The extent of constructive game labellings

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Abstract. We develop a theory of labellings for infinite trees, define the notion of a combinatorial labelling, and show that $\boldsymbol{\Delta}^0_2$ is the largest boldface pointclass in which every set admits a combinatorial labelling.

1 Introduction

Labellings are at the core of the theory of infinite games. The first example of a game labelling was the Zermelo "backward induction technique" from [Zer13] used by Gale and Stewart in [GaSt53] to prove the determinacy of all open sets. This theorem is one of the gems of game theory. Its proof is conceptually clear and arguably constructive.

The Gale-Stewart result is an example of a **constructive determinacy proof**. Such proofs, in particular those using the Cantor-Bendixson method, were investigated by Büchi and Landweber in their seminal paper on games and finite automata [BüLa69]. Büchi describes his fascination with constructive determinacy proofs:[1].

> "The [constructive] proof '*actually presents*' a winning strategy. The [nonconstructive] proofs do no such thing; all you know at the end is existence of a winning strategy.[2]

Although Büchi offers a general idea of what it means for a determinacy proof to be constructive, he doesn't give specific criteria. In this paper, we develop a notion of *combinatorial labelling* that is a possible formalization of "constructive proofs": A game that is analyzed by a combinatorial labelling uses the combinatorial structure of the payoff set and no additional background information. We prove that $\boldsymbol{\Delta}^0_2$ is the largest boldface pointclass in which every set admits a combinatorial labelling (in the sense that every set in $\boldsymbol{\Delta}^0_2$ admits a combinatorial

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[1] The term "constructive determinacy" is used by Gurevich in [Gu90] whereas Büchi more technically writes "CB-proof of determinacy".

[2] [Bü83, p. 1171]; italics in the original.

labelling, and any boldface class strictly bigger than $\boldsymbol{\Delta}^0_2$ contains a set without a combinatorial labelling).

The history of set-theoretic game theory has seen determinacy proofs using more complicated arguments, *e.g.*, Davis' argument for the $\boldsymbol{\Sigma}^0_2$ games [Da63], Wolfe's argument for the $\boldsymbol{\Sigma}^0_3$-games [Wol55], Paris' argument for the $\boldsymbol{\Sigma}^0_4$-games [Pa72], and in general Martin's inductive proof for Borel games [Ma75,Ma85]. Harvey Friedman proved [Fr71] that the increase in complexity is not avoidable: higher determinacy proofs cannot be done in second order number theory.[3]

2 Notation & Definitions

We use ω^ω to denote infinite sequences of natural numbers, and $\omega^{<\omega}$ to denote finite sequences of natural numbers. For $s \in \omega^{<\omega}$, we use the notation $[s] := \{x \in \omega^\omega\,;\, s \subset x\}$ to denote the set of infinite extensions of x. The notation $(s) := \{u \in \omega^{<\omega}\,;\, s \subseteq u\}$ denotes the set of finite extensions of s. If $s, t \in \omega^{<\omega}$ amd $t = s^\frown\langle k\rangle$ for some $k \in \omega$, then we say that t is a **successor** of s. If $A \subseteq \omega^\omega$, then define $A_{\llcorner s \lrcorner} := \{x \in \omega^\omega : s^\frown x \in A\}$.

2.1 The Hausdorff Difference Hierarchy

As usual, we call an ordinal α **even** (**odd**) if it is of the form $\lambda+2n$ ($\lambda+2n+1$) for some limit ordinal λ and some natural number n. For a sequence $\langle A_\gamma\,;\, \gamma < \alpha\rangle$, we define the **Hausdorff difference**, $\mathrm{Diff}(\langle A_\gamma\,;\, \gamma < \alpha\rangle)$, to be the set:

$$\{x \in \bigcup_{\gamma<\alpha} A_\alpha\,;\, \min\{\gamma\,;\, x \in A_\gamma\} \text{ has different parity from } \alpha\}.$$

If $A = \mathrm{Diff}(\langle A_\gamma\,;\, \gamma < \alpha\rangle)$, we call $\langle A_\gamma\,;\, \gamma < \alpha\rangle$ a **presentation of** A. In general, the presentation of a set need not be unique.

The **Hausdorff difference classes** are defined as follows: $A \in \alpha$-$\boldsymbol{\Sigma}^0_1$ if there is an increasing sequence $\langle A_\gamma\,;\, \gamma < \alpha\rangle$ of open sets such that $A = \mathrm{Diff}(\langle A_\gamma\,;\, \gamma < \alpha\rangle)$.

The following theorem expresses the Hausdorff difference classes in terms of the arithmetical hierarchy (for a proof, cf. [Kec94, Theorem (22.27)]):

Theorem 1 (Hausdorff-Kuratowski) $\bigcup_{\alpha<\omega_1} \alpha$-$\boldsymbol{\Sigma}^0_1 = \boldsymbol{\Delta}^0_2$.

2.2 Games

We consider games with two players, Player I and Player II, and a payoff set $A \subseteq \omega^\omega$. Player I begins the game by playing a natural number x_0. Then, Player II plays a natural number x_1. The players alternate moves for ω rounds, which

[3] The fact that Büchi conjectures that there is a constructive proof of Borel determinacy [Bü83, Problem 1] suggests that his notion of "constructive" is extremely liberal, at least more liberal than our notion of a combinatorial labelling.

produces an element $x = \langle x_0, x_1, x_2, ...\rangle \in \omega^\omega$. If x is an element A, then Player I wins. If not, then Player II wins.

We use M_0 to denote the set of finite sequences of even length, and M_1 to denote the set of finite sequences of odd length. A **strategy for Player I** is a function $\sigma : \omega^{<\omega} \cap M_0 \to \omega^{<\omega}$ such that $\sigma(s)$ is a successor of s. Similarly, a **strategy for Player II** is a function $\tau : \omega^{<\omega} \cap M_1 \to \omega^{<\omega}$ such that $\tau(s)$ is a successor of s. If σ is a strategy for Player I and τ is a strategy for Player II, we denote by $\sigma * \tau$ the unique element of ω^ω that is produced if Player I follows σ and Player II follows τ.

We call a strategy σ for Player I **winning** if for all counterstrategies τ, $\sigma * \tau \in A$. Similarly, a strategy τ for Player II is **winning** if for all counterstrategies σ, $\sigma * \tau \notin A$. Clearly, at most one player can have a winning strategy, in which case the set A is called **determined**.

For a position $s \in \omega^{<\omega}$, consider the variant of the game beginning at s. An **s-strategy for Player I** is a function $\sigma : (s) \cap M_0 \to \omega^{<\omega}$ such that $\sigma(u)$ is a successor of u. We define an **s-strategy for Player II** in the analogous way, as well as the notion of a **winning s-strategy**.

3 Labellings I: Soundness

We say that $\mathbf{L} = \langle L_{\mathrm{I}}, <_{\mathrm{I}}, L_{\mathrm{II}}, <_{\mathrm{II}}\rangle$ is a **labelling system** if L_{I} and L_{II} are disjoint sets, $<_{\mathrm{I}}$ is a well-ordering on L_{I}, and $<_{\mathrm{II}}$ is a well-ordering on L_{II}. The elements of L_{I} are called **I-labels** and the elements of L_{II} are called **II-labels**. We will sometimes write $\mathbf{L}$ for the set $L_{\mathrm{I}} \cup L_{\mathrm{II}}$. We call any partial function $\ell : \omega^{<\omega} \to \mathbf{L}$ a **labelling**.

Fix a labelling ℓ and a position s. We say that an s-strategy σ for Player I is **ℓ-good** if it satisfies the following property: if $t \in \mathrm{dom}(\sigma)$ and there exists a $j \in \omega$ such that $\ell(t^\frown\langle j\rangle)$ is a I-label, then $\ell(\sigma(t))$ is the $<_{\mathrm{I}}$-least element of the set $\{\ell(t^\frown\langle j\rangle)\,;\, j \in \omega\} \cap L_{\mathrm{I}}$. In other words, if there are I-labelled successors of t, $\sigma(t)$ is a I-labelled successor with the smallest possible label.

The Player II case is handled analogously.

Letting A be the payoff set, we say that ℓ is **A-sound at s** if either $\ell(s)$ is a I-label and every ℓ-good s-strategy for Player I is winning, or if $\ell(s)$ is a II-label and every ℓ-good s-strategy for Player II is winning.

Proposition 2 *Let $A \subseteq \omega^\omega$ and $s \in \omega^{<\omega}$. Then Player I has a winning s-strategy if and only if there is a labelling ℓ such that $\ell(s) \in L_{\mathrm{I}}$ and ℓ is A-sound at s. Similarly, Player II has a winning s-strategy if and only if there is a labelling ℓ such that $\ell(s) \in L_{\mathrm{II}}$ and ℓ is A-sound at s.*

Proposition 3 *For any $A \subseteq \omega^\omega$, A is determined if and only if there is a labelling that is A-sound at $\varnothing$.*

We say that a labelling is **globally A-sound** if it is A-sound at every $s \in \omega^{<\omega}$. Note that every globally sound labelling must be total. Proposition 3 becomes false if we consider globally A-sound labellings instead of A-sound at $\varnothing$ labellings, but the result still holds classwise for boldface pointclasses.

Proposition 4 *Suppose $\boldsymbol{\Gamma}$ is a boldface pointclass. Then, using the Axiom of Choice, the following are equivalent:*

1. *Every set in $\boldsymbol{\Gamma}$ is determined.*
2. *For every set $A \in \boldsymbol{\Gamma}$, there is a labelling that is globally A-sound.*

4 Labellings II: Combinatorial Labellings

In this section, we will formalize the notion of combinatorial equivalence. We begin with some background information about bisimulations. If $\mathbf{G} = \langle G, E_G \rangle$ and $\mathbf{H} = \langle H, E_H \rangle$ are directed graphs, then we call a relation $R \subseteq G \times H$ a **bisimulation** if the following conditions ("back and forth") hold:

$$\forall g, g^* \in G \forall h \in H \begin{pmatrix} \text{if } \langle g, h \rangle \in R \ \& \ \langle g, g^* \rangle \in E_G \text{ then there is an} \\ h^* \in H \text{ such that } \langle g^*, h^* \rangle \in R \ \& \ \langle h, h^* \rangle \in E_H \end{pmatrix}$$
$$\forall g \in G \forall h, h^* \in H \begin{pmatrix} \text{if } \langle g, h \rangle \in R \ \& \ \langle h, h^* \rangle \in E_H \text{ then there is a} \\ g^* \in G \text{ such that } \langle g^*, h^* \rangle \in R \ \& \ \langle g, g^* \rangle \in E_G \end{pmatrix}$$

Let $s \in \omega^{<\omega}$. We can see (s) as a directed graph E such that $\langle u, v \rangle \in E :\Leftrightarrow v$ is a successor of u. If $A \subseteq \omega^\omega$ and R is a bisimulation between (s) and (t), we say that R is A**-preserving** if for every $x, y \in \omega^\omega$, the following holds:

$$\text{if } \forall n \in \omega[R(s^\frown(x{\restriction}n), t^\frown(y{\restriction}n))], \text{ then } s^\frown x \in A \iff t^\frown y \in A.$$

Let $s, t \in \omega^{<\omega}$ such that $s, t \in M_0$ or $s, t \in M_1$. We say that s and t are A**-bisimilar** if there is an A-preserving bisimulation R between (s) and (t) such that $\langle s, t \rangle \in R$. Furthermore, we say that a labelling ℓ is A**-combinatorial** if any two A-bisimilar nodes get the same ℓ-label. In other words, ℓ is combinatorial if any two bisimilar nodes have the same label.

Proposition 5 *The labellings ℓ_0 and ℓ_1 constructed in the proof of Propositions 3 and 4, respectively, are not in general A-combinatorial.*

Proposition 6 *There is a $\boldsymbol{\Sigma}^0_2$ set A such that no A-sound labelling at $\varnothing$ is A-combinatorial.*

Proof. Define A as follows:

$$x \in A \ :\iff \ \exists n \forall k \geq n \, (x(k) = 0).$$

It is clear that A is $\boldsymbol{\Sigma}^0_2$ and that Player II has a winning strategy. Note the following key fact:

(*) For every $s, t \in \omega^{<\omega}$, $A_{\llcorner s \lrcorner} = A_{\llcorner t \lrcorner}$.

Suppose that ℓ is A-combinatorial. It will be shown that ℓ is not A-sound at $\varnothing$. If $\varnothing$ is unlabeled, then we are done. If $\ell(\varnothing) \in L_{\mathrm{I}}$, then we are done by Proposition 2. Suppose $\ell(\varnothing) \in L_{\mathrm{II}}$. Since ℓ is combinatorial, it follows from (*) that $\ell(u) = \ell(v) \in L_{\mathrm{II}}$ for all u, $v \in M_1$. Therefore, any strategy τ for Player II is ℓ-good. In particular, the strategy $\tau(s) := s^\frown\langle 0 \rangle$ is ℓ-good for Player II. But τ is not winning for Player II: let σ be the strategy for Player I defined by $\sigma(s) := s^\frown\langle 0 \rangle$, then $\sigma * \tau \notin A$. It follows that ℓ is not A-sound at $\varnothing$. □

Theorem 7 *Let $A \in \alpha$-$\boldsymbol{\Sigma}^0_1$. Then there is a labelling ℓ that is globally A-sound and A-combinatorial.*

This is the main technical theorem of this paper. Its proof proceeds by defining the labelling with a modified Gale-Stewart technique level-by-level along the defining sequence of open sets used to define a given α-$\boldsymbol{\Sigma}^0_1$ set. The proof is modelled closely after known proofs of $\boldsymbol{\Delta}^0_2$ determinacy.

Theorem 8 *The pointclass $\boldsymbol{\Delta}^0_2$ is the largest boldface pointclass in which every set set has a labelling that is globally sound and combinatorial.*

Proof. By Theorem 7, all sets in $\boldsymbol{\Delta}^0_2$ have a labelling that is globally sound and combinatorial. Let $\boldsymbol{\Gamma}$ be any boldface pointclass containing $\boldsymbol{\Delta}^0_2$. Any boldface pointclass that is a proper superset of $\boldsymbol{\Delta}^0_2$ contains either all $\boldsymbol{\Sigma}^0_2$ sets or all $\boldsymbol{\Pi}^0_2$ sets. If $\boldsymbol{\Gamma}$ contains all $\boldsymbol{\Sigma}^0_2$ sets, then we are done by Proposition 6. If $\boldsymbol{\Gamma}$ contains all $\boldsymbol{\Pi}^0_2$ sets, then the result follows from the fact that there exists a $\boldsymbol{\Pi}^0_2$ set B such that no B-sound labelling at $\varnothing$ is B-combinatorial. Namely, if A is the $\boldsymbol{\Sigma}^0_2$ set from Proposition 6, take $B = \omega^\omega \setminus A$. Then a similar argument to the proof of Proposition 6 shows that B has the desired property. □

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