THE GENERAL INDUCTIVE ARGUMENT FOR MEASURE ANALYSES WITH ADDITIVE ORDINAL ALGEBRAS

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In [BoLö∞] we gave a survey of measure analyses under AD, discussed the general theory of order measures and gave a simple inductive argument for a measure analysis with just two measures that reached the first ω^2 cardinals after a strong partition cardinal. In this article we show that the restriction to two measures can be lifted under the right prerequisites. As a corollary we get a full measure analysis for sums of order measures. This allows us to inductively reduce the measure analysis of an additive ordinal algebra $\langle \mathfrak{A}, \oplus \rangle$ with a set of generators $\mathfrak{V}$ to the analysis of $\mathfrak{V}$. The notion of an ordinal algebra is introduced in [JaLö∞]. This article is to be understood as a technical report, showing the progress we made in generalizing the results from [BoLö∞], so we refer the reader to that paper for more detailed explanations and definitions of the notions and notations used in this paper.

1. Necessary Lemmas and Definitions

Lemma 1. Let $\kappa < \lambda$ be cardinals, μ a measure on κ and $\mathrm{cf}(\lambda) > \kappa$. Then $\mathrm{cf}(\lambda^\kappa/\mu) = \mathrm{cf}(\lambda)$.

Proof. Cf. [BoLö∞, Lemma 6]. □

Lemma 2. Let κ be a strong partition cardinal and let μ, η and ν be order measures on κ. Then

(1) $\kappa^\kappa/\mu \leq \kappa^\kappa/\mu \oplus \nu$,
(2) $\kappa^\kappa/\nu \leq \kappa^\kappa/\mu \oplus \nu$, and
(3) $\kappa^\kappa/\mu \oplus \nu \leq \kappa^\kappa/\mu \oplus \eta \oplus \nu$.

Proof. Cf. [BoLö∞, Lemma 12]. □

Lemma 3. Let κ be a strong partition cardinal and let μ and ν be order measures, both on κ. Let $\lambda \geq \kappa$ be a cardinal. Then

$$\lambda^\kappa/(\mu \oplus \nu) \leq (\lambda^\kappa/\nu)^\kappa/\mu.$$

Proof. Cf. [BoLö∞, Lemma 13]. □

Theorem 4 (Ultrapower Shifting Lemma)**.** Let β and γ be ordinals and let μ be a κ-complete ultrafilter on κ with $\kappa^\kappa/\mu = \kappa^{(\gamma)}$. If for all cardinals $\kappa < \nu \leq \kappa^{(\beta)}$

- either ν is a successor and $\mathrm{cf}(\nu) > \kappa$,
- or ν is a limit and $\mathrm{cf}(\nu) < \kappa$,

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then $(\kappa^{(\beta)})^\kappa/\mu \leq \kappa^{(\gamma+\beta)}$.

Proof. Cf. [Lö02, Lemma 2.7]. □

Definition 5. If $\langle \mu_i\,;\, i < m+1\rangle$ is a finite sequence of measures on κ we write $\mathrm{iUlt}_\alpha(\mu_0, \cdots, \mu_m)$ for the corresponding iterated Ultrapower:

$$\mathrm{iUlt}_\alpha(\mu_0, \cdots, \mu_m) := (\ldots(\alpha^\kappa/\mu_m)^\kappa/\ldots)^\kappa/\mu_0.$$

We write $\mathrm{iUlt}(\mu_0, \cdots, \mu_m)$ for $\mathrm{iUlt}_\kappa(\mu_0, \cdots, \mu_m)$.

Definition 6. Let $\gamma < \varepsilon_0 (= \sup_{n<\omega} \mathbf{e}_n)$ be an ordinal and $\langle \theta_\alpha\,;\, \alpha \in \gamma\rangle$ the unique sequence of ordinals such that $\theta_0 = 1$, $\theta_{\alpha+1} = \theta_\alpha \cdot \omega + 1$ for $\alpha < \gamma$ and $\theta_\lambda = (\sup_{\alpha<\lambda} \theta_\alpha)+1$ for limit ordinals $\lambda < \gamma$. For every successor ordinal $\xi < \sup_{\alpha\in\gamma} \theta_\alpha \cdot \omega$ the **$\vec{\theta}$-Cantor normal form of** ξ is a decomposition of ξ into a finite sum of elements of $\langle \theta_\alpha\,;\, \alpha \in \gamma\rangle$, i.e., $\xi = \theta_{\alpha_0} + \cdots + \theta_{\alpha_m}$, where $m \in \omega$. It is defined by

- $\alpha_0 := \min\{\alpha \in \gamma\,;\, \xi < \theta_{\alpha+1}\}$ and
- $\alpha_{i+1} := \min\{\alpha \in \gamma\,;\, \xi < \theta_{\alpha_0} + \cdots + \theta_{\alpha_i} + \theta_{\alpha+1}\}$.

By wellfoundedness of $<$ the relativized Cantor normal form of ξ is welldefined and unique.

The **$\vec{\theta}$-Cantor normal form** of a limit ordinal $\xi < \sup_{\alpha\in\gamma} \theta_\alpha \cdot \omega$ is $\theta_{\alpha_0} + \cdots + \theta_{\alpha_m} - 1$, where $\theta_{\alpha_0} + \cdots + \theta_{\alpha_m}$ is the relativized Cantor normal form of $\xi + 1$.

2. The abstract combinatorial computation

Theorem 7. Let κ be a strong partition cardinal and $\gamma < \varepsilon_0$ an ordinal. Let $\langle \mu_\alpha\,;\, \alpha \in \gamma\rangle$ be a sequence of measures on κ and $\langle \theta_\alpha\,;\, \alpha \in \gamma\rangle$, $\langle \iota_\alpha\,;\, \alpha \in \gamma\rangle$ sequences of ordinals such that

i) $\kappa^\kappa/\mu_0 = \kappa^{(\theta_0)} = \kappa^+$,
ii) $\kappa^\kappa/\mu_{\alpha+1} = \kappa^{(\theta_{\alpha+1})} = \kappa^{(\theta_\alpha\cdot\omega+1)}$ for $\alpha < \gamma$,
iii) $\kappa^\kappa/\mu_\lambda = \kappa^{(\theta_\lambda)} = \kappa^{((\sup_{\alpha<\lambda}\theta_\alpha)+1)}$ for limit ordinals $\lambda < \gamma$,
iv) $(\kappa^\kappa/\nu)^{(\theta_\alpha)} \leq \kappa^\kappa/\nu \oplus \mu_\alpha$ for order measures ν and $\alpha < \gamma$, and
v) $\mathrm{cf}(\kappa^\kappa/\mu_\alpha) = \iota_\alpha > \kappa$ for $\alpha < \gamma$.

Then for all $\xi < \sup_{\alpha<\gamma}(\theta_\alpha \cdot \omega)$ the following is true:

(1) If $\xi > 0$ is a limit ordinal and $\theta_{\alpha_0} + \cdots + \theta_{\alpha_m} - 1$ its $\vec{\theta}$-Cantor normal form then, with $\zeta = \theta_{\alpha_m} - 1$,

$$\kappa^{(\xi)} = \mathrm{iUlt}_{\kappa^{(\zeta)}}(\mu_{\alpha_0}, \cdots, \mu_{\alpha_{m-1}}) = (\kappa^{(\zeta)})^\kappa/(\mu_{\alpha_0} \oplus \cdots \oplus \mu_{\alpha_{m-1}}).$$

(2) If ξ is a successor ordinal and $\theta_{\alpha_0} + \cdots + \theta_{\alpha_m}$ its $\vec{\theta}$-Cantor normal form then

$$\kappa^{(\xi)} = \mathrm{iUlt}(\mu_{\alpha_0}, \cdots, \mu_{\alpha_m}) = \kappa^\kappa/(\mu_{\alpha_0} \oplus \cdots \oplus \mu_{\alpha_m}).$$

(3)

$$\mathrm{cf}(\kappa^{(\xi)}) := \begin{cases} \kappa & \text{if } \xi = 0, \\ \omega & \text{if } \xi > 0 \text{ is a limit,} \\ \iota_{\alpha_m} & \text{if } \xi = \theta_{\alpha_0} + \cdots + \theta_{\alpha_m} \text{ is a successor.} \end{cases}$$

Proof. By assumption κ is a strong partition cardinal, thus regular. Also, for all limit ordinals $\xi < \sup_{n<\omega} \mathbf{e}_n$, the cofinality of $\kappa^{(\xi)}$ is ω. So the first two parts of (3) are trivial.

We proceed by induction on $\xi > 0$, using the following induction hypothesis:

$$(\mathrm{IH}_\xi)\left[\begin{array}{l} \text{For all } 0 < \beta \leq \xi \text{, the following three conditions hold:} \\ \text{1. If } \beta \text{ is a limit and } \beta + 1 = \theta_{\alpha_0} + \cdots + \theta_{\alpha_m} \text{, then} \\ \kappa^{(\beta)} = \mathrm{iUlt}_{\kappa^{(\zeta)}}(\mu_{\alpha_0}, \cdots, \mu_{\alpha_{m-1}}) \text{, where } \zeta = \theta_{\alpha_m} - 1 \\ \text{2. If } \beta \text{ is a successor and } \beta = \theta_{\alpha_0} + \cdots + \theta_{\alpha_m} \text{ then} \\ \kappa^{(\beta)} = \mathrm{iUlt}(\mu_{\alpha_0}, \cdots, \mu_{\alpha_m}) = \kappa^\kappa/\mu_{\alpha_0} \oplus \cdots \oplus \mu_{\alpha_m} \\ \text{3. } \mathrm{cf}(\kappa^{(\beta)}) := \begin{cases} \omega & \text{if } \beta > 0 \text{ is a limit,} \\ \iota_{\alpha_m} & \text{if } \beta = \theta_{\alpha_0} + \cdots + \theta_{\alpha_m} \text{ is a successor.} \end{cases} \end{array}\right.$$

Obviously, if all (IH_ξ) (for $\xi < \sup_{\alpha<\gamma}(\theta_\alpha \cdot \omega)$) hold, the theorem is proven.

By assumption we have $\kappa^\kappa/\mu_0 = \kappa^+$ and $\mathrm{cf}(\kappa^+) = cf(\kappa^\kappa/\mu_0) = \iota_0$, so IH_1 holds.

For the successor step we assume that IH_ξ holds and prove $\mathrm{IH}_{\xi+1}$. If $\xi+1 = \theta_\alpha$ for some $\alpha < \gamma$, we have by assumption $\kappa^{(\xi+1)} = \kappa^\kappa/\mu_\alpha$ and $\mathrm{cf}(\kappa^{(\xi+1)}) = \mathrm{cf}(\kappa^\kappa/\mu_\alpha) = \iota_\alpha$ and thus $\mathrm{IH}_{\xi+1}$ holds. Otherwise let $\theta_{\alpha_0} + \cdots + \theta_{\alpha_m}$ be the $\vec{\theta}$-Cantor normal form of $\xi + 1$. Then

$$\begin{array}{rcll} \kappa^{(\xi+1)} & = & (\kappa^{(\theta_{\alpha_0}+\cdots+\theta_{\alpha_{m-1}})})^{(\theta_{\alpha_m})} & \\ & = & (\kappa^\kappa/\mu_{\alpha_0} \oplus \cdots \oplus \mu_{\alpha_{m-1}})^{(\theta_{\alpha_m})} & \text{IH} \\ & \leq & \kappa^\kappa/\mu_{\alpha_0} \oplus \cdots \oplus \mu_{\alpha_m} & \text{Assumption iv)} \\ & \leq & (\kappa^\kappa/\mu_{\alpha_m})^\kappa/\mu_{\alpha_0} \oplus \cdots \oplus \mu_{\alpha_{m-1}} & \text{Lemma 3} \\ & \vdots & & \\ & \leq & \mathrm{iUlt}(\mu_{\alpha_0}, \cdots, \mu_{\alpha_m}) & \text{Lemma 3} \\ & = & (\mathrm{iUlt}(\mu_{\alpha_1}, \cdots, \mu_{\alpha_m})^\kappa/\mu_{\alpha_0} & \\ & = & (\kappa^{(\theta_{\alpha_1}+\cdots+\theta_{\alpha_m})})^\kappa/\mu_{\alpha_0} & \text{IH} \\ & \leq & \kappa^{(\theta_{\alpha_0}+\cdots+\theta_{\alpha_m})} & \text{Theorem 4} \\ & = & \kappa^{(\xi+1)}. & \end{array}$$

Using $\kappa^{(\xi+1)} = \mathrm{iUlt}(\mu_{\alpha_0}, \cdots, \mu_{\alpha_{m-1}})$ and Lemma 1 (repeatedly) we get

$$\iota_{\alpha_{m-1}} = \mathrm{cf}(\kappa^\kappa/\mu_{\alpha_m}) = \cdots = \mathrm{cf}((\mathrm{iUlt}(\mu_{\alpha_1}, \cdots, \mu_{\alpha_m}))^\kappa/\mu_{\alpha_0}) = \mathrm{cf}(\kappa^{(\xi+1)}),$$

which proves $\mathrm{IH}_{\xi+1}$.

Now for the limit case. We assume that IH_β holds for all $\beta < \xi$ and prove IH_ξ. Let $\theta_{\alpha_0} + \cdots + \theta_{\alpha_m} - 1$ be the $\vec{\theta}$-Cantor normal form of ξ. If α_m is a successor we have $\theta_{\alpha_m} - 1 = \sup_{n\in\omega} \theta_{\alpha_m-1} \cdot n$ and so we get

$$\begin{array}{rlll}
\kappa^{(\xi)} & = & \sup_{n\in\omega}\left(\kappa^{(\theta_{\alpha_0}+\cdots+\theta_{\alpha_{m-1}}+\theta_{\alpha_m-1}\cdot n)}\right) & \\
& = & \sup_{n\in\omega}\big(\kappa^\kappa/\mu_{\alpha_0}\oplus\cdots\oplus\mu_{\alpha_{m-1}}\oplus\mu_{\alpha_m-1}\otimes n\big) & \text{IH} \\
& \leq & \sup_{n\in\omega}\big((\kappa^\kappa/\mu_{\alpha_m-1}\otimes n)^\kappa/\mu_{\alpha_0}\oplus\cdots\oplus\mu_{\alpha_{m-1}}\big) & \text{Lemma 3} \\
& = & \sup_{n\in\omega}\big((\kappa^{(\theta_{\alpha_m-1}\cdot n)})^\kappa/\mu_{\alpha_0}\oplus\cdots\oplus\mu_{\alpha_{m-1}}\big) & \text{IH} \\
\big[& = & (\kappa^{(\theta_{\alpha_m}-1)})^\kappa/\mu_{\alpha_0}\oplus\cdots\oplus\mu_{\alpha_{m-1}} & \big] \\
& \leq & \sup_{n\in\omega}\big(\mathrm{iUlt}_{\kappa^{(\theta_{\alpha_m-1}\cdot n)}}(\mu_{\alpha_0}\oplus\cdots\oplus\mu_{\alpha_{m-2}},\mu_{\alpha_{m-1}})\big) & \text{Lemma 3} \\
& \vdots & & \\
& \leq & \sup_{n\in\omega}\big(\mathrm{iUlt}_{\kappa^{(\theta_{\alpha_m-1}\cdot n)}}(\mu_{\alpha_0},\cdots,\mu_{\alpha_{m-1}})\big) & \text{Lemma 3} \\
\big[& = & \mathrm{iUlt}_{\kappa^{(\theta_{\alpha_m}-1)}}(\mu_{\alpha_0},\cdots,\mu_{\alpha_{m-1}}) & \big] \\
& \leq & \sup_{n\in\omega}\left(\mathrm{iUlt}_{\kappa^{(\theta_{\alpha_{m-1}}+\theta_{\alpha_m-1}\cdot n)}}(\mu_{\alpha_0},\cdots,\mu_{\alpha_{m-2}})\right) & \text{Theorem 4} \\
& \vdots & & \\
& \leq & \sup_{n\in\omega}\big(\kappa^{(\theta_{\alpha_0}+\cdots+\theta_{\alpha_m-1}\cdot n)}\big) & \text{Theorem 4} \\
& = & \kappa^{(\xi)} &
\end{array}$$

On the other hand, if α_{m-1} is a limit we have $\theta_{\alpha_m} - 1 = \sup_{\beta\in\alpha_m} \theta_\beta$ and so we get

$$\begin{array}{rlll}
\kappa^{(\xi)} & = & \sup_{\beta\in\alpha_m}\left(\kappa^{(\theta_{\alpha_0}+\cdots+\theta_{\alpha_{m-1}}+\theta_\beta)}\right) & \\
& = & \sup_{\beta\in\alpha_m}\big(\kappa^\kappa/\mu_{\alpha_0}\oplus\cdots\oplus\mu_{\alpha_{m-1}}\oplus\mu_\beta\big) & \text{IH} \\
& \leq & \sup_{\beta\in\alpha_m}\big((\kappa^\kappa/\mu_\beta)^\kappa/\mu_{\alpha_0}\oplus\cdots\oplus\mu_{\alpha_{m-1}}\big) & \text{Lemma 3} \\
& = & \sup_{\beta\in\alpha_m}\big((\kappa^{(\theta_\beta)})^\kappa/\mu_{\alpha_0}\oplus\cdots\oplus\mu_{\alpha_{m-1}}\big) & \text{IH} \\
\big[& = & (\kappa^{(\theta_{\alpha_m}-1)})^\kappa/\mu_{\alpha_0}\oplus\cdots\oplus\mu_{\alpha_{m-1}} & \big] \\
& \leq & \sup_{\beta\in\alpha_m}\big(\mathrm{iUlt}_{\kappa^{(\theta_\beta)}}(\mu_{\alpha_0}\oplus\cdots\oplus\mu_{\alpha_{m-2}},\mu_{\alpha_{m-1}})\big) & \text{Lemma 3} \\
& \vdots & & \\
& \leq & \sup_{\beta\in\alpha_m}\big(\mathrm{iUlt}_{\kappa^{(\theta_\beta)}}(\mu_{\alpha_0},\cdots,\mu_{\alpha_{m-1}})\big) & \text{Lemma 3} \\
\big[& = & \mathrm{iUlt}_{\kappa^{(\theta_{\alpha_m}-1)}}(\mu_{\alpha_0},\cdots,\mu_{\alpha_{m-1}}) & \big] \\
& \leq & \sup_{\beta\in\alpha_m}\left(\mathrm{iUlt}_{\kappa^{(\theta_{\alpha_{m-1}}+\theta_\beta)}}(\mu_{\alpha_0},\cdots,\mu_{\alpha_{m-2}})\right) & \text{Theorem 4} \\
& \vdots & & \\
& \leq & \sup_{\beta\in\alpha_m}\big(\kappa^{(\theta_{\alpha_0}+\cdots+\theta_\beta)}\big) & \text{Theorem 4} \\
& = & \kappa^{(\xi)} &
\end{array}$$

As we mentioned at the beginning of this proof, the cofinality of $\kappa^{(\xi)}$ for a limit ordinal $0 < \xi < \varepsilon_0$ is ω, which concludes the proof. □

Corollary 8. Let κ be a strong partition cardinal and $\gamma < \varepsilon_0$ an ordinal. If $\langle\mu_\alpha\,;\,\alpha\in\gamma\rangle$ is a sequence of measures on κ and $\langle\theta_\alpha\,;\,\alpha\in\gamma\rangle$, a sequence of ordinals that fulfill the requirements of Theorem 7, then for all $\xi < \sup_{\alpha<\gamma}(\theta_\alpha\cdot\omega)$ and finite sequences $\langle\alpha_i\,;\,i\leq m\rangle\in\gamma^{m+1}$ we have

$$\kappa^{(\theta_{\alpha_0}+\cdots+\theta_{\alpha_m}+\xi)} = \mathrm{iUlt}_{\kappa^{(\xi)}}(\mu_{\alpha_0},\cdots,\mu_{\alpha_m}) = \left(\kappa^{(\xi)}\right)^\kappa/(\mu_{\alpha_0}\oplus\cdots\oplus\mu_{\alpha_m}).$$

Proof. If $\theta_{\beta_0}+\cdots+\theta_{\beta_n}$ is the $\vec{\theta}$-Cantor normal form of $\theta_{\alpha_0}+\cdots+\theta_{\alpha_m}+\xi$, then the $\vec{\theta}$-Cantor normal form of ξ is an end segment of $\theta_{\beta_0}+\cdots+\theta_{\beta_n}$, i.e. there is a $k\geq 0$ such that $\theta_{\beta_k}+\cdots+\theta_{\beta_n}$ is the $\vec{\theta}$-Cantor normal form of ξ. And for all $i<k$

there is a $j \leq m$ such that $\theta_{\beta_i} = \theta_{\alpha_j}$. So by Theorem 7

$$\kappa^{(\theta_{\alpha_0}+\cdots+\theta_{\alpha_m}+\xi)} = \kappa^{(\theta_{\beta_0}+\cdots+\theta_{\beta_n})} = \kappa^\kappa/\mu_{\beta_0} \oplus \cdots \oplus \mu_{\beta_n},$$

using Lemma 2 we can insert the missing elements of the sequence $\langle \mu_{\alpha_i}\,;\, i \leq m\rangle$ and then apply Lemma 3 to get

$$\kappa^\kappa/\mu_{\beta_0} \oplus \cdots \oplus \mu_{\beta_n} \leq \kappa^\kappa/\mu_{\alpha_0} \oplus \cdots \oplus \mu_{\alpha_m} \oplus \mu_{\beta_k} \oplus \cdots \oplus \mu_{\beta_n}$$
$$\leq (\kappa^\kappa/\mu_{\beta_k} \oplus \cdots \oplus \mu_{\beta_n})^\kappa/\mu_{\alpha_0} \oplus \cdots \oplus \mu_{\alpha_m} = \left(\kappa^{(\xi)}\right)^\kappa/\mu_{\alpha_0} \oplus \cdots \oplus \mu_{\alpha_m}.$$

And finally we can use Lemma 3 and Theorem 4, both repeatedly as we did before in the proof of Theorem 7, to reach equality:

$$\left(\kappa^{(\xi)}\right)^\kappa/\mu_{\alpha_0} \oplus \cdots \oplus \mu_{\alpha_m} \leq \mathrm{iUlt}_{\kappa^{(\xi)}}(\mu_{\alpha_0}, \cdots, \mu_{\alpha_m}) \leq \kappa^{(\theta_{\alpha_0}+\cdots+\theta_{\alpha_m}+\xi)}.$$

□

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