Axioms for card games

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March 27, 2000

Abstract

We axiomatize two different game states for card games, the state where cards have been dealt over players but where they haven't picked up their cards from the table yet, and the state where they have picked up their cards. The first is mainly interesting for its use in indirect description proofs. The second is extensively illustrated by the example of three players and three cards. We prove that the axiomatizations describe the respective models underlying the game states, in the technical sense that all other models are bisimilar to them. We show that our results correspond to those of fixed point computations of the description of modal models.

1 Introduction

A dealing of cards over players defines the initial state of a knowledge game. We represent that state by a pointed multiagent *S5* model. All the players' knowledge is encoded in this model by way of the accessibility relations for the players between dealings that are relevant given the actual dealing of cards. Why is this the correct model for the initial state of the game?

We answer this question as follows: First, we axiomatize the knowledge that players have about the game and about each other, and we show that our preferred model is indeed a model of this theory. Second, we show that this model is the 'only' model of the theory, because all other models are bisimilar to it. This strengthens our conviction that we have both the right model and the right theory for the state of the game under consideration.

The axiomatization of a given finite modal model can also be computed in a standard way by means of a fixed point construction. We relate our results, that are derived from analyzing agent behaviour, to those from applying this technique on multiagent $S5$ models.

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†Apart from the input of my supervisors Gerard Renardel and Johan van Benthem, I am much in debt to both Wiebe van der Hoek and Barteld Kooi for their willingness to discuss previous versions innumerable times.

We proceed as follows: based on an informal analysis of agent properties, first in section 2 we present the theory $\mathsf{33}^+$, describing the initial state for the knowledge game for three persons and three cards. We show that hexa is a model of $\mathsf{33}^+$. We proceed by proving some dependencies among the axioms, by presenting different versions for some axioms, and by presenting models of restrictions of $\mathsf{33}^+$ that are 'countermodels' in the sense that they clearly do not model the initial game state we attempt to describe. This also serves as a further justification of our informal analysis. We then present a compact but equivalent version of $\mathsf{33}^+$, that we name $\mathsf{33}^-$, or just $\mathsf{33}$. Yet another alternative is to characterize the model hexa by the (exclusive) disjunction of a partial description of its worlds. We show the equivalence of that formula to $\mathsf{33}$. Next, we prove that $\mathsf{33}$ *describes* hexa: we show that all $S5EC_3$ models of $\mathsf{33}$ are bisimilar to hexa.

In section 3 we then continue with the general case: the knowledge game for a given parameter dealing $\mathbf{d}$ from m cards to n players. The intended $S5EC_n$ model $I_\mathbf{d}$ for its initial game state is also described in [vD00c]. At first sight, it seems less clear what its axiomatization is: although some of the axioms from $\mathsf{33}^+$ have obvious generalizations, this is not obvious for the agents' ignorance. We illustrate the difficulties by presenting rejected candidate axioms. It turns out that we can characterize ignorance in three different ways, that are all equivalent to each other. The resulting axioms make up the theory $\mathsf{kgames}^+_\mathbf{d}$. We also present a shortened version $\mathsf{kgames}_\mathbf{d}$. We then prove that each model of $\mathsf{kgames}_\mathbf{d}$ is bisimilar to the intended model $I_\mathbf{d}$.

We can also describe game models indirectly by relating different models with actions. See [vD00a]. Because bisimulation is invariant under action execution, it simplifies our bisimulation proof obligations. We refer to it in section 4 and give details in [vD00b].

In section 4 we discuss the game situation where the cards have been dealt but where players haven't picked up and looked into their own cards. It has a simpler intended model $preI_\mathbf{d}$ and a simpler axiomatization $\mathsf{prekgames}_\mathbf{d}$. Again, we prove that $preI_\mathbf{d}$ is unique. The model $I_\mathbf{d}$ results from $preI_\mathbf{d}$ by executing the action of 'turning cards', thus providing the indirect proof that was mentioned above.

In section 5 we discuss other issues. We compute the descriptions of $I_\mathbf{d}$ and $preI_\mathbf{d}$ by a fixed point construction for finite models. We discuss hypercubes, models for distributed systems that seem much related to models for card games. Finally, we present some ideas on the belief revision that seems necessary for the efficient computation of the axiomatizations of other game states, resulting from action execution. For that, we need pre- and postconditions of actions.

1.1 Logical preliminaries

We use logical notions and terminology as in [MvdH95]. An $S5_n$ model is an *S5* model for a set of n agents. An $S5EC_n$ model is a multiagent *S5* model plus access computed for general and common knowledge operators.

The $\mathbf{S5_n}$ proof system ($\mathbf{S5EC_n}$ proof system) is the axiomatic proof system consisting of the **S5** axiom schemata and rules for all agents (and for all modal operators). We write $\varphi \vdash \psi$ for $\vdash \varphi \Rightarrow \vdash \psi$. We use soundness and completeness of these systems without restriction, see [MvdH95] for proofs. Instead of axiomatic proof we generally use a more informal natural deduction style of proof. It allows for a more natural presentation of cases, and it introduces modal operators by the derivation rule $\varphi_1, ..., \varphi_m \vdash \psi \Rightarrow \Box\varphi_1, ..., \Box\varphi_m \vdash \Box\psi$.

Unless specifically stated otherwise, we assume that $\models$ denotes $\models_{S5EC_n}$ and $\vdash$ denotes $\vdash_{\mathbf{S5EC_n}}$. In section 2 we more specifically assume $\models$ denotes $\models_{S5EC_3}$ and $\vdash$ denotes $\vdash_{\mathbf{S5EC_3}}$.

Axioms

In our epistemic language, we distinguish axiom schemata, constraints ('axioms') and contingencies. Any instance of an axiom scheme, such as $K_1\varphi \rightarrow \varphi$, is an axiom. Differently put, these instances are closed under uniform substitution. A constraint, such as $r_1 \rightarrow K_1 r_1$ (for 'if player 1 holds the red card, he knows it') is also an axiom. However, constraints are not closed under uniform substitution: $r_2 \rightarrow K_1 r_2$ is *not* an axiom, because player 1 doesn't know the card of player 2 in the initial state of the game. Instead of constraints, we *still* call them axioms, as long as it is understood that they are not instances of schemata. Formulas that are neither axioms nor deductive consequences of axioms are contingencies. Contingencies may hold in specific worlds of a model only, like r_1 if player 1 holds red in the actual dealing of cards.

Common knowledge

It is not only the case that player 1 holds (at least) a card – $r_1 \vee w_1 \vee b_1$ –, but this is also commonly known – $C_{123}(r_1 \vee w_1 \vee b_1)$ –. How explicit do we need to be about such common knowledge? Mostly, it suffices to leave it explicit. Axioms are commonly known, because we have necessitation for common knowledge operators. Indeed it will the the case, that not just the axiom (constraint) but also knowledge of it, is essential in order to prove equivalences and dependencies among axioms. From a semantic point of view, observe that axioms hold in all worlds of a model. Therefore, from a given world, they hold in all $(\bigcup_{a \in \mathbf{A}} \sim_a)^*$-accessible worlds. Therefore they are commonly known in that world.

Exclusive disjunction

We sometimes use 'exclusive or' $\triangledown$ and therefore define it here, as an n-ary operation, for each $n \geq 2$:

$$\triangledown_{i=1}^{n} p_i \quad := \quad (p_1 \wedge \neg p_2 ... \wedge \neg p_n) \vee (\neg p_1 \wedge p_2 ... \wedge \neg p_n) \vee ... \vee (\neg p_1 \wedge \neg p_2 ... \wedge p_n)$$

Instead of $\triangledown_{i=1}^{n} p_i$ we also write $p_1 \triangledown ... \triangledown p_n$. Observe that if one defines exclusive disjunction as a *binary* operation *only*, we do not get the desired truth-functionality for the n-ary case. Although $(p_1 \triangledown p_2) \triangledown p_3$ is equivalent to $p_1 \triangledown (p_2 \triangledown p_3)$, neither of those is equivalent to $\triangledown_{i=1}^{3} p_i$.

Model and state descriptions

Our motivation for this investigation was the following: we described a model for a game state but had some doubts on whether it the right model and some doubts on whether it is unique. To remove those doubts, we axiomatize game states and investigate how the resulting theories correspond to our preferred model. It then turns out that they *describe* the model, in the standard logical sense that all models of the theory are bisimilar to the preferred model, see e.g. [vB98, BM96]. As the preferred model is finite, we can compute its description in epistemic logic with common knowledge operators, in a straightforward way. These issues are discussed separately in section 5.1. We should not forget, however, that we started with doubts about both the models and the theories, that validates our approach of first axiomatizing agent behaviour and subsequently reducing those axioms because of interdependencies.

2 Axioms for three players each holding one card

In this section, we present the theory $\mathsf{33}^+$, and its shortened but equivalent version $\mathsf{33}$, that describe the $S5_3$ ($S5EC_3$) model hexa, see figure 1 (reflexive arrows are not drawn in the figure). The knowledge state (hexa, rwb) has been introduced in [vD00c]. It is a (pointed) model of the initial state of the knowledge game for 3 players (1, 2, 3) and three cards (r, w, b), where 1 holds red, 2 holds white and 3 holds blue. Observe that *any* further refinement of access in hexa, symmetrically for all agents, results in the fully refined model consisting of six singleton worlds.

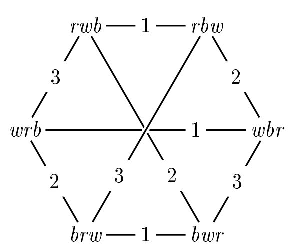


Figure 1: The model hexa for three players each holding a card

2.1 The theory 33^+

What information do the players have about this initial game state? They know how many cards there are, namely three. They know that the cards are all different, namely one red, one white and one blue. That all cards are different, means that the dealing of cards over players is a function. They know that each of them holds one card. Beyond that, if they hold a card, they know it, and if they don't hold a card, they also know that they do not hold it. They don't know anything else, and there seem to be two sides of that ignorance. First, a player doesn't know that another player holds a specific card. Second, he also doesn't know another player *not* to hold a specific card, unless it is his own card; in other words: apart from his own card, a player can imagine any card to be in possession of another player.

The theory $\mathsf{33}^+$ is the set of axioms (constraints) formalizing this information. The axioms are listed in table 1. We call it $\mathsf{33}$ because there are 3 players and 3 cards. We index it with large because we will later present an equivalent but smaller version of the theory. In $\mathsf{33}^+$, the terms in (sansserif) roman are to be interpreted as the conjunction the set of sentences following them, e.g. $\mathsf{see33} = \bigwedge_{a\in\mathbf{A}}\bigwedge_{c\in\mathbf{C}}(c_a \rightarrow K_a c_a)$. Further, $\mathsf{33}^+$ is the conjunction of all its axioms. We may also think of common knowledge $C_{123}\mathsf{33}^+$ as the requested formalization, and we implicitly assume distribution of the common knowledge operator C_{123} over conjunction. In that way, e.g., we derive $C_{123}(r_1 \rightarrow K_1 r_1)$ (it is commonly known, that if player 1 holds the red card, he knows it).

First we show that hexa satisfies $\mathsf{33}^+$. Then we relate the intended meaning of the axioms to their formulation, and present some alternative formulations of axioms. We demonstrate the axioms' independence by showing that for all axioms, the theory without that axiom has a countermodel of that axiom. This also provides circumstantial evidence, so to speak, that our preferred model is likely to be the right model.

2.2 Hexa is a model of 33^+

Fact 1
$\mathsf{hexa} \models \mathsf{33}^+$

For all conjuncts φ of all axioms of $\mathsf{33}^+$ we have to show $\mathsf{hexa} \models \varphi$, i.e. for all worlds $w \in \mathsf{hexa} : \mathsf{hexa}, w \models \varphi$. Because of symmetry in the model, it suffices to show that, e.g., $\mathsf{hexa}, rwb \models \varphi$. For a proof, see the appendix on page 29. Having proven that $\mathsf{hexa} \models \mathsf{33}^+$, we have also proven that $\mathsf{hexa} \models C_{123}\mathsf{33}^+$: because of the definition of the interpretation of common knowledge, and because access for C_{123} on hexa is universal, $\mathsf{hexa} \models C_{123}\mathsf{33}^+$ is equivalent to $\forall w \in \mathsf{hexa} : \mathsf{hexa}, w \models \mathsf{33}^+$.

see33			*players see their own cards*
$r_1 \to K_1 r_1$	$r_2 \to K_2 r_2$	$r_3 \to K_3 r_3$	
$w_1 \to K_1 w_1$	$w_2 \to K_2 w_2$	$w_3 \to K_3 w_3$	
$b_1 \to K_1 b_1$	$b_2 \to K_2 b_2$	$b_3 \to K_3 b_3$	
dontsee33			*players only see their own cards*
$\neg r_1 \to K_1 \neg r_1$	$\neg r_2 \to K_2 \neg r_2$	$\neg r_3 \to K_3 \neg r_3$	
$\neg w_1 \to K_1 \neg w_1$	$\neg w_2 \to K_2 \neg w_2$	$\neg w_3 \to K_3 \neg w_3$	
$\neg b_1 \to K_1 \neg b_1$	$\neg b_2 \to K_2 \neg b_2$	$\neg b_3 \to K_3 \neg b_3$	
atmost33			*there is at most one card of each colour*
$\neg(r_1 \wedge r_2)$	$\neg(w_1 \wedge w_2)$	$\neg(b_1 \wedge b_2)$	
$\neg(r_1 \wedge r_3)$	$\neg(w_1 \wedge w_3)$	$\neg(b_1 \wedge b_3)$	
$\neg(r_2 \wedge r_3)$	$\neg(w_2 \wedge w_3)$	$\neg(b_2 \wedge b_3)$	
atleast33			*there is at least one card per player*
$r_1 \vee w_1 \vee b_1$			
$r_2 \vee w_2 \vee b_2$			
$r_3 \vee w_3 \vee b_3$			
dontknowthat33			*players don't know others' cards*
$\neg K_2 r_1$	$\neg K_2 w_1$	$\neg K_2 b_1$	
$\neg K_3 r_1$	$\neg K_3 w_1$	$\neg K_3 b_1$	
$\neg K_1 r_2$	$\neg K_1 w_2$	$\neg K_1 b_2$	
$\neg K_3 r_2$	$\neg K_3 w_2$	$\neg K_3 b_2$	
$\neg K_1 r_3$	$\neg K_1 w_3$	$\neg K_1 b_3$	
$\neg K_2 r_3$	$\neg K_2 w_3$	$\neg K_2 b_3$	
dontknownot33			*players can imagine others to hold other cards*
$\neg r_2 \to \neg K_2 \neg r_1$	$\neg w_2 \to \neg K_2 \neg w_1$	$\neg b_2 \to \neg K_2 \neg b_1$	
$\neg r_3 \to \neg K_3 \neg r_1$	$\neg w_3 \to \neg K_3 \neg w_1$	$\neg b_3 \to \neg K_3 \neg b_1$	
$\neg r_1 \to \neg K_1 \neg r_2$	$\neg w_1 \to \neg K_1 \neg w_2$	$\neg b_1 \to \neg K_1 \neg b_2$	
$\neg r_3 \to \neg K_3 \neg r_2$	$\neg w_3 \to \neg K_3 \neg w_2$	$\neg b_3 \to \neg K_3 \neg b_2$	
$\neg r_1 \to \neg K_1 \neg r_3$	$\neg w_1 \to \neg K_1 \neg w_3$	$\neg b_1 \to \neg K_1 \neg b_3$	
$\neg r_2 \to \neg K_2 \neg r_3$	$\neg w_2 \to \neg K_2 \neg w_3$	$\neg b_2 \to \neg K_2 \neg b_3$	

Table 1: The theory $\mathsf{33}^+$

2.3 Meaning of and dependencies between axioms

Some proofs are provided in the running text. Other proofs are found in the appendix, on page 29. We remind the reader that we sometimes prove $\varphi \vdash \psi$ from the equivalent, in $\mathbf{S5EC_n}$, $C_{123}\varphi \vdash \psi$.

2.3.1 See33 and dontsee33

Theorem see33 says that in all worlds accessible to a player he should hold the same cards. Theorem dontsee33 says that in all worlds accessible to a player he should hold the same cards *not*: if a player does not hold a card, he doesn't hold that card in all worlds accessible to him. Both dontsee33 and see33 separately follow from the other axioms of 33^+. We will retain see33 and delete dontsee33. Informal proof:

33^+ − dontsee33 $\vdash$ dontsee33:

We prove the case $\neg r_1 \rightarrow K_1 \neg r_1$. Suppose $\neg r_1$. From $\neg r_1$ and $r_1 \vee w_1 \vee b_1$ follows $w_1 \vee b_1$.

From w_1 and see33 follows $K_1 w_1$. From atleast and atmost (or equivalently: from dealings33, as defined below) follows $w_1 \rightarrow \neg r_1$. From w_1 and $w_1 \rightarrow \neg r_1$ follows $\neg r_1$. Therefore, from $K_1 w_1$ and $K_1(w_1 \rightarrow \neg r_1)$ (which holds because 33^+ and therefore also dealings33 are commonly known) follows $K_1 \neg r_1$.

Similarly as for w_1, from b_1 and see33 follows $K_1 b_1$, and from b_1 and $b_1 \rightarrow \neg r_1$ follows $\neg r_1$. Continuing as before, we derive $K_1 \neg r_1$.

Therefore $w_1 \vee b_1 \rightarrow K_1 \neg r_1$, and therefore $\neg r_1 \rightarrow K_1 \neg r_1$. ■

33^+ − see33 $\vdash$ see33: see the appendix.

2.3.2 Atmost33 and atleast33

Axiom atmost33 says that the same card cannot be held by two different players. This is obvious, as a dealing d is a *function* from the set of cards $\mathbf{C}$ to the set of players $\mathbf{A}$. As the propositional language we use to describe game states doesn't have functions, we have to be explicit about it: atmost33 states that a dealing is a function. It even states that a dealing is a *partial* function, as it doesn't require all cards to be dealt. We can express that a dealing is a total function, by the proposition that each card is held by exactly one person:

$$\textsf{function33} := (r_1 \nabla r_2 \nabla r_3) \wedge (w_1 \nabla w_2 \nabla w_3) \wedge (b_1 \nabla b_2 \nabla b_3).$$

It holds that atmost33, atleast33 $\vdash$ function33. For a proof see the appendix.

Axiom atleast33 says that every player holds *at least* one card. We actually wanted to express that every player holds *exactly* one card:

$$\textsf{exactly33} := (r_1 \nabla w_1 \nabla b_1) \wedge (r_2 \nabla w_2 \nabla b_2) \wedge (r_3 \nabla w_3 \nabla b_3)$$

Because of atmost33, atleast33 $\vdash$ exactly33 the former is already sufficient. For a proof see the appendix.

The axiom dealings33 expresses that exactly one of six different dealings of cards can be the case. Let:

$$\delta_{abc} = a_1 \wedge \neg b_1 \wedge \neg c_1 \wedge \neg a_2 \wedge b_2 \wedge \neg c_2 \wedge \neg a_3 \wedge \neg b_3 \wedge c_3$$

then:

$$\mathsf{dealings33} := \delta_{rwb} \vee \delta_{rbw} \vee \delta_{wrb} \vee \delta_{wbr} \vee \delta_{brw} \vee \delta_{bwr}$$

Dealings33 is equivalent to atleast33 and atmost33. For proofs of dealings33 $\vdash$ atleast33 $\wedge$ atmost33 and atleast33 $\wedge$ atmost33 $\vdash$ dealings33, see the appendix.

2.3.3 Dontknowthat33 and dontknownot33

Axiom dontknowthat33 says that a player doesn't know another player's card. We might weaken it with the precondition that the first player doesn't hold that card himself, because that more properly expresses what an agent knows:

$$\mathsf{dontknowother33} := \bigwedge_{a \neq b \in \mathbf{A}} \bigwedge_{c \in \mathbf{C}} (\neg c_a \rightarrow \neg K_a c_b).$$

From this weaker axiom dontknowother33 we can deduce dontknowthat33: for a proof of atmost33, dontknowother33 $\vdash$ dontknowthat33, see the appendix. Of course, the reverse, dontknowthat33 $\vdash$ dontknowother33, holds trivially. We prefer dontknowthat33 over dontknowother33, because it is shorter.

Axiom dontknownot33 says that a player can imagine another player to hold any card that he doesn't hold himself. In this case the antecedent is essential, as one cannot imagine other players to have the same cards as oneself: $\bigwedge_{a \neq b \in \mathbf{A}} \bigwedge_{c \in \mathbf{C}} M_a c_b$ obviously doesn't hold.

Both dontknowthat33 and dontknownot33 separately follow from the other axioms. For proofs, see the appendix. We will retain dontknowthat33 and delete dontknownot33.

2.4 Nonintended models for restrictions of the theory

To understand why all axioms are indispensable for formalizing the game state of three persons each holding a card, it is instructive to present countermodels of restrictions of the theory 33^+. Because these models clearly do not model that state, this strenghtens our case for the preferred model hexa.

In this subsection we also use the following notation for card dealings: let C, D, E be subsets of cards, then $C|D|E$ is the dealing where player 1 holds all cards in C, player 2 all in D, and player 3 all in E. For convenience we write a set as the string of its elements, where an empty string is represented by ε. Thus $r|wbr|\varepsilon$ is the dealing of cards where player 1 holds a red card, player 2 a red, a white and a blue card, and player 3 holds no card.

See33

Figure 2 (where rwb stands for $r|w|b$, etc.) is a model of $\mathsf{33} - \mathsf{see33} - \mathsf{dontsee33}$. This model M doesn't satisfy $\mathsf{see33}$, as $M, rwb \not\models r_1 \rightarrow K_1 r_1$. None of the players can distinguish between any of the six dealings. Incidentally, it is the model of the state of the game where the cards have already been dealt but where the players haven't looked up their own cards yet. See also section 4. Other models of $\mathsf{33} - \mathsf{see33} - \mathsf{dontsee33}$ are those resulting from a permutation of access for 1, 2 and 3 in the model hexa (see figure 1).

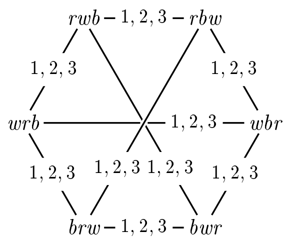


Figure 2: Nobody has looked in his cards (universal access for all agents). Assume transitivity of links.

Atmost33

Without $\mathsf{atmost33}$ there are more dealings relevant ($\{1, 2, 3\}$-accessible) to a given dealing. In the theory $\mathsf{33}^+$, delete $\mathsf{atmost33}$ and replace $\mathsf{atleast33}$ for the stronger axiom $\mathsf{exactly33}$. The theory $\mathsf{33}^+ - \mathsf{atmost33} - \mathsf{atleast33} + \mathsf{exactly33}$ has a model M_{27} containing 27 worlds, where a world is characterized by *any* distribution of three cards of any three colours red, white and blue over three players (thus there are $3 \cdot 3 \cdot 3 = 27$ possibilities). E.g. $r|r|w$ is such a world, where 1 and 2 hold a red card and three holds a white card. Observe that $M_{27} \not\models \mathsf{atmost33}$, as $M_{27}, r|r|w \not\models \neg(r_1 \wedge r_2)$.

The theory $\mathsf{33}^+ - \mathsf{atmost33}$ reveals an implicit constraint that is imposed by the language. That theory has a model M' containing much more worlds than M_{27}, namely any world corresponding to a dealing of between three and nine cards of any of three colours red, white and blue, where every player holds at least one card, but with the restriction that a player cannot hold more than one card of the same colour. The last is, because our language *cannot express* that a player holds more than one card of the same colour! Some cases in more detail:

Total of four cards: There are 3^4 worlds where one of the three players holds two different cards and the others hold one card.[1]

Total of five cards: We have to consider both the case where one player holds three different cards and the others both one, and the case where there are two players holding two different cards. There is only one combination of three different cards: rwb. Thus the total is $3^2 \cdot 1 + 3 \cdot (3 \cdot 3 \cdot 3) = 90$ possibilities.

Total of six, seven, eight, and nine cards: similarly to four and five cards.

Atleast33

Without $\mathsf{atleast33}$ there are $27+18+9+1 = 55$ different dealings of at most three different cards over three players.[2] Given the model M_{55} with all these dealings as worlds, $M_{55} \not\models \mathsf{atleast33}$ as $M_{55}, rwb|\varepsilon|\varepsilon \not\models r_2 \vee w_2 \vee b_2$. Somewhat similarly to above, a model of special interest is the model M'' containing only 27 worlds (different from those in M_{27}, above!), for all the different dealings of *exactly* three different cards (as formalized by $\mathsf{function33}$), although not necessarily dealt to different players. Model M_{55} is not a model of this slightly strengthened theory $\mathsf{33}^+ - \mathsf{atleast33} - \mathsf{atmost33} + \mathsf{function33}$, whereas model M'' is.

Dontknowthat33

Figure 3 is a model M of $\mathsf{33}^+ - (\mathsf{dontknowthat33} + \mathsf{dontknownot33})$. Incidentally, this is the model resulting from the action of player 1 showing player 2 his red card, given a request for his card. Obviously, $M \not\models \mathsf{dontknowthat33}$, as $M, \mathsf{rwb} \not\models \neg K_2 r_1$. Player 2 has become less ignorant. Actually, any model resulting of any game action on hexa results in a decrease of ignorance, as measured in terms of the dealings still imaginable for a player, and therefore satisfies $\mathsf{33}^+ - (\mathsf{dontknowthat33} + \mathsf{dontknownot33})$.

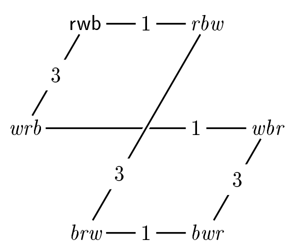


Figure 3: Any 3/3 game state satisfies $\mathsf{33} - (\mathsf{dontknowthat33} + \mathsf{dontknownot33})$

[1] There are three different combinations of two different cards for a player: rw, bw, br. Any of the three players can hold two cards. Thus the total is: $3 \cdot 3 \cdot 3 + 3 \cdot 3 \cdot 3 + 3 \cdot 3 \cdot 3 = 81$.

[2] 27 combinations of three cards, e.g. $rwb|\varepsilon|\varepsilon$ and $r|\varepsilon|wb$, 18 of two, 9 of one, and 1 of zero.

2.5 The theory 33

Given the equivalences proven in the previous subsection, we define the theory $\mathsf{33}$ as the conjunction of the three axioms in table 2. Obviously, $\vdash \mathsf{33} \leftrightarrow \mathsf{33}^+$ Observe that $\vdash \mathsf{atmost33} \wedge \mathsf{atleast33} \leftrightarrow \mathsf{dealings33}$, as shown before. An initial state (hexa, d) of the game for 3 persons and 3 cards is described by the conjunction its atomic description and common knowledge of the theory $\mathsf{33}$. For example, $\delta_{rwb} \wedge C_{123}\mathsf{33}$ describes the initial state of the game where 1 holds red, 2 holds white and 3 holds blue.

$$
\begin{array}{rcl}
\mathsf{see33} & := & \bigwedge_{a\in\{1,2,3\}} \bigwedge_{c\in\{r,w,b\}}(c_a \rightarrow K_a c_a) \\
\mathsf{dealings33} & := & \delta_{rwb} \vee \delta_{rbw} \vee \delta_{wrb} \vee \delta_{wbr} \vee \delta_{brw} \vee \delta_{bwr} \\
\mathsf{dontknowthat33} & := & \bigwedge_{a\neq b\in\{1,2,3\}} \bigwedge_{c\in\{r,w,b\}} \neg K_a c_b
\end{array}
$$

Table 2: Theory $\mathsf{33}$

In the previous subsection we have shown that we cannot substantially *weaken* the theory (by deleting axioms), because it then would model structures of different game states. In the next subsection 2.6, we show that we do not need to *strengthen* the theory, because in a technical sense hexa already is its only model. Together, this shows that we have chosen the right model, and the right axioms, for describing the game state of three players each holding a card. Before we continue with that subsection, we digress on other versions of $\mathsf{33}$ that are equivalent to it:

2.5.1 Summing up worlds

Instead of formalizing observed agent properties, we might describe the model hexa as the (exclusive) disjunction of the modal description of its worlds. Let δ_{xyz} be the formula describing the valuation corresponding to the dealing of cards xyz, as before. Let $\delta^1_{xyz} := x_1 \wedge \neg y_1 \wedge \neg z_1$, $\delta^2_{xyz} := \neg x_2 \wedge y_2 \wedge \neg z_2$, $\delta^3_{xyz} := \neg x_3 \wedge \neg y_3 \wedge z_3$. Now define:

$$
\begin{array}{rcl}
\sigma_{xyz} & := & K_1\delta^1_{xyz} \wedge K_2\delta^2_{xyz} \wedge K_3\delta^3_{xyz} \wedge M_1\delta_{xzy} \wedge M_2\delta_{zyx} \wedge M_3\delta_{yxz} \\
\sigma^{33} & := & \sigma_{rwb} \vee \sigma_{rbw} \vee \sigma_{wrb} \vee \sigma_{wbr} \vee \sigma_{brw} \vee \sigma_{bwr}
\end{array}
$$

In σ^{33} we do not have to make exclusive disjunction explicit, as the disjuncts exclude each other anyway. It is now easy to prove (see the appendix) that both $\sigma^{33} \vdash \mathsf{33}$ and $\mathsf{33} \vdash \sigma^{33}$. The axiomatization of hexa as summing up worlds relates to the 'state descriptions' in [vB98] and [BM96]. In subsection 5.1 we will discuss the general procedure for computing these descriptions from given finite models, such as hexa.

2.6 Any S5 model of theory 33 is bisimilar to hexa

Proposition 1
Let $M = \langle W, \{\sim_1, \sim_2, \sim_3\}, V\rangle$ be an $S5_3$ model of 33, i.e. $M \models 33$. Then M is bisimilar to hexa.

As we also present a more general version of this proposition, namely for any number of players and cards, we have moved the proof of the underlying proposition to the appendix. As compared to the proof for the general proposition, the underlying proof is more explicit in what axioms are (only) needed in what direction of the bisimulation: see33 is essential in the *forth* part of the proof, dontknowthat33 is essential in the *back* part of the proof.

3 Axioms for players holding cards

In the previous section we have axiomatized the initial state of a game for three players each holding a card. In this section we generalize our results for any number of players and cards. Let $\mathbf{d} \in \mathbf{A}^{\mathbf{C}}$ be a dealing of a (nonempty) finite set $\mathbf{C}$ of cards over a finite set $\mathbf{A}$ of more than two players.

Model of the initial state of a knowledge game

In [vD00c] we presented a pointed $S5_n$ model for the information state of a game where these cards have been dealt and where everybody has (only) looked in his cards (to every combination of an agent a and a card c corresponds an atomic proposition c_a, for 'a holds c'):

$$I_{\mathbf{d}} = \langle D_{\mathbf{d}}, (\sim_a)_{a\in\mathbf{A}}, V\rangle$$

where:

$$D_{\mathbf{d}} = \{d \in \mathbf{A}^{\mathbf{C}} \mid \forall a \in \mathbf{A} : |\mathbf{d}^{-1}(a)| = |\mathbf{d}^{-1}(a)|\}$$

$$\forall d_1, d_2 \in D_{\mathbf{d}} : \forall a \in \mathbf{A} : d_1 \sim_a d_2 \Leftrightarrow d_1^{-1}(a) = d_2^{-1}(a)$$

$$\forall c \in \mathbf{C} : \forall a \in \mathbf{A} : \forall d \in D_{\mathbf{d}} : V_d(c_a) = 1 \Leftrightarrow d(c) = a, \text{ and else } 0$$

Previous to the presentation of the axiomatization, we introduce two central concepts: *type of a dealing*, and *description of a dealing*.

Type of a dealing

Parameters for the axiomatization are: the number of agents, the number of cards, and the actual dealing of cards. So in an indirect manner, as a dealing is a function from cards to agents, only that dealing of cards. We actually need less than that, namely only *the number of cards that each player holds in a dealing d*. We call this the *type* of d.

The type $type(d)$ of a dealing d is the sequence $|d^{-1}(1)|, |d^{-1}(2)|, ..., |d^{-1}(n)|$. Just as for dealings, we use vertical bars as separators. For an example the type of the dealing where 1 holds red, 2 holds white and 3 holds blue: $type(rwb) = type(r|w|b) = 1|1|1$. Another example: $type(ab|cde|f|\varepsilon|gh) = 2|3|1|0|2$. Observe that for any dealing d, $D_d = \{d' \mid type(d') = type(d)\}$, in words: D_d is the set of dealings of type $type(d)$.

Description of a dealing

Let $d \in \mathbf{A}^{\mathbf{C}}$ be a dealing of cards, let $\mathbf{P}$ be the set of all atomic propositions c_a, with $a \in \mathbf{A}$ and $c \in \mathbf{C}$. Define, for all $c_a \in \mathbf{P}$:

$$\begin{array}{rcll} sign_d(c_a) & = & c_a & \text{if } d(c) = a \\ sign_d(c_a) & = & \neg c_a & \text{if } d(c) \neq a \end{array}$$

then:

$$\delta_d = \bigwedge_{c_a \in \mathbf{P}} sign_d(c_a)$$

The formula δ_d is called *the description of the dealing of cards d* or, in a model such where worlds are dealings, *the atomic description of the world d*. The generalization of the axiom $\mathsf{dealings33}$ will be:

$$\mathsf{dealings} := \bigvee_{d' \in D_d} \delta_{d'}$$

We further define:

$$\delta_d^a = \bigwedge_{c \in \mathbf{C}} sign_d(c_a)$$

This formula is called *the description of the cards of player a*. In $\mathbf{S5_n} + \mathsf{dealings}$ we can derive some equivalences that will appear to be useful in the continuation:

$$\forall d \in \mathbf{A}^{\mathbf{C}} : \delta_d \leftrightarrow \bigwedge_{a \in \mathbf{A}} \delta_d^a$$

$$\forall d \in \mathbf{A}^{\mathbf{C}} : \delta_d^a \leftrightarrow \bigvee_{d' \sim_a d} \delta_{d'}$$

3.1 The theory $\mathbf{kgames_d^+}$

As in the previous section, we are looking for the $\mathbf{S5EC_n}$ axiomatization of the model underlying this initial state (where $n = |\mathbf{A}|$). Also as before, we profess uncertainty on the adequacy of $I_{\mathbf{d}}$ as a model for this state, and therefore discuss equivalences among and various weaker versions of the axioms, and the countermodels they have.

Our first try at axiomatization is to generalize the axioms in table 1. We have to keep in mind that the generalization of 'player a holding a (one) card' is 'player a holding $|d^{-1}(a)|$ cards'. We will extensively comment on the generalization of ignorance. Table 3 presents the propositional axiomatization $\mathsf{kgames}^+_{\mathbf{d}}$.

The formula $\delta_{\mathbf{d}} \wedge C_{\mathbf{A}} \mathsf{kgames}^+_{\mathbf{d}}$ describes the knowledge of the players in the initial state of a game for $\mathbf{d}$. Unless confusion would otherwise result, instead of $\mathsf{kgames}^+_{\mathbf{d}}$ or $\mathsf{kgames}^+_{type(\mathbf{d})}$ we sometimes write kgames^+. Similarly, axioms from $\mathsf{dealings}$ are sometimes given an index $\mathbf{d}$. Further we will write, unless confusion results: $\sharp a := |d^{-1}(a)|$, the number of cards of agent $a \in \mathbf{A}$ in dealing d. In axiom $\mathsf{dontknowthat}$ of table 3 we write $\sharp\neg ab$ for $|\mathbf{C}| - \sharp a - \sharp b$ ('the number of cards not held by a or b').

		players see their own cards
see	:=	$\bigwedge_{a\in\mathbf{A}} \bigwedge_{c\in\mathbf{C}} (c_a \rightarrow K_a c_a)$
		players only see their own cards
dontsee	:=	$\bigwedge_{a\in\mathbf{A}} \bigwedge_{c\in\mathbf{C}} (\neg c_a \rightarrow K_a \neg c_a)$
		all cards are different
		(there is at most one card of each colour)
atmost	:=	$\bigwedge_{a\neq b\in\mathbf{A}} \bigwedge_{c\in\mathbf{C}} \neg(c_a \wedge c_b)$
		each player has (at least) a known number of cards
atleast	:=	$\bigwedge_{a\in\mathbf{A}} \bigvee_{c^1\neq\ldots\neq c^{\sharp a}\in\mathbf{C}} \bigwedge_{i=1}^{\sharp a} c^i_a$
		players don't know the cards of others
dontknowthat	:=	$\bigwedge_{a\neq b\in\mathbf{A}} \bigwedge_{c^1\neq\ldots\neq c^{\sharp\neg ab}\in\mathbf{C}} M_a(\bigwedge_{i=1}^{\sharp\neg ab} \neg c^i_b)$
		players can imagine others to hold other cards
dontknownot	:=	$\bigwedge_{a\neq b\in\mathbf{A}} \bigwedge_{c^1\neq\ldots\neq c^{\sharp b}\in\mathbf{C}} (\bigwedge_{i=1}^{\sharp b} \neg c^i_a \rightarrow M_a(\bigwedge_{i=1}^{\sharp b} c^i_b))$

Table 3: The theory $\mathsf{kgames}^+_{\mathbf{d}}$ for dealing $\mathbf{d} \in \mathbf{A}^{\mathbf{C}}$

3.2 $\mathbf{I_d}$ is a model of $\mathsf{kgames}^+_{\mathbf{d}}$

Fact 2
$I_{\mathbf{d}}$ is a model of $\mathsf{kgames}_{\mathbf{d}}$.

The proof is along the same lines as that of fact 1 on page 5.

3.3 Dependencies among axioms

As in the previous section, we establish axiomatic dependencies informally, with natural deduction style proofs. Further, to illustrate nonequivalence of axioms, we sometimes 'go semantical' and give countermodels.

3.3.1 See and dontsee

The following six axioms can all be seen as generalizations of see33 and dontsee33 respectively:

$$
\begin{array}{rcl}
\textsf{see} & := & \bigwedge_{a\in\mathbf{A}}\bigwedge_{c\in\mathbf{C}}(c_a \rightarrow K_a c_a) \\
\textsf{dontsee} & := & \bigwedge_{a\in\mathbf{A}}\bigwedge_{c\in\mathbf{C}}(\neg c_a \rightarrow K_a \neg c_a) \\
\textsf{seeall} & := & \bigwedge_{a\in\mathbf{A}}\bigwedge_{c^1\neq\ldots\neq c^{\sharp a}\in\mathbf{C}}(\bigwedge_{i=1}^{\sharp a} c_a^i \rightarrow K_a \bigwedge_{i=1}^{\sharp a} c_a^i) \\
\textsf{dontseeall} & := & \bigwedge_{a\in\mathbf{A}}\bigwedge_{c^1\neq\ldots\neq c^{\sharp\neg a}\in\mathbf{C}}(\bigwedge_{i=1}^{\sharp\neg a} \neg c_a^i \rightarrow K_a \bigwedge_{i=1}^{\sharp\neg a} \neg c_a^i) \\
\textsf{seedeal} & := & \bigwedge_{a\in\mathbf{A}}\bigwedge_{d\in D_\mathbf{d}}(\delta_d^a \rightarrow K_a \delta_d^a) \\
\textsf{dontseedeal} & := & \bigwedge_{a\in\mathbf{A}}\bigwedge_{d\in D_\mathbf{d}}(\neg\delta_d^a \rightarrow K_a \neg\delta_d^a)
\end{array}
$$

Axioms see and dontsee seem to be the most straightforward generalizations: for every agent and for every single card, if a player holds it he knows that, and if he doesn't hold it, he knows that too. They are therefore listed in kgames$^+$. Instead, seeall and dontseeall express that, if a player holds a given number of cards, he knows them all, and that if he holds all others not, he knows that too. Axiom seedeal (for parameter dealing **d**) expresses that, if a player holds a given number of cards and all others not, he knows that, or in other words: he knows his local state. Axiom dontseedeal expresses that for all local states he doesn't have, he knows that too.

Somewhat surprisingly, all six axioms are equivalent in kgames$^+$. The proofs are simple and use the axiom dealings. Although see appears to be the most straightforward of all six, for other reasons we will retain seedeal instead.

3.3.2 Atmost and atleast

Similar to atleast, the following expresses that each player holds a *fixed* number of cards:

$$\textsf{exactly} := \bigwedge_{a\in\mathbf{A}} \bigtriangledown_{c^1\neq\ldots\neq c^{\sharp a}\in\mathbf{C}} \bigwedge_{i=1}^{\sharp a} c_a^i$$

It holds that $\textsf{atmost}, \textsf{atleast} \vdash \textsf{exactly}$. Informal proof:

Suppose not exactly. Then for some player a both $\bigwedge_{i=1}^{\sharp a} c_a^i$ and $\bigwedge_{j=1}^{\sharp a} {c^*}_a^j$ where $\exists i \leq \sharp a : \forall j \leq \sharp a : c^i \neq c^{*j}$ (we may assume that $\sharp a > 0$). Then a holds more than $\sharp a$ cards. That implies that some other player b holds less than $\sharp b$ cards, using the axiom atmost that all cards are different. Therefore, for no selection of $\sharp b$ cards does $\bigwedge_{i=1}^{\sharp b} c_b^i$ hold. Therefore atleast doesn't hold. Contradiction. Therefore exactly. ■

Just as for three players and three cards, we can derive $\textsf{atleast}, \textsf{atmost} \vdash$ dealings and vice versa, similarly to the proof in the previous section. We will prefer dealings over atmost and atleast.

3.3.3 Ignorance: dontknowthat, dontknownot, and dontknow

The form of ignorance in table 3 is the outcome of a process of gradual generalization of dontknowthat33 and dontknownot33. We will repeat this process, in order to convince the reader of the inevitability of this outcome. Apart from dontknowthat and dontknownot, there is a third way to express ignorance as well, more related to the parameter dealing **d**: this is the axiom dontknow. Fortunately, all three versions of ignorance are equivalent.

Dontknowthat

In 33^+ we only had to take into account one card per player. How to generalize to more than one card? Starting from our original 'observations' that for all different agents a and b and for all cards c we demand $\neg K_a c_b$, we might, instead, now claim that a doesn't know any combination of b's cards: for any $c^1, ..., c^{\sharp b}$: $\neg K_a \bigwedge_{i=1}^{\sharp b} c_b^i$. However, it is obvious that a is not only ignorant about that combination of cards, but also about any single card of the combination, which is a stronger claim, as e.g.:

$$\neg K_a c_b^1 \vdash \neg K_a \bigwedge_{i=1}^{\sharp b} c_b^i.$$

Unfortunately, unlike in the game for three players and three cards, the axiom that $\neg K_a c_b$ for all cards c and different players a and b, isn't strong enough. A counterexample is the game for three players $1, 2, 3$ each holding two cards, with dealing $kl|mn|op$, where 2 has told the others that he holds one of m and n. For any single card, it still holds that a player doesn't know another player to have that card. But players 1 and 3 now clearly are less ignorant than they were initially. So we have to demand the players to be less informed than $\neg K_a c_b$, i.e. $\neg K_a(c_b^1 \vee ... \vee c_b^r)$, for some $r > 1$. This is a stronger claim, as, e.g.:

$$\neg K_a(\bigvee_{i=1}^{r} c_b^i) \vdash \neg K_a c_b^1$$

What is r, or, in other words: how large is our ignorance? We argue that, for a given dealing d, $r = |\mathbf{C}| - |d^{-1}(a)| - |d^{-1}(b)|$. We start to illustrate that with an example:

Again, we look at the game for three players and six cards. Player 1 doesn't know of any *two* cards that player 2 has one of those, but of some combinations of *three* cards he does: e.g. $K_1(m_2 \vee n_2 \vee o_2)$. Why? Suppose $\neg K_1(m_2 \vee n_2 \vee o_2)$, then $M_1(\neg m_2 \wedge \neg n_2 \wedge \neg o_2)$, in other words: it would be conceivable for player 1 that player 2 didn't have all those cards. If that were true, and given that player 1 holds k and l himself, there would be only one card left for 2 to hold: p. But then player 2 would hold only 1 card. This contradicts atleast (or dealings).

The extent of player a's ignorance therefore is, that he doesn't know that another player, b, has one from any of $|\mathbf{C}| - |d^{-1}(a)| - |d^{-1}(b)|$ cards (as before, we write $\sharp\neg ab$ for $|\mathbf{C}| - |d^{-1}(a)| - |d^{-1}(b)|$):

$$\bigwedge_{a\neq b\in\mathbf{A}}\ \bigwedge_{c^1\neq\ldots\neq c^{\sharp\neg ab}\in\mathbf{C}} \neg K_a \bigvee_{i=1}^{\sharp\neg ab} c_b^i$$

This, of course, is equivalent to:

$$\mathsf{dontknowthat} = \bigwedge_{a\neq b\in\mathbf{A}}\ \bigwedge_{c^1\neq\ldots\neq c^{\sharp\neg ab}\in\mathbf{C}} M_a \bigwedge_{i=1}^{\sharp\neg ab} \neg c_b^i$$

Just as for dontknowthat33, we might have considered weakening the axiom dontknowthat with the precondition 'if player 1 doesn't hold these cards'. This is dontknowother:

$$\mathsf{dontknowother} = \bigwedge_{a\neq b\in\mathbf{A}}\ \bigwedge_{c^1\neq\ldots\neq c^{\sharp\neg ab}\in\mathbf{C}} (\bigwedge_{i=1}^{\sharp\neg ab} \neg c_a^i \rightarrow M_a \bigwedge_{i=1}^{\sharp\neg ab} \neg c_b^i)$$

As previously, in the theory $\mathsf{kgames}^+_{\mathbf{d}}$ the version without precondition is provably equivalent to the version with precondition:

$\mathsf{kgames}^+_{\mathbf{d}} \vdash \mathsf{dontknowother}$: Trivial.

$\mathsf{kgames}^+_{\mathbf{d}} - \mathsf{dontknowthat} + \mathsf{dontknowother} \vdash \mathsf{dontknowthat}$: Let $a \neq b \in \mathbf{A}$, and $c^1 \neq \ldots \neq c^{\sharp\neg ab} \in \mathbf{C}$. If $\bigwedge_{i=1}^{\sharp\neg ab} \neg c_a^i$ then from that and dontknowother follows $M_a \bigwedge_{i=1}^{\sharp\neg ab} \neg c_b^i$. Otherwise, $\neg \bigwedge_{i=1}^{\sharp\neg ab} \neg c_a^i$. That is equivalent to $\bigvee_{i=1}^{\sharp\neg ab} c_a^i$, i.e. a holds some r, at most $\sharp a$, of the cards c^i. For all of those, there are (different) cards in $|\mathbf{C}| \setminus \{c^1, \ldots, c^{\sharp\neg ab}\}$ that a doesn't hold. Note that this is possible, because $|\mathbf{C}| - \sharp\neg ab = \sharp a + \sharp b$: remove $\sharp b$ cards from that, and there still remain $\sharp a$ to choose from. Once again, we now have a conjunction of $\sharp\neg ab$ cards $ca^1, \ldots, ca^{\sharp\neg ab}$ that a doesn't hold, and from that and dontknowother follows $M_a \bigwedge_{i=1}^{\sharp\neg ab} \neg ca_b^i$ (i). Now for the r cards that a holds from $c^1, \ldots, c^{\sharp\neg ab}$, we have that a knows that b doesn't hold them (ii). Combining (i) with (ii), we get that $M_a \bigwedge_{i=1}^{\sharp\neg ab} \neg c_b^i$. This proves dontknowthat. ■

Dontknownot

A similar process of gradual generalization leads from dontknownot33 to dontknownot.

By itself, $\neg c_a \rightarrow M_a c_b$ for all cards c and different players a and b, is not strong enough. We cannot derive that a can imagine b to hold a combination of two cards: just as in general we cannot derive $M_a(\varphi \wedge \psi)$ from $M_a\varphi$ and $M_a\psi$. E.g. in the example above for three players each holding two cards, 1 can imagine an atom m_2 to be both false – $M_1\neg m_2$ – and true – $M_1 m_2$, but not at the same time – $\neg M_1(m_2 \wedge \neg m_2)$.

On the other hand, in the same example we want, e.g., that $M_1(m_2 \wedge n_2)$ holds. So, similarly to the generalization leading to dontknowthat, we have to find the largest conjunction still conceivable. Obviously, we cannot imagine another player to hold *more* cards than the number we know him to hold. This number indeed is the required maximum:

$$\mathsf{dontknownot} = \bigwedge_{a \neq b \in \mathbf{A}} \bigwedge_{c^1 \neq \ldots \neq c^{\sharp b} \in \mathbf{C}} (\bigwedge_{i=1}^{\sharp b} \neg c_a^i \rightarrow M_a \bigwedge_{i=1}^{\sharp b} c_b^i)$$

We are about to show that dontknownot is equivalent to dontknowthat, but before that we introduce a third way to describe ignorance: dontknow.

Dontknow

The axioms dontknownot and dontknowthat are unsatisfactory, because they are too much in terms relations *between* players. In the special case of three players and three cards, dontknownot33 and dontknowthat33 were more satisfactory. Because the players only had one card each, it appeared that the formulation of ignorance was for arbitrary (single) cards, and not strictly related to the number of cards of player.

Instead of referring to the amounts of cards of two different players, we might as well refer more directly to the entire dealing of cards. This is the case in dontknow. The following explanation might help to make it appear plausible: Prior to the state of the game where the cards have been dealt and where players have looked into their cards, is the state the cards have been dealt but the players *haven't* seen their own cards yet. In that stage, *all* dealings in $D_{\mathbf{d}}$ are possible for all players. (See also section 4.) Looking up cards then corresponds to revising that maximum ignorance $\bigwedge_{a \in \mathbf{A}} \bigwedge_{d \in D_{\mathbf{d}}} M_a \delta_{d'}$. This can be done by conditionalizing on it. The condition is that, after they have looked up their cards, players only consider dealings that correspond with their own cards. This is the axiom dontknow:

$$\mathsf{dontknow} := \bigwedge_{a \in \mathbf{A}} \bigwedge_{d \in D_{\mathbf{d}}} (\delta_d^a \leftrightarrow M_a \delta_d)$$

Fortunately, all three forms of ignorance are equivalent. This is surprising, because dontknownot and dontknowthat appear to describe complementary kinds of ignorance.

dontknownot $\vdash$ dontknowthat:

Proof: Suppose not. Then there are players a, b and cards $c^1, ..., c^{\sharp\neg ab}$, where $\sharp\neg ab = |\mathbf{C}| - \sharp a - \sharp b$, such that $K_a \bigvee_{i=1}^{\sharp\neg ab} c_b^i$.

Regardless of whether a holds some of these cards $c^1, ..., c^{\sharp\neg ab}$ himself, because $r = |\mathbf{C}| - \sharp a - \sharp b$ there must be at least $\sharp b$ cards other than those, that a doesn't hold, suppose: $ca^1, ..., ca^{\sharp b}$. In other words, we have that: $\neg ca_a^1 \wedge ... \wedge \neg ca_a^{\sharp b}$. Applying dontknownot we get $M_a \bigwedge_{i=1}^{\sharp b} ca_b^i$.

Formula $M_a \bigwedge_{i=1}^{\sharp b} ca_b^i$ means that a considers b to hold the $\sharp b$ cards $ca^1, ..., ca^{\sharp b}$. Formula $K_a \bigvee_{i=1}^{m} c_b^i$ means that a knows that b holds at least one more card, namely from the (other!) cards $c^1, ..., c^m$. Therefore, a considers b to hold more than $\sharp b$ cards. On the other hand, axioms atmost and atleast (or, equivalently, dealings) express that b holds exactly $\sharp b$ cards. Contradiction. ■

dontknow $\vdash$ dontknownot:

Proof: Suppose player a doesn't have any of the cards $c^1, ..., c^{\sharp b}$: $\bigwedge_{i=1}^{\sharp b} \neg c_a^i$. Instead, a has cards $c^{\sharp b+1}, ..., c^{\sharp b+\sharp a}$. Let d^* be a dealing of cards where a has those cards and such that b has all the cards $c^1, ..., c^{\sharp b}$, thus $\bigwedge_{i=1}^{\sharp b} c_b^i$. Formula $c_a^{\sharp b+1} \wedge ... \wedge c_a^{\sharp b+\sharp a}$ is the subformula of $\delta_{d^*}^a$ consisting of all positive literals (ie., atoms). Therefore, from $c_a^{\sharp b+1} \wedge ... \wedge c_a^{\sharp b+\sharp a}$ and dealings follows $\delta_{d^*}^a$. From that and dontknow follows $M_a \delta_{d^*}$. Because $\bigwedge_{i=1}^{\sharp b} c_b^i$ is a subformula of the conjunction δ_{d^*}, and because in general $\Diamond\varphi, \varphi \rightarrow \psi \vdash \Diamond\psi$, it follows that $M_a \bigwedge_{i=1}^{\sharp b} c_b^i$. Therefore $\bigwedge_{i=1}^{\sharp b} \neg c_a^i \rightarrow M_a \bigwedge_{i=1}^{\sharp b} c_b^i$. As the cards $c^1, ..., c^{\sharp b}$ were arbitrary, we have shown that dontknownot. ■

dontknowthat $\vdash$ dontknow:

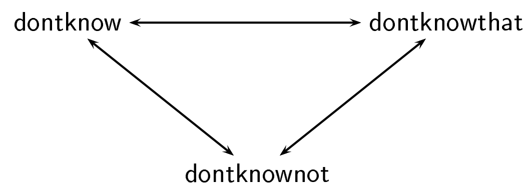
Proof: Let $a \in \mathbf{A}$, $d \in D_{\mathbf{d}}$. Assume δ_d^a. From that and see follows $K_a \delta_d^a$. Suppose $\neg M_a \delta_d$. We have that $\neg M_a \delta_d \leftrightarrow K_a \neg \delta_d \leftrightarrow K_a(\neg \delta_d^1 \vee ... \vee \neg \delta_d^n)$. We show that, for an arbitrary agent b, $\neg \delta_d^b$ leads to a contradiction, so that also $K_a(\neg \delta_d^1 \vee ... \vee \neg \delta_d^n)$ leads to a contradiction.

Suppose $\neg \delta_d^b$, i.e., for some ordering of cards: $\neg c_b^1 \vee \neg c_b^2 \vee ... \vee \neg c_b^{\sharp b} \vee c_b^{\sharp b+1} \vee c_b^{|\mathbf{C}|}$. We may assume that $a \neq b$, because otherwise we have a direct contradiction. Agent a now knows a disjunction of $|\mathbf{C}| - \sharp a - \sharp b$ cards: a has $\sharp a$ cards himself. These cards b doesn't have. Also these cards are all different from the $\sharp b$ cards $c^1, c^2, ..., c^{\sharp b}$, that b also doesn't have. Therefore b must have one of the $|\mathbf{C}| - \sharp a - \sharp b$ remaining cards, and a knows that. This contradicts dontknowthat. Therefore $\delta_d^a \rightarrow M_a \delta_d$.

We prove $M_a \delta_d \rightarrow \delta_d^a$ by contraposition. Assume $\neg \delta_d^a$, i.e. $\neg \bigvee_{d' \sim_a d} \delta_{d'}$. From that, with dealings, follows $\bigvee_{d' \not\sim_a d} \delta_{d'}$. Suppose that $\delta_{d''}^a$ (and keep in mind that $\neg(\delta_d^a \leftrightarrow \delta_{d''}^a)$, because $d'' \not\sim_a d$). From that and see follows, again, $K_a \delta_{d''}^a$, and similarly to the previous argument, and because $d'' \not\sim_a d$, we derive $K_a \neg \delta_d^a$. This is equivalent to $\neg M_a \delta_d^a$. The last implies $\neg M_a \delta_d$ (because of contraposition of the general scheme $\Diamond(\varphi \wedge \psi) \rightarrow \Diamond\varphi$). ■

We have now shown that dontknow $\vdash$ dontknownot, dontknownot $\vdash$ dontknowthat, and dontknowthat $\vdash$ dontknow. Therefore, all three versions of ignorance are equivalent in the theory kgames$^+$:[3]

[3]For better understanding we give an informal argument that from dontknowthat follows dontknownot, even though we don't need it, as we have proven it indirectly: Assume dontknowthat. Suppose dontknownot doesn't hold. I.e. suppose that $\neg c_a^1 \wedge ... \neg c_1^{\sharp b}$ for some players



3.3.4 Boundary case in $\mathbf{kgames_d^+}$

If there are only two players (1 and 2), both players have full knowledge of the dealing of cards. The axiom dontknowthat 'disappears', as $|\mathbf{C}| - \sharp 1 - \sharp 2 = 0$. In some of the proofs (namely, those about ignorance) we have essentially used that there are more than two players. Dontknownot still holds, but now 1 cannot only *imagine* 2 to have some cards he does not have, but *knows* it, because those are all the other cards.

If there are more than two players, it still can be the case that the cards are dealt over only two of those players, i.e. $\exists a, b \in \mathbf{A} : |d^{-1}(a)| + |d^{-1}(b)| = |C|$. Suppose that is the case for players 1 and 2. Now, dontknowthat disappears for just the combination of 1 and 2. Although 1 and 2 still have full knowledge of the dealing of cards, the other players haven't, and 1 and 2 know that.

3.4 The theory $\mathbf{kgames_d}$

Given all the dependencies between axioms, that we have proven in the previous subsection, we are now left with the choice how to simplify the theory $\mathsf{kgames}_\mathbf{d}^+$. The axioms seedeal, dealings and dontknow suffice. Given that we prove in $\mathbf{S5_n}$, so that for all a and d: $K_a\delta_d^a \rightarrow \delta_d^a$, we further propose to combine seedeal $= \bigwedge_{a\in\mathbf{A}} \bigwedge_{d\in D_\mathbf{d}} (\delta_d^a \rightarrow K_a\delta_d^a)$ and dontknow $= \bigwedge_{a\in\mathbf{A}} \bigwedge_{d\in D_\mathbf{d}} (\delta_d^a \leftrightarrow M_a\delta_d)$ into one axiom seedontknow. This appears to be the most elegant formulation of the theory. We therefore present the following as the theory $\mathsf{kgames}_\mathbf{d}$, a shorter but equivalent version of $\mathsf{kgames}_\mathbf{d}^+$:

dealings	:=	$\bigvee_{d\in D_\mathbf{d}} \delta_d$
seedontknow	:=	$\bigwedge_{a\in\mathbf{A}} \bigwedge_{d\in D_\mathbf{d}} (K_a\delta_d^a \leftrightarrow M_a\delta_d)$

Table 4: The theory $\mathsf{kgames}_\mathbf{d}$, for dealing $\mathbf{d} \in \mathbf{A}^\mathbf{C}$

$a \neq b$ and cards $c^1, ..., c^{\sharp b}$, and that $\neg M_a(c_b^1 \wedge ... \wedge c_b^{\sharp b})$. The last is equivalent to: a knowing that b doesn't hold at least one of these cards. As neither a himself holds one of those cards, there must be a third player holding one of them. Now simplify the game by throwing all other players on one heap, so to speak: give all other cards in the hands of an imaginary player b'. Then a knows that b' holds one of $c^1, ..., c^{\sharp b}$: $K_a(c_{b'}^1 \vee ... \vee c_{b'}^{\sharp b})$. As $|C| - \sharp a - \sharp b' = \sharp b$, this contradicts dontknowthat. Therefore dontknownot.

Just as for the case of three persons and three cards, we have shown that we cannot substantially *weaken* the theory, by deleting axioms or weakening axioms. We now show that we also do not need to *strengthen* the theory, because $\mathbf{I_d}$ is its only model. Together, this shows that we have chosen the right model, and the right axioms, for describing the game state where a finite amount of cards are dealt over a finite amount of players.

3.5 All models of $\mathbf{kgames_d}$ are bisimilar to $\mathbf{I_d}$

Proposition 2
Let M be an $S5_n$ model of theory $\mathsf{kgames}_{\mathbf{d}}$. Then M is bisimilar to $I_{\mathbf{d}}$.

Proof Write $M = \langle W^M, (\sim_a^M)_{a\in\mathbf{A}}, V^M\rangle$. We have that $M \models \mathsf{kgames}_{\mathbf{d}}$. Write $I_{\mathbf{d}} = \langle D_{\mathbf{d}}, (\sim_a)_{a\in\mathbf{A}}, V\rangle$, for the intended initial model $I_{\mathbf{d}}$. Observe that, because $M \models \mathsf{dealings}$, each world $w \in M$ has a valuation $V_w = V_d$ for some $d \in D_{\mathbf{d}}$. Define relation $\mathfrak{R} \subseteq (M \times I_{\mathbf{d}})$ as follows:

$$\forall w \in M : \forall d \in D_{\mathbf{d}} : \mathfrak{R}(w,d) \Leftrightarrow V_w = V_d$$

We prove that $\mathfrak{R}$ is a bisimulation between M and $I_{\mathbf{d}}$.

Forth:
Let $w, w' \in M$, let $d \in D_{\mathbf{d}}$. Suppose that $\mathfrak{R}(w,d)$ and that, for an arbitrary $a \in \mathbf{A}$: $w \sim_a w'$. We find an $\mathfrak{R}$-image of w', in $D_{\mathbf{d}}$, as follows:

Observe that $I_{\mathbf{d}}, d \models \delta_d$. As $V_w = V_d$, also $M, w \models \delta_d$. Therefore $M, w \models M_a\delta_d$. From that and $M, w \models \mathsf{seedontknow}$ follows $M, w \models K_a\delta_d^a$. From that and $w \sim_a^M w'$ follows $M, w' \models \delta_d^a$, i.e.: $M, w' \models \bigvee_{d' \sim_a d} \delta_{d'}$. Therefore there is a $d' \sim_a d$ such that $M, w' \models \delta_{d'}$. *That* d' is the required $\mathfrak{R}$-image of w': note that $d \sim_a d'$, and that $V_{w'}^M = V_{d'}$, because also, obviously, $I_{\mathbf{d}}, d' \models \delta_{d'}$.

Back:
Let $d, d' \in I_{\mathbf{d}}$, let $w \in M$. Suppose that $\mathfrak{R}(w,d)$ and that, for an arbitrary $a \in \mathbf{A}$, $d \sim_a d'$. We find an $\mathfrak{R}$-original of d', in M, as follows:

$M, w \models \delta_d$
$\Rightarrow$ reflexivity
$M, w \models M_a\delta_d$
$\Leftrightarrow$ from $\mathsf{seedontknow}$
$M, w \models K_a\delta_d^a$
$\Leftrightarrow$
$\forall w'' \sim_a w : M, w'' \models \delta_d^a$
$\Leftrightarrow$
$\forall w'' \sim_a w : M, w'' \models \bigvee_{d'' \sim_a d} \delta_{d''}$
$\Leftrightarrow$
$\forall w'' \sim_a w : \exists d'' \sim_a d : M, w'' \models \delta_{d''}$

$\Leftrightarrow$ as $d \sim_a d'$

$\forall w'' \sim_a w : \exists d'' \sim_a d' : M, w'' \models \delta_{d''}$

$\Leftrightarrow$

$\forall w'' \sim_a w : M, w'' \models \bigvee_{d'' \sim_a d'} \delta_{d''}$

$\Leftrightarrow$

$\forall w'' \sim_a w : M, w'' \models \delta^a_{d''}$

$\Leftrightarrow$

$M, w \models K_a \delta^a_{d'}$

$\Leftrightarrow$ from seedontknow

$M, w \models M_a \delta_{d'}$

$\Rightarrow$

$\exists w' \sim_a w : M, w' \models \delta_{d'}$

Any w' satisfying the last statement is an $\mathfrak{R}$-original of d', as $M, w' \models \delta^{d'}$ says that $V_{w'} = V_{d'}$.

Note that in the forth part of the proof, we have only essentially used that, for any agent a and dealing d', $M_a \delta_{d'} \rightarrow K_a \delta^a_{d'}$, whereas in the back part of the proof, we have also essentially used the reverse: $K_a \delta^a_{d'} \rightarrow M_a \delta_{d'}$. Further, note that, in the proof, we only use reflexivity of models; the proposition therefore holds for all T_n models. ■

Instead of this direct proof, there is also an indirect proof. The indirect proof uses that the model $I_{\mathbf{d}}$ can be constructed by executing an action in a simpler model for card games. In the next section, 4, we now present this simpler model.

4 Axioms for players not seeing their cards

We have described and axiomatized the state of the game where the cards have been dealt and where players have looked into their cards. We called it the initial state of a knowledge game. This game state is preceded by a state where the cards have been dealt but where the players haven't looked into their cards yet. Assume that everybody sees how many cards lie upside down in front of each player. Therefore players know the *type* of the actual dealing. They know how many cards they have, and how many cards everybody else, but they do not what these cards are. Still, it is enough for all players to know (compute) the set of relevant dealings given actual dealing $\mathbf{d}$, as $D_{\mathbf{d}} = D_{type(\mathbf{d})}$.

We call the model underlying this state the pre-initial model $preI_{\mathbf{d}}$. In the state $(preI_{\mathbf{d}}, \mathbf{d})$, *all* dealings in $D_{\mathbf{d}}$ are possible for all players, i.e. each player's access on the set of relevant dealings is the universal relation:

$$preI_{\mathbf{d}} = \langle D_{\mathbf{d}}, (\sim_a)_{a \in \mathbf{A}}, V \rangle$$

where

$$\forall a \in \mathbf{A} : \forall d, d' \in D_{\mathbf{d}} : d \sim_a d'$$

and, as before:

$$\forall d \in D_{\mathbf{d}} : \forall c_a \in \mathbf{P} : V_d(c_a) = 1 \text{ iff } d(c) = a.$$

We continue with axiomatizing this model $preI_{\mathbf{d}}$.

4.1 The theory prekgames$_{\mathbf{d}}$

Let $\mathbf{d}$ be a dealing of cards. The theory $\mathsf{prekgames}_{\mathbf{d}}$ is the conjunction of the axioms in table 5.

$$\begin{array}{rcl} \mathsf{dealings} & := & \bigvee_{d \in D_{\mathbf{d}}} \delta_d \\ \mathsf{dontknowany} & := & \bigwedge_{a \in \mathbf{A}} \bigwedge_{d \in D_{\mathbf{d}}} M_a \delta_d \end{array}$$

Table 5: The theory $\mathsf{prekgames}_{\mathbf{d}}$, for parameter dealing $\mathbf{d}$

It will be obvious that $preI_{\mathbf{d}}$ is a model of theory prekgames for parameter dealing $\mathbf{d}$. We now proceed as follows: first we prove that the theory describes this model, or, in other words, that all other models of prekgames are bisimilar to $preI_{\mathbf{d}}$. Then we show that the model $I_{\mathbf{d}}$, describing the initial state of a knowledge game, can be constructed from $preI_{\mathbf{d}}$ by executing a knowledge action type (as defined in [vD00a]). We can then prove the uniqueness of $I_{\mathbf{d}}$ without having to worry about its axiomatization! This is also convenient, because the bisimilarity proof to establish the uniqueness of $I_{\mathbf{d}}$ is more complex than the one below, to establish the uniqueness of the pre-initial state $preI_{\mathbf{d}}$.

4.2 All models of prekgames$_{\mathbf{d}}$ are bisimilar to preI$_{\mathbf{d}}$

Proposition 3
Let $\mathbf{d} \in \mathbf{A}^{\mathbf{C}}$ be a dealing of cards. Let M be a model of prekgames for parameter $\mathbf{d}$. Then M is bisimilar to $preI_{\mathbf{d}}$.

Proof Let M be a model of prekgames for parameter $\mathbf{d}$. Write $M = \langle W^M, (\sim_a^M)_{a \in \mathbf{A}}, V^M \rangle$. We remind the reader that $preI_{\mathbf{d}} = \langle D_{\mathbf{d}}, (\sim_a)_{a \in \mathbf{A}}, V \rangle$.

First observe that, because $M \models \mathsf{dealings}$, each world $w \in M$ has a valuation $V_w = V_d$ for some $d \in D_{\mathbf{d}}$.

Define relation $\mathfrak{R} \subseteq (M \times preI_{\mathbf{d}})$ as follows:

$$\forall w \in M : \forall d \in D_{\mathbf{d}} : \mathfrak{R}(w, d) \Leftrightarrow V_w = V_d$$

We prove that $\mathfrak{R}$ is a bisimulation between M and $preI_{\mathbf{d}}$.

Forth: Let $w, w' \in M$. Let $d \in D_{\mathbf{d}}$. Suppose that $\mathfrak{R}(w, d)$ and that, for an arbitrary agent $a \in \mathbf{A}$, $w \sim_a^M w'$. From our observation on valuations in M, it follows that there is a dealing $d' \in D_{\mathbf{d}}$ such that $V_{w'} = V_{d'}$. This dealing d' is our required $\mathfrak{R}$-image in $D_{\mathbf{d}}$: because $\sim_a$ is universal on $D_{\mathbf{d}}$ it trivially holds that $d \sim_a d'$, and because $V_{w'} = V_{d'}$ we have that $\mathfrak{R}(w', d')$.

Back: Let $d, d' \in D_{\mathbf{d}}$. Let $w \in M$. Suppose that $\mathfrak{R}(w, d)$ and that, for an arbitrary agent $a \in \mathbf{A}$, $d \sim_a d'$.

Suppose there is no $w' \in M$ such that $V_{w'} = V_{d'}$. In other words: there is no $w' \in M$ such that $M, w' \models \delta_{d'}$. Then, in particular there is no $w' \in M$ such that $w \sim_a^M w'$ and $M, w' \models \delta_{d'}$, thus $M, w \not\models M_a \delta_{d'}$, thus $M, w \not\models \mathsf{dontknowany}$. Contradiction.

Therefore there is such a $w' \in M$, and, as we have shown, there is even a w' that is $\sim_a^M$-related to w. This world w' is our required $\mathfrak{R}$-original in M: we have that $w \sim_a^M w'$, and because $V_{w'} = V_{d'}$ we have that $\mathfrak{R}(w', d')$. ■

4.3 Looking up cards

In [vD00a], see also [vD99], we presented the language DKL of dynamic knowledge logic, containing dynamic modal operators $[\alpha]$ for actions $\alpha \in \mathsf{KA}$ and action types $\alpha \in \mathsf{KT}$, and its interpretation. The action of all players looking up (turning) their cards, in a state where those cards have been dealt, is a KT action type. We call this action type $\mathsf{lookup}_{\mathbf{A}}$. KT actions and KA action types have a precise formal interpretation $[\![\cdot]\!]$, that is a relation between $S5$ models. If the relation is functional, we can use $[\![\cdot]\!]$ as a postfix unary operator. We now have that:

Fact 3
$preI_{\mathbf{d}}[\![\mathsf{lookup}_{\mathbf{A}}]\!] = I_{\mathbf{d}}$

In other words: the model $I_{\mathbf{d}}$ is the unique model resulting from executing an action of type $\mathsf{lookup}_{\mathbf{A}}$ in the model $preI_{\mathbf{d}}$. Using fact 3, we have an 'indirect' proof of proposition 2 that $I_{\mathbf{d}}$ is the unique model of kgames:

Indirect proof of proposition 2: Let M be a model of kgames. For every agent a, add access for a between all $\mathbf{A}$-related worlds in M that are not a-related. The resulting model M' is a model of $\mathsf{prekgames}$. It holds that $M'[\![\mathsf{lookup}_{\mathbf{A}}]\!] = M$. Because M' is bisimilar to $preI_{\mathbf{d}}$, $M'[\![\mathsf{lookup}]\!]$ is bisimilar to $preI_{\mathbf{d}}[\![\mathsf{lookup}]\!]$, i.e., using fact 3, M is bisimilar to $I_{\mathbf{d}}$. ■

In the proof, we used that bisimilarity is invariant under execution of action types with a functional interpretation. For details, see [vD00a]. For the definition of $\mathsf{lookup}_{\mathbf{A}}$ and the proof that its interpretation is functional, see [vD00b].

5 Further observations

5.1 Modal fixed points

Even though we considered both models and axiomatizations for card game states, we ended up in axiomatizing two different $S5_n$ models, and their states. For modal models and states in general, their description (or, as it is called, 'characteristic formula') in an infinitary (propositional) modal logic can be computed by a fixed point construction. See [vB98], relating to [BM96]. In this subsection we show that our specific results are in accordance with this general construction.

We apply the construction in [vB98], chapter 5 (Modal Fixed Points and Bisimulation), to the initial model $I_{\mathbf{d}}$ of a knowledge game for dealing $\mathbf{d}$. A given model M can be described with the following fixed-point construction:

$$E(M) = \bigwedge_{w \in M} E(M, w) = \bigwedge_{w \in M} (p_w \rightarrow (\delta_w \wedge \bigwedge_{Rwv} \Diamond p_w \wedge \Box \bigvee_{Rwv} p_w))$$

Here δ_w is the atomic description of world w, as usual, and all p_w are fresh atoms.

Beyond that, if M is finite, we can replace the atomic variables p_w by a unique modal definition Δ_w of w in M. Indeed, this is the case for knowledge game states. Initial (and pre-initial) states $(I_{\mathbf{d}}, d)$ of knowledge games, such as (hexa, rwb), are finite, and all worlds even differ in their *atomic* description δ_d. So δ_d already serves as a unique modal definition Δ_d of worlds $d \in I_{\mathbf{d}}$. We can then describe a solution for the equation above by replacing all p_w by δ_w (i.e. p_d by δ_d). Also switching to a multiagent epistemic language we get the equation:

$$E(I_{\mathbf{d}})[p_d := \delta_d] = E^{\delta}(I_{\mathbf{d}}) = \bigwedge_{d \in I_{\mathbf{d}}} \bigwedge_{a \in \mathbf{A}} (\delta_d \rightarrow (\delta_d \wedge (\bigwedge_{d \sim_a d'} M_a \delta_{d'}) \wedge K_a \bigvee_{d \sim_a d'} \delta_{d'}))$$

As $\delta_d^a \leftrightarrow \bigvee_{d \sim_a d'} \delta_{d'}$, and as δ_d in the consequent is superfluous, we get:

$$E^{\delta}(I_{\mathbf{d}}) = \bigwedge_{d \in I_{\mathbf{d}}} \bigwedge_{a \in \mathbf{A}} (\delta_d \rightarrow ((\bigwedge_{d \sim_a d'} M_a \delta_{d'}) \wedge K_a \delta_d^a))$$

For $I_{\mathbf{d}} = \mathsf{hexa}$ and $w = rwb$ we get the following (in the consequent, we also delete formulas expressing reflexivity):

$$E^{\delta}(\mathsf{hexa}, rwb) = \delta_{rwb} \rightarrow (M_1 \delta_{rbw} \wedge M_2 \delta_{bwr} \wedge M_3 \delta_{wrb} \wedge K_1 r_1 \wedge K_2 w_2 \wedge K_3 b_3)$$

What is the relation between $\mathsf{kgames}_{\mathbf{d}}$ and $E^{\delta}(I_{\mathbf{d}})$? Because in $E^{\delta}(I_{\mathbf{d}})$ we replaced atomic (fresh) variables p_d by atomic descriptions δ_d, we have to assume explicitly that we 'live' in one of these worlds/dealings, i.e. that one of these descriptions holds. Differently said: $\mathsf{dealings} = \bigvee_{d \in D_{\mathbf{d}}} \delta_d$, given parameter dealing $\mathbf{d}$, is an axiom. It holds that $\mathsf{kgames}_{\mathbf{d}} = \mathsf{dealings} + \mathsf{seedontknow}$ is logically equivalent to $\mathsf{dealings} + E^{\delta}(I_{\mathbf{d}})$.

Proposition 4
$\mathsf{kgames}_{\mathbf{d}} \leftrightarrow \mathsf{dealings} + E^{\delta}(I_{\mathbf{d}})$

It suffices to prove that in $\mathbf{S5_n}$ plus dealings, seedontknow is equivalent to $E^{\delta}(I_{\mathbf{d}})$:

$$\underbrace{\bigwedge_{d \in I_{\mathbf{d}}} \bigwedge_{a \in \mathbf{A}} (\delta_d \rightarrow (\bigwedge_{d \sim_a d'} M_a \delta_{d'}) \wedge K_a \delta^a_d)}_{(i)} \quad \Leftrightarrow \quad \underbrace{\bigwedge_{d \in I_{\mathbf{d}}} \bigwedge_{a \in \mathbf{A}} (K_a \delta^a_d \leftrightarrow M_a \delta_d)}_{(ii)}$$

For the proof, see the appendix.

Models versus states

We close this subsection with applying another observation from [vB98], on the relation between model and state descriptions. Given the finite model description $E^{\Delta}(M) = E(M)[p_w := \Delta_w]$, where Δ_w is a unique modal description of world w, a state from M is described by the formula

$$\Delta_w \wedge \Box^* E^{\Delta}(M).$$

Given that $\mathsf{kgames}_{\mathbf{d}}$ is the description of the model $I_{\mathbf{d}}$, we can therefore describe any of its states $(I_{\mathbf{d}}, d)$ by

$$\delta_d \wedge C_{\mathbf{A}}(\mathsf{kgames}_{\mathbf{d}}).$$

We even can replace the common knowledge operator by the iteration $E_{\mathbf{A}}^{\mathsf{max}}$ (with E the general knowledge operator, and not the 'description operator' of a model), where max is the maximin length of a path between two dealings. (Except for some examples, we do not know the actual value of max; we conjecture that $\mathsf{max} \leq |\mathbf{A}|$.)

An example: the state (hexa, rwb) can be described by $\delta_{rwb} \wedge C_{123}\mathsf{33}$. As any world of hexa can be reached by a $\{1,2,3\}$-path ($\bigcup_{i \in \{1,2,3\}} \sim_i$-path)of at most length 2, we can replace this by $\delta_{rwb} \wedge E_{123}E_{123}\mathsf{33}$.

Similarly, we can compute that $\mathsf{prekgames}_{\mathbf{d}} = \mathsf{dealings} + \mathsf{dontknowany}$ is equivalent to $\mathsf{dealings} + E^{\delta}(preI_{\mathbf{d}})$. Other than kgames, where we had to prove an equivalence, computing $E^{\delta}(preI_{\mathbf{d}})$ directly results in dontknowany.

5.2 Frame characteristics

We have axiomatized models for knowledge games. One might wonder whether the frames underlying these models cannot be characterized directly. Frames for knowledge games for n players can be seen as distributed systems for n processors, 'with holes'. In [Lom99, LvdMR00] a distributed system for n processors that only know their own states is called an n-dimensional hypercube. Hypercubes have the property of being *weakly-directed*: for any world w, and for any n (possibly different) worlds that the n agents can access from w, there is a world w' accessible from those n worlds for those n agents (possibly in a different order). They are characterized by a multiagent modal axiom **WD**. See also [vD00c].

Figure 4, below, illustrates how hexa relates to a three-dimensional hypercube, i.e., a cube. On the left, the model ('hypercube') of the distributed system for three processors 1, 2 and 3 each having three possible local state values r, w and b. On the right, hexa, the model of an initial state of a game for three players 1, 2, and 3 and three cards r, w and b. In the hypercube on the left, we have highlighted the worlds and the access that we find in hexa, on the right.

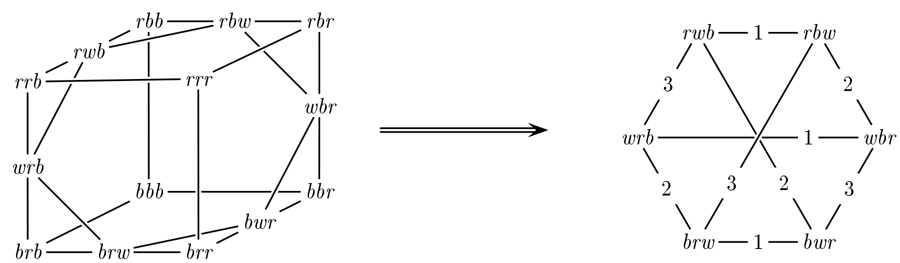


Figure 4: On the left, a three-dimensional hypercube. On the right, hexa.

5.3 Belief revision

In [vD00a] we define a multiagent modal language with dynamic operators for actions ('knowledge changing operations'). These actions are interpreted as a relation between *S5* models and *S5* states. Instead, here we have axiomatized two different models, $I_{\mathbf{d}}$ and $preI_{\mathbf{d}}$. We have seen that $preI_{\mathbf{d}}$ can be transformed into $I_{\mathbf{d}}$ by a lookup action type. (Similarly game states $(preI_{\mathbf{d}}, d')$ can be transformed into game states $(I_{\mathbf{d}}, d')$ by a lookup action.) We now have a relation between the axiomatizations of these models:

$\models \mathsf{prekgames} \rightarrow [\mathsf{lookup}]\mathsf{kgames}$

$\Leftrightarrow$ by definition

$\forall M : M \models \mathsf{prekgames} \rightarrow [\mathsf{lookup}]\mathsf{kgames}$

$\Leftrightarrow$ as only $preI_{\mathbf{d}}$ is a model of prekgames

$preI_{\mathbf{d}} \models [\mathsf{lookup}]\mathsf{kgames}$

$\Leftrightarrow$ by definition of interpretation in DKL

$preI_{\mathbf{d}}[\![\mathsf{lookup}]\!] \models \mathsf{kgames}$

$\Leftrightarrow$ by fact 3

$I_{\mathbf{d}} \models \mathsf{kgames}$

Complementary to our semantic approach, that relates theories by the unique models that they describe, one might consider to relate these theories in a more direct way by a process of *belief revision*. Intuitively, we have already done that. An example: In $preI_{\mathbf{d}}$, axiom

$$\mathsf{dontknowany} = \bigwedge_{a \in \mathbf{A}} \bigwedge_{d' \in \mathbf{D}_d} M_a \delta_{d'}$$

describes that all dealings are relevant. In the transition from $preI_{\mathbf{d}}$ to $I_{\mathbf{d}}$, where all players look up their cards, we have to weaken this axiom: no longer all dealings are conceivable to all players. An obvious way to weaken it, is to conditionalize the $M_a \delta_{d'}$: after 'lookup', a dealing is only conceivable if your known private state corresponds to it, and vice versa:

$$\mathsf{seedontknow} = \bigwedge_{a \in \mathbf{A}} \bigwedge_{d' \in \mathbf{D}_d} (K_a \delta^a_{d'} \leftrightarrow M_a \delta_{d'}).$$

Can these 'intuitions' be made more precise? We imagine axiom seedontknow being computed from something like $\mathsf{dontknowany}[\mathsf{lookup}_{\mathbf{A}}]$, where $\varphi[\alpha]$ stands for: 'the revision of axiom φ as a consequence of the execution of action α'. The process of revising prekgames would then be:

$$\begin{aligned} \mathsf{prekgames}_{\mathbf{d}}[\mathsf{lookup}_{\mathbf{A}}] &= (\mathsf{dealings} \wedge \mathsf{dontknowany})[\mathsf{lookup}_{\mathbf{A}}] \\ &= \mathsf{dealings}[\mathsf{lookup}_{\mathbf{A}}] \wedge \mathsf{dontknowany}[\mathsf{lookup}_{\mathbf{A}}] \\ &= \mathsf{dealings} \wedge \mathsf{seedontknow} \end{aligned}$$

This topic of 'direct' theory revision is also discussed in [vB00]. A procedure is given for the special case of actions that are public announcements. In [vB00], the process is not called 'theory revision', as we do, but *syntactic relativization of a formula*.

5.3.1 Axiomatizing other game states

In order to axiomatize other game states, and in general other states resulting from action execution, we obviously need a way of systematic belief revision.

An example: in a knowledge game state we can execute a show action (see [vD99]). We have to revise once more our ignorance, and only consider dealings that are consistent with both our private state *and* with that what we now know from the private state of others (the card that we have seen). An extra complication is that this involves *subgroup* common knowledge.

Just as for $preI_{\mathbf{d}}$ and $I_{\mathbf{d}}$, any game state resulting from executing a game action in $I_{\mathbf{d}}$ will still satisfy dealings and see. Only ignorance has to be revised. Further, any game state will be *finite*, as it is constructed by a KT action type from the finite model $I_{\mathbf{d}}$, and it is also *unique*, because $I_{\mathbf{d}}$ is unique and bisimilarity is invariant under action execution.

We have not pursued this course any further. The area of knowledge games seems to be a fruitful playing field for interactions between belief revision and model updating. Because our action semantics is clear and simple, we have an easily check proposed beliefs revisions.

6 Conclusion

We have axiomatized two different game states for card games. We started with the state where some cards are dealt over players and where players hold the cards in their hands, i.e. where they can see their own cards. For the state of three players and three cards, we showed that the preferred model hexa is described by the theory 33. For the state of any finite number of players and cards, in other words for any dealing $\mathbf{d}$ of cards over players, we showed that the preferred model $I_{\mathbf{d}}$ is described by the theory $\mathsf{kgames}_{\mathbf{d}}$. In particular we have described various equivalent versions of the axiom see that expresses that a player knows the cards that he holds, and we have described three different axioms that express ignorance, that are all equivalent to each other. Prior to the state where players have picked up their cards from the table, is the state where cards have been dealt over players but where they haven't picked them up yet. We have proven that its preferred model $preI_{\mathbf{d}}$ is described by the axiomatization $\mathsf{prekgames}_{\mathbf{d}}$. We have shown that our results correspond to those of fixed point computations of the description of modal models.

Appendix

Proof from section 2.2

Proof of fact 1: $\mathsf{hexa} \models 33^+$. From all axioms we prove a typical case. In the proofs, read $\mathsf{hexa}, w \models \varphi$ for $w \models \varphi$.

$\mathsf{hexa} \models \mathsf{see33}$:

$rwb \models r_1$ and $rbw \models r_1$
$\Leftrightarrow$ as $[rwb]_{\sim_1} = \{rwb, rbw\}$
$rwb \models K_1 r_1$
$\Rightarrow$ as $rwb \models r_1$
$rwb \models r_1 \rightarrow K_1 r_1$

$\mathsf{hexa} \models \mathsf{dontsee33}$:

$rwb \not\models w_1$ and $rbw \not\models w_1$
$\Leftrightarrow$
$rwb \models \neg w_1$ and $rbw \models \neg w_1$
$\Leftrightarrow$ as $[rwb]_{\sim_1} = \{rwb, rbw\}$
$rwb \models K_1 \neg w_1$
$\Rightarrow$ as $rwb \models \neg w_1$
$rwb \models \neg w_1 \rightarrow K_1 \neg w_1$

$\mathsf{hexa} \models \mathsf{atmost33}$:

$rwb \models r_1$ and $rwb \not\models r_2$
$\Rightarrow$
$rwb \not\models r_1 \wedge r_2$
$\Leftrightarrow$
$rwb \models \neg(r_1 \wedge r_2)$

hexa $\models$ atleast33:

$rwb \models r_1$
$\Rightarrow$
$rwb \models r_1 \vee w_1 \vee b_1$

hexa $\models$ dontknowthat33:

$bwr \not\models r_1$
$\Rightarrow$
$bwr \not\models r_1$ or $rwb \not\models r_1$
$\Leftrightarrow$ as $rwb \models K_2 r_1 \Leftrightarrow (rwb \models r_1$ and $bwr \models r_1)$
$rwb \not\models K_2 r_1$
$\Leftrightarrow$
$rwb \models \neg K_2 r_1$

hexa $\models$ dontknownot33:

$rwb \models r_1$
$\Rightarrow$ as all access is reflexive
$rwb \models M_2 r_1$
$\Rightarrow$ as $rwb \models \neg r_2$
$rwb \models \neg r_2 \rightarrow M_2 r_1$
$\blacksquare$

Proofs from section 2.3

33^+ − see33 $\vdash$ see33:

We prove the case $r_1 \rightarrow K_1 r_1$.
Suppose r_1.
From dealings33 follows $r_1 \rightarrow \neg w_1$. From r_1 and $r_1 \rightarrow \neg w_1$ follows $\neg w_1$. From $\neg w_1$ and dontsee33 follows $K_1 \neg w_1$.
From dealings33 follows $r_1 \rightarrow \neg b_1$. From r_1 and $r_1 \rightarrow \neg b_1$ follows $\neg b_1$. From $\neg b_1$ and dontsee33 follows $K_1 \neg b_1$.
From $K_1 \neg w_1$ and $K_1 \neg b_1$ follows $K_1(\neg w_1 \wedge \neg b_1)$. From atleast33 follows $r_1 \vee w_1 \vee b_1$. From $\neg w_1 \wedge \neg b_1$ and $r_1 \vee w_1 \vee b_1$ follows r_1. Therefore from $K_1(\neg w_1 \wedge \neg b_1)$ and $K_1(r_1 \vee w_1 \vee b_1)$ (as atleast33 is commonly known) follows $K_1 r_1$.
Therefore $r_1 \rightarrow K_1 r_1$. $\blacksquare$

$\mathsf{atmost33}, \mathsf{atleast33} \vdash \mathsf{function33}$:

We show by informal proof that $\mathsf{atmost33}, \mathsf{atleast33} \vdash \mathsf{function33}$, for the case $r_1 \nabla r_2 \nabla r_3$. First we show that there is *at most* one player holding the red card, then we show that there is *at least* one player holding the red card.

Suppose more than one player holds the red card, e.g. $r_1 \wedge r_2$. This contradicts $\neg(r_1 \wedge r_2)$, which is a conjunct from $\mathsf{atmost33}$. Therefore at most one player holds the red card.

Suppose nobody holds the red card, i.e. $\neg r_1 \wedge \neg r_2 \wedge \neg r_3$. From $r_1 \vee w_1 \vee b_1$ and $\neg r_1$ follows $w_1 \vee b_1$. From $r_2 \vee w_2 \vee b_2$ and $\neg r_2$ follows $w_2 \vee b_2$. From $r_3 \vee w_3 \vee b_3$ and $\neg r_3$ follows $w_3 \vee b_3$.

Either w_1 or $\neg w_1$.

Suppose $\neg w_1$. From $\neg w_1$ and $w_1 \vee b_1$ follows b_1. From b_1 and $\neg(b_1 \wedge b_2)$ follows $\neg b_2$. From $\neg b_2$ and $w_2 \vee b_2$ follows w_2. From b_1 and $\neg(b_1 \wedge b_3)$ follows $\neg b_3$. From $\neg b_3$ and $w_3 \vee b_3$ follows w_3. From w_2, w_3 and $\neg(w_2 \wedge w_3)$ follows a contradiction.

Suppose w_1. From w_1 and $\neg(w_1 \wedge w_2)$ follows $\neg w_2$. From $\neg w_2$ and $w_2 \vee b_2$ follows b_2. From b_2 and $\neg(b_2 \wedge b_3)$ follows $\neg b_3$. From $\neg b_3$ and $w_3 \vee b_3$ follows w_3. From w_1, w_3 and $\neg(w_1 \wedge w_3)$ follows a contradiction.

Therefore the assumption that nobody holds the red card leads to a contradiction. Therefore at least one player holds the red card.

Therefore exactly one player holds the red card: $r_1 \nabla r_2 \nabla r_3$. ■

$\mathsf{atmost33}, \mathsf{atleast33} \vdash \mathsf{exactly33}$:

Suppose $\neg(r_1 \nabla w_1 \nabla b_1)$. Then either player 1 doesn't hold any cards, or player 1 holds more than 1 card. Player 1 *not* holding any cards is a contradiction with $\mathsf{atleast33}$. Therefore suppose that he holds more than 1: e.g. that $r_1 \wedge w_1$. From r_1 and $\neg(r_1 \wedge r_2)$ follows $\neg r_2$. From w_1 and $\neg(w_1 \wedge w_2)$ follows $\neg w_2$. From $\neg r_2$ and $\neg w_2$ and $r_2 \vee w_2 \vee b_2$ follows b_2. From r_1 and $\neg(r_1 \wedge r_3)$ follows $\neg r_3$. From w_1 and $\neg(w_1 \wedge w_3)$ follows $\neg w_3$. From $\neg r_3$ and $\neg w_3$ and $r_3 \vee w_3 \vee b_3$ follows b_3. From b_2 and b_3 and $\neg(b_2 \wedge b_3)$ follows a contradiction. Therefore $r_1 \nabla w_1 \nabla b_1$. ■

$\mathsf{dealings33} \vdash \mathsf{atmost33} \wedge \mathsf{atleast33}$.

That $\mathsf{dealings33} \vdash \mathsf{atleast33}$ is obvious, e.g. $\delta_{rwb} \vdash r_1 \vdash r_1 \vee w_1 \vee b_1$, similarly for all other cases φ_d, etc. That $\mathsf{dealings33} \vdash \mathsf{atmost33}$ is also obvious, e.g. $\varphi_{rwb} \vdash r_1 \wedge \neg r_2 \wedge \neg r_3 \vdash \neg(r_1 \wedge r_2)$. Similarly for all other cases δ_d, etc. ■

$\mathsf{atmost33}, \mathsf{atleast33} \vdash \mathsf{dealings33}$:

Also $\mathsf{atleast33}, \mathsf{atmost33} \vdash \mathsf{dealings33}$. This can be proven by reasoning from all different cases from atleast33 that are consistent with atmost33:

It holds that atleast33 $\vdash r_1 \vee w_1 \vee b_1$. Suppose r_1. It holds that r_1, atmost33 $\vdash \neg r_2$ and r_1, atmost33 $\vdash \wedge \neg r_3$ Thus $r_1 \wedge \neg r_2 \wedge \neg r_3$.

Also atleast33 $\vdash r_2 \vee w_2 \vee b_2$. Assumption r_2 is contradictory. Therefore $w_2 \vee b_2$. Suppose w_2. Similarly to above, it follows that $\neg w_1 \wedge w_2 \wedge \neg w_3$.

Also atleast33 $\vdash r_3 \vee w_3 \vee b_3$. Only b_3 is consistent with the previous. Similarly to above, it follows that $\neg b_1 \wedge \neg b_2 \wedge b_3$.

The conjunction of $r_1 \wedge \neg r_2 \wedge \neg r_3$, $\neg w_1 \wedge w_2 \wedge \neg w_3$, and $\neg b_1 \wedge \neg b_2 \wedge b_3$ is the formula δ_{rwb}. Similarly for other cases. Thus dealings33. ■

atmost33, dontknowother33 $\vdash$ dontknowthat33:

We prove the case $\neg K_2 r_1$. Either r_2 or $\neg r_2$. If $\neg r_2$ then from that and from $\neg r_2 \rightarrow \neg K_2 r_1$ follows $\neg K_2 r_1$. If r_2 then from that and from $\neg(r_1 \wedge r_2)$ follows $\neg r_1$. If $K_2 r_1$ held than, because of reflexivity, r_1 would hold. Contradiction with $\neg r_1$. Therefore $\neg K_2 r_1$. ■

$33^+ -$ dontknownot33 $\vdash$ dontknownot33:

We prove the case $\neg r_2 \rightarrow \neg K_2 \neg r_1$. Suppose it doesn't hold. Then both $\neg r_2$ and $K_2 \neg r_1$. From dontsee33 and $\neg r_2$ follows $K_2 \neg r_2$. From function33 and $\neg r_1$ and $\neg r_2$ follows r_3. Therefore, from $K_2 \neg r_1$ and $K_2 \neg r_2$ follows $K_2 r_3$. From dontknowthat33 follows $\neg K_2 r_3$. Contradiction. Therefore $\neg r_2 \rightarrow \neg K_2 \neg r_1$. ■

$33^+ -$ dontknowthat33 $\vdash$ dontknowthat33:

We prove the case $\neg K_2 r_1$. Assume $K_2 r_1$. We derive a contradiction. From $K_2 r_1$ and reflexivity follows r_1. A conjunct from atmost33 is $\neg(r_1 \wedge r_2)$. From r_1 and $\neg(r_1 \wedge r_2)$ follows $\neg r_2$. From $\neg r_2$ and dontknownot33 follows $\neg K_2 \neg r_3$. Also, from r_1 and $\neg(r_1 \wedge r_3)$ follows $\neg r_3$, and therefore: from $K_2 r_1$ and atmost33 follows $K_2 \neg r_3$. Contradiction. Therefore $\neg K_2 r_1$. ■

Proofs from section 2.5.1

$\sigma^{33} \vdash$ 33:

First observe that for any dealing xyz: $\sigma_{xyz} \vdash \delta_{xyz}$.

$\sigma^{33} \vdash$ see33: Obvious.

$\sigma^{33} \vdash$ dealings33: Obvious.

$\sigma^{33} \vdash$ dontknowthat33: Suppose not. Then there are agents a and b and a card c such that $K_a c_b$. E.g. $K_1 r_2$. Then r_2. Therefore either δ_{wrb} or δ_{brw}. Therefore either σ_{wrb} or σ_{brw}.

Suppose σ_{wrb}. Then $M_1\delta_{wbr}$. Therefore $M_1 b_2$, thus (as $\sigma^{33} \vdash$ dealings33) $M_1 \neg r_2$, i.e. $\neg K_1 r_2$. Contradiction.

Similarly for σ_{brw}. Then $M_1\delta_{bwr}$. Therefore $M_1 w_2$, thus (as $\sigma^{33} \vdash$ dealings33) $M_1 \neg r_2$, i.e. $\neg K_1 r_2$. Therefore, again a contradiction. ■

$33 \vdash \sigma^{33}$:

Proof by cases from dealings33: Suppose δ_{rwb}. From δ_{rwb} and see33 follows $K_1 r_1$; and from that and from dealings33 follows $K_1\delta^1_{rwb}$. Similarly for $K_2\delta^2_{rwb}$ and $K_3\delta^3_{rwb}$.

Now suppose $\neg M_1\delta_{rbw}$. Because of r_1, it is obvious that $\neg M_1\delta_{brw}$, $\neg M_1\delta_{bwr}$, $\neg M_1\delta_{wrb}$, $\neg M_1\delta_{wbr}$. Therefore $K_1(\neg\delta_{brw} \wedge \neg\delta_{bwr} \wedge \neg\delta_{wrb} \wedge \neg\delta_{wbr})$. From that and dealings33 follows $K_1\delta_{rwb}$, therefore $K_1 w_2$. From dontknowthat33 follows $\neg K_1 w_2$. Contradiction. Therefore $M_1\delta_{rbw}$. Similarly for $M_2\delta_{bwr}$ and $M_3\delta_{wrb}$. Therefore σ^{33}.

Similarly for other cases from dealings33. ■

Proof from section 2.6

Proof of proposition 1: Let $M = \langle W, \{\sim_1, \sim_2, \sim_3\}, V\rangle$ be an $S5_3$ model of 33, i.e. $M \models 33$. Then M is bisimilar to hexa:

Proof In our proof we use the notation: hexa $= \langle W^h, \sim^h, V^h\rangle$, where $W^h = \{rwb, rbw, brw, bwr, wrb, wbr\}$, $\sim^h_1 = \{\{rwb, rbw\}, \{brw, bwr\}, \{wrb, wbr\}\}$, $\sim^h_2 = \{\{rwb, bwr\}, \{rbw, wbr\}, \{wrb, brw\}\}$, $\sim^h_3 = \{\{rwb, wrb\}, \{wbr, bwr\}, \{rbw, brw\}\}$, $V^h_{ijk} = V_{ijk}$ such that: $V_{ijk}(i_1) = V_{ijk}(j_2) = V_{ijk}(k_3) = 1$ and $V_{ijk}(p) = 0$ for all other (six) atomic propositions p.

First an observation on valuations of worlds in M. Because $M, w \models$ dealings33, and because each one of the six exclusive alternatives in dealings33 correspond to a valuation, any world $w \in M$ has one of six different valuations V_{rwb}, V_{rbw}, V_{brw}, V_{bwr}, V_{wbr}, V_{wrb}.

Now define relation $\mathfrak{R} \subseteq (M \times \text{hexa})$ as follows:

$$\forall w \in M : \forall w^h \in \text{hexa} : \mathfrak{R}(w, w^h) \Leftrightarrow V_w = V^h_{w^h}$$

We prove that $\mathfrak{R}$ is a bisimulation between M and hexa.

Forth:

Let $w, w' \in M$, let $w^h \in$ hexa. Suppose $w \sim_1 w'$ and $\mathfrak{R}(w, w^h)$. We find an $\mathfrak{R}$-image of w' for every valuation V_w on w. First suppose $V_w = V_{rwb}$. From $\mathfrak{R}(w, w^h)$ follows $V^h_{w^h} = V_w = V_{rwb}$. Therefore $w^h = rwb$.

As M is a model of 33, $M \models$ see33. From $M, w \models$ see33 follows $M, w \models r_1 \rightarrow K_1 r_1$. From $V_w(r_1) = V_{rwb}(r_1) = 1$ and $M, w \models r_1 \rightarrow K_1 r_1$ follows $M, w \models K_1 r_1$. From $M, w \models K_1 r_1$ and $w \sim_1 w'$ follows $M, w' \models r_1$. Therefore $V_{w'} = V_{rwb}$ or $V_{w'} = V_{rbw}$. If $V_{w'} = V_{rwb}$, choose rwb as the $\mathfrak{R}$-image of w' in hexa: obviously $rwb \sim_1^h rwb$ and also $\mathfrak{R}(w', rwb)$. If $V_{w'} = V_{rbw}$, choose rbw as the $\mathfrak{R}$-image of w' in hexa: we now have $rwb \sim_1^h rbw$ and $\mathfrak{R}(w', rbw)$.

Similarly for the five other valuations V_w on w. Similarly for $i = 2$ and $i = 3$.

Back:

Let $w^h, w_*^h \in$ hexa, let $w \in M$. Suppose $w^h \sim_1^h w_*^h$ and $\mathfrak{R}(w, w^h)$. We find an $\mathfrak{R}$-original of w_*^h for every valuation $V_{w^h}^h$ on w^h. First suppose $V_{w^h}^h = V_{rwb}$. Obviously $w^h = rwb$.

From $rwb \sim_1^h w_*^h$ follows $w_*^h = rwb$ or $w_*^h = rbw$. If $w_*^h = rwb$ choose w itself as the required $\mathfrak{R}$-original of w_*^h. As M is an *S5* model, $w \sim_1 w$, and we already assumed $\mathfrak{R}(w, rwb)$.

Otherwise $w_*^h = rbw$. We derive a contradiction from the assumption that there is *no* $w' \in M$ such that $w \sim_1 w'$ and $V_{w'} = V_{rbw}$.

Suppose so. In other words: for all $w' \in M : w \sim_1 w' \Rightarrow V_{w'} \neq V_{rbw}$. Suppose $w \sim_1 w'$. As before, from see33 follows $M, w \models K_1 r_1$ and from that and $w \sim_1 w'$ follows $M, w' \models r_1$ and therefore $V_{w'} = V_{rwb}$ or $V_{w'} = V_{rbw}$. From that and the assumption follows $V_{w'} = V_{rwb}$, thus $M, w' \models w_2$, and thus, as w' is an arbitrary 1-accessible world from $w \in M$, $M, w \models K_1 w_2$. However, also $M \models$ dontknowthat33, thus $M, w \models \neg K_1 w_2$. Contradiction.

Therefore there is a $w' \in M$ such that $w \sim_1 w'$ and $V_{w'} = V_{rbw}$. By definition we have $\mathfrak{R}(w', rbw)$. So we have found the required $\mathfrak{R}$-original of rbw.

Similarly for the five other valuations $V_{w^h}^h$ on w^h. Similarly for $i = 2$ and $i = 3$. ■

Proof from section 5.1

Proof of proposition 4: $\mathsf{kgames}_{\mathbf{d}} \leftrightarrow \mathsf{dealings} + E^\delta(I_{\mathbf{d}})$. It suffices to prove that in $\mathbf{S5_n}$ plus dealings, seedontknow is equivalent to $E^\delta(I_{\mathbf{d}})$:

$$\bigwedge_{d \in I_{\mathbf{d}}} \bigwedge_{a \in \mathbf{A}} (\delta_d \to (\bigwedge_{d \sim_a d'} M_a \delta_{d'}) \wedge K_a \delta_d^a) \quad \Leftrightarrow \quad \bigwedge_{d \in I_{\mathbf{d}}} \bigwedge_{a \in \mathbf{A}} (K_a \delta_d^a \leftrightarrow M_a \delta_d)$$
$$(i) \qquad\qquad (ii)$$

Proof: As usual we assume a somewhat informal, natural-deduction like, proof style.

(i) $\Rightarrow$ (ii)

Let $a \in \mathbf{A}, d \in D_{\mathbf{d}}$. First, we prove that $K_a \delta_d^a \to M_a \delta_d$:

$K_a \delta_d^a$
$\Rightarrow$
δ_d^a
$\Leftrightarrow$
$\bigvee_{d' \sim_a d} \delta_{d'}$

Let $d' \sim_a d$ be an arbitrary dealing such that $\delta_{d'}$. Then:

$\delta_{d'}$
$\Rightarrow$ applying (i)
$\bigwedge_{d'' \sim_a d'} M_a \delta_{d''}$
$\Rightarrow$ as $d \sim_a d'$
$M_a \delta_d$

Therefore $K_a \delta_d^a \rightarrow M_a \delta_d$.

Next, we prove that $M_a \delta_d \rightarrow K_a \delta_d^a$, by contraposition. Start by observing that:

$\neg K_a \delta_d^a$
$\Leftrightarrow$
$\neg K_a (\bigvee_{d' \sim_a d} \delta_{d'})$
$\Leftrightarrow$
$M_a (\bigwedge_{d' \sim_a d} \neg \delta_{d'})$

Suppose $M_a \delta_d$. Either δ_d or $\neg \delta_d$. If δ_d, then apply (i) and $K_a \delta_d^a$ follows. If $\neg \delta_d$ then from dealings it follows that $\delta_{d''}$ for some $d'' \neq d$. Again with (i), follows $K_a \delta_{d''}^a$. We can have either $d'' \sim_a d$ or $d'' \not\sim_a d$. If $d'' \sim_a d$ then $\delta_{d''}^a \leftrightarrow \delta_d^a$ and from that and $K_a \delta_{d''}^a$ follows $K_a \delta_d^a$. If $d'' \not\sim_a d$ we derive a contradiction:

$K_a \delta_{d''}^a$
$\Leftrightarrow$
$K_a \bigvee_{d^* \sim_a d''} \delta_{d^*}$
$\Leftrightarrow$ because dealings is an exclusive disjunction
$K_a \bigwedge_{d \not\sim_a d''} \neg \delta_{d^*}$
$\Rightarrow$
$K_a \neg \delta_d$
$\Leftrightarrow$
$\neg M_a \delta_d$

(ii) $\Rightarrow$ (i)

Suppose δ_d. Then $M_a \delta_d$. From that and (ii) follows $K_a \delta_d^a$. From that, and because for all $d' \sim_a d$: $\delta_{d'}^a \leftrightarrow \delta_d^a$, follows that for all $d' \sim_a d$: $K_a \delta_{d'}^a$. Using (ii) for all $d' \sim_a d$, we get: for all $d' \sim_a d$: $M_a \delta_{d'}$. Thus $\bigwedge_{d' \sim_a d} M_a \delta_{d'}$. ■

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