Real Blackwell Determinacy

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Abstract. This bachelor thesis investigates the possibility to extend Blackwell Determinacy to the reals and discusses two possibilities. One uses measures with countable support—the other endows the reals with the Euclidean topology. At the end we propose a possible application.

1 Introduction

Axioms of determinacy are statements about set-theoretic games. Set-theoretic games aren't games in the sense that you can play them for recreation—but as normal games they do consist of players making moves such that at the end one of the players wins.

Gale and Stewart introduced their set-theoretic games in 1953 in their [1]. They were two-player, perfect information, zero-sum games of infinite length. Perfect information means that both players know exactly in what situation they are, i.e., they know what they did and what the opponent did in every turn. Zero-sum means that one player wins and the other one loses. And infinite length means that there are ω many turns before the game ends.

An important notion in games, is that of a **strategy**. This is a function telling a player what to do in every situation. A strategy is **winning** if a player playing by that strategy wins against all strategies the opponent could choose. A game is called **determined** if one of the players has a winning strategy. A lot of effort has been done to show that certain classes of games are determined. Determinacy of some larger classes can't be proven but is used as **axioms** to get different forms of set-theory. This works well for games of perfect information. But for imperfect information games it's not appropriate as we can see in the following example:

Consider the two-player game where both player pick a number from $\{0, 1\}$. Player I wins if they picked the same number and player II wins if they picked a different number. Neither player knows what the other picks so we have imperfect information. The only two strategies are "pick 0" and "pick 1". They are both 'bad' because when you know which one your opponent is playing it's easy to beat him. A better strategy seems to randomly pick one of the number—that way at least you have a fifty-fifty chance of winning, and this is as good as it gets.

So with imperfect information games it seems right to add probability to the strategies. John von Neumann and Oskar Morgenstern first did this in [7] for finite games. In [8] David Blackwell extended this concept to games of infinite length. He introduced what he (in [2]) called "Games with Slightly Imperfect Information" and are now known as **Blackwell games**, where players play by so called mixed strategies. In those games every turn is of the form of the above example but after each turn both players are told again in which situation they are. This is why there is just *slightly imperfect information*. Of course determinacy has to be defined in a different way because now both players can have a reasonable chance of winning.

A nice property of perfect information games is that you can think of them as trees. A game-state is a node in the tree and a strategy can be seen as a subtree. You lose this when you go to imperfect information games. But however imperfect information was the motivation to use mixed strategies, it's also possible to use the notion of mixed strategies with perfect information games.

A surprising result by Tony Martin is that the determinacy of all perfect information Blackwell games is equivalent to the determinacy of all (imperfect information) Blackwell games [5, p. 1579]. This allows us to think of the games as trees again.

Another advantage of perfect information is that it allows to have an infinite set of possible moves. With imperfect information you can't have determinacy of all games if you allow infinitely many moves. We didn't define determinacy for Blackwell games yet but in short the problem is that it's possible that neither player has a 'good' strategy as we can see in the following example: Let both players pick a natural number and player I wins if he picked the larger number. Then for any mixed strategy for one of the two players the other player can always pick a number such that he wins with probability arbitrarily close to one.

For perfect information Blackwell games with countably infinitely many possible moves you don't have the above problem and determinacy has been defined. The goal of this bachelor thesis however is to investigate the possibility to define determinacy for Blackwell games on the reals.

In section 2 we give the definitions and introduce determinacy and Blackwell determinacy. In section 3 we first explain why you can't just replace ω with $\mathbb{R}$ to define Blackwell determinacy for the reals. Then we discuss two ways of defining real Blackwell determinacy and at the end of section 3 we have our main theorem that real Blackwell determinacy follows from real determinacy. The main work though is not the proof of this theorem but to show that the definitions actually make sense. In section 4 we state some open questions and address a theorem that might help answering the question whether real Blackwell determinacy is really stronger than Blackwell determinacy.

2 Definitions

So determinacy is about games. The games are between two players, I and II, who play in turns. Every turn, the active player plays one element from a given set X after which the turn goes to the other player. After ω many turns the game is over. I and II then have created an element x of X^ω where player I chose the even numbered elements and player II the odd numbered elements. I wins this game if x is an element of the given payoff set A. II wins iff I doesn't. The game on X with payoff set A is denoted with $G_X(A)$. Usually X is a countable set.

2.1 The Axiom of Determinacy

A strategy for player I is a function $\sigma : X^{\text{Even}} \to X$ telling the player what to do when it is his turn. A strategy for player II is a similar function $\tau : X^{\text{Odd}} \to X$. With X^{Even} (X^{Odd}) we mean the set of sequences with even (odd) length and elements from X. Given two strategies σ and τ, the outcome $\sigma * \tau \in X^\omega$ of the game is recursively defined by

$$\begin{aligned} \sigma * \tau(0) &= \sigma(\varnothing) \\ \sigma * \tau(2k+1) &= \tau(x \upharpoonright 2k+1) \\ \sigma * \tau(2k+2) &= \sigma(x \upharpoonright 2k+2) \end{aligned}$$

Now a strategy σ for player I in $G_X(A)$ is called a **winning strategy** if for every strategy τ for player II, $\sigma * \tau$ is an element of A. Vice versa a strategy τ for player II is winning if $\sigma * \tau \notin A$ for every strategy σ for player I. A game $G_X(A)$ is said to be **determined** if either player has a winning strategy for $G_X(A)$.

The **Axiom of Determinacy** (AD) states that $G_\omega(A)$ is determined for every $A \subseteq \omega^\omega$. AD is inconsistent with AC (the Axiom of Choice) but is consistent with ZF and implies $\mathsf{AC}_\omega(\mathbb{R})$, the Axiom of Countable Choice for sets of reals. $\mathsf{AD}_\mathbb{R}$ is the axiom that says that $G_\mathbb{R}(A)$ is determined for every $A \subseteq \mathbb{R}^\omega$.

2.2 Blackwell Determinacy

In Blackwell games the players can have mixed strategies. A mixed strategy is a function $\sigma : X^{\text{Even}} \to \text{Prob}(X)$ where $\text{Prob}(X)$ is the set of probability measures on X. So in this case, what a player does in a certain position is not fixed but determined randomly. In the original Blackwell games the players move simultaneously so that neither player knows what the other did in the n^{th} turn when he decides what measure he plays in the same turn. But we will only be talking about **perfect information Blackwell games** where the players just take turns as before. So a mixed strategy for player I will be a function $\sigma : X^{\text{Even}} \to \text{Prob}(X)$ and a mixed strategy for player II will be a function $\tau : X^{\text{Odd}} \to \text{Prob}(X)$.

Given σ and τ, $\sigma * \tau$ isn't defined anymore. We don't get just one outcome now. Instead we get a probability measure.

It will be useful to define $\nu_{\sigma,\tau}$ as follows:

$$\nu_{\sigma,\tau}(x_0, \ldots, x_{n-1}) = \begin{cases} \sigma(x_0, \ldots, x_{n-1}) & \text{if } n \text{ is even,} \\ \tau(x_0, \ldots, x_{n-1}) & \text{if } n \text{ is odd.} \end{cases}$$

So in a sense we can simulate a game between σ and τ by letting $\nu_{\sigma,\tau}$ play against itself.

Let $s \in X^{<\omega}$ be a finite sequence. Then with $|s|$ we denote the length of s and with $[s]$ we denote the basic open set of all sequences extending s:

$$[s] := \{x \in X^\omega : s \subset x\}$$

Then we can say that the probability of ending up in $[s]$ is equal to the probability that the players start with playing the elements of s. We write:

$$\mu_{\sigma,\tau}([s]) = \prod_{i<|s|} \nu_{\sigma,\tau}(s \upharpoonright i)(\{s(i)\})$$

For countable X this will generate a Borel probability measure on X and for Borel sets A we get $\mu_{\sigma,\tau}(A)$ as the probability that the game ends in A, i.e., player I wins. Then we assign two values to a pair of strategies:

$$\begin{aligned} \text{val}^-(\sigma, \tau, A) &:= \sup\{\mu_{\sigma,\tau}(B) : B \subseteq A \text{ Borel}\} \\ \text{val}^+(\sigma, \tau, A) &:= \inf\{\mu_{\sigma,\tau}(B) : B \supseteq A \text{ Borel}\} \end{aligned}$$

which are upper and lower bounds for the probability of ending up in A. For a given strategy for a player we can then look at how well the other player can possibly do against it and we define:

$$\begin{aligned} \text{val}_{\text{I}}(\sigma, A) &:= \inf\{\text{val}^-(\sigma, \tau, A) : \tau \text{ a strategy for II}\} \\ \text{val}_{\text{II}}(\tau, A) &:= \inf\{\text{val}^+(\sigma, \tau, A) : \sigma \text{ a strategy for I}\} \end{aligned}$$

Finally we define values for how well both players could do:

$$\begin{aligned} \text{val}^-(A) &:= \sup\{\text{val}_{\text{I}}(\sigma, A) : \sigma \text{ a strategy for player I}\} \\ \text{val}^+(A) &:= \sup\{\text{val}_{\text{II}}(\tau, A) : \tau \text{ a strategy for player II}\} \end{aligned}$$

Now a set A is said to be **Blackwell determined** if $\text{val}^-(A) = \text{val}^+(A)$ and the **Axiom of** (perfect information[1]) **Blackwell determinacy** (Bl-AD) states that every set $A \subseteq X^\omega$ is Blackwell determined.

[1]Remember that Martin showed that the Axiom of prefect information Blackwell determinacy is equivalent to the original Axiom of Blackwell determinacy.

3 Real Blackwell games

Our goal is to extend the definition of Blackwell determinacy to games on the reals. The problem is that we can't just define the measure $\mu_{\sigma,\tau}$ as before because for a probability measure P on $\mathbb{R}$ we don't necessarily have

$$\sum_{x\in\mathbb{R}} P(x) = 1$$

so it's not enough to say what does $\mu_{\sigma,\tau}$ on basic open sets $[s]$ for $s \in \mathbb{R}^{<\omega}$. And thus we have to think of something else.

We play games of length ω on $\mathbb{R}$. A mixed strategy σ for player I is a function

$$\sigma : \mathbb{R}^{\text{Even}} \to \text{Prob}(\mathbb{R})$$

assigning a measure on $\mathbb{R}$ to an even sequence of plays (so II made the last move).

In the same way a mixed strategy τ for player II is a function

$$\tau : \mathbb{R}^{\text{Odd}} \to \text{Prob}(\mathbb{R})$$

assigning a measure on $\mathbb{R}$ to an odd sequence of plays (so I made the last move).

Now we want to define $\mu_{\sigma,\tau}$ to be a measure on $\mathbb{R}^\omega$ such that $\mu_{\sigma,\tau}(A)$ measures the probability that the sequence of plays will end up in A when I and II play against each other using their mixed strategies σ and τ.

We will consider two different definitions which differ in the strategies they allow and in the subsets of $\mathbb{R}^\omega$ they're defined on. Both definition will give rise to a (possibly different) **Axiom of Real Blackwell Determinacy**.

3.1 Countable support

In the first definition we only allow the players to play probability measures on $\mathbb{R}$ with a countable support. There's a remark in [9, p. 619] where they say it's possible to extend Blackwell Determinacy in this way but they don't show it explicitly. We will do that in this section.

So for every $s \in \mathbb{R}^{<\omega}$ there exists a countable set $S \subset \mathbb{R}$ such that

$$\nu_{\sigma,\tau}(s)(S) = 1$$

In this case we can choose S such that $\nu_{\sigma,\tau}(s)(\{a\}) > 0$ for every $a \in S$. So suppose σ and τ are fixed but arbitrary mixed strategies which only play measures with a countable support. For every basic open set $[s] \subseteq \mathbb{R}^\omega$ we define

$$\mu_{\sigma,\tau}([s]) = \prod_{i=0}^{|s|-1} \nu_{\sigma,\tau}(s \restriction i)(\{s_i\})$$

and we show that this defines a measure on the Borel σ-algebra of the product topology on $\mathbb{R}^\omega$ where $\mathbb{R}$ has the discrete topology. To this end we will use the following theorem without proof. The theorem can be found in [6, Appendix 2].

Theorem (Carathéodory Extension Theorem).
Let $\mathcal{R}$ be a ring of subsets of Ω and let $\mu : \mathcal{R} \to [0, \infty]$ be σ-additive. Then μ extends to a measure μ' on the σ-algebra $\sigma(\mathcal{R})$ generated by $\mathcal{R}$.

We define the following ring that includes all basic open sets:

$$\mathcal{R} = \{\bigcup_{s\in A} [s] : A \subseteq \mathbb{R}^n, n \in \omega\}$$

This is the collection of (disjoint) unions of basic open sets that share the number of coordinates they are fixed on. To show that this is a ring let $C \subseteq \mathbb{R}^n$ and $D \subseteq \mathbb{R}^m$ so that $A = \bigcup_{s \in C}[s]$ and $B = \bigcup_{s\in D}[s]$ are arbitrary elements of $\mathcal{R}$. If $n = m$ then $A \backslash B = \bigcup_{s \in C\backslash D}[s]$ and $A \cup B = \bigcup_{s\in C\cup D}[s]$ are again elements of $\mathcal{R}$. Otherwise assume without loss of generality that $n < m$. Then we can write $A = \bigcup\{[t] : t \in \mathbb{R}^m \wedge \exists s \in C(s \subset t)\}$ so that we are in the above situation again. So $\mathcal{R}$ is a ring.

For $C \subseteq \mathbb{R}^n$, $A = \bigcup_{s\in C}[s] \in \mathcal{R}$ let

$$\mu_{\sigma,\tau}(A) = \sum_{s\in C} \mu_{\sigma,\tau}([s]).$$

Notice that because we are in the countable support situation, this uncountable sum is actually a countable sum since all but countable many terms are equal to zero. We have to show that this definition is well defined in the sense that different notations for a set $A \in \mathcal{R}$ give the same measure. This means we have to show that for $n < m, C \subseteq \mathbb{R}^n$:

$$\mu_{\sigma,\tau}\left(\bigcup_{s\in A}[s]\right) = \mu_{\sigma,\tau}\left(\bigcup_{s\in A, t\in\mathbb{R}^{m-n}}[s^\frown t]\right)$$

First notice that for $s \in \mathbb{R}^n$ we have that

$$\begin{aligned}
\sum_{x\in\mathbb{R}} \mu_{\sigma,\tau}([s^\frown x]) &= \sum_{x\in\mathbb{R}} \prod_{i=0}^{|s|-1} \nu_{\sigma,\tau}(s \restriction i)(\{s_i\}) \cdot \nu_{\sigma,\tau}(s)\{x\} \\
&= \prod_{i=0}^{|s|-1} \nu_{\sigma,\tau}(s \restriction i)(\{s_i\}) \cdot \sum_{x\in\mathbb{R}} \nu_{\sigma,\tau}(s)(\{x\}) \\
&= \prod_{i=0}^{|s|-1} \nu_{\sigma,\tau}(s \restriction i)(\{s_i\}) \\
&= \mu_{\sigma,\tau}([s]).
\end{aligned}$$

($\sum_{x\in\mathbb{R}} \nu_{\sigma,\tau}(s)\{x\} = 1$ because $\nu_{\sigma,\tau}(s)$ has countable support.)
So for $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^{n+1}$, if $\bigcup_{s\in A}[s] = \bigcup_{s\in B}[s]$ we have

$$\begin{aligned}
\mu_{\sigma,\tau}\left(\bigcup_{s\in B}[s]\right) &= \mu_{\sigma,\tau}\left(\bigcup_{s\in A, x\in\mathbb{R}}[s^\frown x]\right) \\
&= \sum_{s\in A}\sum_{x\in\mathbb{R}} \mu_{\sigma,\tau}([s^\frown x]) \\
&= \sum_{s\in A} \mu_{\sigma,\tau}([s]) \\
&= \mu_{\sigma,\tau}\left(\bigcup_{s\in A}[s]\right)
\end{aligned}$$

and with induction we have that $\mu_{\sigma,\tau}$ is well defined on $\mathcal{R}$.

Now if we show that $\mu_{\sigma,\tau}$ is σ-additive on $\mathcal{R}$ we can use the theorem to extend $\mu_{\sigma,\tau}$ to $\sigma(\mathcal{R})$.

So let $\mathcal{A}$ be a countable disjoint collection of elements of $\mathcal{R}$ and $A = \bigcup \mathcal{A} \in \mathcal{R}$ and suppose towards a contradiction that

$$\mu_{\sigma,\tau}(A) \neq \sum_{B\in\mathcal{A}} \mu_{\sigma,\tau}(B).$$

$A \in \mathcal{R}$ so there are $n \in \mathbb{N}$ and $C \subseteq \mathbb{R}^n$ such that $A = \bigcup_{s\in C}[s]$. Choose n and C such that n is minimal.

For every $B \in \mathcal{A}$ there are n_B and $D_B \subseteq \mathbb{R}^{n_B}$ such that $B = \bigcup_{s\in D_B}[s]$. Let $D = \bigcup_{B\in\mathcal{A}} D_B$. Because n was minimal, for every $t \in D$ there is an $s \in C$ such that $s \subseteq t$.

If for every $s \in C$ we have that $\sum\{\mu_{\sigma,\tau}([t]) : t \in D \wedge s \subseteq t\} = \mu_{\sigma,\tau}(s)$, then by definition

$$\begin{aligned}
\mu_{\sigma,\tau}(A) &= \sum_{s\in C} \mu_{\sigma,\tau}([s]) \\
&= \sum_{s\in C} \sum \{\mu_{\sigma,\tau}([t]) : t \in D \wedge s \subseteq t\} \\
&= \sum_{t\in D} \mu_{\sigma,\tau}([t]) \\
&= \sum_{B\in\mathcal{A}} \sum_{t\in B_D} \mu_{\sigma,\tau}([t]) \\
&= \sum_{B\in\mathcal{A}} \mu_{\sigma,\tau}(B)
\end{aligned}$$

So by assumption the set S_s of s_0 for which $\sum\{\mu_{\sigma,\tau}([t]) : t \in D \wedge s \subseteq t\} \neq \mu_{\sigma,\tau}(s_0)$ is non-empty. Proceeding in this way, for every element $s_{n+1} \in S_{s_n}$ we have that the set $S_{s_{n+1}} = \{s_{n+1} : \mu_{\sigma,\tau}(s_{n+1}) \neq \sum\{\mu_{\sigma,\tau}([t]) : t \in D \wedge s_{n+1} \subset t\}$ is non-empty.

Because the measures have countable support we get countably many of of these non-empty sets so using $\mathsf{AC}_\omega(\mathbb{R})$ we get a sequence $x \in \mathbb{R}^\omega$ such that for every finite initial segment $s \subset x$ we have $\mu_{\sigma,\tau}([s]) \neq \sum\{\mu_{\sigma,\tau}([t]) : t \in D \wedge s \subseteq t\}$. Since $x \in A$ there is an $s \in D$ such that $x \in [s]$. Then $s \subset x$ so $\mu_{\sigma,\tau}([s]) \neq \sum\{\mu_{\sigma,\tau}([t]) : t \in D \wedge s \subseteq t\}$. But in this case $\{t : t \in D \wedge s \subseteq t\}$ is just $\{s\}$ so we get $\mu_{\sigma,\tau}([s]) \neq \mu_{\sigma,\tau}([s])$, a contradiction.

So $\mu_{\sigma,\tau}$ is σ-additive and can be extended to $\sigma(\mathcal{R})$. We denote this measure with $\mu^{\mathrm{C}}_{\sigma,\tau}$.

3.2 Euclidean topology

In our second definition we allow the players to play any Borel probability measure in every turn but $\mu_{\sigma,\tau}$ will only be defined on the Borel sets of $\mathbb{R}^\omega$ where $\mathbb{R}^\omega$ has the product topology of the Euclidean topology on the reals $\mathbb{R}$.

Let σ and τ be arbitrary but fixed strategies playing Borel probability measures.

For $k \geq 1$ let the measure P_k be recursively defined by (B is a Borel set of dimension k):

$$\begin{aligned}
P_1(B) &= \nu_{\sigma,\tau}(\varnothing)(B) \\
P_{n+1}(B) &= \int_{s\in\mathbb{R}^n} \nu_{\sigma,\tau}(s)(B_s)\mathrm{d}P_n(s)
\end{aligned}$$

where $B_s = \{x \in \mathbb{R} : s^\frown x \in B\}$. We will now show that P_k is σ-additive. P_1 is a measure by definition so suppose $k = n+1$. Let $\{X_i : i \in \omega\}$ be a countable collection of disjoint subsets of $\mathbb{R}^k$. Then

$$
\begin{aligned}
P_k\left(\bigcup\{X_i : i \in \omega\}\right) &= \int_{s\in\mathbb{R}^n} \nu_{\sigma,\tau}(s)\left(\left(\bigcup\{X_i : i \in \omega\}\right)_s\right) \mathrm{d}P_n(s) \\
&= \int_{s\in\mathbb{R}^n} \sum_{i\in\omega} \nu_{\sigma,\tau}(s)((X_i)_s)\mathrm{d}P_n(s) \\
&= \sum_{i\in\omega} \int_{s\in\mathbb{R}^n} \nu_{\sigma,\tau}(s)((X_i)_s)\mathrm{d}P_n(s) \\
&= \sum_{i\in\omega} P_k(X_i)
\end{aligned}
$$

So P_k is a measure for every $k \geq 1$. We will now want to use the following theorem which can be found in [6, Appendix 7]:

Theorem (Consistency theorem of Kolmogorov).
For every $k \in \mathbb{N}$ let P_k be a probability measure on $(\mathbb{R}^k, \mathrm{Bor}(\mathbb{R}^k))$ such that the sequence $(P_k)_{k\in\mathbb{N}}$ is consistent. Then there exists a measure P on $(\mathbb{R}^\omega, \mathrm{Bor}(\mathbb{R}^\omega))$ such that for every measurable $B \in \mathbb{R}^k$

$$P(B \times \mathbb{R}^\omega) = P_k(B)$$

The sequence $(P_k)_{k\geq 1}$ is said to be consistent if for every $k \in \mathbb{N}$ and B a k-dimensional Borel set we have

$$P_{k+1}(B \times \mathbb{R}) = P_k(B).$$

So we show that our sequence is consistent:

$$
\begin{aligned}
P_{k+1}(B \times \mathbb{R}) &= \int_{s\in\mathbb{R}^k} \nu_{\sigma,\tau}(s)((B \times \mathbb{R})_s)\mathrm{d}P_k(s) \\
&= \int_{s\in\mathbb{R}^k} 1_B(s)\mathrm{d}P_k(s) \\
&= P_k(B)
\end{aligned}
$$

Thus we can apply the theorem giving us a measure P on $\mathbb{R}^\omega$. We then use this as our measure and denote it with $\mu^{\mathrm{E}}_{\sigma,\tau}$.

3.3 The Axiom of Real Blackwell Determinacy

For either of the two definitions of $\mu_{\sigma,\tau}$ we we can define values as before. Let X be either C or E, then we define

$$
\begin{aligned}
\mathrm{val}^{\mathrm{X}-}(\sigma,\tau,A) &:= \sup\{\mu^{\mathrm{X}}_{\sigma,\tau}(B) : B \subseteq A \text{ Borel}\} \\
\mathrm{val}^{\mathrm{X}+}(\sigma,\tau,A) &:= \inf\{\mu^{\mathrm{X}}_{\sigma,\tau}(B) : B \supseteq A \text{ Borel}\} \\
\mathrm{val}^{\mathrm{X}}_{\mathrm{I}}(\sigma,A) &:= \inf\{\mathrm{val}^{\mathrm{X}-}(\sigma,\tau,A) : \tau \text{ a strategy for II}\} \\
\mathrm{val}^{\mathrm{X}}_{\mathrm{II}}(\tau,A) &:= \sup\{\mathrm{val}^{\mathrm{X}+}(\sigma,\tau,A) : \sigma \text{ a strategy for I}\} \\
\mathrm{val}^{\mathrm{X}-}(A) &:= \sup\{\mathrm{val}^{\mathrm{X}}_{\mathrm{I}}(\sigma,A) : \sigma \text{ a strategy for player I}\} \\
\mathrm{val}^{\mathrm{X}+}(A) &:= \inf\{\mathrm{val}^{\mathrm{X}}_{\mathrm{II}}(\tau,A) : \tau \text{ a strategy for player II}\}
\end{aligned}
$$

Here Borel means Borel in the product topology of either the discrete topology or the Euclidean topology, depending on which measure we use. We say a set $A \subseteq \mathbb{R}^\omega$ is **real C-Blackwell determined** if $\mathrm{val}^{\mathrm{C}-}(A) = \mathrm{val}^{\mathrm{C}+}(A)$ and it is **real E-Blackwell determined** if $\mathrm{val}^{\mathrm{E}-}(A) = \mathrm{val}^{\mathrm{E}+}(A)$. And we define the Axiom of Real Blackwell Determinacy to mean that every set $A \subseteq \mathbb{R}^\omega$ is real Blackwell determined.

This gives us two different axioms. One for the measure that allows countable support strategies and one for the measure that uses the Euclidean topology. We denote them with $\mathsf{Bl\text{-}AD}_{\mathbb{R}}^{\mathrm{C}}$ and $\mathsf{Bl\text{-}AD}_{\mathbb{R}}^{\mathrm{E}}$.

So we've seen how we can define real Blackwell determinacy. But are the axioms consistent?

Theorem
Let X be either C or E, then

$$\mathsf{AD}_{\mathbb{R}} \implies \mathsf{Bl\text{-}AD}_{\mathbb{R}}^{\mathrm{X}}$$

Proof
Assume $\mathsf{AD}_{\mathbb{R}}$ and let $A \subseteq \mathbb{R}^\omega$ be arbitrary. One of the two players has a winning strategy for $G_{\mathbb{R}}(A)$. Suppose σ is a winning strategy for player I. We can interpret σ as a mixed strategy such that for every mixed strategy τ for player II we get

$$\mathrm{val}^{\mathrm{X}-}(\sigma, \tau, A) = \mathrm{val}^{\mathrm{X}+}(\sigma, \tau, A) = 1$$

So $\mathrm{val}_{\mathrm{I}}^{\mathrm{X}}(\sigma, A) = 1$ and $\mathrm{val}_{\mathrm{II}}^{\mathrm{X}}(\tau, A) = 1$ for every mixed strategy τ for player II. Therefore $\mathrm{val}^{\mathrm{X}-}(A) = \mathrm{val}^{\mathrm{X}+}(A) = 1$.

Similarly, if τ is a winning strategy for player II, we can interpret it as a mixed strategy and we get $\mathrm{val}^{\mathrm{X}-}(A) = \mathrm{val}^{\mathrm{X}+}(A) = 0$. So every set $A \subseteq \mathbb{R}^\omega$ is real X-Blackwell determined. □

So the Axioms of real Blackwell determinacy are implied by the Axiom of real Determinacy. Thus they are consistent if $\mathsf{AD}_{\mathbb{R}}$ is consistent.

4 Open questions

Some interesting question about Real Blackwell Determinacy are:

- Are $\mathsf{Bl\text{-}AD}_{\mathbb{R}}^{\mathrm{C}}$ and $\mathsf{Bl\text{-}AD}_{\mathbb{R}}^{\mathrm{E}}$ stronger than $\mathsf{Bl\text{-}AD}$?
- Are $\mathsf{Bl\text{-}AD}_{\mathbb{R}}^{\mathrm{C}}$ and $\mathsf{Bl\text{-}AD}_{\mathbb{R}}^{\mathrm{E}}$ equivalent?
- What is their consistency strength?

In [3, p. 6–7] Robert Solovay proves that if you assume $\mathsf{AD}_{\mathbb{R}}$, there exists a normal measure on the the set of countable sets of reals. It might be possible to prove this under $\mathsf{Bl\text{-}AD}_{\mathbb{R}}^{\mathrm{C}}$ and this could help us towards answering the first question. The idea of the proof is as follows:

Let Ω be the set of countable subsets of the reals. Now for $A \subseteq \Omega$ consider the game $G(A)$ where player I and player II subsequently play finite sets of reals and player II wins iff the union of the played sets is an element of A. Solovay proves that the set

$$U = \{A \subseteq \Omega : \text{ player II has a winning strategy for } G(A)\}$$

is a normal ultrafilter on Ω. He proves this by modifying and combining winning strategies to create winning strategies for player II to show that certain sets are in U and winning strategies for player I to show that certain sets are not in U.

Using the Martin-Vervoort Zero-One Law [4, p 41–42] we can create a similar set U using mixed strategies:

$$U = \{A \subseteq \Omega : \text{ the game } G(A) \text{ has value } 0\}$$

And the expectation is that we can do the same modifications and combinations as Solovay did with the pure strategies. However, it seems that this requires a lot of coding and checking and this is outside the scope of this bachelor thesis. But if it works it could give us some more understanding of $\mathsf{Bl}\text{-}\mathsf{AD}^{\mathbb{C}}_{\mathbb{R}}$ and its relation to other axioms.

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