x_1,x_1,x_2,\ldots,x_m;y_1,\ldots,y_n)\longrightarrow I(x_1,x_2,\ldots,x_m;y_1,\ldots,y_n)$,

(Weak") $I_{mn+1}(x_1,\ldots,x_m;y_1,y_1,y_2\ldots,y_n)\longrightarrow I_{mn}(x_1,\ldots,x_m;y_1,y_2\ldots,y_n)$,

(Cut) If $I_{m+1n}(x_1,\ldots,x_m,a;y_1,\ldots,y_n)$ and $I_{mn+1}(x_1,\ldots,x_m;a,y_1,\ldots,y_n)$ then $I(x_1,\ldots,x_m;y_1,\ldots,y_n)$.

It is clear now that C-systems are first-order structures.

Theorem 2.4.

First-order theory of all informational relations in P-systems coincides with the first-order theory of C-systems.

Proof - a direct consequence of the above first-order reformulation of C-systems and the characterization theorem for C-systems /theorem 3.3 in ch. 2.1/. ■

Let us note that the problem of characterization of the first- order theory of all n-place informational relations for n≥3 is an open problem.

3. Similarity relations in attribute systems and their counterparts in bi-consequence systems

In A-systems we can easily extend the definitions of the similarity relations Σ, N and the inclusion relation ≤ but in two different ways, obtaining weak and strong versions. The formal definitions are the following.

Let S be an A-system and $x,y\in Ob_S$. We introduce the following six relations in S defined in the following way:

- Weak positive similarity
  $x\Sigma_S y$ iff $(\exists a\in At_S) f_S(x,a)\cap f_S(y,a)\neq\emptyset$,
- Weak negative similarity
  $xN_S y$ iff $(\exists a\in At_S) \overline{f}_S(x,a)\cap \overline{f}_S(y,a)\neq\emptyset$,
- Weak informational inclusion
  $x<_S y$ iff $(\exists a\in At_S) f_S(x,a)\subseteq f_S(y,a)$,
- Strong positive similarity
  $x\sigma_S y$ iff $(\forall a\in At_S) f_S(x,a)\cap f_S(y,a)\neq\emptyset$,
- Strong negative similarity
  $x\nu_S y$ iff $(\forall a\in At_S) \overline{f}_S(x,a)\cap \overline{f}_S(y,a)\neq\emptyset$,
- Strong informational inclusion
  $x\leq_S y$ $(\forall a\in At) f_S(x,a)\subseteq f_S(y,a)$.

Lemma 3.1.

(i) Let $S=A(W,V)$ be a set-theoretical A-system over the pair (W,V). Then for any $x,y\in W$ the following is true:

- $x\Sigma_S y$ iff $(\exists a\in V)(\exists A\in a)(x\in A \;\&\; y\in A)$,
- $xN_S y$ iff $(\exists a\in V)(\exists A\in a)(x\notin A \;\&\; y\notin A)$,
- $x<_S y$ iff $(\exists a\in V)(\forall A\in a)(x\in A \longrightarrow y\in A)$,
- $x\sigma_S y$ iff $(\forall a\in V)(\exists A\in a)(x\in A \;\&\; y\in A)$,
- $x\nu_S y$ iff $(\forall a\in V)(\exists A\in a)(x\notin A \;\&\; y\notin A)$,
- $x\leq_S y$ iff $(\forall a\in V)(\forall A\in a)(x\in A \longrightarrow y\in A)$.

(ii) Let S and S' be two A-systems and let h be a strong homomorphism from S into S'. Then for any $x,y\in Ob_S$ and $R\in\{\Sigma, N, <, \sigma, \nu, \leq\}$ the following is true: $xR_S y$ iff $h(x)R_{S'}h(y)$.

(iii) Let S be an A-system and let $|S|$ be the set-theoretical A-system associated with S. Then for any $x,y\in Ob_S$ and $R\in\{\Sigma, N, <, \sigma, \nu, \leq\}$ the following is true: $xR_S y$ iff $xR_{|S|}y$.

Proof - straightforward.■

Lemma 3.2.

(i) Let S be an A-system and $S'=B(S)$ be the B-system over S . Then for any $x,y\in Ob_S$ the following is true:

- $x\Sigma_S y$ iff $x,y \nvdash_{S'} \emptyset$,
- $xN_S y$ iff $\emptyset \nvdash_{S'} x,y$,
- $x<_S y$ iff $x \succ_{S'} y$,
- $x\sigma_S y$ iff $x,y \not\succ_{S'} \emptyset$,
- $x\nu_S y$ iff $\emptyset \not\succ_{S'} x,y$,
- $x\leq_S y$ iff $x \vdash_{S'} y$.

(ii) Let S' be a B-system and $S=A(S')$ be the canonical A-system over S'. Then for any $x,y\in Ob_S$ the following is true:

- $x\Sigma_S y$ iff $x,y \nvdash_{S'} \emptyset$,
- $xN_S y$ iff $\emptyset \nvdash_{S'} x,y$,

- $x <_S y$ iff $x \not\succ_S y$,
- $x \sigma_S y$ iff $x, y \not\succ_S \emptyset$,
- $x \nu_S y$ iff $\emptyset \not\succ_S x, y$,
- $x \leq_S y$ iff $x \vdash_S y$.

Proof - straightforward, following the relevant definitions.■

Lemma 3.2 attaches a new meaning of the relations Σ, N, $<$, σ, ν, $\leq$ and suggests how to define them in B-systems. Namely we have the following definition.

Let S be a B-system. Then for $x, y \in Sen_S$ we define:

- $x \Sigma_S y$ iff $x, y \not\vdash_S \emptyset$,
- $x N_S y$ iff $\emptyset \not\vdash_S x, y$,
- $x <_S y$ iff $x \succ_S y$,
- $x \sigma_S y$ iff $x, y \not\succ_S \emptyset$,
- $x \nu_S y$ iff $\emptyset \not\succ_S x, y$,
- $x \leq_S y$ iff $x \vdash_S y$.

Lemma 3.3.

Let S be an A-system /B-system/. Then the following conditions are true for any $x, y, z \in Ob_S$ /$x, y, z \in Sen_S$/: /the subscript S is omitted/

S1.	$x \leq x$,	/ $x \vdash x$,	(Ref $\vdash$)
S2.	$x \leq y$ & $y \leq z \rightarrow x \leq z$,	/ $x \vdash y$ & $y \vdash z \rightarrow x \vdash z$,	(Cut $\vdash$)
S3.	$x \Sigma y \rightarrow y \Sigma x$,	/ $y, x \vdash \emptyset \rightarrow x, y \vdash \emptyset$,	(Permutation $\vdash$)
S4.	$x \Sigma y \rightarrow x \Sigma x$,	/ $x \vdash \emptyset \rightarrow x, y \vdash \emptyset$,	(Mono $\vdash$)
S5.	$x \Sigma y$ & $y \leq z \rightarrow x \Sigma z$,	/ $x, z \vdash \emptyset$ & $y \vdash z$ then $x, y \vdash \emptyset$,	(Cut $\vdash$)
S6.	$x \Sigma x$ or $x \leq y$,	/ $x \vdash \emptyset \rightarrow x \vdash y$,	(Mono $\vdash$)
S7.	$x N y \rightarrow y N x$,	/ $\emptyset \vdash y, x \rightarrow \emptyset \vdash x, y$,	(Permutation $\vdash$)
S8.	$x N y \rightarrow x N x$,	/ $\emptyset \vdash x \rightarrow \emptyset \vdash x, y$,	(Mono $\vdash$)
S9.	$x \leq y$ & $y N z \rightarrow x N z$,	/ $x \vdash y$ & $\emptyset \vdash x, z \rightarrow \emptyset \vdash y, z$	(Cut $\vdash$)
S10.	$y N y$ or $x \leq y$,	/ $\emptyset \vdash y \rightarrow x \vdash y$,	(Mono $\vdash$)
S11.	$x \Sigma z$ or $y N z$ or $x \leq y$,	/ $x, z \vdash \emptyset$ & $\emptyset \vdash z, y \rightarrow x \vdash y$,	(Cut $\vdash$)
S12.	$x < x$,	/ $x \succ x$	(Ref $\succ$)
S13.	$x \leq y$ & $y < z \rightarrow x < z$,	/ $x \vdash y$ & $y \succ z \rightarrow x \succ z$	(Cut $\succ$)
S14.	$x < y$ & $y \leq z \rightarrow x < z$,	/ $x \succ y$ & $y \vdash z \rightarrow x \succ z$	(Cut $\succ$)
S15.	$x \sigma y \rightarrow y \sigma x$,	/ $x, y \succ \emptyset \rightarrow y, x \succ \emptyset$	(Permutation $\succ$)
S16.	$x \sigma y \rightarrow x \sigma x$,	/ $x \succ \emptyset \rightarrow x, y \succ \emptyset$	(Mono $\succ$)
S17.	$x \sigma y$ & $y \leq z \rightarrow x \sigma z$,	/ $x, z \succ \emptyset$ & $y \vdash z \rightarrow x, y \succ \emptyset$	(Cut $\succ$)
S18.	$x \sigma x$ or $x < y$,	/ $x \succ \emptyset \rightarrow x \succ y$	(Mono $\succ$)
S19.	$x \sigma y$ & $y < z \rightarrow x \Sigma z$,	/ $x, z \vdash \emptyset$ & $y \succ z \rightarrow x, y \succ \emptyset$	(Cut $\succ$)
S20.	$x \sigma z$ or $y N z$ or $x < y$	/ $x, z \succ \emptyset$ & $\emptyset \vdash y, z \rightarrow x \succ y$	(Cut $\succ$)
S21.	$x \nu y \rightarrow y \nu x$,	/ $\emptyset \succ y, x \rightarrow \emptyset \succ x, y$	(Permutation $\succ$)
S22.	$x \nu y \rightarrow x \nu x$,	/ $\emptyset \succ x \rightarrow \emptyset \succ x, y$	(Mono $\succ$)
S23.	$x \leq y$ & $y \nu z \rightarrow x \nu z$,	/ $\emptyset \succ x, z$ & $x \vdash y \rightarrow \emptyset \succ y, z$	(Cut $\succ$)
S24.	$y \nu y$ or $x < y$,	/ $\emptyset \succ y \rightarrow x \succ y$	(Mono $\succ$)
S25.	$x < y$ & $y \nu z \rightarrow x N z$,	/ $\emptyset \vdash x, z$ & $x \succ y \rightarrow \emptyset \succ y, z$	(Cut $\succ$)
S26.	$x \Sigma z$ or $y \nu z$ or $x < y$	/ $x, z \vdash \emptyset$ & $\emptyset \succ y, z \rightarrow x \succ y$	(Cut $\succ$).

Proof - straightforward verification.■

Lemma 3.3. suggests the following definition, which is the first step in the abstract characterization of the relations $\leq$, Σ, N, $<$, σ, ν.

Let $\underline{W} = (W, \leq, \Sigma, N, <, \sigma, \nu)$ be a relational system with $W \neq \emptyset$ and $\leq$, Σ, N, $<$, σ, ν be binary relations in W. We call $\underline{W}$ a bi- similarity structure if it satisfies the conditions S1-S26 from lemma 3.3.

Obviously each bi-similarity structure is a similarity structure with respect to ≤, Σ, N. This is the reason why for these relations we preserve the notations from similarity structures.

Now lemma 3.3. says that if S is an A-system /B-system/ then the system BiSim(S)= $(W, \leq_S, \Sigma_S, N_S, <_S, \sigma_S, \nu_S)$ with $W=Ob_S$ /$W=Sen_S$/ is a bi-similarity structure, called the bi-similarity structure over S.

The next lemma states some consequences from the axioms of a bi-similarity structure.

Lemma 3.4.

The following conditions hold in each bi-similarity structure

(i) Let $S\in\{\Sigma, N, \sigma, \nu\}$. Then xSy and $u,v\in\{x,y\}$ imply uSv.

(ii) a. xSy & $x\leq u$ & $y\leq v \longrightarrow uSv$, $S\in\{\Sigma,\sigma\}$,

b. xSy & $u\leq x$ & $v\leq y \longrightarrow uSv, S\in\{N,\nu\}$,

c. $x\sigma y$ & $((x\leq u$ & $y<v)$ or $(x<u$ & $y\leq v)) \longrightarrow u\Sigma v$,

d. $x\nu y$ & $((u\leq x$ & $v<y)$ or $(u<x$ & $v\leq y)) \longrightarrow uNv$,

e. $x\sigma x$ & $x<y \longrightarrow x\Sigma x$,

f. $y\nu y$ & $x<y \longrightarrow xNx$.

(iii) a. $x\sigma y \longrightarrow x\Sigma y$,

b. $x\nu y \longrightarrow xNy$,

c. $x\leq y \longrightarrow x<y$.

(iv) a. $x\bar{\Sigma}x$ & $x\bar{N}x \longrightarrow \forall uv\ u\leq v, u<v, u\bar{\Sigma}v, u\bar{N}v, u\bar{\sigma}v, u\bar{\nu}v$,

b. $x\bar{\Sigma}x$ & $x\bar{N}y \longrightarrow (y\bar{\Sigma}y$ & $y\nu y)$ & $(y\sigma y$ & $y\bar{N}y)$.

c. $(x\bar{\Sigma}x$ & $x\nu x)$ or $(x\sigma x$ & $x\bar{N}x) \longrightarrow \forall uv\ u<v, u\bar{\sigma}v, u\bar{\nu}v$

Proof. Conditions (i) and (ii) are easy. Let us prove (iii). For (iiia). suppose $x\sigma y$. By S12 we have $y<y$ and by S19 we get $x\Sigma y$. In a similar way one can prove (iiib) and (iiic).

For (iva) suppose $x\bar{\Sigma}x$ & $x\bar{N}x$. Then by lemma 1.5.Triv we have $\forall u,v\ u\leq v$, $u\bar{\Sigma}v$ and $u\bar{N}v$. Then by (iii) and we get $u\bar{\sigma}v$, $u\bar{\nu}v$ and $u<v$.

For (ivb) suppose first $x\bar{\Sigma}x$ and $x\nu x$. From $x\bar{\Sigma}x$ we obtain $x\leq v$. From $x\bar{\nu}x$ we obtain $u<x$, and then by S14 we get $u<v$. For $u\bar{\sigma}v$ we proceed as follows. From $x\bar{\Sigma}x$ we get by S16 $x\bar{\Sigma}v$. From $x\nu x$ we obtain by S24 $u<x$. Then by (iic) we obtain $u\sigma v$. For $u\nu v$ we proceed in a similar way: $x\bar{\Sigma}x$ gives $x\leq u$ /by S6/, $x\nu x$ gives $x\nu v$ /by S22/, and from here we obtain /by (iib)/ $u\nu v$. In a similar way we proceed with the assumption $(x\bar{\sigma}x$ & $x\bar{N}x)$.■

Lemma 3.5.

(i) Let S be an A-system and B(S) be the B-system over S. Then BiSim(S)=BiSim(B(S)).

(ii) Let S be a B-system and A(S) be the canonical A-system over S. Then BiSim(S)=BiSim(A(S)).

Proof - direct consequence of lemma 3.3 and lemma 3.4.■

Theorem 3.6./Characterization theorem for bi-similarity structures/

Let $\underline{W}=(W, \leq, \Sigma, N, <, \sigma, \nu)$ be a bi-similarity structure. Then:

(i) There exists a B-system S such that $\underline{W}$=BiSim(S).

(ii) There exists an A-system S' such that $\underline{W}$=BiSim(S').

Proof. (i) Since we want to have $\underline{W}$=BiSim(S) then we have to put $Sen_S=W$. For the relation ⊢ we adopt the definition from theorem 1.7. Namely for $X,Y\in\mathbf{P}_{fin}(W)$ define:

X⊢Y iff (I) $\exists x\in X\ \exists y\in Y\ x\leq y$, or

(II) $\exists x,y\in X\ x\bar{\Sigma}y$, or

(III) $\exists x,y\in Y\ x\bar{N}y$, or

(IV) $\exists a\in W\ a\bar{\Sigma}a\&a\bar{N}a$.

For the relation ≻ we take the following definition: for $X,Y\in\mathbf{P}_{fin}(W)$ we define:

$X \succ\!\!- Y$ iff (J) $\exists x \in X\ \exists y \in Y\ x < y$, or
(JJ) $\exists x, y \in X\ x \bar{\sigma} y$, or
(JJJ) $\exists x, y \in Y\ x \bar{\nu} y$, or
(JV) $\exists x\ ((x \bar{\Sigma} x\ \&\ x \bar{\nu} x)$ or $(x \bar{\sigma} x\ \&\ x \bar{N} x))$.

The proof of (i) will follow from the following two assertions.

Assertion 1.

$(W, \vdash, \succ\!\!-)$ is a B-system.

Assertion 2.

For any $x, y \in W$ the following holds:

(a) $u \leq v$ iff $u \vdash_{S'} v$,

(b) $u \Sigma v$ iff $u, v \not\vdash_{S'} \emptyset$,

(c) $u N v$ iff $\emptyset \not\vdash_{S'} u, v$,

(d) $u < v$ iff $u \succ\!\!- v$,

(e) $u \sigma v$ iff $u, v \not\succ\!\!-_{S'} \emptyset$,

(f) $u \nu v$ iff $\emptyset \not\succ\!\!-_{S'} u, v$.

Proof of assertion 1. Since $\underline{W}$ is also a similarity structure with respect to $\leq$, Σ and N, then by theorem 5.7. $(W, \vdash)$ is a C- system. So we have only to show that $\succ\!\!-$ satisfies the remaining axioms of a B-system.

For (Refl$\succ\!\!-$) suppose $X \cap Y \neq \emptyset$, then $\exists x \in X\ x \in Y$. By S12 we have $x < x$ and by (J) of the definition of $\succ\!\!-$ we obtain $X \succ\!\!- Y$. Axiom (Mono $\succ\!\!-$) do not present difficulties and (Incl) follows from lemma 3.4.(iiiabc)-(ivb).

For (Cut$\succ\!\!-$1) suppose

(1) $X \vdash a, Y$,

(2) $X, a \succ\!\!- Y$

and proceed to show

(3) $X \succ\!\!- Y$.

We have to consider the possible cases for (1) and (2) and then to combine all cases of (1) with all cases of (2).

For (1) $X \vdash a, Y$ we have the following cases.

(1I): $\exists x \in X$, $\exists y \in \{a\} \cup Y\ x \leq y$. Then we have /by lemma 3.4.(iiic)/ $x < y$. If $y \in Y$ then by (J) we obtain (3). So for the combination with other cases of (2) we suppose only one subcase:

(i) $y = a$: $x \in X$, $x \leq a$ and $x < a$.

(1II): $\exists x, y \in X\ x \bar{\Sigma} y$. Then by lemma 3.4.(iiia) we obtain $x \bar{\sigma} y$ and by (JJ) we get (3). So this case will not be combined with other cases of (2).

(1III): $\exists x, y \in \{a\} \cup Y\ x \bar{N} y$. By lemma 3.4.(iiib) we obtain $x \bar{\nu} y$. If $x, y \in Y$ then by (JJJ) we get (3). So, for the combination with other cases we consider only the following sub cases:

(i) $x = a$: $y \in Y$, $a \bar{N} y$, $a \bar{N} y$,

(ii) $y = a$: $x \in Y$, $x \bar{N} a$, $x \bar{\nu} a$,

(iii) $x = y = a$: $a \bar{N} a$, $a \bar{\nu} a$.

(1IV): $\exists x\ x \bar{\Sigma} x\ \&\ x \bar{N} x$. Then by lemma 3.4.(ivb) and (IJ) we get (3). So this case will not be combined with other cases of (2).

For (2) $X, a \succ\!\!- Y$ we have the following cases.

(2J): $\exists u \in X \cup \{a\}\ \exists v \in Y\ u < v$. If $u \in X$ then by (J) we get (3), so for the combination with other cases of (1) we consider only one sub case:

(j) $u = a$: $a < v$, $v \in Y$.

(2JJ): $\exists u, v \in X \cup \{a\}\ u \bar{\sigma} v$. If $u, v \in X$ then by (JJ) we obtain (3) so for the combinations with other cases of (1) we will consider only the following:

(j) $u = a$: $a \bar{\sigma} v$, $v \in X$,

(jj) $v = a$: $u \bar{\sigma} a$, $u \in X$,

(jjj) $u = v = a$: $a \bar{\sigma} a$.

(2JJJ): $\exists u,v\in Y\ u\bar{\nu}v$. By (JJJ) we directly obtain (3), so this case will not be combined with other cases of (1).

(2JV): $\exists x\ ((x\bar{\Sigma}x\ \&\ x\nu x)$ or $(x\bar{N}x\ \&\ x\bar{\sigma}x))$. By (JV) we directly obtain (3), so this case will not be combined with the other cases for (1).

Now we start with the combination of the possible cases for (1) and (2).

Case (1I)(2J), (i)(j): $x\in X$, $x\le a$; $a<v$, $v\in Y$. Then by S13 we obtain $x<v$ and by (J) - (3).

Case (1I)(2JJ). We have to combine (i) with (j), (jj) and (jjj).

(i)(j): $x\in X$, $x\le a$; $a\sigma v$, $v\in X$. By lemma 3.4.(iia) we get $x\sigma v$ and by (JJ) we obtain (3).

(i)(jj): $x\in X$, $x\le a$; $u\sigma a$, $u\in X$. We treat this case as above.

(i)(jjj): $x\in X$, $x\le a$; $a\sigma a$. By lemma 3.4.(iia) we obtain $x\sigma x$ and by (JJJ) we get (3).

Case (1III)(2J). We have to combine (i), (ii) and (iii) with (j).

(i)(j): $a\bar{N}y$, $y\in Y$; $a<v$, $v\in Y$. By S25 we obtain $v\nu y$ and by (JJJ) we obtain (3).

(ii)(j): $x\in Y$, $x\bar{N}a$; $a<v$, $v\in Y$. The proof is as above.

(iii)(j): $a\bar{N}a$; $a<v$, $v\in Y$. By lemma 3.4.(iif) we get $v\bar{\nu}v$ and by (JJJ) we obtain (3).

Case (1III)(2JJ) We have to combine (i)-(iii) with (j)-(jjj).

(i)(j) $y\in Y$, $a\bar{N}y$; $a\sigma v$, $v\in X$. By S20, S15 and S7 we get $v<y$ and by (J) we obtain (3).

The other combinations except (iii)(jjj) are similar to the above one.

(iii)(jjj) $a\bar{N}a$; $a\sigma a$. By (JV) we get (3).

Thus we have completed the combinations of the possible cases for (1) and (2) and (Cut$\succ\!\!-$1) is verified.

The proof of (Cut$\succ\!\!-$2) can be given by the same manner. This completes the proof of assertion 1.

Proof of assertion 2. The proofs of (a), (b) and (c) are the same as the corresponding proofs in theorem 1.7.

(d)($\longrightarrow$) Suppose $u<v$. Then by (J) we obtain $u\succ\!\!-v$.

(d)($\longleftarrow$) Suppose $u\succ\!\!-v$. To show $u<v$ we will inspect the cases (J)-(JV) of the definition of $\succ\!\!-$.

Case (J): $\exists x\in\{u\}\ \exists y\in\{v\}\ x<y$. Then obviously we have $u<v$.

Case (JJ): $\exists x,y\in\{u\}\ x\sigma y$. Then we obtain $u\sigma u$ and by S18 we get $u<v$.

Case (JJJ): $\exists x,y\in\{v\}\ x\nu y$. Then we obtain $v\nu v$ and by S24 we get $u<v$.

Case (JV): (JV) $\exists x\ ((x\bar{\Sigma}x\ \&\ x\nu x)$ or $(x\sigma x\ \&\ x\bar{N}x))$. By lemma 3.4.(ivc) we obtain $u<v$.

(e) We shall prove $u\bar{\sigma}v$ iff $u,v\succ\!\!-\emptyset$.

($\longrightarrow$) Suppose $u\bar{\sigma}v$. Then by (JJ) we get $u,v\succ\!\!-\emptyset$.

($\longleftarrow$) Suppose $u,v\succ\!\!-\emptyset$. We shall prove $u\bar{\sigma}v$ inspecting the possible cases (J)-(JV) of the definition of $\succ\!\!-$. The cases (J) and (JJJ) are impossible.

Case (JJ): $\exists x,y\in\{u,v\}\ x\bar{\sigma}y$. Then by lemma 7.4.(i) we obtain $u\bar{\sigma}v$.

Case (JV): $\exists x((x\bar{\Sigma}x\ \&\ x\nu x)$ or $(x\sigma x\ \&\ x\bar{N}x))$. Then by lemma 3.4.(ivc) we obtain $u\bar{\sigma}v$.

(f) We shall prove $u\bar{\nu}v$ iff $\emptyset\succ\!\!-u,v$.

($\longrightarrow$) Suppose $u\bar{\nu}v$. Then by (JJJ) we have $\emptyset\succ\!\!-u,v$.

($\longleftarrow$) Suppose $\emptyset\succ\!\!-u,v$. The cases (J) and (JJ) are impossible, so we have to inspect only the cases (JJJ) and (JV).

Case (JJJ): $\exists x,y\in\{u,v\}\ x\bar{\nu}y$. By lemma 7.4.(i) we obtain $u\bar{\sigma}v$.

Case (JV): $\exists x((x\bar{\Sigma}x\ \&\ x\nu x)$ or $(x\sigma x\ \&\ x\bar{N}x))$. Then by lemma 3.4.(ivc) we obtain $u\bar{\nu}v$.

This completes the proof of assertion 2 and the part (i) of the theorem.

(ii) Put S'=A(S). Then S' is an A-system. By lemma 7.5 BiSim(S)=BiSim(A(S)). By (i) we have $\underline{W}$=BiSim(S), hence $\underline{W}$=BiSim(S'). This ends the proof of the theorem. ■

As we have done for similarity structures, we can give an abstract characterization of any subfamily of the relations ≤, Σ, N, <, σ, ν formulating an independent set of axioms and a theorem similar to theorem 7.6. We left this work to the reader.

4. Similarity relations in single-valued attribute systems

As we have defined, an A-system S is a single-valued if for all $x \in Ob_S$ and $a \in At_S$ the cardinality of the set $f_S(x,a)$ is less or equal to 1. The main aim of this section is to give an abstract characterization theorem for the relations from the set Re={≤, Σ, N, <, σ, ν} in single-valued A-systems. However, such a characterization for all relations together is an open problem. We shall give characterization theorems only for some concrete subsets of the set Re. A pair (W, R) with R⊆Re, satisfying some relevant axioms, is called an R-structure. The notion of a ρ-structure /ρ⊆{≤, Σ, N}/ introduced in sec. 5. is an R-structure with R=ρ. If a relational system W̲ can be obtained from a bi- similarity structure W̲' by omitting some of the relations of W̲' we will say that W̲ is a reduct of W̲'. Consequently, each R-structure is a reduct of some bi-similarity structure.

Let $\mathfrak{A}$ be the class of all bi-similarity structures over A- systems and $\mathfrak{A}_{sv}$ be the class of all bi-similarity structures over single-valued A-systems. Let α be a first-order sentence build from some relations from Re considered as binary predicates. We say that α separates $\mathfrak{A}$ from $\mathfrak{A}_{sv}$ if α is true in $\mathfrak{A}_{sv}$ but not in $\mathfrak{A}$. The conditions S27-S35 from the next lemma are examples of such sentences.

Lemma 4.1

Let S be a single-valued A-system. Then the following conditions are satisfied for any $x,y,z,u,v \in Ob_S$:

(i) S27. $x\sigma_S y \rightarrow x\leq_S y$,

S28. $x\Sigma_S y \rightarrow x<_S y$,

S29. $x\nu_S u \;\&\; u\sigma_S v \;\&\; v\nu_S y \rightarrow x\nu_S y$,

S30. $x\nu_S y \;\&\; x\sigma_S z \rightarrow u\nu_S u$

S31. $x\sigma_S u \;\&\; u\nu_S v \;\&\; v\sigma_S y \rightarrow x\nu_S y$,

S32. $x\sigma_S y \;\&\; y\sigma_S z \rightarrow x\sigma_S z$,

S33. $x<_S y \rightarrow y<_S x$ or $x<_S z$,

S34. $x\sigma_S y \;\&\; y\Sigma_S z \rightarrow x\Sigma_S z$,

S35. $x\Sigma_S y \;\&\; y\nu_S z \rightarrow xN_S z$.

(ii) On the ground of S1-S26:

- S31 follows from S27,
- S32 follows from S27,
- S33 follows from S28,
- S34 follows from S27 or S28,
- S35 follows from S28.

Proof. (i) follows directly from the definition of a single valued A-system.

(ii) As an example we shall show how S33 follows from S28. Suppose $x<_S y$ and not $x<_S z$ and proceed to show $y<_S x$. Then by S18 we obtain $x\sigma_S x$ and applying S19 we get $x\Sigma_S y$. Then by S3 we have $y\Sigma_S x$ and finally by S28 we conclude that

$y<_S x$. ■

Looking at the sentences S27 - S35 we can see that in each of them at least one of the relations $<$, σ and ν is used. We shall show that there is no a first-order formula, build by $\leq$, Σ and N, which separates $\mathfrak{A}$ from $\mathfrak{A}_{sv}$. First we shall prove the following theorem.

Theorem 4.2./ A characterization theorem for similarity structures in single-valued A-systems/

Each similarity structure is a reduct of some bi-similarity structure over some single-valued A-system.

Proof. By theorem 1.7./the characterization theorem for similarity structures/ there exists a P-system $S'=(Ob_{S'}, Pr_{S'}, f_{S'})$ such that $\underline{W}$ coincides with the similarity structure over S'. This means that $W=Ob_{S'}$ and the relations $\leq$, Σ and N coincide with the relations $\leq_{S'}$, $\Sigma_{S'}$ and $N_{S'}$ respectively, that is, for any $x,y\in W$ $x\leq y$ iff $f_{S'}(x)\subseteq f_{S'}(y)$, $x\Sigma y$ iff $f_{S'}(x)\cap f_{S'}(y)\neq\emptyset$, xNy iff $\overline{f}_{S'}(x)\cap\overline{f}_{S'}(y)\neq\emptyset$. We define a single-valued A-system S in the following way. Put $Ob_S=Ob_{S'}=W$, $At_S=Pr_{S'}$, for each $a\in Pr_{S'}$ put $Val_S(a)=\{a\}$ and for each $x\in Ob_S=Ob_{S'}$ and $a\in At_S=Pr_{S'}$ define $f_S(x,a)=f_{S'}(x)\cap\{a\}$. Obviously S is a single valued A-system. It is only a routine check to see that for any $x,y\in Ob_{S'}$ we have: $x\leq_{S'}y$ iff $x\leq_S y$, $x\Sigma_{S'}y$ iff $x\Sigma_S y$ and $xN_{S'}y$ iff $xN_S y$, which proves the theorem. ■

Corollary 4.3.

There is no a first-order formula build from the relations $\leq$, Σ and N, considered as predicates, which separates $\mathfrak{A}$ and $\mathfrak{A}_{sv}$.

Proof. Suppose for the sake of contradiction that such a formula exists and let it be α. then α is true in $\mathfrak{A}_{sv}$ but not in $\mathfrak{A}$. So there exists a bi-similarity structure $\underline{W}=(W, \leq, \Sigma, N, <, \sigma, \nu)$ in which α is not true. Since α is build by the relations $\leq$, Σ and N, then α will not be true in the similarity structure $(W, \leq, \Sigma, N)$, which is a reduct of $\underline{W}$. By theorem 8.2. there exists a single valued A-system S such that $Ob_S=W$ and the relations $\leq$, Σ and N coincide with the relations $\leq_S$, Σ_S, and N_S respectively. So α will not be true in the bi-similarity structure Sim(S) over S. Since S is a single-valued A system then $Sim(S)\in\mathfrak{A}_{sv}$. By the assumption α is true in $\mathfrak{A}_{sv}$, so α is true in Sim(S) - a contradiction. This completes the proof. ■

Theorem 4.4. /Characterization theorem for r-structures in single- valued A-systems/

Each ρ-structure $\underline{W}=(W, \rho)$ /$\rho\subseteq\{\leq, \Sigma, N\}$/ is a reduct of some bi- similarity structure over some single-valued A-system.

Proof. Let $\underline{W}=(W,\rho)$ be a ρ-structure. By theorem 5.8 ρ can be extended to the set $\{\leq, \Sigma, N\}$ so that the system $(W, \leq, \Sigma, N)$ becomes a similarity structure. Then by theorem 8.2 $(W, \leq, \Sigma, N)$ is a reduct of some bi-similarity structure over some single- valued A-system S, which proves the theorem. ■

Corollary 4.5.

Each quasi-ordered set $(W, \leq)$ can be represented as a $\leq$-structure over some single-valued A-system S with $Ob_S=W$.

By a positive bi-similarity structure we mean any system $\underline{W}=(W, \leq, \Sigma, <, \sigma)$ satisfying the conditions S1-S6, S12-S19. If $\underline{W}$ satisfies also S27 and S28 it will be called a strong positive bi- similarity structure.

let us note that in positive bi-similarity structures we consider only positive similarities. The following theorem is one of the main results in this section.

Theorem 4.6. /Characterization theorem for strong positive bi-similarity structures in single-valued A-systems/

Let $\underline{W}=(W, \leq, \Sigma, <, \sigma)$ be a strong bi-similarity structure. Then $\underline{W}$ is a reduct of some bi-similarity structure over a single-valued A-system.

The proof of this theorem will be given using a method, different from those in the proof of the characterization theorems for similarity structures and bi-similarity structures. It was applied for the first time in [Vak 87] and later in [Vak, 87a, 88, 89, 91].

First we shall introduce some notions and prove some lemmas.

Let $\underline{W}=(W, \leq)$ be a $\leq$-structure. A subset $A\subseteq W$ is called a $\leq$-filter in W if it satisfies the following condition

$(\forall x,y\in W)(x\in A \ \&\ x\leq y \longrightarrow y\in A)$.

Let for $x\in W$ define $[x)=\{y\in W/x\leq y\}$.

Lemma 4.7.

(i) $[x)$ is the smallest $\leq$-filter in W containing x.

(ii) Arbitrary union and intersection of any number of $\leq$-filters is a $\leq$-filter.

Proof - straightforward. ■

Let $S\in\{\Sigma,\sigma\}$ and $\underline{W}=(W, \leq, S)$ be $\leq S$-structure . A subset $A\subseteq W$ is called an S-filter in W if it is a $\leq$-filter and for any $x,y\in A$ we have xSy.

Lemma 4.8.

(i) If xSy then $[x)\cup[y)$ is the smallest S-filter containing x and y.

(ii) Each σ-filter is a Σ-filter.

(iii) Let $\underline{W}=(W, \leq, \Sigma, <, \sigma)$ be a strong positive bi-similarity structure, and A and B be σ-filters in W. Then $A\cap B\neq\emptyset$ implies A=B.

Proof. The proof of (i) and (ii) is straightforward. To prove (iii) suppose that A and B are σ-filters and that $A\cap B\neq\emptyset$. Then for some x we have $x\in A$ and $x\in B$. To prove A=B suppose that $y\in A$. then we have $x\sigma y$ and by S27 $x\leq y$. Since $x\in B$ we obtain that $y\in B$. This shows that $A\subseteq B$. In the same manner we show that $B\subseteq A$. ■

Let $\underline{W}$ be a strong positive bi-similarity structure. A set a of subsets of W is called a good set in $\underline{W}$ if the following conditions are satisfied for any $x,y\in W$:

(g1) $x\leq y \longrightarrow (\forall A\in a)(x\in A \longrightarrow y\in A)$,

(g2) $x\bar{\Sigma}y \longrightarrow (\forall A\in a)(x\notin A \ \&\ y\notin A)$

(g3) not $x<y \longrightarrow (\exists A\in a)(x\in A \ \&\ y\notin A)$

(g4) $x\sigma y \longrightarrow (\exists A\in a)(x\in A \ \&\ y\in A)$,

(g5) $(\forall A,B\in a)(A\cap B\neq\emptyset \longrightarrow A=B)$.

The set of all good sets in $\underline{W}$ is denoted by V. Note that conditions (g1) and (g2) say that all elements of a are Σ-filters.

The following lemma is important for the proof of theorem 4.6.

Lemma 4.9.

(i) $x\leq y \longleftrightarrow (\forall a\in V)(\forall A\in a)(x\in A \longrightarrow y\in A)$,

(ii) $x\Sigma y \longleftrightarrow (\exists a\in V)(\exists A\in a)(x\in A \ \&\ y\in A)$,

(iii) $x<y \longleftrightarrow (\exists a\in V)(\forall A\in a)(x\in A \longrightarrow y\in A)$,

(iv) $x\sigma y \longleftrightarrow (\forall a\in V)(\exists A\in a)(x\in A \ \&\ y\in A)$.

Proof. (i)($\longrightarrow$) Suppose $x\leq y$, $a\in V$, $A\in a$ and $x\in A$. Then by (g1) we obtain that $y\in A$.

($\longrightarrow$) Suppose not $x\leq y$. We shall show that $(\exists a\in V)(\exists A\in a)(x\in A \ \&\ y\notin A)$. We put

$A=[x)$ $a_1=\{A\}$, $a_2=\{[u)/\ u\sigma u\ \&\ u\notin A\}$ and $a=a_1\cup a_2$. Since we have not $x\leq y$ we get $x\in A$ and $y\notin A$, so there exists $A\in a$ such that $x\in A$ and $y\notin A$. By x y and S6 we have $x\Sigma x$. Then by lemma 4.8.(i) A is a Σ-filter. By lemma 4.8.(i) all elements of a_2 are σ-filters and by lemma 4.8.(ii) they are Σ-filters. Consequently conditions (g1) and (g2) of good sets are fulfilled for a.

For (g3) suppose not $u<v$ and proceed to show that there exists $B\in a$ such that $u\in B$ and $v\notin B$. From not $u<v$ by S18 we get $u\sigma u$ and by lemma 4.8.(i) and (ii) that $[u)$ is a Σ filter. We have to consider two cases.

Case 1: $u\notin A$. Then $[u)\in a_2$ and $u\in[u)$. We shall show that $v\notin[u)$. Suppose that $v\in[u)$. Since $[u)$ is a Σ-filter then we get $u\Sigma v$ and by S28 $u<v$, contrary to the assumption not $u<v$. So, in this case $B=[u)$.

Case 2: $u\in A$. Suppose $v\in A$. Since A is a Σ-filter we have $u\Sigma v$ and by S28 we get $u<v$ - a contradiction with the assumption. So $u\notin A$. In this case $B=A$.

For the condition (g4) suppose $u\sigma v$ and proceed to show that there exists $B\in a$ such that $u,v\in B$. From $u\sigma v$ we obtain by S16 $u\sigma \upsilon$ and by S27 $u\leq v$. So $u,v\in[u)$. We have to consider two cases.

Case 1: $u\notin A$. Then $[u)\in a_2$. In this case $B=[u)$.

Case 2: $u\in A$. Then by $u\leq v$ we obtain that $v\in A$. In this case $B=A$.

For (g5) suppose $B,C\in a$, $B\cap C\neq\emptyset$ and proceed to show that $B=C$. We shall consider several cases.

Case 1: $B,C\in a_2$. Since all elements of a_2 are σ-filters, the assertion follows from lemma 4.8.(iii).

Case 2: $B,C\in a_1$. Then obviously $B=C$.

Case 3: $B\in a_1$, $C\in a_2$. Then $B=[x)$ and $C=[u)$ with $u\sigma u$ and not $x\leq u$. From $B\cap C\neq\emptyset$ we obtain that for some y: $y\in[x)$ and $y\in[u)$. Since $[u)$ is a σ-filter we get $y\sigma u$ and by S27 $y\leq u$. Then we obtain $u\in[x)$, so $x\leq u$ - a contradiction. This contradiction shows that the case is impossible. This completes the proof that $a\in V$ and the proof of (i).

(ii)($\longrightarrow$) Suppose $x\Sigma y$ and proceed to show that $(\exists a\in V)(\exists A\in a)(x\in A\ \&\ y\in A)$. We put $A=[x)\cup[y)$, $a_1=\{A\}$, $a_2=\{[u)/u\sigma u\ \&\ u\notin A\}$ and $a=a_1\cup a_2$. Hence there exists $A\in a$ such that $x,y\in A$. As in the above case we can show that all elements of a are Σ-filters, so conditions (g1) and (g2) of a good set are fulfilled.

To verify (g3) suppose not $u<v$ and proceed to show that there exists $B\in a$ such that $u\in B$ and $v\notin B$. From not $u<v$ we get by S18 $u\sigma u$, so $[u)$ is a σ-filter. We have to consider two cases.

Case 1: $u\notin A$. Then $[u)\in a_2$. We shall show that $v\notin[u)$. Suppose that $v\in[u)$, then we get $u\leq v$ and hence $u<v$ - a contradiction with the assumption. So $v\notin[u)$. In this case $B=[u)$.

Case 2: $u\in A$. We shall show that $v\notin A$. Suppose for the sake of contradiction that $v\in A$. Since A is a Σ-filter we obtain $u\Sigma v$ and by S28 - $u<v$ - a contradiction with the assumption. So $v\notin A$. In this case $B=A$.

For (g4) suppose $u\sigma v$ and proceed to show that there exists $B\in a$ such that $u,v\in B$. From $u\sigma v$ we obtain by S16 $u\sigma u$, so $[u)$ is a σ-filter. We have to consider two cases.

Case 1: $u\notin A$. Then $[u)\in a_2$. By S27 we have $u\leq v$ and from here we get $u,v\in[u)$. In this case $B=[u)$.

Case 2: $u\in A$. By $u\leq v$ we obtain that $v\in A$. In this case $B=A$.

For (g5) suppose $B,C\in a$ and $B\cap C\neq\emptyset$. As in (i)($\longleftarrow$) we can conclude that $B=C$. This completes the proof that $a\in V$.

($\longleftarrow$) We have to show that $x\bar{\Sigma}y \longrightarrow (\forall a\in V)(\forall A\in a)(x\notin A$ or $y\notin A)$. This follows directly by (g2). This completes the proof of (ii).

(iii)($\longrightarrow$) Suppose $x<y$ and proceed to show that $(\exists a\in V)(\forall A\in a)(x\in A \longrightarrow y\in A)$. We

shall consider two cases.

Case 1: $x\Sigma y$. In this case we put $a_1=\{[x)\cup[y)\}$, $a_2=\{[u)/u\sigma u \ \& \ u\notin[x)\cup[y)\}$ and $a=a_1\cup a_2$. We shall show that $(\forall A\in a)(x\notin A$ or $y\in A)$. For $A\in a_1$ this is obvious. Suppose $A\in a_2$, $x\in A$ and for the sake of contradiction that $y\notin A$. Then $A=[u)$ for some $u\sigma u$ and $u\notin[x)\cup[y)$, $x\in[u)$ and $y\notin[u)$. From $u\sigma u$ we obtain that $[u)$ is a σ-filter, containing u. From $x,u\in[u)$ we obtain $x\sigma u$ and by S27 $x\le u$, so $u\in[x)\cup[y)$ - a contradiction. As in (ii) we can show that $a\in V$.

Case 2: $x\bar{\Sigma}y$. We shall show that for any $u\in W$: if $u\sigma u$ then not $u\le x$. Suppose that for some $u\in W$ we have $u\sigma u$ and $u\le x$. From here we obtain $x\sigma x$. Since we have $x<y$ then by S19 we get $x\Sigma y$, contrary to the assumption $x\bar{\Sigma}y$. In this case we put $a=\{[u)/u\sigma u\}$ and by the above property we have that for any $A\in a$ we have that $x\notin A$. The proof that $a\in V$ presents no difficulties.

($\leftarrow$) Suppose not $x<y$. Then by (g3) $(\forall a\in V)(\exists A\in a)(x\in A \ \& \ y\notin A)$ which have to be proved.

(iv)($\rightarrow$) Suppose $x\sigma y$, $a\in V$. Then by (g4) $(\exists A\in a)(x\in A \ \& \ y\in A)$.

($\leftarrow$) Let $x\bar{\sigma}y$ and proceed to show that $(\exists a\in V)(\forall A\in a)(x\notin A$ or $y\notin A)$. Put $a=\{[x)/x\sigma x\}$. Then all elements of a are σ-filters, so if $x,y\in A\in a$ then $x\sigma y$, contrary to the assumption. Thus, for any $A\in a$, either $x\notin A$ or $y\notin A$. The proof that $a\in V$ can be given as in the above statements. This completes the proof of the lemma. ■

Proof of theorem 4.6. Let $\underline{W}=(W, \le, \Sigma, <, \sigma)$ be a strong positive bi-similarity structure. Let V be the set of all good sets of $\underline{W}$ and let S be the A-system over the pair (W, V). By lemma 2.1.(iii) from ch.2.1 S is a single-valued A-system and by theorem 3.1 and lemma 4.9 we have $x\le y$ iff $x\le_S y$, $x\Sigma y$ iff $x\Sigma_S y$, $x<y$ iff $x<_S y$, $x\sigma y$ iff $x\sigma_S y$, which had to be proved. ■

5. Informational relations in attribute systems

In this section we will generalize the definitions of the similarity relations in A-systems in order to obtain a more general notion of informational relations in A-systems.

Let S be an A-system. Then for each attribute $a\in At_S$ the system $S_a=(Ob_S, Val_S(a), f_S^a)$ is a P-system, where f_S^a is a function from Ob_S to the set $\mathbf{P}(Val_S(a))$ defined as $f_S^a(x)=f_S(x,a)$. Then the relation Σ_a in S_a have the following definition.

$x\Sigma_a y$ iff $f_S^a(x)\cap f_S^a(y)\neq\emptyset$ iff $f_S(x,a)\cap f_S(y,a)\neq\emptyset$,

Now the relations Σ_S and σ_S can be defined as follows:

$x\Sigma_S y$ iff $(\exists a\in At_S)f_S(x,a)\cap f_S(y,a)\neq\emptyset$ iff $(\exists a\in At_S)x\Sigma_a y$, so

$\Sigma_S=\bigcup\{\Sigma_a/a\in At_S\}$

$x\sigma_S y$ iff $(\forall a\in At_S)f_S(x,a)\cap f_S(y,a)\neq\emptyset$ iff $(\forall a\in At_S)x\Sigma_a y$, so

$\Sigma_S=\bigcap\{\Sigma_a/a\in At_S\}$

This analysis suggests the following definition. Let R be a type of informational relation in P-systems / like Σ/ and let S be an A-system. Denote by R_a the informational relation in each P- system S_a for $a\in At_S$. Then the relations $\bigcup\{R_a/a\in At_S\}$ and $\bigcap\{R_a/a\in At_S\}$ are called respectively weak informational relation in S of type R and strong informational relation S of type R.

The general study of informational relations in A-systems is an open area, which is out of the scope of this dissertation. Let us note that the developed theory in the paper is suitable to study only the similarity relations in A systems. We include this material only to show the place of the similarity relations in the family of all informational relations in A-systems.

CHAPTER 2.3

MODAL LOGICS FOR SIMILARITY RELATIONS IN INFORMATION SYSTEMS

Overview. In this chapter we introduce several modal logics containing modalities corresponding to the similarity relations in P-systems and A-systems. We discuss the query meaning of the corresponding modal languages. The main results are the completeness and decidability theorems for the introduced logics.

Introduction

In this chapter we introduce modal logics for similarity relations in P-systems and A-systems. The aim of these logics is to provide a formal account for reasoning about objects in an information system, including different modalities corresponding to some similarity relations and other kinds of informational relations. The first systems of this type have been introduced by Orlowska and Pawlak in [O&P 84] and [O&P 84a]. Later this line of investigations was continued by Orlowska [Or 84], [Or 85], [Or 85a], [Or 90], Farinas and Orlowska [F&O 85], Vakarelov [Vak 87], [Vak 87a], [Vak 89], [Vak 91] and [Vak 91a].

The main results are some completeness theorems for the introduced modal logics with respect to its standard semantics. The proofs of these completeness theorems are essentially based on the characterization theorems for similarity relations in P- systems and in A-systems, proved in chapter 2.2. Another kind of results are the completeness theorems with respect to the finite models of the considered systems, which yield their decidability. Let us mention also the discussion in sec. 1 of the possibility to give a "query" meaning of the considered modal languages, in which atomic formulas can be considered as simple queries and compound formulas as compound queries. Then modal formulas form a special kind of queries, which can be named as "modal queries".

The structure of the chapter is the following. In sec. 1 we introduce the logic SIM-1 and some related subsystems, connected with P-systems. Section 2 is devoted to the logic SIM-2, connected with A-systems. In sec. 3 we consider some logics interpreted in single-valued A-systems. The results of the chapter have been published in [Vak 94,95,95a].

1. SIM-1 - a modal logic for similarity relations in property systems

In this section we will introduce a modal logic, named SIM-1, with standard interpretation in similarity structures. This logic is a slight generalization of the logic Sim, introduced in [Vak 91a]. The main difference is that Sim is interpreted in similarity structures over non-trivial P-systems, while here SIM-1 is interpreted in similarity structures over arbitrary P-systems. The logic SIM-1 can be used to reason about similarity relations between data in terms of properties. The language of SIM-1 with a given interpretation can be used also as a special query language in which special modal queries can be expressed. We shall discuss this in more details after giving the semantics of SIM-1.

Syntax of SIM-1

The language of SIM-1 contains the following primitive symbols:

- VAR - an infinite set, whose elements are called propositional variables,
- $\wedge$, $\vee$, $\neg$ - the classical Boolean connectives,
- [≤], [≥] [Σ], [N], [U] - modal operations,
- ( ) - parentheses.

The notion of formula and some standard abbreviations are the usual.

Semantics of SIM-1

We interpret the language of SIM-1 in similarity structures as in the usual Kripke semantics. We shall repeat the relevant definitions. Let $\underline{W}=(W,\Sigma,N,\leq)$ be a similarity structure, for $x,y\in W$ we put $x\geq y$ iff $y\leq x$. A function $v:VAR\rightarrow\mathbf{P}(W)$ is called a valuation if it assigns to each variable A a subset $V(A)\subseteq W$. Then the pair $M=(\underline{W},v)$ is called a model over $\underline{W}$. The satisfiability relation $x\Vdash_v A$ /the formula A is true in a point $x\in W$ at the valuation v/ is defined inductively:

$x\Vdash_v A$ iff $x\in v(A)$ for $a\in VAR$,

$x\Vdash_v \neg A$ iff $x\nVdash_v A$,

$x\Vdash_v A\wedge B$ iff $x\Vdash_v A$ and $x\Vdash_v B$,

$x\Vdash_v A\vee B$ iff $x\Vdash_v A$ or $x\Vdash_v B$,

$x\Vdash_v [R]A$ iff $(\forall y\in W)(xRy\rightarrow y\Vdash_v A)$ for $R\in\{\Sigma,N,\leq,\geq\}$,

$x\Vdash_v [U]A$ iff $(\forall y\in W)y\Vdash_v A$.

Using the definition of $\langle R\rangle$ we obtain the interpretation of $\langle R\rangle A$ and $\langle U\rangle A$:

$x\Vdash_v \langle R\rangle A$ iff $(\exists y\in W)(xRy$ and $y\Vdash_v A)$,

$x\Vdash \langle U\rangle A$ iff $(\exists y\in A)(y\Vdash A)$.

We say that a formula A is true in a similarity structure $\underline{W}$ if for any valuation v and $x\in W$ we have $x\Vdash_v A$.

Let S be a P-system, $\underline{W}$ be the similarity structure over S and $M=(\underline{W}, v)$ be a model over $\underline{W}$. For any formula A we put $v(A)=\{x\in W/x\Vdash_v A\}$. The set v(A) may have different meanings. One is that it is the set of all objects from $W=Ob_S$ for which A is true (at v). Another meaning is that v(A) may be considered also as a query to S: "give the set of all objects $x\in Ob_S$, for which A is true". This meaning leads to consider interpreted propositional variables in a given model as a simple queries and formulas as compound queries. Them modal formulas will be "modal queries". Let us consider the following example. Suppose in the above model M that A is a propositional variable such that $v(A)=\{x_0\}$. Then for $v(\langle\Sigma\rangle A)$ we can compute: $v(\langle\Sigma\rangle A)=\{x\in Ob_0/(\exists y\in Ob_S)(x\Sigma y$ and $y\in\{x_0\}) = \{x\in Ob_S/x\Sigma x_0\}$. This is the following query to S: "give all objects of S which are positively similar to x_0".

This informational meaning of models of SIM-1 can be given to the models of all logics later on in the part, so we will not discuss it any more.

Axiomatization of SIM-1.

Axiom schemes:

(Bool) All Boolean tautologies,

(K) $[R](A\Rightarrow B)\Rightarrow([R]A\Rightarrow[R]B)$, $R\in\{\Sigma,N,\leq,\geq,U\}$

A0. $\langle\leq\rangle[\geq]A\Rightarrow A$, $\langle\geq\rangle[\leq]A\Rightarrow A$, $[U]A\Rightarrow A$, $\langle U\rangle[U]A\Rightarrow A$,
$[U]A\Rightarrow[U][U]A$, $[U]A\Rightarrow[R]A$, $R\in\{\leq,\geq,\Sigma,N\}$,

A1. $[\leq]A\Rightarrow A$,

A2. $[\leq]A\Rightarrow[\leq][\leq]A$,

A3. $\langle\Sigma\rangle[\Sigma]A\Rightarrow A$,

A4. $\langle\Sigma\rangle 1\Rightarrow([\Sigma]A\Rightarrow A)$,

A5. $[\Sigma]A\Rightarrow[\Sigma][\leq]A$,

A6. $[\leq]A\Rightarrow([U]A\vee([\Sigma]B\Rightarrow B))$,

A7. $\langle N\rangle[N]A\Rightarrow A$,

A8. $\langle N\rangle 1\Rightarrow([N]A\Rightarrow A)$,

A9. $[N]A\Rightarrow[N][\geq]A$,

A10. $[\geq]A\Rightarrow([U]A\vee([N]B\Rightarrow B))$,

A11. $[\leq]A\wedge[\Sigma]B\Rightarrow([U]B\vee[U]([N]B\Rightarrow A))$,

Rules of inference: modus ponens (MP) $A, A\Rightarrow B/B$,
necessitation (N) $A/[R]A$, $R\in\{\Sigma,N,\leq,\geq,U\}$

The logic SIM-1 is the smallest set of formulas, containing all axiom schemes and closed under the rules of inference. A formula A is a theorem of SIM-1 if there exists a finite sequence of formulas $A_1,\dots,A_n=A$ such that for each $i=1,\dots,n$ the formula A_i is ether an axiom of SIM-1, or can be obtained from one or two formulas with smaller indices by one of the rules of inference.

Let us note that using the standard modal definability theory /see [Ben 86]/ the axioms in A0 say that ≥ is the converse relation of ≤ and that U is an equivalence relation, containing the relations Σ, N, ≤ and ≥. In our semantics we have taken U to be the universal relation W×W, which gives the names of [U] and <U> as universal modalities /see [G&P 90]. The axioms A1-A11 are modal translations of the conditions S1-S11. Let us note that one purpose to introduced the universal modality is to be able to find modal translations of S6, S10 and S11. We introduce the modality [≥] as to obtain the following duality: [Σ]-[N], [≤]-[≥] and [U]- [U].

Theorem 1.1. /Completeness theorem for SIM-1/

For any formula A of SIM-1 the following conditions are equivalent:

(i) A is a theorem of SIM-1,

(ii) A is true in all similarity structures,

(iii) A is true in all similarity structures over P-systems.

Proof. (i)⟶(ii) in a standard way by showing the validity of all axioms and that the rules preserve validity.

(ii)⟷(iii) - by the characterization theorem for similarity structures - theorem 1.7. ch.2.2.

(ii)⟶(i). The proof can be done by the standard canonical- model-construction / for the relevant definitions and facts see [H&C 84] or [Seg 71]/.

Let W be the set of all maximal consistent sets of SIM-1. Define for $x,y\in W$ and $R\in\{\Sigma,N,\le,\ge,U\}$, $[R]x=\{A\in FOR/[R]A\in x\}$, xRy iff $[R]x\subseteq y$. It is easy to show by A0 that U is an equivalence relation containing Σ, N, ≤, ≥. For $a\in W$ let $W_a=\{x/aUx\}$ and Σ_a, N_a, $\le_a$, $\ge_a$ and U_a be the restrictions of Σ, N, ≤ and ≥ in the set W_a. Since U is an equivalence relation, U_a is the universal relation in W_a. Then, using the axioms of SIM-1, one can prove in a standard way the following

Lemma 1.2.

For any $a\in W$ the system $\underline{W}_a=(W_a,\Sigma_a,N_a,\le_a,\ge_a)$ is a similarity structure.

Now, suppose that A is not a theorem of SIM-1. Then there exists a maximal consistent set $a\in W$ such that $A\notin a$. Take the canonical valuation $v(p)=\{x\in W_a/p\in x\}$, $p\in VAR$. Then in a standard way one can prove by induction on the construction of B that for any $x\in W_a$: $x\Vdash_v B$ iff $B\in x$. From here we get $a\not\Vdash_v A$, so A is not true in the similarity structure W_a, which ends the proof of the theorem.■

Now we shall prove by means of the filtration method /see [Seg,71]/ that SIM-1 possesses finite model property in a sense that each non-theorem of SIM-1 can be falsified in a finite model. First we shall formulate some basic definitions and facts about filtration, adapted to the logic SIM-1.

Let Γ be a finite set of formulas, closed under sub formulas and M=((W, ≤, Σ, N) v) be a model over some similarity structure $\underline{W}$ Define an equivalence relation ~ in W in the following way:

$x\sim y$ iff $(\forall A\in\Gamma)(x\Vdash_v A \longleftrightarrow y\Vdash_v A)$.

Let for $x\in W$ $|x|=\{y\in W/x\sim y\}$, $|W|=\{|x|/x\in W\}$ and $v'(p)=\{|x|/x\in v(p)$ for $p\in VAR$.

We say that the model $M'=((|W|, \leq', \Sigma', N'), v')$ is a filtration of M through Γ if M' is a model over similarity structure and the following conditions are satisfied for any $x,y\in W$ and $R\in\{\Sigma, N, \leq, \geq, U\}$:

(FR1) If xRy then $|x|R'|y|$,

(FR2) If $|x|R'|y|$ then $(\forall[R]A\in\Gamma)(x\Vdash_v A \rightarrow y\Vdash_v A)$.

Lemma 1.3. /Filtration lemma/

(i) The following is true for any formula $A\in\Gamma$ and $x\in W$: $x\Vdash_v A$ iff $|x|\Vdash_{v'} A$.

(ii) The set $|W|$ has at most 2^n elements, where n is the number of the elements of Γ.

The proof of (i) is the same as in the standard modal logic /see [Seg 71]/ and can be done by induction on the complexity of A. Conditions (FR1) and (FR2) are used when A is in the form [R]B.

For (ii) let f be a function from $|W|$ to the set $\mathbf{P}(\Gamma)$ of all subsets of Γ defined as follows: $f(|x|)=\{B\in\Gamma/x\Vdash_v B\}$. It is easy to see that f is 1-1-function from $|W|$ into $\mathbf{P}(\Gamma)$. Since $\mathbf{P}(\Gamma)$ has 2^n elements then $|W|$ has no more than 2^n elements. ■

Theorem 1.4. /Filtration of SIM-1/

For any model $M=((W,\leq, \Sigma, N), v)$ over a similarity structure $\underline{W}$ and formula A' there exist a finite set Γ of formulas, containing A' and closed under sub formulas and a filtration $M'=((|W|, \leq', \Sigma', N'), v')$ of M through Γ.

Proof. Let A' be a given formula and define Γ to be the smallest set of formulas, containing A', $\langle\Sigma\rangle 1$, $\langle N\rangle 1$, closed under sub formulas and satisfying the following closure condition

(γ) For any formula A, if one of the formulas $[\Sigma]A$, $[N]A$, $[\leq]A$, $[\geq]A$ is in Γ, then the others are also in Γ.

Obviously Γ is a finite set of formulas. Define $|W|$ and v' as in the definition of filtration and the relations Σ', N', $\leq'$ and $\geq'$ as follows:

(1) $|x|\leq'|y|$ iff $(\forall[\leq]A\in\Gamma)(x\Vdash_v [\leq]A \rightarrow y\Vdash_v [\leq]A)$ &

$(y\Vdash_v [\geq]A \rightarrow x\Vdash_v [\geq]A)$ &

$(y\Vdash_v [\Sigma]A \rightarrow x\Vdash_v [\Sigma]A)$ &

$(x\Vdash_v [N]A \rightarrow y\Vdash_v [N]A)$ &

$(x\Vdash_v \langle\Sigma\rangle 1 \rightarrow y\Vdash_v \langle\Sigma\rangle 1)$ &

$(y\Vdash_v \langle N\rangle 1 \rightarrow x\Vdash_v \langle N\rangle 1)$,

(2) $|x|\geq'|y|$ iff $|y|\leq'|x|$,

(3) $|x|\Sigma'|y|$ iff $(\forall[\Sigma]A\in\Gamma)(x\Vdash_v [\Sigma]A \rightarrow y\Vdash_v [\leq]A)$ &

$(y\Vdash_v [\Sigma]A \rightarrow x\Vdash_v [\leq]A)$ &

$x\Vdash_v \langle\Sigma\rangle 1$ & $y\Vdash_v \langle\Sigma\rangle 1$,

(4) $|x|N'|y|$ iff $(\forall[N]A\in\Gamma)(x\Vdash_v [N]A \rightarrow y\Vdash_v [\geq]A)$ &

$(y\Vdash_v [N]A \rightarrow x\Vdash_v [\geq]A)$ &

$x\Vdash_v \langle N\rangle 1$ & $y\Vdash_v \langle N\rangle 1$.

The proof that the model $M'=(|W|,\Sigma',N',\leq',\geq',v')$ is a filtration of M through Γ follows from the following lemmas.

Lemma 1.5.

(i) If $x\leq y$ then $|x|\leq'|y|$,

(ii) If $|x|\leq'|y|$ then $(\forall[\leq]\forall\in\Gamma)(x\Vdash_{v}[\leq]A \rightarrow y\Vdash_{v}A)$,

(iii) S1 and S2 are satisfied.

Proof. (i) The proof follows from the following easy to prove implications:

$x\leq y$ & $x\Vdash_{v}[\leq]A$ & $y\leq z \rightarrow z\Vdash_{v}A$,

$x\leq y$ & $y\Vdash_{v}[\geq]A$ & $x\geq z \rightarrow z\Vdash_{v}A$,

$x\leq y$ & $x\Vdash_{v}[N]A$ & $yNz \rightarrow z\Vdash_{v}A$,

$x\leq y$ & $y\Vdash_{v}[\Sigma]A$ & $x\Sigma z \rightarrow z\Vdash_{v}A$,

$x\leq y$ & $x\Vdash_{v}\langle\Sigma\rangle 1 \rightarrow y\Vdash_{v}\langle\Sigma\rangle 1$,

$x\leq y$ & $y\Vdash_{v}\langle N\rangle 1 \rightarrow x\Vdash_{v}\langle N\rangle 1$.

(ii) Suppose $|x|\leq'|y|$, $[\leq]A\in\Gamma$ and $x\Vdash_{v}[\leq]A$. Then by (1) we get $y\Vdash_{v}[\leq]A$ and since $y\leq y$ we obtain $y\Vdash_{v}A$.

(iii) Conditions S1 and S2 follow directly from (1).■

Lemma 1.6.

(i) If $x\geq y$ then $|x|\geq'|y|$,

(ii) If $|x|\geq'|y|$ then $(\forall[\geq]A\in\Gamma)(x\Vdash_{v}[\geq]A \rightarrow y\Vdash_{v}A)$.

Proof. (i) Suppose $x\geq y$. Then we have $y\leq x$ and by lemma 1.5(i) we obtain $|y|\leq'|x|$. By (2) we have $|x|\geq'|y|$.

(ii) Suppose $|x|\geq'|y|$, $[\geq]A\in\Gamma$ and $x\Vdash_{v}[\geq]A$. Then by (γ) we have $[\leq]A\in\Gamma$ and by (2) - $|y|\leq'|x|$. Thus by (1) we get $y\Vdash_{v}[\geq]A$ and by $y\geq y$ - $y\Vdash_{v}A$.■

Lemma 1.7.

(i) If $x\Sigma y$ then $|x|\Sigma'|y|$,

(ii) If $|x|\Sigma'|y|$ then $(\forall[\Sigma]A\in\Gamma)(x\Vdash_{v}[\Sigma]A \rightarrow y\Vdash_{v}A)$,

(iii) Conditions S3-S6 are satisfied.

Proof. (i) The proof follows from the following easy to prove implications:

$x\Sigma y$ & $y\leq z$ & $x\Vdash_{v}[\Sigma]A \rightarrow z\Vdash_{v}A$,

$x\Sigma y$ & $x\leq z$ & $y\Vdash_{v}[\Sigma]A \rightarrow z\Vdash_{v}A$,

$x\Sigma y \rightarrow x\Vdash_{v}\langle\Sigma\rangle 1$ & $y\Vdash_{v}\langle\Sigma\rangle 1$.

(ii) Suppose $|x|\Sigma'|y|$, $[\Sigma]A\in\Gamma$ and $x\Vdash_{v}[\Sigma]A$. By (3) we get $y\Vdash_{v}[\leq]A$ and by $y\leq y$ we obtain $y\Vdash_{v}A$.

(iii) To proof S3 suppose $|x|\Sigma'|y|$. Then by (3) we obviously have $|y|\Sigma'|x|$.

For S4 suppose $|x|\Sigma'|y|$ and proceed to show $|x|\Sigma'|x|$. This means by (3) that: $(\forall[\Sigma]A\in\Gamma)(x\Vdash_{v}[\Sigma]A \rightarrow x\Vdash_{v}[\leq]A)$ & $x\Vdash_{v}\langle\Sigma\rangle 1$.

First from $|x|\Sigma'|y|$ we get $x\Vdash_{v}\langle\Sigma\rangle 1$, so $\exists z$ such that $x\Sigma z$ and by S3 - $x\Sigma x$.

Suppose $[\Sigma]A\in\Gamma$, $x\Vdash_{v}[\Sigma]A$ and $x\leq z$. Then by $x\Sigma x$ and S5 we obtain $x\Sigma z$ and consequently $x\Vdash_{v}A$, which has to be proved.

In a similar way we can verify S5.

For S6 suppose $|x|\bar{\Sigma}'|x|$. Then we have $x\bar{\Sigma}x$ and and by S6 we obtain $x\leq y$. Then by lemma 1.5.(i) we obtain $|x|\leq'|y|$. ■

Lemma 1.8.

(i) If xNy then $|x|N'|y|$,

(ii) If $|x|N'|y|$ then $(\forall[N]A\in\Gamma)(x\Vdash_{v}[N]A \rightarrow y\Vdash_{v}A)$,

(iii) Conditions S7-S11 are true.

The proof of this lemma is similar to the proof of lemma 1.7.

Theorem 1.9. /Finite completeness theorem for SIM-1/

The following conditions are equivalent for any formula A of SIM-1:

(i) A is a theorem of SIM-1,

(ii) A is true in all finite similarity structures.

Proof. (i)⟶(ii) is obvious.

(ii)⟶(i). Suppose A is not a theorem of Sim. Then by the completeness theorem of Sim there exist a similarity structure $\underline{W}$, a valuation v in $\underline{W}$ and x∈W such that $x \nVdash_{v} A$. By theorem 1.4 there exist a finite set Γ, containing A and closed under sub formulas, and a filtration M'=((|W|,≤', Σ',N'), v') of the model M=($\underline{W}$,v) through Γ. Then by the filtration lemma |W| is a finite set and $|x| \nVdash_{v'} A$. So A is not true in the finite similarity structure (|W|, ≤', Σ',N'). This completes the proof of theorem.■

Corollary 1.10.

The logic SIM-1 possesses the finite model property and is decidable.

In section 1 ch. 2.2 we introduced the notion of ρ-structure for ρ⊆{≤, Σ, N}. Let ρ be given. We can axiomatize the modal logic SIM-1(ρ) of ρ-structures in the following way. The language Lρ of this logic is a restriction of the language of SIM-1 by omitting the modal operations, which do not correspond to the relations of ρ. The axioms of SIM-1(ρ) can be obtained as follows. First add to the axioms of SIM-1 the following two axioms, which corresponds to the conditions ΣN1 and ΣN2 from lemma 1.5 of ch. 2.2

(AΣN1) ([Σ]A⇒[Σ]¬[Σ]B)v[U]Bv[U]([N]B⇒A),

(AΣN2) ([N]A⇒[N]¬[N]B)v[U]Bv[U]([Σ]B⇒A),

Then take from the axioms of SIM-1 only those axioms and rules of inference, which contain modalities from Lρ. Then we can proof the following theorem.

Theorem 1.11. /Completeness theorem for SIM-1(ρ)/

Let ρ⊆{≤, Σ, N} be given. Then the following conditions are true for any formula A of Lρ:

(i) A is a theorem of SIM-1(ρ),

(ii) A is true in any ρ-structure,

(iii) A is true in any ρ structure over a P-system,

(iv) A is true in any finite ρ-structure.

Proof. The proof of this theorem is similar to the proof of theorem 1.1 with combination with theorem 1.4. The equivalence (ii)⟷(iii) follow from theorem 1.8 of ch. 2.2. We left the details to the reader. ■

Let us note that the logic SIM-1(≤) is the well-known logic S4. So we have the following

Corollary 1.12.

The modal logic S4 is complete in the class of all ≤-structures over P-systems.

This corollary presents an information semantics for S4.

As a corollary of lemma 1.8 of ch. 2.2 we have

Corollary 1.13.

Let to the logic SIM-1 add the formulas (AΣN1) and (AΣN2) as new axioms. Then the logic SIM-1 is a conservative extension of the logic SIM-1(ρ) for any ρ⊆{≤, Σ, N}.

2. SIM-2 - a modal logic for similarity relations in attribute systems

connected with A-systems.

The language of SIM-2 is an extension of the language of SIM-1 by new modalities, denoted by $[<]$, $[>]$, $[\sigma]$, $[\nu]$. These modalities correspond to the relations $<$, $>$ – the converse of $<$, σ and ν in A-systems.

The standard semantics of SIM-2 is a Kripke semantics of the language of SIM-2 in the class of bi-similarity structures.

Now we shall give an axiomatic system of SIM-2.

Axiomatization of SIM-2

Axiom schemes:

(Bool) All Boolean tautologies,

(K) $[R](A\Rightarrow B)\Rightarrow([R]A\Rightarrow[R]B)$, $R\in\{\leq,\geq,\Sigma,N,<,>,\sigma,\nu,U\}$

A0. $\langle\leq\rangle[\geq]A\Rightarrow A$, $\langle\geq\rangle[\leq]A\Rightarrow A$, $\langle<\rangle[>]A\Rightarrow A$, $\langle<\rangle[>]A\Rightarrow A$ $[U]A\Rightarrow A$, $\langle U\rangle[U]A\Rightarrow A$, $[U]A\Rightarrow[U][U]A$, $[U]A\Rightarrow[R]A$, $R\in\{\leq,\geq,\Sigma,N,<,>,\sigma,\nu,U\}$

A1. $[\leq]A\Rightarrow A$,

A2. $[\leq]A\Rightarrow[\leq][\leq]A$,

A3. $\langle\Sigma\rangle[\Sigma]A\Rightarrow A$,

A4. $\langle\Sigma\rangle 1\Rightarrow([\Sigma]A\Rightarrow A)$,

A5. $[\Sigma]A\Rightarrow[\Sigma][\leq]A$,

A6. $[\leq]A\Rightarrow([U]A\vee([\Sigma]B\Rightarrow B))$,

A7. $\langle N\rangle[N]A\Rightarrow A$,

A8. $\langle N\rangle 1\Rightarrow([N]A\Rightarrow A)$,

A9. $[N]A\Rightarrow[N][\geq]A$,

A10. $[\geq]A\Rightarrow([U]A\vee([N]B\Rightarrow B))$,

A11. $[\leq]A\wedge[\Sigma]B\Rightarrow([U]B\vee[U]([N]B\Rightarrow A))$,

A12. $[<]A\Rightarrow A$,

A13. $[<]A\Rightarrow[\leq][<]A$,

A14. $[<]A\Rightarrow[<][\leq]$

A15. $\langle\sigma\rangle[\sigma]A\Rightarrow A$,

A16. $\langle\sigma\rangle 1\Rightarrow([\sigma]A\Rightarrow A)$,

A17. $[\sigma]A\Rightarrow[\sigma][\leq]A$,

A18. $[<]A\Rightarrow([U]A\vee([\sigma]B\Rightarrow B))$,

A19. $[\sigma]A\Rightarrow[\sigma][<]A$,

A20. $[<]A\wedge[\sigma]B\Rightarrow([U]B\vee[U]([N]B\Rightarrow A))$,

A21. $\langle\nu\rangle[\nu]A\Rightarrow A$,

A22. $\langle\nu\rangle 1\Rightarrow([\nu]A\Rightarrow A)$,

A23. $[\nu]A\Rightarrow[\nu][\geq]A$,

A24. $[>]A\Rightarrow([U]A\vee([\nu]B\Rightarrow B))$,

A25. $[N]A\Rightarrow[\nu][>]A$,

A26. $[<]A\wedge[\Sigma]B\Rightarrow([U]B\vee[U]([\nu]B\Rightarrow A))$.

Rules of inference: modus ponens (MP) $A, A\Rightarrow B/B$,
necessitation (N [U]) $A/[U]A$,

Note that the necessitation rule for $[R]$, $R\in\{\leq,\geq,\Sigma,N,<,>,\sigma,\nu,U\}$ follows from (N [U]) and axiom $[U]A\Rightarrow[R]A$.

The logic SIM-2 is the smallest set of formulas, containing all axiom schemes and closed under the rules of inference.

Let us note that the axioms A1-A26 are modal translations of the conditions S1-S26. Obviously SIM-2 is an extension of SIM-1.

Theorem 2.1. /Completeness theorem for SIM-2/

For any formula A of SIM-2 the following conditions are equivalent:

(i) A is a theorem of SIM-2,
(ii) A is true in all bi-similarity structures,
(iii) A is true in all bi-similarity structures over A-systems.

Proof. (i)⟶(ii) in a standard way by showing the validity of all axioms and that the rules preserve validity.

(ii)⟷(iii) - by the characterization theorem for bi- similarity structures - theorem 3.6 ch. 2.2.

(ii)⟶(i). The proof can be done by the standard canonical-model-construction as in theorem 1.1. ■

Theorem 2.2.

The logic SIM-2 is a conservative extension of SIM-1.

Proof. We have to prove the following: for any formula A in the language of SIM-1, if A is a theorem of SIM-2 then A is a theorem of SIM-1.

Let A be a formula in the language of SIM-1. Suppose, for the sake of contradiction, that A is a theorem of SIM-2 but not of SIM-1. Then, by the completeness theorem for SIM-1, there exists a similarity structure $\underline{W}=(W, \leq, \Sigma, N)$ in which A is not true. Define in W $<=\leq$, $\sigma=\Sigma$ and $\nu=N$. It can be verified easily that the system $\underline{W}'=(W, \leq, \Sigma, N, <, \sigma, \nu)$ is a bi-similarity structure. Hence, since A do not contain $<$, σ and ν, then A is not true in $\underline{W}'$. But by the assumption A is true in $\underline{W}'$ - a contradiction. This proves the theorem. ■

Now we shall prove, using filtration, that the logic SIM-2 possesses finite model property and that it is decidable.

Theorem 2.3. /Filtration theorem for SIM-2/

For any model $M=((W, \leq, \Sigma, N, <, \sigma, \nu), v)$ over a bi-similarity structure $\underline{W}$ and formula A' there exist a finite set Γ of formulas, containing A' and closed under sub formulas and a and a model $M'=((|W|, \leq', \Sigma', N', <', \sigma', \nu'), v')$ over finite bi-similarity structure, which is a filtration of M through Γ.

Proof. Let the model M and the formula A' be given. Let Γ be the smallest set of formulas containing A', $\langle\Sigma\rangle 1$, $\langle N\rangle 1$, $\langle\sigma\rangle 1$, $\langle\nu\rangle 1$, closed under subformulas and satisfying the following closure condition

(γ) For any formula A, if one of the formulas $[\Sigma]A$, $[N]A$, $[\leq]A$, $[\geq]A$, $[\sigma]A$, $[\nu]A$, $[<]A$, $[>]A$ is in Γ, then the others are also in Γ.

Obviously Γ is a finite set of formulas. Define $|W|$ and v' as in the definition of filtration. For $|x|, |y| \in |W|$ define:

(1) $|x| \leq' |y|$ iff
$$(\forall[\leq]A\in\Gamma)(x \Vdash_v [\leq]A \rightarrow y \Vdash_v [\leq]A)\ \&$$
$$(y \Vdash_v [\geq]A \rightarrow x \Vdash_v [\geq]A)\ \&$$
$$(y \Vdash_v [\Sigma]A \rightarrow x \Vdash_v [\Sigma]A)\ \&$$
$$(x \Vdash_v [N]A \rightarrow y \Vdash_v [N]A)\ \&$$
$$(x \Vdash_v [<]A \rightarrow y \Vdash_v [<]A)\ \&$$
$$(y \Vdash_v [>]A \rightarrow x \Vdash_v [>]A)\ \&$$
$$(y \Vdash_v [\sigma]A \rightarrow x \Vdash_v [\sigma]A)\ \&$$
$$(x \Vdash_v [\nu]A \rightarrow y \Vdash_v [\nu]A)\ \&$$
$$(x \Vdash_v \langle\Sigma\rangle 1 \rightarrow y \Vdash_v \langle\Sigma\rangle 1)\ \&$$
$$(y \Vdash_v \langle N\rangle 1 \rightarrow x \Vdash_v \langle N\rangle 1)\ \&$$
$$(x \Vdash_v \langle\sigma\rangle 1 \rightarrow y \Vdash_v \langle\sigma\rangle 1)\ \&$$
$$(y \Vdash_v \langle\nu\rangle 1 \rightarrow y \Vdash_v \langle\nu\rangle 1),$$

(2) $|x| \geq' |y|$ iff $|y| \leq' |x|$,

(3) $|x| \Sigma' |y|$ iff $(\forall[\Sigma]A\in\Gamma)(x \Vdash_v [\Sigma]A \rightarrow y \Vdash_v [\leq]A)\ \&$

$$(y \Vdash_v [\Sigma]A \rightarrow x \Vdash_v [\leq]A)\ \&$$
$$x \Vdash_v \langle\Sigma\rangle 1\ \&\ y \Vdash_v \langle\Sigma\rangle 1,$$

(4) $|x|N'|y|$ iff $(\forall[N]A\in\Gamma)(x \Vdash_v [N]A \rightarrow y \Vdash_v [\geq]A)\ \&$
$$(y \Vdash_v [N]A \rightarrow x \Vdash_v [\geq]A)\ \&$$
$$x \Vdash_v \langle N\rangle 1\ \&\ y \Vdash_v \langle N\rangle 1,$$

(5) $|x|<'|y|$ iff $(\forall[<]A\in\Gamma)(x \Vdash_v [<]A \rightarrow y \Vdash_v [\leq]A)\ \&$
$$(y \Vdash_v [>]A \rightarrow y \Vdash_v [\geq]A)\ \&$$
$$(y \Vdash_v [\Sigma]A \rightarrow x \Vdash_v [\sigma]A)\ \&$$
$$(x \Vdash_v [N]A \rightarrow y \Vdash_v [\nu]A)\ \&$$
$$(x \Vdash_v \langle\sigma\rangle 1 \rightarrow y \Vdash_v \langle\Sigma\rangle 1)\ \&$$
$$(y \Vdash_v \langle\nu\rangle 1 \rightarrow x \Vdash_v \langle N\rangle 1)\ \&$$

(6) $|x|>'|y|$ iff $|y|<'|x|$,

(7) $|x|\sigma'|y|$ iff $(\forall[\sigma]A\in\Gamma)(x \Vdash_v [\sigma]A \rightarrow y \Vdash_v [\leq]A)\ \&$
$$(y \Vdash_v [\sigma]A \rightarrow x \Vdash_v [\leq]A)\ \&$$
$$(x \Vdash_v [\Sigma]A \rightarrow y \Vdash_v [<]A)\ \&$$
$$(y \Vdash_v [\Sigma]A \rightarrow x \Vdash_v [<]A)\ \&$$
$$x \Vdash_v \langle\sigma\rangle 1\ \&\ y \Vdash_v \langle\sigma\rangle 1,$$

(8) $|x|\nu'|y|$ iff $(\forall[\nu]A\in\Gamma)(x \Vdash_v [\nu]A \rightarrow y \Vdash_v [\geq]A)\ \&$
$$(y \Vdash_v [\nu]A \rightarrow x \Vdash_v [\geq]A)\ \&$$
$$(x \Vdash_v [N]A \rightarrow y \Vdash_v [>]A)\ \&$$
$$(y \Vdash_v [N]A \rightarrow x \Vdash_v [>]A)\ \&$$
$$x \Vdash_v \langle\nu\rangle 1\ \&\ x \Vdash_v \langle\nu\rangle 1,$$

The required model is $M'=((|W|, \leq', \Sigma', N', <', \sigma', \nu'), v')$. The proof that M' is a filtration of the model M through Γ is similar to the corresponding proof for SIM-1 and therefore is left to the reader. Let us note that the above filtration is an extension of the filtration of SIM-1.■

Let us mention that the filtration theorem of SIM-1 can be obtained as a corollary of Theorem 2.2 and theorem 2.3.

Corollary 2.4.
The logic SIM-2 has finite model property and is decidable.

3. Modal logics for similarity relations in single-valued attribute systems

In this section we shall consider several modal logics complete in single-valued A-systems.

The first system is SIM-1. Since the language of SIM-1 is a part of the language of SIM-2 we can interpret SIM-1 also in bi- similarity structures using only the relations $\leq$, Σ and N. We have the following completeness theorem for SIM-1.

Theorem 3.1. /Completeness theorem of SIM-1 in bi-similarity structures over single-valued A-systems/
The following conditions are equivalent for any formula A of SIM-1:

(i) A is a theorem of SIM-1,

(ii) A is true in all bi-similarity structures over single- valued A-systems.

Proof. The implication (i)⟶(ii) is trivial because bi-similarity structures satisfy the conditions S1-S11, which are needed to verify the axioms of SIM-1.

For the converse implication (ii)⟶(i) we shall proceed contrapositionally: supposing that A is not a theorem of SIM-1. We shall show that A is falsified in some bi-similarity structure over a single-valued A system. Now, let A be a non-theorem of SIM- 1. By the completeness theorem of SIM-1 A is falsified in some similarity structure $\underline{W}=(W, \leq, \Sigma, N)$. Then by theorem 4.2 ch. 2.2there exists a single-valued A-system S such that $\underline{W}$ is a reduct of the bi-similarity structure over S, which proves the theorem.■

Corollary 3.2.

Let $\rho\subseteq\{\leq, \Sigma, N\}$. Then the following conditions are equivalent for any formula A of the logic SIM-1(ρ):

(i) A is a theorem of SIM-1(ρ),

(ii) A is true in all bi-similarity structures over a single- valued A-systems.

Corollary 3.3./ A completeness theorem for the modal logic S4 in single valued A-systems/

The modal logic S4 is complete in the class of all bi- similarity structures over a single-valued A-systems.

Let L+ be a modal language containing only the positive modal operations [≤], [≥], [Σ], [<], [>], [σ] and [U]. The standard interpretation of this language will be in the class of all strong positive bi-similarity structures and in the class of all bi- similarity structures over single-valued A-systems (the definitions are given in ch. 2.2). The corresponding logic will be denoted by SIM-2+. The axioms and rules of SIM-2+ are the following:

Axioms of SIM-2+

(Bool), (K) and (A0) for $R\in\{\leq,\geq,\Sigma,<,>,\sigma,U\}$, (A1)-(A6), (A12)-(A19) (A27) $[\leq]A\Rightarrow[\sigma]A$,

(A28) $[<]A\Rightarrow[\Sigma]A$.

Rules of inference: modus ponens (MP)$A, A\Rightarrow B/B$,
necessitation (N[U]) $A/[U]A$.

Theorem 3.4./Completeness theorem for SIM-2+/

The following conditions are equivalent for any formula A of SIM-2+:

(i) A is a theorem of SIM-2+,

(ii) A is true in all strong positive bi-similarity structures,

(iii) A is true in all bi-similarity structures over single- valued A-systems.

Proof. The equivalence (i)⟷(ii) can be proved by the canonical construction as the corresponding proof of the completeness theorem for SIM-1. The equivalence (ii)⟷(iii) follows from the characterization theorem for strong positive bi- similarity structures /theorem 4.6. ch. 2.2./.■

Theorem 3.5. /Filtration theorem for SIM-2+/

For any model $M=((W, \leq, \Sigma, <, \sigma), v)$ over a strong bi- similarity structure $\underline{W}$ and formula A' there exist a finite set Γ of formulas, containing A' and closed under sub formulas and a filtration $M'=((|W|, \leq', \Sigma', <', \sigma'), v')$ of M through Γ.

Proof. Let A' and M be given. Define Γ as in theorem 11.3. Define |W| and v' as in the definition of filtration. Then for any $|x|$, $|y| \in |W|$ define:

(1) $$|x| \le' |y| \text{ iff } (\forall [\le]A \in \Gamma)(x \Vdash_v [\le]A \rightarrow y \Vdash_v [\le]A)\ \&$$
$$(y \Vdash_v [\ge]A \rightarrow x \Vdash_v [\ge]A)\ \&$$
$$(y \Vdash_v [\Sigma]A \rightarrow x \Vdash_v [\Sigma]A)\ \&$$
$$(x \Vdash_v [<]A \rightarrow y \Vdash_v [<]A)\ \&$$
$$(y \Vdash_v [>]A \rightarrow x \Vdash_v [>]A)\ \&$$
$$(y \Vdash_v [\sigma]A \rightarrow x \Vdash_v [\sigma]A)\ \&$$
$$(x \Vdash_v \langle\Sigma\rangle 1 \rightarrow y \Vdash_v \langle\Sigma\rangle 1)\ \&$$
$$(x \Vdash_v \langle\sigma\rangle 1 \rightarrow y \Vdash_v \langle\sigma\rangle 1),$$

(2) $|x| \ge' |y|$ iff $|y| \le' |x|$,

(3) $$|x| <' |y| \text{ iff } (\forall [<]A \in \Gamma)(x \Vdash_v [<]A \rightarrow y \Vdash_v [\le]A)\ \&$$
$$(y \Vdash_v [>]A \rightarrow y \Vdash_v [\ge]A)\ \&$$
$$(y \Vdash_v [\Sigma]A \rightarrow x \Vdash_v [\sigma]A)\ \&$$
$$(x \Vdash_v \langle\sigma\rangle 1 \rightarrow y \Vdash_v \langle\Sigma\rangle 1),$$

(4) $|x| >' |y|$ iff $|y| <' |x|$,

(5) $$|x| \Sigma' |y| \text{ iff } |x| <' |y|\ \&\ |x| >' |y|\ \&$$
$$(\forall [\Sigma]A \in \Gamma)(x \Vdash_v [\Sigma]A \rightarrow y \Vdash_v [\le]A)\ \&$$
$$(y \Vdash_v [\Sigma]A \rightarrow x \Vdash_v [\le]A)\ \&$$
$$x \Vdash_v \langle\Sigma\rangle 1\ \&\ y \Vdash_v \langle\Sigma\rangle 1,$$

(6) $$|x| \sigma' |y| \text{ iff } |x| \le' |y|\ \&\ |x| \ge' |y|\ \&$$
$$(\forall [\sigma]A \in \Gamma)(x \Vdash_v [\sigma]A \rightarrow y \Vdash_v [\le]A)\ \&$$
$$(y \Vdash_v [\sigma]A \rightarrow x \Vdash_v [\le]A)\ \&$$
$$(x \Vdash_v [\Sigma]A \rightarrow y \Vdash_v [<]A)\ \&$$
$$(y \Vdash_v [\Sigma]A \rightarrow x \Vdash_v [<]A)\ \&$$
$$x \Vdash_v \langle\sigma\rangle 1\ \&\ y \Vdash_v \langle\sigma\rangle 1,$$

The required model is $M' = ((|W|, \le', \Sigma', <', \sigma'), v')$. The verification that M' is a filtration of the model M through Γ and that $(|W|, \le', \Sigma', <', \sigma')$ is a strong positive bi-similarity structure is similar to the corresponding proof for SIM-1 /theorem 1.4/. The details are left to the reader. ■

Corollary 3.6.
The logic SIM-2+ has finite model property and is decidable.

CHAPTER 2.4

MODAL LOGICS FOR INDISCERNIBILITY RELATIONS IN INFORMATION SYSTEMS

Overview. In this chapter we introduce several modal logics for indiscernibility relations relations in information systems: IND-1 as an extension of SIM-1 for property systems, IND-2 as an extension of SIM-2 with modality for strong indiscernibility relation for attribute systems and IND-3 for strong and weak indiscernibility relations in attribute systems. The main results are the completeness theorems and decidability for the corresponding modal logics.

Introduction

This chapter is devoted to modal logics for information systems containing modalities corresponding to indiscernibility relations. A modal logic for indiscernibility relations have been studied for the first time by Orlowska [Or 85a] but not in combination with other modalities.

The main aim of this chapter is to extend the logics SIM-1 and SIM-2 logics with modalities corresponding to indiscernibility relations.

In sec. 1 we introduce the logic IND-1 as extension of SIM-1 with the modality [≡] corresponding to the indiscernibility relation ≡ in P-systems. In the semantical structures for SIM-1 the relation ≡ is definable by the informational inclusion ≤ by the equivalence

(≡) $x \equiv y$ iff $x \leq y$ and $y \leq x$.

This equivalence is not modally definable which presents serious difficulties in the axiomatization of IND-2. We do this by an application of the copying construction. Decidability of IND-1 is proved by the method of filtration.

In sec. 2 we introduce the logic IND-2 as an extension of SIM-2 with modality for strong indiscernibility relation in A-systems. This relation is defined by the relation of the strong information inclusion ≤ in A-systems by the same equivalence (≡) as in IND-1. Since IND-2 is an extension also of IND-1, the completeness theorem and the decidability for IND-2 are obtained by an extension of the proofs and constructions for IND-1.

In sec. 3. we study a modal logic IND-3 containing both weak and strong indiscernibility relations in A-systems - ≅ and ≡. Let us note that the weak indiscernibility relation for A systems is not definable by the other relations in the semantical structures for SIM-2. So to extend SIM-2 with both ≅ and ≡ is a difficult problem which is left open. IND-3 is an axiomatization of the modalities [≅], [≡] together with weak and strong positive similarities - [Σ] and [σ]. For that purpose a relevant abstract characterization theorem for these four relations is given, which then implies an easy axiomatization. The decidability of IND-3 is also proved. Under a different terminology the results of sec. 3 are included in [Vak 91].

1. IND-1 - a modal logic for indiscernibility relation in property systems

Let S=(Ob, Pr, f) be a P-system. As we mentioned in ch. 2.2 the indiscernibility relation $\equiv_S$ in S is defined as follows:

$x \equiv_S y$ iff $f(x)=f(y)$

So x and y are indiscernible in S if they have equal sets of properties. Let us note that $\equiv_S$ is definable by the relation of informational inclusion $\leq_S$:

(≡) $x \equiv_S y$ iff $x \leq_S y$ & $y \leq_S x$

Since we have an abstract characterization of the relation ≤ together with Σ and N in the definition of similarity structure, it is natural to study indiscernibility relation ≡ together with the relations ≤, Σ and N using the above definition (≡). So the standard semantical structure for the indiscernibility relation, called ind-1-structure, is the following - $\underline{S}$=(W, ≤, Σ, N, ≡), where (W, ≤, Σ, N) is a similarity structure and ≡ satisfies the equivalence

(≡) $x \equiv y$ iff $x \leq y$ and $y \leq x$.

The corresponding modal logic will be an extension of the logic SIM-1 with an extra modality [≡] with standard semantics in ind-1- structures by the relation ≡. The obtained logic is denoted by IND-1. Let us note that in the language of IND-1 the equivalence (≡) is not modally definable, which will present some difficulties in the way of axiomatization of IND-1. To overcome these difficulties we shall introduce more general, nonstandard semantics for IND-1, in which IND-1 is easy to axiomatize, and then, using the copying method, we shall prove that the obtained axiomatization is complete also with respect to its standard semantics.

The general semantics for IND-1 is obtained in the following way. Extend the signature of similarity structure with one more relation, denoted by $\equiv$ and add to the axioms of similarity structure enough number of modally definable conditions for $\equiv$, $\leq$, Σ and N, which follow from the equivalence ($\equiv$) ("enough" here means to guarantee then a successful copying construction). The following definition appears to be appropriate.

By a generalized ind-1-structure we mean any relational system $\underline{W}=(W, \leq, \Sigma, N, \equiv)$, satisfying the following conditions:

(i) $(W, \leq, \Sigma, N)$ is a similarity structure, i.e. satisfies the axioms S1-S11, namely

S1. $x\leq x$,
S2. $x\leq y \ \&\ y\leq z \longrightarrow x\leq z$,
S3. $x\Sigma y \longrightarrow y\Sigma x$,
S4. $x\Sigma y \longrightarrow x\Sigma x$,
S5. $x\Sigma y \ \&\ y\leq z \longrightarrow x\Sigma z$,
S6. $x\Sigma x$ or $x\leq y$,
S7. $xNy \longrightarrow yNx$,
S8. $xNy \longrightarrow xNx$,
S9. $x\leq y \ \&\ yNz \longrightarrow xNz$,
S10. yNy or $x\leq y$,
S11. $x\Sigma z$ or yNz or $x\leq y$,

(ii) the relation $\equiv$ satisfies the following first-order axioms

($\equiv$1) $x\bar{\Sigma}x \ \&\ y\leq x \longrightarrow x\equiv y$,
($\equiv$2) $x\bar{N}x \ \&\ x\leq y \longrightarrow x\equiv y$,
($\equiv$3) $x\bar{\Sigma}z \ \&\ y\bar{N}z \ \&\ y\leq x \longrightarrow x\equiv y$,
($\equiv$4) $x\equiv x$,
($\equiv$5) $x\equiv y \longrightarrow y\equiv x$,
($\equiv$6) $x\equiv y \ \&\ y\equiv z \longrightarrow x\equiv z$,
($\equiv$7) $x\equiv y \longrightarrow x\leq y$.

Let us note that all of the conditions ($\equiv$1)-($\equiv$7) are true in any ind-1-structure and if a generalized ind-1-structure $\underline{W}$ satisfies the condition

(*) $x\leq y \ \&\ \ y\leq x \longrightarrow x\equiv y$

then it is an ind-1-structure.

Let $\underline{W}$ be a generalized ind-1-structure. Define the following equivalence relation:

$x\cong y$ iff $x\leq y \ \&\ y\leq x$.

The equivalence classes with respect to $\equiv$ and $\cong$ will be called $\equiv$-clusters and $\cong$-clusters respectively. The clusters generated by an element x will be denoted by $\equiv(x)$ and $\cong(x)$ respectively.

By axiom ($\equiv$7) we obtain that

$x\equiv y \longrightarrow x\cong y$.

so each $\equiv$-cluster is contained in some $\cong$-cluster, or in other words, each $\cong$-cluster is an union of $\equiv$-clusters.

A $\cong$-cluster α is called normal if it contains only one $\equiv$-cluster, i.e. if α is itself a $\equiv$-cluster. Obviously a $\cong$-cluster α is normal if for any $x,y\in\alpha$ we have $x\equiv y$.

Lemma 1.1.

(i) If $x\bar{\Sigma}x$ and $y\bar{\Sigma}y$ then $\cong(x)=\cong(y)$ and $\cong(x)$ is a normal $\cong$-cluster.

(ii) If $x\bar{N}x$ and $y\bar{N}y$ then $\cong(x)=\cong(y)$ and $\cong(x)$ is a normal $\cong$-cluster.

(iii) Let $x\bar{\Sigma}z$ and $y\bar{N}z$ and α be a $\cong$-cluster containing x and y. Then α is a normal $\cong$-cluster.

Proof. (i) Suppose $x\bar{\Sigma}x$ and $y\bar{\Sigma}y$, then by axiom S6 we obtain $x\leq y$ and $y\leq x$ and

by axiom (≡1) - $x\equiv y$, which implies $x\cong y$ and $\cong(x)=\cong(y)$. Let $z\in\cong(x)$, then by axiom (≡1) we obtain $x\equiv z$, which shows that $\cong(x)$ is a normal ≅-cluster.
In a similar way one can prove (ii) and (iii).■

Proposition 1.2.
Let $\underline{W}=(W, \leq, \Sigma, N, \equiv)$ be a generalized ind-1-structure. Then there exists an ind-1-structure $\underline{W}'=(W', \leq', \Sigma', N', \equiv')$ and a copying I from $\underline{W}$ to $\underline{W}'$.

Proof. Let $I=\{1,2,\dots\}$ be the set of natural numbers. We will consider elements of I as functions over W as follows: for $x\in W$ and $f\in I$ define

$$f(x)=\begin{cases} x & \text{if} \cong(x) \text{ is anormal} \cong\text{-cluster} \\ (x,f) & \text{otherwise} \end{cases}$$

Obviously $f(x)=g(y)$ implies $x=y$.

Denote by $W'=\{f(x)/x\in W, f\in I\}$. Then the functions defined above are mappings from W to W', which obviously satisfy the conditions (I1) and (I2) of the definition of copying. It remains to define the relations ≤', Σ', N' and ≡' in W'. For $x,y\in W$ and $f,g\in I$

$f(x)\Sigma'g(y)$ iff $x\Sigma y$,

$f(x)N'g(y)$ iff xNy.

For the relation ≤' let « be a well ordering of all ≡-clusters in W. The definition of $f(x)\leq'g(y)$ will be in two cases.

Case 1: $\cong(x)$ is normal ≅-cluster or $\cong(y)$ is normal ≅-cluster or the relation $x\cong y$ does not hold. Then we put

$f(x)\leq'g(y)$ iff $x\leq y$.

Case 2: the opposite. Then $f(x)=(x,f)$, $g(y)=(y,g)$ and $\alpha=\cong(x)=\cong(y)$ is not a normal ≅-cluster. So the ≡-clusters $\equiv(x)$ and $\equiv(y)$ are contained in α and different from α. Define

$f(x)\leq'g(y)$ iff $f<g$ or $f=g$ & $\equiv(x) \ll \equiv(y)$,

The definition of ≡' is this:

$f(x)\equiv'g(y)$ iff $f(x)\leq'g(y)$ & $g(y)\leq'f(x)$.

First we shall show that the conditions (R1) and (R2) $R\in\{\leq,\Sigma,N,\equiv\}$of copying are satisfied:

(R1) $xRy\rightarrow(\forall f\in I)(\exists g\in I)f(x)R'g(y)$,

(R2) $f(x)R'g(y)\rightarrow xRy$.

The case $R=\Sigma,N$ is trivial.

For (≤1) suppose $x\leq y$ and let f be given. For the case 1 of the definition of ≤' g is arbitrary. For the case 2 take $g=f$ if $\equiv(x) \ll \equiv(y)$ and $g>f$ if not $\equiv(x) \ll \equiv(y)$. The condition (≤2) is obviously fulfilled.

For (≡1) suppose $x\equiv y$ and let f be given. We have to find $g\in I$ such that $f(x)\leq'g(y)$ and $g(y)\leq'f(x)$. From $x\equiv y$ by axiom (≡7) we get $x\cong y$, so $\cong(x)=\cong(y)=\alpha$.

Case 1: α is normal. Then any g will give $f(x)\leq'g(y)$ and $g(y)\leq'f(x)$.

Case 2: α is not normal. From $x\equiv y$ we get $\equiv(x)=\equiv(y)$ so we have $\equiv(x) \ll \equiv(y)$ and $\equiv(y) \ll \equiv(x)$. Then $g=f$ implies $f(x)\leq'g(y)$ and $g(y)\leq'f(x)$.

For the case (≡2) suppose $f(x)\leq'g(y)$ and $g(y)\leq'f(x)$. We have to show $x\equiv y$. By (≤1) we obtain $x\leq y$ and $y\leq x$, so $x\cong y$ and consequently $\cong(x)=\cong(y)=\alpha$ and $x,y\in\alpha$

Case 1: α is normal - then $x\equiv y$.

Case 2: α is not normal. Then from $f(x)\leq'g(y)$ and $g(y)\leq'f(x)$ we obtain

($f<g$ or $f=g$ and $\equiv(x) \ll \equiv(y)$) and

($g<f$ or $g=f$ and $\equiv(y) \ll \equiv(x)$.

From here we get $\equiv(x) \ll \equiv(y)$ and $\equiv(y) \ll \equiv(x)$. Since « is antisimmetric relation we obtain $\equiv(x)=\equiv(y)$, which implies $x\equiv y$.

Now it remains to show that $(W', \leq', \Sigma', N', \equiv')$ is an ind-1- structure. Obviously ≤' is reflexive relation and S1 is fulfilled.

For the axiom S2 suppose $f(x)\leq'g(y)$ and $g(y)\leq'h(z)$. We shall show $f(x)\leq'g(z)$. By (≤1) we have $x\leq y$ and $y\leq z$ which implies $x\leq z$. If $\cong(x)$ is normal or $\cong(z)$ is normal or the relation $x\cong z$ is not fulfilled then we have

$f(x)\le' g(z)$. Let we have $x\cong z$ and $\cong(x)=\cong(z)$ is not a normal $\cong$-cluster. From $x\cong z$ we get $z\le x$ which with $x\le y$ and $y\le z$ imply $x\cong y$ and $y\cong z$, so $\cong(x)=\cong(y)=\cong(z)=\alpha$ and α is not normal $\cong$-cluster. Then from $f(x)\le' g(y)$ and $g(y)\le' h(z)$ we obtain:

$(f<g$ or $f=g$ and $\equiv(x) \ll \equiv(y))$ and $(g<h$ or $g=h$ and $\equiv(y) \ll \equiv(z))$.

From here we get $(f<h$ or $f=h$ and $\equiv(x) \ll \equiv(z))$, which implies $f(x)\le' h(z)$.

The axioms S3, S4, S5, S7, S8 and S9 follow directly from the definitions for $\le'$, Σ' and N' and the corresponding axioms for $\le$, Σ and N.

For S6 suppose $f(x)\bar{\Sigma}f(x)$ and proceed to show that $f(x)\le' g(y)$.

From $f(x)\bar{\Sigma}f(x)$ we obtain $x\bar{\Sigma}x$ and by S6 - $x\le y$. By lemma 1.1 $\cong(x)$ is a normal $\cong$-cluster and by the definition of $\le'$ we obtain $f(x)\le g(y)$.

In the same way one can verify the axiom S10.

For S11 suppose $f(x)\bar{\Sigma}h(z)$ and $g(y)\bar{N}h(z)$. Then by (Σ1) and (N1) we get $x\bar{\Sigma}z$ and $y\bar{N}z$, which by S11 implies $x\le y$. If for x and y we have the case 1 from the definition of $\le'$ then we obtain directly $f(x)\le' g(y)$. Suppose now that we have the case 2: $x\cong y$ and $\alpha=\cong(x)=\cong(y)$ is not a normal $\cong$-cluster, such that $x,y\in\alpha$. But by lemma 1.1(ii) α should be normal. This contradiction shows that the case 2 is not possible.

It remains to show that W' is ind-1-structure. But this follows directly from the definition of $\equiv'$: $f(x)\equiv' g(y)$ iff $f(x)\le' g(y)$ & $g(y)\le' f(x)$.■

Corollary 1.3

The classes of generalized ind-1-structures and ind-1- structures determine one and the same logic, namely IND-1.

Now the axiomatization of IND-1 is easy - extend SIM-1 with axioms for [≡] corresponding to the conditions (≡1)-(≡7).

Axioms and rules for IND-1

(I) Axioms and rules for SIM-1

(II)

(Ax≡1) $[\le]([\equiv]B\Rightarrow([\Sigma]A\Rightarrow A))\vee B$,

(Ax≡2) $[\ge]([\equiv]B\Rightarrow([N]A\Rightarrow A))\vee B$,

(Ax≡3) $\blacksquare C\vee([\equiv]D\wedge[\Sigma]C\Rightarrow[\ge]([N]C\Rightarrow D))$,

(Ax≡4) $[\equiv]A\Rightarrow A$,

(Ax≡5) $A\vee[\equiv]\neg[\equiv]A$,

(Ax≡6) $[\equiv]A\Rightarrow[\equiv][\equiv]A$,

(Ax≡7) $[\le]A\Rightarrow[\equiv]A$.

Theorem 1.4 (Completeness theorem for IND-1)

The following conditions are equivalent for any formula A of IND-1:

(i) A is a theorem of IND-1,

(ii) A is true in all generalized ind-1-structures,

(iii) A is true in all ind-1-structures.

Proof. Note that (ii)⟷(iii) is just corollary 2.3. (i)⟶(ii) is trivial and (ii)⟶(i) can be proved by the canonical construction as in the proof of the completeness theorem for SIM- 1. Note that in this case the axioms for SIM-1 and (Ax≡1)-(Ax≡7) guarantee that the generated canonical model is a generalized ind- 1-structure.■

Now we shall proof that IND-1 admits filtration with respect to its generalized semantics.

Theorem 1.5.

IND-1 admits filtration with respect to its generalized semantics and hence is decidable.

Proof. Let $M=(W,\le,\Sigma,N,\equiv, v)$ be a model over a generalized ind- 1-structure and A' be a formula define the set Γ of formulas to be the smallest set of formulas which contains A', closed under subformulas and satisfying the following closure condition:

(γ) if one of the formulas $[\leq]B$, $[\Sigma]B$, $[N]B$, $[\equiv]B$ is in Γ then the others are also in Γ.

Obviously Γ is finite set of formulas. Define a filtration $M'=(W',\leq',\Sigma',N',\equiv')$ as for the filtration for SIM-1 extended for $\equiv$ as follows:

$|x|\equiv'|y|$ iff $(\forall[\equiv]A\in\Gamma)(x\Vdash_{\overline{v}}A \longleftrightarrow y\Vdash_{\overline{v}}A)$ & $|x|\leq'|y|$ & $|x|\geq'|y|$

It is easy to show that $\equiv'$ satisfies the conditions of filtration and that $\equiv'$ is equivalence relation contained in $\leq'$. So the conditions ($\equiv$4)-($\equiv$7) are fulfilled. It remains to verify the conditions ($\equiv$1)-($\equiv$3).

Proof of ($\equiv$1): $|x|\overline{\Sigma}'|x|$ & $|y|\leq'|x| \longrightarrow |x|\equiv'|y|$.

Suppose $|x|\overline{\Sigma}'|x|$ & $|y|\leq'|x|$.

Case 1: $x\equiv y$. Then by the condition of filtration we obtain $|x|\equiv'|y|$.

Case 2: $x\not\equiv y$. Then by ($\equiv$1) we get $x\Sigma x$ or $y\not\leq x$.

Case 2.1: $x\Sigma x$. From here we obtain $|x|\Sigma'|x|$, which contradicts the assumption.

Case 2.2: $y\not\leq x$. Then by S6 we obtain $y\Sigma y$ and then $|y|\Sigma'|y|$. From here and $|y|\leq'|x|$ we obtain by application of the axioms of similarity structure $|x|\Sigma'|x|$ - again a contradiction, which completes the proof of ($\equiv$1).

The proof of ($\equiv$2) is similar.

Proof of ($\equiv$3): $|x|\overline{\Sigma}'|z|$ & $|y|\overline{N}'|z|$ & $|y|\leq'|x| \longrightarrow |x|\equiv'|y|$.

Suppose $|x|\overline{\Sigma}'|z|$ & $|y|\overline{N}'|z|$ & $|y|\leq'|x|$.

Case 1: $x\equiv y$. Then we obtain $|x|\equiv'|y|$.

Case 2: $x\not\equiv y$. Then by ($\equiv$3) we get $x\Sigma z$ or yNz or $y\not\leq x$.

Case 2.1: $x\Sigma z$. From here we obtain $|x|\Sigma'|z|$ - a contradiction with the assumption.

Case 2.2: yNz. From here we obtain $|x|N'|z|$ - again a contradiction with the assumption.

Case 2.3: $y\not\leq x$. Then by Axiom S11 we obtain $y\Sigma z$ or xNz.

Case 2.3.1: $y\Sigma z$. From here we obtain $|y|\Sigma'|z|$ and by $|y|\leq'|x|$ we obtain $|x|\Sigma'|z|$ - a contradiction with the assumption.

Case 2.3.2: xNz. This implies $|x|N'|z|$ and by $|y|\leq'|x|$ we obtain $|y|N'|z|$ - again a contradiction, which completes the proof of ($\equiv$3).

Thus $(W',\leq',\Sigma',N',\equiv')$ is a finite ind-1-structure, which completes the proof of the theorem.∎

2. IND-2 - a modal logic for strong indiscernibility relation in A-systems

In this section we shall extend the logic SIM-2 with the modality for strong indiscernibility relation.

Let S=(Ob, At, {VALa /a∈AT}, f) be an A-system. The strong indiscernibility relation in S is defined as follows:

$x\equiv_S y$ iff $(\forall a\in AT)f(x,a)=f(y,a)$ iff $x\leq_S y$ & $y\leq_S x$

where $\leq_S$ is the strong information inclusion in S. So indiscernibility relation in A-systems is definable by the relation $\leq$ of strong informational inclusion.

Since we have an abstract characterization of the relation $\leq$ together with Σ, N, σ, ν, and $<$ in the definition of bi-similarity structure, it is natural to study the strong indiscernibility relation $\equiv$ together with these relations, using the above definition of $\equiv$. So the standard semantical structure for the strongindiscernibility relation, called ind-2-structure, is the following - $\underline{W}=(W,\leq,\Sigma,N,<,\sigma,\nu,\equiv)$, where $(W,\leq,\Sigma,N,<,\sigma,\nu)$ is a bi- similarity structure and $\equiv$ satisfies the equivalence

($\equiv$) $x\equiv y$ iff $x\leq y$ and $y\leq x$.

The corresponding modal logic will be an extension of the logic SIM-2 with an extra modality [≡] with standard semantics in ind-2- structures by the relation ≡. The obtained logic is denoted by IND-2. Let us note that in the language of IND-2 the equivalence (≡) is not modally definable, which will present the same difficulties in the way of axiomatization of IND-2 as those for IND-1. To overcome these difficulties we shall introduce general (nonstandard) semantics for IND-2, in which IND-2 is easy to axiomatize, and then, using the copying method, we shall prove that the obtained axiomatization is complete also with respect to its standard semantics.

By a generalized ind-2-structure we mean any relational system $\underline{W}=(W,\leq,\Sigma,N,<,\sigma,\nu,\equiv)$, such that $(W,\leq,\Sigma,N,<,\sigma,\nu)$ is a bi-similarity structure and and $(W,\leq,\Sigma,N,\equiv)$ is a generalized ind-1-structure. So the axioms for the generalized ind-2-structure are S1-S26, (≡1)-(≡7).

Proposition 2.1

Let $\underline{W}=(W,\leq,\Sigma,N,<,\sigma,\nu,\equiv)$ be a generalized ind-2-structure. Then there exists an ind-2-structure $\underline{W}'=(W',\leq',\Sigma',N',<',\sigma',\nu',\equiv')$ and a copying I from $\underline{W}$ to $\underline{W}'$.

Proof. Let I={1,2,...} be the set of natural numbers. Let I be the copying defined for the generalized ind-1-structure $\underline{W}=(W,\leq,\Sigma,N,\equiv)$ as in the proof of proposition 1.2. and let $\underline{W}'=(W',\leq',\Sigma',N',\equiv')$ be the resulting ind-1-structure. For $R\in\{<,\sigma,\nu\}$ $x,y\in W$ and $f,g\in I$ put

$f(x)R'g(y)$ iff xRy.

The proof that I is a copying from $\underline{W}=(W,\leq,\Sigma,N,<,\sigma,\nu,\equiv)$ to $\underline{W}'=(W',\leq',\Sigma',N',<',\sigma',\nu',\equiv')$ and that $\underline{W}'$ is an ind-2-structure is straightforward.■

Corollary 2.2

The classes of generalized ind-2-structures and ind-2- structures determine one and the same logic, namely IND-2.

Now the axiomatization of IND-2 is easy - extend SIM-2 with the axioms (Ax≡1)-(Ax≡7) taken from the logic IND-1. Namely:

Axioms and rules for IND-1

(I) Axioms and rules for SIM-2

(II)

(Ax≡1) $[\leq]([\equiv]B\Rightarrow([\Sigma]A\Rightarrow A))\vee B$,

(Ax≡2) $[\geq]([\equiv]B\Rightarrow([N]A\Rightarrow A))\vee B$,

(Ax≡3) $\blacksquare C\vee([\equiv]D\wedge[\Sigma]C\Rightarrow[\geq]([N]C\Rightarrow D))$,

(Ax≡4) $[\equiv]A\Rightarrow A$,

(Ax≡5) $A\vee[\equiv]\neg[\equiv]A$,

(Ax≡6) $[\equiv]A\Rightarrow[\equiv][\equiv]A$,

(Ax≡7) $[\leq]A\Rightarrow[\equiv]A$.

Theorem 2.3 (Completeness theorem for IND-2)

The following conditions are equivalent for any formula A of IND-2:

(i) A is a theorem of IND-2,

(ii) A is true in all generalized ind-2-structures,

(iii) A is true in all ind-2-structures.

Proof. (ii)⟷(iii) is just corollary 2.2. (i)⟶(ii) is trivial and (ii)⟶(i) is proved by the canonical construction as in the proof of the completeness theorem for SIM-2.■

Now we shall proof that IND-2 admits filtration with respect to its generalized semantics.

Theorem 2.4.

IND-2 admits filtration with respect to its generalized semantics and hence is decidable.

Proof. Let $M=(W,\leq,\Sigma,N,<,\sigma,\nu,\equiv,\ v)$ be a model over a generalized ind-2-structure. Let A' be a formula and define the set of formulas Γ to be the smallest set of formulas, closed under sub- formulas containing A' and satisfying the following closure condition

(γ) if one of the formulas $[\leq]B$, $[\Sigma]B$, $[N]B$, $[<]B$, $[\sigma]B$, $[\nu]B$ and $[\equiv]B$ is in Γ then the others are also in Γ.

For the bi-similarity structure $\underline{W}=(W,\leq,\Sigma,N,<,\sigma,\nu)$ define the same filtration as for the logic SIM-2 and define the relation $\equiv'$ as in theorem 1.5, namely

$|x|\equiv'|y|$ iff $(\forall[\equiv]A\in\Gamma)(x\Vdash_{v}A \longleftrightarrow y\Vdash_{v}A)$ & $|x|\leq'|y|$ & $|x|\geq'|y|$

Let the resulting system be $\underline{W}'=(W',\leq',\Sigma',N',<',\sigma',\nu',\equiv')$. The proof that the conditions for the filtration are fulfilled and that $\underline{W}'$ is an ind-2-structure is straightforward.■

3. IND-3 - a modal logic for strong and weak indiscernibility relations in A-systems

Let $S=(Ob,\ AT,\ \{VALa/a\in AT\},f)$ be an A-system. The strong and weak indiscernibility relations in S - $\equiv$ and $\cong$ are defined as follows:

$x\equiv y$ iff $(\forall a\in AT)f(x,a)=f(y,a)$,

$x\cong y$ iff $(\exists a\in AT)f(x,a)=f(y,a)$.

In sec. 2 we extended the logic SIM-2 with a modality for the strong indiscernibility relation, which is definable by the relation of strong informational inclusion $\leq$. However the weak indiscernibility relation $\cong$ can not be defined by the relations $\leq,\Sigma,N,<,\sigma,\nu$ from bi-similarity structure. So to extend the logic SIM-2 with both strong and weak indiscernibility relations is not easy. It requires a new characterization theorem for the relevant structures and bi-consequence systems in this case are not suitable. We formulate as an open problem the axiomatization of an extension of SIM-2 with modalities for strong and weak indiscernibility relations. We conjecture that this logic has no normal axiomatization by a finite set of axiom schemes.

In this section we shall give characterization theorem for the strong and weak indiscernibility relations $\equiv$ and $\cong$ together with weak and strong positive similarity relations Σ and σ:

$x\Sigma y$ iff $(\exists a\in AT)f(x,a)\cap f(y,a)\neq\emptyset$,

$x\sigma y$ iff $(\forall a\in AT)f(x,a)\cap f(y,a)\neq\emptyset$.

The obtained abstract structure, called ind-3-structure will be the semantic base for the corresponding modal logic, called here IND-3. We shall prove the completeness and decidability of IND-3 with respect to its standard semantics.

Lemma 3.1.

Let S be an A-system. Then the relations $\equiv$, $\cong$, σ and Σ satisfy the following first-order conditions:

I1. $x\equiv x$,

I2. $x\equiv y \longrightarrow y\equiv x$,

I3. $x\equiv y$ & $y\equiv z \longrightarrow x\equiv z$,

I4. $x\cong x$,

I5. $x\cong y \longrightarrow y\cong x$,

I6. $x\equiv y$ & $y\cong z \longrightarrow x\cong z$,

I7. $x\sigma y \longrightarrow y\sigma x$,

I8. $x\sigma y \longrightarrow x\sigma x$,

I9. $x\equiv y$ & $y\sigma z \longrightarrow x\sigma z$,

I10. $x\Sigma y \longrightarrow y\Sigma x$,

I11. $x\Sigma y \longrightarrow x\Sigma x$,
I12. $x\equiv y$ & $y\Sigma z \longrightarrow x\Sigma z$,
I13. $x\equiv y$ or $x\Sigma x$ or $y\Sigma y$.
I14. $x\cong y \longrightarrow x\Sigma y$ or $x\cong z$ or $z\Sigma z$
I15. $x\cong y$ & $y\sigma z \longrightarrow x\Sigma z$,
I16. $x\cong y$ or $x\sigma x$ or $y\Sigma y$.

This lemma suggests the following definition. A relational system $\underline{W}=(W, \equiv, \cong, \sigma, \Sigma)$, with $W\neq\emptyset$, is called an ind-3-structure if the relations $\equiv$, $\cong$, σ, Σ satisfy the conditions I1-I16 from lemma 3.1. If S is an A-system then the system $(OB, \equiv_S, \cong_S, \sigma_S, \Sigma_S)$ is an ind-3-structure, called an ind-3-structure over S. $\underline{W}$ is called a *standard* ind-3-structure if it is an ind-3-structure over some A-system S.

Lemma 3.2.

In each in-3-structure the following conditions are satisfied:

I6' $x\cong y$ & $y\equiv z \longrightarrow x\cong z$,
I6'' $x\equiv y$ & $y\cong z$ & $z\equiv t \longrightarrow x\cong t$,
I8' $x\sigma y \longrightarrow x\sigma x$ & $y\sigma y$,
I9' $x\sigma y$ & $y\equiv z \longrightarrow x\sigma z$,
I9'' $x\equiv y$ & $y\sigma z$ & $z\equiv t \longrightarrow x\sigma t$,
I11' $x\Sigma y \longrightarrow x\Sigma x$ & $y\Sigma y$,
I12' $x\Sigma y$ & $y\equiv z \longrightarrow x\Sigma z$,
I12'' $x\equiv y$ & $y\Sigma z$ &$z\equiv t \longrightarrow x\Sigma t$,
I15' $x\sigma y$ & $y\cong z \longrightarrow x\Sigma z$,
I15'' $x\cong y$ & $y\sigma z$ & $z\equiv t \longrightarrow x\Sigma t$,
I15''' $x\equiv y$ & $y\sigma z$ & $z\cong t \longrightarrow x\Sigma t$,
I17 $x\equiv y \longrightarrow x\cong y$,
I18 $x\sigma y \longrightarrow x\Sigma y$.

The easy proof is left to the reader.■

The main aim of this section is the following

Theorem 3.3.

/Abstract Characterization Theorem for standard ind-3-structures/ Every ind-3-structure is a standard ind-3-structure.

The proof of this theorem will go trough a long series of special constructions.

Directly from the definition of set-theoretical A-system we obtain the following

Lemma 3.4.

Let S=(W, V) be a set-theoretical A-system. Then the following conditions are satisfied for any $x,y\in W$:

(i) $x\equiv_S y$ iff $(\forall a\in V)(\forall A\in a)(x\in A \longleftrightarrow y\in A)$,
(ii) $x\cong_S y$ iff $(\exists a\in V)(\forall A\in a)(x\in A \longleftrightarrow y\in A)$,
(iii) $x\sigma_S y$ iff $(\forall a\in V)(\exists A\in a)(x\in A$ & $y\in A)$,
(iv) $x\Sigma_S y$ iff $(\exists a\in V)(\exists A\in a)(x\in A$ & $y\in A)$.

Now we shall study some special sets of elements of a similarity structure.
Let $\underline{W}=(W, \equiv, \cong, \sigma, \Sigma)$ be an ind-3-structure. A subset $A\subseteq W$ is called:
$\equiv$-monotonic set, if $(\forall x,y\in W)(x\in A$ & $x\equiv y \longrightarrow y\in A)$,
σ-set, if it is a $\equiv$-monotonic set and $(\forall x,y\in A)(x\sigma y)$,
Σ-set if it is a $\equiv$-monotonic set and $(\forall x,y\in A)(x\Sigma y)$.
For $x\in W$ define $[x]=\{y\in W/x\equiv y\}$.

Lemma 3.5.

(i) The set of all $\equiv$-monotonic subsets of W forms a complete Boolean algebra.

(ii) $[x]$ is the smallest $\equiv$-monotonic set containing x.

(iii) Any σ-set is a Σ-set.

(iv) Let $S\in\{\sigma, \Sigma\}$, then:

1. If pSp then $[p]$ is an S-set.
2. If $A\neq\emptyset$ is an S-set, $p\in W$ and for any $x\in A$ we have xSp, then $A\cup[p]$ is an S-set.
3. If pSq then $[p]\cup[q]$ is an S-set.
4. If pSq, qSr and pSr then $[p]\cup[q]\cup[r]$ is an S-set.

Proof - by an easy calculation, using the properties of the relations $\equiv$, $\cong$, σ and Σ, listed in the definition of similarity structure and lemma 3.2. ■

Lemma 3.6.

If $x\not\equiv y$ then there exists a Σ-set $A(x\not\equiv y)$ containing exactly one of the elements x and y.

Proof. Suppose $x\not\equiv y$. Then by I13 we have $x\Sigma x$ or $y\Sigma y$. Define $A(x\not\equiv y)$ as follows

$$A(x\not\equiv y)=\begin{cases}[x] & \text{if} x\Sigma x\\ [y] & \text{if} x\overline{\Sigma}x\end{cases}$$

The set $A(x\not\equiv y)$ is either $[x]$ or $[y]$. In the second case we have $y\Sigma y$ and then in both cases we have by lemma 3.5.iv.1 that $A(x\not\equiv y)$ is a Σ-set. Since $x\not\equiv y$ then $A(x\not\equiv y)$ contains exactly one of the elements x and y. ■

Lemma 3.7.

If $p\ncong q$ then there exists a Σ-set $A(p\ncong q)$ containing exactly one of the elements p and q.

Proof. Suppose $p\ncong q$. Then by I17 we have $p\not\equiv q$. Put $A(p\ncong q)=A(p\not\equiv q)$ and apply lemma 3.6. ■

Lemma 3.8.

If $x\cong y$ and $p\ncong q$, then there exists a Σ-set $A=A(x\cong y, p\ncong q)$, satisfying the following conditions:

(i) $x\in A$ iff $y\in A$,

(ii) A contains exactly one of the elements p and q.

Proof. Suppose $x\cong y$ and $p\ncong q$. Then we have by I17 $p\not\equiv q$. We shall construct A considering several cases.

Case 1: $p\Sigma p$. Then by lemma 3.5.iv.i $[p]$ is a Σ-set and since $p\not\equiv q$ we have $q\notin[p]$.

Case 1.1: $x\in[p]$ iff $y\in[p]$. Put in this case $A=[p]$. Conditions (i) and (ii) are obviously fulfilled.

Case 1.2: $x\in[p]$, $y\notin[p]$. We have in this case $p\equiv x$.

Case 1.2.1: $p\Sigma y$. Then by lemma 3.5.iv.3 $[p]\cup[y]$ is a Σ-set, containing x and y. Put $A=[p]\cup[y]$. Condition (i) is obviously fulfilled. Since $p\in[p]$ we have $p\in A$. We shall show that $q\notin A$. Suppose for the sake of contradiction that $q\in A$. Since $q\notin[p]$ the only place for q is in $[y]$, so $y\equiv q$. Then $p\equiv x$, $x\cong y$ and $y\equiv q$ imply by I6'' $p\cong q$, which contradicts $p\ncong q$. Consequently $q\notin A$ and the condition (ii) is fulfilled.

Case 1.2.2: $p\overline{\Sigma}y$. From $p\equiv x$ and $x\cong y$ we obtain by I6 $p\cong y$. Then $p\cong y$, $p\overline{\Sigma}y$, $p\ncong q$ imply by I14 $q\Sigma q$. By lemma 3.5.iv.1 $[q]$ is a Σ-set. We put in this case $A=[q]$. We have $q\in A$ and since $p\not\equiv q$ $p\notin A$, so condition (ii) is fulfilled. To prove (i) it will suffice to show that $x\notin A$ and $y\notin A$.

Suppose $x\in A$. Then $x\equiv q$ and by $p\equiv x$ we obtain $p\equiv q$, which contradicts $p\not\equiv q$. Consequently $x\notin A$.

Suppose $y\in A$. Then $y\equiv q$ and by $p\equiv x$, $x\cong y$ we obtain $p\cong q$, which contradicts $p\ncong q$.

Consequently $y \notin A$.

Case 1.3: $x \notin [p]$, $y \in [p]$. We proceed in this case as in case 1.2, replacing the role of x and y.

Case 2: $p\overline{\Sigma}p$. Then from $p \not\equiv q$ we obtain by I13 $q\Sigma q$. So $[q]$ is a Σ-set.

Case 2.1: $x \in [q]$ iff $y \in [q]$. Put in this case $A=[q]$. Obviously the conditions (i) and (ii) are fulfilled.

Case 2.2: $x \in [q]$, $y \notin [q]$. From $x \in [q]$ we get $q \equiv x$ and by $x \cong y$ we obtain $q \cong y$. Then $q \cong y$, $q \not\equiv p$ and $p\overline{\Sigma}p$ imply by S14 $q\Sigma y$. Then by lemma 3.5.iv.3 we obtain that $[q] \cup [y]$ is a Σ-set. Put in this case $A=[q] \cup [y]$. Obviously q, x, y $\in A$, which shows that (i) is fulfilled. Since $p \notin [q]$, to prove (ii) it will suffice to show that $p \notin [y]$. Otherwise, from $y \equiv p$ $x \cong y$ and $q \equiv x$ we get $p \cong q$, which is a contradiction.

Case 2.3. $x \notin [q]$, $y \in [q]$. In this case we proceed as in the case 2.2, replacing the role of x and y.■

Lemma 3.9.

If $x \cong y$ and $p\sigma q$ then there exists a Σ-set $A=A(x \cong y, p\sigma q)$, satisfying the following conditions:

(i) $x \in A$ iff $y \in A$,

(ii) $p, q \in A$.

Proof. Suppose $x \cong y$ and $p\sigma q$ and define

$$A=\begin{cases} [p]\cup[q] & \text{if } (x\in[p]\cup[q] \text{ iff } y\in[p]\cup[q]) \\ [p]\cup[q]\cup[x] & \text{if } x\notin[p]\cup[q] \text{ and } y\in[p]\cup[q] \\ [p]\cup[q]\cup[y] & \text{if } x\in[p]\cup[q] \text{ and } y\notin[p]\cup[q] \end{cases}$$

By the definition of the set A we have that $x \in A$ iff $y \in A$. Since $[p]\cup[q] \subseteq A$ we have that $p, q \in A$. So the conditions (i) and (ii) are fulfilled. It remains to show that A is a Σ-set. We will consider several cases.

Case 1: $x \in [p]u[q]$ iff $y \in [p]u[q]$. In this case $A=[p]\cup[q]$ and since $p\sigma q$, then, by lemma 3.5.iv.3 $[p]\cup[q]$ is a σ-set and by lemma 3.5.iii $[p]\cup[q]$ is a Σ-set.

Case 2: $x \notin [p]u[q]$ and $y \in [p]u[q]$. In this case $A=[p]\cup[q]\cup[x]$. Since $[p]\cup[q]$ is a σ-set, then from $y \in [p]\cup[q]$ we get $y\sigma p$ and $y\sigma q$. From $x \cong y$, $y\sigma p$ and $y\sigma q$ we obtain by I15 $x\Sigma p$ and $x\Sigma q$. From $p\sigma q$ we get $p\Sigma q$. From $p\Sigma q$, $x\Sigma p$ and $x\Sigma q$ we get by lemma 3.4.iv.4 that $[p]\cup[q]\cup[x]$ is a Σ-set.

Case 3: $x \in [p]u[q]$ and $y \notin [p]u[q]$. In this case the proof can be obtained from the proof of case 2 by replacing the role of the elements x and y.■

Lemma 3.10.

If $x\overline{\sigma}y$ and $p \not\equiv q$ then there exists a Σ-set $A=A(x\overline{\sigma}y, p \not\equiv q)$, satisfying the following conditions:

(i) $x \notin A$ or $y \notin A$,

(ii) A contains exactly one of the elements p and q.

Proof. Suppose $x\overline{\sigma}y$ and $p \not\equiv q$. We shall construct A considering several cases.

Case 1: $q\overline{\Sigma}q$. Then by S16 we obtain $p\sigma p$. This shows that $[p]$ is a σ-set and hence - a Σ-set. Define $A=[p]$. We have $p \in A$ and from $p \not\equiv q$ we obtain that $q \notin A$, so condition (ii) is fulfilled. Since A is a σ-set, then by $x\overline{\sigma}y$ we get that $x \notin A$ or $y \notin A$, so (i) is fulfilled.

Case 2: $q\Sigma q$.

Case 2.1: $x \not\equiv q$ or $y \not\equiv q$. Take in this case $A=[q]$. Obviously the conditions (i) and (ii) are fulfilled.

Case 2.2: $x \equiv q$ and $y \equiv q$. Then from $x\overline{\sigma}y$ we get $q\overline{\sigma}q$. From $p \not\equiv q$ we get $q \not\equiv p$. Then from $q \not\equiv p$ and $q\overline{\sigma}q$ we get by I16 $p\Sigma p$. Take $A=[p]$. Then A is a Σ-set, containing p and not containing q, x and y. This implies that the conditions (i) and (ii) are fulfilled.■

Let $\underline{W}=(W\ \equiv, \cong, \sigma, \Sigma)$ be an ind-3-structure and let a be a set of subsets of W. We say that a is a good $\underline{W}$-set if it satisfies the following conditions:

(gs.1) $p \not\equiv q \longrightarrow (\exists A \in a)(p \in A \ \&\ q \notin A \text{ or } p \notin A \ \&\ q \in A)$,

(gs.2) $p \sigma q \longrightarrow (\exists A \in a)(p \in A \ \&\ q \in A)$,

(gs.3) all elements of a are Σ-sets.

Let V be the set of all good W-sets.

Lemma 3.11.

For any $x, y \in U$ the following conditions are true:

(i) $x \equiv y$ iff $(\forall a \in V)(\forall A \in a)(x \in A \longleftrightarrow y \in A)$,

(ii) $x \cong y$ iff $(\exists a \in V)(\forall A \in a)(x \in A \longleftrightarrow y \in A)$,

(iii) $x \sigma y$ iff $(\forall a \in V)(\exists A \in a)(x \in A \ \&\ y \in A)$,

(iv) $x \Sigma y$ iff $(\exists a \in V)(\exists A \in a)(x \in A \ \&\ y \in A)$.

Proof. (i)($\longrightarrow$) Suppose $x \equiv y$, $a \in V$ and $A \in a$. By (gs.3) A is a Σ-set, so A is a $\equiv$-monotonic set. Consequently $x \in A \longleftrightarrow y \in A$.

($\longleftarrow$) Suppose $x \not\equiv y$, then we have to prove that

$(\exists a \in V)(\exists A \in a)(x \in A \ \&\ y \notin A \text{ or } x \notin A \ \&\ y \in A)$.

Define a as follows: $a = a_1 \cup a_2 \cup a_3$, where

$a_1 = \{A(x \not\equiv y)\}$, $a_2 = \{A(p \not\equiv q) / p \not\equiv q\}$, $a_3 = \{[p] \cup [q] / p \sigma q\}$

By lemma 3.6 the set $A(x \not\equiv y)$ contains exactly one of the elements x and y. It remains to show that $a \in V$.

Condition (gs.1) is fulfilled by the construction of a_2 and lemma 3.7. Condition (gs.2) is fulfilled by the construction of a_3 and lemma 3.5.iv.3. By this lemma all elements of a_3 are σ-sets, so they are Σ-sets. By lemma 3.6 and lemma 3.7 the elements of a_1 and a_2 are Σ-sets, hence (gs.3) is fulfilled and $a \in V$.

(ii) ($\longrightarrow$) Suppose $x \cong y$, then we have to prove that

$(\exists a \in V)(\forall A \in a)(x \in A \longleftrightarrow y \in A)$

Define $a = a_1 \cup a_2$, where

$a_1 = \{A(x \cong y, p \not\equiv q) / p \not\equiv q\}$, $a_2 = \{A(x \cong y, p \sigma q) / p \sigma q\}$.

Lemma 3.8 and the construction of a_1 guarantee (gs.1), and lemma 3.9 and the construction of a_2 guarantee (gs.2). By these two lemmas all elements of a_1 and a_2 are Σ-sets, so (gs.3) is fulfilled.

($\longleftarrow$) Suppose $x \not\cong y$. Then we have to show that

$(\forall a \in V)(\exists A \in a)(x \in A \ \&\ y \notin A \text{ or } x \notin A \ \&\ y \in A)$.

This is true by (gs.1).

(iii) ($\longrightarrow$) Suppose $x \sigma y$. Then we have to prove that

$(\forall a \in V)(\exists A \in a)(x \in A \ \&\ y \in A)$.

This condition is guaranteed by (gs.2).

($\longleftarrow$) Suppose $x \bar{\sigma} y$. We have to show that

$(\exists a \in V)(\forall A \in a)(x \notin A \text{ or } y \notin A)$.

Define $a = a_1 \cup a_2$, where

$a_1 = \{A(x \bar{\sigma} y, p \not\equiv q) / p \not\equiv q\}$ and $a_2 = \{[p] \cup [q] / p \sigma q\}$.

By lemma 3.10 the elements of a_1 are Σ-sets, and by lemma 3.5.iv.3 the elements of a_2 are σ-sets, hence Σ-sets. If $A \in a_1$ then by lemma 3.10 $x \notin A$ or $y \notin A$. If $A \in a_2$ then $x \notin A$ or $y \notin A$, because $x \bar{\sigma} y$ and A is a σ-set. Condition (gs.1) is guaranteed by lemma 3.10 and condition (gs.2) is guaranteed by lemma 3.5.iv.3.

(iv) ($\longrightarrow$) Suppose $x \Sigma y$. Then we have to prove that

$(\exists a \in V)(\exists A \in a)(x \in A \ \&\ y \in A)$.

Define $a=a_1 \cup a_2 \cup a_3$, where

$a_1=\{[x]\cup[y]\}$, $a_2=\{A(p\not\cong q)/p\not\cong q\}$, $a_3=\{[p]\cup[q]/p\sigma q\}$.

Obviously $a\in V$ and $x,y\in[x]\cup[y]\in a$.

($\longleftarrow$) Suppose $x\bar{\Sigma}y$. Then we have to show that

$(\forall a\in V)(\forall A\in a)(x\notin A$ or $y\notin A)$.

This is guaranteed by (gs.1).■

Now we are ready to prove the Abstract Characterization Theorem for standard ind-3-structures.

Proof of theorem 3.3. Let $\underline{W}=(W, \equiv, \cong, \sigma, \Sigma)$ be an ind-3- structure. We shall show that there is an A-system S with the set OB=W such that $\underline{W}$ is the ind-3-structure over the S. We take S=(W, V) where V is the set of all good $\underline{W}$-sets. Since $x\cong x$, then by lemma 3.11 $(\exists a\in V)(\forall A\in a)(x\in A \longleftrightarrow y\in A)$, so $a\in V$ and $V\neq\varnothing$. This shows that S is a set-theoretical A-system. Then by lemma 3.4 and lemma 3.11 we obtain the following equivalences for any $x,y\in U$:

$x\equiv_S y$ iff $(\forall a\in V)(\forall A\in a)(x\in A \longleftrightarrow y\in A)$ iff $x\equiv y$,

$x\cong_S y$ iff $(\exists a\in V)(\forall A\in a)(x\in A \longleftrightarrow y\in A)$ iff $x\cong y$,

$x\sigma_S y$ iff $(\forall a\in V)(\exists A\in a)(x\in A$ & $y\in A)$ iff $x\sigma y$,

$x\Sigma_S y$ iff $(\exists a\in V)(\exists A\in a)(x\in A$ & $y\in A)$ iff $x\Sigma y$.

This shows that the ind-3- structures $(W, \equiv_S, \cong_S, \sigma_S, \Sigma_S)$ and $(W, \equiv, \cong, \sigma, \Sigma)$ coincide, which ends the proof of the theorem.■

Now we introduce the logic IND-3.

The language of IND-3 is an extension of the language of propositional calculus with five modal connectives:

$[\equiv]$, $[\cong]$, $[\sigma]$, $[\Sigma]$ and ■ - the universal modality.

The standard semantics of IND-3 is in the class of all standard ind-3-structures

Consider the following table. On the left side we have the list of the axioms I1-I16 of similarity structures and on the right side are their modal translations in the sense of the next theorem. The letters A and B in the formulas Ai are propositional variables.

I1.	$x\equiv x$	A1.	$[\equiv]A\Rightarrow A$
I2.	$x\equiv y \longrightarrow y\equiv x$	A2.	$A\vee[\equiv]\neg[\equiv]A$
I3.	$x\equiv y$ & $y\equiv z \longrightarrow x\equiv z$	A3.	$[\equiv]A\Rightarrow[\equiv][\equiv]A$
I4.	$x\cong x$	A4.	$[\cong]A\Rightarrow A$
I5.	$x\cong y \longrightarrow y\cong x$	A5.	$A\vee[\cong]\neg[\cong]A$
I6.	$x\equiv y$ & $y\cong z \longrightarrow x\cong z$	A6.	$[\cong]A\Rightarrow[\equiv][\cong]A$
I7.	$x\sigma y \longrightarrow y\sigma x$	A7.	$A\vee[\sigma]\neg[\sigma]A$
I8.	$x\sigma y \longrightarrow x\sigma x$	A8.	$\langle\sigma\rangle 1\wedge[\sigma]A\Rightarrow A$
I9.	$x\equiv y$ & $y\sigma z \longrightarrow x\sigma z$	A9.	$[\sigma]A\Rightarrow[\equiv][\sigma]A$
I10.	$x\Sigma y \longrightarrow y\Sigma x$	A10.	$A\vee[\Sigma]\neg[\Sigma]A$
I11.	$x\Sigma y \longrightarrow x\Sigma x$	A11.	$\langle\Sigma\rangle 1\wedge[\Sigma]A\Rightarrow A$
I12.	$x\equiv y$ & $y\Sigma z \longrightarrow x\Sigma z$	A12.	$[\Sigma]A\Rightarrow[\equiv][\Sigma]A$
I13.	$x\equiv y$ or $x\Sigma x$ or $y\Sigma y$	A13.	$([\Sigma]A\Rightarrow A)\vee■([\equiv]A\Rightarrow([\Sigma]B\Rightarrow B))$
I14.	$x\cong y \longrightarrow x\Sigma y$ or $x\cong z$ or $z\Sigma s$	A14.	$([\Sigma]A\Rightarrow A)\vee■([\cong]A\Rightarrow([\Sigma]B\Rightarrow[\cong]B)$
I15.	$x\cong y$ & $y\sigma z \longrightarrow x\Sigma z$	A15.	$[\Sigma]A\Rightarrow[\cong][\Sigma]A$
I16.	$x\cong y$ or $x\sigma x$ or $y\Sigma y$	A16.	$([\Sigma]A\Rightarrow A)\vee■([\cong]A\Rightarrow([\sigma]B\Rightarrow B))$

(S5). $■A\Rightarrow A$, $■A\Rightarrow■■A$, $A\vee■\neg■ A$,
$■A\Rightarrow[\equiv]A$, $■A\Rightarrow[\cong]A$, $■A\Rightarrow[\sigma]A$, $■A\Rightarrow[\Sigma]A$

Theorem 3.12 /Modal Definability Theorem for ind-3-structures/

Let $\underline{U}=(U, \equiv, \cong, \sigma, \Sigma)$ be a relational structure. Then the axioms S5 are true in $\underline{U}$ and for any i=1,2,...,16 the formula Ai is true in $\underline{U}$ iff the condition Ii holds for $\underline{U}$.

Proof- standard. ■

This theorem suggests the following axiomatization for the logic IND-3.

Axiom schemes and rules for IND-3

(Bool) All or enough Boolean tautologies,

(K■) $\blacksquare(A\Rightarrow B)\Rightarrow(\blacksquare A\Rightarrow\blacksquare B)$,

(K[R]) $[R](A\Rightarrow B)\Rightarrow([R]A\Rightarrow[R]B)$, $R\in\{\equiv,\cong,\sigma,\Sigma\}$

(S5) and A1 - A16 from theorem 3.12.

(MP) $A,\ A\Rightarrow B/B$, (N■) $A/\blacksquare A$, (N[R]) $A/[R]A$ $R\in\{\equiv,\cong,\sigma,\Sigma\}$

Let us note that the modality ■ satisfies all the axioms of the modal logic S5 and since it is interpreted here in the universal relation $W\times W$, it is called universal modality. We adopt this modality in our language, because of the conditions I13, I14 and I16, which are not modally definable in the restricted language.

We identify the logic IND-3 with the set of all theorems of the above formal system.

Theorem 3.13./Completeness Theorem for IND-3/

The following conditions are equivalent for any formula A of IND-3:

(i) A is a theorem of IND-3,

(ii) A is true in all ind-3-structures,

(iii) A is true in all standard ind-3-structures.

Proof. (i) ⟶ (ii) By the Modal Definability Theorem.

(ii) ⟷ (iii). This equivalence is true by the Abstract Characterization Theorem for standard ind-3-structures.

(ii)⟶(i) - by the standard canonical construction - the generated canonical model is an ind-3-structure.■

Theorem 3.14 /Filtration Theorem for IND-3/

IND-3 admits a filtration.

Proof. Let A_0 be a formula and let Γ be the smallest set of formulas, containing A_0, closed under subformulas and satisfying the following conditions:

(γ1) $\langle\Sigma\rangle 1, \langle\sigma\rangle 1\in\Gamma$,

(γ2) if one of the formulas $[\equiv]A$, $[\cong]A$, $[\sigma]A$ and $[\Sigma]A$ is in Γ then the same will be for the others.

Obviously Γ is a finite set of formulas. Let $\underline{W}=(W, \equiv, \cong, \sigma, \Sigma)$ be an ind-3-structure and $(\underline{W}, v)$ be a model over $\underline{W}$. Define by means of Γ the set $|W|$ and the following relations in $|W|$:

$$
\begin{aligned}
|x|\equiv'|y| \text{ iff } (\forall[\equiv]A\in\Gamma)&(x\Vdash_v[\equiv]A \longleftrightarrow y\Vdash_v[\equiv]A)\ \&\\
&(x\Vdash_v[\cong]A \longleftrightarrow y\Vdash_v[\cong]A)\ \&\\
&(x\Vdash_v[\sigma]A \longleftrightarrow y\Vdash_v[\sigma]A)\ \&\\
&(x\Vdash_v[\Sigma]A \longleftrightarrow y\Vdash_v[\Sigma]A)\ \&\\
&(x\Vdash_v\langle\Sigma\rangle 1 \longleftrightarrow y\Vdash_v\langle\Sigma\rangle 1)\ \&\\
&(x\Vdash_v\langle\sigma\rangle 1 \longleftrightarrow y\Vdash_v\langle\sigma\rangle 1).
\end{aligned}
$$

$$
\begin{aligned}
|x|\cong'|y| &\text{ iff } (\exists p,q\in U)(|x|\equiv'|p|\ \&\ |y|\equiv'|q|\ \&\ p\cong q),\\
|x|\sigma'|y| &\text{ iff } (\exists p,q\in U)(|x|\equiv'|p|\ \&\ |y|\equiv'|q|\ \&\ p\sigma q),\\
|x|\Sigma'|y| &\text{ iff } (\exists p,q\in U)((|x|\equiv'|p|\ \&\ |y|\equiv'|q|\ \&\ p\Sigma q) \text{ or}\\
&\qquad(|x|\cong'|p|\ \&\ |y|\equiv'|q|\ \&\ p\sigma q) \text{ or}\\
&\qquad(|x|\equiv'|p|\ \&\ |y|\cong'|q'|\ \&\ p\sigma q))\ \&\\
&\qquad x\Vdash_v\langle\Sigma\rangle 1\ \&\ y\Vdash_v\langle\Sigma\rangle 1.
\end{aligned}
$$

It is easy to see that the definitions of the relations $\equiv'$, $\cong'$ σ', Σ' are correct in the following sense: for any $R\in\{\equiv,\cong,\sigma,\Sigma\}$ and $x,x',y,y'\in U$: if $x\sim x'$, $y\sim y'$ and $|x|R'|y|$ then $|x'|R'|y'|$.

The proof of the theorem follows from the following lemmas.

Lemma 3.15. (i) $x\equiv y \longrightarrow |x|\equiv'|y|$,

(ii) $\models|x|\equiv'|y| \longrightarrow (\forall[\equiv]A\in\Gamma)(x\Vdash_{v}[\equiv]A \longrightarrow y\Vdash_{v}A)$.

(iii) The relation $\equiv'$ satisfies the conditions S1, S2 and S3.

Proof. (i) For the case of $\equiv$ suppose $x\equiv y$ and $[\equiv]A\in\Gamma$. Then we have to prove the following equivalences:

(1) $x\Vdash_{v}[\equiv]A \longleftrightarrow y\Vdash_{v}[\equiv]A$,

(2) $x\Vdash_{v}[\cong]A \longleftrightarrow y\Vdash_{v}[\cong]A$,

(3) $x\Vdash_{v}[\sigma]A \longleftrightarrow y\Vdash_{v}[\sigma]A$,

(4) $x\Vdash_{v}[\Sigma]A \longleftrightarrow y\Vdash_{v}[\Sigma]A$.

(5) $x\Vdash_{v}\langle\Sigma\rangle 1 \longleftrightarrow y\Vdash_{v}\langle\Sigma\rangle 1$.

(6) $x\Vdash_{v}\langle\sigma\rangle 1 \longleftrightarrow y\Vdash_{v}\langle\sigma\rangle 1$.

(1)($\longrightarrow$) Suppose $x\Vdash_{v}[\equiv]A$, $y\equiv z$ and proceed to show that $z\Vdash_{v}A$. From $x\equiv y$ and $y\equiv z$ by **I1** we obtain $x\equiv z$ and by $x\Vdash_{v}[\equiv]A$ we get $z\Vdash_{v}A$.

($\longleftarrow$) Suppose $y\Vdash_{v}[\equiv]A$, $x\equiv z$ and proceed to show that $z\Vdash_{v}A$. From $x\equiv y$ we obtain by **I2** $y\equiv x$ and by $x\equiv z$ and **I3** we get $y\equiv z$. Then from $y\Vdash_{v}[\equiv]A$ and $y\equiv z$ we get $z\Vdash_{v}A$.

(2)($\longrightarrow$) Suppose $x\Vdash_{v}[\cong]A$, $y\cong z$ and proceed to show that $z\Vdash_{v}A$. From $x\equiv y$ and $y\cong z$ we get by **I6** $x\cong z$ and since $x\Vdash_{v}[\cong]A$ we obtain $z\Vdash_{v}A$.

($\longleftarrow$) Suppose $y\Vdash_{v}[\cong]A$, $x\cong z$ and proceed to show that $z\Vdash_{v}A$. From x we get by **I2** $y\equiv x$ and since $x\cong z$, we obtain by **I6** $y\cong z$. Then from $y\Vdash_{v}[\cong]A$ and $y\cong z$ we get $z\Vdash_{v}A$.

For (3), (4), (5) and (6) we can proceed in a similar way.

(ii) Suppose $|x|\equiv'|y|$, $[\equiv]A\in\Gamma$ and $x\Vdash_{v}[\equiv]A$. Then $y\Vdash_{v}[\equiv]A$ and by A1 - $y\Vdash_{v}A$.

(iii) Directly from the definition of $\equiv'$ we see that $\equiv'$ is an equivalence relation, hence **I1**, **I2** and **I3** are fulfilled.■

Lemma 3.16.

(i) $x\cong y \longrightarrow |x|\cong'|y|$,

(ii) $|x|\cong|y| \longrightarrow (\forall[\cong]A\in\Gamma)(x\Vdash_{v}[\cong]A \longrightarrow A)$.

(iii) The relation $\cong'$ satisfies the conditions **I4**, **I5** and **I6**.

Proof. (i) Suppose $x\cong y$. Then since $|x|\equiv'|x|$, $|y|\equiv'|y|$ we get $|x|\cong'|y|$.

(ii) Let $|x|\cong'|y|$, $[\cong]A\in\Gamma$, $x\Vdash_{v}[\cong]A$ and proceed to show that $y\Vdash_{v}A$. From here we obtain that for some $|p|$ and $|q|$ $|x|\equiv'|p|$, $|y|\equiv'|q|$ $p\cong q$, and $[\equiv]A\in\Gamma$. Then, from $|x|\equiv'|p|$, $[\equiv]A\in\Gamma$ and $x\Vdash_{v}[\cong]A$ we get $p\Vdash_{v}[\cong]A$. It is easy to see from the complete- ness theorem that $[\cong]A\Rightarrow[\cong][\equiv]A$ is a theorem of **IND-3**. So $p\Vdash_{v}[\cong][\equiv]A$. Then by $p\cong q$ we get $q\Vdash_{v}[\equiv]A$. Since $[\equiv]A\in\Gamma$ and $|y|\equiv'|q|$ we obtain $y\Vdash_{v}[\equiv]A$, and by $y\equiv y$ - $y\Vdash_{v}A$.

(iii) From $x\cong x$ we obtain $|x|\cong'|x|$, so **I4** is fulfilled.

For **I5** suppose $|x|\cong'|y|$. Then we have for some $|p|$ and $|q|$: $|x|\equiv'|p|$,

$|y|\equiv'|q|$, $p\cong q$ and by I5 - $q\cong p$. From here, by the definition of $\cong'$ we obtain $|y|\cong'|x|$, which had to be proved.

For **I6** suppose $|x|\equiv'|y|$, $|y|\cong'|z|$ and proceed to show $|x|\cong'|z|$. From $|y|\cong'|z|$ we get $|y|\equiv'|p|$, $|z|\equiv'|q|$ and $p\cong q$. From $|x|\equiv'|y|$ and $|y|\equiv'|p|$ we obtain by **I3** $|x|\equiv'|p|$. Then from $|x|\equiv'|p|$, $|z|\equiv'|q|$ and $p\cong q$ we get $|x|\cong'|z|$.■

Lemma 3.17.

(i) $x\sigma y \longrightarrow |x|\sigma'|y|$,

(ii) $|x|\sigma'|y| \longrightarrow (\forall[\sigma]A\in\Gamma)(x\Vdash_{v}[\sigma]A \longrightarrow y\Vdash_{v}A)$.

(iii) The relation σ' satisfies the conditions S7, S8 and S9.

The proof of this lemma is similar to that of lemma 3.16. ■

Lemma 3.18.

(i) $x\Sigma y \longrightarrow |x|\Sigma'|y|$,

(ii) $|x|\Sigma'|y| \longrightarrow (\forall[\Sigma]A\in\Gamma)(x\Vdash_{v}[\Sigma]A \longrightarrow y\Vdash_{v}A)$.

(iii) The relation Σ' satisfies the conditions I10 -I16.

Proof. (i) Suppose $x\Sigma y$. Then we have $x\Vdash_{v}\langle\Sigma\rangle 1$ and $y\Vdash_{v}\langle\Sigma\rangle 1$. From here and $|x|\equiv'|x|$, $|y|\equiv'|y|$ and $x\Sigma y$ we infer $|x|\Sigma'|y|$.

(ii) Suppose $|x|\Sigma'|y|$, $[\Sigma]A\in\Gamma$, $x\Vdash_{v}[\Sigma]A$ and proceed to show that $y\Vdash_{v}A$. We have to consider three cases.

Case 1: $(\exists p,q)(|x|\equiv'|p|\ \&\ |y|\equiv'|q|\ \&\ p\Sigma q)$.

From $[\Sigma]A\in\Gamma$ we get $[\equiv]A\in\Gamma$. Then $|x|\equiv'|p|$, $[\equiv]A\in\Gamma$ and $x\Vdash_{v}[\Sigma]A$ we obtain $p\Vdash_{v}[\Sigma]A$. It is easy to see/for instance by the completeness theorem for IND-3 that $[\Sigma]A\Rightarrow[\Sigma][\equiv]A$ is a theorem of IND-3. Then $p\Vdash_{v}[\Sigma][\equiv]A$, and by $p\Sigma q$ - $q\Vdash_{v}[\equiv]A$. Since $|y|\equiv'|q|$, we get $y\Vdash_{v}[\equiv]A$ and by A1 - $y\Vdash_{v}A$.

Case 2: $(\exists p,q)(|x|\cong'|p|\ \&\ |y|\equiv'|q|\ \&\ p\sigma q)$.

From $|x|\cong'|p|$ we obtain $(\exists u,v)(|x|\equiv'|u|\ \&\ |p|\equiv'|v|\ \&\ u\cong v)$. From $|x|\cong'|u|$, $[\equiv]A\in\Gamma$ and $x\Vdash_{v}[\Sigma]A$ we get $u\Vdash_{v}[\Sigma]A$, and by the axiom $[\Sigma]A\Rightarrow[\cong][\sigma]A$ we obtain $u\Vdash_{v}[\cong][\sigma]A$. Then, since $u\cong v$, we get $v\Vdash_{v}[\sigma]A$. From here and $|p|\equiv'|v|$ we obtain $p\Vdash_{v}[\sigma]A$. It is easy to check /by the completeness theorem, for instance/ that $[\sigma]A\Rightarrow[\sigma][\equiv]A$ is a theorem of IND-3. Consequently $p\Vdash_{v}[\sigma][\equiv]A$, and by $p\sigma q$ - $q\Vdash_{v}[\equiv]A$. Then from $|y|\equiv'|q|$ we get $y\Vdash_{v}[\equiv]A$ and by A1 - $y\Vdash_{v}A$.

Case 3: $(\exists p,q)(|x|\equiv'|p|\ \&\ |y|\cong'|q|\ \&\ p\sigma q)$.

This case can be treated in the same way as case 2.

(iii) **Proof of I10.** The symmetry of Σ' follows directly from the definition of Σ' and the symmetry of σ and Σ.

Proof of I11. Suppose $|x|\Sigma'|y|$. Then $x\Vdash_{v}\langle\Sigma\rangle 1$, so $x\Sigma t$ for some t and by I10 we obtain $x\Sigma x$. Then by (i) of this lemma we obtain $|x|\Sigma'|x|$.

Proof of I12. Suppose $|x|\equiv'|y|$, $|y|\Sigma'|z|$ and proceed to show that $|x|\Sigma'|z|$. From $|y|\Sigma'|z|$ we obtain $z\Vdash_{v}\langle\Sigma\rangle 1$, $y\Vdash_{v}\langle\Sigma\rangle 1$ and by $|x|\equiv'|y|$ we get $x\Vdash_{v}\langle\Sigma\rangle 1$. For the remaining part of the proof we have to consider three cases.

Case 1: $(\exists p,q)(|y|\equiv'|p|\ \&\ |z|\equiv'|q|\ \&\ p\Sigma q)$.

From $|x|\equiv'|y|$ and $|y|\equiv'|p|$ we get $|x|\equiv'|p|$. Then $|x|\equiv'|p|$, $|z|\equiv'|q|$, $p\Sigma q$, $x\Vdash_{v}\langle\Sigma\rangle 1$ and $z\Vdash_{v}\langle\Sigma\rangle 1$ imply $|x|\Sigma'|z|$.

Case 2: $(\exists p,q)(|y|\cong'|p|\ \&\ |z|\equiv'|q|\ \&\ p\sigma q)$.

From $|x|\cong'|y|$ and $|y|\cong'|p|$ we get by I6 $|x|\cong'|p|$. Then $|x|\cong'|p|$, $|z|\equiv'|q|$,

$p\sigma q$, $x\|\underset{v}{\longrightarrow}\langle\Sigma\rangle 1$ and $z\|\underset{v}{\longrightarrow}\langle\Sigma\rangle 1$ imply $|x|\Sigma'|z|$.

Case 3: $(\exists p,q)(|y|{\cong}'|p|\ \&\ |z|{\cong}'|q|\ \&\ p\sigma q)$. This case can be treated as case 1.

Proof of I13. By I13 we have $x\equiv y$ or $x\Sigma x$ or $y\Sigma y$. Then by lemma 3.15.i and 3.18.i we obtain $|x|{\equiv}'|y|$ or $|x|\Sigma'|x|$ or $|y|\Sigma'|y|$.

Proof of I14. Suppose $|x|{\cong}'|y|$, $|z|\bar{\Sigma}'|z|$, $|x|\bar{\Sigma}'|y|$, and proceed to show that $|x|{\cong}'|z|$.

From $|x|{\cong}'|y|$ we get $(\exists p,q)(|x|{\equiv}'|p|\ \&\ |y|{\equiv}'|q|\ \&\ p\cong q)$.

From $|z|\bar{\Sigma}'|z|$ we get by (i) $z\bar{\Sigma}z$.

From $|x|\bar{\Sigma}'|y|$ we get $(\forall p,q)(|x|{\equiv}'|p|\ \&\ |y|{\equiv}'|q| \longrightarrow p\bar{\Sigma}q)$ or $x\|\underset{v}{\not\longrightarrow}\langle\Sigma\rangle 1$ or $y\|\underset{v}{\not\longrightarrow}\langle\Sigma\rangle 1$. We have to consider three cases.

Case 1: $(\forall p,q)(|x|{\equiv}'|p|\ \&\ |y|{\equiv}'|q| \longrightarrow p\bar{\Sigma}q)$. From here we obtain $p\bar{\Sigma}q$. Then $p\cong q$, $p\bar{\Sigma}q$, $z\bar{\Sigma}z$ imply by S14 $p\cong z$. From $|x|{\equiv}'|p|$, $|z|{\equiv}'|z|$ and $p\cong z$ we obtain $|x|{\cong}'|z|$.

Case 2: $x\|\underset{v}{\not\longrightarrow}\langle\Sigma\rangle 1$. Then from $|x|{\equiv}'|p|$ we get $p\|\underset{v}{\not\longrightarrow}\langle\Sigma\rangle 1$. This implies $p\bar{\Sigma}q$. From $p\cong q$, $p\bar{\Sigma}q$, $z\bar{\Sigma}z$ we get by I14 $p\cong z$. Then, as in case 1, we infer $|x|{\cong}'|z|$.

Case 3: $y\|\underset{v}{\not\longrightarrow}\langle\Sigma\rangle 1$. Then from $|y|{\equiv}'|q|$ we get $q\|\underset{v}{\not\longrightarrow}\langle\Sigma\rangle 1$. From here we obtain $p\bar{\Sigma}q$. Then as in case 2 we infer $|x|{\cong}|z|$.

Proof of I15. Suppose $|x|{\cong}'|y|$, $|y|\sigma'|z|$ and proceed to show $|x|\Sigma'|z|$.

From $|y|\sigma'|z|$ we obtain $(\exists p,q)(|y|{\equiv}'|p|\ \&\ |z|{\equiv}'|q|\ \&\ p\sigma q)$.

From $|x|{\cong}'|y|$ and $|y|{\equiv}'|p|$ we get by lemma 3.16.iii $|x|{\cong}|p|$. From $p\sigma q$ we get $p\|\underset{v}{\longrightarrow}\langle\sigma\rangle 1$ and $q\|\underset{v}{\longrightarrow}\langle\sigma\rangle 1$. Then by $|y|{\equiv}'|p|$ and $|z|{\equiv}'|q|$ we obtain $y\|\underset{v}{\longrightarrow}\langle\sigma\rangle 1$ and $z\|\underset{v}{\longrightarrow}\langle\sigma\rangle 1$. From $z\|\underset{v}{\longrightarrow}\langle\sigma\rangle 1$ we get $z\|\underset{v}{\longrightarrow}\langle\Sigma\rangle 1$.

We shall show that $|x|{\cong}'|y|$ and $y\|\underset{v}{\longrightarrow}\langle\sigma\rangle 1$ imply $x\|\underset{v}{\longrightarrow}\langle\Sigma\rangle 1$. From $|x|{\cong}'|y|$ we have $(\exists r,s)(|x|{\equiv}'|r|\ \&\ |y|{\equiv}'|s|\ \&\ r\cong s)$. Then from $|y|{\equiv}'|s|$ and $y\|\underset{v}{\longrightarrow}\langle\sigma\rangle 1$ we get $s\|\underset{v}{\longrightarrow}\langle\sigma\rangle 1$, so $s\sigma t$ for some t. From $r\cong s$ and $s\sigma t$ we get by I15 $r\Sigma t$, so $r\|\underset{v}{\longrightarrow}\langle\Sigma\rangle 1$. Then from $|x|{\equiv}'|r|$ we obtain $x\|\underset{v}{\longrightarrow}\langle\Sigma\rangle 1$.

We have obtained: $|x|{\cong}'|p|$, $|z|{\equiv}'|q|$, $p\sigma q$, $x\|\underset{v}{\longrightarrow}\langle\Sigma\rangle 1$ and $z\|\underset{v}{\longrightarrow}\langle\Sigma\rangle 1$. This implies $|x|\Sigma'|z|$.

Proof of I16. From I16 we have $x\cong y$ or $x\sigma x$ or $y\Sigma y$. Then by lemma 3.16, 3.17 and 3.18 we obtain $|x|{\cong}'|y|$ or $|x|\sigma'|x|$ or $|y|\Sigma'|y|$.■

Corollary 3.19.
IND-3 has the finite model property and is decidable.

CHAPTER 2.5

A MODAL LOGIC FOR ATTRIBUTE SYSTEMS WITH CONSTANT FOR SINGLE-VALUEDNESS

Overview. In this chapter we introduce a modal logic for A- systems with propositional constant D, interpreted in the set of single-valued objects. This makes possible to consider modal logics for arbitrary A-systems and single-valued A-systems in an uniform way.

Introduction

In chapter 2.3 we introduced a number of modal logics for A- systems some of them interpreted only in single-valued A-systems. In this chapter we shall introduce a special propositional constant D, which makes possible to consider modal logics for arbitrary A-systems and single-valued A-systems in an uniform way.

First we shall introduce the set $D=D_S$ of single-valued elements in an A-system S:

$D_S=\{x\in OB/(\forall a\in AT_S)(Card f(x,a)\leq 1)\}$

We say that an object $x\in OB_S$ is single-valued if the set of his values f(x,a) has cardinality ≤ 1 for any attribute $a\in AT_S$. Obviously S is single-valued system if $D_S=OB_S$ and this makes possible to consider together the general and the single-valued case.

The set D can be used for interpretation of a special propositional constant, denoted by **D** as follows:

$x\Vdash_v$ **D** iff $x\in D$

If we have a modal logic for A-systems containing the constant **D**, then putting **D** as a theorem for the logic we automatically obtain the logic, corresponding to the class of single-valued A- systems. This, however, makes the problem of the axiomatization of a given set of modalities with D much more difficult. For instance the axiomatization of SIM-2, IND-2 and IND-3 with **D** is an open problem. In this chapter we shall consider a particular example of a logic for A-systems with the constant **D**. We denote the logic shortly by IL - information logic. It contains modalities for strong informational inclusion [≤]A, for strong indiscernibility [≡]A and for strong similarity [σ]A.

The results of this chapter are contained, with a slightly different terminology, in [Vak 89].

1. Abstract characterization theorem for ≤, ≡, σ and D

Let S=(OB, AT, {VALa/a∈AT}, f) be an A-system. Let us remind the definitions of the relations $\equiv_S$, $\leq_S$, σ_S and D_S:

$x\equiv_S y$ iff $(\forall a\in AT)(f(x,a)=f(y,a))$ - indiscernibility relation, *$x\leq_S y$ iff $(\forall a\in AT)(f(x,a)\subseteq f(y,a))$ - informational inclusion,*

$x\sigma_S y$ iff $(\forall a\in AT)(f(x,a)\cap f(y,a)\neq\emptyset)$ - similarity relation,

$D_S=\{x\in OB/(\forall a\in AT)(Card f(x,a)\leq 1)\}$ - the set of single-valued objects in S. S is deterministic A-system iff $D_S=OB_S$.

Sometimes, when this will not cause a confusion, we will omit the subscript S in the above relations.

An A-system S is called *separable* if the following condition is satisfied for any a∈AT:

$(\forall A,B\in VALa)((\forall x\in OB)(A\in f(x,a)$ iff $B\in f(x,a))\rightarrow A=B)$

Now we shall remind the construction of set-theoretical A- system, which will give a method of construction of separable A-systems.

Let (U,V) be a pair, U≠∅ and V⊆P(P(U)), i.e. V is a set whose elements are collections of subsets of U. Set OB=U, AT=V, for each a∈AT define VALa=a, and for x∈OB and a∈AT define f(x,a)={A∈a/x∈A}. Then obviously f(x,a)⊆a=VALa and the constructed system S=(OB, AT, {VALa/a∈AT}, f) is an A-system, which we will call *set-theoretical A-system* under the pair (U,V) and will denote it also by (U,V).

Lemma 1.1.

Let S=(U,V) be a set-theoretical A-system. Then:

(i) *S is a separable A-system.*

(ii) $x\le_S y$ *iff* $(\forall a\in V)(\forall A\in a)(x\in A \longrightarrow y\in A)$.

(iii) $x\equiv_S y$ *iff* $(\forall a\in V)(\forall A\in a)(x\in A \longleftrightarrow y\in A)$ *iff* $x\le_S y$ *and* $y\le_S x$.

(iv) $x\sigma_S y$ *iff* $(\forall a\in V)(\exists A\in a)(x\in A$ *and* $y\in A)$.

(v) $x\in D_S$ *iff* $(\forall a\in V)(\forall A,B\in a)(x\in A$ *and* $x\in B \longrightarrow A=B)$.

Lemma 1.2.

(i) Let S be an A-system. Then the following conditions hold:

S1. $x\le x$,

S2. $x\le y$ and $y\le z \longrightarrow x\le z$,

S3. $x\sigma y \longrightarrow y\sigma x$,

S4. $x\sigma y \longrightarrow x\sigma x$,

S5. $x\sigma y$ and $x\le z \longrightarrow z\sigma y$,

S6. $y\in D$ and $x\le y \longrightarrow x\in D$,

S7. $x\in D$ and $x\sigma y \longrightarrow x\le y$,

S8. $x\equiv x$,

S9. $x\equiv y \longrightarrow y\equiv x$,

S10. $x\equiv y$ and $y\equiv z \longrightarrow x\equiv z$,

S11. $x\equiv y \longrightarrow x\le y$,

S12. $x\in D$, $y\in D$, $x\sigma y \longrightarrow x\equiv y$,

Sa. $x\le y$ and $y\le x \longrightarrow x\equiv y$.

(ii) If S is a separable A-system *then*

Sb. $x\notin D \longrightarrow (\exists y\in OB)(x\not\le y)$.

Proof. As an example we shall verify S7 and Sb.

For S7 suppose x∈D and xσy. We have to show that $(\forall a\in AT)(f(x,a)\subseteq f(y,a))$. Suppose for that purpose that $a\in AT$, $A\in VALa$ and $A\in f(x,a)$. We will show that $A\in f(y,a)$. Since x∈D then $f(x,a)$ has at most one element and since $A\in f(x,a)$ then $f(x,a)=\{A\}$. Since xσy then $f(x,a)\cap f(y,a)\neq\emptyset$, so $\{A\}\cap f(x,a)\neq\emptyset$ and hence $A\in f(y,a)$.

For Sb suppose S is separable and x∉D. Then there exists $a\in AT$ such that $f(x,a)$ has at least two elements A≠B. Since S is separable then there exists y∈OB such that

$$\text{not}(A\in f(y,a) \longleftrightarrow B\in f(y,a)).$$

Then $(A\in f(y,a)\ \&\ B\notin f(y,a))$ or $(A\notin f(y,a)\ \&\ B\in f(y,a))$. Since $A,B\in f(x,a)$ then in both cases we obtain that $f(x,a)\not\subseteq f(y,a)$ and hence - not x≤y. ■

Let **U**=(U, ≡, ≤, σ, D) be an abstract relational system with U≠∅, ≡, ≤, σ - binary relations in U, and D⊆U. We say that U is a *D-structure* if it satisfies the conditions S1 - S12, Sa and Sb from lemma 1.2.; **U** is a *generalized D-structure* if it satisfies the conditions S1-S12.

Let S=(OB, AT, {VAL*a*/*a*∈AT}, f) be an *A*-system. Then the relational system $(OB, \equiv_S, \le_S, \sigma_S, D_S)$ is called a *standard D-structure over S*.

From Lemma 1.2 we obtain the following

Lemma 1.3.

(i) Let S be an arbitrary A-system*. Then the standard* D-*structure over S is a generalized* D-*structure satisfying the condition Sa.*

(ii) If S is a separable A-system *then the standard* D-*structure over S is a* D-*structure.*

Now we shall show that any D-structure is a D-structure over some separable

A-system. To prove this we first introduce some constructions in A-systems.

Let **U**=(U, ≡, ≤, σ, D) be a D-structure. A subset A⊆U is called:

a *≤-monotonic set* if (∀x,y∈U)(x∈A and x≤y ⟶ y∈A).

a *σ-set* if it is a ≤-monotonic set and for any x,y from A we have xσy.

We introduce the following operation: [x]={y∈U/x≤y}, x∈U.

Lemma 1.4.

(i) For any x∈U the set [x] is the smallest ≤-monotonic set containing x.

(ii) For any u,v∈U the set [u]∪[v] is the smallest ≤-monotonic set containing u and v.

(iii) If uσv then [u]∪[v] is a σ-set.

(iv) If u∈D∩A and A is a σ-set then A=[u].

(v) If A is a σ-set and B is ≤-monotonic set and A∩B∩D≠∅ then A⊆B.

(vi) If A and B are σ-sets and A∩B∩D≠∅ then A=B.

Proof - The proof of (i)-(iii) is straightforward.

Proof of (iv). Suppose u∈D∩A and A is a σ-set. Then by (i) [u]⊆A. To show A⊆[u] suppose x∈A. Since A is a σ-set then uσx and since u∈D then by S7 we have u≤x and hence x∈[u]. So A=[u].

Proof of (v). Suppose A is a σ-set, B is a ≤-monotonic set and A∩B∩D≠∅. Then for some x: x∈A, x∈B and x∈D. By (iv) A=[x], by (i) [x]⊆B, and so A⊆B.

Proof of (vi) - apply (v). ■

Theorem 1.5 /*Abstract characterization theorem*/

Any D-structure is a standard D-structure over separable A-system.

Proof. Let **U**=(U, ≡, ≤, σ, D) be a *D*-structure. We shall construct a set-theoretical A-system S=(U,V) /which is separable!/ in such a way that **U** and the standard D-structure over S will coincide.

Since U is given it remains to construct V. Let a∈P(P(U)), i.e. a is a set of subsets of U We shall say that a is a *good set* if it satisfies the following conditions:

(*) x≤y ⟶ (∀A∈a)(x∈A ⟶ y∈A),

(**) xσy ⟶ (∃A∈a)(x∈A and y∈A),

(***) x∈D ⟶ (∀A,B∈a)(x∈A and x∈B ⟶ A=B).

Then define V={*a*∈P(P(U))/*a* is a good set}. The remaining part of the proof follows from the following lemma.

Lemma 1.6.

(i) x≤y ⟷ (∀a∈V)(∀A∈a)(x∈A ⟶ y∈A),

(ii) x≡y ⟷ (∀a∈V)(∀A∈a)(x∈A ⟷ y∈A),

(iii) xσy ⟷ (∀a∈V)(∃A∈a)(x∈A and y∈A),

(iv) x∈D ⟷ (∀a∈V)(∀A,B∈a)(x∈A and x∈B ⟶ A=B).

Proof. (i)(⟶). Suppose x≤y, A∈*a* and x∈A. Then by (*) y∈A.

(⟵). Suppose not x≤y. We shall show that there exist *a*∈V and A∈a such that x∈A and y∉A. Take A=[x] and put *a*={A}∪{[u]∪[v]/ uσv and A∩D∩([u]∪[v])=∅}. Obviously A∈*a*, x∈A and y∉A. It remains to show only that *a*∈V. By lemma 1.4(i) and (ii) we see that all elements of the set *a* are ≤-monotonic sets and hence (*) is fulfilled. To show that (**) is fulfilled suppose uσv. We have two cases:

Case 1. A∩D∩([u]∩[v])=∅. Then by the definition of set *a* we have [u]∪[v]∈*a* and u,v∈[u]∪[v].

Case 2. A∩D∩([u]∪[v])≠∅. Since uσv then by lemma 1.4(iii) [u]∪[v] is a σ-set. Then by lemma 1.4(v) [u]∪[v]⊆A, so u,v∈A∈*a*.

To show (***) suppose u∈D; B,C∈*a*, u∈B and u∈C. From here we obtain B∩C∩D≠∅. If B≠A and C≠A /remember that A=[x]/ then B=[u']∪[v'] for u'σv' and C=[u"]∪[v"] for u"σv". Then by lemma 1.4(iii) B and C are σ-sets and by lemma 1.4(vi) we have B=C. By the definition of the set *a* and the condition B∩C∩D≠∅ the cases (B=A,C≠A) and (B≠A,C=A) are not possible. So the final case is B=A and C=A, which also gives B=C.

Proof of (ii). $x\equiv y$ iff $x\le y$ and $y\le x$ iff $(\forall a\in V)(\forall A\in a)(x\in A\longleftrightarrow y\in A)$.

Proof of (iii). $(\longrightarrow)$. Suppose $x\sigma y$ and $a\in V$. Then by (**) there exists $A\in a$ such that $x\in A$ and $y\in A$.

$(\longleftarrow)$ Suppose now that not $x\sigma y$. We shall show that there exists $a\in V$ such that for any $A\in a$ we have $x\notin A$ or $y\notin A$.

Define $a=\{[u]\cup[v]/u\sigma v\}$. By lemma 1.4(iii) all elements of a are σ-sets. Since we have not $x\sigma y$ then $x\notin A$ or $y\notin A$ for any $A\in a$, because otherwise we will have $x\sigma y$ - a contradiction. It remains to show that $a\in V$. Condition (*) is fulfilled, because all elements of a are $\le$-monotonic sets; condition (**) follows by definition, and condition (***) follows from lemma 1.4(iii) and (vi).

Proof of (iv). $(\longrightarrow)$. Suppose $x\in D$, $a\in V$, $A,B\in a$ and $x\in A, x\in B$. Then by (***) we have A=B.

$(\longleftarrow)$ Suppose $x\notin D$. We shall show that there exist $a\in V$ and $A,B\in a$ such that $x\in A$, $x\in B$ and $A\ne B$. Since $x\notin D$ then by Sb there exists $y\in U$ such that not $x\le y$. Define $A=[x]$, $B=[x]\cup[y]$ and $a=\{A,B\}\cup\{[u]\cup[v]/u\sigma v$ and $D\cap[y]\cap([u]\cup[v])=\emptyset\}$. Obviously $x\in[x]$, $y\notin[x]$ /because not $x\le y$/, $x,y\in[x]\cup[y]$, so $x\in A$, $x\in B$, $y\in B$, $y\notin A$ and hence $A\ne B$. For $a\in V$ we see that (*) is fulfilled. To show (**) suppose $u\sigma v$. We have two cases:

Case 1: $D\cap[y]\cap([u]\cup[v])=\emptyset$. Then by the definition of a we have $[u]\cup[v]\in a$ and $u,v\in [u]\cup[v]$.

Case 2: $D\cap[y]\cap([u]\cup[v])\ne\emptyset$. Since $[u]\cup[v]$ is a σ-set and $[y]$ is a $\le$-monotonic set then by lemma 1.4(iv) we have $[u]\cup[v]\subseteq[y]\subseteq[x]\cup[y]=B$, so $u,v\in B\in a$.

To show (***) suppose $u\in D$, $P,Q\in a$, $u\in P$, $u\in Q$. We have to show that P=Q. It is not possible to have P=A or Q=A. Otherwise $u\in[x]$, and since $u\in D$, by S6 we will obtain $x\in D$, contrary to $x\notin D$. So $P\ne A$ and $Q\ne A$. We shall show that if P=B then Q=B and so P=Q. Suppose P=B and for the sake of contradiction that $Q\ne B$. Since $Q\ne A$ then by the definition of a we have $Q=[u']\cup[v']$ with $u'\sigma v'$, and $D\cap[y]\cap([u']\cup[v'])=\emptyset$. From here $u\notin[y]$, because $u\in D$ and $u\in Q$. But $u\in B=[x]\cup[y]$, so $u\in[x]$, $x\le u$ and since $u\in D$, we obtain by S6 that $x\in D$ - a contradiction.

The remaining case is $P,Q\notin\{A,B\}$. Then by the definition of set a we have $P=[u']\cup[v']$ with $u'\sigma v'$ and $Q=[u'']\cup[v'']$ with $u''\sigma v''$ and hence P and Q are σ-sets such that $P\cap Q\cap D\ne\emptyset$. Then by lemma 1.4(vi) we have P=Q. This ends the proof of lemma 1.6 and the proof of theorem 1.5.■

2. The Information Logic IL

The language of IL is an extension of the language of propositional logic with three modal connectives:

$[\equiv]$, $[\le]$, $[\sigma]$

D - propositional constant for single-valuedness.

The set FOR of formulas is defined in the usual way - it is the least set containing VAR and **D** and closed under Boolean and modal connectives.

We will use the standard abbreviations: $\mathbf{1}=p\vee\neg p$, $\mathbf{0}=\neg\mathbf{1}$, $A\Rightarrow B=\neg A\vee B$, $A\Leftrightarrow B=(A\Rightarrow B)\wedge(B\Rightarrow A)$, $\langle\equiv\rangle=\neg[\equiv]\neg$, $\langle\le\rangle=\neg[\le]\neg$, $\langle\sigma\rangle=\neg[\sigma]\neg$.

The semantics of IL will be in relational structures of the type of D-structures. If v is a valuation in a structure $\mathbf{U}=(U, \equiv, \le, \sigma, D)$, the satisfiability relation $x\Vdash_{v} A$ is defined inductively as in the usual Kripke semantics. For instance the clauses for **D** and the modalities are the following:

$x\Vdash_{v}\mathbf{D}$ iff $x\in D$,

$x\Vdash_{v}[R]A$ iff $(\forall y\in U)(xRy\longrightarrow y\Vdash_{v}A)$ for $R\in\{\equiv,\le,\sigma\}$.

The set of all formulas, true in a class Σ of modal structures will be denoted by $L(\Sigma)$ - the *logic of* Σ. We will be interested in the following classes: Σ_0 - generalized D-structures, Σ_1 - standard D-structures over arbitrary A-systems, Σ_2 - standard D- structures over separable A-systems, Σ_3 - D-structures. Models based on structures from Σ_1 and Σ_2 will be called standard and models over structures from Σ_0 will be called non-standard models. We know from theorem 1.5 that $\Sigma_2=\Sigma_3$ and hence that $L(\Sigma_2)=L(\Sigma_3)$. Natural logics are $L(\Sigma_1)$ and $L(\Sigma_2)$ and we intend to define the logic IL to be $L(\Sigma_1)$. Our aim is to axiomatize $L(\Sigma_1)$. We will do this in the following order: first we will axiomatize $L(\Sigma_0)$ and then we will show that $L(\Sigma_0)=L(\Sigma_1)=$ $=L(\Sigma_2)=L(\Sigma_3)$.

The first step in the axiomatization of $L(\Sigma_0)$ is to find a modal translation of axioms S1 - S12 of Σ_0. We will do this in the next section.

3. Modal translations of the axioms of generalized D-structures

Consider the following table. On the left side we have the list of the axioms S1 - S12 of generalized D-structures, and on the right side A1 -A12 with $A\in VAR$ are the corresponding modal translations in the sense of the next theorem.

S1.	$x\leq x$	A1.	$[\leq]A\Rightarrow A$
S2.	$x\leq y$ & $y\leq z \longrightarrow x\leq z$	A2.	$[\leq]A\Rightarrow[\leq][\leq]A$
S3.	$x\sigma y \longrightarrow y\sigma x$	A3.	$A\vee[\sigma]\neg[\sigma]A$
S4.	$x\sigma y \longrightarrow x\sigma x$	A4.	$\langle\sigma\rangle\mathbf{1}\wedge[\sigma]A \Rightarrow A$
S5.	$x\sigma y$ & $x\leq z \longrightarrow z\sigma y$	A5.	$\langle\leq\rangle[\sigma]A\Rightarrow[\sigma]A$
S6.	$y\in D$ & $x\leq y \longrightarrow x\in D$	A6.	$\langle\leq\rangle\mathbf{D}\Rightarrow\mathbf{D}$
S7.	$x\in D$ & $x\sigma y \longrightarrow x\leq y$	A7.	$\mathbf{D}\wedge[\leq]A\Rightarrow[\sigma]A$
S8.	$x\equiv x$	A8.	$[\equiv]A\Rightarrow A$
S9.	$x\equiv y \longrightarrow y\equiv x$	A9.	$A\vee[\equiv]\neg[\equiv]A$
S10.	$x\equiv y$ & $y\equiv z \longrightarrow x\equiv z$	A10.	$[\equiv]A\Rightarrow[\equiv][\equiv]A$
S11.	$x\equiv y \longrightarrow x\leq y$	A11.	$[\leq]A\Rightarrow[\equiv]A$
S12.	$x\in D$ & $y\in D$ & $x\sigma y \longrightarrow x\equiv y$,	A12.	$\mathbf{D}\wedge[\equiv]A\Rightarrow[\sigma](\mathbf{D}\Rightarrow A)$

Theorem 3.1.

Let $\mathbf{U}=(U, \equiv, \leq, \sigma, D)$ *be relational system similar to generalized D-structure. Then for any i=1,2,...,12 the formula Ai is true in* **U** *iff the condition Si holds for* **U**.

Proof - standard.

4. Axiomatic definition of IL. Canonical model

Theorem 3.1 suggests the following axiomatization of $L(\Sigma_0)$.

Axiom schemes:

(Bool) All or enough Boolean tautologies,

(Mod) $[R](A\Rightarrow B)\Rightarrow([R]A\Rightarrow[R]B)$ for $R\in\{\equiv,\leq,\sigma\}$,

A1-A12 from theorem 3.1.

Rules of inference:

(MP) $\dfrac{A,\ A\Rightarrow B}{B}$, (N) $\dfrac{A}{[R]A}$ for $R\in\{\equiv,\leq,\sigma\}$.

We define IL to be the smallest subset of FOR containing all axioms and closed under (MP), (N) and substitution. Note that A1 and A2 are the well known axioms for the modal logic S4 and A8, A9 and A10 are the axioms for the modal logic S5. So IL contains the logics S4 and S5.

Theorem 4.1. */Completeness theorem for IL with respect to generalized D-structures/*

The following conditions are equivalent for any formula of IL:

(i) A is a theorem of IL,

(ii) A is true in all generalized D-structures.

Proof. (i) ⟶ (ii). Use lemma 3.1.

(ii) ⟶ (i) - by the standard canonical construction. Note that in the canonical structure D is defined by the set of all maximal consistent sets, which contain the constant **D**. The axioms of IL guarantee that the canonical structure of IL is a generalized D-structure. ∎

5. Completeness theorem for IL with respect to its standard semantics

The main aim of this section is to prove that IL=L(Σ_1) - the logic of the class Σ_1 of all standard D-structures over arbitrary A- systems. For that purpose we shall show first that the logic L(Σ_0) of all generalized D-structures coincides with the logic L(Σ_3) of all D-structures. To this aim we shall use the copying construction. We will give the corresponding definition adapted for the language of IL.

Let **U**=(U, ≡, ≤, σ, D) and **U'**=(U', ≡', ≤', σ', D') be two relational systems and **M**=(**U**, v), **M'**=(**U'**, v') be models over **U** and **U'** respectively. Let I≠∅ be a set of functions from U into U' and let for any i∈I and x∈U the applications of i to x be denoted by x_i. We say that I is a *copying from* **U** *to* **U'** if the following conditions are satisfied for any x,y∈U, y'∈U' and i,j∈I:

(I) $U' = \bigcup_{i\in I} U_i$, where $U_i=\{x_i / x\in U\}$, and if $x_i=y_j$ then x=y,

For any R∈{≡,≤,σ} we have

(CR1) If xRy then $(\forall i\in I)(\exists j\in I)(x_i R' y_j)$,

(CR2) If $x_i R' y'$ then $(\exists j\in I)(\exists y\in U)(y_j=y'$ and xRy),

(CD) x∈D iff $x_i \in D'$.

We say that I is a *copying from the model M to the model M'* if I is a copying from **U** to **U'** and for any A∈VAR, x∈U and i∈I we have

(Cv) x∈v(A) iff $x_i \in v'(A)$.

The importance of the copying construction is in the following

Lemma 5.1. */Copying Lemma/*

*Let M=(**U**,v) and M'=(**U'**,v') be two models and I is a copying from M to M'. Then for any formula A, x∈U and i∈I the following equivalence holds:*

$x \Vdash_{v} A$ *iff* $x_i \Vdash_{v'} A$.

Proof. The proof goes by induction of the complexity of the formula A.∎

Proposition 5.2.

*Let **U**=(U, ≡, ≤, σ, D) be a generalized D-structure. Then there exist a D-structure **U'**=(U', ≡', ≤', σ', D') and a copying I from **U** to **U'**.*

Proof. Let I=N={0, 1, 2, ...}, U'=U×I and x_i=(x, i).

Define: $\equiv' = \{(x_i, y_j)/x\equiv y \text{ and } i=j\}$

$\leq_o = \{(x_i, y_j)/(j=i \ \& \ x\leq y \ \& \ (y\leq x \longrightarrow x\equiv y) \text{ or } (j=i+1 \ \& \ x\leq y \ \& \ y\leq x \ \& \ x\not\equiv y)\}$.

$\leq'$ is the transitive closure of $\leq_o$.

$\sigma' = \{(x_i, y_j)/(\exists z_k, t_k)(z_k\leq' x_i \ \& \ t_k\leq' y_j \ \& \ z\sigma t)\}$.

$D' = \{x_i/x\in D \ \& \ i\in N\}$.

The proof of the proposition will follow from the following lemmas.

Lemma 5.3.

(i) a/ $x_i\equiv' y_j \longrightarrow x\equiv y$,

b/ $x\equiv y \longrightarrow \forall i \ x_i\equiv' y_i$,

(ii) a/ $x_i\leq' y_j \longrightarrow x\leq y \ \& \ i\leq j$,

b/ $x\leq y \longrightarrow \forall i\exists j \ x_i\leq' y_j$,

c/ $x_i\leq' x_i$,

(iii) a/ $x_i\sigma' y_j \longrightarrow x\sigma y$,

b/ $x\sigma y \longrightarrow \forall i \ x_i\sigma' y_i$,

(iv) *Let* $R\in\{\equiv, \leq, \sigma\}$, *then:*

a/ $x_iR' y_j \longrightarrow xRy$,

b/ $xRy \longrightarrow \forall i\exists j \ x_iR' y_j$.

Proof. Conditions (i), (iia) and (iic) follow directly from the definitions. For (iib) suppose $x\leq y$. We have two cases:

Case 1: $y\leq x$ and $x\not\equiv y$. Then by the definition of $\leq_o$ we have $x_i\leq_o y_{i+1}$ and hence $x_i\leq' y_{i+1}$. In this case $j=i+1$.

Case 2: not $y\leq x$ or $x\equiv y$. Then we have $x_i\leq_o y_i$ and hence $x_i\leq' y_i$. In this case $j=i$.

To prove (iiia) suppose $x_i\sigma' y_j$. Then for some z_k, t_k we have $z_k\leq' x_i$, $t_k\leq' y_j$ and $z\sigma t$. But from (iia) we obtain $z\leq x$ and $t\leq y$. Then $z\sigma t$ and $z\leq x$ give by S5 $x\sigma t$ and by S3 $t\sigma x$. In the same way from $t\leq y$ and $t\sigma x$ we get $x\sigma y$.

For (iiib) suppose $x\sigma y$. Then by (iic) $x_i\leq' x_i$ and $y_i\leq' y_i$, which by the definition of σ' give $x_i\sigma' y_i$.

The proof of (iv) follows from (i), (ii) and (iii).∎

Lemma 5.4.

The relational system $(U', \equiv', \leq', \sigma', D')$ is a D-structure.

Proof. Conditions S1 - S4, S8-S11 follow directly from the definitions. It remains to verify S6, S7, Sa, Sb and S12.

Proof of S6: $y_j\in D' \ \& \ x_i\leq' y_j \longrightarrow x_i\in D'$. Suppose $y_j\in D'$ and $x_i\leq' y_j$. Then $y\in D$ and by lemma 5.3(iva) - $x\leq y$. Then by S6 $x\in D$ and hence $x_i\in D'$.

The proof of S7 follows from the next lemma.

Lemma 5.5

(i) $x\in D$, $z_i\leq' x_i$, $z\sigma t \longrightarrow x_i\leq' t_i$,

(ii) $x\in D$, $z_k\leq' x_i$, $z\sigma t \longrightarrow k=i$,

(iii) S7: $x_i\in D'$, $x_i\sigma' y_j \longrightarrow x_i\leq' y_j$.

Proof of (i). Suppose $x\in D$, $z_i\le' x_i$ and $z\sigma t$. From $z_i\le' x_i$ by lemma 5.3(iva) we have $z\le x$ which by S5 and $z\sigma t$ give $x\sigma t$. From $x\in D$ and $x\sigma t$ by S7 we obtain $x\le t$. We shall show that $t\le x \rightarrow x\equiv t$. Suppose $t\le x$. Then, because $x\in D$, we obtain by S6 $t\in D$. Conditions $x\in D$, $t\in D$ and $x\sigma t$ imply by S12 $x\equiv t$. Then by the definition of $\le_o$ we have $x_i\le_o t_i$ and hence $x_i\le' t_i$.

Proof of (ii). Suppose $x\in D$, $z_k\le' x_i$ and $z\sigma t$. From the definition of $\le'$ we have $k\le i$. Then by the definition of $\le'$ as transitive closure of $\le_o$ there exists y such that $z_k\le' y_{i-1}\le_o x_i$. By lemma 5.3(iva) we obtain $z\le y\le x$. From $z\le y$ and $z\sigma t$ we have by S5 $y\sigma t$ and by S4 - $y\sigma y$. From $y\le x$ and $y\sigma y$ we obtain by S5 $x\sigma y$. By S6 $y\le x$ and $x\in D$ give $y\in D$. From $x\in D$, $y\in D$ and $x\sigma y$ we obtain by S12 $x\equiv y$. But $y_{i-1}\le_o x_i$ implies by the definition of $\le_o$ that $x\not\equiv y$ - a contradiction. So we have not $k<i$ and since $k\le i$ we obtain $k=i$.

Proof of (iii). Suppose $x_i\in D'$ and $x_i\sigma' y_j$. Then $x\in D$ and for some z_k and t_k we have : $z_k\le' x_i$, $t_k\le' y_j$ and $z\sigma t$. From $x\in D$, $z_k\le' x_i$, and $z\sigma t$ we obtain by (ii) $k=i$, so $z_i\le' x_i$ and $t_i\le' y_j$. From $x\in D$, $z_i\le' x_i$ and $z\sigma t$ we obtain by (i) $x_i\le' t_i$. From $x_i\le' t_i$ and $t_i\le' y_j$ we have $x_i\le' y_j$, which had to be proved.■

Now we continue the proof of lemma 5.4.

Proof of Sa: $x_i\le' y_j,\ y_j\le' x_i \rightarrow x_i\equiv y_j$. Suppose $x_i\le' y_j$ and $y_j\le' x_i$. Then by lemma 5.3(iia) we have $x\le y$, $i\le j$, $y\le x$ and $j\le i$ which imply $i=j$ and by the definition of $\le'$ - $x\equiv y$. Consequently $x_i\equiv' y_j$. **Proof of** Sb: $x_i\notin D' \rightarrow (\exists y_j)(\text{not } x_i\le' y_j)$. From the definition of $\le_o$ we see that for any $x\in U$ and $i\in N$ we have not $x_i\le_o x_{i+1}$, and this implies Sb.

Proof of S12. Suppose $x_i\in D'$, $y_j\in D'$ and $x_i\sigma' y_j$. Then by S7 we obtain $x_i\le' y_j$ and $y_j\le' x_i$. Then by Sa we get $x_i\equiv' y_j$. This ends the proof of lemma 5.4 and the proof of proposition 5.2. ■

Now we can proof the main theorem in this section.

Theorem 5.6 */Completeness theorem for IL/*

For any formula A_o the following conditions are equivalent:

(i) A_o is a theorem of IL,

(ii) A_o is true in the class Σ_0 of all generalized D-structures,

(iii) A_o is true in the class Σ_1 of all standard D-structures over arbitrary A-systems,

(iv) A_o is true in the class Σ_2 of all standard D-structures over separable A-systems,

(v) A_o is true in the class Σ_3 of all D-structures.

Proof. (i) $\longleftrightarrow$ (ii) - by theorem 4.3.

(ii) $\longrightarrow$ (iii) - by lemma 1.3(i).

(iii) $\longrightarrow$ (iv) - trivial.

(iv) $\longleftrightarrow$ (v) - by theorem 1.5.

(v) $\longrightarrow$ (ii) To prove this implication suppose for the sake of contradiction that (v) holds but (ii) does not hold. Then there exist a generalized D-structure **U**=(U, ≡, ≤, σ, D) in which A_o is not true, i.e. there exist $c\in U$ and valuation v such that $c\Vdash\!\!\!\!/_{v}\ A_o$. Then by proposition 5.2 there exists a D-structure **U'**=(U', ≡', ≤', σ', D') and a copying I from **U** to **U'**.

Define for $A \in VAR$ $v'(A)=\{x_i / x \in v(A),\ i \in I\}$. Then by the copying lemma we have $c \nVdash_{v'} A_0$. Hence A_0 is not true in the class Σ_3 of al D-structures, which ends the proof of the theorem.■

Corollary 5.7.
$IL=L(\Sigma_0)=L(\Sigma_1)=L(\Sigma_2)=L(\Sigma_3)$

Let us note that by the proof of theorem 5.6 we always obtain infinite standard counter models for the non-theorems of IL. One can think that this depends only on the special copying construction used in the proof of proposition 5.2, In the next section we shall show that the cause is in the logic IL and that there exists non-theorems of IL which can not be falsified in finite standard models., i.e. that IL is not complete in the class of its finite standard models.

6. The problem of finite standard models for IL

The main aim of this section is the following theorem :

Theorem 6.1.
Let Σ_1^{fin} be the class of all finite standard D-structures over A-systems. Then $IL \neq L(\Sigma_1^{fin})$.

Let gGrz */generalized Grzegorczik formula/* be the following formula:
gGrz: $[\leq]([\leq]([\equiv]A \Rightarrow [\leq]A) \Rightarrow A) \Rightarrow A$.
/Note. The Grzegorczik formula is Grz: $\Box(\Box(A \Rightarrow \Box A) \Rightarrow A) \Rightarrow A$ see [Seg 72]/.
The proof of the theorem follows from the following lemma.

Lemma 6.2.
(i) $gGrz \in L(\Sigma_1^{fin})$.
(ii) $gGrz \notin IL$

Proof. (i) We shall show that if gGrz is not true in some D- structure **U** then **U** must be infinite. Then gGrz should be true in all finite D-structures and by theorem 1.5 - in all standard D- structures over A-systems and consequently - gGrz $\in L(\Sigma_1^{fin})$. To show this suppose $x_0 \nVdash_v$ gGrz for some $x_0 \in U$. Then $x \nVdash_v A$ and

(*) $x_0 \Vdash [\leq]([\leq]([\equiv]A \Rightarrow [\leq]A) \Rightarrow A)$.

Since $x_0 \leq x_0$ then $x_0 \Vdash [\leq]([\equiv]A \Rightarrow [\leq]A) \Rightarrow A$ and by $x_0 \nVdash_v A$ we obtain that $x_0 \nVdash_v [\leq]([\equiv]A \Rightarrow [\leq]A)$. Then there exists x_1 such that $x_0 \leq x_1$, $x_1 \Vdash [\equiv]A$ and $x_1 \nVdash_v [\leq]A$ and hence there exists x_2 such that $x_1 \leq x_2$ and $x_2 \nVdash_v A$. It is not possible $x_0 \equiv x_1$ because then $x_1 \equiv x_0$ and since $x_1 \Vdash [\equiv]A$ we will obtain $x_0 \Vdash A$ - a contradiction. So $x_0 \not\equiv x_1$. In the same way we obtain that $x_1 \not\equiv x_2$. Define $x<y$ iff $x \leq y$ and $x \not\equiv y$. It is easy to see that $<$ is a transitive and irreflexive relation. Thus we obtain $x_0 < x_1 < x_2$. From here we have $x_0 \leq x_2$ and by (*) we obtain $x_2 \Vdash [\leq]([\equiv]A \Rightarrow [\leq]A) \Rightarrow A$ and since $x_2 \nVdash_v A$ we can repeat the above reasoning and to obtain x_3 and x_4 such that $x_2 < x_3 < x_4$. Repeating infinitely many times this procedure we obtain an infinite sequence $x_0 < x_1 < \ldots$ Because of the transitivity and irreflexivity of the relation $<$ all members of this sequence are different and hence the model is infinite.

(ii) Take U={0, 1}, $\le = \sigma = U \times U$, $\equiv = \{(x,x)/x \in U\}$, $D=\varnothing$. It is easy to see that $(U, \equiv, \le, \sigma, D)$ thus defined is a generalized D-structure. Let $v(A)=\{0\}$ Then it can be seen that $0 \Vdash\!\!\!\!\not\;\;_{v}$ gGrz and by the completeness theorem that gGrz $\notin$ IL. ■

7. Finite non-standard models for IL. Decidability of IL

In this section we will prove a theorem, which contrasts theorem 6.1.

Theorem 7.1.

IL admits filtration with respect to generalized D-structures and hence is decidable.

Proof. *Let* **U**$=(U, \equiv, \le, \sigma, D)$ *be a generalized* D-*structure, v be a valuation and* A_0 *be a formula. Define* Γ *to be the smallest set of formulas satisfying the following conditions:*

γ_0. Γ *is closed under subformulas,*

γ_1. $A_0 \in \Gamma$,

γ_2. $\langle\sigma\rangle\mathbf{1} \in \Gamma$,

γ_3. $[\le]\neg\mathbf{D} \in \Gamma$,

γ_4. $[\sigma]A \in \Gamma \longrightarrow [\le]A \in \Gamma$,

γ_5. $[\le]A \in \Gamma \longrightarrow [\sigma]A \in \Gamma$,

γ_6. $[\equiv]A \in \Gamma \longrightarrow [\sigma](\mathbf{D} \Rightarrow [\equiv]A) \in \Gamma$.

Let the relations σ', $\le'$ and $\equiv'$ and the set D' be defined as follows:

$|x|\sigma'|y|$ *iff* $(\forall\ [\sigma]A \in \Gamma)(x \Vdash_{v} [\sigma]A \longrightarrow y \Vdash_{v} [\le]A)$ &
$(y \Vdash_{v} [\sigma]A \longrightarrow x \Vdash_{v} [\le]A)$ &
$x \Vdash_{v} \langle\sigma\rangle\mathbf{1}$ & $y \Vdash_{v} \langle\sigma\rangle\mathbf{1}$,

$|x|\le'|y|$ *iff* $(\forall\ [\le]A \in \Gamma)(x \Vdash_{v} [\le]A \longrightarrow y \Vdash_{v} [\le]A)$ &
$(\forall |z| \in U')(|x|\sigma'|z| \longrightarrow |y|\sigma'|z|)$,

$|x|\equiv'|y|$ *iff* $(\forall\ [\equiv]A \in \Gamma)(x \Vdash_{v} [\equiv]A \longleftrightarrow y \Vdash_{v} [\equiv]A)$ &
$|x|\le'|y|$ & $|y|\le'|x|$,

$D' = \{|x| / x \in D\}$.

Then: (i) Γ *is a finite set,*

(ii) $(U', \equiv', \le', \sigma', D', v')$ *is a filtration of* $(U, \equiv, \le, \sigma, D, v)$ *trough* Γ.

Proof. (i). Let Γ^0 be the set of all sub formulas of the formulas A_0, $\langle\sigma\rangle\mathbf{1}$, $[\le]\neg\mathbf{D}$,. Then Γ^0 is finite. Let $\Gamma^1 = \Gamma^0 \cup \{[\le]A / [\sigma]A \in \Gamma^0\}$, $\Gamma^2 = \Gamma^1 \cup \{[\sigma]A / [\le]A \in \Gamma^1\}$, $\Gamma^3 = \Gamma^2 \cup \{[\sigma](\mathbf{D} \Rightarrow [\equiv]A) / [\equiv]A \in \Gamma^2\}$, $\Gamma^4 = \{B/B$ is a sub formula of some formula of $\Gamma^3\}$. Obviously Γ^4 is a finite set of formulas satisfying the conditions γ_0-γ_6 and γ^4 is the smallest set of formulas satisfying these properties. Thus $\Gamma = \Gamma^4$ and hence Γ is a finite set of formulas.

The proof of (ii) follows from the following lemmas.

Lemma 7.2.

The definitions of σ', $\le'$ *and* $\equiv'$ *are correct, namely for* $R \in \{\equiv, \le, \sigma\}$ *we have:* $x \sim x'$, $y \sim y'$ *and* $|x|R'|y| \longrightarrow |x'|R'|y'|$.

Proof. Suppose $x \sim x'$, $y \sim y'$ and $|x|R'|y|$. We have three cases.

Case $R=\sigma$. To show that $|x'|\sigma'|y'|$ suppose $[\sigma]A\in\Gamma$ and for the first implication of the definition of σ' suppose $x'\Vdash_V [\sigma]A$. We have to show that $y'\Vdash_V[\leq]A$. Since $[\sigma]A\in\Gamma$ and $x\sim x'$ then $x\Vdash_V [\sigma]A$ and from $|x|\sigma'|y|$ we have $y\Vdash_V[\leq]A$. Since $[\sigma]A\in\Gamma$ then by γ_4 $[\leq]A\in\Gamma$ and by $y\sim y'$ we obtain $y'\Vdash_V[\leq]A$. The second implication of σ' can be proved in the same way. For the remaining part of the definition of σ' we proceed as follows. From the assumption $|x|\sigma'|y|$ we have $x\Vdash_V\langle\sigma\rangle\mathbf{1}$, $y\Vdash_V\langle\sigma\rangle\mathbf{1}$. By γ_2 $\langle\sigma\rangle\mathbf{1}\in\Gamma$ and from $x\sim x'$, $y\sim y'$ we obtain $x'\Vdash_V\langle\sigma\rangle\mathbf{1}$ and $y'\Vdash_V\langle\sigma\rangle\mathbf{1}$.

Case $R=\leq$, $\equiv$. The proof is similar to that of $R=\sigma$. Case $R=\leq$ makes use of the case $R=\sigma$ and the case $R=\equiv$ makes use of the case $R=\leq$. ■

Lemma 7.3.

(i) $x\sigma y \longrightarrow |x|\sigma'|y|$,

(ii) $x\leq y \longrightarrow |x|\leq'|y|$,

(iii) $x\equiv y \longrightarrow |x|\equiv'|y|$.

Proof of (i). We have to prove the following implications:

(i1) $x\sigma y \longrightarrow (\forall[\sigma]A\in\Gamma)(x\Vdash_V[\sigma]A \longrightarrow y\Vdash_V[\leq]A)$,

(i2) $x\sigma y \longrightarrow (\forall[\sigma]A\in\Gamma)(y\Vdash_V[\sigma]A \longrightarrow x\Vdash_V[\leq]A)$,

(i3) $x\sigma y \longrightarrow x\Vdash_V\langle\sigma\rangle\mathbf{1}$ *and* $y\Vdash_V\langle\sigma\rangle\mathbf{1}$.

To prove (i1) suppose $x\sigma y$, $[\sigma]A\in\Gamma$, $x\Vdash_V[\sigma]A$ and $y\leq z$. Then from $x\sigma y$ and $y\leq z$ we obtain $x\sigma z$ and by $x\Vdash_V[\sigma]A$ - $z\Vdash_V A$. Condition (i2) can be proved in the same way.

For (i3) suppose $x\sigma y$. Then we have $y\sigma x$, $x\Vdash_V\mathbf{1}$, $y\Vdash_V\mathbf{1}$, which give $x\Vdash_V\langle\sigma\rangle\mathbf{1}$ and $y\Vdash_V\langle\sigma\rangle\mathbf{1}$. ■

Proof of (ii). We have to prove the following implications:

(ii1) $x\leq y \longrightarrow (\forall[\leq]A\in\Gamma)(x\Vdash_V[\leq]A \longrightarrow y\Vdash_V[\leq])$,

(ii2) $x\leq y \longrightarrow (\forall|z|)(|x|\sigma'|z| \longrightarrow |y|\sigma'|z|)$.

Proof of (ii1). Suppose $x\leq y$, $[\sigma]A\in\Gamma$, $x\Vdash_V[\sigma]A$ and $y\leq z$. Then $x\leq z$ and from $x\Vdash_V[\leq]A$ we obtain $z\Vdash_V A$, which had to be proved.

Proof of (ii2). To prove (ii2) we have to prove first the following implications:

(ii21) $x\leq y,\ |x|\sigma'|z| \longrightarrow (\forall\ [\sigma]A\in\Gamma)(y\Vdash_V[\sigma]A \longrightarrow z\Vdash_V[\leq]A)$,

(ii22) $x\leq y,\ |x|\sigma'|z| \longrightarrow (\forall\ [\sigma]A\in\Gamma)(z\Vdash_V[\sigma]A \longrightarrow y\Vdash_V[\leq]A)$,

(ii23) $x\leq y,\ |x|\sigma'|z| \longrightarrow y\Vdash_V\langle\sigma\rangle\mathbf{1}$ *and* $z\Vdash_V\langle\sigma\rangle\mathbf{1}$.

Proof of (ii21). Suppose $x\leq y$, $|x|\sigma'|z|$, $y\Vdash_V[\sigma]A$, $[\sigma]A\in\Gamma$ and for the sake of contradiction that $z\nVdash_V [\leq]A$. Then $|x|\sigma'|z|$, $z\nVdash_V [\leq]A$ and $[\sigma]A\in\Gamma$ imply $x\nVdash_V [\sigma]A$. Then there exists t such that $x\sigma t$ and $t\nVdash_V A$. From $x\leq y$ and $x\sigma t$ we obtain $y\sigma t$ and by $y\Vdash_V[\sigma]A$ we have $t\Vdash_V A$ - a contradiction.

Proof of (ii22). Suppose $x\leq y$, $|x|\sigma'|z|$, $z\Vdash_V[\sigma]A$, $[\sigma]A\in\Gamma$ and $y\nVdash_V [\leq]A$. From $|x|\sigma'|z|$ and $z\Vdash_V[\sigma]A$ we obtain $x\Vdash_V [\leq]A$. Since $y\nVdash_V [\leq]A$ we have that for some t: $t\nVdash_V A$ and $y\leq t$. But $x\leq y$ and $y\leq t$ imply $x\leq t$ and by $x\Vdash_V[\leq]A$ we obtain $t\Vdash_V A$ - a contradiction.

Proof of (ii23). Suppose $x\leq y$ and $|x|\sigma'|z|$. Then $z\Vdash_V\langle\sigma\rangle\mathbf{1}$ and $x\Vdash_V\langle\sigma\rangle\mathbf{1}$, which imply that for some t: $x\sigma t$ and $t\Vdash_V \mathbf{1}$. From $x\leq y$ and $x\sigma t$ we have $y\sigma t$, which by $t\Vdash_V \mathbf{1}$ give $y\Vdash_V <\sigma>\mathbf{1}$.■

Proof of (iii). We have to prove the following implications:

(iii1) $x\equiv y \longrightarrow (\forall[\equiv]A\in\Gamma)(x\Vdash_V [\equiv]A \longrightarrow y\Vdash_V [\equiv]A)$,

(iii2) $x\equiv y \longrightarrow (\forall[\equiv]A\in\Gamma)(y\Vdash_V [\equiv]A \longrightarrow x\Vdash_V [\equiv]A)$,

(iii3) $x\equiv y \longrightarrow |x|\leq'|y|$ *and* $|y|\leq'|x|$.

Proof of (iii1). Suppose $x\equiv y$, $x\Vdash_V [\equiv]A$, $[\equiv]A\in\Gamma$ and $y\equiv z$. Then $x\equiv z$ and by $x\Vdash_V [\equiv]A$ we obtain $z\Vdash_V A$.

Proof of (iii2). - similar to the above one.

Proof of (iii3). Suppose $x\equiv y$. Then by S11 we have $x\leq y$ and by S9 and S11 - $y\leq x$. Applying (ii) we obtain $|x|\leq'|y|$ and $|y|\leq'|x|$.■

Lemma 7.4.

Let $R\in\{\equiv,\leq,\sigma\}$ *then*

$$|x|R'|y| \longrightarrow (\forall[R]A\in\Gamma)(x\Vdash_V [R]A \longrightarrow y\Vdash_V A).$$

Proof. Case $R=\sigma$. Suppose $|x|\sigma'|y|$ and $x\Vdash_V [R]A$, $[R]A\in\Gamma$. Then $y\Vdash_V [\leq]A$ and since $y\leq y$ we obtain $y\Vdash_V A$. The remaining cases can be proved in a similar way.■

Lemma 7.5.

The system $(U', \equiv', \leq', \sigma', D')$ *is a generalized* D-*structure.*

Proof. We have to verify the axioms S1 - S12 for generalized D-structures. The conditions S1, S2, S3, S5, S8, S9, S10 and S11 follow directly from the definitions of the relations $\equiv'$, $\leq'$ and σ' and D'. We shall show that S4, S6, S7 and S12 are also true.

Proof of S4: $|x|\sigma'|y| \longrightarrow |x|\sigma'|x|$. To do this we have to prove the following implications:

(i) $|x|\sigma'|y| \longrightarrow x\Vdash_V <\sigma>\mathbf{1}$,

(ii) $|x|\sigma'|y| \longrightarrow (\forall[\sigma]A\in\Gamma)(x\Vdash_V [\sigma]A \longrightarrow x\Vdash_V [\leq]A)$.

Condition (i) follows directly from the definition of σ'. For (ii) suppose $|x|\sigma'|y|$, $x\Vdash_V [\sigma]A$, $[\sigma]A\in\Gamma$ and $x\leq z$. We have to show that $z\Vdash_V A$. From $x\Vdash_V <\sigma>\mathbf{1}$ we have $x\sigma t$ for some t and by S4 - $x\sigma x$. From $x\leq z$ and $x\sigma x$ we obtain by S5 $z\sigma x$ and by S3 $x\sigma z$, and since $x\Vdash_V [\sigma]A$ we have $z\Vdash_V A$.■

Proof of S6: $|y|\in D'$, $|x|\leq'|y| \longrightarrow |x|\in D'$. Suppose $|y|\in D'$, $|x|\leq'|y|$ and $|x|\notin D'$. Then $y\in D'$, $x\notin D$, $y\Vdash_V \mathbf{D}$, $x\nVdash_V \mathbf{D}$ and by A6: $<\leq>\mathbf{D}\Rightarrow\mathbf{D}$ we obtain that $x\nVdash_V <\leq>\mathbf{D}$, so $x\Vdash_V [\leq]\neg\mathbf{D}$. By γ_3 $[\leq]\neg\mathbf{D}\in\Gamma$, then from $|x|\leq|y|$ by lemma 7.4 we obtain $y\Vdash_V \neg\mathbf{D}$, so $y\nVdash_V \mathbf{D}$ - a contradiction.■

Proof of S7: $|x|\in D'$, $|x|\sigma'|y| \longrightarrow |x|\leq'|y|$. For that purpose we have to prove the following two implications:

(i) $|x|\in D'$, $|x|\sigma'|y| \longrightarrow (\forall[\leq]A\in\Gamma)(x\Vdash_V [\leq]A \longrightarrow y\Vdash_V [\leq]A)$.

(ii) $|x|\in D'$, $|x|\sigma'|y| \longrightarrow (\forall|z|\in U')(|x|\sigma'|z| \longrightarrow |y|\sigma'|z|)$.

Proof of (i). Suppose $|x|\in D'$, $|x|\sigma'|y|$ $x\Vdash_V [\leq]A$, $[\leq]A\in\Gamma$ and for the sake of contradiction - that $y\nVdash_V [\leq]A$. By γ_5 from $[\leq]A\in\Gamma$ we have $[\sigma]A\in\Gamma$. Then $|x|\sigma'|y|$, $y\nVdash_V [\leq]A$ and $[\sigma]A\in\Gamma$ imply $x\nVdash_V [\sigma]$. Then there exists t such that $x\sigma t$ and $x\nVdash_V A$. From $|x|\in D'$ we have $x\in D$, which with $x\sigma t$ give $x\leq t$. From

$x \leq t$ and $t \not\Vdash_V A$ we obtain $x \not\Vdash_V [\leq]A$ - a contradiction.

Proof of (ii). We have to prove the following implications:

(ii1) $|x| \in D'$, $|x|\sigma'|y|$, $|x|\sigma'|z| \longrightarrow$
$(\forall[\sigma]A \in \Gamma)(y \Vdash_V [\sigma]A \longrightarrow z \Vdash_V [\leq]A)$,

(ii2) $|x| \in D'$, $|x|\sigma'|y|$, $|x|\sigma'|z| \longrightarrow$
$(\forall[\sigma]A \in \Gamma)(z \Vdash_V [\sigma]A \longrightarrow y \Vdash_V [\leq]A)$,

(ii3) $|x|\sigma'|y|$, $|x|\sigma'|z| \longrightarrow y \Vdash_V \langle\sigma\rangle\mathbf{1}$ *and* $z \Vdash_V \langle\sigma\rangle\mathbf{1}$.

Proof of (ii1). Suppose $|x| \in D'$, $|x|\sigma'|y|$, $|x|\sigma'|z|$, $y \Vdash_V [\sigma]A$, $[\sigma]A \in \Gamma$ but $z \not\Vdash_V [\leq]A$. From $[\sigma]A \in \Gamma$, $|x|\sigma'|z|$ and $z \not\Vdash_V [\leq]A$ we obtain $x \not\Vdash_V [\sigma]A$ and from here - $x\sigma t$ and $t \not\Vdash_V A$ for some t. From $|x| \in D'$ we have $x \in D$, which together with $x\sigma t$ give $x \leq t$. Conditions $x \leq t$ and $t \not\Vdash_V A$ give $x \not\Vdash_V [\leq]A$. This with $|x|\sigma'|y|$ and $[\sigma]A \in \Gamma$ give $y \not\Vdash_V [\sigma]A$ - a contradiction.

Proof of (ii2). In the above proof exchange x and y.

Proof of (ii3) - directly from the definition of σ'.

Proof of S12: $|x| \in D'$ & $|y| \in D'$ & $|x|\sigma'|y| \longrightarrow |x| \equiv' |y|$. For that purpose we have to prove the following implications:

(i) $|x| \in D'$, $|y| \in D'$, $|x|\sigma'|y| \longrightarrow$
$(\forall[\equiv]A \in \Gamma)(x \Vdash_V [\equiv]A \longrightarrow y \Vdash_V [\equiv]A)$,

(ii) $|x| \in D'$, $|y| \in D'$, $|x|\sigma'|y| \longrightarrow$
$(\forall[\equiv]A \in \Gamma)(y \Vdash_V [\equiv]A \longrightarrow x \Vdash_V [\equiv]A)$,

(iii) $|x| \in D'$ & $|y| \in D'$ & $|x|\sigma'|y| \longrightarrow |x| \leq' |y|$ *and* $|y| \leq' |x|$.

Proof of (i). Suppose $|x| \in D'$, $|y| \in D'$, $|x|\sigma'|y|$, $x \Vdash_V [\equiv]A$, $[\equiv]A \in \Gamma$. We have to show that $y \Vdash_V [\equiv]A$. From $|x| \in D'$ and $|y| \in D'$ we have $x \Vdash_V \mathbf{D}$ and $y \Vdash_V \mathbf{D}$. We shall show that $x \Vdash_V [\sigma](\mathbf{D} \Rightarrow [\equiv]A)$. Suppose the contrary. Then for some z we have $x\sigma z$, $z \in D$ and $z \not\Vdash_V [\equiv]$, and hence for some t we have $z \equiv t$ and $t \not\Vdash_V A$. From $x \in D$, $y \in D$ and $x\sigma z$ by S12 we have $x \equiv z$ and by $z \equiv t$ - $x \equiv t$, which with $x \Vdash_V [\equiv]A$ give $t \Vdash_V A$ - a contradiction. Hence $x \Vdash_V [\sigma](\mathbf{D} \Rightarrow [\equiv]A)$. Since $[\equiv]A \in \Gamma$ then by γ_6 we have that $[\sigma](\mathbf{D} \Rightarrow [\equiv]A) \in \Gamma$. This together with $|x|\sigma'|y|$ by lemma7.4 give $y \Vdash_V \mathbf{D} \Rightarrow [\equiv]A$ and since $y \Vdash_V \mathbf{D}$ we obtain $y \Vdash_V [\equiv]A$.

Proof of (ii). By S3 $|x|\sigma'|y|$ implies $|y|\sigma'|x|$. Then apply (i).

Proof of (iii). Apply conditions S7 and S3.

This ends the proof of lemma 7.5 and the proof of theorem 7.1.■

PART III. APPROXIMATION LOGICS BASED ON ROUGH SETS THEORY

This part is devoted to a study of modal logics connected with approximation theory based on Rough-sets Theory. Rough-sets theory have been introduced by Pawlak [Paw 82,84,86,91] as an alternative to fuzzy-sets approach and has been applied in several branches in theoretical computer science and Artificial Intelligence: theory of rough concepts, learning from examples, inductive reasoning, approximate information, rough classification, decision theory, logics for approximate reasoning and others.

The main idea of rough-sets approach is very simple and can be described as follows. Suppose we are given a set $W \neq \emptyset$ of some objects. In many real cases, due to lack of enough information or some other limitations, we are not always able to distinguish each pair $x \neq y$ of different elements of W. This fact determines certain indiscernibility relation R in W, which for many reasons can be assumed to be an equivalence relation in W. Then, observing some element $x \in W$ we actually see not x but the R-equivalence class $|x|_R$ determined by x. So $|x|_R$ can be treated as a "rough" approximation of x by means of R. In this way all subsets of W we can see are unions of R-equivalence classes and let us call such sets R- definable. Now two "rough" approximations of a given subset $X \subseteq W$ can be defined:

$\underline{R}X$ - the lower rough approximation of X, which is the biggest R-definable set contained in X, and

$\overline{R}X$ - the upper rough approximation of X, which is the smallest R-definable set containing X.

If Q is an n-place relation in W, then rough approximations $\underline{R}Q$ and $\overline{R}Q$ can be defined in an analogous way (see [Paw 86]).

This part contains only one chapter - 3.1 Modal logics for rough approximation. We introduced the simplest logic of this kind, denoted by $RPML^n$ based on frames having one equivalence relation R, one n+1-place relation and its upper and lower approximations. Then we study other approximation logics, including Rough Boolean Logic. The main results are completeness theorems for the logics in question and decidability of them by means of the filtration method, generalized here for the polyadic modalities. Let us mention that for the completeness theorem we use a generalization of the canonical construction, based on the notion of co-theory, introduced by the author in [Vak 89a]. The results of this part have been published in [Vak 91b].

CHAPTER 3.1

MODAL LOGICS FOR ROUGH APPROXIMATION

Overview. In this chapter we introduce several modal logics for approximation of sets and relations based on Rough-set Theory. Rough-set Theory is developed as an alternative to the fuzzy-set approach, and has many applications in different branches in AI and theoretical computer science.

Introduction

In this chapter we consider propositional modal logics containing polyadic modalities, namely, formulas of the following kinds: $\Box(A_1,\ldots,A_n)$ and $\Diamond(A_1,\ldots,A_n)$. The first study of such operators in an algebraic context was given by Jonsson and Tarski [J&T 51] and, in some sense, it can be considered as the origin of the famous relational Kripke semantics. The following relational interpretation of polyadic modal operators can be derived from Jonsson-Tarski's paper. Let $W \neq \emptyset$ be a set and $Q \subseteq W^{n+1}$ be an n+1- place relation in W. Extending the usual Kripke interpretation of formulas in W, for the case of $(A_1,\ldots,A_n)$ and $\Diamond(A_1,\ldots,A_n)$ we put

$x \Vdash_v \Box(A_1,\ldots,A_n)$ iff

$$(\forall y_1 \ldots y_n \in W)(Q(x,y_1,\ldots,y_n) \rightarrow y_1 \Vdash_v A_1 \text{ or}\ldots\text{or } y_n \Vdash_v A_n),$$

$x \Vdash_v \Diamond(A_1,\ldots,A_n)$ iff

$$(\exists y_1 \ldots y_n \in W)(Q(x,y_1,\ldots,y_n) \;\&\; y_1 \Vdash_v A_1 \;\&\ldots\&\; y_n \Vdash_v A_n).$$

In this chapter we introduce a class of polyadic modal logics, called Rough Polyadic Modal Logics, aiming to formalize some aspects of approximate reasoning based on the theory of rough sets, investigated by Pawlak [Paw 82,84,86,91].

Pawlak's rough-sets theory has been developed as an alternative to fuzzy-sets approach and has been applied in several branches in theoretical computer science and Artificial Intelligence: theory of rough concepts, learning from examples, inductive reasoning, approximate information, rough classification, decision theory, logics for approximate reasoning and others.

The main idea of rough-sets approach is very simple and can be described as follows. Suppose we are given a set $W \neq \emptyset$ of some objects. In many real cases, due to lack of enough information or some other limitations, we are not always able to distinguish each pair $x \neq y$ of different elements of W. This fact determines certain indiscernibility relation R in W, which for many reasons can be assumed to be an equivalence relation in W. Then, observing some element $x \in W$ we actually see not x but the R-equivalence class $|x|_R$ determined by x. So $|x|_R$ can be treated as a "rough" approximation of x by means of R. In this way all subsets of W we can see are unions of R-equivalence classes and let us call such sets R- definable. Now two "rough" approximations of a given subset $X \subseteq W$ can be defined:

$\underline{R}X$ - the lower rough approximation of X, which is the biggest R-definable set contained in X, and

$\overline{R}X$ - the upper rough approximation of X, which is the smallest R-definable set containing X.

If Q is an n-place relation in W, then rough approximations $\underline{R}Q$ and $\overline{R}Q$ can be defined in an analogous way (see [Paw 86]).

A first order logic of approximate reasoning, based on rough approximation of sets and relations and called Rough Concepts Logic, was introduced by Rasiowa and Skowron [R&S 88]. A generalization of this system was given by Rasiowa [Ras 86]. A logic with rough quantifiers was investigated by Szczerba [Sz 86].

The Rough Polyadic Modal Logics, can be considered as a propositional analog of the Rough Concepts Logic of Rasiowa and Skowron [R&S 88]. One of the advantages of the propositional approach is that the obtained logical systems are sometimes decidable, while the first order ones are normally undecidable.

The language of the simplest rough polyadic modal logic, denoted by $RPML^n$, contains one monadic operator $\blacksquare$ and three n- place polyadic operators $\Box$, $\underline{\Box}$ and $\overline{\Box}$. The standard semantic structure is in the form (W, R, Q, S, T), where $W \neq \emptyset$, R is an equivalence relation in W, Q is an n+1-place relation in W, $S=\underline{R}Q$ and $T=\overline{R}Q$. These relations are used in the interpretation of $\blacksquare$, $\Box$, $\underline{\Box}$, and $\overline{\Box}$ respectively. The main aim of this paper is to give an axiomatization and completeness theorem for $RPML^n$ and to prove its decidability. The main difficulty in the completeness proof is that the condition $S=\underline{R}Q$ is not modally definable in a sense of modal correspondence theory /Benthem [Ben 86]/. To avoid this difficulty we first introduce a nonstandard semantics for $RPML^n$, which is modally definable. The completeness of $RPML^n$ with respect to nonstandard models is given by a modification of the canonical- model- construction, adapted for the polyadic case. Then, by a special construction, called copying, we transform each nonstandard model into a standard one, which gives the completeness of $RPML^n$ with respect to its standard semantics.

The proof of decidability of $RPML^n$ is given by showing that it posseses the finite model property. The method of the proof uses a generalization of the Segerberg's [Seg 71] definition of filtration, adapted for the polyadic case.

If Q is a ternary relation in W such that Q(x,y,z) iff x=y=z then $\Box(A,B) \equiv A \vee B$ and $\Diamond(A,B) \equiv A \wedge B$. So classical disjunction and conjunction are special kinds of polyadic modalities. In the chapter we study their rough analogs $\underline{\Box}(A,B)$, $\overline{\Box}(A,B)$, $\underline{\Diamond}(A,B)$ and $\overline{\Diamond}(A,B)$. We call the obtained system Rough Boolean Logic.

Rough analogs of monadic modalities have also been investigated.

1. Approximation spaces

Following Pawlak [Paw 82] we say that a pair (W, R) is an approximation space if $W \neq \emptyset$ and R is an equivalence relation in W, called indiscernibility relation.

The R-equivalence class determined by an element $x \in W$ will be denoted by $|x|_R$. A set $X \subseteq W$ is called R-definable if it is an union of R-equivalence classes. Obviously the set of all R- definable subsets of W forms a complete Boolean algebra.

For $X \subseteq W$ define:

$\underline{R}X=\{x \in W/|x|_R \subseteq X\}=\{x \in W/(\forall u \in W)(xRu \longrightarrow u \in X)\}$,

$\overline{R}X=\{x \in W/|x|_R \cap X \neq \emptyset\}=\{x \in W/(\exists u \in W)(xRu \ \&\ u \in X)\}$.

Obviously $\underline{R}X$ is the greatest R-definable set contained in X and $\overline{R}X$ is the smallest R-definable set containing X. $\underline{R}X$ is called rough lower approximation of X and $\overline{R}X$ is called rough upper approximation of X. Obviously $\underline{R}X=-\overline{R}-X$ and $\overline{R}X=-\underline{R}-X$, where $-X=W-X$.

The relation R can be extended in W^n in the following way

$(x_1,\ldots,x_n)R(y_1,\ldots,y_n)$ iff $x_1Ry_1\&\ldots\&x_nRy_n$

Obviously R is an equivalence relation in W^n. The R-equivalence class in W^n determined by $(x_1,\ldots,x_n)$ will be denoted by $|(x_1,\ldots,x_n)|_R$.

Let $Q \subseteq W^n$. Following Pawlak [Paw 86] we define:

$\underline{R}Q=\{(x_1,\ldots,x_n) \in W^n/|(x_1,\ldots,x_n)|_R \subseteq Q\}=$

$\{(x_1,\ldots,x_n) \in W^n/(\forall u_1\ldots u_n \in W)(x_1Ru_1\&\ldots\&x_nRu_n \longrightarrow Q(u_1,\ldots,u_n)\}$,

$\overline{R}Q=\{(x_1,\dots,x_n)\in W^n / |(x_1,\dots,x_n)|_R \cap Q\neq\emptyset\}=$

$\{(x_1,..,x_n)\in W^n/(\exists u_1..u_n\in W)(x_1Ru_1 \ \&\ldots\&\ x_nRu_n \ \&\ Q(u_1,\dots,u_n))\}.$

The relations $\underline{R}Q$ and $\overline{R}Q$ are called rough lower and upper approximations of Q respectively.

2. Syntax and semantics for the rough Polyadic Modal Logic - RPML^n

The language $\mathcal{L}^n$ of RPML^n consists of the following symbols:

VAR - an infinite denumerable set of propositional variables,

$\neg$, $\wedge$, $\vee$ - Boolean operations,

$\blacksquare$ - one monadic modal operation,

$\Box$, $\underline{\Box}$, - two n-argument polyadic modal operations,

(), - parentheses and coma.

The set FOR of all formulas is the smallest set, satisfying the following clauses:

(i) VAR$\subseteq$FOR,

(ii) If A$\in$FOR and B$\in$FOR then $\neg$A$\in$FOR, (A$\wedge$B)$\in$FOR and (A$\vee$B)$\in$FOR,

(iii) If A$\in$FOR then $\blacksquare$A$\in$FOR,

(iv) If $A_1\in$FOR,...., $A_n\in$FOR then $\Box(A_1,\dots,A_n)\in$FOR and $\underline{\Box}(A_1,\dots,A_n)\in$FOR.

Abbreviations: 1=(A$\vee\neg$A), 0=$\neg$1, A$\Rightarrow$B=$\neg$A$\vee$B, A$\Leftrightarrow$B=(A$\Rightarrow$B)$\wedge$(B$\Rightarrow$A),

$\overline{\Box}(A_1,\dots,A_n)=\blacksquare\Box(\blacksquare A_1,\dots,\blacksquare A_n)$, $\blacklozenge A=\neg\blacksquare\neg A$, $\Diamond(A_1,\dots,A_n)=\neg\Box(\neg A_1,\dots,\neg A_n)$, and $\underline{\Diamond}$, $\overline{\Diamond}$

- in an obvious way.

We adopt the usual omission of parentheses for Boolean formulas.

We will introduce several types of semantics of this language.

By a frame we will mean any relational structure $\underline{W}$=(W, R, Q, S) where W$\neq\emptyset$, R$\subseteq W^2$ and Q,S$\subseteq W^{n+1}$. By Σ^n and Σ^n_{fin} we denote the class of all frames and the class of all finite frames respectively. By a model over a frame $\underline{W}$ we mean any pair $\underline{M}$=($\underline{W}$, v) where v is a function, called a valuation, which associates to each propositional variable A a subset v(A)$\subseteq$W. The satisfiability relation $x\Vdash_v A$ /the formula A is true in x$\in$W at the valuation v/ is defined inductively as in the usual Kripke semantics:

$x\Vdash_v A$ iff x$\in$v(A) for A$\in$VAR,

$x\Vdash_v \neg A$ iff $x\not\Vdash_v A$ /$x\not\Vdash_v A$ means not $x\Vdash_v A$/,

$x\Vdash_v A\vee B$ iff $x\Vdash_v A$ or $x\Vdash_v B$,

$x\Vdash_v A\wedge B$ iff $x\Vdash_v A$ and $x\Vdash_v B$,

$x\Vdash_v \blacksquare A$ iff $(\forall y\in W)(xRy \rightarrow y\Vdash_v A)$,

$x\Vdash_v \Box(A_1,\dots,A_n)$ iff

$(\forall y_1\dots y_n\in W)(Q(x,y_1,\dots,y_n) \rightarrow y_1\Vdash_v A_1$ or...or $y_n\Vdash_v A_n)$,

$x\Vdash_v \underline{\Box}(A_1,\dots,A_n)$ iff

$(\forall y_1\dots y_n\in W)(S(x,y_1,\dots,y_n) \rightarrow y_1\Vdash_v A_1$ or...or $y_n\Vdash_v A_n)$.

From this definition we obtain the following clause for $\Diamond$/and analogous for $\Diamond$ and $\overline{\Diamond}$ /:

$x\Vdash_v \Diamond(A_1,\dots,A_n)$ iff $(\exists y_1\dots y_n\in W)(Q(x,y_1,\dots,y_n)$ & $y_1\Vdash_v A_1$ &...& $y_n\Vdash_v A_n)$

Let Φ be a class of frames. The set $L(\Phi)$ of all formulas true in each member of Φ is called a logic of Φ. A set of formulas L is sound in a class of frames Φ if $L\subseteq L(\Phi)$, L is called complete in Φ if $L(\Phi)\subseteq L$, L is characterized by Φ if $L=L(\Phi)$.

We say that $\underline{W}=(W, R, Q, S)$ is a standard frame /for RPLM^n/ if (W, R) is an approximation space, $Q\subseteq W^{n+1}$ and $S=\underline{R}Q$. By Σ^n_s and Σ^n_{sfin} we denote the class of all standard and all finite standard frames respectively. Semantically the logic RPML^n is identified with $L(\Sigma^n_s)$. Our first aim is to find an axiomatization of the logic $L(\Sigma^n_s)$.

The next lemma establishes why we introduce $\overline{\Box}$ as an abbreviation.

Lemma 2.1.

Let $\underline{W}=(W, R, Q, S)$ be a standard frame and let $T=RQ$. Then for any valuation v and $x\in W$ we have the following equivalence

$x\Vdash_v \overline{\Box}(A_1,\dots,A_n)$ iff

$(\forall y_1\dots y_n\in W)(T(x,y_1,\dots,y_n)\rightarrow y_1\Vdash_v A_1$ or...or $y_n\Vdash_v A_n)$.

Lemma 2.2.

(i) The following formulas are true in the class Σ^n of all frames for any $i\in\{1,\dots n\}$

(K$\blacksquare$) $\blacksquare(B\Rightarrow C)\Rightarrow(\blacksquare B\Rightarrow\blacksquare C)$

(K$\Box$i) $\Box(A_1,..,\underset{i}{B\Rightarrow C},..,A_n)\Rightarrow(\Box(A_1,..,\underset{i}{B},..,A_n)\Rightarrow\Box(A_1,..,\underset{i}{C},..,A_n))$

(K$\underline{\Box}$i) $\underline{\Box}(A_1,..,\underset{i}{B\Rightarrow C},..,A_n)\Rightarrow(\underline{\Box}(A_1,..,\underset{i}{B},..,A_n)\Rightarrow\underline{\Box}(A_1,..,\underset{i}{C},..,A_n))$

(ii) If the formula A_i is true in a model $\underline{M}$ then for any $A_1,..,A_{i-1},A_{i+1},..,A_n\in FOR$ the formulas $\Box(A_1,..,A_i,..,A_n)$ and $\underline{\Box}(A_1,..A_i,..,A_n)$ are true in $\underline{M}$.

The easy proof of these two lemmas is omitted.

Consider the following table, where $A, A_1,..,A_n\in VAR$

S1. xRx	A1. $\blacksquare A\Rightarrow A$
S2. $xRy\rightarrow yRx$	A2. $\blacklozenge\blacksquare A\Rightarrow A$
S3. $xRy\&yRz\rightarrow xRz$	A3. $\blacksquare A\Rightarrow\blacksquare\blacksquare A$
S4. $S(x_0,..,x_n)\rightarrow Q(x_0,..,x_n)$	A4. $\Box(A_1,..,A_n)\Rightarrow\underline{\Box}(A_1,..,A_n)$
S5. $x_0Ry_0\&...\&x_nRy_n\&$ $\&S(x_0,..x_n)\rightarrow S(y_0,..,y_n)$	A5. $\blacklozenge\underline{\Box}(A_1,..,A_n)\Rightarrow\underline{\Box}(\blacksquare A_1,..,\blacksquare A_n)$

S6. $(\forall y_0..y_n)(x_0Ry_0\&...\&x_nRy_n\rightarrow Q(y_0,..,y_n))\rightarrow S(x_0,..,x_n)$-------

The next lemma can be considered as a new definition of the notion of a standard frame.

Lemma 2.3.

Let $\underline{W}$ be a frame. then $\underline{W}$ is a standard frame iff it satisfies the conditions S1-S6.

Proof - exercise.

Lemma 2.4.

Let $\underline{W}$ be a frame. Then for any i, $1\le i\le 5$, the formula Ai is true in $\underline{W}$ iff $\underline{W}$ satisfies the condition Si.

Proof. The cases i=1-4 are well known. For i=5 the implication $(\longrightarrow)$ can be proved in a standard way.

$(\longleftarrow)$ Suppose that $\underline{W}$ does not satisfy S5 and proceed to show that A5 is not true in $\underline{W}$. We have x_0Ry_0, x_1Ry_1, $\dots, x_nRy_n$, $S(x_0, x_1 \dots, x_n)$ and $\bar{S}(y_0, y_1, \dots, y_n)$. Define for $A_1 \dots, A_n \in VAR$ a valuation v as follows: $v(A_i)=\{z\in W/z\ne y_i\}$, i=1...n. From here it can easily be obtained that $x_0 \not\Vdash_v A5$. Thus A5 is not true in $\underline{W}$. ■

Lemma 2.4 can be considered as a simple fact from the modal correspondence theory for the language $\mathcal{L}^n$. Roughly speaking the lemma says that the conditions S1-S5 are modally definable by the formulas A1-A5. One can ask whether the condition S6 is modally definable. In the section 4 we shall show that S6 is not modally definable in a stronger sense. This will imply that the class Σ^n_s of standard frames is not modally definable, which is one of the difficulties in the axiomatization of the logic $L(\Sigma^n_s)$. For that purpose we introduce a more wide class of frames for RPML^n, called general or nonstandard. Namely we have the following definition.

A frame $\underline{W}$=(W, R, Q, S) is called a general frame for RPML^n if it satisfies the conditions S1 - S5. By Σ^n_g and Σ^n_{gfin} we denote the class of all general and all finite general frames respectively. Now Lemma 2.4 says that Σ^n_g is modally definable by the formulas A1 - A5.

Lemma 2.2 and lemma 2.4 suggest the following axiomatization of RPML^n.

Axiom schemes and rules for RPML^n.

(Bool) All or enough Boolean tautologies,

(K$\blacksquare$) $\blacksquare(B\Rightarrow C)\Rightarrow(\blacksquare B\Rightarrow\blacksquare C)$,

(K$\Box$i) $\Box(A_1,\dots,\underset{i}{B\Rightarrow C},\dots,A_n)\Rightarrow(\Box(A_1,\dots,\underset{i}{B},\dots,A_n)\Rightarrow\Box(A_1,\dots,\underset{i}{C},\dots,A_n))$, $i\le n$,

(K$\underline{\Box}$i) $\underline{\Box}(A_1,\dots,\underset{i}{B\Rightarrow C},\dots,A_n)\Rightarrow(\underline{\Box}(A_1,\dots,\underset{i}{B},\dots,A_n)\Rightarrow\underline{\Box}(A_1,\dots,\underset{i}{C},\dots,A_n))$, $i\le n$,

(A1) $\blacksquare A\Rightarrow A$,

(A2) $\blacklozenge\blacksquare A\Rightarrow A$,

(A4) $\blacksquare A\Rightarrow\blacksquare\blacksquare A$,

(A5) $\blacklozenge\underline{\Box}(A_1,\dots,A_n)\Rightarrow\underline{\Box}(\blacksquare A_1,\dots,\blacksquare A_n)$,

(MP) A, A⇒B/B,

(N$\blacksquare$) A/$\blacksquare$A, (N$\Box$i) $A_i/\Box(A_1,\dots,A_n)$, $1\le i\le n$,

(N$\underline{\Box}$i) $A_i/\underline{\Box}(A_1,\dots,A_n)$, $1\le i\le n$.

A set L of formulas is called a polyadic modal logic over the language $\mathcal{L}^n$ if it contains the axioms (BooL), (K$\blacksquare$), (K$\Box$i), (K$\underline{\Box}$i), $1\le i\le n$, and is closed under the rules (MP), (N$\blacksquare$), (N$\Box$i), (N$\underline{\Box}$i) and the rule of substitution for propositional variables. The smallest polyadic modal logic over $\mathcal{L}^n$ will be denoted by PML^n.

A set L of formulas is called a rough polyadic modal logic over the language $\mathcal{L}^n$ if it is a polyadic modal logic, containing the axioms A1 - A5. The smallest rough polyadic modal logic will be denoted by RPML^n.

From lemma 2.2 and lemma 2.4 we obtain

Corollary 2.5.

$RPML^n \subseteq L(\Sigma^n_g) \subseteq L(\Sigma^n_s)$

In sections 3 and 4 we shall show that $RPML^n = L(\Sigma^n_g)$ and that $L(\Sigma^n_g) = L(\Sigma^n_s)$, which will imply that $RPML^n = L(\Sigma^n_s)$.

Lemma 2.6.

Let L be any polyadic modal logic over $\mathscr{L}^n$, then:

(i) if $B \Rightarrow C \in L$ then $\blacksquare B \Rightarrow \blacksquare C \in L$,

(ii) $\blacksquare(B \wedge C) \Leftrightarrow \blacksquare B \wedge \blacksquare C \in L$,

(iii) if $B \Rightarrow C \in L$ and $A_1, .., A_n \in FOR$ and $1 \le i \le n$, then

$$\square(A_1, .., \underset{i}{B}, .., A_n) \Rightarrow \square(A_1, .., \underset{i}{C}, .., A_n) \in L \text{ and}$$

$$\underline{\square}(A_1, .., \underset{i}{B}, .., A_n) \Rightarrow \underline{\square}(A_1, .., \underset{i}{C}, .., A_n) \in L,$$

(iv) $\square(A_1, .., \underset{i}{B \wedge C}, .., A_n) \Leftrightarrow \square(A_1, .., \underset{i}{B}, .., A_n) \wedge \square(A_1, .., \underset{i}{C}, .., A_n) \in L$,

(v) $\underline{\square}(A_1, .., \underset{i}{B \wedge C}, .., A_n) \Leftrightarrow \underline{\square}(A_1, .., \underset{i}{B}, .., A_n) \wedge \underline{\square}(A_1, .., \underset{i}{C}, .., A_n) \in L$.

3. Completeness theorem for $RPML^n$ with respect to its general semantics

In this section we shall prove that $RPML^n = L(\Sigma^n_g)$. We shall use in this completeness proof a certain modification of the standard canonical-model constructions, adapted here for the polyadic case.

a. Canonical constructions. Let L be any polyadic modal logic over the language $\mathscr{L}^n$. A finite set of formulas $\{A_1, ..., A_n\}$ is called L-consistent if $\neg(A_1 \wedge ... \wedge A_n) \notin L$. An infinite set x of formulas is L-consistent if each finite subset of x is L- consistent. A set of formulas x is a maximal L- consistent set if x is L-consistent and for any L-consistent set y: if $x \subseteq y$ then x=y. The following lemma summarizes all known facts about maximal consistent sets.

Lemma 3.1.

(i)/Lindenbaum Lemma/ Let x be an L-consistent set. Then there exists a maximal L-consistent set y such that $x \subseteq y$.

(ii) If x is a maximal consistent set then for any $A, B \in FOR$:

$\neg A \in x$ iff $A \notin x$,

$A \wedge B \in x$ iff $A \in x$ and $B \in x$,

$A \vee B \in x$ iff $A \in x$ or $B \in x$,

if $A \in x$ and $A \Rightarrow B \in x$ then $B \in x$,

$1 \in x$, $L \subseteq x$, $0 \notin x$.

A set of formulas is called a theory in L if it satisfies the following conditions:

(1) $1 \in x$,

(mon) if $A \in x$ and $A \Rightarrow B \in L$ then $B \in x$,

($\wedge$) if $A \in x$ and $B \in x$ then $A \wedge B \in x$.

A set of formulas is called a co-theory in L if it satisfies the following conditions:

(0) $0 \notin x$,

(mon) if $A \in x$ and $A \Rightarrow B \in L$ then $B \in x$,

($\vee$) if $A \vee B \in x$ then $A \in x$ or $B \in x$.

The notion of a co-theory was introduced in [Vak 89a].

A set of formulas is called a prime theory in L if it is a theory and at the same tame a co-theory in L. The set of all prime theories of L is denoted by W_L. For $A\in FOR$ and $x\subseteq FOR$ denote

$[A]=\{B\in FOR/A\Rightarrow B\in L\}$,

$]A[=\{B\in FOR/B\Rightarrow A\notin L\}$,

$\neg x=\{A\in FOR/\neg A\notin x\}$.

Lemma 3.2.

(i) $[A]$ is the smallest theory in L containing A,

(ii) $]A[$ is the greatest co-theory in L not containing A,

(iii) if x is a co-theory in L then $\neg x$ is a theory in L,

(iv) if x and y are theories in L then $x\cup y$ is an L-consistent set iff for no $A\in FOR$ we have $\neg A\in x$ and $A\in y$,

(v) x is a maximal L-consistent set iff it is a prime theory in L,

(vi) L is a theory in L.

Proof. (i)-(iv) and (vi) follow from the definitions and (v) from lemma 3.1. ∎

Lemma 3.3.

/ Interpolation Lemma/ If x is a theory in L, z is a co-theory in L and $x\subseteq z$ then there exists a prime theory y in L such that $x\subseteq y\subseteq z$.

Proof. Suppose x is a theory in L, z is a co-theory in L and $x\subseteq z$. Then by lemma 3.2.iii $\neg z$ is a theory. We shall show that $x\cup\neg z$ is an L-consistent set. Suppose the contrary. Then by lemma 3.2.iv we have $\neg A\in x$, $A\in\neg z$ for some $A\in FOR$. From $\neg A\in x$ and $x\subseteq z$ we get $\neg A\in z$ and hence $A\notin\neg z$ - a contradiction. Now, applying the Lindenbaum lemma we find a maximal L-consistent set y such that $x\cup\neg z\subseteq y$, so $x\subseteq y$ and $\neg z\subseteq y$. We shall show that $y\subseteq z$. Suppose $A\in y$. Then by lemma 3.1.ii $\neg A\notin y$ and since $\neg z\subseteq y$ we obtain that $\neg A\notin\neg z$, so $\neg\neg A\in z$ and hence $A\in z$, which shows that $y\subseteq z$. So we have $x\subseteq y\subseteq z$ and by lemma 3.2.v y is a prime theory. ∎

For $x,y,x_0,x_1,\dots,x_n\subseteq FOR$ define

xR_Ly iff $(\forall A\in FOR)(\blacksquare A\in x\longrightarrow A\in y)$,

$Q_L(x_0,x_1,\dots,x_n)$ iff

$$(\forall A_1..A_n\in FOR)(\square(A_1,\dots,A_n)\in x_0\longrightarrow A_1\in x_1 \text{ or}\dots\text{or } A_n\in x_n),$$

$S_L(x,x,\dots,x)$ iff

$$(\forall A_1..A_n\in FOR)(\underline{\square}(A_1,\dots,A_n)\in x_0\longrightarrow A_1\in x_1 \text{ or}\dots\text{or } A_n\in x_n),$$

$\blacksquare x=\{A\in FOR/\blacksquare A\in x\}$,

$$\square_i(x_0,x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)=\{A_i\in FOR/\exists A_1..A_{i-1}A_{i+1}..A_n\in FOR: \square(A_1,\dots,A_i,\dots,A_n)\in x_0\ \&\ \forall j\ 1\le j\le n,\ j\ne i\ A_j\notin x_j\},\ i=1\dots n,$$

$$\underline{\square}_i(x_0,x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)=\{A_i\in FOR/\exists A_1..A_{i-1}A_{i+1}..A_n\in FOR: \underline{\square}(A_1,\dots,A_i,\dots,A_n)\in x\ \&\ \forall j\ 1\le j\le n,\ j\ne i\ A_j\notin x\},\ i=1\dots n.$$

Lemma 3.4.

(i) xR_Ly iff $\blacksquare x\subseteq y$.

(ii) If x is an L-theory then $\blacksquare x$ is an L-theory too.

(iii) Let $x\in W_L$ and $A\in FOR$. Then $\blacksquare A\in x$ iff $(\forall y\in W_L)(xR_Ly\longrightarrow A\in y)$.

Proof. (i) is obvious and the proof of (ii) is straightforward. The "if" part of (iii) is easy. For "the only if" part suppose $\blacksquare A\notin x$, $x\in W_L$. Then $A\notin\blacksquare x$. Then by lemma 3.2.ii $\blacksquare x\subseteq]A[$, $\blacksquare x$ is a theory and $]A[$ is a co-theory. By the Interpolation Lemma we can find a prime theory y such that $\blacksquare x\subseteq y\subseteq]A[$. So we have xR_Ly and since $A\notin]A[$ we have that $A\notin y$.■

Lemma 3.5.

(i) For any $i\in\{1,..,n\}$ and $x_0,x_1,..,x_n\subseteq FOR$:

$$Q_L(x_0,x_1,..,x_n) \text{ iff } {}_i(x_0,x_1,..,x_{i-1},x_{i+1},..,x_n)\subseteq x_i,$$

(ii) if x_0 is a theory in L and $x_1,..,x_{i-1},x_{i+1},..x_n$ are co- theories in L then $\Box_i(x_0,x_1,..,x_{i-1},x_{i+1},..,x_n)$ is a theory in L.

(iii) Let $x_0\in W_L$ and $A_1,...,A_n\in FOR$. Then

$\Box(A_1,..,A_n)\in x_0$ iff

$$(\forall x_1...x_n\in W_L)(Q_L(x_0,x_1,..,x_n)\longrightarrow A_1\in x_1 \text{ or..or } A_n\in x_n)$$

Proof. The proof of (i) is straightforward. The proof of (ii) is long but easy and uses lemma 2.6.

(iii) $(\longrightarrow)$ The implication follows directly from the definition of Q_L.

$(\longleftarrow)$ Suppose $\Box(A_1,..,A_n)\notin x_0$. We shall show that

$$(\exists x_1...x_n\in W_L)(Q_L(x_0,x_1,..,x_n) \;\&\; A_1\notin x_1 \;\&...\&\; A_n\notin x_n).$$

Let $y_j=]A_j[$, $j\in\{1,..,n\}$. We shall prove that $Q_L(x_0,y_1,..,y_n)$. Suppose, for the sake of contradiction that this is not true. Then for some $B_1,..,B_n\in FOR$ we have $\Box(B_1,..,B_n)\in x_0$ and $B_1\notin y_1$ &...& $B_n\notin y_n$. From the definitions of y_j we obtain that $B_1\Rightarrow A_1\in L,...,$ $B_n\Rightarrow A_n\in L$. Applying n times lemma 2.6.iii we obtain that $\Box(A_1,..,A_n)\in x_0$ which is a contradiction.

From $Q_L(x_0,y_1,..,y_n)$, by (i) we get that $\Box_1(x_0,y_2,..y_n)\subseteq y_1$. By lemma 3.2.ii all y_j are co-theories in L, so by (ii) $\Box_1(x_0,y_2,..,y_n)$ is a theory. Then by the Interpolation Lemma we find a prime theory x_1 such that $\Box_1(x_0,y_2,..,y_n)\subseteq x_1\subseteq y_1$. By (i) we get $Q_L(x_0,x_1,y_2,..,y_n)$ and again by (i) we obtain $\Box_2(x_0,x_1,y_3,..,y_n)\subseteq y_2$. By the interpolation lemma we find a prime theory x_2 such that $\Box_2(x_0,x_1,y_3,..y_n)\subseteq x_2\subseteq y_2$. Then by (i) we get $Q_L(x_0,x_1,x_2,y_3,..,y_n)$.

Repeating this procedure n times we finally find $x_1,...,x_n\in W_L$ such that $Q(x_0,x_1,..,x_n)$ and $x_1\subseteq y_1$, ..., $x_n\subseteq y_n$. Since $A_i\notin y_i$ we get that $A_i\notin x_i$ for any $i\in\{1,..n\}$. This completes the proof of the lemma.■

Lemma 3.6.

Let $x_0\in W_L$ and $A_1,..,A_n\in FOR$. Then

$\underline{\Box}(A_1,..,A_n)\in x_0$ iff

$$(\forall x_1...x_n\in W)(S_L(x_0,x_1,..,x_n)\longrightarrow A_1\in x_1 \text{ or...or } A_n\in x_n).$$

Proof - the same as the proof of lemma 3.5.■

Consider the relations R_L, Q_L and S_L in the set W_L of all prime theories in L. Then the frame $\underline{W}_L=(W_L, R_L, Q_L, S_L)$ is called a canonical frame for the logic L. Let for $A\in VAR$ $v_L(A)=\{x\in W_L/A\in x\}$, then the model $\underline{M}_L=(\underline{W}_L,v_L)$ is called the canonical model for L.

Lemma 3.7./Truth Lemma/

The following equivalence holds for any $x \in W_L$ and $A \in FOR$:

$$x \Vdash_{v_L} A \text{ iff } A \in x.$$

Proof. The proof is carried out by induction on the complexity of the formula A. When A is in the form $\blacksquare B$, $\Box(B_1,..,B_n)$ and $\underline{\Box}(B_n,..,B_n)$ use lemma 3.4.iii, lemma 3.5.iii and lemma 3.6 respectively.■

Lemma 3.8.

Let $L \neq FOR$ be any rough polyadic modal logic. Then the canonical frame $\underline{W}_L$ of L is a general frame for $RPML^n$.

Proof - straightforward.

Theorem 3.9.

$RPML^n = L(\Sigma^n_g)$.

Proof. Denote $RPML^n$ by L. By corollary 2.5 we have $L \subseteq L(\Sigma^n_g)$. To prove the converse inclusion suppose $A \notin L$. Then by lemma 3.2.ii $L \subseteq]A[$. We have that L is a theory $]A[$ is a co-theory and hence by the Interpolation Lemma we can find a prime theory x such that $L \subseteq x \subseteq]A[$. Since $A \notin]A[$ we have that $A \notin x$. Then by the Truth Lemma $x \nVdash_{v_L} A$. So A is not true in the canonical frame for L, which by lemma 3.8 is a general frame for $RPML^n$. Thus $A \notin L(\Sigma^n_g)$.■

Theorem 3.10.

$PML^n = L(\Sigma^n)$.

The proof is the same as the proof of theorem 3.9.■

4. Equivalence of the general and standard semantics for $RPML^n$

In this section we shall show that the logic $L(\Sigma^n_g)$ of all general frames coincides with the logic $L(\Sigma^n_s)$ of all standard frames. To this end we shall use the copying construction. The general definition, adapted for the case of polyadic modalities and $RPML^n$ is the following.

Let $\underline{W}=(W, R, Q, S)$ and $\underline{W}'=(W', R', Q', S')$ be two frames and $\underline{M}=(\underline{W}, v)$, $\underline{M}'=(\underline{W}', v')$ be models over $\underline{W}$ and $\underline{W}'$ respectively. Let $I \neq \varnothing$ be a set of functions from W into W' and let for any $i \in I$ and $x \in W$ the application of i to x be denoted by x_i. We say that I is a copying from $\underline{W}$ to $\underline{W}'$ if the following conditions are satisfied for any $x, y, x^0,..,x^n \in W$ and $i, j, i_1, ..., i_n \in I$:

(I1) $W' = \bigcup_{i \in I} W_i$, where $W_i = \{x_i / x \in W\}$

(I2) If $x_i = y_j$ then $x=y$,

(CR1) If xRy then $\exists j \in I: x_i R' y_j$,

(CR2) If $x_i R' y_{j_1}$ then xRy,

(CQ1) If $Q(x^0, x^1, .., x^n)$ then $\exists j_1 ... j_n \in I: Q'(x^0_i, x^1_{j_1}, .., x^n_{j_n})$,

(CQ2) If $Q'(x^0_{i_0}, .., x^n_{i_n})$ then $Q(x^0, .., x^n)$,

(CS1) and (CS2) analogous to (CQ1) and (CQ2) respectively.

We say that I is a copying from the model $\underline{M}$ to $\underline{M}'$ if I is a copying from $\underline{W}$

to $\underline{W}'$ and for any A∈VAR, x∈W and i∈I:
(Cv) x∈v(A) iff x_i∈v'(A).

The importance of the copying construction is in the following **Lemma 4.1.** /Copying Lemma/ Let $\underline{M}=(\underline{W},v)$ and $\underline{M}'=(\underline{W}',v')$ be two models and I is a copying from $\underline{M}$ to $\underline{M}'$. Then for any formula A, x∈W and i∈I the following equivalence holds:

$$x \Vdash_{v} A \text{ iff } x_i \Vdash_{v'} A.$$

The easy proof by induction on the complexity of the construction of the formula A is omitted. Note that the clauses (CR1)-(CS2) are used when A is in the form ■B, $\square(B_1,..,B_n)$ and $\underline{\square}(B_1,..,B_n)$.■

Corollary 4.2.

Let Σ_1 and Σ_2 be two classes of frames and for each frame $\underline{W}_1 \in \Sigma_1$ there exists a frame $\underline{W}_2 \in \Sigma_2$ and a copying I from $\underline{W}_1$ to $\underline{W}_2$. Then $L(\Sigma_2) \subseteq L(\Sigma_1)$.

Let $\underline{W}=(W, R, Q, S)$ be a general frame and for every $x^0,...,x^n \in W$ define a relation $D \subseteq W^{n+1}$ as follows:

$$D(x^0,..,x^n) \text{ iff } (\forall y^0..y^n \in W)(x^0Ry^0\&..\&x^nRy^n \rightarrow Q(y^0,..,y^n))\&\bar{S}(x^0,..,x^n)$$

We say that the sequence $x^0,..,x^n$ is a defective one if $D(x^0,..,x^n)$.

Lemma 4.3.

Let $\underline{W}$ be a general frame. then:

(i) $\underline{W}$ is a standard frame iff W has no defective sequences.

(ii) The relation D in $\underline{W}$ satisfies the following condition

$$x^0Ry^0\&...\&x^nRy^n\& D(x^0,..,x^n) \rightarrow D(y^0,..,y^n)$$

Proof. (i) is obvious. Condition (ii) follows from the following two conditions

(a) $x^0Ry^0\&...\&x^nRy^n\&D(x^0,..,x^n) \rightarrow \bar{S}(y^0,..,y^n)$,

(b) $x^0Ry^0\&...\&x^nRy^n\&D(x^0,..,x^n) \rightarrow (\forall z^0..z^n)(y^0Rz^0\&..y^nRz^n \rightarrow Q(z^0,..,z^n))$

Condition (a) follows from the symmetry of R and S5 and (b) follows from the transitivity of R and the definition of D.■

Lemma 4.4.

Let $\underline{W}=(W, R, Q, S)$ be a general frame and let for each x∈W, |x| be the R-equivalence class determined by x. Then there exists a function, which associates to each class |x| an element |x|'∈|x|. This function has the following properties:

(i) |x|'Rx,

(ii) if xRy then |x|'=|y|',

(iii) ||x|'|'=|x|',

(iv) $D(x^0,..,x^n) \rightarrow D(|x^0|',..,|x^n|')$.

Proof. The existence of such a function is guaranteed by the Axiom of Choice. The properties (i)-(iii) are obvious and (iv) follows from lemma 4.3.ii.■

Lemma 4.5.

Let $\underline{W}=(W, R, Q, S)$ be a general frame. Then there exists a general frame $\underline{W}'=(W',R',Q',S')$ and a copying I from $\underline{W}$ to $\underline{W}'$ such that each R'-equivalence class of W' has at least two different elements and if $\underline{W}$ is finite frame then $\underline{W}'$ is a finite frame too.

Proof. Put I={1,2}, W'=W×I and let for x∈W and i∈I $x_i=(x,i)$. Define for any $x,y,x^0,..,x^n \in W$ and $i,j,i_0,..,i_n \in I$:

$x_i R' y_j$ iff xRy,

$Q'(x^0_{i_0},..,x_{i_n})$ iff $Q(x^0,..,x^n)$,

$S'(x^0_{i_0},..,x^n_{i_n})$ iff $S(x^0,..,x^n)$.

Obviously $\underline{W}'=(\underline{W}',R',Q',S')$ is a general frame and I is a copying from $\underline{W}$ to $\underline{W}'$. Since for any $x\in W$ $x_1\neq x_2$ and $x_1R'x_2$ we get that each R'-equivalence class of W' has at least two different elements. Obviously if W is a finite set, then W' is a finite set too.■

Lemma 4.6.

Let $\underline{W}=(W, R, Q, S)$ be a general frame such that each R- equivalence class of W has at least two different elements. Then there exist a standard frame $\underline{W}'=(W',R',Q',S')$ and a copying I from $\underline{W}$ to $\underline{W}'$. If $\underline{W}$ is a finite frame then $\underline{W}'$ is a finite frame too.

Proof. Put $I=\{0,1,..,n\}$, $W'=W\times I$ and let for $x\in W$ and $i\in I$ $x_i=(x,i)$. Define for $x,y,x^0,..,x^n\in W$ and $i,j,i_0,..,i_n\in I$:

(1) $x_i R' y_j$ iff xRy & $i=j$,

(2) $S'(x^0_{i_0},..,x^n_{i_n})$ iff $S(x^0,..,x^n)$ & $i_0=...=i_n$,

(3) $P(x^0,..,x^n)$ iff $D(x^0,..,x^n)$ & $(x^0\neq|x^0|'$ or...or $x^n\neq|x^n|')$,
$\neq(i_0,..,i_n)$ iff $i_0,..,i_n$ are different elements of I,

(4) $Q'(x^0_{i_0},..,x^n_{i_n})$ iff $Q(x^0,..,x^n)$ & $(i_0=...=i_n$ & $\overline{P}(x^0,..,x^n)$ or $\neq(i_0,..,i_n)$ & $P(x^0,..,x^n))$.

Obviously I is a copying from $\underline{W}$ to $\underline{W}'=(W',R',Q',S')$. Also if W is a finite set then W' is a finite set too. It remains to show that $\underline{W}'$ is a standard frame. For that purpose we shall verify the conditions S1-S6 from lemma 2.3.

Since R is an equivalence relation, then it follows from (1) that R' is also an equivalence relation, so S1-S3 are satisfied.

S4. $S'(x^0_{i_0},..,x^n_{i_n})\longrightarrow Q'(x^0_{i_0},..,x^n_{i_n})$.

Suppose $S'(x^0_{i_0},..,x^n_{i_n})$. Then $S(x^0,..,x^n)$ and $i_0=...=i_n$. By S4 we get $Q(x^0,..,x^n)$. From $S(x^0,..,x^n)$ we get $\overline{D}(x^0,..,x^n)$ and $\overline{P}(x^0,..,x^n)$. Then by (4) we obtain $Q'(x^0_{i_0},..,x^n_{i_n})$.

S5. $x^0_{i_0}R'y^0_{j_0}$ &..& $x^n_{i_n}R'y^n_{j_n}$ & $S'(x^0_{i_0},..,x^n_{i_n})\longrightarrow S'(y^0_{j_0},..,y^n_{j_n})$.

Suppose $x^0_{i_0}R'y^0_{j_0}$ &..& $x^n_{i_n}R'y^n_{j_n}$ & $S'(x^0_{i_0},..,x^n_{i_n})$. Then by (1) and (2) we get x^0Ry^0&...& x^nRy^n & $S(x^0,..,x^n)$ and $i_0=...=i_n=j_0=..=j_n$. Then by S5 we get $S(y^0,..,y^n)$ and by (2) - $S'(y^0_{j_0},..,y^n_{j_n})$.

S6. $(\forall y^0_{j_0}..y^n_{j_n})(x^0_{i_0}R'y^0_{j_0}$ &..&$x^n_{i_n}R'y^n_{j_n}\longrightarrow Q'(y^0_{j_0},..,y^n_{j_n}))\longrightarrow$
$S'(x^0_{i_0},..,x^n_{i_n})$

Suppose

(5) $(\forall y^0_{j_0}..y^n_{j_n})(x^0_{i_0}R'y^0_{j_0}$ &..&$x^n_{i_n}R'y^n_{j_n}\longrightarrow Q'(y^0_{j_0},..,y^n_{j_n}))$.

We shall prove the following assertions:

(I) $i_0=\ldots=i_n$,

(II) $(\forall y^0..y^n)(x^0Ry^0\&..\& x^nRy^n \rightarrow Q(y^0,..,y^n))$,

(III) $\overline{D}(x^0,..,x^n)$,

(IV) $S(x^0,..,x^n)$,

(V) $S'(x^0_{i_0},...,x^n_{i_n})$.

(I). Since we have $x^0R|x^0|'\&...\& x^nR|x^n|'$ we get by (1) $x^0_{i_0}R'|x^0|'_{i_0}\&...x^n_{i_n}R'|x^n|'_{i_n}$. Then by (5) we obtain $Q'(|x^0|'_{i_0},...,|x^n|'_{i_n})$. Since $|x^0|'=||x^0|'|'\&..\&|x^n|'=||x^n|'|'$ /lemma 4.4/ we obtain by (4) $\overline{P}(|x^0|',..,|x^n|')$. This condition, together with $Q'(|x^0|'_{i_0},...,|x^n|'_{i_n})$, imply by (3) that $i_0=..=i_n$.

(II). Suppose $x^0Ry^0\&..\& x^nRy^n$. Then by (1) we get $x^0_{i_0}R'y^0_{i_0}\&..\&x^n_{i_n}R'y^n_{i_n}$, which by (5) implies $Q'(y^0_{i_0},...,y^n_{i_n})$, and by (3) – $Q(y^0,..,y^n)$.

(III). Suppose for the sake of contradiction that we have $D(x^0,..,x^n)$. Since each R-equivalence class in W contains at least two different elements we can find $u^0\neq|x^0|'$ such that $u^0\in|x^0|$. Then by lemma 4.4 we obtain $D(u^0,x^1,..,x^n)$ and $|u^0|'\neq u^0$, so by (3) we have $P(u^0,x^1,..,x^n)$. Since $x^0Ru^0\&$ $x^1Rx^1\&..\& x^nRx^n$ we get by (1)
$x^0_{i_0}R'u^0_{i_0}\& x^1_{i_1}R'x^1_{i_1}\&..\& x^n_{i_n}R'x^n_{i_n}$. Then by (5) we get $Q'(u^0_{i_0},x^1_{i_1},...,x^n_{i_n})$. Since we have $P(u^0,x^1,..,x^n)$ this implies $\neq(i_0,..,i_n)$ contrary to $i_0=..=i_n$. This contradiction shows that we have $\overline{D}(x^0,..,x^n)$.

(IV). This is a direct consequence of (II) and (III).

(V). This follows directly from (I), (V) and (1).

Thus (5) implies $S'(x^0_{i_0},..,x^n_{i_n})$, which completes the proof of the condition S6.■

Corollary 4.7.

(i) $L(\Sigma^n_g)=L(\Sigma^n_s)$, (ii) $L(\Sigma^n_{gfin})=L(\Sigma^n_{sfin})$.

Proof. (i) Since $\Sigma^n_s\subseteq\Sigma^n_g$ we have that $L(\Sigma^n_g)\subseteq L(\Sigma^n_s)$. By lemma 4.5 lemma 4.6 and corollary 4.2 we obtain that $L(\Sigma^n_s)\subseteq L(\Sigma^n_g)$, which implies the equality $L(\Sigma^n_g)=L(\Sigma^n_s)$.

In the same way we can prove (ii).■

Theorem 4.8.

$RPML^n=L(\Sigma^n_s)=L(\Sigma^n_g)$.

Proof. The theorem follows directly from theorem 3.9 and corollary 4.7. ■

Theorem 4.9.

There is no a set of formulas F such that for any general frame $\underline{W}$: $\underline{W}$ is a standard frame iff each formula A from F is true in $\underline{W}$, i.e. the class of standard frames is not modally definable.

Proof. Suppose that such a set F exists. Then $F\subseteq L(\Sigma^n_s)$. It is not difficult to find a general frame $\underline{W}_0$ which is not a standard one. Then for some formula

$A\in F$ we obtain that A is not true in $\underline{W}_0$ so $A\notin L(\Sigma^n_g)$. By corollary 4.7.i $A\notin L(\Sigma^n_s)$, so $A\notin F$ - a contradiction.■

5. Finite model property for $RPML^n$

In this section we shall show that $RPML^n=L(\Sigma^n_{gfin})=L(\Sigma^n_{sfin})$. This will show that $RPML^n$ possesses the finite model property /f.m.p./ with respect to its general and standard models, and, consequently, is decidable. This will be done by the method of filtration. For that purpose we shall modify the Segerberg's [Seg 71] definition of filtration to the case of the language of $RPML^n$.

Let $\underline{W}=(W, R, Q, S)$ be a frame and $\underline{M}=(\underline{W},v)$ be a model over $\underline{W}$. Let Γ be a finite set of formulas, closed under subformulas. For any $x,y\in W$ define

$x\sim y$ iff $(\forall A\in\Gamma)(x\Vdash_v A \longleftrightarrow y\Vdash_v A)$.

Then for $x\in W$ let $|x|=\{y\in W/x\sim y\}$, $W'=\{|x|/x\in W\}$ and for $A\in VAR$ let $v'(A)=\{|x|\in W'/x\in v(A)$. Let $R'\subseteq W'^2$ and $Q',S'\subseteq W'^{n+1}$. We say that the model $\underline{M}'=(\underline{W}',v')$ over the frame $\underline{W}'=(W',R',Q',S')$ is a filtration of the model $\underline{M}$ through Γ if the following conditions are satisfied:

(FR1) $xRy \longrightarrow |x|R'|y|$,

(FR2) $|x|R'|y| \longrightarrow (\forall\blacksquare A\in\Gamma)(x\Vdash_v \blacksquare A \longrightarrow y\Vdash_v A)$,

(FQ1) $Q(x_0,..,x_n) \longrightarrow Q'(|x_0|,..,|x_n|)$,

(FQ2) $Q'(|x_0|,|x_1|,..,|x_n|) \longrightarrow (\forall\Box(A_1,..,A_n)\in\Gamma)$
$(x_0\Vdash_v \Box(A_1,..,A_n) \longrightarrow x_1\Vdash_v A_1$ or...or $x_n\Vdash_v A_n)$,

(FS1) $S(x_0,..,x_n) \longrightarrow S'(|x_0|,..,|x_n|)$,

(FS2) $S'(|x_0|,|x_1|,..,|x_n|) \longrightarrow (\forall\underline{\Box}(A_1,..,A_n)\in\Gamma)$

$(x_0\Vdash_v \underline{\Box}(A_1,..,A_n) \longrightarrow x_1\Vdash_v A_1$ or...or $x_n\Vdash_v A_n)$.

Lemma 5.1./ Filtration Lemma/

Let $\underline{M}=(\underline{W}, v)$ be a model and $\underline{M}'=(\underline{W}',v')$ be a filtration of $\underline{M}$ through Γ. Then

(i) For any $A\in\Gamma$ and $x\in W$: $x\Vdash_v A$ iff $|x|\Vdash_{v'} A$.

(ii) If $Card\Gamma=k$ then $CardW'\leq 2^k$.

Proof. The proof of (i) is by induction on the complexity of the formula A and is almost the same as in Segerberg [Seg 71]. Note that the clauses (FR1)-(FS2) are used when A is in the form $\blacksquare B$, $\Box(B_1,..,B_n)$ and $\underline{\Box}(B_1,..,B_n)$. The proof of (ii) is also as in Segerberg [Seg 71].■

Theorem 5.2. /Filtration Theorem for $RPML^n$/

Let $\underline{W}=(W, R, Q, S)$ be a general frame and $\underline{M}=(\underline{W}, v)$ be a model over $\underline{W}$ and A_0 be a formula. Then there exist a finite set Γ of formulas, containing A_0 and closed under subformulas and a filtration $\underline{M}'=((W', R', Q',S'),v')$ of $\underline{M}$ through Γ such that $\underline{W}'=(W',R',Q',S')$ is a general frame.

Proof. Let Γ be the smallest set of formulas containing A_0, closed under subformulas and satisfying the following conditions

(γ1) if $\Box(A_1,..,A_n)\in\Gamma$ then $\underline{\Box}(A_1,..,A_n)\in\Gamma$,

(γ2) if $\underline{\Box}(A_1,..,A_n)\in\Gamma$ then $\blacksquare A_i\in\Gamma$ for $i=1,..,n$.

Define W' and v' as in the definition of filtration. For any $|x|, |y|, |x_0|, \dots, |x_n| \in W'$ define:

(1) $|x|R'|y|$ iff $(\forall \blacksquare A \in \Gamma)(x \Vdash_v \blacksquare A \longleftrightarrow y \Vdash_v \blacksquare A)$ &

$(\forall \underline{\Box}(A_1,\dots,A_n) \in \Gamma)(x \Vdash_v \underline{\Box}(A_1,\dots,A_n) \longleftrightarrow y \Vdash_v \underline{\Box}(A_1,\dots,A_n))$,

(2) $Q'(|x_0|, |x_1|, \dots, |x_n|)$ iff $(\forall \Box(A_1,\dots,A_n) \in \Gamma)$

$(x_0 \Vdash_v \Box(A_1,\dots,A_n) \rightarrow x_1 \Vdash_v A_1 \text{ or..or } x_n \Vdash_v A_n)$,

(3) $S'(|x_0|, |x_1|, \dots, |x_n|)$ iff $(\forall \underline{\Box}(A_1,\dots,A_n) \in \Gamma)$

$(x_0 \Vdash_v \underline{\Box}(A_1,\dots,A_n) \rightarrow x_1 \Vdash_v \blacksquare A_1 \text{ or..or } x_n \Vdash_v \blacksquare A_n)$,

Lemma 5.3.

The definitions of R', Q' and S' are correct in the following sense:

(i) if $x \sim x'$ & $y \sim y'$ & $|x|R'|y|$ then $|x'|R'|y'|$,

(ii) if $x_0 \sim y_0$ &..& $x_n \sim y_n$ & $Q'(|x_0|, \dots, |x_n|)$ then $Q'(|y_0|, \dots, |y_n|)$,

(iii) if $x_0 \sim y_0$ &..& $x_n \sim y_n$ & $S'(|x_0|, \dots, |x_n|)$ then $S'(|y_0|, \dots, |y_n|)$.

Proof. Let us show for example (i).

Suppose $x \sim x'$ & $y \sim y'$ & $|x|R'|y|$. We have to show the following implications:

(a) If $\blacksquare A \in \Gamma$ and $x' \Vdash_v \blacksquare A$ then $y' \Vdash_v \blacksquare A$,

(b) If $\blacksquare A \in \Gamma$ and $y' \Vdash_v \blacksquare A$ then $x' \Vdash_v \blacksquare A$,

(c) If $\underline{\Box}(A_1,\dots,A_n) \in \Gamma$ and $x' \Vdash_v \underline{\Box}(A_1,\dots,A_n)$ then $y' \Vdash_v \underline{\Box}(A_1,\dots,A_n)$,

(d) If $\underline{\Box}(A_1,\dots,A_n) \in \Gamma$ and $y' \Vdash_v \underline{\Box}(A_1,\dots,A_n)$ then $x' \Vdash_v \underline{\Box}(A_1,\dots,A_n)$.

Let us prove for example (c). Suppose $\underline{\Box}(A_1,\dots,A_n) \in \Gamma$ and $x' \Vdash_v \underline{\Box}(A_1,\dots,A_n)$. Since $x \sim x'$ and $\underline{\Box}(A_1,\dots,A_n) \in \Gamma$ we get that $x \Vdash_v \underline{\Box}(A_1,\dots,A_n)$ and since $|x|R'|y|$ we obtain $y \Vdash_v \underline{\Box}(A_1,\dots,A_n)$. Then, since $y \sim y'$, we get $y' \Vdash_v \underline{\Box}(A_1,\dots,A_n)$. ■

The intended frame and model are $\underline{W}' = (W', R', Q', S')$ and $\underline{M}' = (\underline{W}', v')$.

Lemma 5.4.

The model $\underline{M}' = (\underline{W}', v')$ is a filtration of the model $\underline{M}$ through Γ.

Proof. We have to verify the conditions (FR1)-(FS2) from the definition of filtration.

(FR1). Suppose xRy . We have to prove the following implications:

(a) If $x \Vdash_v \blacksquare A$ then $y \Vdash_v \blacksquare A$,

(b) If $y \Vdash_v \blacksquare A$ then $x \Vdash_v \blacksquare A$,

(c) If $x \Vdash_v \underline{\Box}(A_1,\dots,A_n)$ then $y \Vdash_v \underline{\Box}(A_1,\dots,A_n)$,

(d) If $y \Vdash_v \underline{\Box}(A_1,\dots,A_n)$ then $x \Vdash_v \underline{\Box}(A_1,\dots,A_n)$.

For (a) suppose $x \Vdash_v \blacksquare A$, yRz and proceed to show $z \Vdash_v A$. From xRy and yRz, by the transitivity of R, we get xRz. Then, since $x \Vdash_v \blacksquare A$, we get $z \Vdash_v A$.

For (b) we proceed in the same way using symmetry and transitivity of R.

For (c) suppose $x \Vdash_v \underline{\Box}(A_1,\ldots,A_n)$, $S(y,z_1,\ldots,z_n)$ and proceed to show that for some i, $1 \le i \le n$, $z_i \Vdash_v A_i$. From xRy we get yRx. From yRx, $z_1Rz_1,\ldots,z_nRz_n$ and $S(y,z_1,\ldots,z_n)$ we get by S5 $S(x,z_1,\ldots,z_n)$. Then, since $x \Vdash_v \underline{\Box}(A_1,\ldots,A_n)$ we get $z_i \Vdash_v A_i$ for some $1 \le i \le n$.

For (d) we proceed in a similar way.

(FR2). Suppose $|x|R'|y|$, $\blacksquare A \in \Gamma$, $x \Vdash_v \blacksquare A$ and proceed to show that $y \Vdash_v A$. From the assumptions we get $y \Vdash_v \blacksquare A$ and since yRy we obtain $y \Vdash_v A$.

(FQ1) and (FQ2) follow directly from the definition of Q'.

(FS1) Suppose $S(x_0,\ldots,x_n)$, $\underline{\Box}(A_1,\ldots,A_n) \in \Gamma$, $x_0 \Vdash_v \underline{\Box}(A_1,\ldots,A_n)$ and proceed to show that $x_i \Vdash_v \blacksquare A_i$ for some $i=1,\ldots,n$. Suppose, for the sake of contradiction, that for any $i=1,\ldots,n$ we have $x \nVdash_v \blacksquare A_i$. Then there exist $y_1,\ldots,y_n$ such that $x_1Ry_1,\ldots,x_nRy_n$ and $y_1 \nVdash_v A_1,\ldots,y_n \nVdash_v A_n$. From x_0Rx_0, $x_1Ry_1,\ldots,x_nRy_n$ and $S(x_0,x_1,\ldots,x_n)$ we get by S5 $S(x_0,y_1,\ldots,y_n)$. Since $x_0 \Vdash_v \underline{\Box}(A_1,\ldots,A_n)$, we obtain that $y_i \Vdash_v A_i$ for some $i=1,\ldots,n$ – a contradiction.

(FS2). Suppose $S'(|x_0|,|x_1|,\ldots,|x_n|)$, $\underline{\Box}(A_1,\ldots,A_n) \in \Gamma$, $x_0 \Vdash_v \underline{\Box}(A_1,\ldots,A_n)$ and proceed to show that $x_i \Vdash_v A_i$ for some $i=1,\ldots,n$. From the assumptions we obtain that $x_i \Vdash_v \blacksquare A_i$ for some $i=1,\ldots,n$. Since x_iRx_i we get that $x_i \Vdash_v A_i$. ■

Lemma 5.5.

The frame $\underline{W}'=(W', R', Q', S')$ is a general frame.

Proof. By the definition of R' we see that R' is an equivalence relation, so the conditions S1-S3 are satisfied.

S4. $S'(|x_0|,\ldots,|x_n|) \rightarrow Q'(|x_0|,\ldots,|x_n|)$.

Suppose, for the sake of contradiction, that this is not true. Then we have $S'(|x_0|,\ldots,|x_n|)$ and $\overline{Q}'(|x_0|,\ldots,|x_n|)$. Then for some $\Box(A_1,\ldots,A_n) \in \Gamma$ we have $x_0 \Vdash_v \Box(A_1,\ldots,A_n)$ and for any $i=1,\ldots,n$ $x_i \nVdash_v A_i$. From $x_0 \Vdash_v \Box(A_1,\ldots,A_n)$ and axiom A4 we obtain that $x_0 \Vdash_v \underline{\Box}(A_1,\ldots,A_n)$. From $\Box(A_1,\ldots,A_n) \in \Gamma$ we obtain by (γ1) that $\underline{\Box}(A_1,\ldots,A_n) \in \Gamma$. Since we have $S'(|x_0|,\ldots,|x_n|)$, then by (3) we obtain that for some $i=1,\ldots,n$ $x_i \Vdash_v \blacksquare A_i$. Since x_iRx_i we get $x_i \Vdash_v A_i$ – a contradiction.

S5. $|x_0|R'|y_0|$ & $|x_1|R'|y_1|$ &..& $|x_n|R'|y_n|$ &

$S'(|x_0|,|x_1|,\ldots,|x_n|) \rightarrow S'(|y|,|y|,\ldots,|y|)$

Suppose

$|x_0|R'|y_0|$ & $|x_1|R'|y_1|$ &..& $|x_n|R'|y_n|$ & $S'(|x_0|,|x_1|,..,|x_n|)$, $\underline{\Box}(A_1,..,A_n)\in\Gamma$, $y_0 \nVdash_v \underline{\Box}(A_1,..,A_n)$ and proceed to show that $y_i \nVdash_v \blacksquare A_i$ for some $i=1,..,n$. From $\underline{\Box}(A_1,..,A_n)\in\Gamma$ we get by (γ2) that $\blacksquare A_i\in\Gamma$ for any $i=1,..,n$. From $|x_0|R'|y_0|$, $\underline{\Box}(A_1,..,A_n)\in\Gamma$ and $y_0 \nVdash_v \underline{\Box}(A_1,..,A_n)$ we get by (3) that

$x_0 \nVdash_v \underline{\Box}(A_1,..,A_n)$.

From here and $S'(|x_0|,|x_1|,..,|x_n|)$ we obtain by (3) that $x_i \nVdash_v \blacksquare A_i$ for some $i=1,..,n$. For that i we have $\blacksquare A_i\in\Gamma$, $x_i \nVdash_v \blacksquare A_i$ and $|x_i|R'|y_i|$, so by (1) we get $y_i \nVdash_v \blacksquare A_i$, which had to be proved.■

Theorem 5.7.

(i) $L(\Sigma^n_g)=L(\Sigma^n_{gfin})$,

(ii) $RPML^n=L(\Sigma^n_g)=L(\Sigma^n_s)=L(\Sigma^n_{gfin})=L(\Sigma^n_{sfin})$,

(iii) $RPML^n$ is decidable.

Proof. (i) Since $\Sigma^n_{gfin}\subseteq\Sigma^n_g$ we have $L(\Sigma^n_g)\subseteq L(\Sigma^n_{gfin})$. Suppose, for the sake of contradiction, that the converse inclusion is not true. So we have $A\in L(\Sigma^n_{gfin})$, $A\notin L(\Sigma^n_g)$ for some formula A. Then there exists a general model $(\underline{W}, v)$ such that $x \nVdash_v A$ for some $x\in W$. By the Filtration Theorem for $RPML^n$ we can find a finite set of formulas Γ containing A and a filtration $(\underline{W}', v')$ of $(\underline{W}, v)$ through Γ. Then by the Filtration Lemma $|x| \nVdash_{v'} A$. So $A\notin L(\Sigma^g_{gfin})$ - a contradiction.

Condition (ii) is a direct consequence of corollary 4.7 and (i), and (iii) follows from (ii).■

Theorem 5.8.

(i) $L(\Sigma^n)=L(\Sigma^n_{fin})$,

(ii) $PML^n=L(\Sigma^n)=L(\Sigma^n_{fin})$.

Proof. To proof the theorem we use a filtration defined as follows. Let $\underline{M}=(\underline{W}, v)$ be a model and $A\in FOR$. Define Γ to be the set of all subformulas of A and W' and v' as in the definition of filtration. For the relations R', Q' and S' we have the following definitions:

$$|x|R'|y| \text{ iff } (\forall\blacksquare A\in\Gamma)(x\Vdash_v \blacksquare A \rightarrow y\Vdash_v A),$$

$$Q'(|x_0|,|x_1|,..,|x_n|) \text{ iff } (\forall\ (A_1,..,A_n)\in\Gamma)$$
$$(x_0\Vdash_v \Box(A_1,..,A_n) \rightarrow x_1\Vdash_v A_1 \text{ or..or } x_n\Vdash_v A_n),$$

$$S'(|x_0|,|x_1|,..,|x_n|) \text{ iff } (\forall\_(A_1,..,A_n)\in\Gamma)$$
$$(x_0\Vdash_v \underline{\Box}(A_1,..,A_n) \rightarrow x_1\Vdash_v A_1 \text{ or..or } x_n\Vdash_v A_n).$$

We left the details for the reader.■

6. Rough Boolean Logic - RBL

Boolean disjunction and conjunction can be considered as diadic modal operations. Indeed, let $\underline{W}=(W, R, Q, S)$ be a frame such that $Q,S\subseteq W^3$ and for any $x,y,z\in W$ $Q(x,y,z)$ iff $x=y=z$. Then the following equivalences are true in $\underline{W}$:

$\Box(A,B) \Leftrightarrow A \vee B$ and $\Diamond(A,B) \Leftrightarrow A \wedge B$

Such frames will be called Boolean, or B-frames. By B_s (B_g, B_{gfin}, B_{sfin}) we denote the class of all standard (general, finite general, finite standard) B-frames. It is then natural to call $\underline{\Box}(A,B)$ and and $\overline{\Box}(A,B)$ rough disjunctions and $\underline{\Diamond}(A,B)$ and $\overline{\Diamond}(A,B)$ - rough conjunctions and to call the logic $L(B_s)$ - Rough Boolean Logic - RBL.

RBL can be axiomatized by adding to the axioms of $RPML^2$ the following formula as an additional axiom:

(B) $\Box(A,B) \Leftrightarrow A \vee B$

Then RBL is the smallest set of formulas from $\mathcal{L}^2$ containing the axioms of RBL and closed under the rules of $RPML^2$.

We shall show that $RBL=L(B_g)=L(B_s)=L(B_{gfin})=L(B_{sfin})$.

Theorem 6.1.

$RBL=L(B_g)$.

Proof. The proof follows from the proof of the fact that $RPML^2=L(\Sigma_g^2)$ and that the canonical frame for RBL satisfies the following condition for any $x,y,z \in W_L$ /L=RBL/

$Q(x,y,z)$ iff $x=y=z$ To prove this let us note that, by the axiom (B), the definition of Q_L is equivalent to the following one:

(1) $Q_L(x,y,z)$ iff $(\forall A,B \in FOR)(A \vee B \in x \longrightarrow A \in y$ or $B \in z)$, $x,y,z \in W_L$.

Suppose that $x=y=z$. Then by (1) and the fact that x is a prime theory we obtain $Q_L(x,y,z)$.

Now suppose

(2) $Q_L(x,y,z)$.

Since for any $A \in FOR$ $A \vee \neg A \in x$ we obtain by (1) that

(3) $(\forall A \in FOR)(A \in y$ or $\neg A \in z)$

Having in mind that z is a maximal consistent set we obtain from (3) the following equivalences:

$(\forall A \in FOR)(A \in y$ or $A \notin z)$ iff $(\forall A \in FOR)(A \in z \longrightarrow A \in y)$ iff $z \subseteq y$ iff $z=y$.

So (2) implies $y=z$.

Since $A \vee A \Leftrightarrow A \in L$ and $y=z$, we obtain from (1) and (2) that $(\forall A \in FOR)(A \in x \longrightarrow A \in y)$. From here we get $x \subseteq y$ and hence $x=y$. Finally we obtain that (2) implies $x=y=z$.

Thus we have $Q_L(x,y,z)$ iff $x=y=z$. ■

Lemma 6.2.

Let $\underline{W}=(W, R, Q, S)$ be a general B-frame. Then there exists a standard B-frame $\underline{W}'=(W',R',Q',S')$ and a copying I from $\underline{W}$ to $\underline{W}'$. If $\underline{W}$ is a finite frame then $\underline{W}'$ is a finite frame too.

Proof. Let $I=\{1,2\}$ and for $i \in I$ define

$$x_i = \begin{cases} x & \text{if} S(x,x,x) \\ (x,i) & \text{if} \overline{S}(x,x,x) \end{cases}$$

Then let $W_i=\{x_i / x \in W\}$ and $W'=W_1 \cup W_2$. For $x,y,z \in$ and $i,j,k \in I$ define:

(1) $x_i R' y_j$ iff xRy,

(2) $Q'(x_i, y_j, z_k)$ iff $x_i = y_j = z_k$,

(3) $S'(x_i, y_j, z_k)$ iff $S(x,y,z)$.

It is easy to see that I is a copying from $\underline{W}$ to $\underline{W}'=(W', R', Q', S')$. It remains to show that $\underline{W}'$ is a standard B- frame.

From (1) we see that R' is an equivalence relation in W', so S1-S3 are fulfilled.

S4. $S'(x_i, y_j, z_k) \rightarrow S'(x_i, y_j, z_k)$.

The following implications prove S4:

$S'(x_i, y_j, z_k) \rightarrow S(x,y,z) \rightarrow x=y=z \;\&\; S(x,y,z) \rightarrow S(x,x,x) \;\&\; S(y,y,y) \;\&\; S(z,z,z) \;\&\; x=y=z \rightarrow x_i = x \;\&\; y_j = y \;\&\; z_k = z \;\&\; x=y=z \rightarrow x_i = y_j = z_k$.

S5. can be proved in an easy way.

S6. $(\forall u_l v_m w_n)(x_i R' u_l \;\&\; y_j R' v_m \;\&\; z_k R' w_n \rightarrow u_l = v_m = w_n) \rightarrow S(x_i, y_j, z_k)$

Suppose for the sake of contradiction that S6 is not true. Then we have:

(4) $(\forall u_l v_m w_n)(x_i R' u_l \;\&\; y_j R' v_m \;\&\; z_k R' w_n \rightarrow u_l = v_m = w_n)$ and

(5) $\overline{S}(x_i, y_j, z_k)$.

From (4) we get

(6) $x_i = y_j = z_k$

and from (5) - $\overline{S}(x_i, x_i, x_i)$, $\overline{S}(y_j, y_j, y_j)$, $\overline{S}(z_k, z_k, z_k)$.

So by (3) $x_i = (x,i)$, $y_j = (y,j)$ and $z_k = (z,k)$. Then by (6) we get $x=y=z$ and $i=j=k$. Take s such that $s \neq i$. By (1) we have $x_i R' x_s$, $x_i R' x_i$, $x_i R' x_i$. By (4) we get $x_s = x_i$, so $(x,s)=(x,i)$, which yields $s=i$ - a contradiction. Thus $\underline{W}'$ satisfies S6.

Obviously, if W is a finite set then W' is a finite set too.∎

Corollary 6.3.

(i) $L(B_g) = L(B_s)$,

(ii) $L(B_{gfin}) = L(B_{sfin})$,

(iii) $RBL = L(B_g) = L(B_s)$.

Now we shall show that RBL posses the f.m.p. and hence is decidable.

Theorem 6.4. /Filtration Theorem for RBL/

Let $\underline{W}=(W, R, Q, S)$ be a general B-frame, $\underline{M}=(\underline{W}, v)$ be a model over $\underline{W}$ and $A \in FOR$. Then there exist a finite set Γ of formulas, containing A and closed under subformulas and a general B-model $\underline{M}'=(\underline{W}', v)$ which is a filtration of $\underline{M}$ through Γ.

Proof. Let Γ be the smallest set of formulas containing A and $\Diamond(1,1)$ and closed under subformulas. Obviously Γ is a finite set. Let W' and v' be as in the definition of filtration and for $|x|, |y|, |z| \in W'$ define

(1) $|x| R' |y|$ iff $(\forall \blacksquare A \in \Gamma)(x \Vdash_v \blacksquare A \longleftrightarrow y \Vdash_v \blacksquare A)$ &

$$(\forall A \in \Gamma)(x \Vdash_v A \wedge \underline{\Diamond}(1,1) \longleftrightarrow y \Vdash_v A \wedge \underline{\Diamond}(1,1)),$$

(2) $Q'(|x|, |y|, |z|)$ iff $|x| = |y| = |z|$,

(3) $S'(|x|, |y|, |z|)$ iff $(\forall \underline{\Box}(A,B) \in \Gamma)(x \Vdash_v \underline{\Box}(A,B) \longrightarrow$

$y\Vdash_v A$ or $z\Vdash_v B$) & $|x|=|y|=|z|$ & $x\Vdash_v \underline{\Diamond}(1,1)$.

We left to the reader the proof that the definitions of R', and S' are correct.

Lemma 6.5.

The model $\underline{M}'=((\underline{W}',R',Q',S'),v')$ is a filtration of the model $\underline{M}$ through Γ.

Proof. (FR1). Suppose xRy. As in lemma 5.4 we show that $x\Vdash_v \blacksquare A$ iff $y\Vdash_v \blacksquare A$. Suppose now that $x\Vdash_v A\wedge\underline{\Diamond}(1,1)$ and proceed to show that $y\Vdash_v A\wedge\underline{\Diamond}(1,1)$. From $x\Vdash_v A\wedge\underline{\Diamond}(1,1)$ we obtain that $x\Vdash_v A$ and $x\Vdash_v \underline{\Diamond}(1,1)$, so for some u,v we have S(x,u,v). Then by S4 S(x,u,v) implies x=u=v, so we have S(x,x,x). By S5 xRy, xRx, xRx and S(x,x,x) implies S(y,x,x), which by S4 gives y=x. Then we have $y\Vdash_v A\wedge\underline{\Diamond}(1,1)$.

The converse implication $y\Vdash_v A\wedge\underline{\Diamond}(1,1) \rightarrow x\Vdash_v A\wedge\underline{\Diamond}(1,1)$ can be proved in the same way.

(FR2) can be proved as in lemma 5.4.

(FQ1). Suppose Q(x,y,z), then we have x=y=z, so $|x|=|y|=|z|$.

(FQ2). Suppose $|x|=|y|=|z|$, $(A,B)\in\Gamma$, $x\Vdash_v \Box(A,B)$ and proceed to show that $y\Vdash_v A$ or $z\Vdash_v B$. By axiom $\Box(A,B)\Leftrightarrow A\vee B$ we get $x\Vdash_v A\vee B$, so $x\Vdash_v A$ or $x\Vdash_v B$. Since $A,B\in\Gamma$ and $|x|=|y|=|z|$ we obtain $y\Vdash_v A$ or $z\Vdash_v B$.

(FS1). Suppose S(x,y,z). We have to prove the following implications:

(a) $x\Vdash_v \underline{\Box}(A,B) \rightarrow y\Vdash_v A$ or $z\Vdash_v B$,

(b) $x\Vdash_v \underline{\Diamond}(1,1)$,

(c) $|x|=|y|=|z|$

The implication (a) is obvious. From S(x,y,z) we get that $x\Vdash_v \underline{\Diamond}(1,1)$, so (b) is fulfilled. From S(x,y,z) we get x=y=z, which implies $|x|=|y|=|z|$, so (c) is fulfilled.

(FS2) follows directly from (3).■

Lemma 6.6.

The frame $\underline{W}'=(W', R',Q',S')$ is a general B-frame.

Proof. By (2) we have that $\underline{W}'$ is a B-frame. Obviously R' is an equivalence relation and $S'\subseteq Q'$, so S1-S4 are fulfilled. Before proving S5 we shall verify the following condition:

(*) $|x|R'|u|$ & $S'(|x|,|x|,|x|) \rightarrow |x|=|u|$.

Suppose $|x|R'|u|$ & $S'(|x|,|x|,|x|)$ and proceed to show that $|x|=|u|$. From $S'(|x|,|x|,|x|)$ we obtain by (3) that $x\Vdash_v \underline{\Diamond}(1,1)$. Then, since $|x|R'|y|$ and $\underline{\Diamond}(1,1)\in\Gamma$ we get by (1) that

$u\Vdash_v \underline{\Diamond}(1,1)$.

To prove $|x|=|u|$ we have to show that

$(\forall A\in\Gamma)(x\Vdash_v A \longleftrightarrow u\Vdash_v A)$.

($\rightarrow$) Suppose $A\in\Gamma$ and $x\Vdash_v A$. Since $x\Vdash_v \underline{\Diamond}(1,1)$ we obtain that $x\Vdash_v$

$A\wedge\underline{\Diamond}(1,1)$ and by $|x|R'|u|$ we get that $u\Vdash_v A\wedge\underline{\Diamond}(1,1)$, so $u\Vdash_v A$.

($\leftarrow$) Suppose $u\Vdash_v A$. Since $u\Vdash_v \underline{\Diamond}(1,1)$ we have

$u\Vdash_v A\wedge\underline{\Diamond}(1,1)$ and by $|x|R'|u|$ we obtain $x\Vdash_v A\wedge\underline{\Diamond}(1,1)$, hence $x\Vdash_v A$.

This ends the proof of (*). Now we shall verify the condition

S5. $|x|R'|u|$ & $|y|R'|v|$ & $|z|R'|w|$ & $S'(|x|,|y|,|z|) \rightarrow S'(|u|,|v|,|w|)$.

Suppose $|x|R'|u|$ & $|y|R'|v|$ & $|z|R'|w|$ & $S'(|x|,|y|,|z|)$. From $S'(|x|,|y|,|z|)$ we get $|x|=|y|=|z|$ and hence $S'(|x|,|x|,|x|)$, $|x|R'|v|$, $|x|R'|w|$. Then by (*) we obtain $|x|=|u|$, $|x|=|v|$, $|x|=|w|$. From here and $S'(|x|,|x|,|x|)$ we get $S'(|u|,|v|,|w|)$. This ends the proof of the lemma.■

The proof of the theorem follows directly from lemma 6.5 and lemma 6.6.■

Corollary 6.7.

(i) $L(B_g)=L(B_{gfin})$,

(ii) $RBL=L(B_g)=L(B_s)=L(B_{gfin})=L(B_{sfin})$.

Lemma 6.8.

Let $C=\Diamond(1,1)$. Then the following is true:

(i) Let $\underline{W}=(W, R, Q, S)$ be a standard B-frame, v is a valuation in W and $x\in W$. Then $x\Vdash_v C$ iff $(\forall y\in W)(xRy \rightarrow x=y)$.

(ii) $\underline{\Box}(p,q)\Leftrightarrow C\Rightarrow(p\vee q)\in RBL$,

$\underline{\Diamond}(p,q)\Leftrightarrow C\wedge(p\wedge q)\in RBL$,

$C\Rightarrow(p\Rightarrow\blacksquare p)\in RBL$.

Proof. (i) Suppose $x\Vdash_v C$ and xRy. Then $\exists u,v\in W: S(x,u,v)$. By S4 x=u=v. From xRy, uRu, vRv and S(x,u,v), by S5 we get S(y,u,v) and by S4 we get y=u=v. This implies x=y.

Suppose $(\forall y\in W)(xRy \rightarrow x=y)$. Suppose xRu, xRv, xRw. Then x=u, x=v, x=w, so u=v=w. Then by S6 we obtain S(x,x,x) and from here - $x\Vdash_v \underline{\Diamond}(1,1)$, so $x\Vdash_v C$.

(ii) Using the fact that $RBL=L(B_s)$ the three assertions can be proved semantically.■

Lemma 6.8 says that RBL is not really a polyadic modal system and can be redefined in a language containing one modal operator ■ and one propositional constant C. The standard semantics for this language are all frames of the form (W, R) where R is an equivalence relation. The interpretation of the constant C is suggested by lemma 6.8.i:

$x\Vdash_v C$ iff $(\forall y\in W)(xRy \rightarrow x=y)$.

The condition $(\forall y\in W)(xRy \rightarrow x=y)$ says that each R-equivalence class determined by x contains only x. So the set $\{x\in W/(\forall y\in W)(xRy \rightarrow x=y)\}$ is the maximal subset of W in which R is the identity relation. The obtained logic is an extension of the modal logic S5 with the constant C. Call this logic IS5.

An axiomatization of IS5 is the following:

Axiom schemes: (Bool), A1-A3, $C\Rightarrow(p\Rightarrow\blacksquare p)$

Rules (MP), (N■).

Lemma 6.8 shows that IS5 is contained in RBL under the definition $C=\Diamond(1,1)$. The system RBL can be represented in IS5 by the following definitions, suggested by lemma 6.8:

$\Box(p,q)=p\vee q$, $\underline{\Box}(p,q)=C\Rightarrow(p\vee q)$

We left to the reader the formal proof that this is so.

7. Monadic rough modalities

Let L be a modal logic containing only one monadic modal operation denoted by $\Box$. Then the formulas of L are also formulas of $RPML_1$. We denote by RL - the rough analog of L - the smallest extension of $RPML_1$ containing L. Under this notation $RPML_1$ should be denoted by RK - the rough analog of the minimal modal logic K.

Theorem 7.1.

(i) RL is a conservative extension of L,

(ii) If L is a consistent logic then RL is a conservative extension of the modal logic S5.

Proof. (i) Define a translation τ from the language of RL into the language of L as follows:

$\tau(A)=A$ for $A\in VAR$, $\tau(\neg A)=\neg\tau(A)$, $\tau(A\wedge B)=\tau(A)\wedge\tau(B)$, $\tau(A\vee B)=\tau(A)\vee\tau(B)$, $\tau(\Box A)=\Box\tau(A)$, $\tau(\blacksquare A)=\tau(A)$, $\tau(\underline{\Box} A)=1$.

The condition (i) follows from the following

Lemma 7.2.

(j) If $A\in L$ then $\tau(A)=A$

(jj) If $A\in RL$ then $\tau(A)\in L$.

Proof. (j) is obvious. (jj) can be proved by induction on the length of the proof of A in RL. An axiomatization of RL is for example the following: take the axioms and rules of $RPML_1$ and add as new axioms all theorems of L.■

Let **Triv** be the "trivial modal system", in which $\Box A=A$ and **Ver** be the "Verum system", in which $\Box A=1$ /see Hughes & Cresswell [H&C 84]/. It is well known fact that if L is a consistent modal logic then L is contained in one of the systems **Triv** and **Ver** / Makinson [Mac 71], Segerberg [Seg 72]/.

For the second part of the theorem we define the following translation σ of the language of RL into the language of S5.

$\sigma(A)=A$ for $A\in VAR$, $\sigma(\neg A)=\neg\sigma(A)$, $\sigma(A\wedge B)=\sigma(A)\wedge\sigma(B)$,
$\sigma(A\vee B)=\sigma(A)\vee\sigma(B)$, $\sigma(\blacksquare A)=\blacksquare\sigma(A)$, $\sigma(\underline{\Box} A)=1$,

$$\sigma(\Box A)=\begin{cases}\sigma(A) \text{if } L\subseteq \mathbf{Triv}\\ 1 \quad \text{otherwise}\end{cases}$$

Condition (ii) of the theorem follows from the following

Lemma 7.3. (j) If $A\in S5$ then $\sigma(A)=A$.

Let L be a consistent logic. Then:

(jj) If $A\in L$ then $\sigma(A)$ is a propositional tautology.

(jjj) If $A\in RL$ then $\sigma(A)\in S5$.

Proof. (j) is obvious.

(jj) Let L be a consistent logic and $A\in L$. Let L' be any of the systems **Triv** and **Ver**. Then $L\subseteq L'$ and hence $A\in L'$. It can be proved by induction on the length of the proof of A in L' that if $A\in L'$ then $\sigma(A)\in L'$, and since in this case $\sigma(A)$ is a Boolean formula we obtain that $\sigma(A)$ is a Boolean tautology.

(jjj) Suppose $A\in RL$. Then by induction on the length of the proof of A in RL we can see that $\sigma(A)\in S5$. Since the members of L can be considered as axioms of RL, then, when $A\in L$ use (jj).■

Corollary 7.4.

If L is undecidable logic then RL is undecidable too.

Let Φ be a class of frames of the form (W, Q), $W\neq\emptyset$, $R\subseteq W^2$. By $r(\Phi)$ we denote the class of all standard frames (W, R, Q, S) such that $(W, Q)\in\Phi$.

Lemma 7.5.

If L is sound in a class of frames Φ then RL is sound in the class $r(\Phi)$.

Proof - by induction on the length of the proofs of the theorems of RL.■

It is not, however, always true that if L is complete in a class of frames Φ then RL is complete in the class of standard frames $r(\Phi)$. A counter example is the logic KB complete in the class of all frames (W, R) in which R is a symmetric relation.

Theorem 7.6.

The logic RKB is not complete in any class of standard frames.

Proof. Suppose that RKB is complete in some class S of standard frames and let $\underline{W}=(W, R, Q, S)\in S$. Then Q is a symmetric relation in W. It is easy to see that $S=\underline{R}Q$ is also a symmetric relation in W. Then the formula $\underline{\Diamond}A\Rightarrow A$ is true in $\underline{W}$. Consequently $\underline{\Diamond}A\Rightarrow A\in L(S)$. Since $L(S)\subseteq RKB$ we have that $\Diamond A\Rightarrow A\in RKB$. We shall show, however, that $\underline{\Diamond}A\Rightarrow A\notin RKB$.

Let $W_0=\{a,b\}$, $R_0=\{(a,a),(b,b)\}$, $Q_0=\{(a,b),(b,a)\}$, $S_0=\{(a,b)\}$. It is easy to see that the frame $\underline{W}_0=(W_0, R_0, Q_0, S_0)$ is a general frame and since Q_0 is a symmetric relation then $\underline{W}_0$ is a frame for RKB. However, the formula $\underline{\Diamond}\,\underline{\Box}A\Rightarrow A$ is not true in $\underline{W}_0$ because S_0 is not a symmetric relation in $\underline{W}_0$. So $\underline{\Diamond\Box}A\Rightarrow A\notin RKB$, which ends the proof of the theorem.■

The logic RKB can easily be extended to a complete logic. Let $rKB=KB+\underline{\Diamond\Box}A\Rightarrow A$, Σ_g^{sym} and Σ_s^{sym} be the classes of all general and standard frames, respectively, in which Q and S are symmetric binary relations.

Theorem 7.7.

$rKB=L(\Sigma_g^{sym})=L(\Sigma_s^{sym})$.

Proof. The equality $rKB=L(\Sigma_g^{sym})$ can be done by the canonical- model-construction showing that Q_L and S_L are symmetric relations in the canonical frame. The proof of the equality $rKB=L(\Sigma_g^{sym})=L(\Sigma_s^{sym})$ use the same copying construction /for the case n=1/ as in the proof of corollary 4.7.■

Other examples of incomplete logics of the type RL are RK4, RS4, RS5. Examples of complete logics of this type are RT, R**Triv**, R**Ver** /we use the notations K, T, KB, K4, S4, S5, Triv and Ver as in Hughes & Cresswell [H&C 84]/. The proofs of this facts are left to the reader.

8. Open problems and generalizations

Let Σ_s^{tr}, (Σ_s^{reftr}, Σ_s^{e}) be the classes of all standard frames (W, R, Q, S) such that $Q,S\subseteq W^2$ and Q is a transitive (reflexive and transitive, equivalence) relation in W and denote $rK4=L(\Sigma_s^{tr})$, $rS4=L(\Sigma_s^{reftr})$ and $rS5=L(\Sigma_s^{e})$. The logics rK4, rS4 and RS5 are rough analogs of the logics K4, S4 and S5.

Problems 1-3. Axiomatize the logics rK4, rS4 and rS5.

Problems 4-6. Do the logics rK4, rS4 and rS5 posses the f.m.p.?

Problem 7. / A. Skowron / Axiomatize an extension of PDL /Propositional Dynamic Logic/ with operations of lower and upper rough approximations of programs.

Rough approximations of sets and relations can be defined not only by means of equivalence relations. Let in the definition of an approximation space (W,R) R be an arbitrary binary relation in W and let for $x \in W$ $|x|_R=\{y \in W/xRy\}$ and for $(x_1,..,x_n) \in W^n$ $|(x_1,..,x_n)|_R=\{(y_1,..,y_n) \in W^n/x_1Ry_1 \ \&...\& \ x_nRy_n\}$. Then take the definitions of lower and upper rough approximations of sets and relations given in section 1 with this new meaning of $| \ |_R$. In this way we obtain a generalization of the notion of an approximation space and rough approximations of sets and relations. Of course, when R is an arbitrary relation this generalization is not very interesting because in general we do not have the following intuitive properties of an approximation: $\underline{R}X \subseteq X \subseteq \overline{R}X$. But when R is reflexive relation this is true. If in addition R is a transitive relation in W, then we have, as this is when R is an equivalence relation in W, that $\underline{R}X$ is the biggest R-definable set contained in X, and that $\overline{R}X$ is the smallest R- definable set containing X. So, when R is a reflexive and transitive relation the notion of an approximation by means of R is very natural. Let us note also that in this case the operations $\underline{R}$ and $\overline{R}$ are topological interior and closure in W.

Problem 8. Develop a theory of rough polyadic modal logics, based on approximations by means of reflexive and transitive relations.

Problem 9. Develop a theory of rough polyadic modal logics, based on an approximations $\underline{R}$ and $\overline{R}$, being arbitrary topological interior and closure of sets and relations.

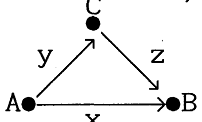
PART IV. ARROW LOGICS

The term "Arrow Logic" has recently become very fashionable, though used by different people with different meaning. I will start with a short history of the subject to explain my point of view.

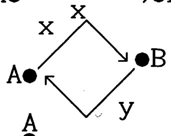
The idea to investigate arrow structures and modal logics based on them, was suggested to me by Johan van Benthem, during the Kleene conference, held in June 1990 in Varna. His idea was that it would be nice to have a simple modal logic, with semantics in two sorted structures, later to be called "arrow structures", which have points /"states"/ and arrows /"transitions"/. Such "arrow structures" must have different models: ordered pairs, directed graphs, categories, vectors, states and transitions, and so on. So, two main problems arose: first, the choice of a good enough mathematical structure of arrows, and second, the corresponding choice of appropriate modal language.

At that time van Benthem [BEN 89] had already introduced one notion of arrow structure /called "arrow frame"/, originating from the algebra of relations. Arrow frame is a relational system of the form $\underline{W}=(W, C, R, I)$, where $W\neq\emptyset$, $C\subseteq W^3$, $R\subseteq W^2$ and $I\subseteq W$, with the following intuition:

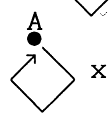
- W - a nonempty set, whose elements are called "arrows",
- C(x,y,z) - x is a "composition" of y and z



- R(x,y) - y is a "reversal" of x



- I(x) - x is an "identity" arrow



In this approach it is not supposed that arrows have some explicitly stated internal structure /to be "ordered pairs", or to have "first point" and "last point", e.t.c./. They are treated as abstract objects, equipped with the above three relations, for which initially nothing is supposed. These structures are used as a semantic base of a modal language, extending the language of the classical propositional logic with one binary modality $A\bullet B$, /composition/, one unary modality $A\check{}$ /reverse/ and a propositional constant I /identity/. The semantics is this:

$x \Vdash A\bullet B$ iff $(\exists y,z\in A)C(x,y,z)$ & $y \Vdash A$ & $z \Vdash B$,

$x \Vdash A\check{}$ iff $(\exists y\in A)R(x,y)$ & $y \Vdash A$,

$x \Vdash I$ iff $I(x)$.

The resulting logic and many of its natural extensions, that aims to approach the desirable properties of composition, reversal and identity from the algebra of relations was studied in [Ben 89]. In [Ben 91] it is called Arrow Logic. Probably this is the first official appearance of the term "Arrow Logic". In [Ben 94] the minimal Arrow Logic /based on frames without any assumptions/ is extended with operation like Kleene star *, called Dynamic Arrow Logic and various possible extensions and variations are discussed. This is in short the van Benthem's approach.

Now some words about my approach. As for the mathematical model of arrows, I decided to take less abstract structure than van Benthem's arrow frames. As The intended "arrow structures" should have "points", some of them connected by "arrows". In Graph theory these are called directed multi graph. I have adopted the following more formal definition; the resulting notion is called "arrow structure", reserving directed multi graphs as one of its models. Arrow structures are systems of the form S=(Ar, Po, 1, 2), where:

- Ar is a nonempty set, whose elements are called arrows,
- Po is a nonempty set, whose elements are called points,
- 1 and 2 are total functions from Ar to Po, called projections. If $x\in Ar$ then 1(x) is called the first end of x and 2(x) is called the second end of x.

- Each point is a first end or second end of some arrow.

Graphically: **1**(x)• —x→ •**2**(x)

So far so good. The next problem was how to associate to arrow structures appropriate modal language and how to interpret this language in such structures. As we know, the standard Kripke semantics requires binary relations in one sorted structure. So one solution of this difficulty is to define in the set of arrows an appropriate set of binary relations in such a way, that the new relational system contains the information of the whole arrow structure. The following four relations in the set Ar proved to have such property: for x,y∈Ar, i,j=1,2

$xR_{ij}y$ iff i(x)=j(y).

The relations R_{ij} , called incidence relations, express the four possible ways for two arrows to have a common end.

Graphically:

$xR_{11}y$: ←x— • —y→ $xR_{22}y$: —x→ • ←y—

$xR_{12}y$: ←x— • ←y— $xR_{21}y$: —x→ • —y→

The relations R_{ij} satisfy the following simple first-order conditions: for x,y,z∈Ar and i,j,k=1,2:

(ρii) $xR_{ii}x$,

(σij) $xR_{ij}y \rightarrow yR_{ji}x$,

(τijk) $xR_{ij}y$ & $yR_{jk}z \rightarrow xR_{ik}z$.

These conditions are characteristic in the following sense: if in a set W we have four relations R_{ij} satisfying the above conditions, then there exists an arrow structure S=(Ar_S, Po_S, 1, 2) such that Ar_S=W and S determines the same relations R_{ij}. So instead of arrow systems, which are two-sorted systems, not convenient for Kripke interpretations, we can use relational systems of the form $\underline{W}=(W,R_{11},R_{12},R_{21},R_{22})$, satisfying the conditions (ρii), (σij) and (τijk). I have called such systems "arrow frames". Arrow frames in this new sense, have two good advantages: first, they are in some sense equivalent to arrow structures, so their abstract elements are real arrows, and second, they have a simple first-order relational definition, suitable for modal purposes. Let us note that the relations C, R and I from van Benthem's arrow frames have natural definitions by the relations R_{ij} :

C(x,y,z) iff $xR_{11}y$ & $yR_{21}z$ & $zR_{22}x$,

R(x,y) iff $xR_{21}y$ & $xR_{12}y$,

I(x) iff $xR_{12}x$

Now the corresponding modal language for the minimal, or Basic Arrow Logic, BAL for short, is easy to define. It extends the language of the propositional logic with four unary modalities [ij] i,j=1,2 with standard Kripke interpretation in arrow frames. Fortunately, our axioms of arrow frame are modally definable and canonical, so BAL is easy to axiomatize. More over it admits filtration, so it is decidable, hence possesses all desirable properties. It is also flexible for extensions with new connectives by definable accessibility relations. For instance an analog of van Benthem's arrow logic will be an extension of BAL with modalities A•B, A˘ and modal constant I. An axiomatization of this system is presented by Andrey Arsov in [Ar 94].

First presentation of the above mentioned results about BAL were summarized in the manuscript [Vak 90], distributed to some people from Amsterdam and Budapest. The full version appears in [VAK 92b] with subtitle Arrow logic I.

In 1990 I met Hungarian logicians Hajnal Andréka and István Németi making a long-term seminar on Algebraic Logic in Warsaw. We had very fruitful discussions, leading us to the conclusion that a good deal of the old Tarskian algebraic logic have "arrow" nature, where arrows are ordered pairs or n-tuples.

At that time István and Hajnal explained to me the importance of the so called "finitization problem in algebraic logic" /see for this problem the excellent survey of algebraic logic given by Németi in [Ne 91]. In short the essence of the finitization problem can be explained as follows. Almost all algebraic systems, studied in algebraic logic, as representable relational algebras, representable cylindric algebras, possess bad properties: their equational theory is undecidable, they do not have finite axiomatization. So the "finitization problem" consists of finding good enough approximations of these logical algebras, as to have finite axiomatization or a decidable equational theory. One solution is to take the so called relativized versions of these algebras. For instance, in the case of relational algebras over some full square $W=U\times U$, to take the appropriate algebra of relations in some subset $W\subseteq U\times U$. In this way, by some results of Maddux [Mad 82] and Kramer [Kra 91] we can obtain varieties, which are finitely axiomatizable and representable in relativized set algebras, and by a result of Németi [Ne 87] we get that their equational theory is decidable.

There is a close analogy between such relativized relational set algebras and the semantics of Basic Arrow Logic. This is because each set $W\subseteq U\times U$ determines an arrow structure and the interpretation of formulas in this structure are subrelations /subsets/ of W and the modal operations on formulas define some relational operations on these subrelations. It is well known fact that completeness theorems in Modal Logic correspond to the representability of some logical algebras and that decidability of the logical calculus corresponds to decidability of equational theory of the corresponding algebraic variety. Thus semantical methods from Modal Logic may help to solve some problems in Algebraic Logic, and conversely, some results of Algebraic Logic may have direct implications for some problems of Modal Logic.

The combination of the algebraic methods of Tarskian Algebraic Logic, and semantical methods coming from Modal Logic proved to be very fruitful. As a result of this combination of methods, Venema [Ven 89,91,92] axiomatizes many modal logics, corresponding to classical systems of algebraic logic, using special modal rules, originated by Gabbay [Ga 85] and a polymodal generalization of Sahlqvist's theorem on first order definability [Sah 75]. Marten Marx [Ma, 95], using the so called graph method, which is in a sense a polymodal and refined version of the Sahlqvist's unraveling construction from modal logic [Sah 75], gave a new, very short and elegant proof of Maddux's theorem [Mad 82] and Kramer's theorem [Kra 91] on representation of relativized relational algebras.

An influence to me from Budapest school of Algebraic Logic is the generalization of arrow structures to n-dimensional arrow structures in [Vak 92a,93], in order to cover the case of n-ary relations. In [Vak 92a,93] I extended the related Basic Arrow Logic of dimension n - BAL^n- with modal operations, corresponding to cylindrifications from cylindric algebras and state the completeness theorem and decidability. Cylindrifications are modal analogs of quantifiers and the first order fragment, corresponding to this logic, presents a decidable version of first order logic.

In [Vak 93] a generalization of arrow structure of dimension n is given, in which the projection functions i, $i\leq n$, assign to each arrow x not a point i.x but a set of points, possibly empty. The obtained arrow structures are called in this dissertation hyper arrow structures of dimension n, and the corresponding arrow logics - hyper arrow logics of dimension n.

Part IV is divided into four chapters.

Chapter 4.1 is devoted to arrow structures and Arrow Logics of dimension 2. Some completeness theorems for the Basic Arrow Logic of dimension 2 - BAL - and some of its extensions are proved. All of the introduced logics possess finite model property and are decidable.

In chapter 4.2 we study systematically arrow structures of dimension n. To each arrow structure of dimension n we associate two relational structures: one over the set of points with one n-place relation, called point frame, and one over the set of arrows, with the incidence relations R_{ij}, called arrow frame. We give an abstract characterization of arrow frames by means of simple first-order conditions. It is proved that the first-order theory of arrow structures, point frames and arrow frames have equal expressive power. Among others this implies that the theory of one n-place relation can be reduced to the theory of n^2 special binary relations. It is proved that the modal logic over point frames is reducible to the modal logic over the arrow frames, which gives an additional motivation of arrow logics. This also shows that the modal logic of polyadic modalities is reducible to a polymodal logic of monadic modalities.

Chapter 4.3 is devoted to the Basic Arrow Logic of dimension n - BAL^n, which corresponds to arrow frames of dimension n. Some completeness and decidability theorems for BAL^n and some of its extensions are proved. Among them is the Arrow logic with operations of cillindrifications - CAL^n, which is a modal version of a decidable fragment of the first-order logic.

In chapter 4.4 we introduce a generalization of the notion of arrow structure of dimension n, putting the projection functions i.x to be multivalued. This means that for each arrow x and i≤n, i.x is not a single point but a set of points, possibly empty. The resulting notion is called hyper arrow structure of dimension n. This notion makes a very close connection with Attribute systems and makes possible to look on the theory of attribute systems from arrow point. Hyper arrow structures of dimension n are also a kind of a Graph-theoretic approach to the theory of set relations and the corresponding modal logic - a logic for set relations.

To each arrow structure of dimension n we associate a relational structure over arrows, called hyper arrow frame of dimension n. We give an abstract characterization of arrow frames, developing a multi-dimension analogs of filters and ideals. The main key is a generalization of the Stone separation theorem of filters and ideals for the distributive lattices and Boolean algebras [Sto 37]. Hyper arrow frames of dimension n are used as a semantic base for a modal logic - $BHAL^n$ - Basic Hyper Arrow Logic of dimension n. The completeness theorem of $BHAL^n$ with respect to its standard semantics is proved. It is shown that $BHAL^n$ possesses finite model property with respect to its non-standard semantics, which implies its decidability. Let us note also that $BHAL^1$ is a slight modification of the logic IND-1 - a modal logic for property systems from chapter 2.4.

CHAPTER 4.1

TWO-DIMENSIONAL ARROW LOGIC

Overview. The notion of two-dimensional arrow structure, called in this chapter arrow structures, is introduced as an algebraic version of the notion of directed multi graph. By means of a special kind of a representation theorem for arrow structures it is shown that the whole information of an arrow structure is contained in the set of his arrows equipped with four binary relations describing the four possibilities for two arrows to have a common point. This makes possible to use arrow structures as a semantic base for a special polymodal logic, called BAL /Basic Arrow Logic/. BAL and various kinds of its extensions are used for expressing in a modal setting different properties of arrow structures. Several kinds of completeness theorems for BAL and some other arrow logics are proved, including completeness with respect to classes of finite models, which implies the decidability of the logic in question.

Introduction

There exist many formal schemes and tools for representing knowledge about different types of data. Sometimes we can better understand this knowledge if it has some graphical representation. In many cases arrows are very suitable visual objects for representing various data structures: different kinds of graphs, binary relations, mappings, categories and so on. An abstract form of this representation scheme is the notion of two-dimensional arrow structure, which in this chapter will be called simply arrow structure. Arrow structure is in some sense an algebraic version of the notion of directed multi graph. Simply speaking, an arrow structure /a.s./ is a two-sorted algebraic system, consisting of a set of arrows Ar, a set of points Po and two functions **1** and **2** from arrows to points, assigning to each arrow x the point **1**(x) - the beginning of x, and the point **2**(x) - the end of x. By means of **1** and **2** we define four relations R_{ij} i,j=1,2 such that $xR_{ij}y$ iff i(x)=j(y). These relations define the four possibilities for a two arrows to have a common point. So each a.s. S determine a relational system W(S)=(Ar, $\{R_{ij}/ij=1,2\}$) called arrow frame /a.f./. It is shown that the whole information of an a.s. S is contained in the arrow frame W(S). Arrow frames as relational systems with binary relations are suitable for interpreting polymodal logics, having modal operations, corresponding to each binary relation in the frame. So we introduce a modal language $\mathcal{L}$ with four boxes [ij] with standard Kripke semantics in arrow frames . We show how different properties of arrow frames are modally definable by means of modal formulas of $\mathcal{L}$. The logic of all arrow frames is axiomatized and called BAL - the Basic Arrow Logic. This chapter is mainly devoted to study BAL and some of its extensions.

The structure of the chapter is the following.

Section 1 is devoted to arrow structures and arrow frames.

In section 2 we introduce semantically the notion of arrow logic as the class of all formulas true in a given class of arrow frames. Some definability and undefinability results are proved there. For instance, applying the copying construction, we show that the logic of all arrow frames coincides with the logic of all normal arrow frames, which correspond to directed graphs, admitting no more than one arrow between an ordered pair of points.

In section 3 we give axiomatization of the logic of all arrow frames - BAL and prove several completeness theorems.

In section 4, applying the filtration technic from ordinary modal logic we prove that BAL and some other arrow logics possess finite model property and are decidable.

In section 5 we study an extension of BAL with a new connective interpreted by an equivalence relation between arrows, stating that two arrows are equivalent if they have common begins and common ends.

In section 6 we study another extension with a modal constant **Loop**, which is true in an arrow if it has common begin and end, i.e. if it forms a loop.

The results of this chapter have been published in [Vak 92b]

1. Arrow structures and arrow frames

By arrow structure (a.s.) we will mean any system S=(Ar, Po, **1**, **2**), where

- Ar is a nonempty set, whose elements are called arrows,
- Po is a nonempty set, whose elements are called points. We assume also that $Ar \cap Po = \emptyset$.
- **1** and **2** are total functions from Ar to Po associated to each arrow x the following two points: **1**(x) - the first point of x (beginning, source, domain), and **2**(x) - the last point of x (end, target, codomain).

Graphically:

$$\mathbf{1}(x)\bullet\xrightarrow{\quad x \quad}\bullet\mathbf{2}(x)$$

If A=**1**(x) and B=**2**(x) we say that x connects A with B, or, that (A,B) is a connected pair of points. It is possible for a pair of points (A,B) to be connected by different arrows.

• For some technical reasons we assume the following axiom for arrow structures:

(Ax) For each point A there exists an arrow x such that A=**1**(x) or A=**2**(x). In other words, each point is ether the first or the last point of some arrow.

Arrow structures are special, two-dimensional case of a more general notion - n-dimensional arrow structure, which differs from the above definition that instead of two projection functions **1** and **2**, it has n projection functions **1**, **2**,...,**n**. n-dimensional arrow structures will be studied in detail in the next chapter.

An a.s. S is called normal if it satisfies the following condition of normality

(Nor) If **1**(x)=**1**(y) and **2**(x)=**2**(y) then x=y.

Sometimes, to denote that Ar, Po, **1**, **2** are from a given a.s. S, we will write Ar_S, Po_S, $\mathbf{1}_S$ and $\mathbf{2}_S$.

The main examples of a.s. structures are directed multi-graphs, and for normal a.s. - directed graphs without isolated points. These are notions studied in Graph theory where graphs are visualized, or sometimes defined, by geometrical notions of a point and arrow. According to the graph-theoretic intuition arrow is a part of a line with some direction, connecting two points. Formally, the notion of an arrow structure coincides with the notion of directed multi-graph without isolated points. We will prefer, however, the term "arrow structure" as it is more neutral, since it has models, not only connected with the graph- theoretic intuitions such as categories and binary relations. Also in this form arrow structures are a special two-dimensional case of the notion of n-dimensional arrow structure, which do not have analogs in Graph theory.

The example of a.s. constructed from a binary relation can be defined as follows. Let R be a nonempty binary relation in a nonempty set W. Define Ar=R, $Po=\{x\in W/(\exists y\in W)(xRy \text{ or } yRx)\}$ and for $(x,y)\in Ar$ define $\mathbf{1}((x,y))=x$ and $\mathbf{2}((x,y))=y$. Then, obviously (Ar, Po, **1**, **2**) is a normal a.s. In some sense this example is typical, because each normal a.s. can be represented as an a.s. determined by a non-empty binary relation.

Let S be an a.s. The following binary relation $\rho=\rho_S$ can be defined in the set Po_S. For each $A,B\in Po_S$:

$A\rho B$ iff $(\exists x\in Ar_S)(\mathbf{1}(x)=A \text{ and } \mathbf{2}(x)=B)$

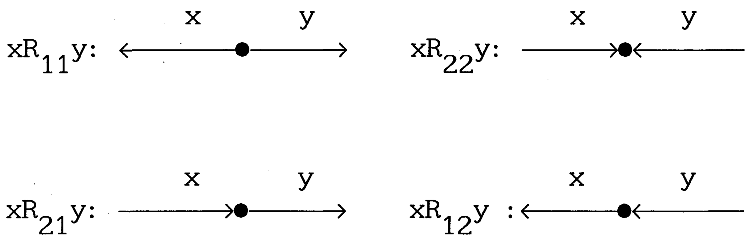
We will consider different kinds of arrow structures, depending on the properties of the relation ρ_S:

- S is serial a.s. if ρ_S is a serial relation (i.e. $\forall A\ \exists B\ A\rho B$),
- S is reflexive a.s. if ρ_S is a reflexive relation,
- S is symmetric a.s. if ρ_S is a symmetric relation,
- S is transitive a.s. if ρ_S is a transitive relation,
- S is total a.s. if ρ_S is a total relation, i.e. $\rho_S=Po_S\times Po_S$.

Let S be a given a.s. The following four relations $R_{ij}=R^S_{ij}$, $i,j\in\{1,2\}$ in Ar_S (called incidence relations in S), will play a fundamental role:

$xR_{ij}y$ iff $\mathbf{i}(x)=\mathbf{j}(y)$

Graphically:



Lemma 1.1.

The relations R_{ij} satisfy the following conditions for any $x,y,z \in Ar_S$ and $i,j,k \in \{1,2\}$:

(ρii) $xR_{ii}x$,

(σij) If $xR_{ij}y$ then $yR_{ji}x$,

(τijk) If $xR_{ij}y$ and $yR_{jk}z$ then $xR_{ik}z$.

Proof. By an easy verification.■

Let $\underline{W}=(W, R_{11}, R_{22}, R_{12}, R_{21})$, $W \neq \varnothing$, be a relational system. $\underline{W}$ will be called arrow frame (a.f.) if it satisfies the axioms (ρii), (σij) and (τijk) for any $i,j,k \in \{1,2\}$and $x,y,z \in W$. The class of all arrow frames will be denoted by ARROW. If S is an a.s. then the a.f. $SAF(S)=(Ar_S, R^S_{11}, R^S_{22}, R^S_{12}, R^S_{21})$ will be called a standard a.f. over S. The class of all standard a.f. will be denoted by (standard)ARROW. One of the main results of this section will be the proof that each a.f. is a standard a.f. over some a.s., i.e. (standard)ARROW=ARROW.

Lemma 1.2.

Let S be an a.s. Then the following equivalences are true, where x,y,z range over Ar_S:

(i) S is normal a.s.iff $\forall xy(xR_{11}y \ \& \ xR_{22}y \rightarrow x=y)$,

(ii) S is serial a.s. iff $\forall x \exists y \ xR_{21}y$,

(iii) S is reflexive a.s. iff $\forall x \exists y(xR_{11}y \ \& \ yR_{21}x)$ and $\forall x \exists y(xR_{21}y \ \& \ yR_{22}x)$,

(iv) S is symmetric a.s. iff $\forall x \exists y(xR_{12}y \ \& \ yR_{12}x)$,

(v) S is transitive a.s. iff $\forall xy \exists z(xR_{21}y \rightarrow xR_{11}z \ \& \ yR_{22}z)$.

(vi) S is total a.s. iff $\forall ij \in \{1,2\} \forall xy \exists z(xR_{i1}z \ \& \ zR_{2j}y)$

Proof. As an example we shall prove (ii).

($\rightarrow$) Suppose S is serial and let $x \in Ar_S$ and $\mathbf{2}(x)=A$. By seriality there exists $B \in Po_S$ such that $A\rho B$. Then for some $y \in Ar_S$ we have $\mathbf{1}(y)=A$ and $\mathbf{2}(y)=B$, so $\mathbf{2}(x)=\mathbf{1}(y)$, which yields $xR_{21}y$. Thus

$\forall x \exists y \ xR_{21}y$.

($\leftarrow$) Suppose $\forall x \exists y \ xR_{21}y$ and let $A \in Po_S$. Then by (Ax) there exists $x \in Ar_S$ such that $A=\mathbf{1}(x)$ or $A=\mathbf{2}(x)$.

Case 1: $A=\mathbf{1}(x)$. Let $B=\mathbf{2}(x)$, then $A\rho B$.

Case 2: $A=\mathbf{2}(x)$. Take y such that $xR_{21}y$. From here we get $\mathbf{2}(x)=\mathbf{1}(y)$ and $A=\mathbf{1}(y)$, Take $B=\mathbf{2}(y)$, then we get $A\rho B$. So in both cases $\forall A \exists B \ A\rho B$.

The remaining conditions can be proved in a similar way. ■ This lemma suggests the following definition concerning arrow frames. Let $\underline{W}=(W, R_{11}, R_{22},$

R_{12}, R_{21}) be an a.f., then $\underline{W}$ is called:

- $\underline{W}$ is normal a.f. iff $\forall xy(xR_{11}y \ \&\ xR_{22}y \rightarrow x=y)$,
- $\underline{W}$ is serial a.f. iff $\forall x\exists y\ xR_{21}y$,
- $\underline{W}$ is reflexive a.f. iff $\forall x\exists y(xR_{21}y \ \&\ xR_{22}y)$ and $\forall x\exists y(xR_{12}y \ \&\ xR_{11}y)$,
- $\underline{W}$ is symmetric a.f. iff $\forall x\exists y(xR_{12}y \ \&\ xR_{21}y)$,
- $\underline{W}$ is transitive a.f. iff $\forall xy\exists z(xR_{21}y \rightarrow xR_{11}z \ \&\ yR_{22}z)$,
- $\underline{W}$ is total a.f. iff $\forall ij\in\{1,2\}\forall xy\exists z(xR_{i1}z \ \&\ zR_{2j}y)$, where the variables

x,y,z range over the set W.

The class of all normal arrow frames will be denoted by (nor)ARROW. Analogously we introduce the notations (ser)ARROW, (ref)ARROW, (sym)ARROW, (tr)ARROW and (total)ARROW for the classes of all serial a.f., reflexive a.f., symmetric a.f., transitive a.f. and total a.f. respectively. We will use also a notation as (ref)(sym)ARROW denoting the class of all reflexive and symmetric arrow frames.

Obviously, if $\underline{W}$ is a total a.f. then $\underline{W}$ is reflexive, symmetric and transitive a.f. An a.f. is called pretotal if it is reflexive, symmetric and transitive. The class of all pretotal a.f. is denoted by (pretotal)ARROW. Using combined notations we have that (pretotal)ARROW=(ref)(sym)(tr)ARROW.

Let $\underline{W}=(W, R_{11}, R_{22}, R_{12}, R_{21})$ be an a.f. and $W'\subseteq W$, $W'\neq\emptyset$ and R'_{ij} are the relations R_{ij} restricted over W'. Then obviously the system $\underline{W}'=(W', R'_{11}, R'_{22}, R'_{12}, R'_{21})$ is an a.f. called subframe of $\underline{W}$. The frame $\underline{W}'$ is called generated subframe of $\underline{W}$ if $\forall ij\in\{1,2\}\forall x\in W'\forall y\in W(xR_{ij}y \rightarrow y\in W')$. If $a\in W$ then by $\underline{W}_a$ we denote the smallest generated subframe of $\underline{W}$, containing a. $\underline{W}_a$ is called an arrow subframe of $\underline{W}$ generated by a. If $\underline{W}$ is an a.f. and there exists some $a\in W$ such that $\underline{W}=\underline{W}_a$ then $\underline{W}$ is called a generated a.f. (by a). If Σ is a class of arrow frames then by Σ_{gen} we denote the class of all generated frames of Σ.

Lemma 1.3.

(i) $((\text{pretotal})\text{ARROW})_{gen}\subseteq(\text{total})\text{ARROW}$,

(ii) $((\text{pretotal})\text{ARROW})_{gen}=(\text{total})\text{ARROW}$

Proof. By an easy verification.■

Let S be an a.s. and for $i\in\{1,2\}$ and $A\in Po_S$ define:

$\mathbf{i}(A)=\{x\in Ar_S/\mathbf{i}(x)=A\}$, $g(A)=(\mathbf{1}(A),\mathbf{2}(A))$.

Lemma 1.4.

The following is true for each $x,y\in Ar_S$ and $i,j\in\{1,2\}$:

(1) If $x\in\mathbf{i}(A)$ and $y\in\mathbf{j}(A)$ then $xR^S_{ij}y$,

(2) If $xR^S_{ij}y$ then $x\in\mathbf{i}(A)$ iff $y\in\mathbf{j}(A)$,

(3) $\mathbf{1}(A)\cup\mathbf{2}(A)\neq\emptyset$.

Proof. By an easy verification.■

Lemma 1.4 suggests the following definition. Let $\underline{W}=(W, R_{11}, R_{22}, R_{12}, R_{21})$ be an a.f. and α_1 and α_2 be subsets of W. The pair (α_1,α_2) will be called a generalized point in $\underline{W}$ if it satisfies the following conditions for each $x,y\in W$ and $i,j\in\{1,2\}$:

(1) If $x\in\alpha_i$ and $y\in\alpha_j$ then $xR_{ij}y$,

(2) If $xR_{ij}y$ then $x\in\alpha_i$ iff $y\in\alpha_j$,

(3) $\alpha_1\cup\alpha_2\neq\emptyset$.

The set of generalized points of an a.f. $\underline{W}$ will be denoted by $Po(\underline{W})$. Lemma 1.2 now means that $g(A)=(\mathbf{1}(A),\mathbf{2}(A))$ is a generalized point in the standard a.f. SAF(S) over S.

For a binary relation R in W and $x\in W$ we define $R(x)=\{y\in W/xRy\}$.

Lemma 1.5.

Let $\underline{U}=(U, U_{11}, R_{22}, R_{12}, R_{21})$ be an a.f.. Then for any $x,y\in U$ and $i,j\in\{1,2\}$: $xR_{ij}y$ iff $R_{i1}(x)=R_{j1}(y)$ and $R_{i2}(x)=R_{j2}(y)$

Proof. By an easy calculation, using the axioms of a.f.■

Lemma 1.6.

Let $\underline{W}$ be an a.f. Then for any $x,y,z\in W$ and $i,j,k\in\{1,2\}$:

(i) The pair $k(z)=(R_{k1}(z), R_{k2}(z))$ is a generalized point in $\underline{W}$.

(ii) For each generalized point (α_1,α_2) there exists $z\in W$ and $k\in\{1,2\}$ such that $k(z)=(\alpha_1,\alpha_2)$.

(iii) $xR_{ij}y$ iff $i(x)=j(y)$.

Proof. (i) Let $i,j\in\{1,2\}$ and $x\in R_{ki}(z)$ and $y\in R_{kj}(z)$. Then we have $zR_{ki}x$ and $zR_{kj}y$. Then by (σki) we obtain $xR_{ik}z$ and by (τikj) we get $xR_{ij}y$. This proves condition (1) from the definition of generalized point. In a similar way one can verify condition (2). By (ρkk) we have $xR_{kk}x$, so $R_{kk}(x)\neq\emptyset$. This shows that $R_{k1}(x)\cup R_{k2}(x)\neq\emptyset$, which proves condition (3).

(ii) Let (α_1,α_2) be a generalized point in $\underline{W}$. Then there exists $z\in W$ such that $z\in\alpha_1\cup\alpha_2$.

Case 1: $z\in\alpha_1$. In this case we will show that k=1, i.e. that $(\alpha_1,\alpha_2)=\mathbf{1}(z)=(R_{11}(z),R_{12}(z))$ i.e. that $\alpha_1=R_{11}(z)$ and that $\alpha_2=R_{12}(z)$.

Let $x\in\alpha_1$. Since $z\in\alpha_1$, then by item (1) of the definition of generalized point we get $xR_{11}z$. So by (σ11) we obtain $zR_{11}x$, which shows that $x\in R_{11}(z)$.

Now let $x\in R_{11}(z)$. Then $zR_{11}x$ and since $z\in\alpha_1$, by item(2) of the definition of generalized point we get $x\in\alpha_1$. This proves the equality $\alpha_1=R_{11}$. In a similar way on can prove that $\alpha_2=R_{12}(z)$.

Case 2: $z\in\alpha_2$. In this case k=2 and we can proceed as in case 1.

(iii) By lemma 1.5. we have: $xR_{ij}y$ iff $R_{i1}(x)=R_{i2}(y)$ and $R_{j1}(x)=R_{j2}(y)$ iff $(R_{i1}(x),R_{i2}(x))=(R_{j1}(y),R_{j2}(y))$ iff $i(x)=j(y)$.■

We will now give a construction of arrow structures from arrow frames. Let $\underline{W}=(W, R_{11}, R_{22}, R_{12}, R_{21})$. Define a system $S=S(\underline{W})$ as follows: $Ar_S=W$, $Po_S=Po(\underline{W})$ - the set of general points of $\underline{W}$, for $k=\mathbf{1},\mathbf{2}$ and $z\in W$ let $k_S(z)=k(z)=(R_{k1}(z),R_{k2}(z))$ as in lemma 1.6. In the next theorem we shall show that $S(\underline{W})$ is an a.s. called the a.s. over $\underline{W}$.

Theorem 1.7.

(i) The system $S(\underline{W})$ defined above is an a.s. More over:

(ii) The standard a.f. $SAF(S(\underline{W}))$ over $S(\underline{W})$ coincides with $\underline{W}$.

(iii) $S(\underline{W})$ is normal (serial, reflexive, symmetric, transitive, total) a.s. iff $\underline{W}$ is normal (serial, reflexive, .. and so on) a.f.

Proof. (i) By lemma 1.6.(i) and (ii) we obtain that the system $S(\underline{W})$ is an a.s.

(ii) By lemma 1.1 and lemma 1.6.(iii) $SAF(S(\underline{W}))$ is a standard a.f. such that for any $x,y\in W$ and $i,j\in\{1,2\}$: $xR_{ij}y$ iff $i(x)=j(y)$ iff $xR^S_{ij}y$, which shows that $SAF(S(\underline{W}))=\underline{W}$.

(iii) By lemma 1.2. $S(\underline{W})$ is normal (serial,...) a.s. iff the corresponding standard a.f. $SAF(S(\underline{W}))$ over $S(\underline{W})$ is normal (serial,...). By (ii) $SAF(S(\underline{W}))=\underline{W}$, which proves the assertion. ■

Corollary 1.8. (standard)ARROW=ARROW

Proof. Immediate from theorem 1.7.■

Let S and S' be two arrow structures. A pair (f,g) of 1one-one functions $f:Ar_S\to Ar_{S'}$ and $g:Po_S\to Po_{S'}$ is called an isomorphism from S onto S' if for any $x\in Ar_S$ and $i=\mathbf{1},\mathbf{2}$ we have $g(\mathbf{i}_S(x))=\mathbf{i}_{S'}(f(x))$.

Lemma 1.9.

Let S be an a.s. $\underline{W}=SAF(S)$ be the standard a.f. over S, $Po(\underline{W})$ be the set of generalized points of $\underline{W}$, and $S'=S(\underline{W})$ be the a.s. over $\underline{W}$. Let for $A\in Po_S$ $g(A)=(\mathbf{1}(A),\mathbf{2}(A))$ be the function defined as before lemma 1.4. and for $x\in Ar_S$ and $i=\mathbf{1},\mathbf{2}$ $\mathbf{i}_{S'}(x)=(R^{S'}_{i1}(x),R^{S'}_{i2}(x))$ be the function defined as in lemma 1.6.(i). Then:

(i) g is a one-one function from Po_S onto $Po(\underline{W})$.

(ii) For any $x\in Ar_S$ and $i=\mathbf{1},\mathbf{2}$: $g(\mathbf{i}(x))=\mathbf{i}_{S'}(x)$.

Proof. Obviously $g(A)$ is a generalized point in $\underline{W}$. Let $g(A)=g(B)$. Then $\mathbf{1}(A)=\mathbf{1}(B)$ and $\mathbf{2}(A)=\mathbf{2}(B)$. For A we can find $x\in Ar_S$ such that $\mathbf{1}(x)=A$ or $\mathbf{2}(x)=A$. Then $x\in\mathbf{1}(A)$ or $x\in\mathbf{2}(A)$. Suppose $x\in\mathbf{1}(A)$. Then $x\in\mathbf{1}(B)$, so $\mathbf{1}(x)=B$. From $\mathbf{1}(x)=A$ and $\mathbf{1}(x)=B$ we get A=B. In the case $x\in\mathbf{2}(A)$ we proceed in the same way and get A=B. This shows that the mapping is injective. To show that it is "onto" suppose that (α_1,α_2) is a generalized point in $\underline{W}$. We will show that for some $A\in Po_S$ it holds that $g(A)=(\mathbf{1}(A),\mathbf{2}(A))=(\alpha_1,\alpha_2)$. Since $\alpha_1\cup\alpha_2\neq\emptyset$ there exists $z\in\alpha_1$ or $z\in\alpha_2$.

Case1: $z\in\alpha_1$. Let $\mathbf{1}(z)=A$, so $z\in\mathbf{1}(A)$. We will show that $\mathbf{1}(A)=\alpha_1$ and that $\mathbf{2}(A)=\alpha_2$. Suppose $x\in\mathbf{1}(A)$. Then $\mathbf{1}(x)=A$, so $\mathbf{1}(x)=\mathbf{1}(z)$ which yields $xR^S_{11}z$. Since $z\in\alpha_1$, then, by the properties of generalized points we get $x\in\alpha_1$, so $\mathbf{1}(A)\subseteq\alpha_1$. Suppose now that $x\in\alpha_1$. Then, since $z\in\alpha_1$, we get $xR_{11}z$, so $\mathbf{1}(x)=\mathbf{1}(z)=A$. Then $x\in\mathbf{1}(A)$, so $\alpha_1\subseteq\mathbf{1}(A)$. Consequently $\mathbf{1}(A)=\alpha_1$. In a similar way one can show that $\mathbf{2}(A)=\alpha_2$. Hence, in this case $g(A)=(\alpha_1,\alpha_2)$.

Case 2: $z\in\alpha_2$. The proof is similar to that of case 1.

(ii) Let $x\in Ar_S$ and $i=1,2$. Since $g(\mathbf{i}(x))=(\mathbf{1}(\mathbf{i}(x),\mathbf{2}(\mathbf{i}(x))$ and $\mathbf{i}_{S'}(x)=(R^S_{i1}(x),R^S_{i2}(x))$, to show that $g(\mathbf{i}(x))=\mathbf{i}_{S'}(x)$, we have to prove that $\mathbf{1}(\mathbf{i}(x))=R^S_{i1}(x)$ and that $\mathbf{2}(\mathbf{i}(x))=R^S_{i1}(x)$. Suppose to this end that $y\in\mathbf{1}(\mathbf{i}(x))$, so $\mathbf{1}(y)=\mathbf{i}(x)$. Thus $xR^S_{i1}y$, which yields $y\in R^S_{i1}(x)$. Consequently $\mathbf{1}(\mathbf{i}(x))\subseteq R^S_{i1}(x)$. The converse inclusion and the second equality can be proved in a similar way. ∎

Theorem 1.10.
Let S be an a.s., $\underline{W}$=SAF(S) be the standard a.f. over S and S($\underline{W}$) be the a.s. **over** $\underline{W}$. Then S is isomorphic with S($\underline{W}$).

Proof. For $x \in Ar_S$ let f(x)=x and for $A \in Po_S$ let g(A)=(**1**(A),**2**(A)). Then lemma 1.8 shows that the pair (f,g) is the required isomorphism. ∎

Theorems 1.10 and 1.7 show that all the information of an a.s. S is contained in the standard $a_S f$. SAF(S) over S and can be expressed in terms of arrows and the relations R_{ij}. An example of such a correspondence is lemma 1.2. As far as first order conditions on the relation ρ are concerned, this correspondence can be defined in an effective way by means of a translation. The intuitive idea of this translation can be explained in the following way. By the axiom (Ax) for each point A there exists $i \in \{\mathbf{1},\mathbf{2}\}$ such that A=i(x). So each variable A for a point is translated by a pair (i,x), where x denotes an arrow and i denotes one of the numbers 1 and 2. Suppose now that we have $A\rho B$, A=i(x) and B=j(y). Then by the definition of ρ we have: $(\exists u)(\mathbf{1}(u)=i(x)\ \&\ \mathbf{2}(u)=j(y))$ which is equivalent to $(\exists u)(xR_{i1}u\ \&\ uR_{2j}y)$. So if A is translated by (i,x) and B by (j,y), then the corresponding translation of $A\rho B$ will be the formula $\varphi = xS_{ij}y = (\exists u)(xR_{i1}u\ \&\ uR_{2j}y)$. Here obviously $S_{ij} = R_{i1} \circ R_{2j}$. The parameters i and j in φ can be eliminated depending on the quantifiers in whose scope A and B occur. If for example A is under the scope of $(\forall A)$, we replace this quantifier by $(\forall i)(\forall x)$ and similarly for $(\exists A)$. Then quantifiers of the type $(\forall i)$ and $(\exists i)$ can be eliminated in a standard way by conjunctions and disjunctions of formulas, substituting 1 and 2 for i. As an example, let us take the formula $(\forall A)(A\rho A)$. First this formula is translated by $(\forall i)(\forall x)xS_{ii}x$. Eliminating $(\forall i)$ we obtain

$(\forall x)(xS_{11}x)\ \&\ (\forall x)(xS_{22}x)$, which is equivalent to

$(\forall x)(\exists y)(xR_{11}y\ \&\ yR_{21}x)\ \&\ (\forall x)(\exists y)(xR_{21}y\ \&\ yR_{22}x)$

which is exactly the condition of reflexivity of ρ from lemma 1.2. The translation of the formula $(\forall A)(\exists B)(A\rho B)$ is the following:

$(\forall i)(\forall x)(\exists j)(\exists y)(xS_{ij}y)$

Eliminating $(\forall i)$ we obtain the conjunction of the following two formulas:

$\varphi_{1j} = (\forall x)(\exists j)(\exists y)(xS_{1j}y)$,

$\varphi_{2j} = (\forall x)(\exists j)(\exists y)xS_{2j}y)$.

Eliminating $(\exists j)$ from φ_{1j} and φ_{2j} we obtain the following formulas φ_1 and φ_2:

$\varphi_1 = (\forall x)((\exists y)(xS_{11}y) \vee (\exists y)(xS_{12}y))$,

$\varphi_2 = (\forall x)((\exists y)(xS_{21}y) \vee (\exists y)(xS_{22}y))$. Substituting here S_{ij} we obtain

$\varphi_1 = (\forall x)((\exists y)(\exists z)(xR_{11}z\ \&\ zR_{21}y) \vee (\exists y)(\exists z)(xR_{11}z\ \&\ zR_{22}y))$,

$\varphi_2 = (\forall x)((\exists y)(\exists z)(xR_{21}z\ \&\ zR_{21}y) \vee (\exists y)(\exists z)(xR_{21}z\ \&\ zR_{22}y)$

The formula φ_1 is always true in a.s. because in the second disjunct we can put y=z=x. Furthermore, the formula $\varphi = (\forall x)(\exists z)(xR_{21}z)$ follows from φ_2, and φ is exactly the condition of seriality from lemma 1.2. It is easy to see that φ implies in a.s. the formula φ_2.

The intuitive idea of translating first order sentences for points in terms of ρ and = in arrow structures into equivalent sentences for arrows in terms of the relations R_{ij} will be given in precise terms in chapter 4.2 for the more general case of n- dimensional arrow structures.

2. Arrow logics - semantic definitions and some definability and nondefinability results

In this section we shall give a semantic definition of a class of modal logics, called arrow logics. To this end we introduce the following modal language $\mathcal{L}$. It contains the following symbols:

- VAR - a denumerable set of proposition variables,
- $\neg$, $\wedge$, $\vee$ - classical propositional connectives,
- $[ij]$, $i,j=1,2$ - four modal operations,
- (,) - parentheses.

The definition of the set of all formulas FOR for $\mathcal{L}$ is defined in the usual way.

Abbreviations: $A\Rightarrow B=\neg A\vee B$, $A\Leftrightarrow B=(A\Rightarrow B)\wedge(B\Rightarrow A)$, $1=A\vee\neg A$, $0=\neg 1$, $\langle ij\rangle A=\neg[ij]\neg A$.

The general semantics of $\mathcal{L}$ is a Kripke semantics over relational structures of the type $\underline{W}=(W, R_{11}, R_{22}, R_{12}, R_{21})$ with $W\neq\emptyset$, called frames. The standard semantics of $\mathcal{L}$ is over the class ARROW of all arrow frames.

Let us remind the basic semantic definitions and notations, which we will use.

Let $\underline{W}=(W, R_{11}, R_{22}, R_{12}, R_{21})$ be a frame. A function $v:VAR\rightarrow 2^W$ assigning to each variable $p\in VAR$ a subset $v(p)$ of W is called a valuation and the pair $M=(\underline{W}, v)$ is called a model over $\underline{W}$. For $x\in W$ and $A\in FOR$ we define a satisfiability relation $x\Vdash_v A$ in M /to be read "A is true in x at the valuation v"/ by induction on the complexity of the formula A as in the usual Kripke definition:

$x\Vdash_v A$ iff $A\in v(A)$ for $A\in VAR$,

$x\Vdash_v \neg A$ iff $x\nVdash_v A$ /not $x\Vdash_v A$/,

$x\Vdash_v A\wedge B$ iff $x\Vdash_v A$ and $x\Vdash_v B$,

$x\Vdash_v A\vee B$ iff $x\Vdash_v A$ or $x\Vdash_v B$,

$x\Vdash_v [ij]A$ iff $(\forall y\in W)(xR_{ij}y \rightarrow y\Vdash_v A)$.

We say that A is true in the model $M=(\underline{W},v)$, or that M is a model for A, if for any $x\in W$ we have $x\Vdash_v A$. A is true in the frame $\underline{W}$, or that W is a frame for A, if A is true in any model over $\underline{W}$. A is true in a class Σ of frames if A is true in any member of Σ. A class of formulas L is true in a model M, or M is a model for L, if any member of L is true in M. L is true in a class of frames Σ if any formula from L is true in Σ. L is called the logic of Σ and denoted by $L(\Sigma)$ if it contains all formulas true in Σ. Obviously, this operation of assigning sets of formulas to classes of frames is antymonotonic in the following sense:

If $\Sigma\subseteq\Sigma'$ then $L(\Sigma')\subseteq L(\Sigma)$.

In this chapter we will study the logics L((standard)ARROW, L(ARROW), L((nor)ARROW), L((ser)ARROW), L((ref)ARROW, L((sym)ARROW), L((tr)ARROW), L((pretotal)ARROW), L((total)ARROW).

The most important logic from this list is L((standard)ARROW). The first result which, can be stated for L(standard)ARROW) and which follows immediately from corollary 1.8. is that

L((standard)ARROW)=L(ARROW).

Lemma 2.1.

/Modal definability of arrow frames/ Let Σ be the class of all frames and

$A \in VAR$. Then in the next table the conditions from the left side are modally definable in Σ by the formulas from the right side: ($i,j,k=1,2$, A is a propositional variable)

(ρii) $(\forall x) xR_{ii}x$,	(Pii) $[ii]A \Rightarrow A$,
(σij) $(\forall xy)(xR_{ij}y \longrightarrow yR_{ji}x)$	(Σij) $A \vee [ij] \neg [ji]A$,
(τijk) $(\forall xyz)(xR_{ij}y$ and $yR_{jk}z \longrightarrow xR_{ik}z)$	($Tijk$) $[ik]A \Rightarrow [ij][jk]A$.

Corollary 2.2.
The class ARROW is modally definable.

Lemma 2.3.
Let Σ=ARROW and $A \in VAR$. Then in the next table the conditions from the left side are modally definable in Σ by the formulas from the right side:

seriality of an a.f.	(ser)	$\langle 21 \rangle 1$,
reflexivity of an a.f.	(ref)	$([11][21]A \Rightarrow A) \wedge ([21][22]A \Rightarrow A)$,
symmetricity of an a.f.	(sym)	$[12][12]A \Rightarrow A$,
transitivity of an a.f.	(tr)	$[11][22]A \Rightarrow [21]A$.

Proof. As an example we shall show the validity of (tr) in an a.f. $\underline{W}$ implies that $\underline{W}$ is a transitive a.f. For the sake of contradiction, suppose that (tr) is true in $\underline{W}$ and that $\underline{W}$ is not transitive a.f. Then for some $x,y,z \in W$ we have $xR_{21}y$ and
not$(\exists z \in W)(xR_{11}z$ & $zR_{22}y)$. Define $v(A)=W \setminus \{y\}$. Then $y \not\Vdash_v A$ and since $xR_{21}y$ we get $x \not\Vdash_v [21]A$. We will show that $x \Vdash_v [11][22]A$. Suppose that this is not true. Then for some $z,t \in W$ we have $xR_{11}z$, $zR_{22}t$ and $t \not\Vdash_v A$, hence $t=y$. So $(\exists z)(xR_{11}z$ & $zR_{22}y)$, which is a contradiction. ∎

Corollary 2.4.
The following classes are modally definable
(ser)ARROW, (ref)ARROW, (sym)ARROW, (tr)ARROW, (pretotal)ARROW.

We will show that the condition of normality of an a.f. is not modally definable and consequently that the class (nor)ARROW is not modally definable. We will show first that the logic L((nor)ARROW) coincides with the logic L(ARROW). To this end we will use the copying construction, adapted here for relational structures in the type of arrow frames.

Let $\underline{W}=(W, R_{11}, R_{22}, R_{12}, R_{21})$ and $\underline{W}'=(W', R'_{11}, R'_{22}, R'_{12}, R'_{21})$ be two frames and $M=(\underline{W}, v)$, $M'=(\underline{W}', v')$ be models over $\underline{W}$ and $\underline{W}'$ respectively. Let I be a nonempty set of mappings from W into W'. We say that I is a copying from $\underline{W}$ to $\underline{W}'$ if the following conditions are satisfied for any $i,j \in \{1,2\}$, $x,y \in W$ and $f,g \in I$:

(I1) $(\forall y' \in W')(\exists y \in W)(\exists g \in I) g(y)=y'$

(I2) If $f(x)=g(y)$ then $x=y$,

($R_{ij}1$) If $xR_{ij}y$ then $(\forall f \in I)(\exists g \in I) f(x)R'_{ij}g(y)$,

($R_{ij}2$) If $f(x)R'_{ij}g(y)$ then $xR_{ij}y$.

We say that I is a copying from M to M' if in addition the following condition is satisfied for any $p \in VAR$, $x \in W$ and $f \in I$:

(V) $x \in v(p)$ iff $f(x) \in v'(p)$.

The importance of the copying construction is in the following **Lemma 2.5.** (i) (Copying lemma) Let I be a copying from the model M to the model M'. Then for any formula $A \in \mathcal{L}$, $x \in W$ and $f \in I$ the following equivalence holds:

$x \Vdash_v A$ iff $f(x) \Vdash_{v'} A$,

(ii) If I is a copying from the frame $\underline{W}$ to the frame $\underline{W}'$ and v is a valuation, then there exists a valuation v' such that I is a copying from the model $M=(\underline{W}, v)$ to the model $M'=(\underline{W}', v')$.

Proof. (i) The proof is by induction on the complexity of the formula A.

(ii) Define for $p \in VAR$:

$v'(p)=\{x' \in W' / (\exists x \in W)(\exists f \in I) f(x)=x' \text{ and } x \in v(p)\}$

We will show that the condition (V) of copying is fulfilled. Let $x \in W$ and $f \in I$ and suppose $x \in v(p)$. Then by the definition of v' we have $f(x) \in v'(p)$. For the converse implication suppose $f(x) \in v'(p)$. Then there exists $y \in W$ and $g \in I$ such that $f(x)=g(y)$ and $y \in v(p)$. By (I2) we get $x=y$, so $x \in v(p)$. ∎

Lemma 2.6.

Let $\underline{W}=(W, R_{11}, R_{22}, R_{12}, R_{21})$ be an arrow frame. Then there exists a normal arrow frame $\underline{W}'=(W', R'_{11}, R'_{22}, R'_{12}, R'_{21})$ and a copying I from $\underline{W}$ to $\underline{W}'$ and if $\underline{W}$ is a finite a.f. the same is $\underline{W}'$.

Proof. Let $\underline{B}(W)=(B(W), 0, 1, +, .)$ be the Boolean ring over the set W, namely B(W) is the set of all subsets of W, $0=\emptyset$, $1=W$, $A+B=(A\setminus B)\cup(B\setminus A)$ and $A.B=AB=A\cap B$.

Note that in Boolean rings $a-b=a+b$.

We put $W'=W\times B(W)$, $I=B(W)$ and for $f \in I$ and $x \in W$ we define $f(x)= (x,f)$. Obviously the conditions (I1) and (I2) from the definition of copying are fulfilled and each element of W' is in the form of f(x) for some $f \in I$ and $x \in W$.

For the relations R'_{ij} we have the following definition:

$f(x)R'_{ij}g(y)$ iff $xR_{ij}y$ & $(f+i.\{x\}=g+j.\{y\})$. Here the indices i, $j\in\{1,2\}$ are considered as elements of B(W): 1 is the unit of B(W) and $2=1+1=1-1=0$.

To verify the condition (R_{ij}1) suppose $xR_{ij}y$ and $f \in I$. Put $g=f+i.\{x\}-j.\{y\}$. Then $f+i.\{x\}=g+j.\{y\}$, which implies $f(x)R'_{ij}g(y)$. Condition (R_{ij}2) follows directly from the definition of R'_{ij}. So I is a copying.

The proof that W' with the relations R'_{ij} is an arrow frame is straightforward. As to the condition of normality suppose $f(x)R'_{11}g(y)$ and $f(x)R'_{22}g(y)$. Then we obtain $xR_{11}y$ & $(f+1.\{x\}=g+1.\{y\})$ and $xR_{22}y$ & $(f+2.\{x\}=g+2.\{y\})$. From this and the fact that $2=0$, we get $f=g$ and $f+\{x\}=g+\{y\}$, which implies $\{x\}=\{y\}$, hence $x=y$ and $f(x)=g(x)$. Thus W' is a normal a.f.

Suppose now that $\underline{W}$ is a finite a.f. Then the Boolean ring over W is finite too and hence $\underline{W}'$ is a finite a.f. ∎

If Σ is a class of a.f. then the class of all finite a.f. from Σ is denoted by Σ_{fin}.

Theorem 2.7.

(i) L((nor)ARROW)=L(ARROW).

(ii) $L(((nor)ARROW)_{fin})=L(ARROW_{fin})$.

Proof. (i) Since (nor)ARROW⊆ARROW we get L(ARROW)⊆ L((nor)ARROW). To prove that L((nor)ARROW)⊆L(ARROW) suppose A∉L(ARROW). Then there exists an a.s. $\underline{W}$, $x \in W$ and a valuation v such that $x \not\Vdash_{v} A$. By lemma 2.6. there exists a normal a.s. $\underline{W}'$ and a copying I from $\underline{W}$ to W'. By lemma 2.5.(ii) there exists a valuation v' in W' such that I is a copying from the model $(\underline{W}, v)$ to the model $(\underline{W}', v')$. Then by the copying lemma we get for any $f \in I$ that $f(x) \not\Vdash_{v'} A$. So A is not true in $\underline{W}'$ and hence A∉L((nor)ARROW). So L((nor)ARROW)⊆L(ARROW).

(ii) The proof is the same as the proof of (i), using the fact that lemma 2.6 guaranties that $\underline{W}'$ is a finite a.f. ■

Corollary 2.8.

The condition of normality of an a.f. is not modally definable.

Proof. Suppose that there exists a formula φ such that for any a.f. $\underline{W}$: φ is true in $\underline{W}$ iff $\underline{W}$ is normal. So $\varphi \in L((nor)ARROW)$. Let $\underline{W}_0$ be an a.f. which is not normal. Then φ is not true in $\underline{W}_0$, so $\varphi \notin L(ARROW)$, hence by theorem 2.7 $\varphi \notin L((nor)ARROW)$, which is a contradiction.■

Another example of modally undefinable condition is totality. First we need the following standard result from modal logic.

Lemma 2.9.

Let Σ be a nonempty class of a.f. closed under subframes and let Σ_{gen} be the class of generated frames of Σ. Then $L(\Sigma)=L(\Sigma_{gen})$.

Corollary 2.10.

(i) $L((pretotal)ARROW)=L(((pretotal)ARROW)_{gen})=L((total)ARROW$.

(ii) $L((pretotal)ARROW)_{fin})=L((((pretotal)ARROW)_{gen})_{fin})=$ $L(((total)ARROW)_{fin})$.

Proof. (i) The first equality follows from lemma 2.9 and the second - from lemma 1.3.

(ii) Use the fact that generated frame of a finite frame is a finite frame too. ■

Corollary 2.11.

The condition of totality of an a.f. is not modally definable.

Proof. Suppose that there exists a formula φ such that for any a.f. $\underline{W}$: φ is true in $\underline{W}$ iff $\underline{W}$ is total a.f. Then $\varphi \in L((total)ARROW)$ and by corollary 2.10 $\varphi \in L((pretotal)ARROW)$. Let $\underline{W}_0$ be a pretotal a.f. which is not total (such frames obviously exist). Then φ is not true in $\underline{W}_0$, so $\varphi \notin L((pretotal)ARROW)$ - a contradiction.■

3. Axiomatization of some arrow logics

In this section we introduce a syntactical definition of arrow logic as sets of formulas containing some formulas as axioms and closed under some rules. The minimal set of axioms which we shall use, contains those from the minimal modal logic for each modality [ij] and the formulas, which modally define arrow frames. The formal system, obtained in this way is denoted by BAL and called Basic Arrow Logic.

Axioms and rules for BAL.

(Bool) All or enough Boolean tautologies,

(K[ij]) $[ij](A \Rightarrow B) \Rightarrow ([ij]A \Rightarrow [ij]B)$,

(Pii) $[ii]A \Rightarrow A$,

(Σij) $A \vee [ij]\neg[ji]A$,

(Tijk) $[ik]A \Rightarrow [ij][jk]A$,

(MP) $\dfrac{A,\ A \Rightarrow B}{B}$, (N[ij]) $\dfrac{A}{[ij]A}$, i,j are any members of{1,2} and A and B are arbitrary formulas.

We identify BAL with the set of its theorems.

By an arrow logic (a.l.) we mean any set L of formulas containing BAL and closed under the rules (MP), (N[ij]) and the rule of substitution of propositional variables. So BAL is the smallest arrow logic. We adopt the following notation. If X is a finite sequence of formulas , (viewed as new axioms) then by BAL+X we denote the smallest arrow logic containing all formulas from X. We shall us the following formulas as additional axioms:

(ser) $\langle 21\rangle 1$,

(ref) $([11][21A\Rightarrow A)\wedge([21][22]A\Rightarrow A)$,

(sym) $[12][12]A\Rightarrow A$,

(tr) $[11][22]A\Rightarrow[21]A$.

Let $X\subseteq\{ser, ref, sym, tr\}$ and let for instance $X=\{ser, tr\}$. Then BAL+X= BAL+ser+tr. We will use also the notation (X)ARROW and for that concrete X (X)ARROW=(ser)(tr)ARROW.

Let L be an a.l. and Σ be a class of arrow frames. We say that L is sound in Σ if $L\subseteq L(\Sigma)$, L is complete in Σ if $L(\Sigma)\subseteq L$, and that L is characterized by Σ, or that $L(\Sigma)$ is axiomatized by L, if L is sound and complete in Σ, i.e. if $L=L(\Sigma)$.

In the completeness proofs we shall use the standard method of canonical models. We shall give a brief description of the method. For more details and some definitions we refer Segerberg [Seg 71] or Hughes & Cresswell [H&C 84].

Let L be an a.l. The frame $\underline{W}_L=(W_L, R^L_{11}, R^L_{22}, R^L_{12}, R^L_{21})$ will be called canonical frame for the logic L if W_L is the set of all maximal consistent sets in L and the relations R^L_{ij} are defined in W_L as follows: $xR^L_{ij}y$ iff $\{A\in FOR/[ij]A\in x\}\subseteq y$. For $p\in VAR$ the function $v_L(p)=\{x\in W_L/p\in x\}$ is called canonical valuation and the pair $M_L=(\underline{W}_L,v_L)$ is called the canonical model for L. The following is a standard result from modal logic.

Lemma 3.1.

(i) Truth lemma for the canonical model for L. The following is true for any formula A and $x\in W_L$: $x\Vdash_{v_L} A$ iff $A\in x$.

(ii) If $A\notin L$ then there exists $x\in W_L$ such that $A\notin x$.

Lemma 3.2.

Let L be an a.l. Then the canonical frame $\underline{W}_L$ of L is an a.f.

Proof. It is well known fact from the standard modal logic that the axiom (Pii) yields the condition (ρii) for the canonical frame. In the same way the axioms (Σij) and (Tijk) yield the conditions (σij) and (τijk) for $\underline{W}_L$. Thus $\underline{W}_L$ is an a.f.∎

Theorem 3.3.

BAL is sound and complete in the class of all arrow frames.

Proof. Soundness follows by lemma 2.1 and the completeness can be proved by the method of canonical models. Let L=BAL. By lemma 3.2 the canonical frame for L is an a.f. To show that $L(ARROW)\subseteq L$ suppose that $A\notin L$. Then by lemma 3.1.(ii) there exists $x\in W_L$ such that $A\notin x$. Then by the truth lemma we have $x\not\Vdash_{v_L} A$, so A is not true in the a.f. $\underline{W}_L$. Then $A\notin L(ARROW)$, which proves the theorem.∎

Corollary 3.4.

BAL=L(ARROW)=L((nor)ARROW).

Proof - immediate from theorem 3.3 and theorem 2.7. ∎

Lemma 3.5.

Let L be an a.l. Then the following conditions are true:

(i) (ser)∈L iff $\underline{W}_L$ is a serial a.f.,

(ii) (ref)∈L iff $\underline{W}_L$ is a reflexive a.f.,

(iii) (sym)∈L iff $\underline{W}_L$ is a symmetric a.f.,

(iv) (tr)∈L iff $\underline{W}_L$ is a transitive a.f.

Proof. As an example we shall show (iv)(⟶). Suppose (tr)= [11][22]A⇒[21]A∈L and proceed to show the condition of transitivity of $\underline{W}_L$: $(\forall xy\in W_L)(\exists z\in W_L)(xR^L_{21}y \longrightarrow xR^L_{11}z \ \&\ zR^L_{22}y)$.

Let $M_1=\{A/[11]A\in x\}$, $M_2=\{A/(\exists B\in y)(\neg A\Rightarrow[22]\neg B\in L)\}$ and $M=M_1\cup M_2$. Then the following claim is true:

Claim

(i) If $A_1,\dots,A_n\in M_i$ then $A_1\wedge\dots\wedge A_n\in M_i$, i=1,2,

(ii) If $A\in M_i$ and A⇒B∈L then $B\in M_i$, i=1,2,

(iii) $M_1\cup M_2$ is L-inconsistent set iff ∃A∈FOR: $A\in M_1$ and $\neg A\in M_2$,

(iv) If $xR^L_{21}y$ then M is L-consistent set of formulas.

(v) Let z be a maximal consistent set. Then $M_2\subseteq z$ implies $zR^L_{22}y$, and $M_1\subseteq z$ implies $xR^L_{11}z$.

(vi) If xRy then $(\exists z\in W_L)(xR^L_{11}z \ \&\ zR^L_{22}y)$.

Proof **of** Claim. The proof of (i) and (ii) is straightforward and (iii) follows from (i) and (ii).

Let us proof (iv). Suppose $xR^L_{21}y$ and that M is not L- consistent. Then by (iii) there exists a formula A such that $A\in M_1$ and $\neg A\in M_2$. Then [11]A∈x and ∃B∈y such that ¬¬A⇒[22]¬B∈L, hence A⇒[22]¬B∈L. Then by the rule (N[11]) we get [11](A⇒[22]¬B)∈L and by axiom (K[11]) and (MP) we obtain that [11]A⇒[11][22]¬B∈L. But [11]A∈x, so [11][22]¬B∈x, Then by the axiom (tr): [11][22]¬B⇒[21]¬B and (MP) we get [21]¬B∈x and since $xR^L_{21}y$ we get ¬B∈y. Since B∈x we obtain a contradiction.

(v) Suppose $M_2\subseteq z\in W_L$. Suppose that $zR^L_{22}y$ does not hold. Then for some formula A we have: [22]B∈z and B∉y, so ¬B∈y. Since ¬¬[22]B⇒[22]¬¬B∈L, then by the definition of M_2 we get that $\neg[22]B\in M_2$, hence ¬[22]B∈z - a contradiction. The second part of (v) follows by the definition of R^L_{11}.

(vi) Suppose $xR^L_{21}y$. Then by (iv) M is an L-consistent set. Then there exists a maximal consistent set z such that M⊆z and by (v) we have $xR^L_{11}z$ and $zR^L_{22}y$. Now the proof of (iv)(⟶) follows directly from claim (vi). ∎

Theorem 3.6.

Let X⊆{ser, ref, sym, tr}. Then BAL+X=L((X)ARROW).

Proof. The consistency part of the theorem follows from lemma 2.3 and the completeness part can be obtained from lemma 3.5. as in the proof of theorem

3.3. ■

Corollary 3.7.

(i) BAL+ref+sym+tr=L((pretotal)ARROW),

(ii) BAL+ref+sym+tr=L((total)ARROW).

Proof. (i) is a direct consequence of theorem 3.6 and (ii) follows from corollary 2.10. ■

4. Filtration and finite model property

In this section we will show that BAL and some of its extensions posses finite model property and are decidable, by applying the filtration techniques. We adopt the Segerberg's definition of filtration, adapted for the language $\mathcal{L}$ of arrow logics (see [SEG 71]).

Let $\underline{W}=(W, R_{11}, R_{22}, R_{12}, R_{21})$ be an a.f. and $M=(\underline{W}, v)$ be a model over $\underline{W}$. Let Ψ be a finite set of formulas, closed under subformulas. For x,y∈W define:

$x\sim y$ iff $(\forall A\in\Psi)(x\Vdash_{v} A \text{ iff } y\Vdash_{v} A)$, $|x|=\{y\in W/x\sim y\}$,

$W'=\{|x|/x\in W\}$, for $p\in VAR$ $v'(p)=\{|x|/x\in v(p)\}$.

Let R'_{ij} i,j=1,2 be any binary relations in W' such that $\underline{W}'=(W', R'_{11}, R'_{22}, R'_{12}, R'_{21})$ be an a.f. We say that the model $M'=(W', v')$ is a filtration of the model M trough Ψ if following conditions are satisfied for any i,j=1,2 and x,y∈W:

$(FR_{ij}1)$ If $xR_{ij}y$ then $|x|R'_{ij}|y|$,

$(FR_{ij}2)$ If $|x|R|y|$ then $(\forall[ij]A\in\Psi)(x\Vdash_{v}[ij]A \rightarrow y\Vdash_{v} A)$.

The following lemma is a standard result in filtration theory.

Lemma 4.1.

([Seg 71])(i)/ Filtration lemma/ For any formula $A\in\Psi$ and x∈W the following is true: $x\Vdash_{v} A$ iff $|x|\Vdash_{v'} A$.

(ii) $CardW'\leq 2^n$, where $n=Card\Psi$.

Let L be an a.l. We say that L admits a filtration if for any frame $\underline{W}$ for L and a model $M=(\underline{W},v)$ over $\underline{W}$ and for any formula A there exist a finite set of formulas Ψ containing A and closed under subformulas and a filtration $M'=(\underline{W}',v')$ of M trough Ψ, such that $\underline{W}'$ is a frame for L.

Corollary 4.2.

(i) Let Σ be a class of arrow frames, let Σ_{fin} be the class of all finite arrow frames from Σ and let $L(\Sigma)$ admits a filtration. Then $L(\Sigma)=L(\Sigma_{fin})$.

(ii) If $L(\Sigma)$ is finitely axiomatizable then it is decidable.

Lemma 4.3.

Let $\underline{W}$ be an a.f., $M=(\underline{W}, v)$ be a model over $\underline{W}$ and $M'=(\underline{W}',v,)$ be a filtration of M trough Ψ. Then:

(i) If $\underline{W}$ is a serial a.f. then $\underline{W}'$is a serial a.f.,

(ii) If $\underline{W}$ is a reflexive a.f. then $\underline{W}'$ is a reflexive a.f.,

(iii) If $\underline{W}$ is a symmetric a.f. the $\underline{W}'$ is a symmetric a.f.,

(iv) If $\underline{W}$ is a total a.f. then $\underline{W}'$is a total a.f.

Proof. As an example we shall prove (iii). We have to show that $(\forall|x|\in W')(\exists|y|\in W')(|x|R'_{12}|y| \ \&\ |y|R'_{12}|x|)$. Suppose $|x|\in W'$. Then there exists y∈W such that $xR_{12}y$ & $yR_{12}x$. Then by the condition $(FR_{12}1)$ of the filtration we obtain $|x|R'_{12}|y|$ & $|y|R'_{12}|x|$. ■

Theorem 4.4.

The logic L(ARROW) admits a filtration.

Proof. Let A_0 be a formula and let Ψ be the smallest set of formulas containing A_0, closed under subformulas and satisfying the following condition

(*) If for some i,j=1,2 $[ij]A\in\Psi$ then for any ij=1,2 $[ij]A\in\Psi$.

It is easy to see that Ψ is finite and if n is the number of subformulas of A then $\mathrm{Card}\Psi\leq 2^{4n}$. Then define W' and v' as in the definition of filtration. We define the relations R'_{ij} in W' as follows:

$|x|R'_{ij}|y|$ iff $(\forall[ij]A\in\Psi)(\forall k\in\{1,2\})(x\Vdash_v [ik]A \leftrightarrow y\Vdash_v [jk]A)$.

First we will show that the frame $\underline{W}'$ is an a.f. The conditions (ρii) and (σij) follow directly from the definition of R'_{ij}. For the condition (τijk) suppose $|x|R'_{ij}|y|$ and $|y|R'_{jk}|z|$. To prove $|x|R'_{ik}|z|$ suppose $[ik]A\in\Psi$, $l\in\{1,2\}$ and for the direction ($\rightarrow$) suppose $x\Vdash_v [il]A$ and proceed to show that $z\Vdash_v [kl]A$. From $[ik]A\in\Psi$ we get $[ij]A, [jl]A\in\Psi$. Then $|x|R'_{ij}|y|$, $[ij]A\in\Psi$ and $x\Vdash_v [il]A$ imply $y\Vdash_v [jl]A$. This and $[jl]A\in\Psi$ and $|y|R'_{jk}|z|$ imply $z\Vdash_v [kl]A$.

The converse direction ($\leftarrow$) can be proved in a similar way.

It remains to show that the conditions of filtration ($FR_{ij}1$) and ($FR_{ij}2$) are satisfied.

For the condition ($FR_{ij}1$) suppose $xR_{ij}y$, $[ij]A\in\Psi$, $k\in\{1,2\}$ and for the direction ($\rightarrow$) suppose $x\Vdash_v [ik]A$, $yR_{jk}z$ and proceed to show that $z\Vdash_v A$. From $xR_{ij}y$ and $yR_{jk}z$ we get $xR_{ik}z$ and since $x\Vdash_v [ik]A$ we get $z\Vdash_v A$. For the direction ($\leftarrow$) suppose $y\Vdash_v [jk]A$, $xR_{ik}z$ and proceed to show that $z\Vdash_v A$. From $xR_{ij}y$ we get $yR_{ji}x$ and by $xR_{ik}z$ we get $yR_{jk}z$. From here and $y\Vdash_v [jk]A$ we obtain $z\Vdash_v A$. This ends the proof of ($FR_{ij}1$).

For the condition ($FR_{ij}2$) suppose $|x|R'_{ij}|y|$, $[ij]A\in\Psi$ and $x\Vdash_v [ij]A$. From here we obtain $y\Vdash_v [jj]A$ and since $yR_{jj}y$ we get $y\Vdash_v A$. This completes the proof of the theorem. ■

Corollary 4.5.

(i) BAL=L(ARROW)=L(ARROW_{fin})=L($((nor)\mathrm{ARROW})_{fin}$).

(ii) BAL is a decidable logic.

Proof. (i) The first two equalities follow from corollary 3.4 and theorem 4.4. The last equality follows from theorem 2.7.

(ii) is a consequence of corollary 4.2 and corollary 3.4. ■

Theorem 4.6.

Let X⊆{ser, ref, sym}. Then the logic L((X)ARROW) admits a filtration.

Proof. Use the same filtration as in theorem 4.4 and apply lemma 4.3. ■

Corollary 4.7.

Let X⊆{ser, ref, sym}. Then:

(i) B+X=L((X)ARROW)= $L(((X)ARROW)_{fin})$

(ii) B+X is a decidable logic.

Theorem 4.8.

The logic L((total)ARROW) admits a filtration.

Proof. Use the same filtration as in theorem 4.4 and apply lemma 4.3.■

Corollary 4.9.

(i) B+ref+sym+tr=L((pretotal)ARROW)= L((total)ARROW= $L(((total)ARROW)_{fin})$=$L(((pretotal)ARROW)_{fin})$

(ii) B+ref+sym+tr is a decidable logic.

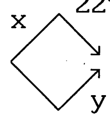
5. An extension of BAL with a modality for equivalent arrows

We have seen that the condition of a normality is not modally definable. This means that the language $\mathcal{L}$ is not strong enough to tell us the difference between normal and non normal a.f. In this section we shall show that there exists a natural extension of the language $\mathcal{L}$ in which normality become modally definable.

Let $\underline{W}$ be an a.f. and for $x,y \in W$ define

(≡) $x \equiv y$ iff $xR_{11}y$ & $xR_{22}y$.

Graphically $x \equiv y$:



The relation ≡ is called an equivalence of two arrows.

In terms of the relation ≡ the normality condition is equivalent to the following one:

(Nor') $(\forall xy \in W)(x \equiv y \longrightarrow x=y)$.

If we extend our language $\mathcal{L}$ with a new modality [≡], interpreted in a.f. with the relation ≡, then (Nor') is modally definable by the formula $p \Rightarrow [\equiv]p$.

Let $\mathcal{L}([\equiv])$ denote the extension of $\mathcal{L}$ with the operator [≡]. The general semantics of $\mathcal{L}([\equiv])$ is defined in the class of all relational structures /called also frames/ of the form $\underline{W}=(W, R_{11}, R_{22}, R_{12}, R_{21}, \equiv)$. The standard semantics of $\mathcal{L}([\equiv])$ is defined in the class of arrow frames with the relation ≡ defined by (≡).

We shall show that the condition (≡) is not modally definable. For that purpose we introduce the following nonstandard semantics of $\mathcal{L}([\equiv])$.

By a nonstandard ≡-arrow frame (≡-a.f.) we mean any system $\underline{W}=(W, R_{11}, R_{22}, R_{12}, R_{21}, \equiv)$ satisfying the following conditions for any $x,y,z \in W$ and $i,j,k \in \{1,2\}$:

(ρii), (σij) and (τijk),

($\equiv\rho$) $x \equiv x$,

($\equiv\sigma$) $x \equiv y \longrightarrow y \equiv x$,

($\equiv\tau$) $x \equiv y$ & $y \equiv z \longrightarrow x \equiv z$,

($\equiv \subseteq R_{ii}$) $x \equiv y \longrightarrow xR_{ii}y$

The class of all nonstandard ≡-arrow frames is denoted by Nonstandard-≡-ARROW.

If a nonstandard ≡-a.f. satisfies the condition

($R_{11} \cap R_{22} \subseteq \equiv$) $xR_{11}y$ & $xR_{22}y \longrightarrow x \equiv y$,

then it is called a standard ≡-a.f. It is easily seen that in any standard ≡-a.f. we have

$x\equiv y \longleftrightarrow xR_{11}y$ & $xR_{22}y$.

The class of all nonstandard and standard ≡-arrow frames are denoted respectively by Nonstandard-≡-ARROW and Standard-≡-ARROW.

The conditions from the definition of nonstandard ≡-a.f. are modally definable by the following formulas respectively:

(≡P) $[\equiv]A\Rightarrow A$,

(≡Σ) $A\vee[\equiv]\neg[\equiv]A$,

(≡T) $[\equiv]A\Rightarrow[\equiv][\equiv]A$,

($\subseteq_{ii}$) $[ii]A\Rightarrow[\equiv]A$, $i=1,2$.

We shall show that the condition ($R_{11}\cap R_{22}\subseteq\equiv$) is not modally definable. To this end we will first proof that

L(Standard-≡-ARROW)=L(Nonstandard-≡-ARROW),

using copying construction.

Lemma 5.1.

Let $\underline{W}=(W, R_{11}, R_{22}, R_{12}, R_{21}, \equiv)$ be a nonstandard ≡-a.f. Then there exists a standard ≡-a.f. $\underline{W}'=(W', R'_{11}, R'_{22}, R'_{12}, R'_{21}, \equiv')$ and a copying from $\underline{W}$ to $\underline{W}'$ and if $\underline{W}$ is a finite a.f. then $\underline{W}'$ is a finite a.f. too.

Proof. Use the same construction as in the lemma 2.6 with the following modification. Let $\equiv(x)=\{y\in W/x\equiv y\}$. Since ≡ is an equivalence relation then $\equiv(x)=\equiv(y)$ implies $x\equiv y$. The definitions of R'_{ij} and ≡' are the following:

$f(x)R'_{ij}g(y)$ iff $xR_{ij}y$ & $(f+i.\equiv(x) = g+j.\equiv(y))$

$f(x)\equiv'g(y)$ iff $x\equiv y$ & $f=g$.

The details that this will do are left to the reader. ■

Corollary 5.2.

L(Standard-≡-ARROW)=L(Nonstandard-≡-ARROW).

Now the axiomatization of L(Nonstandard-≡-ARROW) is easy.

Denote by [≡]BAL the following axiomatic system:

Axioms and rules for [≡]BAL

(I) All axioms and rules of BAL,

(II) The following new axioms:

(≡P) $[\equiv]A\Rightarrow A$,

(≡Σ) $A\vee[\equiv]\neg[\equiv]A$,

(≡T) $[\equiv]A\Rightarrow[\equiv][\equiv]A$,

(⊆ii) $[ii]A\Rightarrow[\equiv]A$, $i=1,2$.

Theorem 5.3.

[≡]BAL is sound and complete in the class Nonstandard-≡-ARROW.

Proof - by the canonical construction. ■

Corollary 5.4.

[≡]BAL=L(Nonstandard-≡-ARROW)= L(Standard-[≡]-ARROW).

Theorem 5.5.

(i) [≡]BAL= L(Nonstandard-≡-ARROW)= L((Nonstandard-≡-ARROW)$_{fin}$)

(ii) [≡]BAL is a decidable logic.

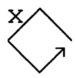
Proof. Apply the filtration technic with the following modification: the definition of the relations R'_{ij} is the same as in theorem 4.4., the definition of ≡' is the following

$|x|\equiv'|y|$ iff $(\forall[\equiv]A\in\Psi)(x\Vdash_v[\equiv]A \longleftrightarrow y\Vdash_v[\equiv]A)$ & $|x|R'_{11}|y|$ & $|x|R'_{22}|y|$.

The details that this definition of filtration will work is left to the reader. ■

6. Extensions of BAL with the propositional constant Loop

We say that an arrow x forms a loop if $xR_{12}x$. Graphically



Let $\underline{W}$ be an a.f. We let $Loop_{\underline{W}}=\{x\in W/xR_{12}x\}$.

Lemma 6.1.

In the language $\mathcal{L}$ Loop is not expressible in a sense that there is no a formula A in $\mathcal{L}$ such that for any a.f. $\underline{W}$, valuation v and $x\in W$: $x\Vdash_v A$ iff $x\in Loop_{\underline{W}}$.

Proof. Let W={a, b, c}, $R_{11}=R_{22}=\{(a,a), (b,b), (c,c)\}$, $R_{12}=R_{21}=\{(a,a), (b,c), (c,b)\}$. It is east to see that W with the relations R_{ij} is an a.f. Let v be a valuation in W such that for any $p\in VAR$ $v(p)=\emptyset$. Then by induction on the complexity of a formula one can see that for any formula A the set $v(A)=\{x\in W/x\Vdash_v A\}$ is ether W or $\emptyset$. Suppose now that there exists a formula A such that $x\Vdash_v A$ iff $x\in Loop$. Then $v(A)=\{a\}$ which contradicts the previous result. ■

Let $\mathcal{L}$(**Loop**) extend the language $\mathcal{L}$ by a new propositional constant **Loop** with the following standard semantics: for any a.f. $\underline{W}$, valuation v and $x\in W$: $x\Vdash_v$ **loop** iff $x\in Loop_{\underline{W}}$. **Loop** has also a nonstandard semantics which can be defined in the following way. By a nonstandard Loop arrow frame we mean any system $\underline{W}=(W, R_{11}, R_{22}, R_{12}, R_{21}, \delta)$ such that $(W, R_{11}, R_{22}, R_{12}, R_{21})$ is an a.f. and δ /sometimes denoted by $\delta_{\underline{W}}$/ is a subset of W. Then the interpretation of **Loop** in a nonstandard Loop a.f. $\underline{W}$ is: $x\Vdash_v$ **Loop** iff $x\in\delta_{\underline{W}}$. A nonstandard Loop a.f. $\underline{W}$ is called a standard one if the following two conditions are satisfied stating together that $Loop_{\underline{W}}=\delta_{\underline{W}}$:

(Loop 1) $(\forall x\in W)(x\in\delta_{\underline{W}} \longrightarrow x\in Loop_{\underline{W}})$,

(Loop 2) $(\forall x\in W)(x\in Loop_{\underline{W}} \longrightarrow x\in\delta_{\underline{W}})$.

The class of all nonstandard Loop arrow frames is denoted by NonstandardLoopARROW. Likewise, the class of all standard Loop a.f. is denoted by StandardLoopARROW. It can be easily shown that the condition (Loop 1) is modally definable in NonstandardLoopARROW by the following formula

(Loop) **Loop**$\Rightarrow$([12]A$\Rightarrow$A). If a nonstandard Loop a.f. satisfies (Loop 1) we call it a general Loop a.f. The class of general Loop arrow frames is denoted by GeneralLoopARROW. We shall show that condition (Loop 2) is not modally definable in GeneralLoopARROW. To this end we will use the copying construction, which for frames with δ contains an additional condition

(δ) For any $x\in W$ and $f\in I$: $x\in\delta$ iff $f(x)\in\delta'$.

The copying lemma for this version of copying is also true.

Lemma 6.2.

Let $\underline{W}$=(W, R_{11}, R_{22}, R_{12}, R_{21}, δ) be a general Loop a.f. Then there exists a standard Loop a.f. $\underline{W}$'=(W', R'_{11}, R'_{22}, R'_{12}, R'_{21}, δ') and a copying I from $\underline{W}$ to $\underline{W}$'and if $\underline{W}$ is a finite then $\underline{W}$' is a finite frame too.

Proof. The construction of I and W' is the same as in the proof of lemma 2.6. To define R'_{ij} we first define the function

$$\delta(x)=\begin{cases}0 & \text{if } x\in\delta\\ 1 & \text{if } x\notin\delta\end{cases}$$

where 0 and 1 are considered as zero and unit of the Boolean ring. Then:

$f(x)R'_{ij}g(y)$ iff $xR_{ij}y$ & $(f+i.\delta(x)=g+j.\delta(y))$.

$\delta'=\{x'/\exists x\in\delta\ \exists f\in I\ f(x)=x'\}$

The proof that this is a copying and that $\underline{W}$ is an a.f. is the same as in lemma 2.6. Let us show that $\underline{W}$' is a standard Loop a.f.

For the condition (Loop 1) suppose x'∈δ'. Then x'=f(x) for some x∈δ and f∈I. So we have $xR_{12}x$, δ(x)=0 and hence f+1.δ(x)=f+2.δ(x). This shows that $f(x)R'_{12}f(x)$, hence $x'R'_{12}x'$.

For the condition (Loop 2) suppose $x'R'_{12}x'$. Then for some x∈W and f∈I we have f(x)=x' and $f(x)R'_{12}f(x)$. Then $xR_{12}x$ and f+δ(x)=f, so δ(x)=0, which yields that x∈δ. Thus x'∈δ'. ■

Lemma 6.2 implies the following

Theorem 6.3.

L(LoopARROW)=L(GeneralLoopARROW).

Corollary 6.4.

Condition (Loop 2) is not modally definable.

Given the above results it is easy to obtain an axiomatization of the logic L(LoopARROW): we simply axiomatize L(GeneralLoopARROW) by adding to the axioms of BAL the axiom

(Loop) **Loop**⇒([12]A⇒A).

The obtained axiomatic system is denoted by LoopBAL. Using the canonical construction one can prove the following

Theorem 6.5.

LoopBAL is sound and complete in GeneralLoopARROW.

Corollary 6.6.

LoopBAL=L(GeneralLoopARROW)=L(LoopARROW).

The constant **Loop** makes possible to distinguish the logics L(LoopARROW) and L((nor)LoopARROW). Namely we have

Lemma 6.7.

Let φ=A∧**Loop**⇒[12](**Loop**⇒A). Then:

(i) φ∉L(LoopARROW),

(ii) φ∈L((nor)LoopARROW),

(iii) L(LoopARROW)≠L((nor)LoopARROW).

Proof - straightforward by the completeness theorem.■

The formula φ from lemma 6.7 modally defines in GeneralLoopARROW the following condition

(nor_0) $(\forall xy)(xR_{11}y\ \&\ x\in\delta\ \&\ y\in\delta \longrightarrow x=y)$

Let $\underline{W}$ be a general Loop a.f. We call $\underline{W}$ quasi-normal if it satisfies the condition Nor_0.

Lemma 6.8.
Let $\underline{W}$=(W, R_{11}, R_{22}, R_{12}, R_{21}, δ) be a quasi-normal general Loop a.f. Then there exists a normal Loop a.f. $\underline{W}'$=(W, R'_{11}, R'_{22}, R'_{12}, R'_{21}, δ') and a copying I from $\underline{W}$ to $\underline{W}'$ and if $\underline{W}$ is finite then $\underline{W}'$ is finite too.

Proof. The construction of I, W' and R'_{ij} is the same as in lemma 6.2 with the following modification of the function δ(x):

$$\delta(x)=\begin{cases} 0 & \text{if } x\in\delta \\ \{x\} & \text{if } x\notin\delta \end{cases}$$

The proof that $\underline{W}'$ is a standard Loop a.f. is the same as in lemma 6.2. Let us show the condition of normality. Suppose $f(x)R'_{11}g(y)$ and $f(x)R'_{22}g(y)$. Then we have $xR_{11}y$ & (f+δ(x)=g+δ(y)) and $xR_{22}y$ & (f=g). From this we get δ(x)=δ(y).

Case 1: δ(x)=∅. Then δ(y)=∅ and hence x,y∈$\delta_{\underline{W}}$. By $xR_{11}y$ and x,y∈$\delta_{\underline{W}}$ we get by (nor_0) x=y and by f=g we obtain f(x)=f(y).

Case 2: δ(x)≠∅. Then δ(y)≠∅ and hence {x}={y}, so x=y and consequently f(x)=g(y). This proves the condition of normality. ■

From lemma 6.8 we obtain the following

Theorem 6.9.
L((nor_0)GeneralLoopARROW)=L((nor)LoopARROW).

Let NorLoopBAL=LoopBAL+A∧**Loop**⇒[12](**Loop**⇒A). Using the canonical method we can easily prove the following

Theorem 6.10.
NorLoopBAL is sound and complete in the class (nor_0)GeneralLoopARROW.

Corollary 6.11.
NorLoopBAL=L((nor)LoopARROW).

Lemma 6.12.
The logics L(GeneralLoopARROW) and L((nor_0)GeneralLoopARROW) admit a filtration and are decidable.

Proof. For the L(generalLoopARROW) use the same filtration as for the logic L(ARROW) with the following definition for δ:
δ'={|x|/x∈δ}.

We have to show that the filtered frame satisfies the condition (Loop 1). Suppose |x|∈δ'. Then x∈δ and by (Loop 1) we get $xR_{12}x$.
Then by the properties of filtration we get $|x|R'_{12}|y|$.

For the logic L((nor_0)GeneralLoopARROW) we modify the definition of R'_{ij} as follows:

$$|x|R'_{ij}|y| \text{ iff } (\forall[ij]A\in\Psi)(\forall k\in\{1,2\})(x\Vdash_{V}[ik]A \longleftrightarrow x\Vdash_{V}[jk]A)\ \& \ (x,y\in\delta' \longrightarrow |x|=|y|).$$

The proof that this definition works is left to the reader. ■

Corollary 6.13.
The logics LoopBAL and NorLoopBAL possess finite model property and are

decidable.

The language $\mathscr{L}([\equiv],\mathbf{Loop})$ is an extension of the language $\mathscr{L}([\equiv])$ with the constant **Loop**. The standard semantics of this language is a combination of the standard semantics of $\mathscr{L}([\equiv])$ and $\mathscr{L}(\mathbf{Loop})$. This semantics is also modally undefinable. To axiomatize it we introduce a general semantics for $\mathscr{L}([\equiv],\mathbf{Loop})$ as follows.

A frame $\underline{W}=(W, R_{11}, R_{22}, R_{12}, R_{21}, \equiv, \delta)$ is called a general Loop-≡ arrow frame if $(W, R_{11}, R_{22}, R_{12}, R_{21})$ is an a.f. and ≡ and δ satisfy the conditions on the left side in the next table:

$(\rho\equiv)$	$x\equiv x$,	$(\equiv P)$	$[\equiv]A\Rightarrow A$,
$(\sigma\equiv)$	$x\equiv y\longrightarrow y\equiv x$,	$(\equiv\Sigma)$	$A\vee[\equiv]\neg[\equiv]A$,
$(\tau\equiv)$	$x\equiv y \& y\equiv z\longrightarrow x\equiv z$,	$(\equiv T)$	$[\equiv]A\Rightarrow[\equiv][\equiv]A$,
$(\equiv\subseteq R_{ii})$	$x\equiv y\longrightarrow xR_{ii}y$, $i=1,2$,	$(\subseteq ii)$	$[ii]A\Rightarrow[\equiv]A$, $i=1,2$,
(Loop 1)	$x\in\delta\longrightarrow xR_{12}x$,	(Loop)	$\mathbf{Loop}\Rightarrow([\equiv]A\Rightarrow A)$,
$(\equiv\delta)$	$x\equiv y$ & $x\in\delta\longrightarrow y\in\delta$,	(≡Loop)	$\mathbf{Loop}\Rightarrow[\equiv]\mathbf{Loop}$,
$(\equiv R_{11}\delta)$	$xR_{11}y$ & $x\in\delta$ & $y\in\delta\longrightarrow x\equiv y$	(≡11Loop)	$\mathbf{Loop}\wedge[\equiv]A\Rightarrow[11](\mathbf{Loop}\Rightarrow A)$.

If in addition $\underline{W}$ satisfies the condition $(R_{11}\cap R_{22}\subseteq\equiv)$ and (Loop2) it is called standard Loop-≡ arrow frame. The classes of all general Loop-≡ arrow frames and standard Loop-≡-arrow frames are denoted by GeneralLoop-≡-ARROW and StandardLoop-≡-ARROW respectively.

All conditions from the left side of the above table are modally definable by the corresponding formulas from the right side.

We axiomatize the logic $\mathscr{L}([\equiv],\mathbf{Loop})$ by adding all these formulas as axioms to the logic BAL The obtained system is /denoted by [≡]LoopBAL.

Theorem 6.14.

The logic [≡]LoopBAL is sound and complete in the class GeneralLoop-≡-ARROW.

Proof - by the canonical construction. ■

Lemma 6.15.

Let $\underline{W}=(W, R_{11}, R_{22}, R_{12}, R_{21}, \equiv, \delta)$ be a general Loop-≡ a.f. Then there exist a standard Loop-≡ a.f. $\underline{W}'=(W', R'_{11}, R'_{22}, R'_{12}, R'_{21}, \equiv', \delta')$ and a copying I from $\underline{W}$ to $\underline{W}'$.

Proof. The set W', I, δ' and R'_{ij} are defined as in lemma 6.8 with the following modification of the function δ(x):

$$\delta(x)=\begin{cases}0 & \text{if } x\in\delta\\ \equiv(x) & \text{if } x\notin\delta\end{cases}$$

The relation ≡' is defined as in lemma 5.1. The proof that this construction works is left to the reader. ■

Corollary 6.16.

[≡]LoopBAL=L(GeneralLoop-≡-ARROW)=L(StandardLoop-≡-ARROW).

7. Further perspectives

A. Extensions of BAL with additional connectives

Sections 5 and 6 can be considered as examples of possible extensions of the language $\mathscr{L}$ with operators having their standard semantics in terms of arrow frames. There are many possibilities of such extensions, depending of what kind of relations between arrows we want to describe in a modal setting. The main scheme is the following: to each n+1-ary relation $R(x_0,x_1,\dots,x_n)$ to introduce an n-place modal box operation $[R](A_1,\dots,A_n)$ with the following semantics, coming from the representation theory of Boolean algebras with operators ([J&T 51], see also [Vak 91b]):

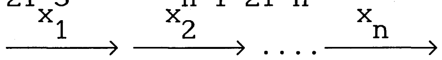
$x_0 \Vdash_v [R](A_1,\dots,A_n)$ iff

$(\forall x_1,\dots,x_n \in W)(R(x_0,x_1,\dots,x_n) \rightarrow x_1 \Vdash_v A_1$ or...or $x \Vdash_v A_n)$

The dual operator $\langle R\rangle(A_1,\dots,A_n)$ is defined by $\neg[R](\neg A_1,\dots,\neg A_n)$.

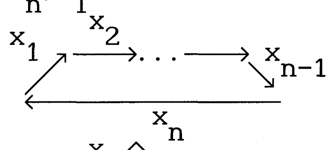
In the following we list some natural relations between arrows, which are candidates for a modal study:

$Path_n(x_1,\dots,x_n)$ iff $x_1R_{21}x_2$ & $x_2R_{21}x_3$ & &$x_{n-1}R_{21}x_n$, $n\geq 2$

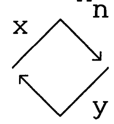


$Path_\infty(x_1,x_2,x_3,\dots)$ iff $(\forall n)Path(x_1,\dots,x_n)$

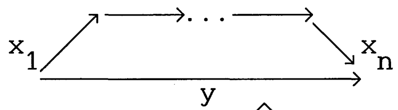
$Loop_n(x_1,x_2,\dots,x_n)$ iff $Path_{n+1}(x_1,\dots,x_n,x_1)$



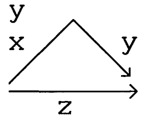
Converse: xSy iff $Loop_2(x,y)$



$Trapezium_n(x_1,\dots,x_n,y)$ iff $Path_n(x_1,\dots,x_n)$ & $x_1R_{11}y$ & $x_nR_{22}y$

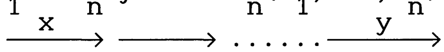


Triangle(x,y,z) iff $Trapezium_2(x,y,z)$

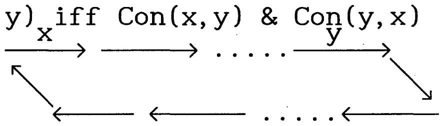


Connectedness: Con(x,y) iff

$\exists n\geq 2 \exists x_1\dots x_n$: $x=x_1$ & $x_n=y$ & $Path_n(x_1,\dots,x_n)$



Double side connectedness: Dcon(x,y) iff Con(x,y) & Con(y,x)



The relations $Path_n$, $Path_\infty$, $Loop_n$ can be used to define also semantics for suitable propositional constants:

$\mathbf{Path_n}$, $\mathbf{Path_\infty}$, $\mathbf{Loop_n}$, **Loop** as follows.

$x_1 \Vdash_v \mathbf{Path_n}$ iff $(\exists x_2,\dots x_n \in W)Path_n(x_1,x_2,\dots,x_n)$,

$x_1 \Vdash_v \mathbf{Path_\infty}$ iff $(\exists x_2,x_3,\dots)Path_\infty(x_1,x_2,x_3,\dots)$,

$x_1 \Vdash_v \mathbf{Loop_n}$ iff $(\exists x_2,\dots,x_n)Loop_n(x_1,x_2,\dots,x_n)$

$x \Vdash_v \mathbf{Loop}$ iff $\exists n$ $x \Vdash_v \mathbf{Loop_n}$.

These considerations motivate the following general problem: develop a modal theory /axiomatization, definability, (un)decidability/ of some extensions of BAL with modal operations corresponding to the above defined relations in arrow structures.

For example, the extension of BAL with the modal operations A•B=<Triangle>(A,B), A^{-1}=[Converse]A and the propositional constant Id=**Loop** is a natural generalization of the modal logic of binary relations ([Ben 89], [Ven 89], [Ven 91]). This logic has a closed connection with various versions of representable relativized relational algebras ([Kra 89], [Ma 82], [Ne 91]). It have been axiomatized by Arsov [Ar 94].

B. Arrow semantics of Lambek Calculus and its generalizations

Let A/B and A\B are "duals" of A•B with the following semantics:

$x \Vdash_v B/A$ iff $(\forall yz \in W)(\text{Triangle}(x,y,z)\ \&\ y \Vdash_v A \rightarrow z \Vdash_v B)$,

$y \Vdash_v A\backslash B$ iff $(\forall xz \in W)(\text{Triangle}(x,y,z)\ \&\ x \Vdash_v A \rightarrow z \Vdash_v B)$

The modal connectives A•B, A\B, and A/B can be considered as the operations in the Lambek Calculus. Mikulás [Mik 92] proves a completeness theorem for the Lambek Calculus with respect to a relational semantics of the above type over transitive normal arrow frames /this is an equivalent reformulation of Mikulás result in "arrow" terminology/.

C. Arrow logics and point logics over arrow systems

With each arrow structure S=(Ar, Po, **1**, **2**) we can associate the following two relational systems: the arrow frame (Ar, R_{11}, R_{22}, R_{12}, R_{21}) and the point frame (Po, ρ). The first system is used as a semantic base of the logic BAL and the later can be used as a semantic base of an ordinary modal language with a modal operator □. So each class Σ of arrow systems determines a class of arrow frames Ar(Σ) and a class Po(Σ) of point frames. A general question, which arises is the problem of comparative study of the corresponding logics L(Ar(Σ)) and L(Po(Σ)). A kind of a correspondence between first order properties of Po(Σ) and Ar(Σ) was shown in section 1.

CHAPTER 4.2

n-DIMENSIONAL ARROW STRUCTURES

Overview. The notion of n-dimensional arrow structure is introduced, which for n=2 coincides with the notion of directed multi graph. With each arrow structure of dimension n we associate two relational structures - the structure of n-point frame and the structure of n-arrow frame. It is proved a special characterization theorem from which it follows that the information in an arrow structure can be equally expressed in the structure of the corresponding n-arrow frame. It is proved that the first-order languages corresponding to n-dimensional arrow structure, the structure of n-point frame and n-arrow frame have equal expressive powers. Some modal languages connected with n- point frames and n-arrow frames are introduced and an effective translation of the point language into the arrow language is constructed.

Introduction

In this chapter we introduce the notion of arrow structure of dimension n. It is noted the connection between arrow structures of dimension n and single-valued attribute systems, studied in part two. The chapter consists of three sections.

In sec. 1 we prove that the relational structure, consisted of the set of arrows of some n-dimensional arrow structure, equipped with the incidence relations R_{ij} i,j≤n, contains in some sense the whole information of the arrow structure. This makes possible to introduce the notion of n-arrow frame as one-sorted equivalent of the notion of n-dimensional arrow structure. With the point part of an n-dimensional arrow structure we associate another relational system, called n-point frame, containing one n-place relation ρ, which holds for the sequence of points $A_1,\dots,A_n$ iff they are points on some arrow.

In sec. 2 we introduce three first order languages, connected with arrow structures - the language Ł(S) of the whole structure, the language Ł(W), connected with the n-arrow frame of the structure, and the language Ł(V), connected with the n-point frame of the structure. The main result here is a theorem stated that all these three languages have equal expressive power. As a consequence of this we obtain that the first-order theory of one n-place relation can be reduced to the first-order theory of n^2 special relations /the relations R_{ij}/.

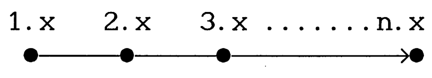
In sec. 3. we introduce two modal languages - an "arrow" language corresponding to the n-arrow frames, extended with the universal modality, and a "point" language, corresponding to the n-point frames, containing n-1-argument modal operations, interpreted with the n-place point relation ρ. The main result here is a translation of the "point" language into the "arrow" language, which translation preserves the corresponding semantical validity. There are several implications from this result. First it is shown that one can associate to arrow structures both "arrow logics", talking about arrows, and "point logics", talking about points. The translation of "point" language into the "arrow" language shows that the second language is at least expressible as the "point" language. In other words this answers the question "why Arrow Logics". Another implication of the result is that polyadic modalities can be reduced to monadic ones.

1. Arrow structures and arrow frames of dimension n

By an arrow structure of dimension n (n-arrow structure for short) we will understand any three-sorted algebraic system S=(Ar, Po, (n),.) where:

- Ar is a non-empty set whose elements are called arrows,
- Po is a non-empty set whose elements are called points,
- (n)={1,...,n} is the set of natural numbers from 1 to n.
- . is a total function from Ar×(n) to Po, called application. If x∈Ar and i∈(n) then i.x is called the i-th point of x, 1.x is also called the beginning of x and n.x is called the end of x, or the head of x.

Graphically:



- We assume that Ar∩Po=∅ and that the following axiom is satisfied (Ax) (∀A∈Po)(∃i∈(n))(∃x∈Ar)(i.x=A).

The meaning of (Ax) is that each point is a point in some arrow, so that there are no isolated points. It is possible to develop a more general theory without this axiom. We take it only because the formulations of some theorems become simpler.

Two-dimensional arrow structures are just directed multi-graphs without isolated points. Thus, in some sense n-arrow structures can be considered as a generalization of the notion of directed multi-graph.

Sometimes to denote that Ar and Po are from a given system S we will write Ar_S and Po_S.

An n-arrow structure S is called normal if it satisfies the following axiom of normality

(Nor) $(\forall x, y \in Ar_S)((\forall i \in (n))(i.x = i.y) \rightarrow x = y)$.

In two dimensional case normal arrow structures correspond to the notion of a directed graph.

There is a close connection between n-arrow structures and single-valued attribute systems, studied in part two. Namely, let S be an n-arrow structure and define an attribute system S' as follows. Put $Ob_{S'} = Ar_S$, $AT_{S'} = (n) = \{1, 2, \dots, n\}$, for $i \in (n)$ $VALi = \{A \in Po_S / (\exists x \in Ar_S) i.x = A\}$, for $x \in Ar_S$ and $i \in (n)$ $f_{S'}(x, i) = i.x$. Obviously S' is a total ($\forall x \in Ob\ \forall a \in AT\ f(x,a) \neq \emptyset$), single-valued attribute system. Conversely, let S' be a total single-valued attribute system with finite number of attributes $AT_{S'} = \{a_1, \dots, a_n\}$. Then S' determines an n-arrow structure S as follows. $Po_S = \{A / (\exists x \in Ob_{S'}, \exists a_i \in AT_{S'}) A \in f(x, a_i)\}$, $Ar_S = Ob_{S'}$, and for $x \in Ar_S$ and $i \in (n)$ define $i.x = f_{S'}(x, a_i)$. Then obviously S is an n- arrow structure.

The following example of normal n-arrow structure is very important. Let $\underline{V} = (V, \rho)$, $V \neq \emptyset$, be a relational system, such that $\rho \subseteq V^n$ be a non-empty n-ary relation in V satisfying the following condition:

$(\forall A \in V)(\exists i \in (n))(\exists A_1, \dots A_n \in V)(\rho(A_1, \dots, A_n)\ \&\ A = A_i)$.

Such system will be called n-point frame. We shall construct an n-arrow structure $S = S(\underline{V}) = (Ar(\underline{V}), Po(\underline{V}), (n), .)$ over $\underline{V}$ in the following way. Put $Ar(\underline{V}) = \rho$, $Po(\underline{V}) = V$ and for $i \in (n)$ and $(A_1, \dots, A_i, \dots, A_n) \in \rho$ define $i.(A_1, \dots, A_i, \dots, A_n) = A_i$. Then obviously the system $S(\underline{V})$ is a normal n-arrow structure, called relational n-arrow structure over the system $\underline{V}$.

Now we shall show that each n-arrow structure S determines a normal n-arrow structure in the following way. Let S be given and define in Po_S the following n-ary relation $\rho = \rho_S$: for $A_1, \dots, A_n \in Po_S$

$(A_1, \dots, A_n) \in \rho_S$ iff $(\exists x \in Ar_S)(1.x = A_1\ \& \dots \&\ n.x = A_n)$

The relational structure $V(S) = (Po_S, \rho_S)$ will be called n-point frame over S. Now V(S) determines a normal n-arrow structure S(V(S)) in a way described above. We shall show that S(V(S)) is a homomorphic image of S and if S is a normal n-arrow structure then S(V(S)) is isomorphic with S in the following sense of homomorphism and isomorphism.

Let S and S' be two n-arrow structures. A pair (f,g) of mappings $f: Ar_S \rightarrow Ar_{S'}$ and $g: Po_S \rightarrow Po_{S'}$ is called a homomorphism from S into S' if for any $x \in Ar_S$ and $i \in (n)$ we have $g(i.x)) = i.f(x)$. If f and g are one-one mappings then (f,g) is called an isomorphism from S onto S'.

Theorem 1.1.

Let S be an n-arrow structure, $V(S) = (Po_S, \rho_S)$ be the n-point frame over S and $S' = S(V(S))$ be the n-arrow structure over V(S). Then S' is a homomorphic image of S and if S is a normal n-arrow structure then S' is an isomorphic image of S.

Proof. Let S be given. For $x \in Ar_S$ and $A \in Po$ we define $f(x)=(1.x,\ldots,n.x)$ and $g(A)=A$.

Then (f,g) is the required homomorphism from S onto S'. Indeed $f(x) \in \rho_S = Po_S$, so f is correctly defined. The following equalities show that (f,g) is a homomorphism:

$i.f(x)=i.(1.x,\ldots,n.x)=i.x=g(i.x)$.

Suppose now that S is a normal n-arrow structure and let $f(x)=f(y)$. Then $(1.x,\ldots,n.x)=(1.y,\ldots,n.y)$, which by the condition of normality implies $x=y$. So f is an injective function. To show that f is a function "onto" let $(A_1,\ldots,A_n) \in \rho_S$. Then by the definition of ρ_S there exists a $x \in Ar_S$ such that $(A_1,\ldots,A_n)=(1.x,\ldots,n.x)=f(x)$. This shows that in this case (f,g) is an isomorphism from S onto S'.■

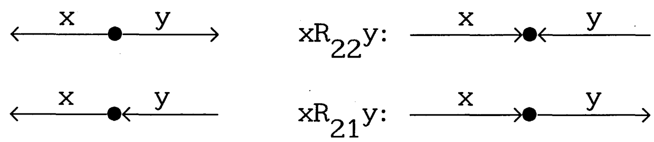
Let S be an n-arrow structure. The following binary relations will play an important role in the theory of n-arrow structure For $x,y \in Ar_S$ and $i,j \in (n)$ we define a relation $R_{ij}=R^S_{ij}$ as follows:

$xR_{ij}y$ iff $i.x=j.y$.

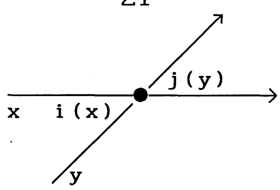
The relation R_{ij} between x and y says that the i-th point of x coincides with the j-th point of y.

Graphically:

for n=2: $xR_{11}y$: $xR_{22}y$: $xR_{12}y$: $xR_{21}y$:



for arbitrary n: $xR_{ij}y$:



Note. For n=2. the relation R_{21} is studied by Kuhn [Ku 89] and called "domino relation".

Lemma 1.2.

Let S be an n-arrow structure Then:

(i) The relations R_{ij} satisfy the following conditions for any $x,y,z \in Ar_S$ and $i,j,k \le n$:

(ρii) $xR_{ii}x$,

(σij) $xR_{ij}y \rightarrow yR_{ji}x$,

(τijk) $xR_{ij}y \;\&\; yR_{jk}z \rightarrow xR_{ik}z$.

(ii) S is normal n-arrow structure iff the following condition is satisfied for any $x,y \in Ar_S$:

(Nor') $xR_{11}y \;\&\; xR_{22}y \;\&\ldots\&\; xR_{nn}y \rightarrow x=y$.

Proof. By an easy verification.■

Let $W \neq \emptyset$ be a set, $n \ge 2$ be a fixed natural number and $\{R_{ij} \subseteq W^2 / i,j \le n\}$ be a set of n^2 binary relations in W. Then the relational system $\underline{W}=(W, \{R_{ij}/i,j \le n\})$ is called an n-dimensional arrow frame (n-arrow frame for short) if the conditions (ρii), (σij) and (τijk) from lemma 1.2 are satisfied for any

$i,j,k\leq n$ and $x,y,z\in W$. If $\underline{W}_S$ satisfies (Nor') then it is called a normal n-arrow frame. If $W=Ar_S$ and $R_{ij}=R^S_{ij}$ for some n-arrow structure S then $\underline{W}=\underline{W}(S)$ is called a standard n-arrow frame (over S).

The aim of this section is to show that each (normal) n-arrow frame is a standard n-arrow frame over some (normal) n-arrow structure S. Namely we have

Theorem 1.3.

Let $\underline{W}=(W, \{R_{ij}/i,j\leq n\})$ be an n-arrow frame Then there exists an n-arrow structure $S=S(\underline{W})$ such that $Ar_S=W$ and for any $i,j\leq n$ $R^S_{ij}=R_{ij}$ and if $\underline{W}$ is normal n-arrow frame, then S is normal n-arrow structure too.

Proof. Let $\underline{W}$ be given. We have to construct $S=S(\underline{W})$. We put $Ar_S=Ar(\underline{W})=W$. To construct Po_S we shall introduce the notion of a generalized point in $\underline{W}$.

Let $\alpha_1,\dots,\alpha_n$ be subsets of W. The n-tuple $(\alpha_1,\dots,\alpha_n)$ will be called a generalized point in $\underline{W}$ if the following conditions are satisfied for any $x,y\in W$ and $i,j\leq n$:

(1) If $x\in\alpha_i$ and $y\in\alpha_j$ then $xR_{ij}y$,

(2) If $xR_{ij}y$ and $x\in\alpha_i$ then $y\in\alpha_j$,

(3) $\alpha_1\cup\dots\cup\alpha_n\neq\emptyset$

The set of all generalized points in $\underline{W}$ will be denoted by $Po(\underline{W})$. We put $Po_S=Po(\underline{W})$.

It remains to define the application function, which here will be denoted by $\circ$. For that purpose we need the following notation: let R be any binary relation in W and $x\in W$, then we put $R(x)=\{y\in W/xRy\}$.

For any $i\in(n)$ and $x\in W$ define:

$i\circ x=(R_{i1}(x),\dots,R_{in}(x))$.

Now we put $S(\underline{W})=(Ar(\underline{W}), Po(\underline{W}), (n),\circ)$.

The proof that $S(\underline{W})$ is an n-arrow structure directly follows from the next lemma.

Lemma 1.4.

(i) For any $i\in(n)$ and $x\in W$ $i.x$ is a generalized point in $\underline{W}$.

(ii) $xR_{ij}y$ iff $i.x=j.y$.

(iii) If $A=(\alpha_1,\dots,\alpha_n)$ is a generalized point in $\underline{W}$ then there exists $x\in W$ and $i\in(n)$ such that $i\circ x=A$.

Proof. The proof of this technical lemma is long but straightforward and consists of many applications of the axioms (ρii), (σij) and (τijk). Let for instance prove (iii). Suppose $A=(\alpha_1,\dots,\alpha_n)$, by condition (3) of the definition of a generalized point $\alpha_1\cup\dots\cup\alpha_n\neq\emptyset$, so there exist $i\in(n)$ and $x\in W$ such that $x\in\alpha_i$. We shall show that $i.x=A$, i.e. that for any $j\in(n)$ $R_{ij}(x)=\alpha_j$. Let $y\in R_{ij}(x)$, then $xR_{ij}y$. From $xR_{ij}y$ and $x\in\alpha_i$, by the condition (2) of the definition of a generalized point we get $y\in\alpha_j$, hence $R_{ij}(x)\subseteq\alpha_j$. For the converse inclusion suppose that $y\in\alpha_j$. Then by (1) and $x\in\alpha_i$ we get $xR_{ij}y$, so $y\in R_{ij}(x)$ and hence $\alpha_j\subseteq R_{ij}(x)$. Thus $R_{ij}(x)=\alpha_j$.■

Let us turn to the proof of the theorem 1.3. Suppose that $\underline{W}$ is a normal n-arrow frame We shall show that $S(\underline{W})$ is a normal n-arrow structure too. Let $x,y\in Ar(W)=W$ and suppose that for any $i\in(n)$ we have $i.x=i.y$. Then for any $i\in(n)$

we get $R_{ii}(x)=R_{ii}(y)$. By (ρii) we have $yR_{ii}y$ so $y\in R_{ii}(y)$, hence $y\in R_{ii}(x)$ and consequently $xR_{ii}y$ for any $i\in(n)$. Then by (Nor') we get x=y. This ends the proof of the theorem.■

The n-arrow structure $S(\underline{W})$, constructed in theorem 1.3 will be called a canonical n-arrow structure over $\underline{W}$.

Note. A theorem similar to theorem 1.3. characterizing domino relations is proved by Kuhn in [Ku 89].

Theorem 1.5.

Let S be an n-arrow structure, $\underline{W}=\underline{W}(S)$ be the standard n-arrow frame over S and $S'=S(\underline{W}(S))$ be the canonical n-arrow structure over $\underline{W}(S)$. Then $S(\underline{W}(S))$ is isomorphic with S.

Proof. Let for $i\in(n)$ and $A\in Po_S$ define $i(A)=\{x\in Ar_S/i.x=A\}$ and $g(A)=(1(A),\dots,n(A))$. Let id be the identity function in Ar_S. Then the pair (id, g) is the required isomorphism. This follows from the next lemma.

Lemma 1.6.

(i) g(A) is a generalized point in $\underline{W}(S)$,

(ii) g is a one-one function from Po_S into $Po(\underline{W}(S))$,

(iii) for any $x\in Ar_S$ and $i\in(n)$: $g(i.x)=i\circ x=i\circ id(x)$.

Proof. (i) The proof that g(A) is a generalized point in $\underline{W}(S)$ is straightforward.

(ii) Supposing g(A)=g(B), we shall show that A=B, which proves that g is an injective function. From g(A)=g(B) we obtain that for any $i\le n$ we have i(A)=i(B). By the axiom (Ax) for the point A there exists $i\le n$ such that i.x=A. From here we get that $x\in i(A)$, and since i(A)=i(B) we obtain $x\in i(B)$, so B=i.x. Thus we have A=B. To prove that g is a function "onto" suppose that $(\alpha_1,\dots,\alpha_n)$ is a generalized point in $\underline{W}$. We shall show that there exists a point $A\in Po_S$ such that $g(A)=(\alpha_1,\dots,\alpha_n)$.

By lemma 1.4(iii) there exist $x\in Ar_S$ and $i\le n$ such that $i\circ x=(R_{i1}(x),\dots,R_{in}(x))=(\alpha_1,\dots,\alpha_n)$. Let i.x=A. We shall show that for any $j\le n$ $j(A)=R_{ij}(x)$, which will prove the equality $g(A)=i\circ x$. Suppose that $y\in j(A)$. Then j.y=A, so i.x=j.y, $xR_{ij}y$ and consequently $y\in R_{ij}(x)$. Hence $j(A)\subseteq R_{ij}(x)$. For the converse inclusion suppose $y\in R_{ij}(x)$, so $xR_{ij}y$, A=i.x=j.y, and finally $y\in j(A)$. Hence $R_{ij}(x)\subseteq j(A)$. Thus $j(A)=R_{ij}(x)$. From here we get $g(A)=i\circ x=(\alpha_1,\dots,\alpha_n)$. This proof contains also the proof of (iii), because we have shown that $g(i.x)=i\circ x=i\circ id(x)$.■

Theorem 1.5 shows that the whole information of an n-arrow structure is contained in the standard n-arrow frame $\underline{W}(S)$ over S and can be expressed in terms of arrows and the relations R_{ij}. An example of such a correspondence is the conditions (Nor) and (Nor'). In the next section we shall examine in more details this correspondence.

2. First-order languages associated with arrow structures

Let $S=(Ar_S, Po_S, (n),.)$ be an n-arrow structure, $V(S)=(Po_S, \rho_S)$ be the n-point frame over S and $W(S)=(Ar_S, \{R^S_{ij}/i,j\le n\})$ be the n- arrow frame over S. We will construct three first-order languages Ł(S), Ł(V) and Ł(W) corresponding to the systems S, V(S) and W(S) respectively. Since V(S) and W(S) are definable subsystems of S, then the corresponding languages Ł(V) and Ł(W) are in some sense definable sublanguages in the language Ł(S) for S. The main result of this section will be the fact that Ł(S), Ł(V) and Ł(W) have equal expressive power. In particular this has an implication that the first order theory of one n-place relation can be reduced to the first order theory of n^2 special binary relations - the relations R_{ij}.

The language Ł(S) for n dimensional arrow structures.

The language Ł(S) is a three-sorted first-order language. It contains symbols for point variables, arrow variables, the set of integer constants (n)={1,...,n} and integer variables, ranging over (n). The formal definition is the following.

The alphabet of Ł(S):

- VARPo=$\{A^1, A^2, A^3, \ldots\}$ - a denumerable sequence of different point variables,
- VARAr=$\{x^1, x^2, x^3, \ldots\}$ - a denumerable sequence of different arrow variables,
- (n)={1,2,...,n} - integer constants,
- VAR(n)=$\{i^1, i^2, i^3, \ldots\}$ - a denumerable sequence of different integer variables /ranging in (n)/,
- . - application symbol,
- = - equality for points
- ¬, ∧, ∃, - logical symbols,
- (,) - parentheses.

Sometimes we will consider also equality symbol for arrows denoted also by =.

Terms:

A^p, $i^p.x^q$, $c.x^q$ for $c \le n$.

Atomic formulas:

(i) If A and B are terms then A=B is an atomic formula,

(ii) $x^p = x^q$ is an atomic formula (if the language contains the symbol of equality = for arrows).

Formulas:

The set FOR(Ł(S)) of all formulas of Ł(S) is build in the usual way from the set of atomic formulas by using the logical symbols and quantifiers over VARPo and VARAr / quantification over integer variables will be defined/.

Abbreviations: the other logical connectives - ∨ ⇒, ⇔, the constants ⊥, ⊤ and the quantifier ∀ are introduced in the standard way. The replacement of all occurrences of an integer variable i^p by a constant $c \le n$ in a formula α will be denoted by $\alpha[i^p/c]$. The quantification over integer variables is defined as follows:

$(\forall i^p)\alpha =_{def} \alpha[i^p/1] \wedge \alpha[i^p/2] \wedge \ldots \wedge \alpha[i^p/n]$,

$(\exists i^p)\alpha =_{def} \neg(\forall i^p)\neg\alpha = \alpha[i^p/1] \vee \alpha[i^p/2] \vee \ldots \vee \alpha[i^p/n]$

We introduce also the following abbreviations:

$x^p R_{ij} x^q =_{def} i.x^p = j.x^q$ where $i, j \in VAR(n) \cup (n)$,

$\rho(A_1, \ldots, A_n) = (\exists x^p)(1.x^p = A_1 \wedge 2.x^p = A_2 \wedge \ldots \wedge n.x^p = A_n)$, where A_1, A_2, ...,A_n are terms.

As an example of a formula in Ł(S) we will write the axiom

(Ax) $(\forall A^p)(\exists i^q)(\exists x^r)(A^p = i^q . x^r)$

Semantics for Ł(S)

The standard semantics for the language Ł(S) will be in n- dimensional arrow structures. This is a standard definition but we will formulate it in order to fix the notations we will use.

Let S be an n-arrow structure and let v be a function, called valuation, which assigns to different kinds of variables from the language corresponding objects from S, namely $v(A^p)\in Po_S$, $v(x^p)\in Ar_S$ and $v(i^p)\le n$. Let v be a valuation, x be any variable and a be an object from S of the same type as the type of the variable x /for instance if x is the point variable A^p then $a\in Po_S$/. Then we define a new valuation v^x_a such that v(x)=a and for any variable y different from x we have $v^x_a(y)=v(y)$. The notation v^{xy}_{ab} will mean $((v^x_a)^y_b$. If S is an n-arrow structure and v is a valuation in S then the pair M=(S, v) will be called a model.

The interpretation of a term at a valuation v is the standard one: $v(i^p.x^q)=v(i^p).v(x^q)$, for $c\le n$ $v(c.x^q)=c.v(x^q)$.

The interpretations of the formulas of Ł(S) in a model M=(S, v) will be given inductively. We shall use the standard notation $(S, v)\models\alpha$ to be read "α is true in the model (S, v)" or "α is true in S at the valuation v". For the atomic formulas we have:

$(S, v)\models x^p=x^q$ iff $v(x^p)=v(x^q)$, (if the language contains the symbol of equality = for arrows).

$(S, v)\models A=B$ iff $v(A)=v(B)$ where A and B are terms,

$(S, v)\models \neg\alpha$ iff $(S, v)\not\models \alpha$,

$(S, v)\models \alpha\wedge\beta$ iff $(S, v)\models \alpha$ and $(S, v)\models \beta$,

$(S, v)\models (\exists A^p)\alpha$ iff $(\exists B\in Po_S)(\ (S, v^{A^p}_B)\models \alpha)$,

$(S, v)\models (\exists x^p)\alpha$ iff $(\exists a\in Ar_S)((S, v^{x^p}_a)\models\alpha)$.

A formula α is true in an n-arrow structure S, in symbols $S\models\alpha$, if for each valuation v we have $(S, v)\models\alpha$. The truth of a closed formula in a model (S, v) do not depend on v, so if α is a closed formula, then $S\models\alpha$ if for some v we have $(S, v)\models\alpha$.

The language Ł(V) of point frames

The alphabet of Ł(V) contains the set of point variables VARPo, the logical symbols, equality = and one n-place predicate symbol ρ. The atomic formulas are of the form

$A^p=A^q$, $\rho(A^{q}{}_1,\ldots,A^{q}{}_n)$.

The set of all formulas FOR(Ł(V)) of Ł(V) is build from atomic formulas in a standard way. Obviously Ł(V) can be considered as a part of Ł(S), namely the part, talking about points in terms of = and the relation ρ /definable in Ł(S)/.

The standard semantics for Ł(V) is in n-point frames V(S) over n-arrow structures S and can be formulated as for the language Ł(S). The relation ρ is interpreted in V(S) by the relation ρ_S.

The language Ł(W) of n-dimensional arrow frames

The alphabet of Ł(W) contains the set VARAr of arrow variables, and for each $i,j \leq n$ a two place predicate symbol R_{ij}. If the language Ł(S) contains equality $\overset{\rightarrow}{=}$ for arrows we will assume that the language Ł(W) contains also equality =. Atomic formulas are $x^p R_{ij} x^q$, $x^p = x^q$ (in the presence of equality) and the set FOR(Ł(W)) is build from the atomic ones in the standard way.

We shall define a translation of the closed formulas of Ł(S) into the closed formulas of Ł(W). But since the definition will go inductively for arbitrary formulas, for the translation of open formulas of Ł(S) we need more rich version Ł'(W) of Ł(W), such that the set of closed formulas of Ł'(W) will be the same as the set of closed formulas of Ł(W).

The definition of Ł'(W) is the following. It contains the sets VARAr, (n), VAR(n), and instead of the predicates R_{ij} from Ł(W), we have now one four-place predicate symbol R with two places for integer variables or integer constants and two places for arrow variables. If we assume that Ł(W) contains equality = the same assumption we do for Ł'(W). The atomic formulas of Ł'(W) are the following:

$x^p R_{ij} x^q$, $i,j \in VAR(n) \cup (n)$ and $x^p = x^q$ if in the language we have equality =.

When $i,j \leq n$ are fixed we will consider R_{ij} as a two-place predicate for arrows and it will be identified with the corresponding predicate R_{ij} from the language Ł(W).

The set of all formulas FOR(Ł'(W)) is defined in the usual way from the set of atomic formulas by means of logical connectives and quantifiers over arrow variables. Quantification over integer variables can be defined as in the language Ł(S). Now it is obvious that the sets of closed formulas of the languages Ł(W) and Ł'(W) coincide.

The standard interpretation of Ł'(W) and Ł(W) is in n-arrow frames over n-arrow structures and can be formulated in a way similar to that for the language Ł(S).

A translation τ from Ł(S) into Ł'(W) and Ł(W)

We define a translation τ from Ł(S) to Ł'(W) by induction on the complexity of the formulas in Ł(S). For closed formulas in Ł(S) τ will be a translation of Ł(S) into Ł(W). First we devide the individual variables of Ł'(W) into two series $x^1, x^2, \ldots$ and $y^1, y^2, \ldots$ and the same we do for the integer variables: $i^1, i^2, \ldots$ and $j^1, j^2, \ldots$. For closed formulas in Ł(S) τ will be a translation of Ł(S) into Ł(W).

1. $\tau(A^p = A^q) =_{def} x^p R_{i^p i^q} x^q$,
2. $\tau(A^p = i^q . x^r) =_{def} x^p R_{i^p j^q} y^r$,
3. $\tau(i^q . x^r = A^p) =_{def} y^r R_{j^q i^p} x^p$,
4. $\tau(A^p = j . x^r) =_{def} x^p R_{i^p j} y^r$, $j \in (n)$, 5. $\tau(j . x^r = A^p) =_{def} y^r R_{j i^p} x^p$, $j \in (n)$
5. $\tau(i^p . x^q = i^r . x^s) =_{def} y^q R_{j^p j^r} y^s$,
6. $\tau(j . x^q = i^r . x^s) =_{def} y^q R_{j j^r} y^s$, $j \in (n)$,
7. $\tau(i^p . x^q = k . x^s) =_{def} y^q R_{j^p k} y^s$, $k \in (n)$,
8. $\tau(j . x^q = k . x^s) =_{def} y^q R_{jk} y^s$, $j,k \in (n)$,
9. $\tau(x^p = x^q) =_{def} y^p = y^q$, (in the presence of equality in Ł(S))

10. $\tau(\neg\alpha) =_{def} \neg\tau(\alpha)$,

11. $\tau(\alpha\wedge\beta) =_{def} \tau(\alpha)\wedge\tau(\beta)$,

12. $\tau((\exists x^p)\alpha) =_{def} (\exists y^p)\tau(\alpha)$,

13. $\tau((\exists A^p)\alpha) =_{def} (\exists i^p)(\exists x^p)\tau(\alpha)$,

14. $\tau((\exists i^p)\alpha) =_{def} (\exists j^p)\tau(\alpha)$.

Let S be an n-arrow structure and $W(S)=(Ar_S, \{R_{ij}/i,j\leq n\})$ be the n-arrow frame over S. Let v be a valuation of the variables of Ł(S) in S and w be a valuation of the variables of Ł'(W) in W(S). We say that v and w are connected if the following conditions are satisfied:

(1) $v(A^p)=w(i^p).w(x^p)$,

(2) $v(x^p)=w(y^p)$,

(3) $v(i^p)=w(j^p)$.

If (S, v) is a model for Ł(S) in S and (W(S), w) is a model of Ł'(W) in W(S) we say that (S, v) and (W(S), w) are connected models if v and w are connected valuations.

Lemma 2.1.

(i) if w is a valuation of Ł'(W) in W(S) then there exists a valuation v of Ł(S) in S such that v and w are connected.

(ii) If v and w are connected valuations then for any $c\leq n$ and $a\in Ar_S$ we have the following:

(1) the valuations $v^{A^p}_{c.a}$ and $w^{i^p x^p}_{c\ a}$ are connected,

(2) the valuations $v^{i^p}_{c}$ and $w^{j^p}_{c}$ are connected,

(3) the valuations $v^{x^p}_{a}$ and $w^{y^p}_{a}$ are connectrd.

Proof (i). Define v by the equations (1)-(3). The proof of (ii) follows dyrectly from the definitions of connected valuations.∎

Lemma 2.2.

Let S be an n-arrow structure and W(S) be the n-arrow frame over S. Then for any formula $\alpha\in$FORŁ(S) and for any two connected models (S, v) and (W(S), w) the following equivalence is true:

$(S, v)\models\alpha$ iff $(W(S), w)\models\tau(\alpha)$.

Proof. The proof is by induction on the complexity of α.

$\alpha=(A^p=A^q)$. Then $(S,v)\models\alpha$ iff $(S, v)\models A^p=A^q$ iff $v(A^p)=v(A^q)$ iff $w(i^p).w(x^p)=w(i^q).w(x^q)$ iff $w(x^p)R_{w(i^p)w(i^q)}w(x^q)$ iff $(W(S), w)\models x^p R_{i^p i^q} x^q$ iff $(W(S),w)\models\tau(A^p=A^q)$ iff $(W(S), w)\models\tau(\alpha)$.

In the cases when α has another form of atomic formula the proof goes in the same manner.

Now suppose as induction hypothesis that the assertion is true for the formulas β and γ. The proof for the cases $\alpha=\neg\beta, \alpha=\beta\wedge\gamma$, $\alpha=(\exists x^p)\beta$ and $\alpha=(\exists i^p)\beta$ is straightforward. The remaining case is $\alpha=(\exists A^p)\beta$. We will proceed for the two directions separately.

($\longrightarrow$) Suppose $(S, v)\models(\exists A^p)\beta$. Then there exists $B\in Po_S$ such that $(S, v^{A^p}_B)\models\beta$. Then by axiom (Ax) there exists $c\leq n$ and $a\in Ar_S$ such that B=c.a. By lemma 2.1.(ii) the valuations $v^{A^p}_B$ and $w^{i^p x^p}_{c\ a}$ are connected and then by the induction hypothesis we have $(W(S), w^{i^p x^p}_{c\ a})\models\tau(\beta)$. This shows that (W(S),

$w)\models(\exists i^p)(\exists x^p)\tau(\beta)$, so $(W(S), w)\models \tau((\exists A^p)\beta)$.

($\leftarrow$) Let $(W(S), w)\models \tau((\exists A^p)\beta)$. Then $(W(S), w)\models (\exists i^p)(\exists x^p)\tau(\beta)$, so for some $c\leq n$ and $a\in Ar_S$ $(W(S), w^{i^p x^p}_{c\ a})\models\tau(\beta)$. Let $B=c.a$. By lemma 2.1.(ii) the valuations $v^{A^p}_B$ and $w^{i^p x^p}_{c\ a}$ are connected. Then by the induction hypothesis we have $(W(S), v^{A^p}_B)\models\beta$. This shows that $(W(S), v)\models (\exists A^p)\beta$, which completes the proof of the lemma.■

Theorem 2.3.

Let S be an arbitrary n-arrow structure and W(S) be the n-arrow frame over S. Then for any closed formula α of Ł(S):

$S\models\alpha$ iff $W(S)\models\tau(\alpha)$.

Proof. Let w be an arbitrary valuation of Ł'(W) in W(S). Then by lemma 2.1.(i) there exists a valuation v of Ł(S) in S such that the valuations v and w are connected. Applying lemma 2.2 we have that $(S, v)\models\alpha$ iff $(W(S), w)\models\tau(\alpha)$. But α and $\tau(\alpha)$ are closed formulas, so we have $S\models\alpha$ iff $W(S)\models\tau(\alpha)$.■

A translation μ of Ł(W) into Ł(V)

We define a translation μ by induction on the complexity of formulas in Ł(W). For that purpose we arrange the point variables in Ł(V) in the following way: $A^1_1, A^1_2, \ldots, A^1_n, \ldots, A^p_1, A^p_2, \ldots, A^p_n, \ldots$

1. $\mu(x^p R_{ij} x^q) =_{def} A^p_i = A^q_j$, $i, j\leq n$,
2. $\mu(x^p = x^q) =_{def} A^p_1 = A^q_1 \wedge \ldots \wedge A^p_n = A^q_n$ /when Ł(W) has =/.
3. $\mu(\neg\alpha) =_{def} \neg\mu(\alpha)$,
4. $\mu(\alpha\wedge\beta) =_{def} \mu(\alpha)\wedge\mu(\beta)$,
5. $\mu((\exists x^p)\alpha) =_{def} (\exists A^p_1)\ldots(\exists A^p_n)(\rho(A^p_1, \ldots, A^p_n)\wedge\mu(\alpha))$.

Let S be an n-arrow structure and $V(S)=(Po_S, \rho_S)$ and $W(S)=(Ar_S, \{R_{ij}/i, j\leq n\})$ be the n-arrow frame over S. Let v be a valuation of the point variables of Ł(V) in V(S) and w be a valuation of arrow variables of Ł(W) in W(S). We will say that v and w are connected /in a new sense/ if for any $i\leq n$ we have $v(A^p_i)=i.w(x^p)$.

Lemma 2.4.

(i) If w is a valuation of Ł(W) in W(S) then there exists a valuation of Ł(V) in V(S) such that v and w are connected.

(ii) if v and w are connected and for $a\in Ar_S$ we have $B_1=1.a, \ldots, B_n=n.a$, then the valuations $v^{A^p_1\ldots A^p_n}_{B_1\ldots B_n}$ and $w^{x^p}_a$ are connected.

Proof - similar to the proof of lemma 2.1.■

Lemma 2.5.

Let S be an n-arrow structure, W(S) be the n-arrow frame over S and V(S) be the n-point frame over S. Then for any formula α and any two connected models (W(S), w) and (V(S), v) we have the following equivalence:

$(W(S), w)\models\alpha$ iff $(V(S), v)\models\mu(\alpha)$.

Proof. We proceed by induction of the complexity of α.

1. $\alpha=x^p R_{ij} x^q$. Then $(W(S), w)\models x^p R_{ij} x^q$ iff $i.w(x^p)=j.w(x^q)$ iff $v(A^p_i)=v(A^q_j)$

iff (V(S), v)$\models$ $A_i^p=A_j^q$ iff (V(S), v)$\models\mu(x^pR_{ij}x^q)$.

2. $\alpha=(x^p=x^q)$ / In this case we assume that Ł(W) has = and that the n-arrow structure S is normal./

We have: (W(S), w)$\models(x^p=x^q)$ iff $w(x^p)=w(x^q)$ iff /by the normality condition/ $1.w(x^p)=1.w(x^q)\&\ldots\&n.w(x^p)=n(x^q)$ iff $v(A_1^p)=v(A_1^q)\&\ldots\&v(A_n^p)=v(A_n^q)$ iff (V(S), v)$\models A_1^p=A_1^q\wedge\ldots\wedge A_n^p=A_n^q$ iff (V(S), v)$\models$ $\mu(x^p=x^q)$.

3. Induction hypothesis: suppose that the assertion is true for β and γ.

4. The proof when $\alpha=\neg\beta$ and $\alpha=\beta\wedge\gamma$ is straightforward.

5. $\alpha=(\exists x^p)\beta$.

($\longrightarrow$) Suppose (W(S), w)$\models(\exists x^p)\beta$. Then for some $a\in Ar_S$ we have (W(S), $w_a^{x^p}$)$\models\beta$. Let $B_1=1.a,\ldots,B_n=n.a$. Then by lemma 2.4.(ii) the valuations $w_a^{x^p}$ and $v^{A_1^p\ldots A_n^p}_{B_1\ldots B_n}$ are connected and by the induction hypothesis we have (V(S), $v^{A_1^p\ldots A_n^p}_{B_1\ldots B_n}$)$\models\mu(\beta)$. We have also $\rho_S(B_1,\ldots,B_n)$, so (V(S) $v^{A_1^p\ldots A_n^p}_{B_1\ldots B_n}$)$\models$ $\rho(A_1^p,\ldots,A_n^p)\wedge\mu(\beta)$ and hence (S(V), v)$\models(\exists A_1^p)\ldots(\exists A_n^p)(\rho(A_1^p,\ldots A_n^p)\wedge\mu(\beta))$. Thus (V(S), v)$\models$ $\mu((\exists x^p)\beta)$.

($\longleftarrow$) Suppose now that (V(S), v)$\models\mu((\exists x^p)\beta)$. Then (V(S), v)$\models$ $(\exists A_1^p)\ldots(\exists A_n^p)(\rho(A_1^p,\ldots,A_n^p)\wedge\mu(\beta)$. This means that for some $B_1,\ldots B_n\in Po_S$ we have (V(S), $v^{A_1^p\ldots A_n^p}_{B_1\ldots B_n}$)$\models$ $\rho(A_1^p,\ldots,A_n^p)\wedge\mu(\beta)$. From here we obtain $\rho_S(B_1,\ldots,B_n)$. By the definition of ρ_S, there exists $a\in Ar_S$ such that $B_1=1.a,\ldots,B_n=n.a$. Then by lemma 2.4.(ii) the valuations $v^{A_1^p\ldots A_n^p}_{B_1\ldots B_n}$ and $w_a^{x^p}$ are connected and by the induction hypothesis we have (W(S), $w_a^{x^p}$)$\models\beta$, so (W(S), v)$\models(\exists x^p)\beta$. This ends the proof of the lemma. ■

Theorem 2.6.

Let S be arbitrary n-arrow structure, V(S) be the n-point frame over S and W(S) be the n-arrow frame over S. Then:

(i) If the language Ł(S) is without equality for arrows =, then for any closed formula α of Ł(W) we have:

W(S)$\models\alpha$ iff V(S)$\models\mu(\alpha)$.

(ii) if Ł(S) has equality for arrows = then the above equivalence is true if S is a normal n-arrow structure.

Proof. By lemma 2.5.■

Corollary 2.7.

If the language Ł(S) does not contain equality for arrows =, then the languages Ł(S), Ł(W) and Ł(V) have one and the same expressive power.

3. Modal languages associated with arrow structures. Point logics and Arrow logics

We have associated with each n-arrow structure S two relational structures:

$W(S)=(Ar_S, \{R^S_{ij}/i,j\leq n\})$ - the n-arrow frame over S, and $V(S)=(Po_S, \rho_S)$ - the n-point frame over S. Both structures can be used for interpretation of corresponding modal languages. The language associated with point frames will be called point language and the language associated with arrow frames will be called arrow language. The main result of this section will be a construction of a translation of a rich enough point language into a certain arrow language which preserves semantic validity. In particular this implies that the modal theory of n-ary modality can be reduced to the theory of special unary modalities.

A modal language for arrow frames. Arrow logics

The most natural way to associate a modal language to n-arrow frames of the type W(S), is to extend the language of classical propositional logic with modal connectives [ij] for each relation R_{ij}, being used to interpret [ij] in a Kripke style manner. We denote this language by $ML^n([ij])$. To be more precise $ML^n([ij])$ contains:

- $VAR(ML^n([ij]))=\{p^1, p^2, p^3,\dots\}$ - a denumerable list of distinct propositional variables,
- $\neg, \wedge, \vee$ - Boolean connectives,
- $[ij]$ $i,j\leq n$ - n^2 one argument modal (box) operations, and
- (,) - parentheses.

The set $FOR(ML^n([ij]))$ of formulas is defined in a usual way: all p^i are formulas, if α and β are formulas so are $\neg\alpha$, $(\alpha\wedge\beta)$, $(\alpha\vee\beta)$ and $[ij]\alpha$ for $i,j\leq n$. We will use the usual omission of parentheses. We abbreviate $\langle ij\rangle\alpha=\neg[ij]\neg\alpha$ and adopt the usual definitions for the other Boolean connectives.

The standard semantics for this language is a Kripke style semantics over n-arrow frames. The general semantics is over arbitrary relational structures of the type $\underline{W}=(W, \{R_{ij}/i,j\leq n\})$, which will be called also frames. Let a frame $\underline{W}$ be given. A function v, which assign to each propositional variable p^i a subset $v(p^i)\subseteq W$ is called a valuation of propositional variables. The pair $M=(\underline{W}, v)$ is called a model. The satisfiability relation $x\Vdash_v\alpha$ /the formula α is true in $x\in W$ at the valuation v/ is defined inductively as in the usual Kripke definition as follows:

$x\Vdash_v p^i$ iff $x\in v(p^i)$,

$x\Vdash_v \neg\alpha$ iff $x\nVdash_v\alpha$ ($x\nVdash_v\alpha$ means not $x\Vdash_v\alpha$),

$x\Vdash_v \alpha\wedge\beta$ iff $x\Vdash_v\alpha$ and $x\Vdash_v\beta$,

$x\Vdash_v \alpha\vee\beta$ iff $x\Vdash_v\alpha$ or $x\Vdash_v\beta$,

$x\Vdash_v [ij]\alpha$ iff $(\forall y\in W)(xR_{ij}y \rightarrow y\Vdash_v\beta)$.

The pair $M=(\underline{W}, v)$ is called a model. We say that a formula α is true in a model $M=(\underline{W}, v)$, or that M is a model for α, in symbols $M\models\alpha$, if for any $x\in W$ we have $x\Vdash_v\alpha$. A formula α is true in a frame $\underline{W}$, or that $\underline{W}$ is a frame for α, in symbols $\underline{W}\models\alpha$, if α is true in all models over $\underline{W}$. A formula α is true in a class Σ of frames if α is true in any member of Σ, in symbols $\Sigma\models\alpha$. A class of formulas L is true in a model M if any member of L is true in M. L is true in a class of frames Σ if any member of L is true in Σ. L is called the logic of Σ and denoted by $L(\Sigma)$ if it contains all formulas true in Σ.

Let $ARROW^n$ denote the class of all n-arrow frames. Then $L(ARROW^n)$ is called the Basic Arrow logic of dimension n and is denoted by BAL^n. The language $ML^n([ij])$ is called the language of BAL^n.

The language of BAL^n can be extended in different ways. In this section we shall study an extension denoted by $ML^n([ij],\blacksquare)$ with a new modal symbol $\blacksquare$, called universal modality /see [G&P 90]/. The semantics of this modality can be given in any set W as follows:

$x \Vdash_v \blacksquare\alpha$ iff $(\forall y \in W)(y \Vdash_v \alpha)$

We abbreviate $\blacklozenge$ as $\neg\blacksquare\neg$.

If Σ is a class of frames then by $L(\Sigma\blacksquare)$ we will denote the class of all formulas from $ML^n([ij],\blacksquare)$ which are true in Σ. Using this notation the logic $L(ARROW^n\blacksquare)$ will be called Basic Arrow logic of dimension n with universal modality and will be denoted also by $BAL^n\blacksquare$.

A modal language for point frames. Point logics

The relational structures of n-point frames are of the form $V(S)=(Po_S, \rho_S)$ for certain n-arrow structure S. Here ρ_S is a n- place relation in Po_S. Let us consider arbitrary relational systems of the form $\underline{V}=(V, \rho)$ with $V\neq\emptyset$, where ρ is an n-place relation in V. A natural way to generalize the Kripke semantics is to use a generalization of $\Box$ and $\Diamond$ with more than one arguments and to use the relation ρ for interpretation of such operations. Since for one argument modality the Kripke definition requires a two place relation, in our case $\Box$ and $\Diamond$ should have n-1 arguments - $\Box(\alpha_2,...,\alpha_n)$ and $\Diamond(\alpha_2,...,\alpha_n)$. The semantic conditions for these operations are the following:

$A \Vdash_v \Diamond(\alpha_2,...,\alpha_n)$ iff

$(\exists A_1 A_2 ... A_n \in V)(\rho(A_1,A_2,...A_n)$ & $A=A_1$ & $A_2 \Vdash_v \alpha_2$ &...& $A_n \Vdash_v \alpha_n)$.

For the $\Box$ we have dually

$A \Vdash_v \Box(\alpha_2,...,\alpha_n)$ iff

$(\forall A_1 A_2 ... A_n \in V)(\rho(A_1,A_2,...,A_n)$&$A=A_1 \rightarrow A_2 \Vdash_v \alpha_2$ or..or $A \Vdash_v \alpha_n)$.

Obviously we have that $\Diamond(\alpha_2,...,\alpha_n)$ is equivalent to $\neg\Box(\neg\alpha_2,...,\neg\alpha_n)$ and similarly for $\Diamond$. Let us note that the semantical definition for $\Diamond$ is taken from the Jonsson-Tarski representation theory of Boolean algebras with operators [J&T 51].

The frames of the form (V, ρ) can be used for interpretation of other (n-1)-argument operations. Let σ be any permutation of the sequence <1,2,...,n> and let ρ^σ be a new n-place relation in V defined by the equivalence

$\rho^\sigma(A_1,A_2,...A_n)$ iff $\rho(A_{\sigma(1)},A_{\sigma(2)},...,A_{\sigma(n)})$.

Now let for each permutation σ of <1,2,...,n>, $[\sigma]$ be an (n-1)- argument modality. Then the interpretation of $[\sigma](\alpha_2,...,\alpha_n)$ is the same as for the $\Box(\alpha_2,...,\alpha_n)$, where instead of ρ we use ρ^σ. It is obvious now that the operation $\Box$ corresponds to [id], where id is the identity permutation.

In (V, ρ) we can also interpret the universal modality $\blacksquare$.

Hence, our point language, denoted by $ML^n([\sigma],\blacksquare)$, contains:

- $VAR(ML^n([\sigma],\blacksquare))=\{p^1,p^2,...\}$ - infinite sequence of different propositional variables,
- $\neg$, $\wedge$, $\vee$ - Boolean connectives,
- $[\sigma]$ - (n-1)-argument box operators for each permutation σ of <1,2,...n>,
- $\blacksquare$ - the universal modality,
- (,) - parentheses.

The set of the formulas $FOR(ML^n([\sigma],\blacksquare))$ is defined now in an obvious way. The standard semantics of this language is a Kripke like semantics, as described above, in n-point frames of the form $V(S)=(Po_S, \rho_S)$ under some n-arrow structure S. The general semantics is in arbitrary relational systems of the form (V, ρ), where $\rho \subseteq V^n$. The notions of a model, validity and so on in this semantics can be defined in the same way as for the arrow case.

Let us note that in this language we have n! modal operators of the form $[\sigma]$ since we have n! different permutations in $<1,2,\ldots,n>$. But not all of these modal operators are independent. For instance let σ be a permutation such that $\sigma(1)=1$ and let σ^{-1} be the converse of σ. Then $[\sigma](\alpha_2,\ldots,\alpha_n)$ is semantically equivalent to the formula $[id](\alpha_{\sigma^{-1}(2)},\ldots,\alpha_{\sigma^{-1}(n)})$. Of course, we can take some basis to define the others, but for the sake of symmetry, we take all of the operations $[\sigma]$. Let us note that in the case n=2, we have only two permutations and hence two modalities, which in tense logic correspond to "future" and "past".

A translation of the point language $ML^n([\sigma],\blacksquare)$ into the arrow language $ML^n([ij],\blacksquare)$

We will define a translation τ of the language $L1=ML^n([\sigma],\blacksquare)$ into the language $L2=ML^n([ij],\blacksquare)$, which translation will preserve semantic validity. It will be composed of several translations.

Let the propositional variables of L1 and L2 be arranged as follows:

$VAR(L1)=<p^1_1,p^2_1,\ldots,p^k_1,\ldots>$

$VAR(L2)=<p_1,p_2,\ldots,p_n,\ldots,p^k_1,p^k_2,\ldots,p^k_n,\ldots>$,

Now we define inductively n different translations of L1 into L2 denoted by $1,\ldots,n$. For $1\le i\le n$ put:

$i(p^k)=p^k_i$,

$i(\neg\alpha)=\neg i(\alpha)$,

$i(\alpha\wedge\beta)=i(\alpha)\wedge i(\beta)$,

$i(<\sigma>(\alpha_2,\ldots,\alpha_n)=<i\sigma^{-1}(1)>(\sigma^{-1}(2)(\alpha_2)\wedge\ldots\wedge\sigma^{-1}(n)(\alpha_n))$

$i(\blacklozenge\alpha)=\blacklozenge(1(\alpha)\vee\ldots\vee n(\alpha))$

$\nu(\alpha)=1(\alpha)\wedge\ldots\wedge n(\alpha)$

Let $VAR(\alpha)$ be the set of the propositional variables occurring in α. Then for each $p^k\in VAR(\alpha)$ and $i,j\le n$ define the formula $<ij>j(p^k)\Rightarrow i(p^k)$ and let $\mu(\alpha)$ be the conjunction of all these formulas, i.e.

$\mu(\alpha)=\bigwedge\{<ij>j(p^k)\Rightarrow i(p^k)/i,j\le n \text{ and } p^k\in VAR(\alpha)\}$

Now the definition of the translation τ is

$\tau(\alpha)=\blacksquare\mu(\alpha)\Rightarrow\nu(\alpha)$

Let S be a given n-arrow structure and Let $V(S)=(Po_S, \rho_S)$ and $W(S)=(Ar_S,\{R^S_{ij}/i,j\le n\}$ be the corresponding n-point frame and n- arrow frame over S and let $(V(S), v)$ and $(W(S), w)$ be two models over $V(S)$ and $W(S)$ respectively. Let $\Phi\subseteq VAR(L1)$ be a set of propositional variables of the language L1 and let $FOR(\Phi)$ be the set of all formulas of L1 with variables taken from Φ. We say that the valuations v and w are Φ-connected if for any $x\in Ar_S$, $i\le n$ and $p^k\in\Phi$ we have

$i.x\in v(p^k)$ iff $x\in w(p^k_i)$.

Lemma 3.1.

If the valuations v and w are Φ-connected then for any $x\in Ar_S$, $i\le n$ and $\alpha\in FOR(\Phi)$ we have

$i.x \Vdash_{v} \alpha$ iff $x \Vdash_{w} i(\alpha)$.

Proof. We shall proceed by induction on α. If α is a propositional variable then the assertion is true by the definition of Φ-connected valuations. Since the translations i commute with Boolean connectives these cases do not present difficulties.

Now suppose that the assertion is true for the formulas $\alpha_2,\ldots,\alpha_n$ from $FOR(\Phi)$ and let $\alpha=\langle\sigma\rangle(\alpha_2,\ldots,\alpha_n)$. Then

$(\longrightarrow)$ $i.x \Vdash_v \langle\sigma\rangle(\alpha_2,\ldots,\alpha_n)$ iff $(\exists A_1A_2\ldots A_n\in Po_S)(\rho^{\sigma}_S(A_1,A_2,\ldots,A_n)$ $\&\ A_1=i.x\ \&\ A_2\Vdash_v\alpha_2\ \&\ldots\ \&\ A_n\Vdash_v\alpha_n)$.

From here we get $\rho_S(A_{\sigma(1)},A_{\sigma(2)},\ldots,A_{\sigma(n)})$. By the definition of ρ_S we have: $(\exists u\in Ar_S)(\forall j\le n)(j.u=A_{\sigma(j)})$. Since σ has a converse σ^{-1} we get that for any $j\le n$ $\sigma^{-1}(j).u=A_j$. So we have $\sigma^{-1}(1).u=A_1=i.x$, $\sigma^{-1}(2).u=A_2\Vdash_v\alpha_2,\ldots,$ $\sigma^{-1}(n).u=A_n\Vdash_v\alpha_n$. Then by the induction hypothesis we get :

$xR^S_{i\sigma^{-1}(1)}u$, $u\Vdash_w\sigma^{-1}(2)(\alpha_2),\ldots,$ $u\Vdash_w\sigma^{-1}(n)(\alpha_n)$.

Hence

$x\Vdash_w\langle i\sigma^{-1}(1)\rangle(\sigma^{-1}(2)(\alpha_2)\wedge\ldots\wedge\sigma^{-1}(n)(\alpha_n))$ and

$x\Vdash_w i(\langle\sigma\rangle(\alpha_2,\ldots,\alpha_n))$.

$(\longleftarrow)$ For the converse implication suppose that

$x\Vdash_w i(\langle\sigma\rangle(\alpha_2,\ldots,\alpha_n))$.

Then there exists $u\in Ar_S$ such that

$xR^S_{i\sigma^{-1}(1)}u$, $u\Vdash_w\sigma^{-1}(2)(\alpha_2),\ldots,$ $u\Vdash_w\sigma^{-1}(n)(\alpha_n)$.

Using the i.h. we obtain:

$i.x=\sigma^{-1}(1).u$ and $\sigma^{-1}(2).u\Vdash_v\alpha_2,\ldots,\sigma^{-1}(n).u\Vdash_v\alpha_n$.

Let for $j=1,\ldots,n$ denote $A_j=\sigma^{-1}(j).u$. From here we obtain $A_{\sigma(j)}=j.u$ for $j=1,\ldots n$. This gives $\rho_S(A_{\sigma(1)},A_{\sigma(2)},\ldots,A_{\sigma(n)})$ and consequently $\rho^{\sigma}_S(A_1,A_2,\ldots,A_n)$. Since we have $A_1=i.x$, $A_2\Vdash_v\alpha_2$, ..., $A_n\Vdash_v\alpha_n$ we obtain that $i.x\Vdash_v\langle\sigma\rangle(\alpha_2,\ldots,\alpha_n)$. This completes the proof of the case.

The case $\alpha=\blacklozenge\beta$ is more easy and can be treated in a similar way.■

Lemma 3.2.

Let $(V(S),\ v)$ and $(W(S),\ w)$ be two models and let $\Phi\subseteq VAR(L1)$.

(i) if v and w are Φ-connected then w satisfies the following condition for any $p^k\in\Phi$, $x,y\in Ar_S$ and $i,j\le n$

(*) $xR^S_{ij}y\ \&\ y\in w(j(p^k))\longrightarrow x\in w(i(p^k))$

(ii) Let $\alpha\in FOR(L1)$ and $\Phi=VAR(\alpha)$. Then the condition (*) is fulfilled for w iff $(W(S),\ w)\models\mu(\alpha)$.

(iii) Let for any $p^k\in\Phi$, $x,y\in Ar_S$ and $i,j\le n$ we have (*) and let for $p^k\in\Phi$ v be defined as follows:

$v(p^k)=\{A\in Po_S/\exists j\le n\exists y\in Ar_S\ A=j.y \text{ and } y\in w(j(p^k))\}$. Then the valuations v and w are Φ-connected.

(iv) Let for any $p^k\in\Phi$ and $i\le n$ the valuation v be defined as follows:

$w(i(p^k))=\{x\in Ar_S/\ i.x\in v(p^k)\}$.

Then the valuations v and w are Φ-connected.

Proof - by straightforward calculations.■

Theorem 3.3.

Let S be a n-arrow structure and V(S) and W(S) be n-point frame and n-arrow frame over S. Then for any formula α we have:

$V(S)\models\alpha$ iff $W(S)\models\tau(\alpha)$.

Proof. ($\longrightarrow$) Suppose that $V(S)\models\alpha$ and for the sake of contradiction that $W(S)\not\models\tau(\alpha)$, so $W(S)\not\models\blacksquare\mu(\alpha)\Rightarrow\nu(\alpha)$. Then there exists a valuation w and $x\in Ar_S$ such that $x\Vdash_w \blacksquare\mu(\alpha)$ and $x\nVdash_w \nu(\alpha)$.

From $x\Vdash_w \blacksquare\mu(\alpha)$ we obtain that for any $y\in Ar_S$ we have $y\Vdash_v \mu(\alpha)$, so $(W(S), w)\models\mu(\alpha)$. Then by lemma 3.2.(ii) we obtain that the valuation w satisfies the condition (*) for Φ=VAR(α). Then by lemma 3.2.(iii) there exists a valuation v of VAR(L1) into V(S) such that v and W are Φ-connected.

From $x\nVdash_w \nu(\alpha)$, i.e. from $x\nVdash_w 1(\alpha)\wedge\ldots\wedge n(\alpha)$ we conclude that for some i, $1\leq i\leq n$, we have $x\nVdash_w i(\alpha)$. Since v and w are Φ-connected, we obtain by lemma 3.1 that $i.x\nVdash_v \alpha$. But $i.x\in Po_S$, so $V(S)\not\models\alpha$, contrary to the assumption.

($\longleftarrow$) Suppose that $W(S)\models\tau(\alpha)$ and for the sake of contradiction that $V(S)\not\models\alpha$. Then there exists $A\in Po_S$ and a valuation v such that $A\nVdash_v \alpha$. Then by lemma 3.2.(iv) there exists a valuation w of VAR(L2) into Ar_S such that v and w are Φ-connected for Φ=VAR(α). Then by lemma 3.2.(i) condition (*) is fulfilled and by lemma 3.2.(ii) we have $(W(S),w)\models\mu(\alpha)$, so for any $y\in Ar_S$ we have $y\Vdash_w \mu(\alpha)$. Thus $x\Vdash_w \blacksquare\mu(\alpha)$.

Let A=i.x, so $i.x\nVdash_v \alpha$. Then by lemma 3.1 $x\nVdash_w i(\alpha)$, so $x\nVdash_w \nu(\alpha)$. We obtain from here that $x\nVdash_v \blacksquare\mu(\alpha)\Rightarrow\nu(\alpha)$, so $x\nVdash_w \tau(\alpha)$. Thus $W(S)\not\models\tau(\alpha)$, which contradicts the assumption. This completes the proof of the theorem.■

Theorem 3.4.

The translation τ preserves first-order definability in a sense that a formula $\alpha\in$L1 is a first order definable in the standard semantics of L1 if and only if its translation $\tau(\alpha)$ is first order definable in the standard semantics of L2.

Proof. ($\longrightarrow$) Let α be a formula, first-order definable in L1 by a first order sentence F . Of course this sentence is in the first-order language Ł(V) of point frames. We have to show that the translation $\tau(\alpha)$ is first-order definable by some sentence in the first order language Ł(W) of arrow frames. By theorem 2.3 we can conclude that there exists a translation * /in the theorem this translation is denoted by τ/ from the language Ł(V) to the language L(W) such that for any n-arrow structure S and sentence G in the language Ł(V) we have: $V(S)\models G$ iff $W(S)\models G^*$. We shall show that $\tau(\alpha)$ is first-order definable by the sentence F*. Suppose that this is not true. Then there exists a n-arrow structure S such that the n-arrow frame W(S) satisfies F* but $W(S)\not\models\tau(\alpha)$. By the above remark we have that V(S) satisfies F. From $W(S)\not\models\tau(\alpha)$ by theorem 3.3 we get that $V(S)\not\models\alpha$. But α is modally definable by the sentence F. From here we obtain that F is not true in V(S) - a contradiction.

($\longleftarrow$) Suppose now that $\tau(\alpha)$ is first order definable by a sentence F in the language Ł(W). By theorem 2.6 there exists a translation + /in the theorem this translation is denoted by ν/ of the language Ł(W) into the language Ł(V) such that for any sentence G, and n-arrow structure S we have $V(S)\models G^+$ iff $W(S)\models G$. We shall show that α is first-order definable by F^+. Suppose that this is not true . Then, as in the above case, applying theorem 3.3 we obtain a contradiction. This completes the proof of the theorem.■

Problem 3.1.

Is it possible to find a translation from the language $ML^n([\sigma])$ into the language $ML^n([ij])$ /■ is dropped/ with properties similar to τ?

Problem 3.2.

Is it possible to define a translation from the language $ML^n([ij])$ to the language $ML^n([\sigma],■)$ or some extension of $ML^n([\sigma],■)$ with properties similar to τ?

CHAPTER 4.3.

n-DIMENSIONAL ARROW LOGICS

Overview. In this chapter we axiomatize the basic arrow logic of dimension n - BAL^n and some of its extensions - $BAL^n\blacksquare$ with universal modality and CAL^n - an extension with operators of cillindrifications, which is a modal version of a fragment of first order logic. Various completeness theorems and decidability results for the introduced logics have been obtained.

Introduction

This chapter is devoted to the axiomatization of some Arrow logics of dimension n. It consists of three sections.

In sec. 1 we axiomatize the Basic Arrow Logic of dimension n - BAL^n - and an extension of BAL^n with the universal modality. We prove for them several completeness results, including finite model property.

In sec. 2 we axiomatize an extension of BAL^n with the operations of cylindrifications, corresponding to a version of relativized set cylindric algebras /without constants δ_{ij} for identity/. We prove the corresponding completeness theorem and decidability. It have to be mention here the essential use of copying construction, consisting of transforming non-intended models into intended ones, preserving semantical validity.

In the final sec. 3 we discuss some open problems.

1. Axiomatization of BAL^n and BAL^n■

In chapter 4.2 We defined the logic BAL^n - the Basic Arrow Logic of dimension n - as the set $L(ARROW^n)$ of all formulas in the language $ML^n([ij])$ true in the class $ARROW^n$ of all n-arrow frames of n. This is a semantic definition of BAL^n. Let us introduce the following classes of arrow frames:

$ARROW^n$FIN - the class of finite arrow frames of dimension n,

$ARROW^n$NOR - the class of normal arrow frames of dimension n,

$ARROW^n$FINNOR - the class of finite normal arrow frames of dimension n.

In this section we shall propose an axiomatization of BAL^n and prove that it is sound and complete in each of the above classes of arrow frames.

We propose the following set of axioms and rules for BAL^n:

Axiom schemes for BAL^n

(Bool) All or enough Boolean tautologies,

(K[ij]) $[ij](A\Rightarrow B)\Rightarrow([ij]A\Rightarrow[ij]B)$, $i,j\leq n$

Ax(ρii) $[ii]A\Rightarrow A$, $i\leq n$

Ax(σij) $A\vee[ij]\neg[ji]A$, $i,j\leq n$,

Ax(τijk) $[ik]A\Rightarrow[ij][jk]A$, $i,j,k\leq n$.

Rules of inference:

modus ponenes (MP) $A, A\Rightarrow B/B$,

Necessitation (N[ij]) $A/[ij]A$, $i,j\leq n$.

We will denote the set of theorems of this formal system also by BAL^n.

Let us note that the axioms Ax(ρii), Ax(σij) and Ax(τijk) are just modal translations in the sense of modal definability theory [Ben 86] of the conditions (ρii), (σij) and (τijk), which immediately shows that BAL^n is sound in its intended semantics.

Theorem 4.1. /Completeness theorem for BAL^n/

The following conditions are equivalent for any formula A of BAL^n:

(i) A is a theorem of BAL^n,

(ii) A is true in the class $ARROW^n$.

Proof. (i)⟶(ii) This follows from the observation that BAL^n is sound with respect to its standard semantics - $ARROW^n$. (ii)⟶(i) This implication can be proved by a standard application of the canonical construction known from Modal Logic.

Corollary 1.2.

$BAL^n=L(ARROW^n)$.

Now, using the method of filtration, known from monomodal logic we shall show that BAL^n is complete in the class $ARROW^n FIN$ of all finite arrow frames of dimension n. For that purpose we shall give the relevant definition of filtration in the form of Segerberg [Seg 71], adapted for the language of BAL^n.

Let Γ be a finite set of formulas, closed under subformulas and let $\underline{W}=(W, \{R_{ij}/i,j\leq n\})$ be any frame and $M=(\underline{W}, v)$ be any model over $\underline{W}$. By means of Γ and M we define an equivalence relation ~ in W in the following way:

for $x,y\in W$ we define $x\sim y$ iff $(\forall A\in\Gamma)(x\Vdash_{v} A$ iff $y\Vdash_{v} A)$. Let $|x|=\{y\in W/x\sim y\}$, $|W|=\{|x|/x\in W\}$ and for $A\in VAR$ $v'(A)=\{|x|\in|W|/x\in v(A)\}$. We say that the model $M'=(|W|, \{R'_{ij}/i,j\leq n\}, v')$ is a filtration of M trough Γ if the relations R'_{ij} satisfy the following two conditions for any $x,y\in W$:

$(FilR_{ij}1)$ if $xR_{ij}y$ then $|x|R'_{ij}|y|$,

$(FilR_{ij}2)$ if $|x|R'_{ij}|y|$ then $(\forall[ij]A\in\Gamma)(x\Vdash_{v}[ij]A \rightarrow y\Vdash_{v} A)$.

In the next section we shall use filtration for another kind of frames. The general definition is that for each relation R we suppose two conditions (FilR1) and (FilR2) like the above.

Lemma 1.3. /Filtration lemma/

(i) For any $A\in\Gamma$ and $x\in W$ the following equivalence is true:

$x\Vdash_{v} A$ iff $|x|\Vdash_{v'} A$,

(ii) $Card|W|\leq 2^m$, where $m=Card\Gamma$.

Lemma 1.4. /Filtration for BAL^n/

Let $\underline{W}=(W, \{R_{ij}/i,j\leq n\})$ be an n-arrow frame, $M=(\underline{W}, v)$ be a model over $\underline{W}$ and A be a formula of BAL^n. Then there exist a finite set Γ of formulas, containing A and closed under subformulas and a filtration $M'=(|W|, \{R'_{ij}/ij\leq n\}, v')$ trough Γ such that $\underline{W}'=(|W|, \{R'_{ij}/ij\leq n\})$ is a finite arrow frame of dimension n such that $Card|W|\leq 2^{n^2.m}$, where m is the number of subformulas of A.

Proof. Let Γ be the smallest set of formulas closed under subformulas and containing A, satisfying the following closure condition

(γ) If for some $ij\leq n$ $[ij]B\in\Gamma$ then for any $i,j\leq n$ $[ij]B\in\Gamma$.

Obviously Γ is a finite set of formulas, containing no more than $n^2.m$ elements, where m is the number of the subformulas of A. Define |W| and v' as in the definition of filtration. By lemma 1.3.(ii) we have $Card\Gamma\leq 2^{n^2.m}$. For the relations R'_{ij} we put:

$|x|R'_{ij}|y|$ iff $(\forall[ij]B\in\Gamma)(\forall k\leq n)(x\Vdash_{v}[ik]B \longleftrightarrow y\Vdash_{v}[jk]B)$

It can be easily proved that the definition of R'_{ij} is correct in the sense that

if $|x|=|x'|$ and $|y|=|y'|$ then $|x|R'_{ij}|y|$ iff $|x'|R'_{ij}|y'|$.

First we shall show that the frame $\underline{W}'=(|W|, \{R'_{ij}/ij\leq n\})$ is an n-arrow frame The conditions (ρii) and (σij) follow directly from the definition of R'_{ij}. For the condition (τijk) suppose $|x|R'_{ij}|y|$ and $|y|R'_{jk}|z|$. To prove $|x|R'_{ik}|z|$ suppose $[ik]A\in\Gamma$, $l\leq n$ and for the direction $(\rightarrow)$ suppose $x\Vdash_{v}[il]A$ and proceed to show that $z\Vdash_{v}[kl]A$. From $[ik]A\in\Gamma$ we get $[ij]A, [jl]A\in\Gamma$. Then $|x|R'_{ij}|y|$, $[ij]A\in\Gamma$ and $x\Vdash_{v}[il]A$ imply $y\Vdash_{v}[jl]A$. This and $[jl]A\in\Gamma$ and

$|y|R'_{jk}|z|$ imply $z\Vdash_v [kl]A$.

The converse direction (⟵) can be proved in a similar way.

It remains to show that the conditions of filtration ($FilR_{ij}1$) and ($FilR_{ij}2$) are satisfied.

For the condition ($FR_{ij}1$) suppose $xR_{ij}y$, $[ij]A\in\Gamma$ $k\le n$ and for the direction (⟶) suppose $x\Vdash_v [ik]A$, $yR_{jk}z$ and proceed to show that $z\Vdash_v A$. From $xR_{ij}y$ and $yR_{jk}z$ we get $xR_{ik}z$ and since $x\Vdash_v[ik]A$ we get $z\Vdash_v A$. For the direction (⟵) suppose $y\Vdash_v[jk]A$, $xR_{ik}z$ and proceed to show that $z\Vdash_v A$. From $xR_{ij}y$ we get $yR_{ji}x$ and by $xR_{ik}z$ we get $yR_{jk}z$. From here and $y\Vdash_v[jk]A$ we obtain $z\Vdash_v A$. This ends the proof of ($FR_{ij}1$).

For the condition ($FR_{ij}2$) suppose $|x|R'_{ij}|y|$, $[ij]A\in\Gamma$ and $x\Vdash_v[ij]A$. From here we obtain $y\Vdash_v[jj]A$ and since $yR_{jj}y$ we get $y\Vdash_v A$. This completes the proof of the lemma.■

Theorem 1.5. /Finite completeness theorem for BAL^n/

The following conditions are equivalent for any formula A of BAL^n:

(i) A is a theorem of BAL^n,

(ii) A is true in all finite n-arrow structure with cardinality $\le 2^{n^2.m}$ where m is the number of the subformulas of A.

Proof - by theorem 1.1 and lemma 1.4.■

Corollary 1.6.

(i) BAL^n=L($ARROW^n$FIN)

(ii) BAL^n possesses finite model property and is decidable.

Now we shall show that BAL^n=L($ARROW^n$NOR). For that purpose we shall use the copying construction, called copying, which can perform each model over an arrow frame into an equivalent model over a normal arrow frame.

Lemma 1.7. (i) (Copying lemma)

Let I be a copying from the model M to the model M'. Then for any formula A, $x\in W$ and $f\in I$ the following equivalence holds:

$x\Vdash_v A$ iff $f(x)\Vdash_{v'} A$,

(ii) If I is a copying from the frame $\underline{W}$ to the frame $\underline{W}'$ and v is a valuation, then there exists a valuation v' such that I is a copying from the model M=($\underline{W}$, v) to the model M'=($\underline{W}'$, v').

Lemma 1.8.

Let $\underline{W}$=(W, $\{R_{ij}/ij\le n\}$) be an n-arrow frame Then

(i) There exists a normal arrow frame $\underline{W}'$=(W', $\{R'_{ij}/ij\le n\}$) and a copying I from $\underline{W}$ to $\underline{W}'$.

(ii) If $\underline{W}$ is a finite n-arrow frame then the frame $\underline{W}'$ from (i) is a finite frame too.

Proof. Let $\underline{B}$(W)=(B(W), 0, 1, +, .) be the Boolean ring over the set W, namely B(W) is the set of all subsets of W, 0=∅, 1=W, A+B=(A\B)∪(B\A) and A.B=AB=A∩B. Note that in Boolean rings a- b=a+b. Let $I=B(W)^n=B(W)\times\ldots\times B(W)$ n times. For $f\in I$ we denote by f_i the i-th coordinate of f, so $f=(f_1,\ldots,f_n)$.

We put $W'=W\times B(W)^n$. The elements of $I=B(W)^n$ can be treated as functions from W into W' as follows: for $f\in I$ and $x\in W$ we put $f(x)=(f,x)$. Obviously the conditions (I1) and (I2) from the definition of copying are fulfilled and each element of W' is in the form of f(x) for some $f \in I$ and $x \in W$.

For the relations R'_{ij} we have the following definition:

$f(x)R'_{ij}g(y)$ iff $xR_{ij}y$ & $(f_1+\ldots+f_n+\{x\}=g_1+\ldots+g_n+\{y\})$ & $f_i=g_j$.

To verify the condition ($R_{ij}1$) suppose $xR_{ij}y$ and $f \in I$. Define g as follows. Put $g_j=f_i$ and choose $g_1,\ldots,g_{j-1},g_{j+1},\ldots,g_n$ in such a way as to satisfy the equation $f_1+\ldots+f_n+\{x\}=g_1+\ldots+g_n+\{y\}$. This is possible because B(W) is a ring and $n \geq 2$.

Condition ($R_{ij}2$) follows directly from the definition of R'_{ij}. So I is a copying.

The proof that W' with the relations R'_{ij} is an arrow frame is straightforward. For the condition of normality suppose $f(x)R'_{ii}g(y)$ for $i=1,\ldots,n$ and proceed to show that $f(x)=g(y)$. By the definition of R'_{ii} we have $f_1+\ldots+f_n+\{x\} = g_1+\ldots+g_n+\{y\}$ and $f_i=g_i$ for $i=1,\ldots,n$, so f=g. Since B(W) is a ring we obtain $\{x\}=\{y\}$, so x=y and hence f(x)=g(y). Thus W' is a normal n-arrow frame

Suppose now that $\underline{W}$ is a finite n-arrow frame. Then the Boolean ring over W is finite too and hence $\underline{W}'$ is a finite n-arrow frame.■

Theorem 1.9. /Completeness theorem for BAL^n in the class of normal frames/

The following conditions are equivalent for any formula A of BAL^n:

(i) A is a theorem of BAL^n,

(ii) A is true in the class $ARROW^n NOR$,

(iii) A is true in the class $ARROW^n FINNOR$.

Proof. The implication (i)⟶(ii) follows from the soundness part of theorem 1.1.

For the converse implication (ii)⟶(i) suppose that A is true in $ARROW^n NOR$ and for the sake of contradiction that A is not a theorem of BAL^n. Then by theorem 1.1 A is falsified in some n- arrow frame $\underline{W}=(W, \{R_{ij}/ij \leq n\})$, so there exists a valuation v and $x \in W$ such that $x \Vdash_v\!\!\!\!\not\;\; A$. By theorem 1.8 there exists a normal n-arrow structure $\underline{W}'=(W', \{R'_{ij}/ij \leq n\})$ and a copying I from $\underline{W}$ to $\underline{W}'$. Then by lemma 1.7.(i) there exists a valuation v' such that I is a copying from the model ($\underline{W}$, v) to the model ($\underline{W}'$, v'). By lemma 1.7.(ii) we obtain that for any $f \in I$ $f(x) \Vdash_v\!\!\!\!\not\;\; A$, so A is not true in the normal n-arrow structure $\underline{W}'$ - a contradiction.

The equivalence (i) ⟷ (iii) can be proved in a similar way, using theorem 1.5. and theorem 1.8.(ii). ■

Corollary 1.10.

$BAL^n=L(ARROW^n NOR)=L(ARROW^n FINNOR)$.

Corollary 1.11.

The normality condition is not modally definable.

The following theorem summarizes the completeness theorems for BAL^n.

Theorem 1.12. /Completeness theorem for BAL^n/

The following conditions are equivalent for any formula A of BAL^n:

(i) A is a theorem of BAL^n,

(ii) A is true in the class $ARROW^n$,

(iii) A is true in the class $ARROW^n FIN$,

(iv) A is true in the class $ARROW^{n}NOR$,
(v) A is true in the class $ARROW^{n}FINNOR$.

Now we turn to the axiomatization of BAL^{n}■. The language of BAL^{n}■ - $ML^{n}([ij],■)$ - is an extension of the language ML([ij]) of BAL^{n} with the universal modality ■. As we have introduced, L($ARROW^{n}$■) is the the semantic definition of the logic BAL^{n}■. We propose the following axiomatization of BAL^{n}■, which is an extension of the axiomatizations of BAL^{n}.

Axiom schemes for BAL^{n}■

(Bool) All or enough Boolean tautologies,
(K[ij]) $[ij](A\Rightarrow B)\Rightarrow([ij]A\Rightarrow[ij]B)$, $i,j\le n$
Ax(ρii) $[ii]A\Rightarrow A$, $i\le n$
Ax(σij) $A\vee[ij]\neg[ji]A$, $i,j\le n$,
Ax(τijk) $[ik]A\Rightarrow[ij][jk]A$, $i,j,k\le n$,
(K■) ■$(A\Rightarrow B)\Rightarrow$(■$A\Rightarrow$■$B)$,
Ax(S5■) ■$A\Rightarrow A$, $A\vee$■$\neg$■A, ■$A\Rightarrow$■■A,
Ax(incl) ■$A\Rightarrow[ij]A$.

Rules of inference:

modus ponenes (MP) $A, A\Rightarrow B/B$,

Necessitation (N■) A/■A

Let us note that the rule (N[ij]) A/[ij]A can be obtained by (N■) and Ax(incl). The axiom Ax(incl) expresses the fact that the relations R_{ij} are contained in the universal relation.

Theorem 4.13. /Completeness theorem for BAL^{n}■/
The following conditions are equivalent for any formula A of BAL^{n}■:
(i) A is a theorem of BAL^{n}■,
(ii) A is true in the class $ARROW^{n}$,
(iii) A is true in the class $ARROW^{n}FIN$,
(iv) A is true in the class $ARROW^{n}NOR$,
(v) A is true in the class $ARROW^{n}FINNOR$.
Proof - similar to the proof of the completeness theorems for BAL^{n} and therefore is left to the reader.■

2. An extension of BAL^{n} with cylindric operators: the logic CAL^{n}

In this section we shall study an extension of the logic BAL^{n} with new modal formulas [i]A and <i>A, $i\le n$, with meaning like $\forall x_i$ and $\exists x_i$ respectively. The modalities <i> are called cylindric operators and the obtained polymodal logic, denoted by CAL^{n}, is called cylindric arrow logic of dimension n. The name "cylindric" comes from the theory of cylindric algebras in algebraic logic, in which the cylindric operations are algebraic analogs of the existential quantifiers.

The formal definition of CAL^{n} is the following. His language, denoted by $ML^{n}([ij],[i])$, extends the language $ML^{n}([ij])$ of BAL^{n} with n box modalities [i] i=1,...,n. Diamonds are defined as usual $\langle i\rangle=\neg[i]\neg$.

The standard semantics of $ML^{n}([ij],[i])$ is defined as follows. A relational system of the form $\underline{W}=(W, \{R_{ij}/ij\le n\}, \{R_i/i\le n\})$ is called a standard frame for CAL^{n} if the reduct $(W, \{R_{ij}/ij\le n\})$ is a normal n-arrow frame and the relations R_i, **$i\le n$,** are defined by the equations

(*) xR_iy iff $(\forall j\neq i)xR_{jj}y$, or in another form

$R_i=R_{11}\cap..\cap R_{i-1i-1}\cap R_{i+1i+1}\cap..\cap R_{nn}$, $i\leq n$.

The relations R_{ij} are used for the interpretation of modalities [ij] as before and the relations R_i are used for the interpretation of the new modalities [i]:

$x\Vdash_v [i]A$ iff $(\forall y\in W)(xR_iy \rightarrow y\Vdash_v A)$, and for the diamonds

$x\Vdash_v \langle i\rangle A$ iff $(\exists y\in W)(xR_iy \ \&\ y\Vdash_v A)$.

By theorem 1.1 from chapter 4.1, each normal arrow structure can be represented as a normal arrow structure over some relational system (U, ρ) where ρ is a nonempty n-place relation in U. Then by theorem 1.3. ch. 4.1 each normal n-arrow frame $\underline{W}=(W, \{R_{ij}/ij\leq n\})$ can be identified with the frames of the following form: there exists $U\neq\emptyset$ and $W\subseteq U^n$ and for $(x_1,...,x_n)$, $(y_1,...,y_n)\in W$ and $i,j\leq n$ the relation R_{ij} is defined by the equivalence

$(x_1,...,x_n)R_{ij}(y_1,...,y_n)$ iff $x_i=y_j$.

Then for the relations R_i we obtain:

$(x_1,...,x_n)R_i(y_1,...,y_n)$ iff $(\forall j\neq i)x_j=y_j$.

Now the semantics of $ML^n([ij],[i])$ can be reformulated as follows. For $(x_1,...,x_n)\in W$

$(x_1,...,x_n)\Vdash_v A$ iff $(x_1,...,x_n)\in v(A)$, for $A\in VAR$,

$(x_1,...,x_n)\Vdash_v \neg A$ iff $(x_1,...,x_n)\nVdash_v A$,

$(x_1,...,x_n)\Vdash_v A\wedge B$ iff $(x_1,...,x_n)\Vdash_v A$ and $(x_1,...,x_n)\Vdash_v B$,

$(x_1,...,x_n)\Vdash_v A\vee B$ iff $(x_1,...,x_n)\Vdash_v A$ or $(x_1,...,x_n)\Vdash_v B$,

$(x_1,...,x_n)\Vdash_v \langle ij\rangle A$ iff

$(\exists y_1..\exists y_{j-1}\exists y_{j+1}...\exists y_n \text{ in } U)(y_1,..,y_{j-1},x_i,y_{j+1},..y_n)\in W\ \&$
$(y_1,..,y_{j-1},x_i,y_{j+1},..y_n)\Vdash_v A$,

$(x_1,...,x_n)\Vdash_v \langle i\rangle A$ iff $(\exists y_i\in U)(x_1,..,x_{i-1},y_i,x_{i+1},..,x_n)\in W\ \&$
$(x_1,..,x_{i-1},y_i,x_{i+1},..,x_n)\Vdash_v A$,

and dually for [ij]A and <i>A. This form of the semantics shows that the meaning of <i> is indeed like the meaning of the existential quantifier $\exists y_i$. The meaning of <jj> is more complex and can be represented by a prefix of n-1 existential quantifiers $\exists y_1..\exists y_{j-1}\exists y_{j+1}...\exists y_n$. The meaning of <ij> is the same quantification, followed by a substitution, changing the value of j-th variable with the value of x_i.

A modal study of operations of the form <i> is given by Venema in [Ven 91]. The standard semantics of the modality <i> in Venema's paper is the same as above, with the difference that Venema considers the more reasonable case $W=U^n$ - the full n-cube of some set U. The Venema's logic contains also propositional constants δ_{ij}, formalizing the equality, with the following semantics:

$(x_1,...,x_n)\Vdash_v \delta_{ij}$ iff $x_i=x_j$.

The obtained in this way n-dimensional modal logic appears to be undecidable and non-finitely axiomatizable by normal modal rules /see [Ven 91] and [Ven 92]/. In contrast with this result, we show in this section that

CAL^n, with the semantics given above, is finitely axiomatizable by the orthodox rules of modus ponens and necessitation and is decidable. Let us note that the price for this result is that we take our semantics to be based not on full n-cube but on a subset of a full cube. An implication of this is that CAL^n determines a natural decidable version of first order logic. The first announcement of this result is in [Vak 92a] and can be considered as a contribution to the so called "finitization problem in algebraic logic", described in [Nem 91]. Similar result was obtained by Németi [Nem 92] for a version of first order logic.

The axiomatization of CAL^n presents some difficulties because the equalities (*) defining R_i are not modally definable, in the sense of modal definability theory [Ben 86]. For that purpose we introduce a nonstandard semantics for CAL^n, for which it is easy to prove the corresponding completeness theorem. Then, using the copying construction, we prove the completeness of CAL^n with respect to its standard semantics.

The nonstandard semantics for CAL^n is defined as follows. A relational structure $\underline{W}=(W, \{R_{ij}/ij\le n\}, \{R_i/i\le n\})$ is called a nonstandard frame for CAL^n if $(W, \{R_{ij}/ij\le n\})$ is an n-arrow frame and the relations R_i satisfy the following conditions for any $x,y,z\in W$ and $i,j\le n$:

(ρi) xR_ix,

(σi) $xR_iy \longrightarrow yR_ix$,

(τi) $xR_iy \;\&\; yR_iz \longrightarrow xR_iz$,

(incl i≠j) $i\ne j, \;\&\; xR_iy \longrightarrow xR_{jj}y$.

Let us note that each standard frame for CAL^n is also a nonstandard frame for CAL^n and that a nonstandard frame $\underline{W}$ is a standard one iff it is normal n-arrow frame and satisfies the following condition for any $x,y\in W$ and $i\le n$:

(St i) $(\forall j\ne i)(xR_{jj}y) \longrightarrow xR_iy$.

Note also that all conditions of standard frames are modally definable except normality and the condition (St i), a fact, which will be proved later.

We propose the following axiomatization of CAL^n.

Axioms schemes for CALn /i,j≤n/

(Bool) All or enough Boolean tautologies,

(K[ij]) $[ij](A\Rightarrow B)\Rightarrow([ij]A\Rightarrow[ij]B)$, i,j≤n

Ax(ρii) $[ii]A\Rightarrow A$, i≤n

Ax(σij) $A\vee[ij]\neg[ji]A$, i,j≤n,

Ax(τijk) $[ik]A\Rightarrow[ij][jk]A$, i,j,k≤n.

(K[i]) $[i](A\Rightarrow B)\Rightarrow([i]A\Rightarrow[i]B)$, i≤n

Ax(ρi) $[i]A\Rightarrow A$, i≤n

Ax(σi) $\langle i\rangle[i]A\Rightarrow A$, i≤n

Ax(τi) $[i]A\Rightarrow[i][i]A$, i≤n

Ax(incl,i≠j) $[jj]A\Rightarrow[i]A$, i≠j, i,j≤n

Rulesofinference:

modus ponenes (MP) $A, A\Rightarrow B/B$,

Necessitation (N[ij]) $A/[ij]A$, i,j≤n,

(N[i]) $A/[i]A$, i≤n.

Let us note that axioms Ax(ρi) - Ax(incl,i≠j) are modal translations of the corresponding conditions of nonstandard frames.

Theorem 2.1./Completeness theorem for CAL^n with respect to nonstandard semantics/

The following conditions are equivalent for any formula A of CAL^n:

(i) A is a theorem of CAL^n,

(ii) A is true in all nonstandard frames for CAL^n.

Proof. (i)$\longrightarrow$(ii) - in a standard way by proving that axioms are true in each nonstandard frame and that the rules of inference preserve the validity.

(ii)$\longrightarrow$(i). Use the canonical construction and the fact that all axioms are canonical in a sense that they determine the corresponding semantic condition in the canonical model.■

To prove the completeness of CAL^n with respect to its standard semantics we shall use the copying construction with an obvious modification of the notion of copying for frames for CAL^n.

Lemma 5.2/Copying for nonstandard frames for CAL^n/

Let $\underline{W}=(W, \{R_{ij}/ij\le n\}, \{R_i/i\le n\})$ be a nonstandard frame for CAL^n. Then:

(i) There exists a standard frame $\underline{W}'=(W',\{R'_{ij}/ij\le n\}, \{R'_i/i\le n\})$ and a copying I from $\underline{W}$ to $\underline{W}'$.

(ii) If $\underline{W}$ is a finite frame then $\underline{W}'$ from (i) is a finite frame too.

Proof. Let $\underline{B}(W)=(B(W), 0, 1, +, .)$ be the Boolean ring over the set W, namely B(W) is the set of all subsets of W, $0=\emptyset$, $1=W$, $A+B=(A\setminus B)\cup(B\setminus A)$ and $A.B=AB=A\cap B$. Let I be the set of all matrices over B(W) in the following form

$$f=\begin{pmatrix} f_1^0 & f_2^0 \ldots f_n^0 \\ f_1^1 & f_2^1 \ldots f_n^1 \\ \ldots & \ldots \\ f_1^n & f_2^n \ldots f_n^n \end{pmatrix}$$

For $f\in I$ and $x\in W$ define

$$f_i=\begin{pmatrix} f_i^0 \\ f_i^1 \\ \vdots \\ f_i^n \end{pmatrix}, \quad f^i[x]=\textstyle\sum_{k=1,k\neq i}^{n} f_k^i+R_i(x), \qquad f[x]=\begin{pmatrix} f^0[x] \\ f^1[x] \\ \vdots \\ f^n[x] \end{pmatrix}$$

where $R_0(x)=\{x\}$ and $R_i(x)=\{y\in W/xR_iy\}$. Since R_i is an equivalence relation, then $R_i(x)$ is just the equivalence class of R_i determined by x. Now define $W'=W\times I$ and for $x\in W$ and $f\in I$ let $f(x)=(f,x)$. Obviously the conditions (I1) and (I2) from the definition of copying are fulfilled. Then each element of W' is in the form f(x) for some $f\in I$ and $x\in W$. Now we define the relations R'_{ij} and R'_i, $i,j\le n$, as follows. Let $f,g\in I$ and $x,y\in W$, then:

(R'_{ij}) $f(x)R'_{ij}g(y)$ iff $xR_{ij}y$ & $f[x]=g[y]$ & $f_i=g_j$,

(R'_i) $f(x)R'_ig(y)$ iff xR_iy & $f[x]=g[y]$ & $(\forall k\neq i)$ $f_i=g_i$.

The remaining part of the **lemma** follows from the next lemmas.

Lemma 2.3.

The relations R'_{ij} and R'_i satisfy the conditions of copying.

Proof. For the conditions of the type (R2) the assertion is obvious. Now we will proceed to prove the condition (R_i2):

(R_i2) $xR_iy \longrightarrow (\forall f\in I)(\exists g\in I)f(x)R'_ig(y)$.

For that purpose it is enough to prove the implication:

$xR_iy \longrightarrow (\forall f\in I)(\exists g\in I)f[x]=g[y]$ & $(\forall k\neq i)(f_k=g_k)$

Suppose xR_iy. Then we have $R_i(x)=R_i(y)$. Let $f\in I$ be given. Put for $k\neq i$ $g_k=f_k$. In this way we have defined all columns of g except g_i. The equality $f[x]=g[y]$ can be considered as a system of n linear equations for the unknown elements of g_i. The j-th equation is $f^j[x]=g^j[y]$, or in more details

$f^j_1+..+f^j_{j-1}+f^j_{j+1}+..+f^j_i..+f^j_n+R_j(x)=g^j_1+..+g^j_{j-1}+g^j_{j+1}+..+g^j_i..+g^j_n+R_j(y)$

In this equation the only unknown element is g^j_i. Since B(W) is a ring we can solve this equation with respect to g^j_i. The same can be done with the other equations, except the i-th one:

$f^i_1+..+f^i_{i-1}+f^i_{i+1}+..+f^i_n+R_i(x)=g^i_1+..+g^i_{i-1}+g^i_{i+1}+..+g^i_n+R_i(y)$.

This equation do not contain unknown elements, but nevertheless it is true, because for all $k\neq i$ we have $f^i_k=g^i_k$ and $R_i(x)=R_i(y)$.

The proof of the condition $(R_{ij}1)$ is more easy and can be done in the same way. ■

Lemma 2.4.

The structure $\underline{W}'$ is a standard frame for CAL^n.

Proof. The proof that $\underline{W}'$ is a nonstandard frame for CAL^n is straightforward and follows directly from the definitions of R'_{ij} and R'_i. We shall show the condition of normality and the condition (St. i) for $i\leq n$.

For the condition of normality let $f,g\in I$, $x,y\in W$ and suppose that for any $i\leq n$ we have $f(x)R'_{ii}g(y)$. Then we have $f[x]=g[y]$ and $\forall i\leq n$ $f_i=g_i$. From here we obtain $f=g$ and from the first equation of $f[x]=g[y]$ we obtain $\{x\}=\{y\}$. Consequently $x=y$ and $f(x)=f(y)$ - the normality is proven.

For the condition (St i) suppose that for any $j\neq i$ $f(x)R'_{jj}g(y)$. and proceed to show that $f(x)R'_ig(y)$. From the assumption we obtain that for any $j\neq i$: $f[x]=g[y]$ and $f_j=g_j$. From here we obtain that for any $j\neq i$ $f^i_j=g^i_j$. Then from the i-th equality of $f[x]=g[x]$ we obtain $R_i(x)=R_i(y)$, so xR_iy. Then by the definition of R'_i we obtain $f(x)R'_ig(y)$, which ends the proof of the lemma. ■

Now the proof of theorem **2**.2. follows from lemma **2**.3. and lemma **2**.4. ■

Theorem 2.5. /Completeness theorem of CAL^n with respect to its standard semantics/

The following conditions are equivalent for any formula A of CAL^n:

(i) A is a theorem of CAL^n,

(ii) A is true in all standard frames for CAL^n.

Proof. The implication (i)$\longrightarrow$(ii) is trivial. To prove the converse implication (ii)$\longrightarrow$(i) suppose (ii) and for the sake of contradiction that A is not a theorem of CAL^n. Then by theorem **2**.1. A is falsified in some nonstandard frame $\underline{W}$ for CAL^n, that is, there exists valuation v in W and $x\in W$ such that

$x \Vdash_{\overline{v}} \not{} A$. Then by theorem **2**.2. there exist a standard frame $\underline{W}'$ and a copying I from $\underline{W}$ to $\underline{W}'$. Then by the copying lemma /a lemma like lemma 1.7, which is true for any copying/ there exists a valuation v' in $\underline{W}'$ such that for any f∈I $f(x) \Vdash_{\overline{v}} \not{} A$. This contradicts the assumption that A is true in all standard frames for CAL^n. ■

Corollary 2.6.
The condition (St i) $(\forall j\neq i)(xR_{jj}y) \longrightarrow xR_iy$ is not modally definable.

Now we will show the decidability of CAL^n, showing that it possesses f.m.p. For that purpose we shall use the method of filtration, applying to the nonstandard models of CAL^n.

Lemma 2.7./Filtration for CAL^n/
Let $\underline{W}=(W, \{R_{ij}/i,j\leq n\},\{R_i/i\leq n\})$ be a nonstandard frame for CAL^n, $M=(\underline{W}, v)$ be a model over $\underline{W}$ and A be a formula of CAL^n. Then there exist a finite set Γ of formulas, containing A and closed under subformulas and a filtration $M'=((|W|, \{R'_{ij}/ij\leq n\},\{R'_i/i\leq n\}),v')$ through Γ such that $\underline{W}'= (|W|, \{R'_{ij}/ij\leq n\}, \{R'_i/i\leq n\}))$ is a finite nonstandard frame for CAL^n, such that $Card|W|\leq 2^{(n^2+n).m}$, where m is the number of the subformulas of A.

Proof. Let the formula A be given and let Γ be the smallest set of formulas containing A, closed under subformulas and satisfying the following condition:
(γ) if [ij]A ∈ Γ then for any ij≤n [ij]A∈Γ.

Obviously Γ is a finite set of formulas, containing no more than $(n^2+n).m$ elements, where m is the number of subformulas of A. Define |W| and the valuation v' as in the definition of filtration and R'_{ij} as in lemma 1.4. The relations R'_i are defined as follows:

$|x|R'_i|y|$ iff $(\forall[i]B\in\Gamma)(x\Vdash_{\overline{v}}[i]B \longleftrightarrow y\Vdash_{\overline{v}}[i]B)$ & $(\forall k\neq i)|x|R_{kk}|y|$.

The proof that the conditions of the filtration are fulfilled and that $(|W|. \{R'_{ij}/ij\leq n\},\{R'_i/i\leq n\})$ is a nonstandard frame for CAL^n is similar to the proof of lemma 1.4, so we left this task to the reader. ■

Theorem 2.8. /Finite completeness theorem for CAL^n/
The following conditions are equivalent for any formula A of CAL^n:
(i) A is a theorem of CAL^n,
(ii) A is true in all finite nonstandard frames $\underline{W}$ of CAL^n with $CardW\leq 2^{(n^2+n).m}$, where m is the number of the subformulas of A,
(iii) A is true in all finite standard frames for CAL^n.

Proof. The implication (i)⟶(ii) is standard, and (ii)⟶(i) follows from lemma 2.7. (i)⟶(iii) is also standard, so we have (ii)⟶(iii). The implication (iii)⟶(ii) follows from lemma 2.2.(ii).■

Corollary 2.9.
(i) CAL^n possesses finite model property,
(ii) CAL^n is decidable.

The following theorem summarizes all completeness results for CAL^n.

Theorem 2.10.
The following conditions are equivalent for any formula A of CAL^n:
(i) A is a theorem of CAL^n,
(ii) A is true in all nonstandard frames for CAL^n,

(iii) A is true in all standard frames for CAL^n,

(iv) A is true in all finite nonstandard frames $\underline{W}$ for CAL^n such that $Card\underline{W} \le 2^{(n^2+n).m}$, where m is the number of subformulas of A,

(v) A is true in all finite standard frames for CAL^n.

3. Open problems

Some open problems have been formulated in the main text. Now we shall discuss some new problems.

1. Extend and axiomatize the logic BAL^n with propositional constants δ_{ij}, $ij \le n$ with the following semantics:

$x \Vdash_{v} \delta_{ij}$ iff $xR_{ij}y$

2. Extend and axiomatize the logic CAL^n with the constants δ_{ij}. The obtained logic will correspond to the cylindric modal logic of Venema [Ven 91]. The standard meaning of δ_{ij} is the equality of i-th and j-th coordinates of a normal n-arrow $x=(x_1,\dots,x_n)$.

3. Extend and axiomatize the logic BAL^n with modalities corresponding to arbitrary intersections of the relations R_{ij}. For instance the $\langle R\rangle$ modality corresponding to the intersection

$R=R_{i_1 i_1} \cap \dots \cap R_{i_k i_k}$

will have the meaning of a quantification prefix, which can be obtained from the prefix $\exists x_1 \dots \exists x_n$ by omitting all quantifiers of the form $\exists x_{i_j}$, $j=1,\dots,k$.

If $R=R_{ij} \cap R_{ji} \cap \bigcap_{k=1,k\ne i,j}^{n} R_{kk}$, then the modality $\langle R\rangle$ will correspond to the permutation of i-th and j-th coordinates of a normal n-arrow x.

CHAPTER 4.4.

HYPER ARROW STRUCTURES AND HYPER ARROW LOGICS OF DIMENSION N

Overview. In this chapter we generalize the notion of n-dimensional arrow structure putting for each $i \le n$ $i.x$ to be not a single point but a set of points. For n=2 this means that each point $A \in 1.x$ is a beginning of x and each $B \in 2.x$ is an end of x. The obtained notion is called hyper arrow structure of dimension n. To each such a structure we associate a relational system over the set of arrows, called hyper arrow frame of dimension n. By means of a kind of characterization theorem we give an abstract definition of hyper arrow frames of dimension n. The corresponding modal logic, called Basic Hyper Arrow Logic of dimension n - $BHAL^n$ - is introduced. A completeness theorem with respect to its standard semantics is proved by the method of copying. $BHAL^n$ possesses finite model property and is decidable.

Introduction

In this chapter we generalize the notion of arrow structure of dimension n putting for each arrow x and i≤n i.x to be not a single point but a set of points. The resulting notion is called hyper arrow structure of dimension n. For n=2 this is a generalization of the notion of multigraph and for n=1 this is a slight generalization of the notion of hypergraph. Hyper arrow structures of dimension n are used as a semantical basis for a modal logic called Basic Arrow Logic of Dimension n - $BHAL^n$. The chapter consists of two sections.

In sec. 1 we study the notion of hyper arrow structure of dimension n and a corresponding relational system, called hyper arrow frame of dimension n. In some sense hyper arrow frames of dimension n contain the information of hyper arrow structure of dimension n up to the so called Boolean isomorphism.

The main result of the section is an abstract characterization of hyper arrow frames of dimension n. This is done by developing a kind of n-dimensional filters and ideals in hyper arrow frames of dimension n, which generalizes the Stone theory for filters and ideals in distributive lattices and Boolean algebras.

In section 2 we introduce the Basic Arrow Logic of Dimension n - $BHAL^n$ and proof the completeness theorem with respect to the class of hyper arrow frames of dimension n. In the completeness proof we have to apply a non-trivial copying construction. We proof decidability of $BHAL^n$ showing that it possesses finite model property with respect to a class of non-standard models.

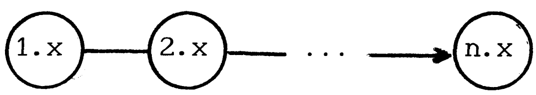
Let us mention that $BHAL^1$ is a slight modification of the logic IND-1 - a modal logic for property systems from chapter 2.4. This shows that there exist closed connections between arrow logics and modal logics for information systems, developed in part II.

1. Hyper arrow structures of dimension n

In this section we will generalize the notion of an n-arrow structure in the following way. A system S=(Po, Ar, (n), .) is called hyper arrow structure of dimension n (hyper n-arrow structure, for short) if

- Po≠∅ is a set, whose elements are called points,
- Ar≠∅ is a set whose elements are called arrows,
- (n)={1,2,...,n}
- . is a total operation, which for each i∈(n) and x∈Ar assigns a subset i.x⊆Ar.

Graphically:



Let A∈i.x, if i=1 then A is called beginning of x, if i=n then A is called end of x and if 1<i<n then A is called intermediate point of x. Obviously an arrow x may have many or zero beginnings (intermediate points, ends).

The difference from the notion of n-arrow structure is that now i.x is not a point but a set of points. For n=1 the notion of a hyper 1-arrow structure is a slight generalization of the notion of hypergraph (see Claude Berge [Ber 89]). For n=1 it just coincides with the notion of Property system with Ob=Ar, Pr=Po and f(x)=1.x.

There is a closed connection between hyper n-arrow structures and attribute systems, studied in part two. Namely, let S be a hyper n-arrow structure. Define an attribute system S' as follows. Put $Ob_{S'}=Ar_{S'}$, $AT_{S'}=(n)$, for i∈(n) define VALi={A∈Po_S/ (∃x∈Ar_S) A∈i.x} and for x∈Ar_S and i∈(n) define

$f_{S'}(x,i)=i.x$. Obviously S' is an attribute system. Conversely, let S' be an attribute system with finite number of attributes: $AT_{S'}=\{a_1,\dots,a_n\}$. then S' determines a hyper n-arrow structure S as follows. Put $Ar_S=Ob_{S'}$, $Po_S=\bigcup\{VALa/a\in AT_{S'}\}$, for $i\in(n)$ and $x\in Ob_{S'}$ define $i.x=f(x,a_i)$. Obviously S is a hyper n-arrow structure. This connection between Attribute systems and hyper n-arrow structures suggests many analogies between the theory of attribute systems and the theory of arrow systems.

Another example of hyper n-arrow structure is the following. Let $W\neq\emptyset$ be a set and ρ be a nonempty n-place relation in the power set 2^W of W i.e. $\rho\subseteq(2^W)^n$. Put Po=W, Ar=ρ and for $x=(\alpha_1,\dots,\alpha_n)\in\rho$ and $i\in(n)$ define $i.x=\alpha_i$. Then obviously (Po, Ar, (n), .) is a hyper n-arrow structure. This example shows that the theory of hyper n-arrow structures may have some implications to the theory of set-relations /relations in power set/.

Let W be a set. By a Boolean relation between two subsets α and β of W we mean any relation R in 2^W for which there exists a Boolean term B(x,y) and $R(\alpha,\beta)\longleftrightarrow B(\alpha,\beta)=\emptyset$ or $R(\alpha,\beta)\longleftrightarrow B(\alpha,\beta)\neq\emptyset$. In chapter 2.2 sec. 2 we show that each Boolean relation of this kind can be expressed by the following three relations:

(*) $\alpha\cap\beta\neq\emptyset$, $\bar{\alpha}\cap\bar{\beta}\neq\emptyset$ and $\alpha\subseteq\beta$, $\bar{\alpha}=W-\alpha$.

We shall use this notions in the following definition.

Let S and S' be two hyper n-arrow structures. A mapping from Ar_S onto $Ar_{S'}$ is called a Boolean isomorphism from S onto S' if f is one-one mapping and for any Boolean relation $R(\alpha,\beta)$, $x,y\in Ar_S$ and $i,j\in(n)$ we have

$R(i.x,j.y)$ iff $R(i.f(x),j.f(y))$

In the above equivalence instead of all Boolean relations we may assume only the relations from (*).

Roughly speaking, Boolean isomorfism treads two hyper n-arrow structures as identical if their pictures as Venn diagrams coincides.

In the theory of n-arrow structures, the n-arrow frame $(Ar_S, \{R_{ij}/i,j\in(n)\})$ determines S up to an isomorfism. We shall introduce now a similar arrow frame, associated with each hyper n-arrow structure, which will determine S up to a Boolean isomorphism.

Let S be a hyper n-arrow structure. Define in Ar_S the following relations: R^S_{ij}, Σ^S_{ij}, N^S_{ij}, $\leq^S_{ij}$, $x,y\in Ar_S$ $i,j\in(n)$

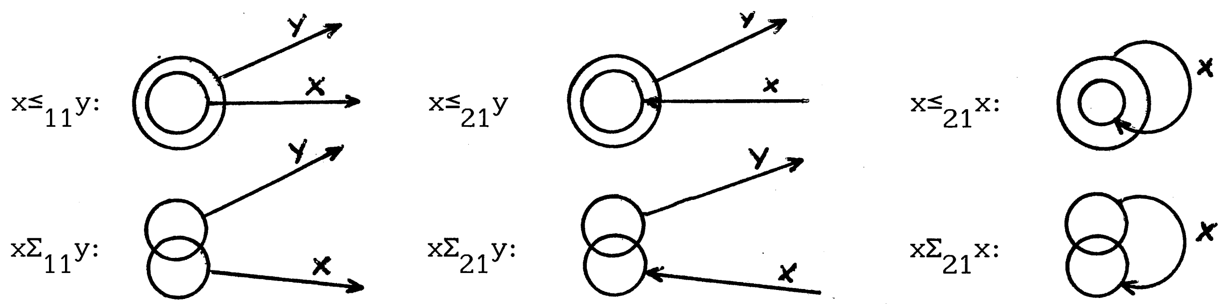
$xR^S_{ij}y$ iff $i.x=j.y$,

$x\Sigma^S_{ij}y$ iff $(i.x)\cap(j.y)\neq\emptyset$,

$xN^S_{ij}y$ iff $\overline{(i.x)}\cap\overline{(j.x)}\neq\emptyset$, where $\overline{(i.x)}=Po-(i.x)$

$x\leq^S_{ij}y$ iff $i.x\subseteq j.y$.

Graphical examples (for n=2):



Obviously we have $xR^S_{ij}y$ iff $x\leq^S_{ij}y$ & $y\leq_{ji}x$.

Sometimes, for simplicity, we will omit the superscript S.

Lemma 1.1.

The relations Σ_{ij}, N_{ij} and $\leq_{ij}$ satisfies the following first-order conditions for any $x,y,z\in Ar$ and $i,j,k\in(n)$:

S1 $x\leq_{ii}x$,

S2 $x\leq_{ij}y$ & $y\leq_{jk}z \longrightarrow x\leq_{ik}z$

S3 $x\Sigma_{ij}y \longrightarrow y\Sigma_{ji}x$,

S4 $x\Sigma_{ij}y \longrightarrow x\Sigma_{ii}x$,

S5 $x\Sigma_{ij}y$ & $y\leq_{jk}z \longrightarrow x\Sigma_{ik}z$,

S6 $x\overline{\Sigma}_{ii}x \longrightarrow x\leq_{ij}y$,

S7 $xN_{ij}y \longrightarrow yN_{ji}x$,

S8 $xN_{ij}y \longrightarrow xN_{ii}x$,

S9 $x\leq_{ij}y$ & $yN_{jk}z \longrightarrow xN_{ik}z$,

S10 $y\overline{N}_{jj}y \longrightarrow x\leq_{ij}y$,

S11 $x\leq_{ij}y$ or $x\Sigma_{ik}z$ or $yN_{jk}z$,

S12 $x\Sigma_{ii}x$ or $xN_{ii}x$.

Proof - straightforward verification. ∎

Lemma 1.1 suggests the following definition.

Let $\underline{W}=(W, \{\leq_{ij},\Sigma_{ij},N_{ij}/i,j\leq n\})$, $W\neq\emptyset$, be a relational system. $\underline{W}$ will be called a hyper arrow frame of dimension n (hyper n-arrow frame, for short) if it satisfies the conditions S1-S12 from lemma 1.1 for any $i,j,k\leq n$ and $x,y,z\in W$. $\underline{W}$ will be called a standard n-arrow frame if there exists a hyper n-arrow structure S such that $W=Ar_S$ and the relations $\leq_{ij},\Sigma_{ij}$ and N_{ij} coincide with the relations $\leq^S_{ij}$, Σ^S_{ij} and N^S_{ij} respectively. The standard hyper n- arrow frame over some hyper n-arrow structure S will be denoted by $\underline{W}(S)$.

Our aim in this section is to prove that each hyper n-arrow frame is a standard hyper n-arrow frame over some hyper n-arrow structure. We will show also that the standard hyper n-arrow frame over an hyper n-arrow structure S describes S up to Boolean isomorphism.

Let $\underline{W}$ be a hyper n-arrow frame and $\alpha=(\alpha_1,\dots,\alpha_n)$, $\beta=(\beta,\dots,\beta)$, $\alpha_i,\beta_i\subseteq W, i\leq n$. We define

$\alpha\subseteq\beta$ iff $\forall i\leq n\ \alpha_i\subseteq\beta_i$, $\alpha\cup\beta=(\alpha_1\cup\beta_1,\dots,\alpha_n\cup\beta_n)$, $(\alpha\cup\beta)_i=\alpha_i\cup\beta_i$,
$\alpha\cap\beta=(\alpha_1\cap\beta_1,\dots,\alpha_n\cap\beta_n)$, $(\alpha\cap\beta)_i=\alpha_i\cap\beta_i$, $-\alpha=(-\alpha_1,\dots,-\alpha_n)$, $(-\alpha)_i=-\alpha_i$,
where $-\alpha_i=W-\alpha_i$; $\mathbf{0}=(\emptyset,\dots,\emptyset)$ and $\mathbf{1}=(W,\dots,W)$.

In an obvious way we may define also arbitrary unions and intersections of such n-tuples of subsets of W.

Let $\alpha=(\alpha_1,\dots,\alpha_n)$, $\alpha_i\subseteq W$. We say that α is a prefilter in $\underline{W}$ if the following condition is satisfied for any $x,y\in W$ and $i,j\leq n$:

(Φ0) if $x\in\alpha_i$ and $x\leq_{ij}y$ then $y\in\alpha_j$,

We say that α is a filter if it is a prefilter and for any $x,y\in W$ and $i,j\leq n$ we have

(Φ1) if $x\in\alpha_i$ and $y\in\alpha_j$ then $x\Sigma_{ij}y$.

We say that α is a preideal in W if for any $x,y\in W$ and $i,j\le n$ the following is satisfied

(I0) if $y\in\alpha_j$ and $x\le_{ij}y$ then $x\in\alpha_i$.

We say that α is an ideal if it is a preideal and for any $x,y\in W$ and $i,j\le n$ we have

(I1) if $x\in\alpha_i$ and $y\in\alpha_j$ then $xN_{ij}y$.

Obviously $\mathbf{0}=(\emptyset,\ldots,\emptyset)$ is both a filter and an ideal.

We say that α is a prime filter if it is a filter and $-\alpha$ is an ideal; α is a prime ideal if it is an ideal and $-\alpha$ is a filter. From this definition it follows that a filter (ideal) α is a prime filter (ideal) if it satisfies the following condition

(Φ3) if $x\notin\alpha_i$ and $y\notin\alpha_j$ then $xN_{ij}y$.

((I3) if if $x\notin\alpha_i$ and $y\notin\alpha_j$ then $x\Sigma_{ij}y$)

The set of all prime filters of a given hyper n-arrow frame $\underline{W}$ will be denoted by $PF(\underline{W})$.

In the rest of this section we will develop a theory of filters and ideals which will generalize the Stone theory of filters and ideals of distributive lattices ([Sto 37]).

We will use the following notations:

$x\ge_{ij}y$ iff $y\le_{ji}x$,

If R is any binary relation in W we denote $R(x)=\{y\in W/xRy\}$. Using this notation we introduce the following operations on elements of W producing prefilters and preideals:

$\le_k[a]=(\le_{k1}(a),\ldots,\le_{kn}(a))$, $\ge_k[a]=(\ge_{k1}(a),\ldots,\ge_{kn}(a))$, $k\le n$.

$(\le_k[a])_i=\le_{ki}(a)$, $(\ge_k[a])_i=\ge_{ki}(a)$, $k,i\le n$.

Lemma 1.2

(i) $\le_k[a]$ is a prefilter containing a in its k component, i.e. $a\in(\le_k[a])_k$.

(i') $\ge_k[a]$ is a preideal containing a in its k component, i.e. $a\in(\ge_k[a])_k$.

(ii) If α is a prefilter and $a\in W$ then $\alpha\cup\le_k[a]$ is a prefilter extending α and containing a in its k component. In particular $\le_i[a]\cup\le_j[b]$ is a prefilter such that $a\in(\le_i[a]\cup\le_j[b])_i$ and $b\in(\le_i[a]\cup\le_j[b])_j$.

(ii') If α is a preideal and $a\in W$ then $\alpha\cup\ge_k[a]$ is a preideal extending α and containing a in its k component. In particular $\ge_i[a]\cup\ge_j[b]$ is a preideal such that $a\in(\ge_i[a]\cup\ge_j[b])_i$ and $b\in(\ge_i[a]\cup\ge_j[b])_j$.

(iii) If $a\Sigma_{ij}b$ then $\le_i[a]\cup\le_j[b]$ is a filter such that $a\in(\le_i[a]\cup\le_j[b])_i$ and $b\in(\le_i[a]\cup\le_j[b])_j$.

(iii') If $aN_{ij}b$ then $\ge_i[a]\cup\ge_j[b]$ is an ideal such that $a\in(\ge_i[a]\cup\ge_j[b])_i$ and $b\in(\ge_i[a]\cup\ge_j[b])_j$.

Proof - straightforward. ■

Lemma 1.3

(i) If α is a filter and $a\in W$ then the prefilter $\alpha\cup\le_k[a]$ is a filter iff $a\Sigma_{kk}a$ and for any $i,j\le n$ and $x,y\in W$: if $x\in\alpha_i$ and $a\le_{kj}y$ then $x\Sigma_{ij}y$.

(i') If α is an ideal and $a\in W$ then the preideal $\alpha\cup\ge_k[a]$ is an ideal iff $aN_{kk}a$ and for any $i,j\le n$ and $x,y\in W$: if $x\in\alpha_i$ and $a\ge_{kj}y$ then $xN_{ij}y$.

Proof. (i) ($\longrightarrow$) Suppose that α is and $\alpha\cup\leq_k[a]$ are filters. Then $a\in(\alpha\cup\leq_k[a])_k$ and hence $a\Sigma_{kk}a$. Let $x\in\alpha_i$ and $a\leq_{kj}y$. Then $x\in(\alpha\cup\leq_k[a])_i$, $y\in(\leq_k[a])_j$ and consequently $y\in(\alpha\cup\leq_k[a])_j$. But $x\in(\alpha\cup\leq_k[a])_i$ and $y\in(\alpha\cup\leq_k[a])_j$ implies $x\Sigma_{ij}y$, which have to be proved.

(ii) ($\longleftarrow$) Suppose α is a filter, $a\Sigma_{kk}a$ and for any $i,j\leq n$ and $x,y\in W$: if $x\in\alpha_i$ and $a\leq_{kj}y$ then $x\Sigma_{ij}y$. By lemma 1.2(ii) $\alpha\cup\leq_k[a]$ is a prefilter, so it remains to show that if $x\in(\alpha\cup\leq_k[a])_i$ and $y\in(\alpha\cup\leq_k[a])_j$ then $x\Sigma_{ij}y$. We have to consider four cases:

Case 1: $x\in\alpha_i$ and $y\in\alpha_j$. Then $x\Sigma_{ij}y$, because α is a filter.

Case 2: $x\in\alpha_i$, $y\in(\leq_k[a])_j$. Then we have $a\leq_{kj}y$ and by the assumption we obtain $x\Sigma_{ij}y$.

Case 3: $x\in(\leq_k[a])_i$, $y\in(\leq_k[a])_j$. Then we have $a\leq_{ki}x$ and $a\leq_{kj}y$ and by $a\Sigma_{kk}a$ we get $x\Sigma_{ij}y$.

Case 4: $x\in(\leq_k[a])_i$, $y\in\alpha_j$. Then we have $a\leq_{ki}x$ and by the assumption we obtain $y\Sigma_{ji}x$, which by axiom S3 implies $x\Sigma_{ij}y$.

The proof of (ii) is similar.■

Lemma 1.4

Let α be a filter, β be an ideal, $\alpha\cap\beta=\mathbf{0}$, $k\leq n$. Then for any $a\in W$, such that $a\notin\alpha_k$ and $a\notin\beta_k$ we have

(i) $(\alpha\cup\leq_k[a])\cap\beta=\mathbf{0}$ and $\alpha\cap(\beta\cup\geq_k[a])=\mathbf{0}$,

(ii) either $\alpha\cup\leq_k[a]$ is a filter or $\beta\cup\geq_k[a]$ is an ideal.

Proof. (i) First we will show that $(\alpha\cup\leq_k[a])\cap\beta=\mathbf{0}$. Suppose that this is not true. Then for some $i\leq n$ we will have $(\alpha\cup\leq_k[a])_i\cap\beta_i\neq\emptyset$ so for some $x\in W$ we have $x\in\beta_i$ and ($x\in\alpha_i$ or $x\in(\leq_k[a])_i$). Since $\alpha\cap\beta=\mathbf{0}$ $x\in\alpha_i$ and $x\in\beta_i$ is impossible, hence we have $x\in(\leq_k[a])_i$, so $a\leq_{ki}x$. Since β is an ideal then from $x\in\beta_i$ and $a\leq_{ki}x$ we obtain that $a\in\beta_k$ - a contradiction with the assumption on a. In the same way one can prove that $\alpha\cap(\beta\cup\geq_k[a])=\mathbf{0}$.

(ii) Suppose that neither $\alpha\cup\leq_k[a]$ is a filter nor $\beta\cup\geq_k[a]$ is an ideal and proceed to obtain a contradiction. Applying lemma 1.3 we obtain the following two conditions (*) and (**):

(*) either (1) $a\overline{\Sigma}_{kk}a$ or (2) $\exists x,y\in W$, $\exists i,j\leq n$ $x\in\alpha_i$ $a\leq_{kj}y$ and $x\overline{\Sigma}_{ij}y$,

(**) either (1') $a\overline{N}_{kk}a$ or (2') $\exists u,v\in W$, $\exists p,q\leq n$ $u\in\beta_p$ $a\geq_{kq}v$ and $u\overline{N}_{pq}v$.

We have to combine (1) and (2) from (*) with (1') and (2') from (**). Hence we have to consider four cases

Case (11'): $a\overline{\Sigma}_{kk}a$ and $a\overline{N}_{kk}a$. This case is impossible because it contradicts axiom S12.

Case (12') $a\overline{\Sigma}_{kk}a$, $u\in\beta_p$, $a\geq_{kq}v$, $u\overline{N}_{pq}v$. By axiom S6 we obtain $a\leq_{kp}u$ and since β is an ideal and $u\in\beta_p$ this implies $a\in\beta_k$ - a contradiction with the assumption for a.

Case (1'2) $a\overline{N}_{kk}a$, $x\in\alpha_i$, $a\leq_{kj}y$, $x\overline{\Sigma}_{ij}y$. By axiom S10 we obtain $x\leq_{ik}a$ and since α is a filter and $x\in\alpha_i$ this implies $a\in\alpha_k$ - a contradiction.

Case (22') $x\in\alpha_i$ $a\leq_{kj}y$, $x\bar{\Sigma}_{ij}y$, $u\in\beta_p$,$a\geq_{kq}v$, $u\bar{N}_{pq}v$. From $a\leq_{kj}y$, $x\bar{\Sigma}_{ij}y$ we obtain by S5 $x\bar{\Sigma}_{ik}a$. From $a\geq_{kq}v$, $u\bar{N}_{pq}v$ we obtain by S9 $u\bar{N}_{pk}a$. From $x\bar{\Sigma}_{ik}a$ and $u\bar{N}_{pk}a$ we obtain by S11 $x\leq_{ip}u$. Since α is a filter then from $x\leq_{ip}u$ and $x\in\alpha_i$ we obtain $u\in\alpha_p$. But we have also that $u\in\beta_p$, which implies that $\alpha_p\cap\beta_p\neq\emptyset$ and consequently - $\alpha\cap\beta\neq\mathbf{0}$, which contradicts the assumption $\alpha\cap\beta=\mathbf{0}$. This completes the proof of the lemma.∎

Let N be a set of filters we say that N is a chain if for any $\alpha,\beta\in N$ we have $\alpha\subseteq\beta$ or $\beta\subseteq\alpha$.

Lemma 1.5.

The union of any nonempty chain of filters (ideals) is a filter (ideal).

Proof - straightforward verification.

Theorem 1.6. (Separation theorem for filters and ideals)

Let $\underline{W}=(W, \{\leq_{ij},\Sigma_{ij},N_{ij}/i,j\leq n\})$ be a hyper n-arrow frame, α' be a filter in $\underline{W}$, β' be an ideal in $\underline{W}$ and $\alpha'\cap\beta'=\mathbf{0}$. Then there exist a prime filter α and a prime ideal β in $\underline{W}$ such that $\alpha'\subseteq\alpha$, $\beta'\subseteq\beta$ and $\alpha\cap\beta=\mathbf{0}$.

Proof. Let $M_1=\{\gamma/y$ is a filter, $\alpha'\subseteq\gamma$ and $\gamma\cap\beta=\mathbf{0}\}$. Applying lemma 1.5 we can see that M_1 satisfies the conditions of the Zorn lemma and hence contains a maximal element, say α. Let $M_2=\{\delta/\delta$ is an ideal, $\beta'\subseteq\delta$ and $\alpha\cap\delta=\mathbf{0}\}$. M_2 also satisfies the conditions of the Zorn lemma and contains a maximal element, say β. So $\alpha\cap\beta=\mathbf{0}$, α is a filter, β is an ideal. We will show that α is a prime filter and β is a prime ideal. Since $\alpha\cap\beta=\mathbf{0}$ it remains to show that $\alpha\cup\beta=\mathbf{1}=(W,\dots,W)$. Suppose that $\alpha\cup\beta\neq\mathbf{1}$. Then there exists $k\leq n$ such that $\alpha_k\cup\beta_k\neq W$, i.e. that there exists $a\in W$ such that $a\notin\alpha_k$ and $a\notin\beta_k$. Define $\alpha''=\alpha\cup\leq_k[a]$ and $\beta''=\beta\cup\geq_k[a]$. By lemma 1.5 ether α'' is a filter or β'' is an ideal and $\alpha''\cap\beta=\mathbf{0}$ and $\alpha\cap\beta''=\mathbf{0}$. We will consider two cases.

Case 1: α'' is a filter. Since α'' is a proper extension of α ($a\notin\alpha_k$), then by the maximality of α in M_1 we have $\alpha''\cap\beta'\neq\mathbf{0}$. But $\beta'\subseteq\beta$, so $\alpha''\cap\beta\neq\mathbf{0}$ - a contradiction.

Case 2: β'' is an ideal. Since β'' is a proper extension of β ($a\notin\beta_k$), then by the maximality of β in M_2 we have that $\alpha\cap\beta''\neq\mathbf{0}$ - a contradiction. This completes the proof of the theorem.∎

Now we are ready to formulate the main theorem of this section.

Lemma 1.7

Let $\underline{W}=(W, \{\leq_{ij},\Sigma_{ij},N_{ij}/i,j\leq n\})$ be a hyper n-arrow frame and $PF(\underline{W})$ be the set of prime filters of $\underline{W}$. Then for any $x,y\in W$ and $i,j\leq n$ the following is true:

(i) $x\leq_{ij}y \longleftrightarrow (\forall\alpha\in PF(\underline{W})(x\in\alpha_i \longrightarrow y\in\alpha_j)$,

(ii) $x\Sigma_{ij}y \longleftrightarrow (\exists\alpha\in PF(\underline{W})(x\in\alpha_i$ and $y\in\alpha_j)$,

(iii) $xN_{ij}y \longleftrightarrow (\exists\alpha\in PF(\underline{W})(x\notin\alpha_i$ and $y\notin\alpha_j)$.

Proof. (i)($\longrightarrow$) Suppose $x\leq_{ij}y$, $\alpha\in PF(\underline{W})$ and $x\in\alpha_i$. Then by the definition of filter $y\in\alpha_j$.

($\longleftarrow$) In this case we will reason by contraposition: suppose $x\not\leq_{ij}y$ and prove that there exists a prime filter α such that $x\in\alpha_i$ and $y\notin\alpha_j$.

From $x\not\leq_{ij}y$ we obtain by axiom S6 and S10 that $x\Sigma_{ii}x$ and $yN_{jj}y$. Then by lemma 1.2.(iii) and (iv) we have that $x\in(\leq_i[x])_i$, $y\in(\geq_j[y])_j$, $\leq_i[x]$ is a filter and $\geq_j[y]$ is an ideal. We will show that $(\leq_i[x])\cap(\geq_j[y])=\mathbf{0}$. Suppose that this is not the case. Then for some $k\leq n$ $(\leq_i[x])_k\cap(\geq_j[y])_k\neq\mathbf{0}$, so there exists $z\in W$ such that $x\leq_{ik}z$ and $y\geq_{jk}z$. Then by S1 we obtain $x\leq_{ij}y$, which contradicts the assumption $x\not\leq_{ij}y$.

Now we have $(\leq_i[x])\cap(\geq_j[y])=\mathbf{0}$, $(\leq_i[x])$ is a filter and $(\geq_j[y])$ is an ideal. Then by the separation theorem there exist a prime filter α and a prime ideal β such that $(\leq_i[x])\subseteq\alpha$, $(\geq_j[y])\subseteq\beta$ and $\alpha\cap\beta=\mathbf{0}$. From here we obtain $x\in\alpha$, $y\in\beta$ and consequently $y\notin\alpha$.

(ii) $(\longrightarrow)$ Suppose $x\Sigma_{ij}y$. Then by lemma 1.2.(iii) $\alpha'=\leq_i[x]\cup\leq_j[y]$ is a filter such that $x\in\alpha_i$ and $y\in\alpha_j$. Let $\beta'=\mathbf{0}$ - β' is an ideal and $\alpha'\cap\beta'=\mathbf{0}$. Then by the separation theorem there exist a prime filter α and a prime filter β such that $\alpha'\subseteq\alpha$, $\beta'\subseteq\beta$ and $\alpha\cap\beta=\mathbf{0}$. Since $x\in\alpha'_i$ and $y\in\alpha'_j$ and $\alpha'\subseteq\alpha$, then $x\in\alpha_i$ and $y\in\alpha_j$.

$(\longleftarrow)$ Suppose that for some prime filter α we have $x\in\alpha_i$ and $y\in\alpha_j$. Then by the definition of a filter we have $x\Sigma_{ij}y$.

(iii) $(\longrightarrow)$ Suppose $xN_{ij}y$. Then by lemma 1.2.(iv) $\beta'=\geq_i[x]\cup\geq_j[y]$ is an ideal such that $x\in\beta'_i$ and $y\in\beta'_j$. Let $\alpha'=\mathbf{0}$ - α' is a filter and $\alpha'\cap\beta'=\mathbf{0}$. By the separation theorem there exist a prime filter α extending α' and a prime ideal β, extending β', such that $\alpha\cap\beta=\mathbf{0}$. Since $x\in\beta'_i$, $y\in\beta'_j$ and $\beta'\subseteq\beta$ we have $x\in\beta_i$, $y\in\beta_j$ and by $\alpha\cap\beta=\mathbf{0}$ we obtain that $x\notin\alpha_i$ and $y\notin\alpha_j$, which have to be proved.

$(\longleftarrow)$ Suppose that for some prime filter α $x\notin\alpha_i$ and $y\notin\alpha_j$. Then by the definition of a prime filter we obtain $xN_{ij}y$. This completes the proof of the lemma.■

Theorem 1.8.
Let $\underline{W}=(W, \{\leq_{ij},\Sigma_{ij},N_{ij}/i,j\leq n\})$ be a hyper n-arrow frame. Then there exists a hyper n-arrow structure $S=S(\underline{W})$ such that $Ar_S=W$ and $\leq_{ij}=\leq^S_{ij}$, $\Sigma_{ij}=\Sigma^S_{ij}$, $N_{ij}=N^S_{ij}$, $i,j\leq n$. In other words, each hyper n-arrow frame is a standard hyper n-arrow frame over some hyper n-arrow structure.

Proof. We define $S=S(\underline{W})$ as follows. Put $Ar_S=W$, $Po_S=PF(\underline{W})$ - the set of prime filters of $\underline{W}$. For $x\in W$ and $i\leq n$ define

$i.x=\{\alpha\in PF(\underline{W})/x\in\alpha_i\}$.

The remaining part of the proof follows from the following

Lemma 1.9.
For any $x,y\in W$ and $i,j\leq n$ the following is true:

(i) $x\leq_{ij}y \longleftrightarrow (i.x)\subseteq(j.y) \longleftrightarrow x\leq^S_{ij}y$

(ii) $x\Sigma_{ij}y \longleftrightarrow (i.x)\cap(j.y)\neq\emptyset \longleftrightarrow x\Sigma^S_{ij}y$,

(iii) $xN_{ij}y \longleftrightarrow (i.x)\cap(j.y)\neq\emptyset \longleftrightarrow xN^S_{ij}y$.

Proof. (i) Applying lemma 1.7.(i) and the definition of $\leq^S_{ij}$ we obtain:

$x\leq_{ij}y \longleftrightarrow (\forall\alpha\in PF(\underline{W})(x\in\alpha_i \longrightarrow y\in\alpha_j) \longleftrightarrow (\forall\alpha\in PF(\underline{W})(\alpha\in i.x \longrightarrow \alpha\in j.x) \longleftrightarrow (i.x)\subseteq(j.y) \longleftrightarrow x\leq^S_{ij}y$.

The proof of (ii) and (iii) is similar. ■

This completes the proof of the theorem. ■

The hyper n-arrow structure $S(\underline{W})$ over the hyper n-arrow frame $\underline{W}$, constructed in theorem 1.8 will be called canonical n-arrow structure over $\underline{W}$.

Theorem 1.10

Let S be a hyper n-arrow structure and let $\underline{W}(S)$ be the standard hyper n-arrow frame over S and let $S(\underline{W}(S))$ be the canonical hyper n-arrow structure over $\underline{W}(S)$. Then S and $S(\underline{W}(S))$ are Boolean isomorphic.

Proof. Let S=(Po, Ar, (n), .) be a hyper n-arrow structure. Then $\underline{W}(S)=(Ar, \{\leq^S_{ij}, \Sigma^S_{ij}, N^S_{ij}/i,j\leq n\})$ is the hyper n-arrow frame over S. The canonical hyper n-arrow structure $S'=S(\underline{W}(S))$over $\underline{W}(S)$ is the following: $S'=(PF(\underline{W}(S)), Ar, (n), \bullet)$ where for $i\in(n)$ and $x\in Ar$ $i\bullet x=\{\alpha\in PF(\underline{W}(S))/x\in\alpha_i\}$. We will prove that the identity mapping from Ar_S to $Ar_{S'}=Ar_S$ is the required Boolean isomorphism. This will follow from the next lemma, which is another formulation of lemma 1.9.

Lemma 1.11.

For any $x,y\in Ar$ and $i,j\in(n)$ the following is true:

(i) $i.x\subseteq j.y \longleftrightarrow (i\bullet x)\subseteq(j\bullet y)$,

(ii) $\underline{(i.x)}\cap\underline{(j.y)}\neq\mathbf{0} \longleftrightarrow \underline{(i\bullet x)}\cap\underline{(j\bullet y)}\neq\mathbf{0}$,

(iii) $\overline{(i.x)}\cap\overline{(j.y)}\neq\mathbf{0} \longleftrightarrow \overline{(i\bullet x)}\cap\overline{(j\bullet y)}\neq\mathbf{0}$,

This ends the proof of the theorem.■

Theorem 1.10 says that hyper n-arrow frames are in some sense equivalent to hyper n-arrow structures - they contain the whole information of hyper n-arrow structures up to Boolean isomorphism.

Hyper n-arrow frames are suitable for modal semantics and in the next section we shall use them to define Hyper Arrow Logic of dimension n.

Let S and S' be two hyper n-arrow structures. In the rest of this section we will introduce the natural notion of an isomorphism from S into S'. A pair (f,g) of mappings $f:Po_S\longrightarrow Po_{S'}$ and $g:Ar_S\longrightarrow Ar_{S'}$ is called an isomorphism from S into S' if f and g are injective mappings and for any $A\in Po$, $x\in Ar$ and $i\in(n)$ the following equivalence holds:

$A\in i.x$ iff $f(A)\in i.g(x)$.

Theorem 1.12.

Let S be a hyper n-arrow structure and $S'=S(\underline{W}(S))$. Then there exists an isomorphism (f,g) from S into S'.

Let S=(Po, Ar, (n), .), then $S'=(PF(\underline{W}(S)), Ar, (n), \bullet)$, where for $x\in Ar$ and $i\in(n)$ $i\bullet x=\{\alpha\in PF(\underline{W}(S))/x\in\alpha_i\}$. Since $Ar_S=Ar_{S'}=Ar$ we put g to be the identity mapping id in Ar. To define f we will proceed as follows.

For $i\in(n)$ and $A\in Po$ define $i.A=\{x\in Ar/A\in i.x\}$. Then put

$f(A)=(1.A,\ldots,n.A)$

Now the proof of the theorem will follow from the following

Lemma 1.13.

(i) For any $A\in Po$ $f(A)\in PF(\underline{W}(S))$,

(ii) For any $A\in Po$, $x\in Ar$ and $i\in(n)$ the following equivalence hold:

$A\in i.x \longleftrightarrow f(A)\in i\bullet x$.

Proof. (i) Let $A\in Po$ then $f(A)=(1.A,\ldots,n.A)$. We will show that f(A) is a prime filter in $\underline{W}(S)$.

(Φ0) Let $x\in i.A$ and $x\leq^S_{ij}y$. Then $A\in i.x$ and $i.x\subseteq j.y$, hence $A\in j.y$ and $y\in j.A$.

(Φ1) Let $x\in i.A$ and $y\in j.A$. Then $A\in i.x$ and $A\in j.y$, so $(i.x)\cap(j.y)\neq\emptyset$ and hence $x\Sigma^S_{ij}y$.

(Φ3) Let $x\notin i.A$ and $y\notin j.A$. Then $A\notin i.x$, $A\notin j.y$ and hence $\overline{(i.x)}\cap\overline{(j.y)}\neq\emptyset$, so $xN^S_{ij}y$.

(ii) $f(A)\in i\bullet x \longleftrightarrow (1.A,\dots,n.A)\in\{\alpha\in PF(\underline{W}(S))/x\in\alpha_i\}\longleftrightarrow x\in i.A \longleftrightarrow A\in i.x$.

This ends the proof of the lemma and the proof of the theorem.■

Let us note that in the above proof f is an embedding. We can not prove in general that for any prime filter α in $\underline{W}(S)$ there exists a point $A\in Po$ such that $f(A)=\alpha$. All this is in accordance with the analogy with representation theory of distributive lattices and Boolean algebras.

2. BHAL^n - Basic Hyper Arrow Logic of Dimension n

In this section we will introduce the Basic Hyper Arrow logic of Dimension n - BHAL^n - semantically based on hyper n-arrow structures. The language of BHAL^n is an extension of the language of classical propositional logic with the following modalities:

[ij], [≤ij], [Σij], [Nij], $i,j\leq n$ and [U] - the universal modality, so this language is also an extension of the language of BAL^n■.

The standard frames of BHAL^n consists of the class of of all hyper n-arrow frames with relations R_{ij} $i,j\leq n$, definable by

(*) $xR_{ij}y \longleftrightarrow x\leq_{ij}y \;\&\; y\leq_{ij}x$

The relations R_{ij} are for the interpretations of the modalities [ij] and the relations $\leq_{ij}$, Σ_{ij}, N_{ij} are used for the interpretation of modalities [≤ij], [Σij], [Nij]. The universal modality ■ will be interpreted by the universal relation of the frame.

The equivalence (*) is not modally definable, so in order to axiomatize the logic BHAL^n we will use nonstandard semantics. By a nonstandard hyper n-arrow frame we will mean any relational system $\underline{W}=(W,\ \{\leq,\Sigma,N/i,j\leq n\},\ \{R_{ij}/i,j\leq n\})$ such that the system $(W,\ \{\leq,\Sigma,N/i,j\leq n\})$ is a hyper n-arrow frame and the relations satisfy the following additional conditions for any $x,y,z\in W$ and $i,j,k\leq n$:

S13 $xR_{ii}x$,

S14 $xR_{ij}y \longrightarrow yR_{ji}x$,

S15 $xR_{ij}y \;\&\; yR_{jk}z \longrightarrow xR_{ik}z$,

S16 $xR_{ij}y \longrightarrow x\leq_{ij}y$,

S17 $x\bar{\Sigma}_{ii}x \;\&\; y\leq_{ji}x \longrightarrow xR_{ij}y$,

S18 $y\bar{N}_{jj}y \;\&\; y_{ji}\leq x \longrightarrow xR_{ij}y$,

S19 $x\bar{\Sigma}_{ik}z \;\&\; y\bar{N}_{jk}z \;\&\; y\leq_{ji}x \longrightarrow xR_{ij}y$.

Let us note that the conditions S13-S19 follow from (*) and the other axioms of hyper n-arrow frames. If we add to S13-S19 the following condition

(*') $x\leq_{ij}y \;\&\; y\leq_{ji}x \longrightarrow xR_{ij}y$

then (*) follows and the frame becomes standard. Note that just (*') is the modally undefinable part of (*) and the conditions S13-S19 are the modally definable consequences of (*). Our aim will be to show that each hyper n-arrow frame with relations R_{ij} definable by (*) can be obtained by a suitable copying from nonstandard hyper n-arrow frames. To this end first we will study the nonstandard hyper n-arrow frames.

Let $\underline{W}$ be a nonstandard hyper n-arrow frame. For $x,y\in W$ and $i,j\le n$ define

$xS_{ij}y \longleftrightarrow x\le_{ij}y$ & $y\le_{ji}x$.

Lemma 2.1.

The relations S_{ij} satisfy the following conditions for any $x,y,z\in W$ and $i,j,k\le n$:

(ρii) $xS_{ii}x$

(σij) $xS_{ij}y \longrightarrow yS_{ji}x$,

(τijk) $xS_{ij}y$ & $yS_{jk}z \longrightarrow xS_{ik}z$,

($R_{ij}\subseteq S_{ij}$) $xR_{ij}y \longrightarrow xS_{ij}y$.

Proof - straightforward.■

Let Q_{ij} be any of the relations R_{ij} and S_{ij}, $i,j\le n$. Ann-tuple $\alpha=(\alpha_1,\dots,\alpha_n)$, $\alpha_i\subseteq W$, $i\le n$, is called a Q-filter in W if it satisfies the following conditions for any $x,y\in W$ and $i,j\le n$:

(i) $x\in\alpha_i$ & $y\in\alpha_j \longrightarrow xQ_{ij}y$,

(ii) $x\in\alpha_i$ & $xQ_{ij}y \longrightarrow y\in\alpha_j$,

(iii) $\alpha_1\cup\dots\cup\alpha_n\neq\emptyset$.

Define $Q_i[x]=(Q_{i1}(x),\dots,Q_{in}(x))$, where $Q_{ij}(x)=\{y\in W/xQ_{ij}y\}$.

Lemma 2.2.

(i) For any $x\in W$ and $i\le n$ $Q_i[x]$ is a Q-filter containing x in its i-th component.

(ii) If α is a Q-filter and $x\in\alpha_i$ then $\alpha=Q_i[x]$.

(iii) If α and β are Q-filters then either $\alpha=\beta$ or $\alpha\cap\beta=\mathbf{0}$.

(iv) $xQ_{ij}y \longleftrightarrow Q_i[x]=Q_j[y]$.

(v) If α is an S-filter and $x\in\alpha_i$ then $R_i[x]\subseteq\alpha$

(vi) If α is an R-filter and β is an S-filter then either $\alpha\cap\beta=\mathbf{0}$ or $\alpha\subseteq\beta$. Each S-filter is an union of R-filters.

Proof. Note that the notion of a Q-filter coincides with the notion of a generalized point for n-arrow frames from chapter 4.2. Then (i) and (ii) and (iv) can be proved as in theorem 1.4. ch.4.2. Condition (iii) follows directly from (ii). Let us prove (v). Let α be an S-filter and $x\in\alpha_i$. Then by (iii) $\alpha=S_i[x]$. We will show that $R_i[x]\subseteq S_i[x]$, i.e. that $R_{ij}(x)\subseteq S_{ij}(x)$. But the later is true by the lemma 2.1 - ($R_{ij}\subseteq S_{ij}$). Condition (vi) follows directly from (v). ■

By lemma 2.2 each S-filter is an union of R-filters. If an S-filter α contains only one R-filter β then $\alpha=\beta$. In this case we will say that the S-filter α is a normal S-filter. So α is a normal S-filter if it is at the same time an R-filter. The next lemma gives important examples of normal S-filters.

Lemma 2.3.

(i) If $x\bar{\Sigma}_{ii}x$ then $S_i[x]$ is a normal S-filter,

(ii) If $x\bar{N}x$ then $S[x]$ is a normal S-filter,

(iii) If $x\bar{\Sigma}_{ij}y$ and $x\bar{N}_{ij}y$ then $S_i[x]$ and $S_j[y]$ are normal S- filters.

Proof. (i) Suppose $x\overline{\Sigma}_{ii}x$. To prove the statement we will show that $S_i[x]=R_i[x]$, i.e. that for any $j\le n$ $S_{ij}(x)=R_{ij}(x)$. By the definition of S_{ij} we have $R_{ij}\subseteq S_{ij}$ so $R_{ij}(x)\subseteq S_{ij}(x)$. To prove the converse inclusion suppose $xS_{ij}y$ and proceed to show $xR_{ij}y$. From $xS_{ij}y$ we have $y\le_{ji}x$, then applying axiom S17 we obtain $xR_{ij}y$.

The proof of (ii) is similar with the help of axiom S18.

(iii) Suppose $x\overline{\Sigma}_{ij}y$ and $x\overline{N}_{ij}y$. First we will show that $S_i[x]=R_i[x]$, i.e. that for any $k\le n$ - $S_{ik}(x)=R_{ik}(x)$. Since $R_{ik}(x)\subseteq S_{ik}(x)$ it is enough to prove the converse inclusion. To this end suppose $xS_{ik}z$ and proceed to show $xR_{ik}z$. From $xS_{ik}z$ we obtain $x\le_{ik}z$, $z\le_{ki}x$. From $x\overline{\Sigma}_{ij}y$ and $z\le_{ik}x$ we obtain $z\overline{\Sigma}_{kj}y$. Then $z\Sigma_{kj}y$, $xN_{ij}y$ and $x\le_{ik}z$ imply by S19 $zR_{ki}x$ and by S14 - $xR_{ik}z$. Replacing x and y and i and j we obtain that $S_j[y]$ is also a normal S-filter. ■

Now we are ready to start with the main theorem in this section.

Theorem 2.4.

Let $\underline{W}=(W, \{\le_{ij},\Sigma_{ij},N_{ij}/i,j\le n\},\{R_{ij}/ij\le n\})$ be a nonstandard hyper n-arrow frame. Then there exist a standard hyper n-arrow frame $\underline{W}'=(W', \{\le'_{ij},\Sigma'_{ij},N'_{ij}/i,j\le n\},\{R'_{ij}/ij\le n\})$ and a copying I from $\underline{W}$ to $\underline{W}'$.

Proof. Define $I=N=\{1,2,3,\dots\}$. For $f\in I$, $x\in W$ and $i\le n$ define

$$f_i(x)=\begin{cases} x & \text{if } S_i[x] \text{ is a normal S-filter} \\ (x,f) & \text{if } S_i[x] \text{ is not a normal S-filter}\end{cases}$$

Let $f(x)=(f_1(x),\dots,f_n(x))$ and $W'=\{f(x)/x\in W \text{ and } f\in I\}$. For the relations Σ'_{ij} and N'_{ij} we take the following definitions:

$f(x)\Sigma'_{ij}g(y) \longleftrightarrow x\Sigma_{ij}y$,

$f(x)N'_{ij}g(y) \longleftrightarrow xN_{ij}y$.

To define the relation $\le'_{ij}$ we first introduce a well-ordering relation $\ll$ in the set of all S-filters in $\underline{W}$ and consider the following two cases.

Case 1: $S_i[x]$ is a normal S-filter, or $S_j[y]$ is a normal S- filter, or $S_i[x]\ne S_j[y]$ (i.e. $x\overline{S}_{ij}y$). Then we define

$f(x)\le'_{ij}g(y)$ iff $x\le y$.

Case 2: - the opposite of case 1, namely $S_i[x]$ is not a normal S-filter and $S_j[y]$ is not a normal S-filter and $S_i[x]=S_j[y]$, i.e. $xS_{ij}y$. In this case we define

$f(x)\le'_{ij}g(y)$ iff $f<g$ or $f=g$ & $R_i[x] \ll R_j[y]$.

For the relation R'_{ij} we take the following definition:

$f(x)R'_{ij}f(y)$ iff $f(x)\le'_{ij}g(y)$ & $g(y)\le'_{ji}f(x)$.

First we shall show that the conditions of copying are fulfilled. This is obvious for the conditions (I1) and (I2). Let us verify the condition (R1). For $R=\Sigma_{ij}$ and $R=N_{ij}$ it follows directly from the definitions of Σ_{ij} and N_{ij}. For $R=\le_{ij}$ suppose $x\le_{ij}y$ and let f be given. We have to find $g\in I$ such that

$f(x)\leq'_{ij}g(y)$.

Case 1: $S_i[x]$ is a normal S-filter, or $S_j[y]$ is a normal S- filter, or $S_i[x]\neq S_j[y]$ (i.e. $x\overline{S}_{ij}y$). In this case any g will do the job.

Case 2: - the opposite of case 1, namely $S_i[x]$ is not a normal S-filter and $S_j[y]$ is not a normal S-filter and $S_i[x]=S_j[y]$, i.e. $xS_{ij}y$. Then if $R_i[x] \ll R_j[y]$ holds take $g=f$ and if $R_i[x] \ll R_j[y]$ does not hold take any g such that $f<g$. Then by the definition of $\leq'_{ij}$ we have $f(x)\leq'_{ij}g(y)$. This ends the proof for the case $R=\leq'_{ij}$.

For the case $R=R_{ij}$ suppose $xR_{ij}y$ and $f\in I$. We have to find $g\in I$ such that $f(x)\leq'_{ij}g(y)$ and $g(y)\leq'_{ji}f(x)$. From $xR_{ij}y$ we obtain $x\leq'_{ij}y$ and $y\leq'_{ji}x$, which yields $xS_{ij}y$ and $S_i[x]=S_j[y]$. We will consider two cases.

Case 1: $S_i[x]$ is a normal S-filter or $S_j[y]$ is a normal S-filter. In this case take any g. Then by the definition of $\leq'_{ij}$and $\leq'_{ji}$ we get $f(x)\leq'_{ij}g(y)$ and $g(y)\leq'_{ji}f(x)$, which gives $f(x)R'_{ij}g(y)$.

Case 2: $S_i[x]$ and $S_j[y]$ are not normal S-filters. From $xR_{ij}y$ we obtain $R_i[x]=R_j[y]$, which gives $R_i[x] \ll R_j[y]$ and $R_i[y] \ll R_j[x]$. In this case take $g=f$. Then by the definition of $\leq'_{ij}$ and $\leq'_{ji}$ we obtain $f(x)\leq'_{ij}g(y)$ and $g(y)\leq'_{ji}f(x)$, which gives $f(x)R'_{ij}g(y)$.

The condition (R2) from the definition of copying is obvious for $R=\Sigma_{ij},N_{ij}$. For $R=\leq'_{ij}$ suppose $f(x)\leq'_{ij}g(y)$ and proceed to prove $x\leq_{ij}y$. We will consider two cases.

Case 1: $S_i[x]$ is a normal S-filter, or $S_j[y]$ is a normal S- filter, or $S_i[x]\neq S_j[y]$ (i.e. $x\overline{S}_{ij}y$). Then from $f(x)\leq'_{ij}g(y)$ we obtain $x\leq y$.

Case 2: - the opposite of case 1, namely $S_i[x]$ is not a normal S-filter and $S_j[y]$ is not a normal S-filter and $S_i[x]=S_j[y]$, i.e. $xS_{ij}y$. Then from $xS_{ij}y$ we obtain $x\leq_{ij}y$.

Now for the case $R=R_{ij}$ suppose $f(x)R'_{ij}g(y)$ and proceed to show $xR_{ij}y$. From $f(x)R'_{ij}g(y)$ we obtain $f(x)\leq'_{ij}g(y)$ and $g(y)\leq'_{ji}f(x)$, which gives $x\leq_{ij}y$ and $y\leq_{ji}x$. From here we obtain $xS_{ij}y$ and $S_i[x]=S_j[y]$. We will consider two cases.

Case 1: $S_i[x]$ is a normal S-filter or $S_j[y]$ is a normal S- filter. Then $S_i[x]=R_i[x]$ and $S_j[y]=R_j[y]$ and from $S_i[x]=S_j[y]$ we get $R_i[x]=R_j[y]$, which yields $xR_{ij}y$.

Case 2: $S_i[x]$ is not a normal S-filter and $S_j[y]$ is not a normal S-filter. From $f(x)\leq'_{ij}g(y)$ we obtain

(1) $f<g$ or (2) $f=g$ and $R_i[x] \ll R_j[y]$.

From $g(y)\leq'_{ji}f(x)$ we obtain

(1') $g<f$ or (2') $g=f$ and $R_j[y] \ll R_i[x]$.

Obviously the only possible combination of the above cases is (22'). Then from $R_i[x] \ll R_j[y]$ and $R_j[y] \ll R_i[x]$ we obtain $R_i[x]=R_j[y]$ ($\ll$ is a well order), which gives $xR_{ij}y$.

This ends the verification of the conditions of copying. It remains to show that the above system is a standard hyper n-arrow frame. Since $f(x)R'_{ij}f(y)$ iff $f(x)\le'_{ij}g(y)$ & $g(y)\le'_{ji}f(x)$ it remains to verify the conditions S1=S12. The conditions S1, S3, S4, S5, S7, S8 and S9 follow directly from the definitions.

For the condition S2 suppose $f(x)\le'_{ij}g(y)$ and $g(y)\le'_{jk}h(z)$ and proceed to show that $f(x)\le'_{ik}h(z)$. By the conditions of copying ($\le_{ij}1$) and ($\le_{jk}1$) we obtain $x\le_{ij}y$ and $y\le_{jk}z$, which by S1 gives $x\le_{ik}z$. We will consider two cases.

Case 1: $S_i[x]$ is a normal S-filter or $S_k[z]$ is a normal S- filter or $S_i[x]\ne S_k[z]$. Then from $x\le_{ik}z$ we obtain $f(x)\le'_{ik}h(z)$.

Case 2: the opposite of case 1 - $S_i[x]$ is not a normal S-filter and $S_k[z]$ is not a normal S-filter and $S_i[x]=S_k[z]$. From $S_i[x]=S_k[z]$ we get $xS_{ik}z$ and $z\le_{ki}x$. From $z\le_{ki}x$ and $x\le_{ij}y$ we obtain by S2 $z\le_{kj}y$. From $z\le_{kj}y$ and $y\le_{jk}z$ we obtain $yS_{jk}z$ and $S_j[y]=S_k[z]$. Then from $S_j[y]=S_k[z]$ and $S_i[x]=S_k[z]$ we obtain $S_i[x]=S_j[y]$. Since $S_i[x]$ is not a normal S-filter then $S_j[y]$ is also a not normal S-filter. Hence we have just obtained the conditions for the case 2 of the definition for $f(x)\le'_{ij}g(y)$ and $g(y)\le'_{jk}h(z)$. From $f(x)\le'_{ij}g(y)$ we have

(1) $f<g$ or (2) $f=g$ & $R_i[x] \ll R_j[y]$.

From $g(y)\le'_{jk}h(z)$ we obtain

(1') $g<h$ or (2') $g=h$ & $R_j[y] \ll R_k[z]$.

We will consider four cases.

Case (11'): $f<g$ and $g<h$. then $f<h$ and hence $f(x)\le'_{ik}h(z)$.

Case (12'): $f<g$ and $g=h$ & $R_j[y] \ll R_k[z]$. Then $f<h$ and hence $f(x)\le'_{ik}h(z)$.

Case (21'): $f=g$ & $R_i[x] \ll R_j[y]$ and $g<h$. Then $f<h$ and hence $f(x)\le'_{ik}h(z)$.

Case (22'): $f=g$ & $R_i[x] \ll R_j[y]$ and $g=h$ & $R_j[y] \ll R_k[z]$. Then we obtain $f=h$ and by the transitivity of $\ll$ - $R_i[x] \ll R_k[z]$. This again yields $f(x)\le'_{ik}h(z)$. This completes the verification of S2.

Condition S6: $f(x)\bar{\Sigma}'_{ii}f(x) \longrightarrow f(x)\le'_{ij}g(y)$.

Suppose $f(x)\bar{\Sigma}'_{ii}f(x)$. Then we have $x\bar{\Sigma}_{ii}x$. By S6 we have $x\le_{ij}y$ and by lemma 2.3.(i) $S_i[x]$ is a normal S-filter. Then by the definition of $\le'_{ij}$ we obtain $f(x)\le'_{ij}g(y)$.

The proof of the condition S10 is similar.

Condition S11: $f(x)\bar{\Sigma}'_{ik}h(z)$ & $g(y)\bar{N}'_{jk}h(z) \longrightarrow f(x)\le'_{ij}g(y)$.

Suppose $f(x)\bar{\Sigma}'_{ik}h(z)$ & $g(y)\bar{N}'_{jk}h(z)$. Then we obtain $x\bar{\Sigma}_{ik}z$, $y\bar{N}_{jk}z$ and by S11 $x\le_{ij}y$. We shall show that $f(x)\le'_{ij}g(y)$.

Case 1: $S_i[x]$ is a normal S-filter, or $S_j[y]$ is a normal S- filter, or $S_i[x]\ne S_j[y]$. Then by $x\le_{ij}y$ we obtain $f(x)\le'_{ij}g(y)$.

Case 2: - the opposite of case 1, namely $S_i[x]$ is not a normal S-filter and $S_j[y]$ is not a normal S-filter and $S_i[x]=S_j[y]$. From $S_i[x]=S_j[y]$ we get $y\le_{ji}x$.

From $y\leq_{ji}x$ and $y\bar{N}_{jk}z$ we get $x\bar{N}_{ik}z$. Then by $x\bar{\Sigma}_{ik}z$, $x\bar{N}_{ik}z$, by lemma 2.3. $S_i[x]$ is a normal S-filter. But this contradicts the condition that $S_i[x]$ is not a normal S-filter. This shows that the case is impossible.

Condition S12: $f(x)\Sigma'_{ii}f(x)$ or $f(x)N'_{ii}f(x)$.

Suppose that for some x and f we have $f(x)\bar{\Sigma}'_{ii}f(x)$ and $f(x)\bar{N}'_{ii}f(x)$. Then we obtain $x\bar{\Sigma}_{ii}x$ and $x\bar{N}_{ii}x$, which contradicts S12 for the original frame. This ends the proof of the theorem.■

Corollary 2.5.

The following conditions are equivalent for any formula A of $BHAL^n$:

(i) A is true in all standard hyper n-arrow frames for $BHAL^n$.

(ii) A is true in all nonstandard hyper n-arrow frames for $BHAL^n$.

Proof - by theorem 2.4. ■

Now the axiomatization of the logic $BHAL^n$ is easy, because the conditions of nonstandard semantics for $BHAL^n$ are modally definable and canonical (in generated canonical structures). We suggest the following axiomatizations of $BHAL^n$.

Axiom schemes and rules for $BHAL^n$

I. All classical tautologies,

II.

(K[R]) $[R](A\Rightarrow B)\Rightarrow([R]A\Rightarrow[R]B)$,

(S5[U] $[U]A\Rightarrow A$, $[U]A\Rightarrow[U][U]A$, $\langle U\rangle[U]A\Rightarrow A$, $[U]A\Rightarrow[R]A$,

$R\in\{\leq_{ij},\Sigma_{ij},N_{ij},R_{ij},U\}$, $i,j\leq n$.

A1. $[\leq ij]A\Rightarrow A$,

A2. $[\leq ik]A\Rightarrow[\leq ij][\leq jk]A$,

A3. $\langle\Sigma ij\rangle[\Sigma ji]A\Rightarrow A$,

A4. $\langle\Sigma ij\rangle\top\Rightarrow([\Sigma ii]A\Rightarrow A)$,

A5. $[\Sigma ik]A\Rightarrow[\Sigma ij][\leq jk]A$,

A6. $[\Sigma ii]A\wedge[\leq ij]A\Rightarrow([U]A\vee B)$,

A7. $\langle Nij\rangle[Nji]A\Rightarrow A$,

A8. $\langle Nij\rangle\top\Rightarrow([Nii]A\Rightarrow A)$,

A9. $[Nik]A\Rightarrow[\leq ij][Njk]A$,

A10. $[\leq ij]A\Rightarrow[U]([Njj]A\Rightarrow A)$,

A11. $[\leq ij]A\wedge[\Sigma ik]B\Rightarrow([U]B\vee[U]([Njk]B\Rightarrow A))$,

A12. $([\Sigma ii]A\Rightarrow A)\vee([Nii]A\Rightarrow A)$,

A13. $[ii]A\Rightarrow A$,

A14. $\langle ij\rangle[ji]A\Rightarrow A$,

A15. $[ik]A\Rightarrow[ij][jk]A$,

A16. $[\leq ij]A\Rightarrow[ij]A$,

A17. $B\vee[\leq ji]([\Sigma ii]A\wedge[ij]B\Rightarrow A)$,

A18. $[Njj]A\Rightarrow([\leq ji]\neg[ij]A\vee A)$,

A19 $[\Sigma ik]A\wedge[ij]B\Rightarrow([U]A\vee[U](Njk]A\Rightarrow B)$.

(MP) $\dfrac{A,\ A\Rightarrow B}{B}$, (N[U]) $\dfrac{A}{[U]A}$

Theorem 2.6. (Completeness theorem for $BHAL^n$)

The following conditions are equivalent for any formula A of $BHAL^n$:

(i) A is a theorem of $BHAL^n$,

(ii) A is true in all nonstandard hyper n-arrow frames,

(iii) A is true in all standard hyper n-arrow frames.

Proof. (i)⟶(ii) - by noticing that all axioms A1-A19 are modal translations of the axioms of generalized n-arrow frame.

(ii)$\longrightarrow$(i) - by the generated canonical model. It is easy to see that the axioms A1-A19 guarantee that the generated canonical structure of $BHAL^n$ is a generalized hyper n-arrow frame.

(ii)$\longleftrightarrow$(iii) - by corollary 2.5. ■

Theorem 2.7.

$BHAL^n$ admits a filtration with respect to its nonstandard semantics.

Proof. Let $M=(W,\{\leq_{ij},\Sigma_{ij},N_{ij}/i,j\leq n\},\{R_{ij}/i,j\leq n\}, v)$ be a model over a nonstandard hyper n-arrow frame $\underline{W}$ and A_0 be a formula. Then there exists a finite set Γ of formulas, containing A_0 and closed under subformulas and a model $M'=(W', \{\leq'_{ij},\Sigma'_{ij},N'_{ij}/i,j\leq n\}, \{R'_{ij}/i,j\leq n\}, v')$ over some finite nonstandard hyper n-arrow frame, which is a filtration of M through Γ.

Proof. Let Γ be the smallest set of formulas satisfying the following conditions

(γ0) $A_0\in\Gamma$, $\{\langle\Sigma ij\rangle\top, \langle Nij\rangle\top/i,j\leq n\}\subseteq\Gamma$ and Γ is closed under subformulas.

(γ1) if one of the formulas $[\leq ij]A$, $[\Sigma ij]A$, $[Nij]A$, $[ij]A$, $i,j\leq n$, is in Γ then the others are also in Γ.

It is easy to see that Γ is a finite set of formulas.

Define the equivalence relation $x\sim y \longleftrightarrow (\forall A\in\Gamma)x\Vdash_v A$ iff $y\Vdash_v A)$ and put $|x|=\{y\in W/x\sim y\}$, $W'=\{|x|/x\in W\}$, $v'(p)=\{|x|/x\in v(p)\}$, $p\in VAR$. For the relations $\leq'_{ij}$, Σ'_{ij}, N'_{ij} and R'_{ij}, $i,j\leq n$ we take the following definitions:

(1) $|x|\leq'_{ij}|y|$ iff $(\forall[\leq ij]A\in\Gamma)(\forall k\leq n)((x\Vdash_v[\leq ik]A \longrightarrow y\Vdash_v[\leq jk]A)$ &
$(y\Vdash_v[\Sigma jk]A \longrightarrow x\Vdash_v[\Sigma ik]A)$ &
$(x\Vdash_v[Nik]A \longrightarrow y\Vdash_v[Njk]A)$ &
$(x\Vdash_v\langle\Sigma ii\rangle\top \longrightarrow y\Vdash_v\langle\Sigma jj\rangle\top)$ &
$(y\Vdash_v\langle Njj\rangle\top \longrightarrow x\Vdash_v\langle Nii\rangle\top))$,

(2) $|x|\Sigma'_{ij}|y|$ iff $(\forall[\Sigma ij]A\in\Gamma)(\forall k\leq n)((x\Vdash_v[\Sigma ik]A \longrightarrow y\Vdash_v[\Sigma jk]A)$ &
$(y\Vdash_v[\Sigma jk]A \longrightarrow y\Vdash_v[\Sigma ik]A)$ &
$x\Vdash_v\langle\Sigma ii\rangle\top$ & $y\Vdash_v\langle\Sigma jj\rangle\top)$,

(3) $|x|N'_{ij}|y|$ iff $(\forall[Nij]A\in\Gamma)(\forall k\leq n)((x\Vdash_v[Nik]A \longrightarrow y\Vdash_v[Njk]A)$ &
$(y\Vdash_v[Njk]A \longrightarrow x\Vdash_v[Nik]A)$ &
$x\Vdash_v\langle Nii\rangle\top$ & $y\Vdash_v\langle Njj\rangle\top)$,

(4) $|x|R'_{ij}|y|$ iff $(\forall[ij]A\in\Gamma)(\forall k\leq n)(x\Vdash_v[ik]A \longleftrightarrow y\Vdash_v[jk]A)$ &
$|x|\leq'_{ij}|y|$ & $|y|\leq'_{ji}|x|$.

The proof that the above definitions satisfy the conditions of the filtration is long but straightforward, so we left it to the reader. In the next we will verify the axioms for the nonstandard hyper n-arrow frame.

S1. $|x|\leq'_{ii}|x|$.

The proof of S1 follows directly from (1).

S2. $|x|\leq'_{ij}|y|$ & $|y|\leq'_{jk}|z| \longrightarrow |x|\leq'_{ik}|z|$.

Suppose $|x|\leq'_{ij}|y|$ & $|y|\leq'_{jk}|z|$ and proceed to show $|x|\leq'_{ik}|z|$.

The proof follows from the following

Claim

For any formula $[\leq ik]A \in \Gamma$ and $l \leq n$ we have:

(i) $(x \Vdash_{v} [\leq il]A \rightarrow z \Vdash_{v} [\leq kl]A)$,

(ii) $(z \Vdash_{v} [\Sigma kl]A \rightarrow x \Vdash_{v} [\Sigma il]A)$,

(iii) $(x \Vdash_{v} [Nil]A \rightarrow z \Vdash_{v} [Nkl]A)$,

(iv) $(x \Vdash_{v} \langle \Sigma ii \rangle \top \rightarrow z \Vdash_{v} \langle \Sigma kk \rangle \top)$,

(v) $(z \Vdash_{v} \langle Njj \rangle \top \rightarrow x \Vdash_{v} \langle Nii \rangle \top)))$.

Proof - directly from (1).

S3. $|x|\Sigma'_{ij}|y| \rightarrow |y|\Sigma'_{ji}|x|$. The proof follows directly from (2).

S4. $|x|\Sigma'_{ij}|y| \rightarrow |x|\Sigma'_{ii}|x|$. Suppose $|x|\Sigma'_{ij}|y|$. Then we have $x \Vdash_{v} \langle \Sigma ii \rangle \top$, so there exists z such that $x\Sigma_{ii}z$. From here we obtain by S4 $x\Sigma_{ii}x$ and by the condition $(\Sigma_{ii}1)$ of the filtration we obtain $|x|\Sigma'_{ii}|x|$.

S5. $|x|\Sigma'_{ij}|y|$ & $|y|\leq'_{jk}|z| \rightarrow |x|\Sigma'_{ik}|z|$.

Suppose $|x|\Sigma'_{ij}|y|$ & $|y|\leq'_{jk}|z|$ and proceed to show $|x|\Sigma'_{ik}|z|$. The proof follows from the following

Claim

For any formula $[\Sigma ik]A \in \Gamma$ and $l \leq n$ we have:

(i) $x \Vdash_{v} [\Sigma il]A \rightarrow z \Vdash_{v} [\Sigma kl]A$,

(ii) $z \Vdash_{v} [\Sigma kl]A \rightarrow x \Vdash_{v} [\Sigma il]A$,

(iii) $x \Vdash_{v} \langle \Sigma ii \rangle \top$ & $z \Vdash_{v} \langle \Sigma kk \rangle \top$.

The proof of the Claim follows directly from (1) and (2).

S6. $|x|\overline{\Sigma}'_{ii}|x| \rightarrow |x|\leq'_{ij}|y|$.

Suppose $|x|\overline{\Sigma}'_{ii}|x|$, then by the condition $(\Sigma_{ii}1)$ of the filtration we obtain $x\overline{\Sigma}_{ii}x$ and by S6 we obtain $x\leq_{ij}y$. Then by the condition $(\leq_{ij}1)$ of the filtration we get $|x|\leq'_{ij}|y|$.

The proof of the conditions S7-S10 is similar to the proof of the conditions S3-S6.

S11. $|x|\leq'_{ij}|y|$ or $|x|\Sigma'_{ik}|z|$ or $|y|N'_{jk}|z|$.

By S11 we have $x\leq_{ij}y$ or $x\Sigma_{ik}z$ or $yN_{jk}z$. Then by the condition (R1) of the filtration we obtain $|x|\leq'_{ij}|y|$ or $|x|\Sigma'_{ik}|z|$ or $|y|N'_{jk}|z|$.

S12. $|x|\Sigma'_{ii}|x|$ or $|x|N'_{ii}|x|$. The proof is similar to the proof of S11.

The proof of the conditions S13-S16 follows directly from (4).

S17. $|x|\overline{\Sigma}'_{ii}|x|$ & $|y|\leq'_{ji}|x| \rightarrow |x|R'_{ij}|y|$.

Suppose $|x|\overline{\Sigma}'_{ii}|x|$ & $|y|\leq'_{ji}|x|$ and proceed to prove $|x|R'_{ij}|y|$. We will consider two cases:

Case 1: $xR_{ij}y$. Then by the condition ($R_{ij}1$) of the filtration we obtain $|x|R'_{ij}|y|$.

Case 2: $x\bar{R}_{ij}y$. Then by S17 we obtain ether $x\Sigma_{ii}x$ or $y\neq_{ji}x$.

Subcase 2.1: $x\Sigma_{ii}x$. Then by the condition ($\Sigma_{ii}1$) of the filtration we get $|x|\Sigma'_{ii}|x|$, which contradicts the assumption.

Subcase 2.2: $y\neq_{ji}x$. Then by S6 we obtain $y\Sigma_{jj}y$ and by the condition ($\Sigma_{jj}1$) of the filtration we obtain $|y|\Sigma'_{jj}|y|$. By the assumption $|y|\leq'_{ji}|x|$ and S3 and S5 we obtain $|x|\Sigma'_{ii}|x|$, which contradicts the assumption $|x|\bar{\Sigma}'_{ii}|x|$.

S18. $|y|\bar{N}'_{jj}|y|$ & $|y|\leq'_{ji}|x|\longrightarrow |x|R'_{ij}|y|$.

The proof is similar to the proof of S17.

S19 $|x|\bar{\Sigma}'_{ik}|z|$ & $|y|\bar{N}'_{jk}|z|$ & $|y|\leq'_{ji}|x| \longrightarrow |x|R'_{ij}|y|$.

Suppose $|x|\bar{\Sigma}'_{ik}|z|$ & $|y|\bar{N}'_{jk}|z|$ & $|y|\leq'_{ji}|x|$ and proceed to prove $|x|R'_{ij}|y|$. We will consider two cases:

Case 1: $xR_{ij}y$. Then by by the condition ($R_{ij}1$) of the filtration we obtain $|x|R'_{ij}|y|$.

Case 2: $x\bar{R}_{ij}y$. Then by S19 we obtain: either $x\Sigma_{ik}z$ or $yN_{jk}z$ or $y\neq_{ji}x$.

Subcase 2.1: $x\Sigma_{ik}z$. By the condition ($\Sigma_{ik}1$) of the filtration we obtain $|x|\Sigma'_{ik}|z|$, which contradicts the assumption $|x|\bar{\Sigma}'_{ik}|z|$.

Subcase 2.2: $xN_{jk}z$. By the condition ($N_{jk}1$) we obtain $|y|N'_{jk}|z|$, which contradicts the assumption $|y|\bar{N}'_{jk}|z|$.

Subcase 2.3: $y\neq_{ji}x$. Then by S11 we obtain either $y\Sigma_{jk}z$ or $xN_{ik}z$.

Subsubcase 2.3.1: $y\Sigma_{jk}z$. From here we obtain $|y|\Sigma'_{jk}|z|$ and by the assumption $|y|\leq'_{ji}|x|$ we obtain $|x|\Sigma'_{ik}|z|$, which contradicts the assumption $|x|\bar{\Sigma}'_{ik}|z|$.

Subsubcase 2.3.2: $xN_{ik}z$. From here we obtain $|x|N'_{ik}|z|$ and by the assumption $|y|\leq'_{ji}|x|$ we obtain $|y|N'_{jk}|z|$, which contradicts the assumption $|y|\bar{N}'_{jk}|z|$. This completes the verification of S19 and the proof of the theorem. ■

Corollary 2.8.
$BHAL^n$ possesses finite model property and is decidable.

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