Student Papers from an Intuitionistic Logic Project

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ML is not finitely axiomatizable over Cheq

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Abstract. We show that the Medvedev logic **ML** is not finitely axiomatizable over the logic **Cheq** of chequered subsets of $\mathbb{R}^\infty$. This gives a negative answer to one of the questions raised by Litak [3].

1 Introduction

In 1962, Medvedev introduced the logic of "finite problems". It became known as the Medvedev logic **ML**. It is known that **ML** has the finite model property, the disjunction property, contains the Kreisel-Putnam and Scott logics, and is contained in the logic of weak excluded middle (see, e.g., [2]). Recently, van Benthem et al. [1] introduced the logic **Cheq** of chequered subsets of $\mathbb{R}^\infty$ and showed that **Cheq** has the finite model property. Litak [3] proved that **Cheq** has the disjunction property, contains the Scott logic and is contained in the Medvedev logic **ML**. He raised a question whether **ML** is finitely axiomatizable over **Cheq**. In this note we give a negative answer to this question. Thus, the connection between the Medvedev logic and **Cheq** is not as strong as it first appeared.

2 ML is not finitely axiomatizable

We assume the reader's familiarity with basics of Kripke semantics for intermediate logics and refer to Chagrov and Zakharyaschev [2] for the details.

Definition 1. [4] For a finite non-empty set D, let $\mathcal{P}^0(D)$ denote the Kripke frame

$$\mathcal{P}^0(D) = \langle \{X \subseteq D | X \neq \emptyset\}, \supseteq \rangle.$$

We call $\mathcal{P}^0(D)$ a *Medvedev frame.* The logic **ML** is the logic of all Medvedev frames. As usual, a frame $\mathcal{F}$ is called an **ML**-*frame* if all the theorems of **ML** are valid in $\mathcal{F}$.

For each natural number $k \neq 0$ and each $i \leq k$, let $\mathcal{G}_k$ and $\mathcal{G}_k^i$ be the frames shown in Figure 1 (a) and (b), respectively. The following lemma is proved in [4].

Lemma 2. (*a*) *For each natural number $k > 0$, the frame $\mathcal{G}_k$ is not an* **ML**-*frame.*

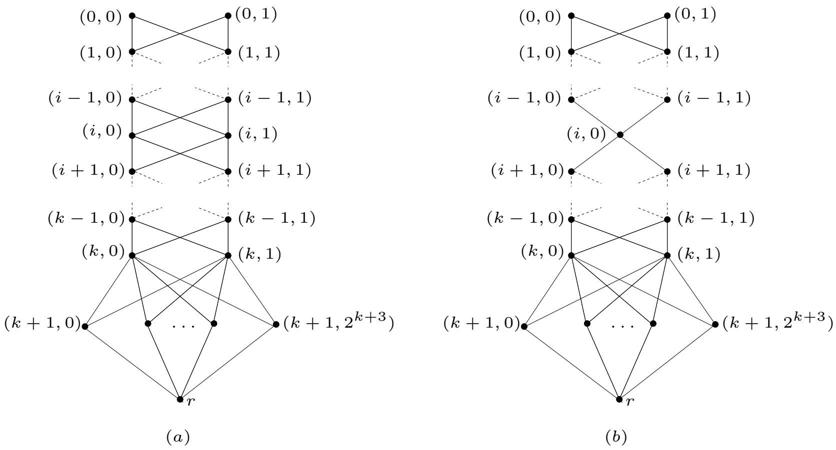

Fig. 1. The frames $\mathcal{G}_k$ and $\mathcal{G}_k^i$.

(b) *For each natural number $k > 0$ and each $i \leq k$, the frame $\mathcal{G}_k^i$ is an* **ML**-*frame.*

(c) *Let φ be a formula with k variables. There exists a natural number $i \leq k$ such that*

$$\mathcal{G}_k \Vdash \phi \text{ iff } \mathcal{G}_k^i \Vdash \phi.$$

It is an easy corollary of Lemma 2 that **ML** is not finitely axiomatizable. Indeed, suppose there is a finite set of formulas axiomatizing **ML**. Without loss of generality we may assume that **ML** is axiomatized by a single formula ϕ with k variables (for some natural number k). By Lemma 2(c), there exists a natural number $i \leq k$ such that ϕ is valid in $\mathcal{G}_k$ iff ϕ is valid in $\mathcal{G}_k^i$. By Lemma 2(b), $\mathcal{G}_k^i$ is an **ML**-frame. Thus, ϕ is valid in $\mathcal{G}_k^i$. Therefore, ϕ is valid in $\mathcal{G}_k$. But $\mathcal{G}_k$ is not an **ML**-frame by Lemma 2(a). This contradiction proves that such a ϕ does not exist. Thus, we arrive at the following theorem.

Theorem 3. *[4] The logic* **ML** *is not finitely axiomatizable.*

3 ML is not finitely axiomatizable over Cheq

Definition 4. [1] Let $\mathcal{F}$ denote the two-fork Kripke frame shown in Figure 2. Let $\mathcal{F}_n = \underbrace{\mathcal{F} \times \cdots \times \mathcal{F}}_{n \text{ times}}$. The logic **Cheq** is the logic of $\{\mathcal{F}_n : n \in \mathbb{N}\}$.

Theorem 5. *[3]* **ML** *is a proper extension of* **Cheq**.

Our main goal is to show that **ML** is not finitely axiomatizable over **Cheq**. For an n-tuple x, let $N_i(x)$ denote the number of w_i that occur in x ($i = 0, 1, 2$). We denote the j-th component of x by $x(j)$. For a Kripke frame $\langle W, \leq \rangle$ and

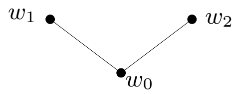

Fig. 2. The frame $\mathcal{F}_1$.

$w, v \in W$, we say that v is an *immediate successor* of w if $w \neq v$, $w \leq v$ and there is no $u \notin \{w, v\}$ such that $w \leq u$ and $u \leq v$. Note that if $x \in \mathcal{F}_n$ is an immediate successor of $(w_0, \ldots, w_0)$, it has only one component that differs from w_0 and we denote it by $\delta(x)$.

For every $k > 1$ and every $l > 0$, let $\mathcal{G}_{k,l}$ denote the frame shown in Figure 3 (note that $\mathcal{G}_k = \mathcal{G}_{k,2^{k+3}-1}$).

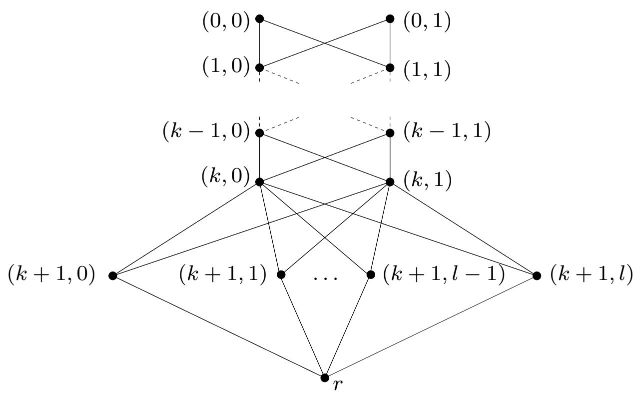

Fig. 3. The frame $\mathcal{G}_{k,l}$.

Proposition 6. *For each $l > 0$, there exists n such that $\mathcal{G}_{2,l}$ is a p-morphic image of $\mathcal{F}_n$. Moreover, there is a p-morphism f from $\mathcal{F}_n$ onto $\mathcal{G}_{2,l}$ such that $f^{-1}\{(3,i) | i \leq l\}$ is the set of all immediate successors of $(w_0, \ldots, w_0)$.*

Proof. Fix $l > 0$ and fix an arbitrary n such that $2n \geq l+1$ and $n > 3$. We show that there is a p-morphism f from $\mathcal{F}_n$ onto $\mathcal{G}_{2,l}$ such that $f^{-1}\{(3,i)|i \leq l\}$ is the set of all immediate successors of $(w_0, \ldots, w_0)$. Since $2n \geq l+1$, there is a map g from the set of immediate successors of $(w_0, \ldots, w_0)$ onto $\{(3,i)|i \leq l\}$.

Define f by

$$f(x) = \begin{cases} r & \text{if } x = (w_0, \dots, w_0) \\ g(x) & \text{if } x \text{ is an immediate successor of } (w_0, \dots, w_0) \\ (2,0) & \text{if } N_0(x) = n-2,\ x(i) = w_1,\ x(j) = w_2 \text{ and } i+j \text{ is even} \\ (2,1) & \text{if } N_0(x) = n-2,\ x(i) = w_1,\ x(j) = w_2 \text{ and } i+j \text{ is odd} \\ (1,0) & \text{if } x \text{ is not maximal, } N_1(x) > 1 \text{ and } N_2(x) \leq 1 \\ (1,1) & \text{if } x \text{ is not maximal, } N_2(x) > 1 \text{ and } N_1(x) \leq 1 \\ (0,0) & \text{if } x \text{ is maximal and either } N_1(x) = 1 \text{ or } N_2(x) = 1 \\ (0,1) & \text{otherwise.} \end{cases}$$

Obviously, f is a well-defined onto map such that $f^{-1}\{(3,i) | i \leq l\}$ is the set of all immediate successors of $(w_0, \dots, w_0)$. We show that f is a p-morphism; that is, if $f(x) \leq u$, then there is y such that $x \leq y$ and $f(y) = u$ and if $x \leq y$, then $f(x) \leq f(y)$. First, we verify that if $f(x) \leq u$, then there is y such that $x \leq y$ and $f(y) = u$.

For $x \in \mathcal{F}_n$ and $u \in \mathcal{G}_{2,l}$, let $f(x) \leq u$. Then we need to find a $y \in \mathcal{F}_n$ such that $x \leq y$ and $f(y) = u$. There are nine cases possible.

1. $x = (w_0, \dots, w_0)$. Take any y such that $f(y) = u$.
2. x is an immediate successor of $(w_0, \dots, w_0)$ and $u = (2,0)$. Without loss of generality we may assume that $x(i_0) = w_1$. Since $n > 3$, there is an index $i_1 \neq i_0$ such that $i_0 + i_1$ is even. Then take y such that $y(i_1) = w_2$ and $y(i) = x(i)$ for all $i \neq i_1$.
3. x is an immediate successor of $(w_0, \dots, w_0)$ and $u = (2,1)$. Then the argument is similar to case (2).
4. $N_0(x) = n-2$, $x(i) = w_1$, $x(j) = w_2$ and $u = (1,0)$. Since $n > 3$, there is an index i_0 such that $x(i_0) = w_0$. Then take y such that $y(i_0) = w_1$ and $y(i) = x(i)$ for all $i \neq i_0$.
5. $N_0(x) = n-2$, $x(i) = w_1$, $x(j) = w_2$ and $u = (1,1)$. Then the argument is similar to case (4).
6. $N_1(x) > 1$, $N_2(x) \leq 1$ and $u = (0,0)$. If $N_2(x) = 1$, then there exists i_0 such that $x(i_0) = w_2$. Then take y such that $y(i_0) = w_2$ and $y(i) = w_1$ for all $i \neq i_0$. If $N_2(x) = 0$, fix an index i_0 such that $x(i_0) = w_0$ and take y such that $y(i_0) = w_2$ and $y(i) = w_1$ for all $i \neq i_0$.
7. $N_2(x) > 1$, $N_1(x) \leq 1$ and $u = (0,0)$. Then the argument is similar to case (6).
8. $N_1(x) > 1$, $N_2(x) \leq 1$ and $u = (0,1)$. If $N_2(x) = 0$, then define y as $(w_1, \dots, w_1)$. If $N_2(x) = 1$, then there exists i_0 such that $x(i_0) = w_0$. Then take y such that $y(i_0) = w_2$ and $y(i) = x(i)$ for all $i \neq i_0$.
9. $N_2(x) > 1$, $N_1(x) \leq 1$ and $u = (0,1)$. Then the argument is similar to case (8).

Finally we verify that if $x \leq y$, then $f(x) \leq f(y)$. Suppose $x, y \in \mathcal{F}_n$ are two distinct points such that $x \leq y$. We show that $f(x) \leq f(y)$. There are six cases possible.

1. $x = (w_0, \ldots, w_0)$. Then $f(x) = r$ and $r \leq f(y)$.
2. x is an immediate successor of $(w_0, \ldots, w_0)$. By the definition of f we have $f(x)$ is equal to some $(3, i)$. Since y is not an immediate successor of $(w_0, \ldots, w_0)$, $f(y)$ is also not an immediate successor of r. Hence, $f(x) \leq f(y)$.
3. $N_0(x) = n-2$, $x(i) = w_1$ and $x(j) = w_2$. By the definition of f $f(x)$ is either $(2, 0)$ or $(2, 1)$. Since $x \leq y$, we can deduce that either $N_1(y) > 1$ or $N_2(y) > 1$. In both cases this implies that $f(y)$ belongs to $\{(1, 0), (1, 1), (0, 0), (0, 1)\}$. So $f(x) \leq f(y)$.
4. x is not maximal, $N_1(x) > 1$ and $N_2(x) \leq 1$. From the definition of f it follows that $f(x) = (1, 0)$. Moreover, since $x \leq y$, we also have that $N_1(y) > 1$. So $f(y)$ belongs to $\{(1, 0), (0, 0), (0, 1)\}$. In any case, $f(x) \leq f(y)$.
5. x is not maximal, $N_2(x) > 1$ and $N_1(x) \leq 1$. Then the argument is similar to case (4).
6. $N_1(x) > 1$ and $N_2(x) > 1$. By the definition of f we have that $f(x) = (0, 1)$. Moreover $x \leq y$ implies $N_1(y) > 1$ and $N_2(y) > 1$. So $f(y)$ is also equal to $(0, 1)$.

Proposition 7. *For each $k > 1$ and for each $l > 0$, there exists $n > 2$ such that $\mathcal{G}_{k,l}$ is a p-morphic image of $\mathcal{F}_n$. Moreover, there is a p-morphism f from $\mathcal{F}_n$ onto $\mathcal{G}_{k,l}$ such that $f^{-1}\{(k+1, i) | i \leq l\}$ is the set of all immediate successors of $(w_0, \ldots, w_0)$.*

Proof. The proof is by induction on k. If $k = 2$, apply Proposition 6. Suppose $k = k' + 1$ and there is a p-morphism f from $\mathcal{F}_n$ onto $\mathcal{G}_{k',l}$ such that $f^{-1}\{(k' + 1, i) | i \leq l\}$ is the set of all immediate successors of $(w_0, \ldots, w_0)$.

Define $g : \mathcal{F}_{n+1} \to \mathcal{G}_{k'+1,l}$ by

$$g(x) = \begin{cases} (k'+2, 0) & \text{if } x = (w_0, \ldots, w_0, w_j) \\ (k'+2, i) & \text{if } x \neq (w_0, \ldots, w_0, w_j),\ N_0(x) = n \text{ and} \\ & f(y) = (k'+1, i) \\ (k'+1, 0) & \text{if } N_0(x) = n-2,\ N_0(y) = n-1 \text{ and } \delta(y) = x(n+1) \\ (k'+1, 1) & \text{if } N_0(x) = n-2,\ N_0(y) = n-1 \text{ and } \delta(y) \neq x(n+1) \\ f(y) & \text{if } N_0(y) < n-1, \end{cases}$$

where j belongs to $\{1, 2\}$ and $y = (x(1), \ldots, x(n))$.

Intuitively, the frame $\mathcal{G}_{k'+1,l}$ is obtained from the frame $\mathcal{G}_{k',l}$ by adding two points between the points of depth $k' + 1$ and the points of depth $k' + 2$. The idea is to consider $(n+1)$-tuple of w_j. In general, if $x = (x', w)$, we just map x to the same point on which x' was mapped before. The only exceptions are when $w \neq w_0$ and x' is either $(w_0, \ldots, w_0)$ or an immediate successor of $(w_0, \ldots, w_0)$. In case x' is equal to $(w_0, \ldots, w_0)$ and w is either w_1 or w_2, we map x to an arbitrary immediate successor of r. In case x' is an immediate successor of $(w_0, \ldots, w_0)$ and w is either w_1 or w_2, we map x to one of the two added points.

Obviously, g is a well-defined onto map such that $g^{-1}\{(k+1, i) | i \leq l\}$ is the set of all immediate successors of $(w_0, \ldots, w_0)$. We check that g is a p-morphism.

For $x \in \mathcal{F}_{n+1}$ and $u \in \mathcal{G}_{k,l}$, let $g(x) \leq u$. Then we need to find a $y \in \mathcal{F}_{n+1}$ such that $x \leq y$ and $g(y) = u$. There are eight cases possible.

1. $x = (w_0, \ldots, w_0)$ and $u = (k' + 2, i)$. By the induction hypothesis, there is a t such that $N_0(t) = 1$ and $f(t) = (k' + 1, i)$. Then put $y = (t, w_0)$.
2. $x = (x', w_0)$, $N_0(x') = 1$ and $u = (k' + 1, 0)$. Then put $y = (x', \delta(x'))$.
3. $x = (x', w_0)$, $N_0(x') = 1$ and $u = (k' + 1, 1)$. Then the argument is similar to case (2).
4. $x = (w_0, \ldots, w_0, w_i)$, i is either 1 or 2 and $u = (k' + 1, 0)$. Then put $y = (w_i, w_0, \ldots, w_0, w_i)$.
5. $x = (w_0, \ldots, w_0, w_i)$, i is either 1 or 2 and $u = (k' + 1, 1)$. Then the argument is similar to case (4).
6. $x = (x', w_i)$, i belongs to $\{1, 2\}$, $N_0(x') = 1$, $u = (i_1, i_2)$ and $i_1 \leq k'$. Recall that $f(x')$ has to be equal to some $(k' + 1, i)$. Since f is a p-morphism, there is $s \in \mathcal{F}_n$ such that $y \leq s$ and $f(s) = u$. We put $y = (s, w_i)$.
7. $x = (x', w_i)$, $N_0(x') < n - 1$, $f(x') = (i_1, i_2)$ and $u = (i_1 - 1, 0)$. By the definition of g, we have that $g(x) = (i_1, i_2)$. Since f is a p-morphism, there is $s \in \mathcal{F}_n$ such that $y \leq s$ and $f(s) = u$. We put $y = (t, w_i)$.
8. $x = (x', w_i)$, $N_0(x') < n - 1$, $f(x') = (i_1, i_2)$ and $u = (i_1 - 1, 1)$. Then the argument is similar to case (7).

Next suppose that $x, y \in \mathcal{F}_{n+1}$ are two distinct points such that $x \leq y$. We show that $g(x) \leq g(y)$. Let x', y', i and i' be such that $x = (x', w_i)$ and $y = (y', w_{i'})$. There are four cases possible.

1. $x = (w_0, \ldots, w_0)$. Then $g(x) = r$ and $r \leq g(y)$.
2. x is an immediate successor of $(w_0, \ldots, w_0)$. It is easy to see that $g^{-1}\{(k' + 2, i) | i \leq l\}$ is the set of all immediate successors of $(w_0, \ldots, w_0)$. Thus, $g(x) \leq g(y)$.
3. $N_0(x) = n - 2$ and $i \in \{1, 2\}$. By the definition of g, $g(x)$ is either $(k' + 1, 0)$ or $(k' + 1, 1)$. Now since $x' \leq y'$ and $x' \neq y'$, we also have that $N_0(y') < n - 1$. So $g(y)$ is equal to $f(y')$ and from our assumption on f, we can deduce that $f(y')$ is equal to some (i_1, i_2), where $i_1 \leq k'$. It follows that $g(x) \leq f(y')$.
4. $N_0(y) > n - 1$. By the definition of g, $g(x)$ is equal to $f(x')$. Also since $x' \leq y'$, we have that $N_0(y') > n - 1$ and so $g(y) = f(y')$. Using the fact that f is a p-morphism, we obtain that $f(x') \leq f(y')$.

Corollary 8. (*a*) *For each $k > 1$, the frame $\mathcal{G}_k$ is a p-morphic image of some $\mathcal{F}_n$.*
(*b*) *For each $k > 1$, $\mathcal{G}_k$ is a* **Cheq**-*frame.*

Proof. Follows from Proposition 7.

Theorem 9. *The logic* **ML** *is not finitely axiomatizable over* **Cheq**.

Proof. Suppose there is a finite set of formulas that axiomatizes **ML** over **Cheq**. Without loss of generality we may assume that there is a single formula φ with k variables such that **ML** = **Cheq** + ϕ. By Lemma 2(c), there exists a natural number $i \leq k$ such that ϕ is valid in $\mathcal{G}_k$ iff ϕ is valid in $\mathcal{G}_k^i$. By Lemma 2(b), $\mathcal{G}_k^i$ is an **ML**-frame. Thus, ϕ is valid in $\mathcal{G}_k^i$. Therefore, ϕ is valid in $\mathcal{G}_k$. By Corollary 8, $\mathcal{G}_k$ is a **Cheq**-frame. Thus, $\mathcal{G}_k$ is a **ML**-frame. But this contradicts Lemma 2(a).

4 Conclusion

We proved that **ML** is not finitely axiomatizable over **Cheq**. Thus, the two logics **ML** and **Cheq** are not as closely related as previously thought. It still remains an open problem whether **Cheq** is finitely axiomatizable and/or decidable. At present we can only show that **Cheq** can not be finitely axiomatized in four variables. Of course, the decidability of **ML** still remains an interesting (but difficult) open problem.

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Frame-based Completeness of Intermediate Logics

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Abstract. In this paper, the duality between descriptive frames and Heyting algebras is proved in detail. This, together with the standard result on the completeness of intermediate logics with respect to Heyting algebras, is used to obtain the completeness of intermediate logics with respect to descriptive frames.

1 Introduction

It is well known that propositional logics are complete with respect to the appropriate algebraic semantics. For example, classical propositional logic (**CPC**) is complete with respect to the class of Boolean algebras and intuitionistic propositional logic (**IPC**) is complete with respect to the class of Heyting algebras. These proofs can be modified to obtain similar results about certain classes of extensions. For example, every modal logic (which can be considered an extension of **CPC**) is complete with respect to a certain class of Boolean algebras with operators. And every intermediate logic (which is an extension of **IPC**) is complete with respect to a certain class of Heyting algebras.

Some would prefer, however, completeness with respect to a frame-based semantics. This can be done easily in the cases of **CPC** (which is complete with respect to the single reflexive point) and **IPC** (which is complete with respect to Kripke frames). Similar results can be obtained for particular extensions. (For example, the modal logic **S4** is complete with respect to reflexive, transitive Kripke frames.) But this cannot be done directly for all extensions, so a new method is needed. The new method relies on a broader notion of frame which provides a link to the algebraic semantics. In the case of modal logics, this notion is that of the general frame, and the link with algebraic semantics comes in the form of the Stone Representation Theorem. The final result is that every modal logic is complete with respect to a certain class of general frames.

The goal of this paper is to prove the result for intermediate logics analogous to the one just mentioned for modal logics. That is, we will prove the completeness of intermediate logics with respect to frame-based semantics. We will do this by introducing the notion of a descriptive frame, relating these to Heyting algebras, and then transferring the completeness from the algebraic side to the frame-based side.

2 Preliminaries

In this section, we introduce the preliminary notions needed for the proof, along with some results to be assumed.

2.1 Logics, Frames, and Algebras

We here recall some familiar definitions, mainly to set the notation for the rest of the paper.

Definition 1. The *intuitionistic propositional calculus* **IPC** is the smallest set of formulas (of a propositional language $\mathcal{L}$ containing $\vee, \wedge, \to, \bot$, and infinitely many propositional letters PROP) containing

1. $p \to (q \to p)$,
2. $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$,
3. $p \wedge q \to p$,
4. $p \wedge q \to q$,
5. $p \to p \vee q$,
6. $q \to p \vee q$,
7. $(p \to r) \to ((q \to r) \to ((p \vee q) \to r)))$,
8. $\bot \to p$,

and closed under modus ponens and substitution.

The logics we will be concerned with in this paper are extensions of intuitionistic logic:

Definition 2. An *intermediate logic* is any consistent (i.e., not containg $\bot$) logic (i.e., set of formulas closed under modus ponens and substitution) of $\mathcal{L}$ containing **IPC**.

The intuitive semantics for modal and intuitionistic logics is based on Kripke frames.

Definition 3. An intuitionistic *Kripke frame* is a pair $\mathfrak{F} = (W, R)$ where R is a partial order on $W \neq \emptyset$. An intuitionistic *Kripke model* is a pair $\mathfrak{M} = (\mathfrak{F}, V)$ where $\mathfrak{F} = (W, R)$ is a Kripke frame and V, an *intuitionistic valuation*, is a map from PROP to

$$Up(\mathfrak{F}) = \{X \in \mathcal{P}(W) : w \in X \wedge wRv \to v \in X\},$$

the *upsets* of $\mathfrak{F}$.

The notions of truth and validity are standard, except for the implication clause of the truth defintion.

Definition 4. We define by recursion φ *is true in* $\mathfrak{M}$ *at* w (notation $\mathfrak{M}, w \models \varphi$):

1. $\mathfrak{M}, w \models p$ iff $w \in V(p)$,

2. $\mathfrak{M}, w \models \varphi \wedge \psi$ iff $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$,
3. $\mathfrak{M}, w \models \varphi \vee \psi$ iff $\mathfrak{M}, w \models \varphi$ or $\mathfrak{M}, w \models \psi$,
4. $\mathfrak{M}, w \models \varphi \to \psi$ iff for all v such that wRv, if $\mathfrak{M}, v \models \varphi$ then $\mathfrak{M}, v \models \psi$,
5. $\mathfrak{M}, w \not\models \bot$.

We say that φ is valid on a frame $\mathfrak{F}$, and write $\mathfrak{F} \models \varphi$, if $(\mathfrak{F}, V), w \models \varphi$ for every valuation V and world w.

With the frame-based semantics in hand, we now recall the algebraic semantics.

Definition 5. A structure $\mathfrak{A} = (A, \vee, \wedge, \to, \bot, \top)$ is a *Heyting algebra* iff $A \neq \emptyset$, $\vee$, $\wedge$, and $\to$ are binary operations on A, and $\bot, \top \in A$ such that for every $a, b, c \in A$:

(i) $\mathfrak{A}$ is a *bounded lattice*:

1.	$a \vee a = a$,	$a \wedge a = a$,
2.	$a \vee b = b \vee a$,	$a \wedge b = a \wedge b$,
3.	$a \vee (b \vee c) = (a \vee b) \vee c$,	$a \wedge (b \wedge c) = (a \wedge b) \wedge c$,
4.	$a \vee \bot = a$,	$a \wedge \top = a$,
5.	$a \vee (b \wedge a) = a$,	$a \wedge (b \vee a) = a$,

(ii) $\mathfrak{A}$ is *distributive*:
  1. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$,
  2. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,

(iii) and $\to$ is *Heyting implication*:
  1. $a \to a = \top$,
  2. $a \wedge (a \to b) = a \wedge b$,
  3. $b \wedge (a \to b) = b$,
  4. $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$.

A useful semantic characterization of Heyting implication is

$$c \leq a \to b \text{ iff } a \wedge c \leq b$$

where $a \leq b$ iff $a \wedge b = a$ (see [3, Theorem 7.10]). To define truth in a Heyting algebra $\mathfrak{A}$, we define a valuation $v : \textsc{Prop} \to A$ and extend it to all formulas of $\mathcal{L}$ by the obvious recursion. Then

Definition 6. φ is *valid* in $\mathfrak{A}$ iff $v(\varphi) = \top$ for every valuation v.

One last notion that will be needed is that of a filter.

Definition 7. Let $\mathfrak{A}$ be a Heyting algebra. A nonempty, proper subset $F \subset A$ is a *filter* of $\mathfrak{A}$ if

1. $a, b \in F$ implies $a \wedge b \in F$,
2. $a \in F$ and $a \leq b$ implies $b \in F$,

and a *prime* filter if in addition:

3. $a \vee b \in F$ implies $a \in F$ or $b \in F$.

A sometimes useful equivalent (see [3, Theorem 7.23]) definition of filter replaces conditions 1 and 2 with $\top \in F$ and

$$a \in F \text{ and } a \to b \in F \text{ implies } b \in F.$$

We state here a result, sometimes referred to as the Prime Filter Theorem, about filters that will be needed in §3. It is a minor generalization of [3, Theorem 7.41].

Proposition 8. *Let F be a filter of $\mathfrak{A}$ and $X \subset A$ such that $F \cap X = \emptyset$. Then there is a prime filter F' of $\mathfrak{A}$ such that $F \subseteq F'$ and $F' \cap X = \emptyset$.*

2.2 Algebraic Completeness

We can associate to each intermediate logic L the class $\mathbf{V}_L$ of those Heyting algebras in which all theorems of L are valid. $\mathbf{V}_L$ will be a variety by Birkhoff's Theorem, which states that a class of algebras is equationally defined iff it is a variety. Then, by a Lindenbaum-Tarski type construction, the following can be proved (as in [3, Theorem 7.73(iv)]).

Theorem 9. *Every intermediate logic L is sound and complete with respect to $\mathbf{V}_L$.*

This gives us the completeness with respect to algebraic semantics that we will try to transfer to the frame-based semantics. Before we can do that, though, we must define the frame-based semantics.

2.3 Descriptive Frames

We define here the notion that will give us an adequate frame-based semantics for completeness. It is a generalization of the Kripke frame:

Definition 10. An intuitionistic *general frame* is a triple $\mathfrak{F} = (W, R, \mathcal{P})$ where (W, R) is a Kripke frame, $\mathcal{P} \subseteq Up(\mathfrak{F})$ containing $\emptyset$ and W, and $\mathcal{P}$ is closed under $\cup$, $\cap$, and $\to$ defined by

$$U_1 \to U_2 := \{w \in W : \forall v(wRv \wedge v \in U_1 \to v \in U_2)\} = W \backslash R^{-1}(U_1 \backslash U_2),$$

where $R^{-1}(U) = \bigcup_{w \in U}\{v \in W : vRw\}$.

Definition 11. An intuitionistic *descriptive frame* is a general frame that is refined and compact, where:

1. $\mathfrak{F}$ is *refined* if for every $w, v \in W$, $\neg(wRv)$ implies that there is $U \in \mathcal{P}$ such that $w \in U$ and $v \notin U$, and
2. $\mathfrak{F}$ is *compact* if for every $\mathcal{X} \subseteq \mathcal{P}$ and $\mathcal{Y} \subseteq \{W \backslash U : U \in \mathcal{P}\}$, if $\mathcal{X} \cup \mathcal{Y}$ has the finite intersection property then $\bigcap(\mathcal{X} \cup \mathcal{Y}) \neq \emptyset$.

Definition 12. An intuitionistic *descriptive model* is a pair $\mathfrak{M} = (\mathfrak{F}, V)$ such that $\mathfrak{F} = (W, R, \mathcal{P})$ is a descriptive frame and $V : \textsc{Prop} \to \mathcal{P}$.

Truth and validity are defined as usual.

3 Duality

In this section, we prove a duality theorem for Heyting algebras and descriptive frames. This will provide us with the link needed to infer frame-based completeness from algebraic completeness.

3.1 From Frames to Algebras

We first define an operator * from descriptive frames to Heyting algebras.

Definition 13. Let $\mathfrak{F} = (W, R, \mathcal{P})$ be a descriptive frame. Then $\mathfrak{F}^* := (\mathcal{P}, \cup, \cap, \to, \emptyset, W)$ (where $\to$ is the operation on $\mathcal{P}$ defined in the previous section).

Lemma 14. *For every descriptive frame $\mathfrak{F}$, $\mathfrak{F}^*$ is a Heyting algebra.*

Proof. That $\mathfrak{F}^*$ is a distributive lattice follows directly from the fact that any set of sets forms a distributive lattice. So we only need to show that $\to$ satisfies the axioms for Heyting implication. Let $X, Y, Z \in \mathcal{P} \subset Up(\mathfrak{F})$.

1. $a \to a = \top$:

$$\begin{aligned} X \to X &= W \backslash R^{-1}(X \backslash X) \\ &= W \backslash R^{-1}(\emptyset) \\ &= W \backslash \emptyset \\ &= W. \end{aligned}$$

2. $a \wedge (a \to b) = a \wedge b$:

$$\begin{aligned} X \cap (X \to Y) &= X \cap (W \backslash R^{-1}(X \backslash Y)) \\ &= X \backslash R^{-1}(X \backslash Y) \\ &= X \cap Y. \end{aligned}$$

   To see that the last equality holds, notice that if $w \in X \cap Y$, then $u \in Y$ if wRu, since Y is an upset. So $\neg wRu$ for any $u \in X \backslash Y$. For the other containment, let $w \in X$ and $w \notin R^{-1}(X \backslash Y)$. Then $w \notin X \backslash Y$, since R is reflexive, showing that $w \in Y$.
3. $b \wedge (a \to b) = b$:

$$\begin{aligned} Y \cap (X \to Y) &= Y \cap (W \backslash R^{-1}(X \backslash Y)) \\ &= Y \backslash R^{-1}(X \backslash Y) \\ &= Y, \end{aligned}$$

   where the last equality holds because $Y \cap R^{-1}(X \backslash Y) = \emptyset$, since Y is an upset.

4. $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$:

$$\begin{aligned} W \backslash R^{-1}(X \backslash (Y \cap Z)) &= W \backslash R^{-1}((X \backslash Y) \cup (X \backslash Z)) \\ &= W \backslash (R^{-1}(X \backslash Y) \cup R^{-1}(X \backslash Z)) \\ &= W \backslash R^{-1}(X \backslash Y) \cap W \backslash R^{-1}(X \backslash Z) \\ &= (X \to Y) \wedge (X \to Z). \end{aligned}$$

The second equality holds because

$$\begin{aligned} w \in R^{-1}((X \backslash Y) \cup (X \backslash Z)) &\text{ iff } \exists u \in (X \backslash Y) \cup (X \backslash Z) : wRu \\ &\text{ iff } \exists u \in X \backslash Y : wRu \text{ or } \exists u \in X \backslash Z : wRu \\ &\text{ iff } w \in R^{-1}(X \backslash Y) \text{ or } w \in R^{-1}(X \backslash Z). \end{aligned}$$

Therefore $\mathfrak{F}^*$ is a Heyting algebra.

Note that we didn't use that $\mathfrak{A}$ was descriptive. This works for general frames as well, but the importance of using descriptive frames will become clear in the proof of Theorem 17.

3.2 From Algebras to Frames

Now we define an operator $_*$ in the other direction.

Definition 15. Let $\mathfrak{A} = (A, \vee, \wedge, \to, \bot, \top)$ be a Heyting algebra.

1. $W_{\mathfrak{A}} = \{F \subset A : F \text{ is a prime filter of } \mathfrak{A}\}$,
2. $FR_{\mathfrak{A}}F'$ iff $F \subseteq F'$,
3. $\mathcal{P}_{\mathfrak{A}} = \{\widehat{a} : a \in A\}$ where $\widehat{a} = \{F \in W_{\mathfrak{A}} : a \in F\}$.

Then $\mathfrak{A}_* := (W_{\mathfrak{A}}, R_{\mathfrak{A}}, \mathcal{P}_{\mathfrak{A}})$.

Lemma 16. *For every Heyting algebra $\mathfrak{A}$, $\mathfrak{A}_*$ is a descriptive frame.*

Proof. That $R_{\mathfrak{A}}$ is a partial order, and hence that $\mathfrak{F}_{\mathfrak{A}} := (W_{\mathfrak{A}}, R_{\mathfrak{A}})$ is a Kripke frame, follows directly from the fact that $\subseteq$ is a partial order on sets. If $F \in \widehat{a}$ and $FR_{\mathfrak{A}}F'$, then $a \in F$ and $F \subseteq F'$, and so $a \in F'$ and $F' \in \widehat{a}$. So each $\widehat{a}$ is an upset of $F_{\mathfrak{A}}$, giving us $\mathcal{P}_{\mathfrak{A}} \subseteq Up(\mathfrak{F}_{\mathfrak{A}})$. Because filters are upsets, $\bot \notin F$ (else $F = A$, contradicting that filters are proper) and $\top \in F$ (else $F = \emptyset$, contradicting that filters are nonempty) for every (prime) filter F. Thus $\widehat{\bot} = \emptyset$ and $\widehat{\top} = W_{\mathfrak{A}}$ are in $\mathcal{P}_{\mathfrak{A}}$.

We next have to check that $\mathcal{P}_{\mathcal{A}}$ is closed under $\cup$, $\cap$, and $\to$. Let $\widehat{a}, \widehat{b} \in \mathcal{P}_{\mathfrak{A}}$. Then

$$\begin{aligned} \widehat{a} \cup \widehat{b} &= \{F \in W_{\mathfrak{A}} : a \in F\} \cup \{F \in W_{\mathfrak{A}} : b \in F\} \\ &= \{F \in W_{\mathfrak{A}} : a \in F \text{ or } b \in F\} \\ &= \{F \in W_{\mathfrak{A}} : a \vee b \in F\} \\ &= \widehat{a \vee b} \\ &\in \mathcal{P}_{\mathfrak{A}}, \end{aligned}$$

where the third equality holds because F is a prime filter. Also

$$\begin{aligned}\widehat{a}\cap\widehat{b} &= \{F\in W_{\mathfrak{A}} : a\in F\}\cap\{F\in W_{\mathfrak{A}} : b\in F\}\\ &= \{F\in W_{\mathfrak{A}} : a\in F \text{ and } b\in F\}\\ &= \{F\in W_{\mathfrak{A}} : a\wedge b\in F\}\\ &= \widehat{a\wedge b}\\ &\in \mathcal{P}_{\mathfrak{A}},\end{aligned}$$

where the third equality holds because F is a filter. And in the definition of descriptive frame, we defined $\to$ precisely to make the following work:

$$\begin{aligned}\widehat{a}\to\widehat{b} &= \{F\in W_{\mathfrak{A}} : \forall F'(FR_{\mathfrak{A}}F'\wedge F'\in\widehat{a}\to F'\in\widehat{b})\}\\ &= \{F\in W_{\mathfrak{A}} : \forall F'(F\subseteq F'\wedge a\in F'\to b\in F')\}\\ &= \widehat{a\to b}\\ &\in \mathcal{P}_{\mathfrak{A}}.\end{aligned}$$

To show the right is contained in the left in the last equality, let $F\in\widehat{a\to b}$ and suppose that $F\subseteq F'$ with $a\in F'$. Then, as $a\to b\in F$, $a\to b\in F'$. And so, as F' is a filter containing a, $b\in F'$. Thus $F\in\{F\in W_{\mathfrak{A}} : \forall F'(F\subseteq F'\wedge a\in F'\to b\in F')\}$. For the reverse containment, let $F\in\{F\in W_{\mathfrak{A}} : \forall F'(F\subseteq F'\wedge a\in F'\to b\in F')\}$. We want to show that $a\to b\in F$. If $b\in F$, then $a\to b\in F$, so we suppose $b\notin F$. If there is $c\in F$ such that $c\wedge a=0$, then $c\wedge a\leq b$, so $c\leq a\to b$ by the semantic characterization of $\to$, and so $a\to b\in F$. So assume there is no such c. Let F_a be the filter generated by F and a.[1] This exists since $c\wedge a\neq 0$ for every $c\in F$ by assumption. If $b\notin F_a$, then there is a prime filter F' extending F_a with $b\notin F'$ (by Proposition 8). Since $F\subseteq F_a\subseteq F'$ and $a\in F'$, $b\in F'$ (by our original supposition about F), a contradiction. So $b\in F_a$. That is, there is a $c\in F$ such that $c\wedge a\leq b$. Then $c\leq a\to b$, by the semantic characterization of $\to$, and so $a\to b\in F$, since filters are upsets. Therefore, $F\in\widehat{a\to b}$.

Thus $\mathfrak{A}_*$ is a general frame. It remains to show that it is descriptive. To see that $\mathfrak{A}_*$ is refined, suppose that $\neg(FR_{\mathfrak{A}}F')$, that is $F\not\subseteq F'$. Then there is an $a\in A$ such that $a\in F\wedge a\notin F'$. So there is an $\widehat{a}\in\mathcal{P}_{\mathfrak{A}}$ such that $F\in\widehat{a}\wedge F'\notin\widehat{a}$.

For compactness, let $\mathcal{X}\subseteq\mathcal{P}_{\mathfrak{A}}$, $\mathcal{Y}\subseteq\{W_{\mathfrak{A}}\backslash\widehat{b} : \widehat{b}\in\mathcal{P}_{\mathfrak{A}}\}$, and $\mathcal{X}\cup\mathcal{Y}$ have the finite intersection property. We want to show $\bigcap(\mathcal{X}\cup\mathcal{Y})\neq\emptyset$. Let $F=[\{a : \widehat{a}\in\mathcal{X}\})$ be the filter in $\mathfrak{A}$ generated by those $a\in A$ such that $\widehat{a}\in\mathcal{X}$ and $I=(\{b : W_{\mathfrak{A}}\backslash\widehat{b}\in\mathcal{Y}\}]$ the ideal[2] generated by those $b\in A$ such that $W_{\mathfrak{A}}\backslash\widehat{b}\in\mathcal{Y}$. To see that F is a proper subset of A, and hence exists, suppose not. Then there

[1] If the closure under $\wedge$ of the set $\{a\in A : \exists x\in X(x\leq a)\}$ is a proper subset of A, then it is a filter called *the filter generated by* X and denoted $[X)$.

[2] An *ideal* is a nonempty, proper subset F of a Heyting algebra $\mathfrak{A}$ such that (1) $a,b\in F$ implies $a\vee b\in F$, and (2) $a\in F$ and $a\geq b$ implies $b\in F$. The ideal generated by a set X, which is defined analogously to the filter generated by a set, but replacing $\wedge$ by $\vee$ and $\leq$ by $\geq$, is denoted by $(X]$.

are $\widehat{a}_1, \ldots, \widehat{a}_n \in \mathcal{X}$ such that $a_1 \wedge \cdots \wedge a_n = 0$. But then $\widehat{a}_1 \cap \cdots \cap \widehat{a}_n = \emptyset$, contradicting that $\mathcal{X} \cup \mathcal{Y}$ has the finite intersection property. Similarly, I is a proper subset since, if not, then there are $W_{\mathfrak{A}} \backslash \widehat{b}_1, \ldots, W_{\mathfrak{A}} \backslash \widehat{b}_m \in \mathcal{Y}$ such that $b_1 \vee \cdots \vee b_m = 1$. But then $\widehat{b}_1 \cup \cdots \cup \widehat{b}_m = W_{\mathfrak{A}}$, and so $W_{\mathfrak{A}} \backslash \widehat{b}_1 \cap \cdots \cap W_{\mathfrak{A}} \backslash \widehat{b}_m = \emptyset$, again contradicting that $\mathcal{X} \cup \mathcal{Y}$ has the finite intersection property.

So F is a filter and I is an ideal. We now show they are disjoint. Suppose $a \in F \cap I$. Then, since $a \in F$, there are $\widehat{a}_1, \ldots, \widehat{a}_n \in \mathcal{X}$ such that $a_1 \wedge \cdots \wedge a_n \leq a$. Also, since $a \in I$, there are $W_{\mathfrak{A}} \backslash \widehat{b}_1, \ldots, W_{\mathfrak{A}} \backslash \widehat{b}_m \in \mathcal{Y}$ such that $b_1 \vee \cdots \vee b_m \geq a$. So

$$\begin{aligned}
\widehat{a}_1 \cap \cdots \cap \widehat{a}_n &= \widehat{a_1 \wedge \cdots \wedge a_n} \\
&\subseteq \widehat{a} \\
&\subseteq \widehat{b_1 \vee \cdots \vee b_m} \\
&= \widehat{b}_1 \cup \cdots \cup \widehat{b}_m \\
&= W_{\mathfrak{A}} \backslash (W_{\mathfrak{A}} \backslash \widehat{b}_1 \cap \cdots \cap W_{\mathfrak{A}} \backslash \widehat{b}_m).
\end{aligned}$$

But then $\widehat{a}_1 \cap \cdots \cap \widehat{a}_n \cap W_{\mathfrak{A}} \backslash \widehat{b}_1 \cap \cdots \cap W_{\mathfrak{A}} \backslash \widehat{b}_m = \emptyset$, contradicting that $\mathcal{X} \cup \mathcal{Y}$ has the finite intersection property. So $F \cap I = \emptyset$, and we can apply the Prime Filter Theorem to get a prime filter F' of $\mathfrak{A}$ such that $F \subseteq F'$ and $F' \cap I = \emptyset$.

It is this F' we will show to be in $\bigcap(\mathcal{X} \cup \mathcal{Y})$, completing the proof. Let $\widehat{a} \in \mathcal{X}$. Then $a \in F$ by the definition of F, and so $a \in F'$ since it contains F, giving us $F' \in \widehat{a}$. Now let $W_{\mathfrak{A}} \backslash \widehat{b} \in \mathcal{Y}$. Then $b \in I$ by the definition of I, and so $b \notin F'$ since it is disjoint from I, giving us $F' \notin \widehat{b}$ and thus $F' \in W_{\mathfrak{A}} \backslash \widehat{b}$. So $F' \in \bigcap(\mathcal{X} \cup \mathcal{Y})$. Thus $\bigcap(\mathcal{X} \cup \mathcal{Y}) \neq \emptyset$, so $\mathfrak{A}_*$ is compact and therefore a descriptive frame.

3.3 Back and Forth

Results analogous to the previous two lemmas hold for regular Kripke frames. But what we gain with the added generality of descriptive frames is that every Heyting algebra can be obtained from a descriptive frame via the * operation, and, vice versa, every descriptive frame can be obtained from a Heyting algebra via $_*$. This fact, plus some extra symmetry, is expressed by the following duality.

Theorem 17. *Let $\mathfrak{A}$ be a Heyting algebra and $\mathfrak{F}$ a descriptive frame. Then*

1. $\mathfrak{A} \cong (\mathfrak{A}_*)^*$,
2. $\mathfrak{F} \cong (\mathfrak{F}^*)_*$.

Proof. For part 1, we define a map $f : \mathfrak{A} \to (\mathfrak{A}_*)^*$ by

$$f(a) = \widehat{a}.$$

f is bijective since the map from a to $\widehat{a}$ is a bijection between A and $\mathcal{P}_{\mathfrak{A}}$, which is the domain of $(\mathfrak{A}_*)^*$. The proof that f is a homomorphism is contained in the proof of Lemma 16. From that proof, we get the second equality in each of the following:

$$\begin{aligned}
f(\bot) &= \widehat{\bot} & \qquad f(\top) &= \widehat{\top} \\
&= \emptyset & &= W_{\mathfrak{A}}
\end{aligned}$$

and

$$f(a \vee b) = \widehat{a \vee b} = \widehat{a} \cup \widehat{b} = f(a) \cup f(b) \qquad f(a \wedge b) = \widehat{a \wedge b} = \widehat{a} \cap \widehat{b} = f(a) \cap f(b) \qquad f(a \to b) = \widehat{a \to b} = \widehat{a} \to \widehat{b} = f(a) \to f(b).$$

So f is an isomorphism.

For part 2, we define a map $g : \mathfrak{F} \to (\mathfrak{F}^*)_*$ by

$$g(w) = \widehat{w} = \{U \in P : w \in U\}.$$

We start by showing that $g(w) \in W_{\mathfrak{F}^*}$, that is, that $\widehat{w}$ is a prime filter of $\mathfrak{F}^*$. By definition, $\widehat{w} \subseteq \mathcal{P}$. Since $w \notin \emptyset$ and $w \in W$, we have $\emptyset \notin \widehat{w}$ and $W \in \widehat{w}$. Let $X, Y \in \widehat{w}$. Then $w \in X, Y$. So $w \in X \cap Y$, giving $X \cap Y \in \widehat{w}$. Now let $X \in \widehat{w}$ and $X \subseteq Y$. Then $w \in X \subseteq Y$, so $Y \in \widehat{w}$. Now let $X \cup Y \in \widehat{w}$. Then $w \in X \cup Y$. So $w \in X$ or $w \in Y$, and hence $X \in \widehat{w}$ or $Y \in \widehat{w}$. So $\widehat{w}$ is an element of $W_{\mathfrak{F}^*}$. To see that g is injective, let $w \neq v$. Then either $\neg(wRv)$ or $\neg(vRw)$, since Kripke frames are partial orders. Without loss of generality, assume the first. Then, since $\mathfrak{F}$ is descriptive and hence refined, there is an upset $U \in \mathcal{P}$ such that $w \in U$ and $v \notin U$. Then $U \in \widehat{w}$ and $U \notin \widehat{v}$, giving us $\widehat{w} \neq \widehat{v}$. We now show that g is a homomorphism: wRv iff every upset in $\mathcal{P}$ containing w contains v iff $\widehat{w} \subseteq \widehat{v}$ iff $\widehat{w} R_{\mathfrak{F}^*} \widehat{v}$ (where the first equivalence holds from right to left because $\mathfrak{F}$ is refined). To show that g is a homomorphism, we must also prove that for every $U \subseteq W$, $U \in \mathcal{P}$ iff $g(U) \in \mathcal{P}_{\mathfrak{F}^*}$. Note that $\mathcal{P}_{\mathfrak{F}^*} = \{\{\widehat{w} : w \in U\} : U \in \mathcal{P}\}$. Then $U \in \mathcal{P}$ implies $g(U) = \{\widehat{w} : w \in U\} \in \mathcal{P}_{\mathfrak{F}^*}$. And $g(U) \in \mathcal{P}_{\mathfrak{F}^*}$ implies $g(U) = \{\widehat{w} : w \in U'\}$ for some $U' \in \mathcal{P}$. But then $g(U) = g(U')$, and so $U = U'$ since g is injective. So $U \in \mathcal{P}$.

It remains to show that g is surjective. This is where that fact that $\mathfrak{F}$ is descriptive, and compact in particular, is essential. We must show that every element of $W_{\mathfrak{F}^*}$ is of the form $\widehat{w}$ for some $w \in W$. So let $\mathcal{X} \in W_{\mathfrak{F}^*}$. Then $\mathcal{X}$ is a prime filter in $\mathfrak{F}^*$. Then $\mathcal{Y} = \mathcal{P} \backslash \mathcal{X}$ is a prime ideal, and so $\bigcup \mathcal{Y}_0 \neq W$ for any finite subset $\mathcal{Y}_0 \subseteq \mathcal{Y}$. We will show that the set $\mathcal{X} \cup \mathcal{Y}'$ has the finite intersection property, where $\mathcal{Y}' = \{W \backslash Y : Y \in \mathcal{Y}\}$. Let $Z = X_1 \cap \cdots \cap X_n \cap Y_1 \cap \cdots \cap Y_m$ where each $X_i \in \mathcal{X}$, $Y_i \in \mathcal{Y}'$. Let X and Y be the intersections of the X_i and Y_i, respectively. Then $X \in \mathcal{X}$, since $\mathcal{X}$ is a filter, and $Y \in \mathcal{Y}'$ since $W \backslash Y \in \mathcal{Y}$ (since $\mathcal{Y}$ is an ideal). If $Z = \emptyset$, then $X \subseteq W \backslash Y \in \mathcal{Y}$. Since $\mathcal{X}$ is a filter, this gives $W \backslash Y \in \mathcal{X}$, yielding $\mathcal{X} \cap \mathcal{Y} \neq \emptyset$, contradicting the definition of $\mathcal{Y}$. So $Z \neq \emptyset$, showing that $\mathcal{X} \cup \mathcal{Y}'$ has the finite intersection property. Since $\mathfrak{F}$ is descriptive and hence compact, there is a $w \in \bigcap(\mathcal{X} \cup \mathcal{Y}')$. Finally, we show that $\mathcal{X} = \widehat{w}$. Let $U \in P$. If $U \in \mathcal{X}$, then $w \in \bigcap(\mathcal{X} \cup \mathcal{Y}') \subseteq U$, so $U \in \widehat{w}$, giving $\mathcal{X} \subseteq \widehat{w}$. Now let $U \in \widehat{w}$. So $w \in U$. Suppose $U \in \mathcal{Y}$. Then $W \backslash U \in \mathcal{Y}'$ giving $w \notin \bigcap \mathcal{Y}'$, contradicting $w \in \bigcap(\mathcal{X} \cup \mathcal{Y}')$. So $U \in \mathcal{X}$, giving $\widehat{w} \subseteq \mathcal{X}$. Thus $\mathcal{X} = \widehat{w}$, establishing that g is surjective, and therefore an isomorphism.

4 Completeness

This section contains the final result. Having obtained a link between the algebraic and frame-based semantics, we can obtain the more intuitive version of completeness we have been looking for.

Lemma 18. *Let $\mathfrak{A}$ be a Heyting algebra. Then*

$$\mathfrak{A}, v \models \varphi \textit{ iff } \mathfrak{A}_*, v_* \models \varphi,$$

where $v_*(p) = \widehat{v(p)}$.

Proof. We first prove that $v_*(\varphi) = \widehat{v(\varphi)}$ for all formulas by induction on φ:

PROP: $v_*(p) = \widehat{v(p)}$ by definition.

$\wedge$:

$$\begin{aligned} \widehat{v(\varphi \wedge \psi)} &= \widehat{v(\varphi) \wedge v(\psi)} \\ &= \widehat{v(\varphi)} \cap \widehat{v(\psi)} \\ &= v_*(\varphi) \cap v_*(\psi) \\ &= v_*(\varphi \wedge \psi) \end{aligned}$$

$\vee$:

$$\begin{aligned} \widehat{v(\varphi \vee \psi)} &= \widehat{v(\varphi) \vee v(\psi)} \\ &= \widehat{v(\varphi)} \cup \widehat{v(\psi)} \\ &= v_*(\varphi) \cup v_*(\psi) \\ &= v_*(\varphi \vee \psi) \end{aligned}$$

$\to$:

$$\begin{aligned} \widehat{v(\varphi \to \psi)} &= \widehat{v(\varphi) \to v(\psi)} \\ &= \widehat{v(\varphi)} \to \widehat{v(\psi)} \\ &= v_*(\varphi) \to v_*(\psi) \\ &= v_*(\varphi \to \psi) \end{aligned}$$

where the second equality of each induction step was proved in Lemma 16. Using this, we get that

$$\begin{aligned} \mathfrak{A}, v \models \varphi &\text{ iff } v(\varphi) = \top \\ &\text{ iff } v_*(\varphi) = \widehat{\top} = \{F \subset W_{\mathfrak{A}} : \top \in F\} = W_{\mathfrak{A}} \\ &\text{ iff } \mathfrak{A}_*, v_* \models \varphi \end{aligned}$$

since every (prime) filter is a nonempty upset and hence contains the top element $\top$.

Notice that this lemma only makes sense given the correspondence between descriptive frames and Heyting algebras proved in the previous section. Combining this lemma with the algebraic completeness theorem of §2 will give us the desired result. For a class of algebras C, we write $C_* := \{\mathfrak{A}_* : \mathfrak{A} \in C\}$.

Theorem 19. *Every intermediate logic L is sound and complete with respect to $(\mathbf{V}_L)_*$.*

Proof. Let L be an intermediate logic. Then

$$\begin{aligned} L \vdash \varphi \text{ iff } & \mathbf{V}_L \models \varphi && (4.1)\\ \text{iff } & \mathfrak{A} \models \varphi \text{ for every } \mathfrak{A} \in \mathbf{V}_L && (4.2)\\ \text{iff } & \mathfrak{A}, v \models \varphi \text{ for every } \mathfrak{A} \in \mathbf{V}_L \text{ and valuation } v && (4.3)\\ \text{iff } & \mathfrak{A}_*, v_* \models \varphi \text{ for every } \mathfrak{A}_* \in (\mathbf{V}_L)_* \text{ and valuation } v_* && (4.4)\\ \text{iff } & \mathfrak{A}_* \models \varphi \text{ for every } \mathfrak{A}_* \in (\mathbf{V}_L)_* && (4.5)\\ \text{iff } & (\mathbf{V}_L)_* \models \varphi. && (4.6) \end{aligned}$$

(1) is just the algebraic completeness theorem from §2.2, (4) follows from the previous lemma, and the rest follow from the definition of $\models$. So provability in L corresponds with validity in the class $(\mathbf{V}_L)_*$.

Thus we have shown that every intermediate logic is complete with respect to a class of descriptive frames. That is, we have found our complete frame-based semantics.

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Intermediate logics and finite frames

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Abstract. An intermediate logic is a consistent set of propositional formulas containing **IPC** which is closed under modus ponens and substitution. In [3], Jankov showed that there are continuum many intermediate logics. In this paper we will discuss a method proving this fact, and also show how the same techniques can be used to prove some results on axiomatizations of intermediate logics of finite depth.

1 IPC

Intuitionistic propositional calculus (**IPC**) is the proposed formal calculus for intuitionistic or constructive mathematics. A characterizing feature of this logic is that the law of excluded middle fails, that is **IPC** $\nvdash p \vee \neg p$. However, this does not uniquely determine the logic per se, as we shall see later.

Our language $\mathcal{L}$ consists of a countable set **Prop** of propositional variables and the following set of logical connectives $\{\bot, \to, \wedge, \vee\}$. We define $\neg\varphi$ as an abbreviation for $\varphi \to \bot$. The set of formulas is defined inductively in the standard way. A *logic* is a set of formulas closed under uniform substitution and modus ponens.

Definition 1. IPC is the smallest set containing the following axiom schemes:

1. $\varphi \to (\psi \to \varphi)$.
2. $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$.
3. $\varphi \wedge \psi \to \varphi$.
4. $\varphi \wedge \psi \to \psi$.
5. $\varphi \to \varphi \vee \psi$.
6. $\psi \to \varphi \vee \psi$.
7. $(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$.
8. $\bot \to \varphi$.

and which is closed under *modus ponens*: if $\varphi, \varphi \to \psi \in$ **IPC** then $\psi \in$ **IPC**.

If $\varphi \notin$ **IPC** and $L =$ **IPC** $\cup \{\varphi\}$ then it is not generally true that L is a logic, since we might have $\varphi \to \psi \in$ **IPC** but $\psi \notin$ **IPC**. However, if we let L' be the deductive closure of L, i.e. L' is closed under modus ponens and uniform substitution, then L' is a logic. It is easy to see that L' is the smallest logic containing both **IPC** and φ. For later use we will introduce a convenient notation.

Definition 2. If L is a logic and Γ a set of formulas of $\mathcal{L}$ we will denote by $L + \Gamma$ the deductive closure of the set $L \cup \Gamma$. We write $L + \varphi$ for $L + \{\varphi\}$.

For example we have that the smallest logic containing both **IPC** and $p \vee \neg p$ is **CPC**, that is, **CPC** $=$ **IPC** $+ p \vee \neg p$. It is easily seen that **CPC** is a logic and, in fact, it is maximally consistent.

Proposition 3. *The only logic strictly stronger than* **CPC** *is the inconsistent one, i.e., the set of all formulas.*

Proof. Let L be a logic which properly contains **CPC** and let φ be an element of $L \setminus$ **CPC**. Then, since $\varphi \notin$ **CPC**, φ is not a tautology, and hence there is a valuation V making φ false. Let φ' be the result of replacing every propositional variable p in φ such that $V(p) = 1$ with $\top$ (i.e., $\bot \to \bot$), and every p such that $V(p) = 0$ with $\bot$. Then $V(\varphi) = V(\varphi')$ (which can be shown with a simple inductive argument), and since φ' does not contain any propositional variables, it must evaluate to "false" under any valuation. But this means that $\varphi' \leftrightarrow \bot$ is a tautology, and hence **CPC** $\vdash \varphi' \leftrightarrow \bot$. Therefore, $\varphi' \leftrightarrow \bot \in L$ and since L is closed under substitution, $\varphi' \in L$. Finally, since L is closed under modus ponens, we have that $\bot \in L$, i.e., L is inconsistent.

2 Kripke structures

In this section we briefly remind the reader of the relational semantics for **IPC**, namely intuitionistic Kripke models. A *Kripke frame* $\mathcal{F}$ is a pair $(W, \leq)$, where $\leq$ is a partial order on W. A *Kripke model* $\mathfrak{M}$ consists of a Kripke frame $\mathcal{F} = (W, \leq)$ (we say that $\mathfrak{M}$ is based on $\mathcal{F}$) with an added *valuation function* $V\colon Prop \to \mathcal{P}(W)$ which is persistent in the sense that if $w \in V(p)$ and $w \leq v$, then $v \in V(p)$. The satisfaction relation, $\models$, is defined as follows:

$$\begin{aligned}
\mathfrak{M}, w &\models p && \iff w \in V(p)\\
\mathfrak{M}, w &\models \varphi \wedge \psi && \iff \mathfrak{M}, w \models \varphi \text{ and } \mathfrak{M}, w \models \psi\\
\mathfrak{M}, w &\models \varphi \vee \psi && \iff \mathfrak{M}, w \models \varphi \text{ or } \mathfrak{M}, w \models \psi\\
\mathfrak{M}, w &\models \varphi \to \psi && \iff \forall v \geq w\, (\mathfrak{M}, v \models \varphi \implies \mathfrak{M}, v \models \psi)\\
\mathfrak{M}, w &\not\models \bot
\end{aligned}$$

and we say that φ is *true* at a point w in a model $\mathfrak{M}$ iff $\mathfrak{M}, w \models \varphi$. If φ is true at every point in a model $\mathfrak{M}$, we say that φ is *valid* on $\mathfrak{M}$, which we write as $\mathfrak{M} \models \varphi$. Similarly we have that φ is valid on a frame $\mathcal{F}$, written $\mathcal{F} \models \varphi$ if φ is valid in every model based on $\mathcal{F}$.

Fact 4. *If* $\mathcal{F}$ *is a frame, then* $\mathrm{Log}(\mathcal{F}) = \{\, \varphi \mid \mathcal{F} \models \varphi \,\}$ *is a logic. Similarly, if* F *is a class of frames, then* $\mathrm{Log}(\mathsf{F}) = \bigcap_{\mathcal{F} \in \mathsf{F}} \mathrm{Log}(\mathcal{F})$ *is a logic.*

In fact, both **IPC** and **CPC** can be defined in this way.

Fact 5. **IPC** *is the logic of the class of* all *Kripke frames.* **CPC** *is the logic of the frame consisting of a single reflexive point.*

The claim about **CPC** is trivial, since for a single reflexive point the satisfaction relation is identic to the classical truth relation. The proof for **IPC** is trickier, the idea being that from an unprovable formula φ we can create a *canonical* model in which φ is false. By being careful with the way one constructs this model it can be made finite, which gives us the following stronger theorem (for a formal proof, see for example Chagrov and Zakharyaschev [**?**]):

Theorem 6. **IPC** $= \mathrm{Log}(\mathsf{F}_{<\omega})$, *where* $\mathsf{F}_{<\omega}$ *is the class of all finite Kripke frames.*

If L is a logic, we let $\mathrm{Fr}(L) = \{\, \mathcal{F} \mid \mathcal{F} \models L \,\}$, where $\mathcal{F} \models L$ if and only if every formula in L is valid in $\mathcal{F}$.

We now introduce two operations on frames which preserve validity.

Definition 7. If $(W, \leq)$ is a frame and u is an element of F, we let

$$\geq(u) = \{\, v \mid v \geq u \,\}$$

and say that $\geq(u)$ is the *upward cone* of u. Similarly, $>(u) = \geq(u) \setminus \{u\}$, is called the *strict upward cone* of u. The definition of $\leq(u)$ (the downward cone of u) and $<(u)$ (the strict downward cone of u) is symmetric.

Definition 8. Let $\mathcal{F} = (W, \leq)$ be a frame, A be a subset of W and $W' = \bigcup_{a \in A} \geq(a)$. If we let $\leq_{W'}$ be the restriction of $\leq$ to W', then the frame $(W', \leq_{W'})$ is called the *subframe of* $\mathcal{F}$ *generated by* A.

Since the truth of a formula at a point w is determined by the truth of its subformulas in the upward cone of w, we get the following fact.

Fact 9. *If* φ *is true at a point* w *in a model* $\mathfrak{M}$, *then* φ *is true at the root of the upward cone of* w.

Fact 9 immediately implies the following corollary.

Corollary 10. *If* φ *is valid on a frame* $\mathcal{F}$ *and* $\mathcal{G}$ *is a generated subframe of* $\mathcal{G}$, *then* φ *is valid on* $\mathcal{G}$ *as well.*

The second validity-preserving operation that we will consider is that of a p-morphism.

Definition 11. A function f from a frame $\mathcal{F}$ to a frame $\mathcal{F}'$ is called a *p-morphism* if the following two conditions hold:

1. f is order-preserving, i.e., if $w \leq_{\mathcal{F}} v$, then $f(w) \leq_{\mathcal{F}'} f(v)$.
2. If $f(w) \leq_{\mathcal{F}'} v$ for some $v \in \mathcal{F}'$ then there is a point $v' \in \mathcal{F}$ such that $f(v') = v$ and $w \leq_{\mathcal{F}} v'$.

We will also consider p-morphisms between models.

Definition 12. Let $\mathfrak{M} = (\mathcal{F}, V)$ and $\mathfrak{M}'{=}(\mathcal{F}', V')$. A function $f \colon \mathfrak{M} \to \mathfrak{M}'$ is called a *p-morphism* if it is a p-morphism from $\mathcal{F}$ to $\mathcal{F}'$ and satisfies the following condition.

(3) f is truth-preserving in the sense that if $f(w) = v$, and p is a propositional letter, then

$$\mathfrak{M}, w \models p \text{ iff } \mathfrak{M}', f(w) \models p.$$

We say that a p-morphism f from $\mathcal{F}$ onto a frame $\mathcal{F}'$ is *proper* if $\mathcal{F} \neq \mathcal{F}'$.

Theorem 13. *If f is a p-morphism from a model $\mathfrak{M}$ to a model $\mathfrak{M}'$ then*

$$\mathfrak{M}, w \models \varphi \textit{ iff } \mathfrak{M}', f(w) \models \varphi.$$

Proof. The proof is a standard proof by induction on the complexity of φ and can be found in most textbooks on modal logic.

We immediately get the following corollary.

Corollary 14. *If $\mathcal{G}$ is a p-morphic image of $\mathcal{F}$, then all formulas valid on $\mathcal{F}$ are valid on $\mathcal{G}$ also.*

There are two specific kinds of p-morphisms which will play a role later on. Consider a frame $\mathcal{F}$ with two points w and w' such that $\geq(w) \setminus w = \geq(w')$ (w' is the only immediate successor of w in a terminology presented later). Then the result of identifying the two points while keeping relations intact will be a p-morphic image of $\mathcal{F}$; the p-morphism being called an *α-reduction.* Next, consider a frame $\mathcal{F}$ with two points w and w' such that $>(w) = >(w')$. Again, identifying the two points while keeping relationships intact will result in a p-morphic image of $\mathcal{F}$, with the p-morphism called a *β-reduction.* These two operations on frames actually generate all p-morphic images of a finite frame $\mathcal{F}$ in the following sense:

Fact 15. *If f is a proper p-morphism from a finite frame $\mathcal{F}$ onto $\mathcal{G}$, then there exists a sequence $f_1, \ldots, f_n$ of α- and β-reductions such that $f = f_1 \circ \ldots \circ f_n$.*

Proof. Assume that f is a p-morphism from $\mathcal{F}$ onto $\mathcal{G}$.

$$\mathcal{F} \xrightarrow{f} \mathcal{G}$$

We show that there is a frame $\mathcal{H}$ such that $\mathcal{F}$ can be reduced to $\mathcal{H}$ by means of a single α- or β-reduction, and such that $\mathcal{G}$ is a p-morphic image of $\mathcal{H}$. The result then follows by induction over $n = |\mathcal{F}| - |\mathcal{G}|$.

For this, take an endpoint w of G such that $|f^{-1}(w)| \geq 2$ (this is possible, since f is proper). Note that the inverse image of an endpoint is always upwards closed by preservation of order. Let v be an endpoint of $f^{-1}(w)$ and let v' be an endpoint in $f^{-1}(w) \setminus \{v\}$. We have two cases, either both v and v' are endpoints in $\mathcal{F}$, or v' is an immediate predecessor of v and v is the only point that v' sees. In both cases we let $\mathcal{H}$ be the result of identifying the two points, which in the first case is a β-reduction and in the second case an α-reduction. Then, as can easily be seen, f restricted to the domain of $\mathcal{H}$ is a p-morphism from $\mathcal{H}$ onto $\mathcal{G}$.

$$\mathcal{F} \xrightarrow{\alpha \text{ or } \beta} \mathcal{H} \xrightarrow{f \restriction \mathcal{H}} \mathcal{G}$$

Definition 16. A frame $\mathcal{G}$ is said to be a *reduction* of a frame $\mathcal{F}$ if $\mathcal{G}$ is a p-morphic image of a generated subframe of $\mathcal{F}$. We also say that $\mathcal{F}$ can be *reduced* to $\mathcal{G}$, and write $\mathcal{G} \leq \mathcal{F}$.

As it turns out we get the same ordering by taking the operations in the inverse order, i.e. we could just as well define $G \leq F$ as G being a generated subframe of a p-morphic image of F.

Theorem 17. *A frame $\mathcal{G}$ is a p-morphic image of a generated subframe of $\mathcal{F}$ if and only if it is a generated subframe of a p-morphic image of $\mathcal{F}$.*

Proof. $\Rightarrow$) Let $A \subseteq \mathcal{F}$ and assume that f is a p-morphism from $\mathcal{I} = \bigcup_{a \in A} \geq (a)$ onto $\mathcal{G}$. The idea is now simple, we let first apply f to $\mathcal{F}$ and then use $f[A]$ as a generating set. The crux of the matter is what to do with the elements which are not in the pre-image of f, but basically we just let them be related to the image of what they were related to before f was applied.

Formally, we construct a p-morphism h, with image $\mathcal{H}$, such that we have $\mathcal{G} = \bigcup_{a \in A} \geq_{\mathcal{H}}(h(a))$, i.e., the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\bigcup_{a \in A} \geq_{\mathcal{F}}(a)"] \arrow[d, "h"'] & \mathcal{I} \arrow[d, "f"] \\
\mathcal{H} \arrow[r, "\bigcup_{a \in A} \geq_{\mathcal{H}}(h(a))"'] & \mathcal{G}
\end{tikzcd}$$

For this, let $h = f \cup (\mathrm{id} \restriction (\mathcal{F} \setminus I))$ and set $W_{\mathcal{H}} = h[\mathcal{F}]$ (i.e., $W_{\mathcal{H}} = \mathcal{G} \cup (G \setminus I)$). We define $\leq_{\mathcal{H}}$ by letting v, v' be arbitrary elements in $\mathcal{H}$ and discriminating between the following four situations:

(1) If $v, v' \in \mathcal{F} \setminus \mathcal{I}$, then $v \leq_{\mathcal{H}} v'$ if and only if $v \leq_{\mathcal{F}} v'$.
(2) If $v \in (\mathcal{F} \setminus \mathcal{I})$ and $v' \in \mathcal{G}$, then $v \leq_{\mathcal{H}} v'$ if and only if there exists a $u \in \mathcal{F}$ such that $v \leq_{\mathcal{F}} u$ and $f(u) = v'$.
(3) If $v \in \mathcal{G}$ and $v' \in (\mathcal{F} \setminus \mathcal{I})$, then $v \not\leq_{\mathcal{H}} v'$.
(4) If both v and v' are in $\mathcal{G}$, $v \leq_{\mathcal{H}} v'$ holds if and only if $v \leq_{\mathcal{G}} v'$.

To prove that $(W_{\mathcal{H}}, \leq_{\mathcal{H}})$ really is a p-morphic image, first assume that $u \leq_{\mathcal{F}} u'$. We have the following cases:

(a) If $u, u' \in (\mathcal{F} \setminus \mathcal{I})$, then $h(u) = u$ and $h(u') = u'$ and so, by (1), $h(u) \leq_{\mathcal{H}} h(u')$.
(b) If $u \in (\mathcal{F} \setminus \mathcal{I})$ and $u' \in \mathcal{I}$, then $h(u) = u$ and $h(u') = f(u')$ and so, by (2), $h(u) \leq_{\mathcal{H}} h(u')$.
(c) If $u, u' \in \mathcal{I}$, then $h(u) = f(u)$ and $h(u') = f(u')$ and since f is a p-morphism, $f(u) \leq_{\mathcal{G}} f(u')$, and so, by (4), $h(u) \leq_{\mathcal{H}} h(u')$.

Note that this is exhaustive, since if $u \in \mathcal{I}$ and $u \leq_{\mathcal{F}} u'$ then $u' \in I$ as well.

Next, assume that $h(u) \leq_{\mathcal{H}} v$. We need to find a u' such that $u \leq_{\mathcal{F}} u'$ and $h(u') = v$. We have the following cases:

(a) If v is a member of $(\mathcal{F} \setminus \mathcal{I})$, then $h(u)$ must be too, so we can let $u' = v$ since then $u \leq_{\mathcal{F}} u'$ and $h(u') = u' = v$.
(b) If $h(u) \in (\mathcal{F} \setminus \mathcal{I})$ and $v \in \mathcal{G}$, then, according to (2), there exists a $u' \in \mathcal{F}$ such that $h(u) = u \leq_{\mathcal{F}} u'$ and $f(u') = v$, which is exactly what we wanted.
(c) If both $h(u)$ and v are in $\mathcal{G}$, then there is some element u' of $\mathcal{I}$ such that $f(u') = v$ and $f(u) \leq_{\mathcal{G}} f(u')$ since f is a p-morphism. But then by (4), $h(u) \leq_{\mathcal{H}} h(u')$ holds as well.

By (3), this list is exhaustive, so h is a p-morphism onto $(W_{\mathcal{H}}, \leq_{\mathcal{H}})$.

The finishing step in the proof is showing that $v \geq_{\mathcal{H}} h(a)$ for some $a \in A$ if and only if $v \in \mathcal{G}$. So, assume that $v \geq_{\mathcal{H}} h(a)$ for some $a \in A$. By definition of h, $h(a) \in \mathcal{G}$, and since $v \geq_{\mathcal{H}} h(a)$, then by definition of $\geq_{\mathcal{H}}$, $v \in \mathcal{G}$.

Next, assume that $v \in \mathcal{G}$. Then $v = f(w)$ for some $w \in \mathcal{I}$, and by definition of $\mathcal{I}$, $w \geq_{\mathcal{I}} a$ for some $a \in A$. So $f(w) = v \leq_{\mathcal{G}} f(a)$, since f is a p-morphism. But then $h(w) = v \geq_{\mathcal{H}} h(a)$ by definition of h and $\geq_{\mathcal{H}}$.

$\Leftarrow$) The proof is somewhat symmetric to the one above. Assume $\mathcal{G}$ is generated by some set A in $\mathcal{H} = h[\mathcal{F}]$ where h is a p-morphism. Then we can let $\mathcal{I}$ be the frame generated by $\bigcup_{a \in A} h^{-1}(a)$ and let f be $h \restriction \mathcal{I}$. It is now routine to show that $f[\mathcal{I}] = \mathcal{F}$.

3 Intermediate logics

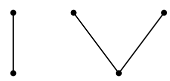

Fig. 1. $\mathcal{F}_1$ and $\mathcal{F}_2$

A logic L such that **IPC** $\subseteq L \subseteq$ **CPC** is called an *intermediate* logic. By Proposition 3, this is equivalent to saying that L is a consistent logic extending **IPC**.

Our first observation is that the set of all intermediate logics constitutes a lattice with regards to set inclusion; the greatest lower bound of two logics L_1 and L_2 is their intersection, and the least upper bound is the deductive closure of their union. Secondly, and more importantly, there actually exist proper intermediate logics, i.e. logics L such that **IPC** $\subset L \subset$ **CPC**. Take the two frames $\mathcal{F}_1$ and $\mathcal{F}_2$ shown in Figure 1. The logic of $\mathcal{F}_1$ is not **IPC**, since it validates $\neg p \vee \neg\neg p$ whereas $\mathcal{F}_2$ does not (remember that, by Fact 6, **IPC** is the set of formulas valid on *all* finite frames). On the other hand, $\mathrm{Log}(\mathcal{F}_1)$ is not equal to **CPC** either, since there is a model on $\mathcal{F}_1$ which refutes $p \vee \neg p$. In fact, there are continuum many $(2^{\aleph_0})$ intermediate logics, the idea of the proof is the following:

Suppose we can find (construct) a countable set of formulas $\{ \varphi_i \mid i \in I \}$ such that whenever I' and I'' are two distinct subsets of I, then **IPC** $+ \{ \varphi_i \mid i \in I' \}$

and **IPC** $+ \{ \varphi_i \mid i \in I'' \}$ are non-equal. Then, since there are continuum many subsets of a countable set, this means that there are at least continuum many distinct logics extending **IPC**. Of course, there cannot be *more* than continuum many intermediate logics either, since there are only continuum many distinct sets of formulas in our language $\mathcal{L}$. Note also that by Prop 3, all (except possibly for one) of these logics must be intermediate.

As it turns out it is possible to construct a set of formulas with the desired properties described above. These formulas are usually referred to as Jankov or de Jongh formulas, we will give a definition of the de Jongh formulas.

However, before turning our attention to these matters we need to introduce some preliminary notions and results. The first notion we need is that of a universal model.

The n-universal model.

In this section we will define the so called n-universal model which intuitively is a model carrying the information of all Kripke models for formulas of a language containing finitely many propositional letters. Note that we write $A \subset B$ if $A \subseteq B$ and $A \neq B$.

Definition 18. Let $\mathcal{L}_n$ be a propositional language consisting of finitely many propositional letters $p_1, \ldots, p_n$ for some $n \in \omega$. Let $\mathfrak{M}$ be an intuitionistic Kripke model. For every point w in $\mathfrak{M}$ we let Φ_w be the subset of $\{ p_1, \ldots, p_n \}$ defined by

$$p_k \in \Phi_w \text{ iff } w \models p_k, \text{ for } k = 1, \ldots, n.$$

Definition 19. Let $\mathcal{F} = (W, \leq)$ be a Kripke frame and let $w, v \in W$ be such that $w \leq v$. We say that v is an *immediate successor* of w if for all $u \in W$ such that $w \leq u$ and $u \leq v$ we have $u = w$ or $u = v$. If S_w is the set of all immediate successors of a point w we write $w \prec S_w$. We will use the shorthand $w \prec v$ for $w \prec \{v\}$.

If $\mathcal{F} = (W, \leq)$ is a frame such that every point $w \in W$ has finitely many successors, then $\leq$ is uniquely defined by the immediate successor relation $\prec$. More precisely, $\leq$ is the reflexive and transitive closure of $\prec$. So to define such a frame $\mathcal{F}$ it is sufficient to define the universe W and relation $\prec$. This is what we will do in defining the n-universal model of **IPC**.

Definition 20. The n-universal model $\mathcal{U}(n) = (U(n), \leq, V)$ is the minimal model satisfying the following three conditions:

(i) To every subset Φ of $\{p_1, \ldots, p_n\}$ there corresponds exactly one endpoint w of $\mathcal{U}(n)$ such that $\Phi = \Phi_w$, and these are the only endpoints of $\mathcal{U}(n)$.

(ii) For every $w \in U(n)$ and every subset $\Phi \subset \Phi_w$, there is a unique point $v \in U(n)$ such that $v \prec w$ and $\Phi_v = \Phi$.

(*iii*) For every finite antichain A in $\mathcal{U}(n)$ and every subset Φ of $\{ p_1, \dots, p_n \}$ such that $\Phi \subseteq \Phi_w$ for all $w \in A$, there is a unique point $v \in \mathcal{U}(n)$ such that $v \prec A$ and $\Phi_v = \Phi$.

The following theorem explains why $\mathcal{U}(n)$ is called the universal model.

Theorem 21.

(*i*) *For every Kripke model $\mathfrak{M} = (\mathcal{F}, V)$, there exists a Kripke model $\mathfrak{M}' = (\mathcal{F}', V')$ such that $\mathfrak{M}'$ is a generated submodel of $\mathcal{U}(n)$ and $\mathfrak{M}'$ is a generated submodel of a p-morphic image of $\mathfrak{M}$ such that for every φ of $\mathcal{L}_n$, we have that*

$$\mathfrak{M} \models \varphi \textit{ iff } \mathfrak{M}' \models \varphi$$

(*ii*) *For every finite Kripke frame $\mathcal{F}$, there exists a valuation V on $\mathcal{L}_n$ and $n \leq |\mathcal{F}|$ such that $\mathfrak{M} = (\mathcal{F}, V)$ is (isomorphic to) a generated submodel of $\mathcal{U}(n)$.*

Proof. For (i) see [**?**], sections 8.6 and 8.7.

(ii) Let $\mathcal{F} = (W, \leq)$ be a finite Kripke frame with $W = \{w_1, \dots, w_n\}$. For every point w_i we introduce a new propositional letter p_i. We define a valuation V on $\mathcal{F}$ as follows:

$$V(p_i) := \geq(w_i).$$

We show that $\mathfrak{M} = (\mathcal{F}, V)$ is isomorphic to a generated submodel of $\mathcal{U}(n)$. The submodel of $\mathfrak{M}$ consisting of all its endpoints is clearly isomorphic to some set of endpoints of $\mathcal{U}(n)$ since every possible valuation is instantiated in some endpoint. Now, assume that we have an isomorphism f between the submodel of $\mathfrak{M}$ consisting of all points with depth $< k$ and some generated submodel of $\mathcal{U}(n)$. Let v be an arbitrary point of depth k. If $v \prec v'$ for some v', then $\Phi_v \subset \Phi_{v'} = \Phi_{f(v')}$. Then, by property (ii), there is a point $u \in \mathcal{U}(n)$ such that $u \prec f(v')$ and $\Phi_u = \Phi_v$. On the other hand, if $v \prec \{v_1, \dots, v_m\}$, then $\Phi_v \subset \bigcap_{i \leq m} \Phi_{v_i} = \bigcap_{i \leq m} \Phi_{f(v_i)}$ and so, by property (iii), there is a point $u \in \mathcal{U}(n)$ such that $u \prec f[\{v_1, \dots, v_m\}]$ and $\Phi_u = \Phi_v$. In both cases we extend f so that $f(v) = u$.

Corollary 22. *For every formula φ in the language $\mathcal{L}_n$, we have that*

$$\vdash_{\mathbf{IPC}} \varphi \quad \textit{iff} \quad \mathcal{U}(n) \models \varphi.$$

Proof. Suppose we are working in the language $\mathcal{L}_n$. Since $\mathcal{U}(n)$ is a Kripke model it is trivial that $\mathcal{U}(n) \models \varphi$ for every formula φ such that $\vdash_{\mathbf{IPC}} \varphi$.

Conversely, suppose $\nvdash_{\mathbf{IPC}} \varphi$. Since **IPC** has the finite model property, there is a finite Kripke model $\mathfrak{M}$ such that $\mathfrak{M} \not\models \varphi$. By Theorem 21 (*i*) there is a generated submodel $\mathfrak{M}'$ of $\mathcal{U}(n)$ such that $\mathfrak{M}$ and $\mathfrak{M}'$ validates the same formulas (in $\mathcal{L}_n$). Then $\mathfrak{M}' \not\models \varphi$ and hence $\mathcal{U}(n) \not\models \varphi$.

Jankov-de Jongh formulas.

Jankov introduce his frame based formulas in an algebraic setting (see e.g. [?]). We will instead introduce the de Jongh formulas (see e.g. [?]) which do the same job as the Jankov formulas but saves us the trouble of introducing Heyting algebras. The formulas are interesting since they define submodels of the n-universal model and this will enable us to get information about the frames for a logic by checking whether the logic contains certain de Jongh formulas.

Definition 23 (Chain)**.** We call a set C of points in a frame $(W, \leq)$ a *chain* if $\leq$ is a linear order on C. By the *length* of a chain, we mean its cardinality as a set.

Definition 24 (Depth)**.** Let w be a point in a frame $\mathcal{F}$ and let $\mathfrak{C}$ be the set of chains C such that w is the least element of C. By the *depth* of w, written $d(w)$, we mean $\max\{\, |C| \mid C \in \mathfrak{C}\,\}$. The depth of a frame $\mathcal{F}$, written $d(\mathcal{F})$, is then defined as $\max\{\, d(w) \mid w \in \mathcal{F}\,\}$.

The de Jongh formulas are defined inductively using the n-universal model. Let w be a point in $\mathcal{U}(n)$. If $d(w) = 1$, we let

$$\varphi_w := \bigwedge\{\, p_k \mid w \models p_k \,\} \wedge \bigwedge\{\, \neg p_j \mid w \not\models p_j \,\} \text{ for each } k, j \in \{1, \ldots, n\}$$
$$\psi_w := \neg\varphi_w.$$

If $d(w) > 1$, we let $\{w_1, \ldots, w_k\}$ be the set of all immediate successors of w. Let

$$prop(w) := \{\, p_k \mid w \models p_k \,\}$$

and

$$newprop(w) := \{\, p_k \mid \; w \not\models p_k \text{ and } w_i \models p_k \text{ for all } i \text{ such that } 1 \leq i \leq m \,\}.$$

Now, let

$$\varphi_w := \bigwedge prop(w) \wedge \left((\bigvee newprop(w) \vee \bigvee_{i=1}^{m} \psi_{w_i}) \to \bigvee_{i=1}^{m} \varphi_{w_i} \right)$$
$$\psi_w := \varphi_w \to \bigvee_{i=1}^{m} \varphi_{w_i}$$

The formulas φ_w and ψ_w are called the *de Jongh formulas*. As we claimed above, the de Jongh formulas define submodels of the n-universal model. More precisely we have:

Theorem 25. *For every $w \in U(n)$ we have that*

- $\geq(w) = V(\varphi_w)$,
- $U(n) \setminus \leq(w) = V(\psi_w)$.

Proof. The proof goes by induction on the depth of w. Suppose the depth of w is 1. This means that w is a maximal point of $\mathcal{U}(n)$. By Definition 20 (i) we have that for every endpoint v of $U(n) \setminus \{w\}$, $\Phi_v \neq \Phi_w$ and thus $v \not\models \varphi_w$. Therefore, if $u \in U(n)$ is such that $u \leq v$ for some maximal point v of $\mathcal{U}(n)$ distinct from w, then $u \not\models \varphi_w$. Finally, suppose that $v < w$ and v is not related to any other maximal point. By the definition of $\mathcal{U}(n)$ this implies that $\Phi_v \subset \Phi_w$, and therefore $v \not\models \varphi_w$. Thus $v \models \varphi_w$ iff $v = w$ and so $V(\varphi_w) = \{w\}$. Similarly, if $v \not\models \psi_w$ there is a u such that $v \leq u$ and $u \models \varphi_w$, but then, by the above, we have $u = w$ and hence $v \in {\leq}(w)$. On the other hand, if $v \models \psi_w$ we cannot have $v \in {\leq}(w)$ since $w \models \varphi_w$ and intuitionistic valuations are persistent. Thus, $v \models \psi_w$ iff $v \in U(n) \setminus {\leq}(w)$, i.e. $V(\psi_w) = U(n) \setminus {\leq}(w)$.

Now suppose the depth of w is greater than 1 and that the theorem holds for all points of depth strictly less than $d(w)$. Then the theorem holds for every immediate successor w_i of w, i.e. for each $i = 1, \ldots, m$ we have $V(\varphi_{w_i}) = {\geq}(w_i)$ and $V(\psi_{w_i}) = U(n) \setminus {\leq}(w_i)$.

By the induction hypothesis, $w \not\models \bigvee_{i=1}^m \psi_{w_i}$. So by the definition of $newprop(w)$, we have $w \not\models \bigvee newprop(w) \vee \bigvee_{i=1}^m \psi_{w_i}$. Since $w \models \bigwedge prop(w)$ (by definition) we must have $w \models \varphi_w$, and then we have that $v \models \varphi_w$ for every $v \in {\geq}(w)$ (since intuitionistic valuations are persistent).

Now, suppose $v \notin {\geq}(w)$. If $v \not\models \bigwedge prop(w)$ then $v \not\models \varphi_w$ and we are done. So let's assume that $v \models \bigwedge prop(w)$. This means that $v \models p_j$ for every $p_j \in \Phi_w$, i.e. $\Phi_w \subseteq \Phi_v$. We get two cases:

(1) $v \in \bigcup_{i=1}^m U(n) \setminus {\leq}(w_i)$.

(2) $v \notin \bigcup_{i=1}^m U(n) \setminus {\leq}(w_i)$.

Suppose we are in case (1). By the induction hypothesis, $v \models \bigvee_{i=1}^m \psi_{w_i}$ and by our assumption above $v \notin {\geq}(w)$, so we have $v \not\models \bigvee_{i=1}^m \varphi_{w_i}$. But then $v \not\models \varphi_w$ and we are done. So, suppose we are in case (2). Then we have that $v \leq w_i$ for every $i = 1, \ldots, m$. Now, if there exists a $v' > v$ such that $v' \not\geq w$ and $v' \in \bigcup_{i=1}^m U(n) \setminus {\leq}(w_i)$, then, as in case (1), we get $v' \not\models \varphi_{w_i}$, and so $v \not\models \varphi_{w_i}$. So, let's assume that for every v' such that $v' \geq v$ and $v' \not\geq w'$ we have that $v' \notin \bigcup_{i=1}^m U(n) \setminus {\leq}(w_i)$, i.e. $v' \leq w_i$ for every i. Since the upward cone of v is finite, we can let u be a maximal point with this property (i.e. $u \geq v$ and $u \not\geq w$). It should be apparent that we either have that $u \prec \{w_1, \ldots, w_m\}$ or $u \prec w$, since, by our assumption on v, it must see *some* immediate predecessor of $\{w_1, \ldots, w_m\}$.

Now, in the former case we have that $u \prec \{w_1, \ldots, w_m\}$ and $u \neq w$ which by Definition 20 (iii) of the universal model means that $\Phi_u \neq \Phi_w$. Since $\Phi_u \supseteq \Phi_v \supseteq \Phi_w$, we have that $\Phi_u \supset \Phi_w$. But then there is a p_j, for some $j = 1, \ldots, n$, such that $u \models p_j$ while $w \not\models p_j$. Then $w_i \models p_j$ for every $i = 1, \ldots, m$, and hence $p_j \in newprop(w)$. Therefore, $u \models \bigvee newprop(w)$ and since $u \not\models \bigvee_{i=1}^m \varphi_{w_i}$ (by the induction hypothesis) we have $u \not\models \varphi_w$. Hence, $v \not\models \varphi_w$.

In the latter case, where we have $u \prec w$, we have by Definition 20 (ii) that $\Phi_u \subset \Phi_w$ which contradicts $\Phi_w \subseteq \Phi_v$ and $v \leq u$.

Hence, for every point v of $U(n)$ we have:

$$v \models \varphi_w \text{ iff } w \leq v$$

which means exactly that $V(\varphi_w) = {\geq}(w)$.

What is left to prove is that ψ_w defines $U(n) \setminus {\leq}(w)$. For every $v \in U(n)$, $v \not\models \psi_w$ iff there exists $u \in U(n)$ such that $v \leq u$ and $u \models \varphi_w$ and $u \not\models \bigvee_{i=1}^{m} \psi_{w_i}$, which holds iff $u \in {\geq}(w)$ and $u \notin \bigcup_{i=1}^{m} {\geq}(w_i)$, which, in turn, holds iff $u = w$. Hence, $v \not\models \psi_w$ iff $v \in {\leq}(w)$.

The following theorem links the de Jongh formulas to the notion of reduction. It was originally stated and proved (using Jankov formulas) by Jankov in [**?**].

Theorem 26. *For every finite rooted frame $\mathcal{F}$ there exists a formula $\chi(\mathcal{F})$ such that for every frame $\mathcal{G}$*

$$\mathcal{G} \not\models \chi(\mathcal{F}) \textit{ iff } \mathcal{G} \textit{ is reducible to } \mathcal{F}.$$

Proof. Let $\mathcal{F}$ be a finite rooted frame. By Theorem 21 (ii) there exists $n \in \omega$ such that $\mathcal{F}$ is isomorphic to a generated subframe of $\mathcal{U}(n)$, i.e. there is $w \in \mathcal{U}(n)$ such that $\mathcal{F}$ is isomorphic to $\mathcal{F}_w$. The formula $\chi(\mathcal{F})$ will be the de Jongh formula ψ_w. By Lemma 17 we are done if we can prove the following:

$$\mathcal{G} \not\models \psi_w \text{ iff } \mathcal{F}_w \text{ is a generated subframe of a p-morphic image of } \mathcal{G}.$$

Suppose that $\mathcal{F}_w$ is a generated subframe of a p-morphic image of $\mathcal{G}$. Since $w \not\models \psi_w$ we have that $\mathcal{F}_w \not\models \psi_w$, and since p-morphisms preserve validity of formulas we have that $\mathcal{G} \not\models \psi_w$.

Now suppose that $\mathcal{G} \not\models \psi_w$. Then there is a model $\mathfrak{M} = (\mathcal{G}, V)$ such that $\mathfrak{M} \not\models \psi_w$, and so by Theorem 21 ($i$) there is a generated submodel $\mathfrak{M}' = (\mathcal{G}', V')$ of $\mathcal{U}(n)$ which is a generated submodel of a p-morphic image of $\mathfrak{M}$ such that $\mathfrak{M}$ and $\mathfrak{M}'$ validates the same formulas of $\mathcal{L}_n$. Since ψ_w is a formula over $\mathcal{L}_n$, we have that $\mathfrak{M}' \not\models \psi_w$. Now, by Theorem 25 $\mathfrak{M}' \not\models \psi_w$ iff there is v in $\mathcal{G}'$ such that $v \leq w$, which holds iff w belongs to $\mathcal{G}'$. It follows that $\mathcal{F}_w$ is a generated subframe of $\mathcal{G}'$, and so $\mathcal{F}$ is a generated subframe of a p-morphic image of $\mathcal{G}$.

4 The structure of intermediate logics.

With the work done in the previous sections we are now ready to prove some results about classes of intermediate logics. These results will help to sketch a better picture of the class of intermediate logics.

Let Δ be the set of finite Kripke frames given in Figure 2. Then we have the following:

Lemma 27. *Δ forms an infinite $\leq$-antichain.*

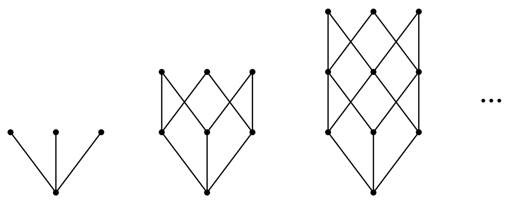

Fig. 2. The antichain Δ

To prove this statement, we will need to show a couple of preservation theorems about reductions of finite frames. Firstly, by the *branching* of a point w in a frame $\mathcal{F}$ we mean the number of immediate successors of w. The branching of a frame is then defined as the maximal branching of its points. Secondly, an *endpoint* is quite simply a point that only sees itself, i.e., if w is an endpoint, then $\geq(w) = \{w\}$.

Lemma 28. *Neither the number of endpoints nor the branching of a frame can increase by taking generated subframes or p-morphic images.*

Proof. The case for generated subframes should in both cases be obvious. Instead, let f be a p-morphism from $\mathcal{F}$ onto $\mathcal{G}$.

Now, take an endpoint v of $\mathcal{G}$ and let w be a maximal element in the preimage of v. Then, if there is a $w' > w$, me have that $f(w') \geq f(w) = v$, and since v was an endpoint, $f(w') = w$ which contradicts the maximality of w.

Next, take a point v in $\mathcal{G}$ with $\{\, a_1, \ldots, a_n \}$ being all of its immediate successors. Let w be a maximal element of $f^{-1}(v)$, then, since f is a p-morphism, w sees at least one element of $f^{-1}(a_i)$ for each i, so we can define b_i to be a minimal element in $f^{-1}(a_i) \cap \geq(w)$. The claim is that the b_i's are immediate successors of w. If they were not, then there would be a w' such that $w < w' < b_i$ for some i. But then $v = f(w) \leq f(w') \leq f(b_i) = a_i$, and since a_i was an immediate successor of v, we have either $f(w') = v$ or $f(w') = a_i$. The former contradicts to the maximality of w, and the later contradicts to the minimality of the b_i's.

We now proceed by showing that Δ is an antichain.

Proof (Proof of Lemma 27). Let $\mathcal{F}$ be a frame in Δ, then any nonidentical generated subframe of Δ has branching ≤ 2, and since branching cannot increase by p-morphisms, we can disregard generated subframes entirely.

Now, let f be a p-morphism from $\mathcal{F}$ onto some frame $\mathcal{G}$. By Fact 15, f is a composition of a finite number of α- and β-reductions. The first reduction in such a composition must be a β-reduction on a couple of endpoints, as can be easily seen. But then the result after the first reduction will be a frame with 2 endpoints, and since the number of endpoints cannot increase by taking p-morphisms, $\mathcal{G}$ cannot be an element of Δ.

Theorem 29. *For every $\Gamma_1, \Gamma_2 \subseteq \Delta$ such that $\Gamma_1 \neq \Gamma_2$ we have* $\mathrm{Log}(\Gamma_1) \neq \mathrm{Log}(\Gamma_2)$.

Proof. We may without loss of generality assume that there is $\mathcal{F} \in \Gamma_1$ such that $\mathcal{F} \notin \Gamma_2$. Let $\chi(\mathcal{F})$ be the de Jongh formula of $\mathcal{F}$. Since every frame is a reduction of itself, we have that $\mathcal{F} \not\models \chi(\mathcal{F})$. Hence, $\Gamma_1 \not\models \chi(\mathcal{F})$ and so $\chi(\mathcal{F}) \in \mathrm{Log}(\Gamma_1)$. Now we show that $\chi(\mathcal{F}) \in \mathrm{Log}(\Gamma_2)$. Suppose for a contradiction that $\chi(\mathcal{F}) \notin \mathrm{Log}(\Gamma_2)$, then there is $\mathcal{G} \in \Gamma_2$ such that $\mathcal{G} \not\models \chi(\mathcal{F})$. By Theorem 26 this implies that $\mathcal{F}$ is a reduction of $\mathcal{G}$. But this contradicts the fact that Δ forms an $\leq$-antichain. We must conclude that $\chi(\mathcal{F}) \notin \mathrm{Log}(\Gamma_1)$ and $\chi(\mathcal{F}) \in \mathrm{Log}(\Gamma_2)$. Thus, $\mathrm{Log}(\Gamma_1) \neq \mathrm{Log}(\Gamma_2)$ as desired.

Corollary 30. *There are continuum many intermediate logics.*

Proof. This follows immediately from Theorem 29 since there are continuum many distinct subsets Γ of Δ.

As noted above the de Jongh formulas can be used to give information about intermediate logics. We will start by looking at logics of finite depth.

A logic $L \supseteq \mathbf{IPC}$ has depth $\leq n$ if every L-frame has depth $\leq n$. As it turns out, every logic L of finite depth has the finite model property (see Segerberg [**?**]) and for such a logic L this is the same as being complete with respect to the class of finite rooted L-frames (see Chagrov and Zakharyaschev [**?**]).

Theorem 31. *Every intermediate logic of finite depth is complete with respect to the class of its finite rooted frames, i.e. if $L \supseteq \mathbf{IPC}$ has depth $\leq n$ then*

$$L = \mathrm{Log}(\{\, \mathcal{F} \mid \mathcal{F} \textit{ is a finite rooted } L\textit{-frame} \,\}).$$

Proof. For a complete proof, see Chagrov and Zakharyaschev [**?**].

In fact, we can define the depth of a logic in terms of de Jongh formulas. Let $\mathcal{C}_n$ denote the Kripke frame consisting of exactly n points $w_1 < w_2 < \ldots < w_n$. Then we have the following:

Theorem 32.

(*i*) *A frame $\mathcal{F}$ has depth $\leq n$ iff $\mathcal{F}$ is not reducible to $\mathcal{C}_{n+1}$.*

(*ii*) *A logic $L \supseteq \mathbf{IPC}$ has depth $\leq n$ iff $\chi(\mathcal{C}_{n+1}) \in L$.*

Proof.

(*i*) If $\mathcal{C}_{n+1} \leq \mathcal{F}$, then clearly the depth of $\mathcal{F}$ is $\geq n+1$. So suppose the depth of $\mathcal{F}$ is $> n$. Then there is a chain $w_0 < w_1 < \ldots < w_n$ of distinct points in $\mathcal{F}$. Let $\mathcal{F}_{w_0}$ be the subframe generated by w_0 and define a map $f \colon \mathcal{F}_{w_0} \to \mathcal{F}_{w_0}$ as follows:

$$f(w) = \begin{cases} w_i, & \text{where } i \text{ is the least such that } w \leq w_i, \\ w_n, & \text{if no such } i \text{ exists.} \end{cases}$$

Then f is a p-morphism. To see why, assume $w \leq w'$ and consider the following cases:

(a) $w' \not\leq w_i$ for any i. Then $f(w') = w_n$, and so $f(w) \leq f(w')$.
(b) There is some least i such that w' sees w_i. Then w sees w_i as well, so $f(w) \leq w_i = f(w')$.

Next, assume $f(w) \leq w_i$ for some i. We must find w' such that $w \leq w'$ and $f(w') = w_i$. Again we have two cases:
(a) If $w \not\leq w_j$ for any j, we have that $f(w) = w_n = w_i$ and so we can choose $w' = w$.
(b) Otherwise, $w \leq w_j$ for some $j \leq i$, so we can let $w' = w_i$.

Now, the f-image of $\mathcal{F}_{w_0}$ is isomorphic to $\mathcal{C}_{n+1}$. Hence, $\mathcal{C}_{n+1}$ is a reduction of $\mathcal{F}$.

(ii) Suppose $\chi(\mathcal{C}_{n+1}) \in L$. Then we have $\mathcal{F} \models \chi(\mathcal{C}_{n+1})$ for every L-frame $\mathcal{F}$. Let $\mathcal{F}$ be an L-frame, by Theorem 26 we have that $\mathcal{F}$ is not reducible to $\mathcal{C}_{n+1}$. Hence, by *(i)* we have that $\mathcal{F}$ has depth $\leq n$, and since $\mathcal{F}$ was arbitrary, L has depth $\leq n$.

Now, suppose L has depth $\leq n$. This means that no L-frame is reducible to $\mathcal{C}_{n+1}$, and hence every L-frame satisfies $\chi(\mathcal{C}_{n+1})$. By Theorem 31,

$$L = \mathrm{Log}(\{\, \mathcal{F} \mid \mathcal{F} \text{ is a finite rooted } L\text{-frame} \,\}),$$

and so we have that $\chi(\mathcal{C}_{n+1}) \in L$.

Now we can give a first example of a logic axiomatized by de Jongh formulas. Let $L_{d(n)}$ denote the least logic of depth $\leq n$, i.e. if $\mathsf{L}_{\leq n}$ is the class of logics of depth $\leq n$ then $L_{d(n)} = \bigcap_{L \in \mathsf{L}_{\leq n}} L$.

Corollary 33. $L_{d(n)} = \mathbf{IPC} + \chi(\mathcal{C}_{n+1})$.

Proof. By Theorem 32 we have that $\mathbf{IPC} + \chi(\mathcal{C}_{n+1}) \subseteq L$ for every logic L of depth $\leq n$. And again by Theorem 32 $\mathbf{IPC} + \chi(\mathcal{C}_{n+1})$ has depth $\leq n$. Hence $L_{d(n)} = \mathbf{IPC} + \chi(\mathcal{C}_{n+1})$.

In fact, we can give axiomatizations of all logics of finite depth by de Jongh formulas.

Theorem 34. *Every logic of finite depth can be axiomatized by de Jongh formulas.*

Proof. Let L be a logic of depth $\leq n$ and let F_L be the class of finite rooted L-frames. Note that F_L is downward closed in the sense that if $\mathcal{F} \leq \mathcal{G}$ and $\mathcal{G} \in \mathsf{F}_L$ then $\mathcal{F} \in \mathsf{F}_L$ (since every reduction of a finite frame is finite), and furthermore that, by Theorem 31, we have $L = Log(\mathsf{F}_L)$. Now, let

$$\mathbf{Fr} = \{\, \mathcal{G} \mid \forall \mathcal{F} \in \mathsf{F}_L : \mathcal{G} \not\leq \mathcal{F} \,\}.$$

We claim that $L = \mathbf{IPC} + \{\, \chi(\mathcal{G}) \mid \mathcal{G} \in \mathbf{Fr} \,\}$. By Theorem 32 we have $\mathcal{C}_{n+1} \in \mathbf{Fr}$, and so $\mathbf{IPC} + \{\, \chi(\mathcal{G}) \mid \mathcal{G} \in \mathbf{Fr} \,\}$ is also of depth $\leq n$, hence it is complete with respect to the class of its finite rooted frames.

Now, suppose that $\mathcal{H}$ is a finite rooted Kripke frame of $\mathbf{IPC} + \{\, \chi(\mathcal{G}) \mid \mathcal{G} \in \mathbf{Fr}\,\}$. Then, $\mathcal{G} \not\leq \mathcal{H}$ for every $\mathcal{G} \in \mathbf{Fr}$, so $\mathcal{H}$ is not in $\mathbf{Fr}$, i.e. $\mathcal{H} \leq \mathcal{F}$ for some $\mathcal{F} \in \mathsf{F}_L$. But then $\mathcal{H}$ is an L-frame, which proves $L \subseteq \mathbf{IPC} + \{\, \chi(\mathcal{G}) \mid \mathcal{G} \in \mathbf{Fr}\,\}$.

On the other hand, if $\mathcal{H}$ is a finite rooted L-frame such that

$$\mathcal{H} \not\models \mathbf{IPC} + \{\, \chi(\mathcal{G}) \mid \mathcal{G} \in \mathbf{Fr}\,\},$$

then we must have $\mathcal{H} \not\models \chi(\mathcal{G})$ for some $\mathcal{G} \in \mathbf{Fr}$. But then we have $\mathcal{G} \leq \mathcal{H}$ contradicting the fact that $\mathcal{G} \in \mathbf{Fr}$. Hence, the finite rooted frames of L and $\mathbf{IPC} + \{\, \chi(\mathcal{G}) \mid \mathcal{G} \in \mathbf{Fr}\,\}$ coincide. This implies that the two logics are equal and we obtain that L is axiomatizable by de Jongh formulas.

Theorem 34 does not in general give us finitely many axioms, but in some cases it is possible to give finite axiomatizations.

We call a logic L *tabular* if $L = Log(\mathcal{F})$ for some finite frame $\mathcal{F}$. Then we have:

Corollary 35. *Every tabular logic is finitely axiomatizable.*

Proof. Just note the following in the proof of Theorem 34: We will have $\mathbf{Fr} = \{\, \mathcal{G} \mid \mathcal{G} \not\leq \mathcal{F}\,\}$ and so $(\mathbf{Fr}, \leq)$ will have finitely many minimal elements. For suppose $\mathcal{G}$ is a frame consisting of more points than $\mathcal{F}$. By Fact 15, we can use α and β-reductions to reduce $\mathcal{G}$ to a frame $\mathcal{G}'$ with $|\mathcal{F}| + 1$ points and then $\mathcal{G}' \not\leq \mathcal{F}$. Hence, the minimal elements of $(\mathbf{Fr}, \leq)$ must have $\leq |\mathcal{F}| + 1$ points and so there can only be finitely many of them. Now, let the minimal elements of $\mathbf{Fr}$ be $\mathcal{G}_1, \ldots, \mathcal{G}_k$ Then we will have

$$L = Log(\mathcal{F}) = \mathbf{IPC} + \chi(\mathcal{G}_1) + \ldots + \chi(\mathcal{G}_k).$$

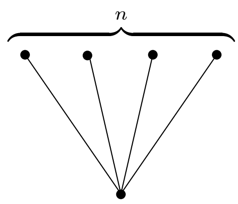

Fig. 3. The finite rooted frame $\mathcal{V}_n$ of depth ≤ 2

Proposition 36. *Every logic of depth ≤ 2 is finitely axiomatizable by de Jongh formulas.*

Proof. Let L be a logic of depth ≤ 2. By Theorem 31 we have that

$$L = \mathrm{Log}(\{\, \mathcal{F} \mid \ \mathcal{F} \text{ is a finite rooted } L\text{-frame of depth} \leq 2\,\}).$$

The only finite rooted frames of depth ≤ 2 are the frames $\mathcal{V}_n$, for $n \in \omega$, consisting of one point with n immediate successors, see Figure 3. Hence $L = \mathrm{Log}(\{\, \mathcal{V}_n \mid n \in N \,\})$ for some finite $N \subseteq \omega$. Clearly $\mathcal{V}_m \leq \mathcal{V}_n$ if $m \leq n$, but then $L = \mathrm{Log}(\mathcal{V}_n)$ for $n = max\{\, x \in N \,\}$ since: if $\mathcal{V}_n \models \varphi$ then $\mathcal{V}_m \models \varphi$ for every $m \leq n$ (p-morphism preserve validity), the other direction is trivial. But then L is tabular and so by Corollary 35 it is finitely axiomatizable

However, as soon as we consider logics of depth ≤ 3 we loose finite axiomatization. In fact, there are continuum many logics of depth ≤ 3 which follows as a simple consequence of the following:

Theorem 37. $\mathsf{F}_{\leq 3}$ *contains an infinite* $\leq$*-antichain.*

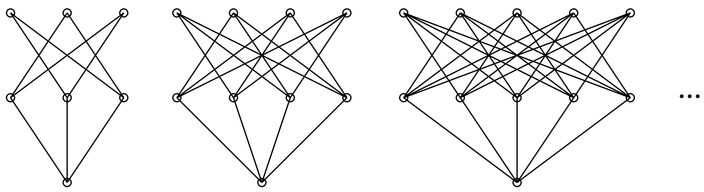

Fig. 4. $\mathcal{F}_3$, $\mathcal{F}_4$ and $\mathcal{F}_5$

The antichain in question is the one depicted in Figure 4. As in Lemma 27 we prove this by showing that all the frames in the antichain has some property that is not preserved in the image of any proper p-morphism — the property in question this time being that every endpoint excludes some non-endpoint from its downward cone.

Lemma 38. *If w is an endpoint in $\mathcal{F}$, and f is a p-morphism from $\mathcal{F}$ to $\mathcal{G}$, then $f(w)$ is an endpoint.*

Proof. Let $\mathcal{F}, \mathcal{G}, f$ and w be as above, and assume, for a contradiction, that there is a $v \in \mathcal{G}$ such that $f(w) < v$. Then, since f is a p-morphism, there is a $v' \in \mathcal{F}$ such that $f(v') = v$ and $w \leq v'$. However, since f is a function $w \neq v'$, so $w < v'$ which contradicts to w being an endpoint.

Theorem 39. *Let $\mathcal{F}_n = (W_n, \leq_n)$ be defined as follows*

- $W_n = \{w_1^1, w_1^2, w_2^2, \ldots, w_n^2, w_1^3, w_2^3, \ldots, w_n^3\}$,
- $w_1^1 \leq w_i^d$ *for all* w_i^d,
- $w_i^2 \leq w_i^2$,
- $w_i^2 \leq w_j^d$ *if and only if* $d = 2$ *and* $i = j$ *or* $d = 3$ *and* $i \neq j$*, and finally*
- $w_i^3 \leq w_j^d$ *if and only if* $d = 3$ *and* $i = j$*. (See Figure 4 for* $\mathcal{F}_3$*,* $\mathcal{F}_4$ *and* $\mathcal{F}_5$*)*

Then $\mathsf{F} = \{\, \mathcal{F}_n \mid n > 3 \,\}$ *is an antichain of depth 3.*

Proof. First of all, notice that if $\mathcal{F}$ is an element of F, then any generated subframe $\mathcal{G}$ of $\mathcal{F}$ is either isomorphic to the original frame or it has depth < 3 and in this latter case, since p-morphisms can never increase depth, any p-morphic image of G is not an element of F. For this reason, we can restrict our attention to p-morphic images only.

Now, assume, for a contradiction, that we have two frames $\mathcal{F}_n, \mathcal{F}_m \in \mathsf{F}$ with $m < n$ and a p-morphism f onto $\mathcal{F}_m$. By Fact 15, we know that f is a composition of some finite collection of α- and β-reductions, i.e., $f = f_1 \circ \ldots \circ f_n$. However, the only choice for f_n is a β-reduction of two endpoints w_i^3 and w_j^3, since every other pair of points differ in their strict upward cones. By preservation of order we have that ${<}(f_n(w_i^3)) = {<}(w_i^3) \cup {<}(w_j^3)$ so, in $f_n[\mathcal{F}]$, every non-endpoint sees $f_n(w_i^3)$. By Lemma 38 $f(w_i^3)$ is an endpoint, and by the same corollary and f being order preserving, every non-endpoint of $\mathcal{F}_m$ sees $f(w_i^3)$. However, every endpoint v_i^3 of $\mathcal{F}$ excludes some non-endpoint from its downward cone, viz., v_i^2; which is a contradiction.

From this we get the following corollary:

Corollary 40. *There are continuum many logics of depth* ≤ 3.

Proof. Let F denote the infinite antichain given in Theorem 39. By Theorem 29 we get distinct logics $\mathrm{Log}(\Gamma)$ for every $\Gamma \subseteq \mathsf{F}$, and hence continuum many of them. Since $\mathsf{F} \subseteq \mathsf{F}_{\leq 3}$ we have $\mathcal{C}_4 \not\leq \mathcal{F}$ for every frame $\mathcal{F} \in \mathsf{F}$ and so by Theorem 32 $\chi(\mathcal{C}_4) \in \mathrm{Log}(\Gamma)$ for every $\Gamma \subseteq \mathsf{F}$. Hence, for every pair $\Gamma_1, \Gamma_2 \subseteq \mathsf{F}$ both $\mathrm{Log}(\Gamma_1)$ and $\mathrm{Log}(\Gamma_2)$ have depth ≤ 3 and $\mathrm{Log}(\Gamma_1) \neq \mathrm{Log}(\Gamma_2)$.

Corollary 41. *Not every logic of depth* ≤ 3 *is finitely axiomatizable.*

Proof. There are only countably many finite subsets of the set of formulas in the language of **IPC** (since this set is countable) so we can at most get countably many logics by adding a finite number of formulas to **IPC**. Since there are continuum many logics of depth ≤ 3 there must be such logics which are not finitely axiomatizable (in fact, there must be continuum many such logics).

Interpolation in Intermediate Logics

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Abstract. We discuss the connection between the interpolation property of intermediate logics and the amalgamation property of Heyting algebras. The main part of this paper is based on [3].

1 Introduction

Let $\mathcal{L}$ be an intermediate logic. $\mathcal{L}$ has the interpolation property if for every formula $\varphi \to \psi \in \mathcal{L}$ there is a χ such that $\text{Var}(\chi) \subseteq \text{Var}(\varphi) \cap \text{Var}(\psi)$ ($\text{Var}(\varphi)$ denotes the set of propositional variables occurring in φ) and $\varphi \to \chi$, $\chi \to \psi \in \mathcal{L}$. It is a celebrated theorem of mathematical logic, proved by William Craig (see [2]), that the classical predicate calculus (with identity) has the interpolation property (with the condition that $\text{Var}(\chi) \subseteq \text{Var}(\varphi) \cap \text{Var}(\psi)$ be replaced by similar conditions on the relation and function symbols, and on the free variables). A straightforward translation of this result gives us that classical propositional logic **CL** has the interpolation property. Kurt Schütte (see [4]) later proved that the intuitionistic predicate calculus has the interpolation property; and as a result, a straighforward translation gives us that intuitionistic propositional logic, **IPC**, has the interpolation property. Both Craig and Schütte use proof-theoretical methods in their proofs. This paper will present a proof due to Larissa Maksimova which shows that algebraic means can be used to prove whether a given intermediate logic has the interpolation property. More precisely, Maksimova proved in [3] that $\mathcal{L}$ has the interpolation property if and only if the variety Var$\mathcal{L}$ of Heyting algebras validating $\mathcal{L}$ has the amalgamation property. In this paper, I intend to lead the reader who already knows the basic notions of intermediate logics and its algebraic interpretation through Maksimova's proof of this result. I will also present Maksimova's proof showing that six intermediate logics have the interpolation property. In fact, Maksimova [3] showed that it is only these six intermediate logics, classical logic and the inconsistent logic that have the interpolation property. However, presenting the whole proof of this fact is beyond the scope of this paper.

I will start by introducing most of the notions we will need to be familiar with.

2 Preliminaries

We will work in a standard propositional language with connectives $\wedge, \vee, \to$ and a constant $\perp$ which is always interpreted as False. When we write $\varphi(p_1, ..., p_n)$

we mean that the variables of φ are among $p_1, ..., p_n$. We abbreviate $p \to \bot$ by $\neg p$, $p \to q \wedge q \to p$ by $p \leftrightarrow q$ and $\bot \to \bot$ by $\top$. We will use $\odot$ as a variable over the logical connectives. It is important to note that in most of the logics we will deal with, none of the logical operations are definable in terms of any of the other, so we really need the whole range of logical connectives. A **logic** is a set of propositional formulas closed under uniform substitution and modus ponens. A logic $\mathcal{L}$ is **intermediate** if $\mathbf{IPC} \subseteq \mathcal{L} \subseteq \mathbf{CL}$. **IPC** is axiomatized[1] by a set of formula schemes securing the basic properties of $\wedge$, $\vee$ and $\to$, plus the explosion rule $\bot \to p$. These rules are strong enough to prove the replacement theorem, i.e the theorem stating that if $\varphi(p)$ and $\psi(q)$ are two propositional formulas, then $\mathbf{IPC} \vdash \varphi \leftrightarrow \psi$ and $\mathbf{IPC} \vdash \alpha \leftrightarrow \beta$ implies that $\mathbf{IPC} \vdash \varphi(\alpha) \leftrightarrow \psi(\beta)$. Since the proof of this theorem only depends on the axioms available, it is easily seen that the replacement theorem holds for every intermediate logic. The same goes for the deduction theorem, i.e the theorem saying that if $\Gamma \cup \varphi \vdash_{\mathcal{L}} \psi$, then $\Gamma \vdash_{\mathcal{L}} \varphi \to \psi$. A famous theorem of Jankov (see, for instance [1]) says that there are 2^ω many intermediate logics, hence not every such logic is specified by adding to **IPC** some finite set of axioms. However, all the logics explicitly mentioned in this paper can be specified in that way. In these cases we denote the logic so obtained by $\mathbf{IPC} + \varphi_1 + ... + \varphi_n$, where $\varphi_1, ..., \varphi_n$ are the axioms added to **IPC**. We call a structure $\mathfrak{A} = \langle A, \wedge, \vee, \to, \bot\rangle$, where $\wedge$, $\vee$, and $\to$ are binary function symbols and $\bot$ is a constant an **algebra**. As in the case of the propositional language we write $\top$ for $\bot \to \bot$. Using the propositional connectives as names for functions enables one to treat propositional formulas as defining functions in $\mathfrak{A}$. A **valuation** $\mathfrak{V}$ in $\mathfrak{A}$ is a mapping of propositional variables into A which is extended to the whole propositional language by letting $\mathfrak{V}(\varphi \odot \psi) = \mathfrak{V}(\varphi) \odot \mathfrak{V}(\psi)$, and $\mathfrak{V}(\bot) = \bot$. We write $\varphi(a_1, ..., a_n)$ for the value of $\varphi(p_1, ..., p_n)$ under a valuation $\mathfrak{V}$ such that $\mathfrak{V}(p_i) = a_i$. An algebra $\mathfrak{A}$ **validates** φ, written $\mathfrak{A} \models \varphi$, if for every valuation $\mathfrak{V}$ in $\mathfrak{A}$ we have $\mathfrak{V}(\varphi) = \top$. For a formula $\varphi(p_1, ..., p_n)$, we write $\mathfrak{A} \models \varphi(a_1, ..., a_n)$ if for every $\mathfrak{V}$ such that $\mathfrak{V}(p_i) = a_i$ for $1 \leq i \leq n$ we have $\mathfrak{V}(\varphi) = \top$. A **lattice** is a structure $\mathfrak{A} = \langle A, \wedge, \vee, \to, \bot\rangle$ such that for every $a, b, c \in A$ the following holds.

1. $a \vee a = a$ and $a \wedge a = a$
2. $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$
3. $a \vee (b \vee c) = (a \vee b) \vee c$ and $(a \wedge b) \wedge c = a \wedge (b \wedge c)$
4. $a \vee \bot = a$ and $a \wedge \bot = a$
5. $a \vee (b \wedge a) = a$ and $a \wedge (b \vee a) = a$

It is a standard fact that if we define $\leq$ in $\mathfrak{A}$ by

$$a \leq b \text{ iff}_{\text{def}}\ a \wedge b = a,$$

the structure $\langle A, \leq\rangle$ is a partially ordered set and the operations $a \wedge b$ and $a \vee b$ coincides with the operations $glb\{a, b\}$ (i.e the greatest lower bound of a and b) and $lub\{a, b\}$ (i.e the least upper bound of a and b) respectively. Also, $\bot$ is the

[1] See Definition 1, p. 19 of this report, for one possible axiomatization.

least element in $\langle A, \leq\rangle$. A **Heyting algebra** is a lattice with an extra operation, $\rightarrow$ (so a Heyting algebra is really an algebra in our sense), defined by

For every $c \in A$, $c \leq a \rightarrow b$ iff $c \wedge a \leq b$.

It then follows easily from the definition of $\rightarrow$ that $\top$ is the greatest element in every Heyting algebra. The reason Heyting algebras are singled out here is the basic fact, left unproved here (for a proof see [1]), that $\varphi \in$ **IPC** iff φ is validated by every Heyting algebra.
For an intermediate logic $\mathcal{L}$ we denote by **Var**$\mathcal{L}$ the class of Heyting algebras $\mathfrak{A}$ such that $\varphi \in \mathcal{L}$ iff $\mathfrak{A} \models \varphi$. Var$\mathcal{L}$ is called the variety of $\mathcal{L}$
Denote by $\textsc{Word}X$ the smallest (with respect to $\subseteq$) set Y such that $X \subseteq Y$, $\perp \in Y$ and if $x, y \in Y$ then $x \odot y \in Y$. For each logic $\mathcal{L}$ we define an equivalence relation $\sim_{\mathcal{L}}$ by

$\varphi \sim_{\mathcal{L}} \psi$ iff $\varphi \leftrightarrow \psi \in \mathcal{L}$,

and we let $\|\varphi\|_{\mathcal{L}}$ denote the equivalence class of φ under $\sim_{\mathcal{L}}$ and $\|\textsc{Word}X\|_{\mathcal{L}}$ the corresponding set of equivalence classes. Define $\|\varphi\|_{\mathcal{L}} \odot \|\psi\|_{\mathcal{L}} = \|\varphi \odot \psi\|_{\mathcal{L}}$. If $\mathcal{L}$ is an intermediate logic, $\odot$ is well-defined, for the replacement theorem holds for $\mathcal{L}$. Let $\mathfrak{A}_{\mathcal{L}}(X) = \langle \|\textsc{Word}X\|_{\mathcal{L}}, \wedge, \vee, \perp\rangle$. If $\mathcal{L}$ is intermediate, $\mathfrak{A}_{\mathcal{L}}(X)$ is a Heyting algebra. Moreover, if the number of variables in φ is less than $|X|$, we have $\varphi \in \mathcal{L}$ iff $\mathfrak{A}_{\mathcal{L}}(X) \models \varphi$. Since $\mathfrak{A}_{\mathcal{L}}(X)$ is a free algebra, it is called the **free algebra in Var$\mathcal{L}$ over X**. Note that two free algebras in Var$\mathcal{L}$ are isomorphic if their respective set of generators has the same cardinality.
Let $\mathfrak{B} = \langle B, \wedge, \vee, \rightarrow, \perp\rangle$ be a Heyting algebra. $\nabla \subseteq B$ is a **filter** iff

1. $\nabla \neq \emptyset$
2. If $b \in \nabla$ and $b \leq a$ for $a \in B$, then $a \in \nabla$
3. If $a, b \in \nabla$ then $a \wedge b \in \nabla$

For $X \subseteq B$ let $[X) = \{b \in B \,|\, \text{there are } b_1, ..., b_n \in X \text{ s.th. } b_1 \wedge ... \wedge b_n \leq b\}$. It is easily seen that $[X)$ is a filter. Hence, every filter can be represented in the form $[X)$ for some X. When we are dealing with several Heyting algebras $\mathfrak{A}_i$ at the same time, we will write $[X)_i$ for the filter in $\mathfrak{A}_i$ generated by $X \subseteq A_i$. A filter ∇ is **prime** if $\nabla \neq B$ (i.e ∇ is a proper filter) and $a \vee b \in \nabla$ implies $a \in \nabla$ or $b \in \nabla$.
The dual of a filter is called an ideal. $\Delta \subseteq B$ is an **ideal** iff

1. $\Delta \neq \emptyset$
2. If $b \in \Delta$ and $a \leq b$ for $a \in B$, then $a \in \Delta$
3. If $a, b \in \Delta$, then $a \vee b \in \Delta$

For $X \subseteq B$, let $(X] = \{b \in B \,|\, \text{there are } b_1, ..., b_n \in X \text{ s.th. } b \leq b_1 \vee ... \vee b_n\}$. It is easily seen that $(X]$ is an ideal. Hence, every ideal can be represented in the form $(X]$ for some X. An ideal Δ is prime if $a \wedge b \in \Delta$ implies $a \in \Delta$ or $b \in \Delta$.
If $\mathfrak{A}$ is a Heyting algebra and $\nabla \subseteq A$ is a filter, we may define an equivalence relation $\sim$ on A by $a \sim_{\nabla} b$ iff $a \leftrightarrow b \in \nabla$, where $x \leftrightarrow y$ is an abbreviation for $x \rightarrow y \wedge y \rightarrow x$. It is easily seen that if $x \rightarrow y \in \nabla$ and $x \in \nabla$, then $y \in \nabla$. Also,

in every Heyting algebra, $(x \leftrightarrow y \wedge x' \leftrightarrow y') \rightarrow (x \odot y \rightarrow x' \odot y') = \top$ and thus $(x \leftrightarrow y \wedge x' \leftrightarrow y') \rightarrow (x \odot y \rightarrow x' \odot y') \in \nabla$. It follows that $\|x\|_\nabla \odot \|y\|_\nabla = \|x \odot y\|_\nabla$ is a sound definition. Hence $a \mapsto \|a\|_\nabla$ is a homomorphism, so $\mathfrak{A}/\nabla$ is a Heyting algebra. Furthermore,

$$\mathfrak{A}/\nabla \models \varphi(a_1, ..., a_n) \text{ iff } \varphi(a_1, ..., a_n) \in \nabla$$

3 Interpolation and Amalgamation

As was stated in the introduction, $\mathcal{L}$ has **the interpolation property** if for every $\varphi \rightarrow \psi \in \mathcal{L}$, there is a χ such that $\mathrm{Var}(\chi) \subseteq \mathrm{Var}(\varphi) \cap \mathrm{Var}(\psi)$ and $\varphi \rightarrow \chi$, $\chi \rightarrow \psi \in \mathcal{L}$. A class $\mathcal{C}$ of algebras is amalgamable if for every $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2$ in $\mathcal{C}$ such that $\mathfrak{A}_0$ can be embedded into both $\mathfrak{A}_1$ and $\mathfrak{A}_2$ via some f_1 and f_2 respectively, there is an $\mathfrak{A}$ in $\mathcal{C}$ and embeddings $g_i : \mathfrak{A}_i \hookrightarrow \mathfrak{A}$ for $i \in \{1, 2\}$ such that $g_1 \circ f_1 = g_2 \circ f_2$. In this case we say that the g_i's embed the $\mathfrak{A}_i$'s over $\mathfrak{A}_0$. In the following I will present an argument proving that an intermediate logic $\mathcal{L}$ has the interpolation property iff $\mathrm{Var}\mathcal{L}$ is amalgamable.

Theorem 1. *Let $\mathcal{L}$ be an intermediate logic. If $\mathcal{L}$ has the interpolation property, then $\mathrm{Var}\mathcal{L}$ is amalgamable.*

Proof. Let $\mathcal{L}$ be an intermediate logic and suppose $\mathfrak{A}_0$, $\mathfrak{A}_1$, $\mathfrak{A}_2 \in \mathrm{Var}\mathcal{L}$ and $f_i : \mathfrak{A}_0 \hookrightarrow \mathfrak{A}_i$ for $i \in \{1, 2\}$ is an embedding. Hence $\mathfrak{A}_0$ is isomorphic both to some subalgebra of $\mathfrak{A}_1$ and to some subalgebra of $\mathfrak{A}_2$, so we may suppose that $\mathfrak{A}_0$ *is* a subalgebra of both of the $\mathfrak{A}_i$'s and thus that the f_i's are identity maps[2]. We want to show that there is an algebra $\mathfrak{A}$ in $\mathrm{Var}\mathcal{L}$ and embeddings g_1 and g_2 of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ into $\mathfrak{A}$ over $\mathfrak{A}_0$. To this end, associate with every $a \in A_i$ a propositional variable x_a^i such that if $a \in A_0$ (and thus in $A_1 \cap A_2$), $x_a^0 = x_a^1 = x_a^2$ and $x_a^i \neq x_b^j$ if $a, b \notin A_0$ and $i \neq j$. For each i, let $\mathcal{F}_i$ be the propositional language based on the variables x_a^i for $a \in A_i$ and let $\mathcal{F}$ be the language based on all of the variables x_a^i. We may suppose that $\mathcal{F}$ is the language of $\mathcal{L}$. Fix a valuation $\mathfrak{V}_i$ of $\mathcal{F}_i$ in $\mathfrak{A}_i$ by $\mathfrak{V}_i(x_a^i) = a$. Then define

$$T_i = \{\varphi \in \mathcal{F}_i \,|\, \mathfrak{V}_i(\varphi) = \top\}.$$

Note that T_i is closed under modus ponens. Indeed, suppose $\mathfrak{V}_i(\varphi \rightarrow \psi) = \mathfrak{V}_i(\varphi) \rightarrow \mathfrak{V}_i(\psi) = \top$ and $\mathfrak{V}_i(\varphi) = \top$. Then $\top \leq \mathfrak{V}_i(\varphi \rightarrow \psi)$ and thus $\top \wedge \mathfrak{V}_i(\varphi) = \top \leq \mathfrak{V}_i(\psi) = \top$. Since $\mathfrak{A}_i \in \mathrm{Var}\mathcal{L}$, we also have $\mathcal{L} \cap \mathcal{F}_i \subseteq T_i$. Now let

$$T = \{\varphi \in \mathcal{L} \,|\, T_1 \cup T_2 \vdash_{\mathcal{L}}^{\mathrm{MP}} \varphi\},$$

where $\vdash_{\mathcal{L}}^{\mathrm{MP}}$ means provable in $\mathcal{L}$ using only modus ponens. The main move in the proof of Theorem 1 is an argument showing that for $\varphi \in \mathcal{F}_i$ and $\psi \in \mathcal{F}_j$ $(i, j \in \{1, 2\})$,

$$(+)\; \varphi \leftrightarrow \psi \in T \text{ iff there exists } \chi \in \mathcal{F}_0 \text{ s.th. } \varphi \rightarrow \chi \in T_i \text{ and } \chi \rightarrow \psi \in T_j$$

[2] We will suppose throughout this paper in verifying the property of amalgamation that $\mathfrak{A}_0$ is a common subalgebra of $\mathfrak{A}_1$ and $\mathfrak{A}_2$.

Letting $\varphi = \top$ in (+) we obtain $\psi \in T$ iff there exists a $\chi \in \mathcal{F}_0$ such that $\chi \in T_i$ and $\chi \to \psi \in T_j$. This means that $\mathfrak{V}_j(\chi) = \mathfrak{V}_i(\chi) = \top$ and since $\mathfrak{V}_j(\chi \to \psi) = \top$, we get $\mathfrak{V}_j(\psi) = \top$, hence $\psi \in T_j$. It follows that for $\varphi \in \mathcal{F}_i$, $i \in \{1, 2\}$

(++) $\varphi \in T_i$ iff $\varphi \in T$.

This equivalence will be crucial for our proof, so we see now why we close T only under modus ponens and not substitution. If we close T also under substitution, we may get a $\varphi \in \mathcal{F}_1$ such that $\varphi \in T$, but $\varphi \notin T_1$, rather $\varphi \in T_2$ since $\mathfrak{A}_2 \models \mathfrak{V}_2(\varphi) = \top$. Now for the proof of (+). The if-part is obvious, for if $\varphi \to \chi \in T_i$ and $\chi \to \psi \in T_j$, then $\varphi \to \psi \in T$ for $\mathcal{L}$. The only if-part is more involved. Suppose $\varphi \to \psi \in T$ for $\varphi \in \mathcal{F}_i$ and $\psi \in \mathcal{F}_j$. There are then finite sets $\Gamma_i \subseteq T_i$ and $\Gamma_j \subseteq T_j$ such that $\Gamma_i \cup \Gamma_j \vdash^{\mathrm{MP}}_{\mathcal{L}} \varphi \to \psi$. By the deduction theorem for $\mathcal{L}$, $\vdash^{\mathrm{MP}}_{\mathcal{L}} \bigwedge \Gamma_i \cup \Gamma_j \to (\varphi \to \psi)$ and this is equivalent to $\vdash^{\mathrm{MP}}_{\mathcal{L}} \bigwedge \Gamma_i \wedge \varphi \to (\bigwedge \Gamma_j \to \psi)$. So, since $\mathcal{L}$ has the interpolation property, there is a $\chi \in \mathcal{F}_0$ such that $\vdash^{\mathrm{MP}}_{\mathcal{L}} \bigwedge \Gamma_i \wedge \varphi \to \chi$ and $\vdash^{\mathrm{MP}}_{\mathcal{L}} \chi \to (\bigwedge \Gamma_j \to \psi)$. Hence $\vdash^{\mathrm{MP}}_{\mathcal{L}} \bigwedge \Gamma_i \to (\varphi \to \chi)$ and so $\varphi \to \chi \in T_i$ since T_i is closed under MP. Similarly, $\vdash^{\mathrm{MP}}_{\mathcal{L}} \bigwedge \Gamma_j \to (\chi \to \psi)$, so $\chi \to \psi \in T_j$.
Now define an algebra $\mathfrak{A}$ as follows. The domain of $\mathfrak{A}$ is the set $\mathcal{F}/\sim$ of equivalence classes under the equivalence relation $\sim$ defined on $\mathcal{F}$ by

$\varphi \sim \psi$ iff $\varphi \leftrightarrow \psi \in T$.

We let the $\bot$ of $\mathfrak{A}$ be $\|\bot\|_\sim$ and define the operations $\odot$ in $\mathfrak{A}$ by $\|\varphi \odot \psi\|_\sim = \|\varphi\|_\sim \odot \|\psi\|_\sim$. That $\odot$ is well-defined follows from the fact that the replacement theorem holds for $\mathcal{L}$: If $T_i \cup T_j \vdash^{\mathrm{MP}}_{\mathcal{L}} (\varphi \leftrightarrow \varphi') \wedge (\psi \leftrightarrow \psi')$ and $T_i \cup T_j \vdash^{\mathrm{MP}}_{\mathcal{L}} \varphi \odot \psi$, then $T_i \cup T_j \vdash^{\mathrm{MP}}_{\mathcal{L}} \varphi' \odot \psi'$ by the replacement theorem. To see that $\mathfrak{A}$ is in Var$\mathcal{L}$, note first that $\mathfrak{A} \models \varphi$ iff $\varphi \leftrightarrow \top \in T$ iff there are finite sets $\Gamma_1 \subseteq T_1$ and $\Gamma_2 \subseteq T_2$ such that $\vdash^{\mathrm{MP}}_{\mathcal{L}} \bigwedge \Gamma_1 \cup \Gamma_2 \to \varphi$. But from the definition of T_i and the fact that $\mathfrak{A}_i \in$Var$\mathcal{L}$, we have $\Gamma_1 \cup \Gamma_2 \subseteq \mathcal{L}$ and so $\vdash^{\mathrm{MP}}_{\mathcal{L}} \bigwedge \Gamma_1 \cup \Gamma_2 \to \varphi$ implies $\varphi \in \mathcal{L}$. On the other hand, if $\varphi \in \mathcal{L}$, we obviously have $\varphi \leftrightarrow \top \in T$ and thus $\mathfrak{A} \models \varphi$.
We now show that $\mathfrak{A}$ is indeed the algebra we are looking for. Consider the maps $g_i : \mathfrak{A}_i \to \mathfrak{A}$; $a \mapsto \|x^i_a\|_\sim$ for $i \in \{1, 2\}$. To prove that g_i and a homomorphism consider the following. If $\varphi(a_1, ..., a_n) = \psi(b_1, ..., b_m)$ for $a_1, ..., a_n, b_1, ..., b_m \in A_i$ then $\varphi(x^i_{a_1}, ..., x^i_{a_n}) \leftrightarrow \psi(x^i_{b_1}, ..., x^i_{b_m}) \in T_i$, so

$$\|\varphi(x^i_{a_1}, ..., x^i_{a_n})\|_\sim = \|\psi(x^i_{b_1}, ..., x^i_{b_m})\|_\sim.$$

Hence, since

$$\|\varphi(x^i_{a_1}, ..., x^i_{a_n})\|_\sim = \|x^i_{\varphi(a_1, ..., a_n)}\|_\sim,$$

we get $\|x^i_{\varphi(a_1,...,a_n)}\|_\sim = \psi(\|x^i_{b_1}\|_\sim, ..., \|x^i_{b_m}\|_\sim)$. To show that g_i is well-defined means to show that if $\varphi(a_1, ..., a_n) = \psi(b_1, ..., b_m)$, then $\|x_{\varphi(x^i_{a_1}, ..., x^i_{a_n})}\|_\sim = \|x_{\psi(x^i_{b_1}, ..., x^i_{b_m})}\|_\sim$. But that follows directly from the argument above.
To show that g_i is a homomorphism suppose that $a = b \odot c$ for $a, b, c \in A_i$. As a special case of the above we get $\|x^i_a\|_\sim = \|x^i_b\|_\sim \odot \|x^i_c\|_\sim$ and so

$$g_i(a) = \|x_a^i\|_\sim = \|x_b^i\|_\sim \odot \|x_c^i\|_\sim = g_i(b) \odot g_i(c).$$

Also, because $\mathfrak{V}_i(x_\perp^i) = \perp$, we have $x_\perp^i \leftrightarrow \perp \in T$, hence $g_i(\perp) = \|x_\perp^i\|_\sim = \|\perp\|_\sim = \perp$. Hence, g_i is a homomorphism.
It remains to show that g_i is injective. For that, we use the right to left direction of (++). Suppose $\|\varphi(x_{a_1}^i, ..., x_{a_n}^i)\|_\sim = \|\psi(x_{b_1}^i, ..., x_{b_n}^i\|_\sim$. To prove the injectivity of g_i, we need to show that $\varphi(a_1, ..., a_n) = \psi(b_1, ..., b_m)$. But since $\varphi(x_{a_1}^i, ..., x_{a_n}^i), \psi(x_{b_1}^i, ..., x_{b_m}^i) \in \mathcal{F}_i$, it follows from (++) that $\varphi(x_{a_1}^i, ..., x_{a_n}^i) \leftrightarrow \psi(x_{b_1}^i, ..., x_{b_m}^i) \in T_i$. By the definition of $\mathfrak{V}_i$ we then get

$$\varphi(a_1, ..., a_n) = \mathfrak{V}_i(\varphi(x_{a_1}^i, ..., x_{a_n}^i)) = \mathfrak{V}(\psi(x_{b_1}^i, ..., x_{b_m}^i)) = \psi(b_1, ..., b_m).$$

In particular $\varphi(a_1, ..., a_n) = \psi(b_1, ..., b_m)$, so g_i is injective. This completes the proof of Theorem 1.

Theorem 1 gives us a way to prove that $\mathcal{L}$ does not have the interpolation property: show that Var$\mathcal{L}$ is not amalgamable. We will not be able to utilize that technique in this paper.
We will now prove that the amalgamability of Var$\mathcal{L}$ is also a sufficient condition for $\mathcal{L}$ having the interpolation property. For that we need the following technical lemma.

Lemma 2. *Let $\mathfrak{A}_0$, $\mathfrak{A}_1$, $\mathfrak{A}_2$ be Heyting algebras and suppose that $\mathfrak{A}_0$ is a subalgebra of both $\mathfrak{A}_1$ and of $\mathfrak{A}_2$. Let $a \in A_1$ and $b \in A_2$ and suppose further that there is no $c \in A_0$ such that $a \leq_1 c \leq_2 b$ ($\leq_i$ is the partial order in $\mathfrak{A}_i$). Then there are prime filters $\nabla_1 \subseteq A_1$ and $\nabla_2 \subseteq A_2$ such that $a \in \nabla_1$ and $b \notin \nabla_2$ and such that $\nabla_1 \cap A_0 = \nabla_2 \cap A_0$.*

Proof. Consider

$$X = \{x \in A_0 \,|\, a \leq_1 x\},\; Y = \{y \in A_0 \,|\, y \leq_2 b\}.$$

By our assumption, $X \cap Y = \emptyset$. Define

$$\Sigma_1 = \{\Delta \subseteq A_2 \,|\, \Delta = (\Delta]_2,\; \{b\} \cup Y \subset \Delta,\; X \cap \Delta = \emptyset\}.$$

Since $(b]_2 \in \Sigma_1$ and since every chain $(\Delta_i)_{i \in \mathcal{I}}$ in Σ_1 has an upper bound (i.e the union of the chain) in Σ_1, Zorn's lemma applies and gives a maximal $\Delta_2 \in \Sigma_1$. The ideal Δ_2 is prime. To see this, suppose $x \vee y \in \Delta_2$ but $x \notin \Delta_2$ and $y \notin \Delta_2$. That Δ_2 is maximal in Σ_1 implies

$$X \cap (\{x\} \cup \Delta_2]_2 \neq \emptyset \neq X \cap (\{y\} \cup \Delta_2]_2.$$

So there are $u, v \in \Delta_2$ such that $a \leq_1 x \vee u$ and $a \leq_1 y \vee v$ and also $x \vee u, y \vee v \in A_0$. It follows that $(x \vee u) \wedge (y \vee v) \in X \cap \Delta_2$, contradicting $X \cap \Delta_2 = \emptyset$.

Let $\nabla_2 = A_2 - \Delta_2$. Since, Δ_2 is a prime ideal, it follows that ∇_2 is a prime filter. Let $\nabla_0 = \nabla_2 \cap A_0$ and $\Delta_0 = \Delta_2 \cap A_0$. Consider

$$\Sigma_1 = \{\nabla \subseteq A_1 \,|\, \nabla = [\nabla)_1,\ \{a\} \cup \nabla_0 \subseteq \nabla,\ \nabla \cap \Delta_0 = \emptyset\}.$$

Suppose that $x \in [\{a\} \cup \nabla_0)_1 \cap \Delta_0$. Then, for some $z \in \Delta_0$, we have $a \leq_1 z \to x$ and $x \in \Delta_0$. Hence, $z \to x \in X \subseteq \nabla_0$, and since $z \in \nabla_0$, $x \in \nabla_0$, contradicting $\nabla_0 \cap \Delta_0 = \emptyset$. It follows that $\Sigma_2 \neq \emptyset$, so by applying Zorn's lemma, we obtain a maximal element ∇_1. As above, we can prove that ∇_1 is a prime filter.
By definition, $a \in \nabla_1$ and $b \notin \nabla_2$. Also, $\nabla_1 \cap A_0 = \nabla_2 \cap A_0$, as is easily seen.

With this lemma at hand, we are ready to go on and state and prove

Theorem 3. *Let $\mathcal{L}$ be an intermediate logic. If Var$\mathcal{L}$ is amalgamable then $\mathcal{L}$ has the interpolation property.*

Proof. Suppose that $\mathcal{L}$ is intermediate, Var$\mathcal{L}$ is amalgamable, but for some $\varphi(p_1, ..., p_n, q_1, ..., q_m)$ and $\psi(q_1, ..., q_m, r_1, ..., r_l)$ there is no $\chi(q_1, ..., q_m)$ such that $\varphi \to \chi \in \mathcal{L}$ and $\chi \to \psi \in \mathcal{L}$. We will show that $\varphi \to \psi \notin \mathcal{L}$. This will be done by defining an algebra $\mathfrak{A} \in$Var$\mathcal{L}$ such that $\mathfrak{A} \not\models \varphi \to \psi$. Let $\mathfrak{A}'_0$, $\mathfrak{A}'_1$ and $\mathfrak{A}'_2$ be the free algebras in Var$\mathcal{L}$ on $\{q_1, ..., q_m\}$, $\{p_1, ..., p_n, q_1, ..., q_m\}$ and $\{q_1, ..., q_m, r_1, ..., r_l\}$ respectively. Note that in every Heyting algebra we have $a \to b$ iff $a \leq b$. Our assumption on φ, ψ and $\mathcal{L}$ is thus equivalent to the assumption that there is no $\chi(q_1, ..., q_n)$ such that $\varphi(p_1, ..., p_n, q_1, ..., q_m) \leq_1 \chi(q_1, ..., q_m) \leq_2 \psi(q_1, ..., q_m, r_1, ..., r_l)$. So by Lemma 2 it follows that there are filters $\nabla_1 \subseteq \mathfrak{A}_1$ and $\nabla_2 \subseteq \mathfrak{A}_2$ such that $\varphi(p_1, ..., p_n, q_1, ..., q_m) \in \nabla_1$ and $\psi(q_1, ..., q_m, r_1, ..., r_l) \notin \nabla_2$ and such that $\nabla_1 \cap A'_0 = \nabla_2 \cap A'_0$. Let $\mathfrak{A}_1 = \mathfrak{A}'_1/\nabla_1$ and $\mathfrak{A}'_2/\nabla_2$. As noted earlier we have

$$\|\varphi(p_1, ..., p_n, q_1, ..., q_m\|_{\nabla_1} = \top \text{ and } \|\psi(q_1, ..., q_m, r_1, ..., r_l\|_{\nabla_2} \neq \top.$$

Define $\mathfrak{A}_0$ by letting $A_0 = \{\|a\|_{\nabla_1} \cap A'_0 \,|\, a \in A'_0\}$. Then $\mathfrak{A}_0$ with the operations $\wedge$, $\vee$ and $\to$ induced by $\mathfrak{A}_1$ is a Heyting algebra and it is naturally embedded in $\mathfrak{A}_1$ by the map $f_1 : \|a\|_{\nabla_1} \cap A'_0 \mapsto \|a\|_{\nabla_1}$. Using amalgamability we will show that $f_2 : \|a\|_{\nabla_1} \cap A'_0 \mapsto \|a\|_{\nabla_2}$ is an embedding. We must first show that f_2 is well-defined. For this, let $a, b \in A'_0$ and consider:

$$\begin{aligned} \|a\|_{\nabla_1} \cap A'_0 = \|b\|_{\nabla_1} \cap A'_0 &\text{ iff } a \leftrightarrow b \in \nabla_1 \\ &\text{ iff } a \leftrightarrow b \in \nabla_2 \\ &\quad (\text{since } \nabla_1 \cap A'_0 = \nabla_2 \cap A'_0) \\ &\text{ iff } \|a\|_{\nabla_2} = \|b\|_{\nabla_2} \\ &\text{ iff } f_2(\|a\|_{\nabla_1} \cap A'_0) = f_2(\|a\|_{\nabla_1} \cap A'_0) \end{aligned}$$

Note that this line of equations also shows f_2 to be injective. That f_2 is homomorphic follows from

$$\begin{aligned} f_2(\|a\|_{\nabla_1} \cap A'_0 \odot \|b\|_{\nabla_1} \cap A'_0) &= f_2(\|a \odot b\|_{\nabla_1} \cap A'_0) \\ &= \|a \odot b\|_{\nabla_2} = \|a\|_{\nabla_2} \odot \|b\|_{\nabla_2} \\ &= f_2(\|a\|_{\nabla_1}) \odot f_2(\|b\|_{\nabla_1}) \end{aligned}$$

Since Var$\mathcal{L}$ is amalgamable, there is an algebra $\mathfrak{A}$ and embeddings g_1 and g_2 from $\mathfrak{A}_1$ and $\mathfrak{A}_2$ into $\mathfrak{A}$ respectively such that for $a \in A_0'$ $g_1(f_1(a)) = g_2(f_2(a))$. To show that $\mathfrak{A} \not\models \varphi \to \psi$, let $\mathfrak{V}$ be a valuation of $\mathcal{L}$ in $\mathfrak{A}$ such that

$$\begin{array}{ll} \mathfrak{V}(p_i) = g_1(\|p_i\|_{\nabla_1}) & \text{for } i = 1, ..., n \\ \mathfrak{V}(q_i) = g_1(\|q_i\|_{\nabla_1}) = g_2(\|q_i\|_{\nabla_2}) & \text{for } i = 1, ..., m \\ \mathfrak{V}(r_i) = g_2(\|r_i\|_{\nabla_2}) & \text{for } i = 1, ..., l \end{array}$$

We then have

$$\mathfrak{V}(\varphi(p_1, ..., p_n, q_1, ..., q_m)) = g_1(\|\varphi(p_1, ..., p_n, q_1, ..., q_m)\|_{\nabla_1}) = g_1(\top) = \top$$

and

$$\mathfrak{V}(\psi(q_1, ..., q_m, r_1, ..., r_l)) = g_2(\|\psi(q_1, ..., q_m, r_1, ..., r_l)\|_{\nabla_2}) \neq g_2(\top) = \top,$$

from which it follows that $\mathfrak{A} \not\models \varphi \to \psi$ and since $\mathfrak{A} \in$Var$\mathcal{L}$, $\varphi \to \psi \notin \mathcal{L}$. This completes the proof.

Note that we didn't use the fact that the filters ∇_1 and ∇_2 are prime, as promised by Lemma 2. But that fact can be used to obtain a useful corollary stated below. An algebra $\mathfrak{A}$ is **well-connected** if for every $a, b \in \mathfrak{A}$, $a \vee b = 1$ implies $a = 1$ or $b = 1$. For a variety $\mathcal{V}$, and a subclass $\mathcal{C} \subseteq \mathcal{V}$, say that $\mathcal{C}$ is amalgamable in $\mathcal{V}$ if for every $\mathfrak{A}_0$, $\mathfrak{A}_1$ and $\mathfrak{A}_2$ in $\mathcal{C}$ such that $\mathfrak{A}_0$ is a common subalgebra of $\mathfrak{A}_1$ and $\mathfrak{A}_2$, there is an $\mathfrak{A}$ in $\mathcal{V}$ and embeddings $g_i : \mathfrak{A}_i \hookrightarrow \mathfrak{A}$ for $i \in \{1, 2\}$.

Corollary 4. *Let $\mathcal{L}$ be an intermediate logic. If the class of well-connected algebras in Var$\mathcal{L}$ is amalgamable in Var$\mathcal{L}$, then $\mathcal{L}$ has the interpolation property.*

Proof. We refer to the proof of Theorem 3. Since the filters ∇_1 and ∇_2 are prime, it follows that the algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are well-connected. For $\|a\|_{\nabla_i} \vee \|b\|_{\nabla_i} = \top$ iff $a \vee b \in \nabla_i$ iff $a \in \nabla_i$ or $b \in \nabla_i$ iff $\|a\|_{\nabla_i} = \top$ or $\|b\|_{\nabla_i} = \top$. Obviously, $\mathfrak{A}_0$ is then also well-connected. So if the class of well-connected algebras in Var$\mathcal{L}$ is amalgamable in Var$\mathcal{L}$, there is an $\mathfrak{A} \in$Var$\mathcal{L}$ verifying $\mathfrak{A} \not\models \varphi \to \psi$.

Corollary 5. *Let $\mathcal{L}$ be an intermediate logic. Then $\mathcal{L}$ has the interpolation property iff the class of well-connected algebras in Var$\mathcal{L}$ is amalgamable in Var$\mathcal{L}$.*

Proof. Immediate from Theorem 1 and Corollary 4.

4 Amalgamable Algebras

We are now going to classify some amalgamable varieties of Heyting algebras (these varieties turn out to be the only ones). In view of Theorem 3 and the correspondence between varieties of Heyting algebras and intermediate logics, by doing this, we classify some logics which have the amalgamation property. In

other words, what follows are some exercises in universal algebra, however, by using Theorem 3 we can draw conclusions about logics from those exercises.
The varieties $\mathcal{V}$ we will look at can be defined by giving a propositional formula φ and saying that $\mathfrak{A} \in \mathcal{V}$ iff $\mathfrak{A} \models \varphi$. We will be concerned with the following varieties of Heyting algebras. We use Maksimova's notation (see [3]) and denote by $\mathcal{H}_1$ the variety of all Heyting algebras and for $i = 2, \ldots 7$ $\mathcal{H}_i$ is the variety characterized by the formulas as listed below.

$\mathcal{H}_2$: $\neg p \vee \neg\neg p$
$\mathcal{H}_3$: $p \vee (p \to (q \vee \neg q))$
$\mathcal{H}_4$: $p \vee (p \to (q \vee \neg q)), (p \to q) \vee (q \to p) \vee (p \leftrightarrow \neg q)$
$\mathcal{H}_5$: $p \vee (p \to (q \vee \neg q)), \neg p \vee \neg\neg p$
$\mathcal{H}_6$: $(p \to q) \vee (q \to p)$
$\mathcal{H}_7$: $p \vee \neg p$

As should be known, $\mathcal{H}_7$ is the variety corresponding to **CL**, i.e the variety of boolean algebras. We take it as a fact that **CL** has the interpolation property (for a proof see [2]) and thus that $\mathcal{H}_7$ is amalgamable. We will thus concentrate on the varieties $\mathcal{H}_1 - \mathcal{H}_6$. The proofs showing that the varieties $\mathcal{H}_1 - \mathcal{H}_5$ are amalgamable all rely on two constructions. One of them turns any partially ordered set $\langle S, \leq \rangle$ into a Heyting algebra $\mathcal{B}(S)$. The other relates to three Heyting algebras $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2$ such that $\mathfrak{A}_0$ is a common subalgebra of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ a partial order $\langle S, \leq \rangle$ which in some cases contains a subset S' such that $\mathcal{B}(S')$ verifies the amalgamability of $\mathfrak{A}_0$, $\mathfrak{A}_1$ and $\mathfrak{A}_2$.
Let $\langle S, \leq \rangle$ be a partially ordered set. We define the structure

$$\mathcal{B}(S) = \langle \mathrm{Up}S, \cap, \cup, \to, \emptyset \rangle,$$

where

$$\mathrm{Up}S = \{U \subseteq S \mid x \in U \text{ and } x \leq y \text{ implies } y \in U\},$$

and

$$U \to V = \{x \in S \mid x \leq y \text{ and } y \in U \text{ implies } y \in V\}.$$

It is easily seen that this definition makes $\mathcal{B}(S)$ into a Heyting algebra.
To each Heyting algebra $\mathfrak{A}$ we can associate a partially ordered set $\langle S_{\mathfrak{A}}, \subseteq \rangle$, where $S_{\mathfrak{A}}$ is the set of prime filters in $\mathfrak{A}$. Hence, to every Heyting algebra $\mathfrak{A}$, there corresponds another Heyting algebra $\mathcal{B}(S_{\mathfrak{A}})$. Indeed, we have the following

Lemma 6. *The mapping $F_{\mathfrak{A}} \colon \mathfrak{A} \to \mathcal{B}(S_{\mathfrak{A}})$ defined by*

$$a \mapsto \{\nabla \in S_{\mathfrak{A}} \mid a \in \nabla\}$$

is an embedding.

Proof. Note first that $F_{\mathfrak{A}}$ is well-defined, that is, $F_{\mathfrak{A}}(a)$ is an up-set for every $a \in A$ ($\emptyset$ is an up-set). For the homomorphism property of $F_{\mathfrak{A}}$, we just compute:

$$F_{\mathfrak{A}}(a \wedge b) = \{\nabla \in S_{\mathfrak{A}} \mid a \wedge b \in \nabla\} = \{\nabla \in S_{\mathfrak{A}} \mid a \in S_{\mathfrak{A}}\} \cap \{\nabla \in S_{\mathfrak{A}} \mid b \in S_{\mathfrak{A}}\},$$

where the last equality follows from the fact that $a \wedge b \in \nabla$ iff $a \in \nabla$ and $b \in \nabla$. Since $S_{\mathfrak{A}}$ only contains prime filters, we also have $a \vee b \in \nabla$ iff $a \in \nabla$ or $b \in \nabla$ for $\nabla \in S_{\mathfrak{A}}$ and hence

$$F_{\mathfrak{A}}(a \vee b) = \{\nabla \in S_{\mathfrak{A}} \mid a \vee b \in \nabla\} = \{\nabla \in S_{\mathfrak{A}} \mid a \in \nabla\} \cup \{\nabla \in S_{\mathfrak{A}} \mid b \in \nabla\}.$$

Since prime filters are assumed to be proper, we also have

$$F_{\mathfrak{A}}(\bot) = \{\nabla \in S_{\mathfrak{A}} \mid \bot \in \nabla\} = \emptyset.$$

For $\rightarrow$, we have to prove

$$F_{\mathfrak{A}}(a \rightarrow b)(= \{\nabla \in S_{\mathfrak{A}} \mid a \rightarrow b \in \nabla\}) = \{\nabla \in S_{\mathfrak{A}} \mid a \in \nabla\} \rightarrow \{\nabla \in S_{\mathfrak{A}} \mid b \in \nabla\}.$$

$\subseteq$: Suppose $\nabla \in S_{\mathfrak{A}}$ is such that $a \rightarrow b \in \nabla$. We want to show that $\nabla \subseteq \nabla'$ and $a \in \nabla'$ implies $b \in \nabla'$. But this is obvious, for $a \rightarrow b \in \nabla'$ and $a \wedge (a \rightarrow b) \leq b$.
$\supseteq$: Suppose $\nabla \in S_{\mathfrak{A}}$ is such that if $\nabla \subseteq \nabla'$ and $a \in \nabla'$ then $b \in \nabla'$. We need to show that $a \rightarrow b \in \nabla$. Suppose otherwise. Then there is no $c \in \nabla$ such that $c \wedge a \leq b$. Consider

$$\varSigma = \{\nabla_0 \mid \nabla_0 \text{ is a filter, } [\nabla, a) \subseteq \nabla_0 \text{ and } b \notin \nabla_0\}.$$

We have $[\nabla, a) \in \varSigma$, so by Zorn's lemma there is a maximal $\nabla' \in \varSigma$. We prove that ∇' is prime. Suppose otherwise. By maximality, there are then $c_0, c_1, c_2 \in \nabla$ and $u, v \in A$ such that

$$c_0 \wedge a \leq u \vee v \text{ and } c_1 \wedge a \wedge u \leq b \text{ and } c_2 \wedge a \wedge v \leq b.$$

This implies that $b \in \nabla'$ contradicting $\nabla' \in \varSigma$. Obviously, $\nabla \subseteq \nabla'$ and $a \in \nabla$ but $b \notin \nabla'$, contradicting the assumption on ∇.

Now for the construction that will help us verify the amalgamation property for certain varieties. Let $\mathfrak{A}_0$, $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be Heyting algebras such that $\mathfrak{A}_0$ is a common subalgebra of $\mathfrak{A}_1$ of $\mathfrak{A}_2$. Consider the following subset of $S_{\mathfrak{A}_1} \times S_{\mathfrak{A}_2}$:

$$\mathcal{S}(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2) = \{\langle \nabla_1, \nabla_2 \rangle \in S_{\mathfrak{A}_1} \times S_{\mathfrak{A}_2} \mid \nabla_1 \cap A_0 = \nabla_2 \cap A_0\}$$

Define $\leq$ on $\mathcal{S}(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2)$ by $\langle \nabla_1, \nabla_2 \rangle \leq \langle \nabla_1', \nabla_2' \rangle$ iff $\nabla_1 \subseteq \nabla_1'$ and $\nabla_2 \subseteq \nabla_2'$. We will now single out the subsets of $\mathcal{S}(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2)$ that will be of interest to us.

Lemma 7. *Suppose $\tilde{S} \subseteq \mathcal{S}(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2)$ satisfies the following conditions:*

1a) *For all $\nabla_1 \in S_{\mathfrak{A}_1}$ there exists $\nabla_2 \in S_{\mathfrak{A}_2}$ such that $\langle \nabla_1, \nabla_2 \rangle \in \tilde{S}$*
1b) *For all $\nabla_2 \in S_{\mathfrak{A}_2}$ there exists $\nabla_1 \in S_{\mathfrak{A}_1}$ such that $\langle \nabla_1, \nabla_2 \rangle \in \tilde{S}$*
2a) *$\langle \nabla_1, \nabla_2 \rangle \in \tilde{S}$ and $\nabla_1 \subseteq \nabla_1' \in S_{\mathfrak{A}_1}$ implies $\langle \nabla_1', \nabla_2' \rangle \in \tilde{S}$ and $\nabla_2 \subseteq \nabla_2'$ for some $\nabla_2' \in S_{\mathfrak{A}_2}$*
2b) *$\langle \nabla_1, \nabla_2 \rangle \in \tilde{S}$ and $\nabla_2 \subseteq \nabla_2' \in S_{\mathfrak{A}_2}$ implies $\langle \nabla_1', \nabla_2' \rangle \in \tilde{S}$ and $\nabla_1 \subseteq \nabla_1'$ for some $\nabla_1' \in S_{\mathfrak{A}_1}$*

Then the map $G_k : \mathfrak{A}_k \to \mathcal{B}(\tilde{S})$ *defined by*

$$a \mapsto \{\langle \nabla_1, \nabla_2 \rangle \in \tilde{S} \,|\, a \in \nabla_k\}$$

for $k \in \{1, 2\}$ *is a homomorphism.*

Proof. To prove this we use Lemma 6. For, immediately from the definitions we have

$$\langle \nabla_1, \nabla_2 \rangle \in G_k(a) \text{ iff } \nabla_k \in F_{\mathfrak{A}_\mathfrak{k}}(a).$$

Using the conditions **1a)–2b)** we prove that G_1 is a homomorphism thus:

$$\begin{aligned}
&\langle \nabla_1, \nabla_2 \rangle \in G_1(a \to b) \Longrightarrow \nabla_1 \in F_{\mathfrak{A}_1}(a \to b) = \\
&F_{\mathfrak{A}_1}(a) \to F_{\mathfrak{A}_1}(b) \Longrightarrow \text{ for all } \nabla_1' \supseteq \nabla_1 : \\
&\nabla_1' \in F_{\mathfrak{A}_1}(a) \text{ implies } \nabla_1' \in F_{\mathfrak{A}_1}(b) \Longrightarrow \text{ for all } \nabla_1' \supseteq \nabla_1, \\
&\text{for all } \nabla_2' \supseteq \nabla_2 : \langle \nabla_1', \nabla_2' \rangle \in \tilde{S} \text{ and} \\
&\langle \nabla_1', \nabla_2' \rangle \in G_1(a) \text{ implies } \langle \nabla_1', \nabla_2' \rangle \in G_2(b) \Longrightarrow \\
&\langle \nabla_1, \nabla_2 \rangle \in G_1(a) \to G_1(b).
\end{aligned}$$

In the other direction, we have:

$$\begin{aligned}
&\langle \nabla_1', \nabla_2' \rangle \notin G_1(a \to b) \Longrightarrow \nabla_1 \notin F_{\mathfrak{A}_1}(a \to b) = \\
&F_{\mathfrak{A}_1}(a) \to F_{\mathfrak{A}_2}(b) \Longrightarrow \text{ there is some } \nabla_1' \supseteq \nabla_1 \text{ such that} \\
&\nabla_1' \in F_{\mathfrak{A}_1}(a) - F_{\mathfrak{A}_1}(b) \Longrightarrow \text{ there are some } \nabla_1' \supseteq \nabla_1, \\
&\nabla_2' \supseteq \nabla_2 \text{ such that } \langle \nabla_1', \nabla_2' \rangle \in \tilde{S} \text{ and} \\
&\langle \nabla_1', \nabla_2' \rangle \in G_1(a) - G_2(b) \Longrightarrow \langle \nabla_1', \nabla_2' \rangle \notin G_1(a) \to G_1(b).
\end{aligned}$$

For $\wedge$, the computation goes:

$$\begin{aligned}
&\langle \nabla_1, \nabla_2 \rangle \in G_1(a \wedge b) \Longleftrightarrow \nabla_1 \in F_{\mathfrak{A}_1}(a) \cap F_{\mathfrak{A}_2}(b) \Longleftrightarrow \\
&\nabla_1 \in \{\nabla \in S_{\mathfrak{A}_1} \,|\, a \in \nabla\} \cap \{\nabla \in S_{\mathfrak{A}_1} \,|\, b \in \nabla\} \Longleftrightarrow \\
&\langle \nabla_1, \nabla_2 \rangle \in G_1(a) \cap G_1(b).
\end{aligned}$$

The $\vee$ is proved analogously, substituting "$\wedge$" with "$\vee$" in the above. And, directly from the definition we have $G_1(\bot) = \bot$ since the ∇'s are proper. Hence, G_1 is a homomorphism. Exchanging "1" with "2" in the above proves that G_2 is a homomorphism.

Lemma 7 suggests a strategy for verifying that a variety $\mathcal{V}$ is amalgamable: Show that for every $\mathfrak{A}_0$, $\mathfrak{A}_1$ and $\mathfrak{A}_2$ in $\mathcal{V}$, there is an $\tilde{S} \subseteq \mathcal{S}(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2)$ which satisfies all of the conditions **1a) - 2b)** and $\mathcal{B}(\tilde{S}) \in \mathcal{V}$.

Lemma 8. *For any non-trivial* $\mathfrak{A}_1$ *and* $\mathfrak{A}_2$ *with common subalgebra* $\mathfrak{A}_0$*, the set* $\mathcal{S}(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2)$ *satisfies conditions **1a) - 2b)**.*

Proof. First, note the following. Let $\mathfrak{A}_0$ be a Heyting algebra and $\nabla_0 \subseteq A_0$ a prime filter. Suppose further that $\mathfrak{A}_0$ is a subalgebra of $\mathfrak{A}$. Define

$$\nabla_1 = \{x \in A \,|\, \text{ there is a } y \in \nabla_0 \text{ such that } y \leq x\},$$

i.e ∇_1 is the filter in $\mathfrak{A}$ generated by ∇_0. We have $\nabla_1 \cap (A_0 - \nabla_0) = \emptyset$, and moreover, ∇_1 is prime. For suppose $a \vee b \in \nabla_1$. There is then some $c \in \nabla_0$ such that $c \leq a \vee b$. This means $c \wedge (a \vee b) = c$ and so $(c \wedge a) \vee (c \wedge b) \in \nabla_0$. Since ∇_0 is prime we have $c \wedge a \in \nabla_0$ or $c \wedge b \in \nabla_0$. Both cases imply $a \in \nabla_1$ or $b \in \nabla_1$, that is ∇_1 is prime.
Now let $\nabla_1 \in S_{\mathfrak{A}_1}$. Note that $A_0 \cap \nabla_1$ is a prime filter in $\mathfrak{A}_0$. By the reasoning above $A_0 \cap \nabla_1$ can be extended to a $\nabla_2 \in S_{\mathfrak{A}_2}$ such that $A_0 \cap \nabla_1 = A_0 \cap \nabla_2$. It follows immediately that $\mathcal{S}(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2)$ has property **1a)** and, by analogy, **1b)**.
For property **2a)**, suppose $\langle \nabla_1, \nabla_2 \rangle \in \mathcal{S}(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2)$ and $\nabla_1 \subseteq \nabla'_1 \in S_{\mathfrak{A}_1}$. Let

$$\Sigma = \{\nabla'_2 \in S_{\mathfrak{A}_2} \mid \nabla_2 \subseteq \nabla'_2 \text{ and } \nabla'_2 \cap A_0 = \nabla'_1 \cap A_0\}$$

Let $\nabla = [\nabla_2 \cup (\nabla'_1 \cap A_0))_2$. In order to use Zorn's lemma, we show that $\nabla \in \Sigma$. So suppose $a \in \nabla \cap A_0$. There is then a $b \in \nabla_2$ and a $c \in \nabla'_1 \cap A_0$ such that $b \wedge c \leq a$. We have $b \leq c \to a$ and so $c \to a \in \nabla_2 \cap A_0 = \nabla_1 \cap A_0 \subseteq \nabla'_1 \cap A_0$. It follows thus that $a \in \nabla'_1 \cap A_0$. On the other hand, $\nabla'_1 \cap A_0 \subseteq \nabla \cap A_0$. Hence $\nabla \in \Sigma$, so by Zorn's lemma there is a maximal $\nabla'_2 \in \Sigma$. We show that ∇'_2 is prime. Suppose otherwise that $a \vee b \in \nabla'_2$ but $a \notin \nabla'_2$ and $b \notin \nabla'_2$ for some $a, b \in A_2$. By maximality of ∇'_2 and definition of Σ, this means that by adding a or b to ∇'_2 we get a filter not in Σ. Hence, there must be $v_a, v_b \in A_0 - \nabla'_1$ and $u_a, u_b \in \nabla'_2 \cap A_0$ such that $u_a \wedge a \leq v_a$ and $u_b \wedge b \leq v_b$. It follows that

$$(u_a \wedge a) \vee (u_b \wedge b) \leq v_a \vee v_b,$$

which is equivalent to

$$(u_a \vee u_b) \wedge (u_a \vee b) \wedge (u_b \vee a) \wedge (a \vee b) \leq v_a \vee v_b.$$

In this inequality, all of the conjuncts on the left are elements of ∇'_2. Hence $v_a \vee v_b \in \nabla'_2 \cap A_0 = \nabla'_1 \cap A_0$. Since ∇'_1 is prime, $v_a \in \nabla'_2$ or $v_b \in \nabla'_2$, contrary to assumption. We conclude that ∇'_2 is prime and thus $\langle \nabla'_1, \nabla'_2 \rangle \in \mathcal{S}(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2)$. This proves that $\mathcal{S}(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2)$ has property **2a)**. An analogous proof shows that $\mathcal{S}(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2)$ also has property **2b)**.

That $\mathcal{H}_1$, the class of all Heyting algebras, has the amalgamation property now follows immediately: As already proved, if $\mathfrak{A}_0$, $\mathfrak{A}_1$, $\mathfrak{A}_2$ are Heyting algebras with $\mathfrak{A}_0$ a common subalgebra of $\mathfrak{A}_1$ and $\mathfrak{A}_2$, the structure $\mathcal{B}(\mathcal{S}(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2))$ is also a Heyting algebra and it verifies the amalgamation property for the triple $\mathfrak{A}_0$, $\mathfrak{A}_1$, $\mathfrak{A}_2$. Since $\mathcal{H}_1$ corresponds to **IPC**, Theorem 3 implies

Theorem 9. IPC *has the interpolation property.*

For the arguments to come we point out that if a Heyting algebra $\mathfrak{A}$ is well-connected, then $S_{\mathfrak{A}}$ has a least element. Indeed, if $\mathfrak{A}$ is well-connected, then $\{\top\}$ is a prime filter and, obviously, with respect to inclusion $\{\top\}$ is the least filter in any Heyting algebra.

Also, a standard use of Zorn's lemma gives that for every $\nabla \in S_{\mathfrak{A}}$ there is a maximal $\nabla' \in S_{\mathfrak{A}}$ such that $\nabla \subseteq \nabla'$. That is, a union of elements of

$$\Sigma = \{\nabla_0 \in S_{\mathfrak{A}} \,|\, \nabla \subseteq \nabla_0\}$$

is again an element of Σ.

Lemma 10. *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be Heyting algebras with a common subalgebra $\mathfrak{A}_0$. Then*

1. *If ∇_{10} and ∇_{20} are the smallest elements of $S_{\mathfrak{A}_1}$ and $S_{\mathfrak{A}_2}$ respectively, then $\langle \nabla_{10}, \nabla_{20} \rangle \in \mathcal{S}(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2)$.*
2. *For any maximal $\nabla_1 \in S_{\mathfrak{A}_1}$, there exists a maximal $\nabla_2 \in S_{\mathfrak{A}_2}$ such that $\langle \nabla_1, \nabla_2 \rangle \in \mathcal{S}(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2)$.*
3. *For any maximal $\nabla_2 \in S_{\mathfrak{A}_2}$ there exists a maximal $\nabla_1 \in S_{\mathfrak{A}_3}$ such that $\langle \nabla_1, \nabla_2 \rangle \in \mathcal{S}(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2)$.*

Proof. 1. By Lemma 7 there is a $\nabla_2 \in S_{\mathfrak{A}_2}$ and a $\nabla_1 \in S_{\mathfrak{A}_1}$ such that $\langle \nabla_{10}, \nabla_2 \rangle \in \mathcal{S}(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2)$ and $\langle \nabla_1, \nabla_{20} \rangle \in \mathcal{S}(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2)$. But then

$$\nabla_{10} \cap A_0 \subseteq \nabla_1 \cap A_0 = \nabla_{20} \cap A_0 \subseteq \nabla_2 \cap A_0 = \nabla_{10} \cap A_0,$$

so $\langle \nabla_{10}, \nabla_{20} \rangle \in \mathcal{S}(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2)$.

2. Let ∇_1 be maximal in $S_{\mathfrak{A}_1}$. By Lemma 8 there is a $\nabla_2' \in S_{\mathfrak{A}_2}$ such that $\langle \nabla_1, \nabla_2' \rangle \in \mathcal{S}(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2)$. There is some maximal $\nabla_2 \in S_{\mathfrak{A}_2}$ such that $\nabla_2' \subseteq \nabla_2$. By property 2a) there is some $\nabla_1' \in S_{\mathfrak{A}_1}$ such that $\nabla_1 \subseteq \nabla_1'$ and $\langle \nabla_1', \nabla_2 \rangle \in \mathcal{S}(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2)$. But since ∇_1 is maximal, $\nabla_1 = \nabla_1'$ and thus $\langle \nabla_1, \nabla_2 \rangle \in \mathcal{S}(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2)$.

3. This is analogous to 2.

In the following we will look at $\mathcal{H}_2$ - $\mathcal{H}_5$ and for each case $\mathcal{H}_i$ prove that for certain triples $\mathfrak{A}_0$, $\mathfrak{A}_1$ and $\mathfrak{A}_2$ in $\mathcal{H}_i$, $S_{\mathfrak{A}_i}$ has properties such that we can find a $\tilde{S} \subseteq \mathcal{S}(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2)$ satisfying conditions **1a.) - 2b.)**. We start with $\mathcal{H}_2$, the variety of Heyting algebras validating $\neg p \vee \neg\neg p$.

Lemma 11. *a.) If $\mathfrak{A}$ is a non-trivial well-connected Heyting algebra such that $\mathfrak{A} \models \neg p \vee \neg\neg p$, then $S_{\mathfrak{A}}$ has a largest element.*
b.) If a partially ordered set $\langle S \leq \rangle$ has a largest element, then $\mathcal{B}(S) \models \neg p \vee \neg\neg p$.

Proof. a.) Let $\nabla = \{x \in A \,|\, a \neq \bot\}$ and suppose $a, b \in \nabla$. Then $\neg a (= a \to \bot) \neq \top$ and $\neg b \neq \top$, by the definition of $\to$. Since $\mathfrak{A} \models \neg p \vee \neg\neg p$ and $\mathfrak{A}$ is well-connected, we have $\neg\neg a = \top$ and $\neg\neg b = \top$. We take it as a fact that $\mathbf{IPC} \vdash \neg(p \wedge q) \leftrightarrow \neg(\neg\neg p \wedge \neg\neg q)$, and thus $\neg(a \wedge b) = \neg(\neg\neg a \wedge \neg\neg b) = \neg\top = \bot$. Since $\mathfrak{A}$ is non-trivial, $\bot \neq \top$, and so $a \wedge b \neq \bot$. It follows immediately that ∇

is a prime filter, and obviously it must be the greatest in $\mathfrak{A}$.
b.) Let m be the greatest element in $\langle S, \leq\rangle$. Let $X \in \mathrm{Up}S$. We consider two cases:
$m \in X$. Then

$$\neg X = X \to \bot = \{x \in S \mid x \leq y \text{ and } y \in X \text{ implies } y \in \emptyset\} = \emptyset,$$

since for every $x \in S$ we have $x \leq m$ and $m \in S$ by assumption. It follows that $\neg\neg X = S$.
$m \notin X$. Since X is an up-set we must have $X = \emptyset$, so $\neg X = S$
In both cases we have $\mathcal{B}(S) \models \neg p \vee \neg\neg p$.

By ordering what we have proved up till now we see that $\mathcal{H}_2$ is amalgamable. First, from Corollary 5 and Theorem 1 we get that a variety $\mathcal{V}$ is amalgamable iff the class of well-connected algebras in $\mathcal{V}$ is amalgamable in $\mathcal{V}$. Now let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be well-connected and in $\mathcal{H}_2$ with common subalgebra $\mathfrak{A}_0$. By Lemma 11 a.), $S_{\mathfrak{A}_1}$ and $S_{\mathfrak{A}_2}$ has largest elements ∇_1 and ∇_2 respectively. By lemma, $\langle\nabla_1, \nabla_2\rangle \in \mathcal{S}(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2)$. Obviously, $\langle\nabla_1, \nabla_2\rangle$ is the greatest element in $\mathcal{S}(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2)$. Hence $\mathcal{B}(\mathcal{S}(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2)) \in \mathcal{H}_2$ by Lemma 11 b.). It follows that $\mathcal{H}_2$ is amalgamable. From Theorem 3 we get

Theorem 12. *The logic* $\mathbf{IPC} + \neg p \vee \neg\neg p$ *has the interpolation property.*

Remember that $\mathcal{H}_3$ is the variety of Heyting algebras validating $p \vee (p \to (q \vee \neg q))$.

Lemma 13. *a.) Let $\mathfrak{A}$ be a non-trivial well-connected Heyting algebra such that $\mathfrak{A} \models p \vee (p \to (q \vee \neg q))$. Then $S_{\mathfrak{A}}$ has a smallest element and all other elements are maximal.*
b.) If $\langle S, \leq\rangle$ is partially ordered set with a minimal element and the rest maximal elements, then $\mathcal{B}(S) \in \mathcal{H}_3$.

Proof. a.) That $S_{\mathfrak{A}}$ has a minimal element follows from $\mathfrak{A}$ being well-connected. Suppose $\nabla \in S_{\mathfrak{A}}$ and $a \in \nabla$ with $a \neq \top$. Let $b \in A$ be arbitrary. By assumption $a \vee (a \to (b \vee \neg b)) = \top$, so by well-connectedness, $a \to (b \vee \neg b) = \top \in A$, so $b \vee \neg b \in \nabla$. Since ∇ is prime $b \in \nabla$ or $\neg b \in \nabla$. Hence, ∇ must be maximal.
b.) Let $\langle S \leq\rangle$ be a partially ordered set with a minimal element and the rest maximal elements. We want to show that for every $X \in \mathrm{Up}S$ either $X = S$ or else $X \subseteq (Y \cup \neg Y)$ for every $Y \in \mathrm{Up}S$. Denote by MAX the set of $\leq$-maximal elements in S. Then $\mathrm{Up}S = \{S\} \cup \wp(MAX)$. Suppose $X \neq \top = S$ and $Y \in \mathrm{Up}S$ arbitrary. There are two possibilities to consider.
$Y = S$. Then, obviously, $X \subseteq (Y \cup \neg Y)$.
$Y \neq S$. Then $Y \subseteq MAX$. It is easy to see that $\neg Y = MAX - Y$. Since also $X \subseteq MAX$, we have $X \subseteq (Y \cup \neg Y)$.

It is now easily seen that $\mathcal{H}_3$ is amalgamable. Suppose $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are non-trivial well-connected algebras in $\mathcal{H}_3$ with a common subalgebra $\mathfrak{A}_0$. Let ∇_{10} and ∇_{20} be the smallest elements of $S_{\mathfrak{A}_1}$ and $S_{\mathfrak{A}_2}$ respectively. Let

$$\tilde{S} = \{\langle \nabla_{10}, \nabla_{20} \rangle\} \cup \{\langle \nabla_1, \nabla_2 \rangle \in \mathcal{S}(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2) \mid \nabla_1 \text{ is maximal in } S_{\mathfrak{A}_1}, \nabla_2 \text{ is maximal in } S_{\mathfrak{A}_2}\}.$$

By Lemma 13 $\mathcal{B}(\tilde{S}) \in \mathcal{H}_3$ and by lemmas 7 a.) and 5, $\tilde{S}$ satisfies the conditions **1a.) - 2.b)**. We immediately get

Theorem 14. *The logic* $\mathbf{IPC}+p \vee (p \to (q \vee \neg q))$ *has the interpolation property.*

$\mathcal{H}_4$ is the variety of algebras validating $p \vee (p \to (q \vee \neg q))$ and $(p \to q) \vee (q \to p) \vee (p \leftrightarrow q)$. For this variety the lemma needed is

Lemma 15. *a.) Suppose* $\mathfrak{A}$ *is a well-connected algebra in* $\mathcal{H}_4$*. Then* $S_{\mathfrak{A}}$ *contains at most three elements, one being minimal and the other maximal.*
b.) If a partially ordered set $\langle S, \leq, \rangle$ *contains at most three elements, one being minimal and the rest maximal, then* $\mathcal{B}(S) \in \mathcal{H}_4$*.*

Proof. Since $\mathfrak{A} \models p \vee (p \to (q \vee \neg q))$, it follows from Lemma 13 that $\mathfrak{A}$ has one minimal element and the rest being maximal. Now, if $\mathfrak{A}$ is linearly ordered, $S_{\mathfrak{A}}$ contains only two elements. So suppose $a, b \in \mathfrak{A}$ are incomparable. Then, since $\mathfrak{A} \models (p \to q) \vee (q \to p) \vee (p \leftrightarrow \neg q)$, remembering that in a Heyting algebra $a \to b = \top$ iff $a \leq b$, we have $a = \neg b$ and $b = \neg a$. Thus, there is no $c \in A$ such that $a \neq c \neq b$ and $a \not\leq c \not\leq a$ and $b \not\leq c \not\leq b$, for this would immediately lead to $c = a$ or $c = b$. Suppose ∇ is a filter in $\mathfrak{A}$ such that $a \notin \nabla$ and $b \notin \nabla$. There are then $v_a, v_b \in \nabla$ such that $a \not\leq v_a$ and $b \not\leq v_b$. But this means that $a \not\leq v_a \wedge v_b \not\leq a$ and $b \not\leq v_a \wedge v_b \not\leq b$, a contradiction.
b.) Let $\langle S \leq \rangle$ be a partially ordered set with at most three elements, one of which is minimal and the rest maximal. By Lemma 13 b.), $\mathcal{B}(S) \models p \vee (p \to (q \vee \neg q))$. And by inspection, if $X, Y \in \mathrm{Up}S$ is such that $X \not\subseteq Y$ and $Y \not\subseteq X$ then $X = \neg Y$, so $\mathcal{B}(S) \models (p \to q) \vee (q \to p) \vee (p \leftrightarrow \neg q)$.

The following argument proves that $\mathcal{H}_4$ is amalgamable. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be well-connected algebras in $\mathcal{H}_4$ with a common subalgebra $\mathfrak{A}_0$. For $i \in \{1, 2\}$, let ∇_{i0} be the smallest element of $\mathfrak{A}_i$ and MAX_i the set of maximal elements in $S_{\mathfrak{A}_i}$. By Lemma 15 a.), $S_{\mathfrak{A}_i} = \{\nabla_{i0}\} \cup MAX_i$ and $|MAX_i| \leq 2$. We now consider the differing cases as to the cardinality of MAX_i and in each case find a subset $\tilde{S} \subseteq \mathcal{S}(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2)$ such that Lemma 15 b.) applies. We may suppose $|MAX_2| \leq |MAX_1|$. The first case is where $MAX_1 = \{\nabla_{11}, \nabla_{12}\}$ with ∇_{11} and ∇_{12} not necessarily distinct. By Lemma 10 b.) there are ∇_{21}, $\nabla_{22} \in MAX_2$ such that $S' = \{\langle \nabla_{11}, \nabla_{21} \rangle, \langle \nabla_{12}, \nabla_{22} \rangle\} \subseteq \mathcal{S}(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2)$. If $\{\nabla_{21}, \nabla_{22}\} = S_2$, we let $\tilde{S} = \{\langle \nabla_{10}, \nabla_{20} \rangle\} \cup S'$. Otherwise, $MAX_2 - \{\nabla_{21}, \nabla_{22}\} = \{\nabla_2\}$. By Lemma 10 c.) there is some $\nabla_1 \in S_{\mathfrak{A}_1}$ such that $\langle \nabla_1, \nabla_2 \rangle \in \mathcal{S}(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2)$. Let $\nabla'_1 \in MAX_1 - \{\nabla_1\}$ and set $\tilde{S} = \{\langle \nabla_{10}, \nabla_{20} \rangle, \langle \nabla_1, \nabla_2 \rangle, \langle \nabla'_1, \nabla_{21} \rangle\}$. Note that in this case $\nabla_{21} = \nabla_{22}$, so conditions **1a.) - 2b).** are satisfied.
In all cases we obtain $\tilde{S} \subseteq \mathcal{S}(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2)$, that Lemma 15 b.) applies to $\tilde{S}$ and that $\tilde{S}$ satisfies conditions **1a) - 2b)**.

Theorem 16. *The intermediate logic*

$$\mathbf{IPC} + p \vee (p \to (q \vee \neg q)) + (p \to q) \vee (q \to p) \vee (p \leftrightarrow \neg q)$$

has the interpolation property.

We don't have to do much now to show that $\mathcal{H}_5$, the class of Heyting algebras validating $p \vee (p \to (q \vee \neg q))$ and $\neg p \vee \neg\neg p$, is amalgamable. For by lemmas 6 and 7 we immediately get $S_{\mathfrak{A}_i} = \{\nabla_{i0}, \nabla_{i1}\}$ where $\nabla_{i0} \subseteq \nabla_{i1}$. We let $\tilde{S} = \{\langle \nabla_{10}, \nabla_{20} \rangle, \langle \nabla_{11}, \nabla_{21} \rangle\}$. From Lemma 10 it follows that $\tilde{S} \subseteq \mathcal{S}(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2)$ and clearly $\tilde{S}$ satisfies the conditions **1a)-2b)**. Furthermore, by lemmas 6 and 7 $\mathcal{B}(\tilde{S}) \in \mathcal{H}_5$.

Theorem 17. *The intermediate logic*

$$\mathbf{IPC} + p \vee (p \to (q \vee \neg q)) + \neg p \vee \neg\neg p$$

has the interpolation property.

This concludes the work on $\mathcal{H}_2$-$\mathcal{H}_5$. For $\mathcal{H}_6$, the variety corresponding to Dummett's logic $\mathbf{IPC} + (p \to q) \vee (q \to p)$, we use less detours to prove that it is amalgamable.
Note first that for every well-connected non-trivial $\mathfrak{A}$ in $\mathcal{H}_6$ and $a, b \in A$, $a \to b \vee b \to a = \top$, so since $\mathfrak{A}$ is well-connected either $a \leq b$ or $b \leq a$. Hence the subclass of well-connected algebras in $\mathcal{H}_6$ is precisely the class of linearly ordered Heyting algebras. So suppose $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are non-trivial linear Heyting algebras with a common subalgebra $\mathfrak{A}_0$. To verify that the class of well-connected Heyting algebras in $\mathcal{H}_6$ is amalgamable in $\mathcal{H}_6$, we construct an $\mathfrak{A}$ as follows. Let $A = A_1 \cup A_2$. Note that any linearly ordered set $\langle S, \leq \rangle$ with a least and a greatest element gives rise to a Heyting algebra by defining $\wedge$ as the *glb*, $\vee$ as the *lub* and $a \to b$ as $\bigvee\{c \in S \mid c \wedge a \leq b\}$. We therefore want to embed $\mathfrak{A}_1$ and $\mathfrak{A}_2$ viewed as linearly ordered sets into A preserving $\top$ and $\perp$ and then extend the order on A induced by this embedding to a linear order securing $\langle A, \leq \rangle$ gives rise to a Heyting algebra in $\mathcal{H}_6$. For $i \in \{1, 2\}$, let $\leq_i$ be the order induced on A_i by $\wedge$. The wanted embedding can then be had by defining $\leq'$ on $A_1 \cup A_2$ by:
$a \leq' b$ iff $(a, b \in A_1$ and $a \leq_1 b)$ or $(a, b \in A_2$ and $a \leq_2 b)$, or
$a \in A_1$ and $b \in A_2$ and there is a $c \in A_1 \cap A_2$ such that
$a \leq_1 c \leq_2 b$, or
$a \in A_2$ and $b \in A_1$ and there is a $c \in A_1 \cap A_2$ such that
$a \leq_2 c \leq_1 b$.
It is easily seen that $\leq'$ defined in this way is a partial order and that $\top$ is the greatest element and $\perp$ the smallest element under $\leq'$. Now extend $\leq'$ to a linear order $\leq$ with the use of Zorn's lemma. Then $\langle A, \leq \rangle$ gives rise to a (well-connected) algebra $\mathfrak{A}$ in $\mathcal{H}_4$ and $\mathfrak{A}_1$ and $\mathfrak{A}_2$ can be embedded into $\mathfrak{A}$ over $\mathfrak{A}_0$.

Theorem 18. *The intermediate logic* $\mathbf{IPC} + (p \to q) \vee (q \to p)$ *has the interpolation property.*

The natural next step in this work is to give necessary conditions for the amalgamability of varieties of algebras. This is done in [3], where it is shown that the varieties $\mathcal{H}_1$-$\mathcal{H}_7$ are the only amalgamable varieties. Using Theorem 1, one can prove

Theorem 19. *The only intermediate logics having the interpolation property are*

– **IPC**
– $\mathbf{IPC} + \neg p \vee \neg\neg p$
– $\mathbf{IPC} + p \vee (p \to (q \vee \neg q))$
– $\mathbf{IPC} + p \vee (p \to (q \vee \neg q)), (p \to q) \vee (q \to p) \vee (p \leftrightarrow \neg q)$
– $\mathbf{IPC} + p \vee (p \to (q \vee \neg q)), (\neg p \vee \neg\neg p)$
– $\mathbf{IPC} + (p \to q) \vee (q \to p)$
– **CL**

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Computational Complexity of Intuitionistic Propositional Logic and Intermediate Logics

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Abstract. In these notes, we summarize some results concerning the computational complexity of the intuitionistic propositional calculus and some intermediate logics. In particular, we give an outline of the original proof of PSPACE-hardness of intuitionistic propositional calculus as it has been done by Statman [7] and present a detailed proof of this result using semantic methods following the proof by Švejdar [8]. We also look at results concerning the complexity of some intermediate logics such as **LC** and tabular logics.

1 Introduction

R. Ladner [7] set the stage for the investigation of the computational complexity of intuitionistic propositional logic (**IPC**), normal modal logics and intermediate logics by determining the complexity of **K**, **T**, **S4** and **S5**. In the same article, Ladner determined an upper bound for the complexity of **IPC**. Later, R. Statman determined a lower bound for the complexity of **IPC** by providing a reduction of the problem of determining whether a quantified Boolean formula is classically valid (or not) to determining whether a given formula of **IPC** is provable (or not). Statman's original reduction, however, is not intuitive and difficult to understand. This motivated V. Švejdar to give an alternative reduction that provides semantic insight into why **IPC** is *PSPACE-hard*. With the complexity of **IPC** established and also the complexity of classical propositional logic (**CPC**) well-known, a natural question to ask is: What are the complexities of logics intermediate between **IPC** and **CPC**? This question has been well-investigated for modal logics between **K** and **S5**, and variations thereof. The purpose of this paper is to present the some background in order to carry out investigations of the analogous results for intermediate logics.

The paper is structured as follows: In Section 2, we outline preliminary knowledge necessary for understanding the proofs that are presented in subsequent sections and define the different complexity classes. In Section 3, we present Ladner's results showing that the provability problem of **IPC** is in the complexity class PSPACE. In Section 4, we begin by presenting Statman's original proof that the provability problem of **IPC** is PSPACE-hard and then go on to present Švejdar's revision of Statman's proof. Next, Section 5 presents some established results related to the complexity of intermediate logics. Finally, we conclude by giving a synopsis of interesting avenues for the investigation of the complexities of intermediate logics.

2 Preliminaries

As explained in the previous section, our paper deals with complexity issues of intuitionistic logic and intermediate logics. In this section we explain the central concepts that will be used in the subsequent sections.

2.1 Logics

Definition 1 (IPC)**.** Intuitionistic propositional calculus (IPC) is specified by the following axioms:

1. $\varphi \to (\psi \to \varphi)$
2. $\varphi \to (\psi \to \theta) \to ((\varphi \to \psi) \to (\varphi \to \theta))$
3. $(\varphi \wedge \psi) \to \varphi;\ (\psi \wedge \varphi) \to \psi$
4. $\varphi \to (\psi \to (\varphi \wedge \psi))$
5. $\varphi \to (\varphi \vee \psi);\ \psi \to (\varphi \vee \psi)$
6. $(\varphi \to \theta) \to ((\psi \to \theta) \to ((\varphi \vee \psi) \to \theta))$
7. $\bot \to \varphi$

Rule of inference: Modus Ponens.

IPC is a subset of **CPC**. The main difference between them is that in intuitionistic logic, one does not use the law of excluded middle. So, in intuitionistic logic, it is not possible, as it is in classical logic, to proof a statement φ by showing that assuming $\neg\phi$ leads to a contradiction.

Definition 2 (CPC)**. CPC** can be obtained from **IPC** by adding the axiom

8. $\neg\neg\varphi \to \varphi$.

Our paper deals with **IPC** and intermediate logics (i.e. logics whose set of axioms is a superset of the set of axioms of **IPC**). For reasoning about the computational complexity of **IPC**, we will use known results about the complexity of other logics (e.g. **CPC, S4** and **QBF**). In order to understand subsequent sections of this paper – in particular Švejdar's proof of the PSPACE-completeness of **IPC** – it is important to understand the semantic structures for **IPC** because several proofs that will be presented in what follows are based on theses structures.

Definition 3. A Kripke frame $\mathfrak{F} = \langle W, R \rangle$ for **IPC** consists of a set of states W and a partial order R on W. A Kripke model $K = \langle \mathfrak{F}, V \rangle$ for **IPC** extends such a frame by a valuation V that specifies the truth values of propositional variables. Formally, V is defined as a function $V : \Phi \longrightarrow \mathfrak{P}(W)$, where Φ is the set of propositional variables of the underlying language and V maps each to the set of states where the respective propositional variable is true.

One important feature of an **IPC** model is that it is persistent, which means that if wRw' and $w \in V(p)$, then $w' \in V(p)$. In a model, formulas are evaluated in the following way:

1. $w \Vdash p$ iff $w \in V(p)$
2. $w \Vdash \varphi \wedge \psi$ iff $w \Vdash \varphi$ and $w \Vdash \psi$
3. $w \Vdash \varphi \vee \psi$ iff $w \Vdash \varphi$ or $w \Vdash \psi$
4. $w \Vdash \varphi \to \psi$ iff $\forall w' \geq w(w' \nVdash \varphi$ or $w \Vdash \psi)$
5. $w \nVdash \bot$

It follows that $w \Vdash \neg\varphi$ iff $\forall w' \geq w(w' \nVdash \varphi)$

A central result concerning the connection between classical and intuitionistic logic is Glivenko's theorem. It will play a crucial role in Statman's proof and the proof that all intermediate logics are NP-hard.

Theorem 4 (Glivenko). *For every formula* $\varphi, \varphi \in$ **CPC** *iff* $\neg\neg\varphi \in$ **IPC**.

For a proof see e.g. [4].

As mentioned in the introduction, the complexity of the validity problem of **QBF** will play a central role in both proofs of PSPACE-hardness of **IPC**. **QBF** is defined more precisely as follows:

Definition 5 (QBF). **QBF** is the smallest set S containing all formulas of propositional calculus such that if $B(p) \in S$ and p is a proposition letter, then both $\forall pB(p)$ and $\exists pB(p) \in S$. The quantifiers range over truth-values 1 (true) and 0 (false), and a quantified Boolean formula without free variables is valid iff it evaluates to 1.
A quantified Boolean formula is said to be in prenex form if it is of the form

$$Q_m p_m \dots Q_1 p_1 B(p_1, \dots, p_m),$$

where Q_i is either $\forall$ or $\exists$ and $B(p_1, \dots, p_m)$ is a formula of classical propositional logic.

If a quantified Boolean formula has free variables, we say that it is valid if and only if it evaluates to 1 for all valuations of the free variables (i.e. for all possible assignments of truth-values to the free variables).

2.2 Computational Complexity

Complexity theory is a branch of (theoretical) computer science that is concerned with the analysis of computational problems. We recall some basic facts from complexity theory from [9]. Such problems usually are decision problems of the following form:

Definition 6 (Decision Problem)**.** Given a set $\mathbf{X}$, the decision problem X is the problem of answering the following question:
Given an element x, is it the case that $x \in \mathbf{X}$?

Typically, by an **instance of a problem** X, one means the question "$x \in \mathbf{X}$?" for some x.

Complexity classes classify problems according to how hard it is to solve them computationally. This is done with respect to an underlying computational model– typically, one takes a Turing machine to be the underlying model. For the definition of Turing machine see e.g. [9]. With the Turing machine model in place, computational problems can be compared with respect to how many steps a Turing machine needs to perform in order to compute the solution, or with respect to the number of *tape-cells* (i.e., the amount of space) the machines needs to write on during the computation. In general, we distinguish between two kinds of computations, deterministic and non-deterministic, where the respective models are deterministic and non-deterministic Turing machines. For a deterministic Turing machine, it holds that for every state it is in, given the symbol it is reading, there is always a unique next step in the computation (if it is the final state, then the computation just stops). On the other hand, for a non-deterministic Turing machine it holds that it can choose between different ways of proceeding in the computation.

In this paper, we are concerned with the computational complexity of different logics.

For a logic $\mathbf{L}$, define the following sets:

$$\mathbf{L}_0 = \{\varphi | \vdash_L \varphi\} \text{ and } \mathbf{L}_1 = \{\varphi | \Vdash_L \varphi\}$$

Then the provability problem of $\mathbf{L}$ is the following:
Given a formula φ, $\varphi \in \mathbf{L_0}$?

And the validity problem of $\mathbf{L}$ is:
Given a formula φ, $\varphi \in \mathbf{L_1}$?

Since all the logics that we consider in this paper are complete, both problems are identical.

In particular, we will investigate this problem for the validity problem and the provability problem (which are equivalent) of **IPC** and the validity problems of **LC** and for tabular logics.

In order to proceed, we also need to define the complexity classes we consider and the relations between them:

Definition 7 (PSPACE)**.** PSPACE is the set of decision problems that can be solved by a deterministic or non-deterministic Turing machine using a polynomial amount of memory (i.e. the number of tape cells it writes on during the computation is bound by a polynomial in the length of the input)[1].

Definition 8 (NP)**.** NP is the set of decision problems solvable in polynomial time on a non-deterministic Turing machine.

If a problem is in a complexity class, an upper bound for the complexity of the given problem is established i.e., we have specified an upper bound for what the most efficient algorithm for solving the given problem is.

We also have a notion for establishing a lower bound for the computational complexity of a given problem:

Definition 9 (Hardness)**.** Let C be a complexity class. If a problem X is at least as hard as all the other problems in C, we say that X is C-hard.

The two concepts that have been defined previously allow us to specify the hardest problems within one complexity class.

Definition 10 (Completeness)**.** Given a complexity class C and a problem X, we say that X is C-complete if X is C-hard and furthermore $X \in \mathsf{C}$.

A central concept in complexity theory that is used to compare the complexity of two problems is that of a reduction.

Definition 11 (Reduction)**.** A reduction is a transformation of one problem into another problem, i.e. a reduction τ from a problem X to a problem Y transforms an instance of X (given in the form "$x \in \mathbf{X}$")into one of Y in such a way that for all x it holds that $x \in \mathbf{X}$ iff $\tau(x) \in \mathbf{Y}$.

Intuitively, if X can be reduced to Y, then the problem Y is at least as hard as X. So, if we already know how to solve Y, then we can solve X by first transforming an instance of X into one of Y and subsequently solving it using the algorithm for solving Y.

It is clear that the notion of reduction should be more precise since we must restrict the complexity of the reduction: If we do not, then one might reduce a hard problem to one that is much easier by choosing a very complex reduction.

Definition 12 (Polynomial-time Reduction)**.** A polynomial-time reduction is a reduction that can be computed by a deterministic Turing machine in polynomial time.

Throughout the rest of this paper we are primarily concerned with showing that the validity problems of different logics are in certain complexity classes and showing their hardness. In order to show that the problem X is in class

[1] Note that has been shown by Savitch that for both deterministic and non-deterministic Turing machines the definition above specifies the same set of decision problems (see eg. [9]).

C, one usually gives an algorithm that solves the problem in restricted time or space as specified by the complexity class C. However, another possibility that we will make use of, is to reduce the problem X to one that is already known to be in C.

On the other hand, in order to show that a problem X is C-hard one has to show that it is at least as hard as all the problems in C. If one knows already a C-hard problem Y, one can use it to show C-hardness of X by reducing Y to X. So, one shows that X is at least as hard as Y, which is already known to be at least as hard as all the problems in C.

3 IPC is in PSPACE

In this section we present the proof due to Ladner [7] that the provability problem of **IPC** is in PSPACE. Ladner's main result was his proof that the provability problem of the modal logic **S4** is in PSPACE. With this result he showed that the provability problem of **IPC** is in PSPACE as a simple consequence by noticing that there is already a well-established translation of **IPC** to **S4** – namely, that of McKinsey and Tarski [8]. This translation in complexity theory is a reduction of **IPC** to **S4**. Moreover, this reduction is trivially seen to be polytime and thus we end up with a proof that the provability problem of **IPC** is in PSPACE. We begin our presentation of the proof that **IPC** is in PSPACE by first presenting Ladner's main result as a lemma and second, presenting the reduction of **IPC** to **S4** as a theorem.

Lemma 13. *The provability problem of **S4** is in PSPACE.*

Proof. In order to show that the provability problem of **S4** is in PSPACE, Ladner provides an algorithm for deciding whether or not an arbitrary formula of **S4** is **S4**-provable. The algorithm is split into two separate procedures: The procedure **S4**-WORLD that encodes how we test the validity of a formula of **S4** by trying to find an **S4**-model that refutes it. Finally, the second procedure determines whether a given formula of **S4** is **S4**-provable. We begin by presenting the procedure **S4**-WORLD:

Let an input of **S4**-WORLD be $(\mathcal{T}, \mathcal{F}, \tilde{\mathcal{T}}, \tilde{\mathcal{F}}, \mathcal{L})$ where each of the arguments $\mathcal{T}, \mathcal{F}, \tilde{\mathcal{T}}, \tilde{\mathcal{F}}$ is a finite set of modal formulas and $\mathcal{L}$ is a sequence

$$(\mathcal{T}_1, \psi_1), (\mathcal{T}_2, \psi_2), ..., (\mathcal{T}_k, \psi_k)$$

where $\mathcal{T}_1 \subseteq \mathcal{T}_2 \subseteq ... \subseteq \mathcal{T}_k$ are sets of modal formulas and $B_1, B_2, ..., B_k$ are modal formulas.

Define the value of **S4**-WORLD$(\mathcal{T}, \mathcal{F}, \tilde{\mathcal{T}}, \tilde{\mathcal{F}}, \mathcal{L})$ to be true if there is an **S4**-model $\langle W, R, V\rangle$ and a sequence of worlds $w_1, w_2, ..., w_k, w$ in W with the following properties:

(i) $w_i + 1$ is accessible from w_i and w is accessible from w_k;
(ii) $V(\bigwedge_{\varphi\in\mathcal{T}_i} \varphi \wedge \neg B_i, w_i)$ is true for each i;

(iii) $V(\bigwedge_{\varphi\in\mathcal{T}}\varphi\wedge\bigwedge_{\varphi\in\mathcal{F}}\neg\varphi\wedge\bigwedge_{\varphi\in\tilde{\mathcal{T}}}\Box\varphi\wedge\bigwedge_{\varphi\in\tilde{\mathcal{F}}}\neg\Box\varphi$ is true.
Then the procedure **S4**-WORLD is as follows:

Algorithm 14 (**S4**-WORLD)**.**
On input $(\mathcal{T},\mathcal{F},\tilde{\mathcal{T}},\tilde{\mathcal{F}},\mathcal{L})$:

If $\mathcal{T}\cup\mathcal{F}\not\subseteq VAR$*, then:*

1. *Choose* $\varphi\in\mathcal{T}\cup\mathcal{F}\setminus VAR$*.*
2. *If* $\varphi=\neg\psi$ and $\varphi\in\mathcal{T}$*, then return*

$$\mathbf{S4}\text{-WORLD}(\mathcal{T}\setminus\{\varphi\},\mathcal{F}\cup\{\psi\},\tilde{\mathcal{T}},\tilde{\mathcal{F}},\mathcal{L});$$

3. *If* $\varphi=\neg\psi$ and $\varphi\in\mathcal{F}$*, then return*

$$\mathbf{S4}\text{-WORLD}(\mathcal{T}\cup\{\psi\},\mathcal{F}\setminus\{\varphi\},\tilde{\mathcal{T}},\tilde{\mathcal{F}},\mathcal{L});$$

4. *If* $\varphi=\psi\wedge\chi$ and $\varphi\in\mathcal{T}$*, then return*

$$\mathbf{S4}\text{-WORLD}((\mathcal{T}\cup\{\psi,\chi\})\{\varphi\},\mathcal{F},\tilde{\mathcal{T}},\tilde{\mathcal{F}},\mathcal{L});$$

5. *If* $\varphi=\psi\wedge\chi$ and $\varphi\in\mathcal{F}$*, then return*

$$\mathbf{S4}\text{-WORLD}(\mathcal{T},\mathcal{F}\cup\{\psi\})\{\varphi\},\tilde{\mathcal{T}},\tilde{\mathcal{F}},\mathcal{L})$$

   or

$$\mathbf{S4}\text{-WORLD}(\mathcal{T},(\mathcal{F}\cup\{\chi\})\{\varphi\},\tilde{\mathcal{T}},\tilde{\mathcal{F}},\mathcal{L});$$

6. *If* $\varphi=\Box\psi$ *and* $\varphi\in\mathcal{T}$*, then return*

$$\mathbf{S4}\text{-WORLD}((\mathcal{T}\cup\{\psi\})\{\varphi\},\mathcal{F},\tilde{\mathcal{T}}\cup\{\psi\},\tilde{\mathcal{F}},\mathcal{L});$$

7. *If* $\varphi=\Box\psi$ and $\varphi\in\mathcal{F}$*, then return*

$$\mathbf{S4}\text{-WORLD}(\mathcal{T},\mathcal{F}\setminus\{\varphi\},\tilde{\mathcal{T}},\tilde{\mathcal{F}}\cup\{\psi\},\mathcal{L});$$

If $\mathcal{T}\cup\mathcal{F}\subseteq VAR$*, then:*

1. *If* $\mathcal{T}\cap\mathcal{F}\neq\varnothing$*, then return false*;
2. *If* $\mathcal{T}\cap\mathcal{F}=\varnothing$ *and* $\tilde{\mathcal{F}}\neq\varnothing$*, then return*

$$\bigwedge_{psi\in\tilde{\mathcal{F}},(\tilde{\mathcal{T}},\psi)\notin\mathcal{L}}\mathbf{S4}\text{-WORLD}(\tilde{\mathcal{T}},\{\psi\},\tilde{\mathcal{T}},\varnothing,\mathcal{L}\cdot(\tilde{\mathcal{T}},\psi));$$

   return true.

The next procedure makes use of **S4**-WORLD to decide given an input φ whether or not φ is **S4**-provable.

Algorithm 15 (Decision Algorithm)**.**
On input φ:
If $\neg$**S4**-WORLD$(\{\varphi\}, \varnothing, \varnothing, \varnothing, \varnothing)$, *then return true* (i.e., φ is **S4**-provable).
Otherwise, return false.

Now, the main result of this section:

Theorem 16. *The provability problem of **IPC** is in PSPACE and the validity problem of **IPC** is in is in PSPACE.*

Proof. Define a reduction, τ, of **IPC** to **S4** inductively as follows:
$\tau(p) = p$ if p is a variable,
$\tau(\varphi \wedge \psi) = \tau(\varphi) \wedge \tau(\psi)$,
$\tau(\varphi \to \psi) = \Box(\tau(\varphi) \to \tau(\psi))$,
$\tau(\neg\varphi) = \Box\neg\tau(\varphi)$.

Then τ is a polytime reduction since φ is **IPC**-provable iff $\tau(\varphi)$ is **S4**-provable (see [8]), and we can compute $\tau(\varphi)$ in the length of φ steps since each step of the recursion translates precisely one unit of the length of φ. Thus, the provability problem of **IPC** is in PSPACE since the provability problem of **S4** is in PSPACE by Lemma 1. Moreover, we also have that the validity problem of **IPC** is in PSPACE since φ is **IPC**-valid iff φ is **IPC**-provable.

We will use the former result (i.e., that the provability problem is in PSPACE) in the next section where we present Statman's original proof that **IPC** is PSPACE-hard; and the latter result (i.e. that the provability problem is in PSPACE) when we present Švejdar's revised proof that **IPC** is PSPACE-hard.

4 IPC is PSPACE-hard

PSPACE-hardness of **IPC** can be proven by reducing a known PSPACE-hard problem to the validity problem of **IPC**. One well-known PSPACE-hard problem is the validity problem of **QBF**, the problem of deciding whether a given QBF with no free variables is valid.

4.1 Understanding Statman's Proof

The syntactic proof of PSPACE-hardness, due to R. Statman [11], uses the fact that the validity problem of **QBF** is PSPACE-hard. In addition, the proof relies on the completeness results of Kripke [5].

Statman shows that for an arbitrary prenex formula A of **QBF**, A is **QBF**-valid iff $g(A)$ is **IPC**-provable (where $g(A)$ is a quantifier-free Boolean formula constructed from A - i.e., a formula of **CPC**) iff $f(A)$ is **IPC**-provable (where $f(A)$ is a formula of **IPC** constructed from A and $g(A)$). More precisely, by **IPC**-provable, we mean a formula is provable using the axioms and rules for **IPC** presented in the preliminaries. Since the first step in the strategy of Statman's proof is to give a polytime reduction – namely, g – of the problem of determining whether or not a formula of **CPC** is **IPC**-provable to the validity problem of **QBF**, we begin with his definition of g:

Definition 17. Let $A = Q_m p_m Q_{m-1} p_{m-1} ... Q_1 p_1 B_0$ be a QBF where B_0 is a quantifier-free Boolean formula, $Q_i = \forall$ or $\exists$, and we set $Q_{k+1} p_{k+1} B_k$. Then the reduction $g : QBF \to BF$ is defined recursively as follows:

$$g(B_0) = \neg\neg B_0$$

$$g(B_{k+1}) = \begin{cases} (p_{k+1} \vee \neg p_{k+1}) \to g(B_k) & \text{if } Q_{k+1} = \forall \\ (p_{k+1} \to g(B_k)) \vee (\neg p_{k+1} \to g(B_k)) & \text{if } Q_{k+1} = \exists \end{cases}$$

To illustrate what is happening in this definition we give a short example:

Example 1. Suppose $A = \forall p_2 \exists p_1 B_0$, then:
$g(A) = g(B_2) = (p_2 \vee \neg p_2) \to g(B_1)$
$g(B_1) = (p_1 \to g(B_0)) \vee (\neg p_1 \to g(B_0))$
$g(B_0) = \neg\neg B_0$
Therefore, by substituting the corresponding values we get:
$g(A) = g(B_2) = (p_2 \vee \neg p_2) \to ((p_1 \to \neg\neg B_0) \vee (\neg p_1 \to \neg\neg B_0))$

Notice that $g(A)$ is a quantifier-free Boolean formula – i.e., a formula of **CPC**. This reduction is an intuitive translation of QBFs into equivalent formulas of **CPC**: We only have two possible evaluations to the propositional variables, 0 and 1, which means that $\forall p_2$ should be translated as, roughly speaking, "$(p_2 \vee \neg p_2)$ implies the remaining part of the formula, $g(B_1)$, holds", to represent the fact that both assignments force $g(B_1)$ to be evaluated to 1. Similarly for the translation of $\exists$.

Now, in order to show that the reduction works, we need to show that: A is **QBF**-valid iff $g(A)$ is **IPC**-provable, and in addition, $g(A)$ is computable in polytime. The latter is clearly the case since for any given input A with m quantifiers, we can compute $g(A)$ in at most $2m+1$ steps. Now, we establish the former as lemma after briefly giving a definition needed for its proof:

Definition 18. Let A be a quantified Boolean formula with propositional variables $p_1, \ldots, p_m$. If we say that V is a valuation of $p_{j+1}, \ldots, p_m$, it means that we assigned fixed truth values to these variables, i.e. each variable $p_{j+1}, \ldots, p_m$ is either assigned 0 or 1.

Lemma 19. *Let A be a QBF and g be as defined in Definition 12. Then A is **QBF**-valid iff $g(A)$ is **IPC**-provable.*

Proof. Let $A = Q_m p_m Q_{m-1} p_{m-1} ... Q_1 p_1 B_0$ be a QBF where B_0 is a quantifier-free Boolean formula, $Q_i = \forall$ or $\exists$, and where $B_{k+1} = Q_{k+1} p_{k+1} B_k$. If Q_i is the ith $\forall$ or $\exists$ from the left in A, then we write $\forall_i$ or $\exists_i$ respectively.

(⇒) Suppose A is **QBF**-valid. That means there are Skolem functions (i.e., valuations similar to those in the above definition) $V_1, ..., V_m$ that witness each existential quantifier in A – that is, if $\exists_{i+1} p_{i+1} B_i$ is a subformula of A, then V_{i+1}, given as input $\exists_{i+1} p_{i+1} B_i$ and an assignment to each of the remaining variables, will return a value to p_{i+1} that makes the formula B_i true. In this way, if $Q_k = \exists_i$, then V_i for a QBF is a function of $p_m, p_{m-1}, ..., p_{k+1}$ where $V_i(l_m, l_{m-1}, ..., l_{k+1}) = l_k$ and $l_j = p_j$ whenever $V_i(p_j) = 1$ (i.e., whenever l_j is assigned the truth-value true) and $l_j = \neg p_j$ whenever $V(p_j) = 0$ (i.e., whenever l_j is assigned the truth-value false).

Next, we construct a tree, $\mathfrak{T}_1$ out of statements of the form $\vdash_{\mathbf{IPC}} g(B_k)$:[2]

The root of $\mathfrak{T}_1$ is $\vdash_I g(A)$, and the leafs of $\vdash_I g(A)$ are constructed by using the using the axioms and rules of **IPC** to derive the subformulas of $g(A)$ into the different branches. Now, if $\{l_m, l_{m-1}, ..., l_{k+1}\} \vdash_{\mathbf{IPC}} g(B_k)$ is a leaf, then construct new nodes:

$$\begin{array}{c} \{l_m, l_{m-1}, ..., l_{k+1}\} \vdash_{\mathbf{IPC}} l_k \to g(B_{k-1}) \\ | \\ \{l_m, l_{m-1}, ..., l_{k+1}, l_k\} \vdash_{\mathbf{IPC}} g(B_{k-1}) \end{array}$$

if $Q_k = \exists_i$ and $C_i(l_m, l_{m-1}, ..., l_{k+1}) = l_k$; or construct new nodes:

$$\begin{array}{cc} \multicolumn{2}{c}{\{l_m, l_{m-1}, ..., l_{k+1}, p_k \vee \neg p_k\} \vdash_{\mathbf{IPC}} g(B_{k-1})} \\ / & | \\ \{l_m, l_{m-1}, ..., l_{k+1}, p_k\} \vdash_{\mathbf{IPC}} g(B_{k-1}) & \{l_m, l_{m-1}, ..., l_{k+1}, \neg p_k\} \vdash_{\mathbf{IPC}} g(B_{k-1}) \end{array}$$

if $Q_k = \forall_i$.

Now by structural induction on $\mathfrak{T}_1$, we show that if $\{l_m, l_{m-1}, ..., l_{k+1}\} \vdash_{\mathbf{IPC}} g(B_k)$ occurs in $\mathfrak{T}_1$, then $g(B_k)$ is a classical consequence of $\{l_m, l_{m-1}, ..., l_{k+1}\}$ (which we denote by $\{l_m, l_{m-1}, ..., l_{k+1}\} \vdash_{\mathbf{CPC}} g(B_k)$):

Base Case Suppose $\mathfrak{T}_1$ contains only its root $\vdash_{\mathbf{IPC}} g(A)$. Then $g(A) = g(B_0) = \neg\neg B_0$. Now, suppose further that $\vdash_{\mathbf{IPC}} g(A)$ holds (i.e., $\vdash_{\mathbf{IPC}} \neg\neg B_0$ holds) and $\nvdash_{\mathbf{CPC}} B_0$. Then $\vdash_{\mathbf{CPC}} \neg B_0 \Rightarrow \vdash_{\mathbf{IPC}} \neg\neg\neg B_0$ by Glivenko's Theorem $\Rightarrow \nvdash_{\mathbf{IPC}} \neg\neg B_0$. But we assumed $\vdash_{\mathbf{IPC}} g(A)$ holds. This is a contradiction. Thus, $\vdash_{\mathbf{CPC}} B_0$.

Inductive Hypothesis Let $\mathfrak{T}_1$ is a tree with root $\vdash_{\mathbf{IPC}} g(A)$ whose leafs are $c_1, c_2, ..., c_s$. Let $\mathcal{T}_1, \mathcal{T}_2, ..., \mathcal{T}_s$ be the subtrees of $\mathfrak{T}_1$ rooted at $c_1, c_2, ..., c_s$ respectively. Assume the following statement holds for each $\mathcal{T}_1, \mathcal{T}_2, ..., \mathcal{T}_s$: If $\{l_m, l_{m-1}, ..., l_{k+1}\} \vdash_{\mathbf{IPC}} g(B_k)$ occurs in $\mathfrak{T}_1$, then $\{l_m, l_{m-1}, ..., l_{k+1}\} \vdash_{\mathbf{CPC}} g(B_k)$.

Inductive Step

[2] Note that although it works, Statman's motivation for constructing trees of this form is lacking: It is not a standard method of complexity theory.

Case 1: $Q_m = \exists$

Let $A = Q_m p_m Q_{m-1} p_{m-1} ... Q_1 p_1 B_0 = \exists_{k+1} p_{k+1} B_k$. Suppose $\vdash_{\mathbf{IPC}} g(A)$ occurs at the root of $\mathfrak{T}_1$, then $\vdash_{\mathbf{IPC}} g(A) \Rightarrow \vdash_{\mathbf{IPC}} g(\exists_{k+1} p_{k+1} B_k) \Rightarrow \vdash_{\mathbf{IPC}} (p_{k+1} \to g(B_k)) \vee (\neg p_{k+1} \to g(B_k))$. This means that $\vdash_{\mathbf{IPC}} (p_{k+1} \to g(B_k))$ or $\vdash_{\mathbf{IPC}} (\neg p_{k+1} \to g(B_k)) \Rightarrow \{p_{k+1}\} \vdash_{\mathbf{IPC}} g(B_k)$ or $\{\neg p_{k+1}\} \vdash_{\mathbf{IPC}} g(B_k) \Rightarrow \{l_{k+1}\} \vdash_{\mathbf{IPC}} g(B_k)$ is a leaf of $\vdash_{\mathbf{IPC}} g(A)$. Now, by the Inductive Hypothesis, $\{l_{k+1}\} \vdash_{\mathbf{CPC}} g(B_k)$ since $\{l_{k+1}\} \vdash_{\mathbf{IPC}} g(B_k)$ occurs in some $\mathcal{T}_j$. But this means that $\vdash_{\mathbf{CPC}} g(B_{k+1}) = g(A)$ since $\{l_{k+1}\} \vdash_{\mathbf{CPC}} g(B_k)$ is equivalent to $\vdash_{\mathbf{CPC}} (p_{k+1} \to g(B_k)) \vee (\neg p_{k+1} \to g(B_k))$.

Case2: $Q_m = \forall$

Let $A = Q_m p_m Q_{m-1} p_{m-1} ... Q_1 p_1 B_0 = \forall_{k+1} p_{k+1} B_k$. Suppose $\vdash_{\mathbf{IPC}} g(A)$ occurs at the root of $\mathfrak{T}_1$, then $\vdash_{\mathbf{IPC}} g(A) \Rightarrow \vdash_{\mathbf{IPC}} g(\forall_{k+1} p_{k+1} B_k) \Rightarrow \vdash_{\mathbf{IPC}} (p_{k+1} \vee \neg p_{k+1}) \to g(B_k))$. This means that $\{p_{k+1} \vee \neg p_{k+1})\} \vdash_{\mathbf{IPC}} g(B_k) \Rightarrow \{p_{k+1}\} \vdash_{\mathbf{IPC}} g(B_k)$ and $\{\neg p_{k+1}\} \vdash_{\mathbf{IPC}} g(B_k) \Rightarrow \{l_{k+1}\} \vdash_{\mathbf{IPC}} g(B_k)$ is a leaf of $\vdash_{\mathbf{IPC}} g(A)$. Now, by the Inductive Hypothesis, $\{l_{k+1}\} \vdash_{\mathbf{CPC}} g(B_k)$ since $\{l_{k+1}\} \vdash_{\mathbf{IPC}} g(B_k)$ occurs in some $\mathcal{T}_j$. But this means that $\vdash_{\mathbf{CPC}} g(B_{k+1}) = g(A)$ since $\{l_{k+1}\} \vdash_{\mathbf{CPC}} g(B_k)$ is equivalent to $\vdash_{\mathbf{CPC}} (p_{k+1} \vee \neg p_{k+1} \to g(B_k)$.

Now, by the result of the induction and by Glivenko's Theorem, we have that each leaf of $\mathfrak{T}_1$ is true and by the axioms and rules of Definition 1, each node of $\mathfrak{T}_1$ is true. Thus, $\vdash_{\mathbf{IPC}} g(A)$.

($\Leftarrow$) Suppose $\vdash_{\mathbf{IPC}} g(A)$. Then we can construct a tree $\mathfrak{T}_2$ of subformulas of g(A) as follows:

The root of $\mathfrak{T}_2$ is $\vdash_{\mathbf{IPC}} g(A)$, and the leafs of $\vdash_{\mathbf{IPC}} g(A)$ are constructed in a similar way as before. Now, if $\{l_m, l_{m-1}, ..., l_{k+1}\} \vdash_{\mathbf{IPC}} g(B_k)$ is a leaf, then construct new nodes:

$$\{l_m, l_{m-1}, ..., l_{k+1}\} \vdash_{\mathbf{IPC}} l_k \to g(B_{k-1})$$

$$\{l_m, l_{m-1}, ..., l_{k+1}, l_k\} \vdash_{\mathbf{IPC}} g(B_{k-1})$$

if $Q_k = \exists_i$ and $\{l_m, l_{m-1}, ..., l_{k+1}\} \vdash_{\mathbf{IPC}} l_k \to B_{k-1}$; or construct new nodes:

$$\{l_m, l_{m-1}, ..., l_{k+1}, p_k \vee \neg p_k\} \vdash_{\mathbf{IPC}} g(B_{k-1})$$

$$\{l_m, l_{m-1}, ..., l_{k+1}, p_k\} \vdash_{\mathbf{IPC}} g(B_{k-1}) \qquad \{l_m, l_{m-1}, ..., l_{k+1}, \neg p_k\} \vdash_{\mathbf{IPC}} g(B_{k-1})$$

if $Q_k = \forall_i$.

Now, we see by the axioms of Definition 1 that if $\{l_m, ..., l_{k+1}\} \vdash_{\mathbf{IPC}} g(B_k)$ occurs in $\mathfrak{T}_2$, then $\{l_m, ..., l_{k+1}\} \vdash_{\mathbf{CPC}} g(B_k)$. Hence, A is **QBF**-valid.

The next step in Statman's proof of PSPACE-hardness of **IPC** is to convert the QBF, A, and the quantifier-free Boolean formula $g(A)$ into a formula of **IPC**, $f(A)$, so that $g(A)$ is **IPC**-provable iff $f(A)$ is **IPC**-provable. In this way, we have a reduction of the provability problem of **IPC** to the problem of determining whether a formula of **CPC** is **IPC**-provable; and given that Lemma 19 establishes that this problem is PSPACE-hard, we indeed have that the provability problem of **IPC** is PSPACE-hard. We now present the definition of this reduction and then establish its correctness via Lemma 21:

Definition 20. Let $A = Q_m p_m Q_{m-1} p_{m-1} ... Q_1 p_1 B_0$ be a QBF where B_0 is a quantifier-free Boolean formula, $Q_i = \forall$ or $\exists$, and $y_0, y_1, ..., y_m$ be new variables. Then we define $f(A)$ as follows:

$$h(B_0) = \neg\neg B_0 \leftrightarrow y_0$$

$$h(B_{k+1}) = \begin{cases} ((p_{k+1} \vee \neg p_{k+1}) \to y_k) \leftrightarrow y_{k+1} & \text{if } Q_{k+1} = \forall \\ (p_{k+1} \to y_k) \vee (\neg p_{k+1} \to y_k) \leftrightarrow y_{k+1} & \text{if } Q_{k+1} = \exists \end{cases}$$

So that $f(A) = h(B_0) \to (...(h(B_m) \to y_m)...)$

To illustrate what is happening in this definition we give a short example:

Example 2. Suppose $A = \forall p_2 \exists p_1 B_0$, then:
$h(B_2) = ((p_2 \vee \neg p_2) \to y_1) \leftrightarrow y_2$ since $Q_2 = \forall$.
$h(B_1) = ((p_1 \to y_0) \vee (\neg p_1 \to y_0)) \leftrightarrow y_1$ since $Q_1 = \exists$.
$h(B_0) = \neg\neg B_0 \leftrightarrow y_0$
Therefore, by substituting the corresponding values we get:
$f(A) = h(B_0) \to (h(B_1) \to ((h(B_2) \to y_2)))$.

Lemma 21. *Let A be a QBF, g be as defined in Definition 12 and f be as defined in Definition 14. Then $\vdash_{\mathbf{IPC}} g(A)$ iff $\vdash_{\mathbf{IPC}} f(A)$*

Proof. $\vdash_{\mathbf{IPC}} g(A) \Leftrightarrow \vdash_{\mathbf{IPC}} f(A)$:

($\Rightarrow$) Suppose $\vdash_{\mathbf{IPC}} g(A)$. Then by induction on k it follows that

$$\{h(B_0), h(B_1), ..., h(B_m)\} \vdash_{\mathbf{IPC}} y_k \leftrightarrow g(B_k):$$

Base Case $k = 0$
$\{h(B_0), h(B_1), ..., h(B_m)\} \vdash_{\mathbf{IPC}} y_0 \leftrightarrow g(B_0)$ since $h(B_0) = y_0 \leftrightarrow \neg\neg B_0 \in \{h(B_0), h(B_1), ..., h(B_m)\}$ and $g(B_0) = \neg\neg B_0$.
Inductive Hypothesis Assume $\{h(B_0), h(B_1), ..., h(B_m)\} \vdash_{\mathbf{IPC}} y_k \leftrightarrow g(B_k)$.
Inductive Step
Case 1: $Q_{k+1} = \exists$

$$\{h(B_0), ..., h(B_k), h(B_{k+1}), ..., h(B_m)\} \vdash_{\mathbf{IPC}} y_{k+1} \leftrightarrow g(B_{k+1})$$

since

$$\begin{aligned} h(B_{k+1}) &= ((p_{k+1} \to y_k) \vee (\neg p_{k+1} \to y_k)) \leftrightarrow y_{k+1} \\ &\in \{h(B_0), ..., h(B_k), h(B_{k+1}), ..., h(B_m)\}, \end{aligned}$$

$g(B_k) \leftrightarrow y_k$ by the Inductive Hypothesis and $g(B_{k+1}) = (p_{k+1} \to g(B_k)) \vee (\neg p_{k+1} \to g(B_k))$.

Case 2: $Q_{k+1} = \forall$

$$\{h(B_0), ..., h(B_k), h(B_{k+1}), ..., h(B_m)\} \vdash_{\mathbf{IPC}} y_{k+1} \leftrightarrow g(B_{k+1})$$

since

$$h(B_{k+1}) = ((p_{k+1} \vee \neg p_{k+1}) \to y_k) \leftrightarrow y_{k+1} \\ \in \{h(B_0), ..., h(B_k), h(B_{k+1}), ..., h(B_m)\},$$

$g(B_k) \leftrightarrow y_k$ by the Inductive Hypothesis and $g(B_{k+1}) = (p_{k+1} \vee \neg p_{k+1}) \to g(B_k)$.

In particular, using the result of the induction, we have that

$$\{h(B_0), h(B_1), ..., h(B_m)\} \vdash_{\mathbf{IPC}} y_m \leftrightarrow g(B_m)$$

and since $y_m \leftrightarrow g(B_m)$, we have that

$$\{h(B_0), h(B_1), ..., h(B_m)\} \vdash_{\mathbf{IPC}} y_m.$$

Moreover, by repeated iteration of (2), we get $\vdash_{\mathbf{IPC}} h(B_0) \to (h(B_1) \to (...(h(B_m) \to y_m)))$ and hence, $\vdash_{\mathbf{IPC}} f(A)$.

($\Leftarrow$) Suppose $\vdash_{\mathbf{IPC}} f(A)$. Then by repeated iteration of (2), we get

$$\{h(B_0), h(B_1), ..., h(B_m)\} \vdash_{\mathbf{IPC}} y_m.$$

Now for each $1 \leqslant k \leqslant m$ substitute $g(B_k)$ for y_k in $h(B_k)$. Then we end up with $\{g(B_0) \leftrightarrow g(B_0), g(B_1) \leftrightarrow g(B_1), ..., g(B_m) \leftrightarrow g(B_m)\}$ and from our first premise we have $\{g(B_0) \leftrightarrow g(B_0), g(B_1) \leftrightarrow g(B_1), ..., g(B_m) \leftrightarrow g(B_m)\} \vdash_{\mathbf{IPC}} g(B_m)$. Thus,$\vdash_{\mathbf{IPC}} g(A)$.

Theorem 22. *The provability problem of* ***IPC*** *is PSPACE-hard.*

Proof. The main theorem follows immediately from Lemma 19 and Lemma 21: We have a reduction from the validity problem of **QBF** to the **IPC**-provability problem of **CPC** by Lemma 19 and that $\vdash_{\mathbf{IPC}} g(A) \Leftrightarrow \vdash_{\mathbf{IPC}} f(A)$. Hence, the provability problem of **IPC** is PSPACE-hard since the validity problem of **QBF** is.

4.2 Švejdar's Proof

As we have seen above, the original proof of PSPACE-hardness of **IPC** is not very intuitive in the sense that it doesn't give us any insights into why **IPC** is PSPACE-hard. Moreover, it uses techniques that are not standard for complexity theory. This motivated Švejdar [12] to prove the same result but using different methods. He does draw on certain aspects of Statman's proof though as we shall see.

The strategy of Švejdar's proof, as presented in [12], is similar to the proofs of PSPACE-hardness of normal modal logics between **K** and **S4** by Ladner [7]. The aim is to define a polynomial time reduction τ such that it holds that for all quantified Boolean formulas A with no free variables, that A is a valid QBF iff $\tau(A)$ is a valid formula of **IPC**. In fact, Švejdar proves the equivalent statement that A is a QBF that is not valid iff there exists a Kripke counterexample for $\tau(A)$, i.e. there exists a model where for the root a it holds that $a \nVdash \tau(A)$.

When taking a closer look at QBFs, we will see that evaluating them leads us to a tree-like structure. When evaluating a QBF, we start by peeling of the outermost quantifier. If this quantifier is $\exists$, we choose one of 0 or 1 and substitute it for the variable that was bound by the quantifier $\exists$. On the other hand, if it is $\forall$, then we must substitute both 0 and 1 for the newly freed propositional variable. This process can be illustrated by a tree: We start with the original formula in the root and each existential quantifier extends the tree by a single branch and each universal quantifier adds a binary branching.

Such trees provide a connection between QBFs and the semantic structures for **IPC**. This connection between **QBF** and **IPC** is used in the proof by Švejdar [12].

Roughly speaking, the reduction we are looking for maps a QBF that is not valid to a formula that can be falsified in a Kripke model. On the other hand, a valid QBF is mapped to a formula that cannot be falsified in a model. Furthermore, we have to make sure that the reduction is poly-time.

The reduction is defined recursively. In case we have a formula without any quantifiers, the reduction is the identity function. In case it is of the form $\exists p_k B(p_1, \ldots, p_k)$, the formula is mapped to one that gives a countermodel consisting of a root and two countermodels for $B(p_1, \ldots, p_k)$, where p_k is false in one and true in the other. Similarly, if the formula is of the form $\forall p_k B(p_1, \ldots, p_k)$, it is mapped to one that gives rise one countermodel for it that contains a countermodel for $B(p_1, \ldots, p_k)$, where p_k is either true or false everywhere in it.

More precisely, the reduction τ that maps a quantified Boolean formula A to an intuitionistic formula $\tau(A)$ is defined as follows:

Definition 23. Let A be a quantified Boolean formula in prenex form, i.e. $A = Q_m p_m \ldots Q_1 p_1 B(p_1, \ldots, p_m)$.
Let p denote $(p_1, \ldots, p_m)$.
Then the formulas $\tau(A_0), \ldots, \tau(A_m)$ are constructed recursively:
$\tau(A_0) = B(p)$.
If $Q_{j+1} = \exists$, then
$\tau(A_{j+1}) =$
$[(\tau(A_j) \to q_{j+1}) \wedge ((p_{j+1} \to q_{j+1}) \to s_{j+1}) \wedge ((\neg p_{j+1} \to q_{j+1}) \to s_{j+1})] \to s_{j+1}$.
If $Q_{j+1} = \forall$, then

$$\tau(A_{j+1}) = [(\tau(A_j) \to q_{j+1}) \wedge ((p_{j+1} \to q_{j+1}) \wedge (\neg p_{j+1} \to q_{j+1}) \to q_{j+1})] \to q_{j+1}.$$

The variables q_{j+1} and s_{j+1} have auxiliary purpose and serve for simplification as we will see in the following.

Finally, let $\tau(A) = \tau(A_m)$.

Now, we have to show that the function τ as defined above is indeed a reduction from the validity problem of **QBF** to the validity problem of **IPC**, i.e. it has to be shown that for all quantified Boolean formulas A: A is **QBF**-valid iff $\tau(A)$ is **IPC**-valid. This is equivalent to: A is not **QBF**-valid iff $\tau(A)$ is not **IPC**-valid.

For understanding the following proof, it is substantial to understand what it means that a formula is not **QBF**-valid. If for a quantified Boolean formula A we have that A is not **QBF**-valid, then it is not the case that the formula is true for all possible valuations of the propositional variables that are contained in it. Therefore, there must be an assignment of truth values that makes the formula false.

In case of **IPC**, if a formula is not in **IPC**-valid, then there is an **IPC** model that refutes the formula.

Let us recall Definition 18 and fix some notation:

Definition 24. Let A be a quantified Boolean formula with propositional variables $p_1, \ldots, p_m$. If we say that V is a valuation of $p_{j+1}, \ldots, p_m$, it means that we assigned fixed truth values to these variables, i.e. each variable $p_{j+1}, \ldots, p_m$ is either assigned 0 or 1.
Given such a valuation V, we say that **in an IPC model M, the propositional variables $\mathbf{p_{j+1}, \ldots, p_m}$ are evaluated according to V** iff for all states in the model it holds that each p_i for $j + 1 \leq i \leq m$ has the same truth value at this state as the one that is assigned to it by V. So, if $V(p_i) = 1$ and in the **IPC** model M, p_i is evaluated according to V, then p_i is true at every state in M. On the other hand, if $V(p_i) = 0$, then p_i is false at every state in the model.

Lemma 25. *Let V be a valuation of the atoms $p_{j+1}, \ldots, p_m$.*
*Then $V \nvDash Q_j p_j \ldots Q_1 p_1 B(p)$ iff there is an **IPC** model where each atom $p_i, i > j$ is evaluated according to V and where for its root a, $a \nVdash \tau(A_j)$.*

Proof. By induction on j.

Base Case $j = 0$

- ($\Rightarrow$) Let $V \nvDash B(p)$. Consider the one-element reflexive frame consisting of a single state a and the model based on this frame, where all propositional variables are evaluated according to V. Then in this model clearly $a \nVdash B(p)$.
- ($\Leftarrow$) Let K be a model **IPC** such that for the root a, $a \nVdash B(p)$ and all propositional variables are evaluated according to V. Now, by induction it can be shown that every subformula of $B(p)$ has the same truth value everywhere in the model, namely the one assigned by V. Thus, $V \nvDash B(p)$.

I.H. Assume that for a j with $0 \leq j \leq m$ the following holds:

Given a valuation V of the atoms $p_{j+1}, \ldots, p_m$, we have that

$$V \not\models Q_j p_j \ldots Q_1 p_1 B(p)$$

iff there is a model where $p_{j+1}, \ldots, p_m$ are evaluated according to V everywhere in the model and where for the root a it holds that $a \nVdash \tau(A_j)$.

Inductive Step

$Q_{j+1} = \exists$

($\Rightarrow$) Let V be an evaluation of $p_{j+2}, \ldots, p_m$ and let

$$V \not\models \exists p_{j+1} Q_j p_j \ldots Q_1 p_1 B(p).$$

The following definition specifies valuations that extend a given valuation V. These extensions will also be used in other parts of this proof.
We define valuations V_0 and V_1: Given a valuation V of propositional variables $p_{j+2}, \ldots, p_m$, let us define valuations $V_0 : \{p_{j+1}, \ldots, p_m\} \to \{0, 1\}$ and $V_1 : \{p_{j+1}, \ldots, p_m\} \to \{0, 1\}$ as follows:

$$V_0(p_i) = \begin{cases} V(p_i) & \text{if } (j+2) \leq i \leq m \\ 0 & \text{if } i = j+1 \end{cases}$$

$$V_1(p_i) = \begin{cases} V(p_i) & \text{if } (j+2) \leq i \leq m \\ 1 & \text{if } i = j+1 \end{cases}$$

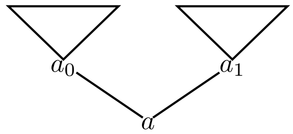
Then it follows from the definition of the quantifiers that

$$V_0 \not\models Q_j p_j \ldots Q_1 p_1 B(p) \text{ and } V_1 \not\models Q_j p_j \ldots Q_1 p_1 B(p)$$

because otherwise we would have that $V \models \exists p_{j+1} Q_j p_j \ldots Q_1 p_1 B(p)$.
By the induction hypothesis, we get two models: K_0 and K_1, where $p_{j+2}, \ldots p_m$ are evaluated according to V and where for the roots a_0 and a_1 respectively, $a_0 \nVdash \tau(A_j)$ and $a_1 \nVdash \tau(A_j)$.
Furthermore, p_{j+1} is false everywhere in K_0 and true everywhere in K_1. Thus, for the respective roots, it holds that $a_0 \Vdash \neg p_{j+1}$ and $a_1 \Vdash p_{j+1}$. Now, our aim is to build a countermodel for $\tau(A_{j+1})$. Recall that since $Q_{j+1} = \exists$, we have that $\tau(A_{j+1})$ is equal to

$$\begin{gathered} ((\tau(A) \to q_{j+1}) \wedge ((p_{j+1} \to q_{j+1}) \to s_{j+1}) \\ \wedge \\ ((\neg p_{j+1} \to q_{j+1}) \to s_{j+1})) \to s_{j+1}. \end{gathered}$$

The model can be built in the following way: We take a new state a and connect it to the two models K_0 and K_1, with roots a_0 and a_1 in such a way that all states of K_0 and K_1 are accessible from a, i.e. our new model consists of a root a and the two submodels generated by a_0 and a_1.



Next, the truth values of all the atomic propositions have to be defined in a. For all the other states in the model, we have to specify the truth values of the newly introduced variables q_{j+1} and s_{j+1}. This is done as follows:

- In a, let $p_{j+2}, \ldots, p_m$ be evaluated according to V.
- The truth values of $p_1, \ldots, p_{j+1}, q_1, \ldots, q_j, s_1, \ldots, s_j$ are set to false in a.
- Everywhere in the model, the truth value of q_{j+1} is defined as being the same as the one of $\tau(A_j)$.
- The truth value of s_{j+1} is defined as being the same as the one of $(p_{j+1} \to q_{j+1}) \vee (\neg p_{j+1} \to q_{j+1})$.

It can easily be checked that the new model is indeed an **IPC** model and that the persistency condition is satisfied.

Now we will show that for the root a it holds that $a \nVdash \tau(A_{j+1})$. Since $\tau(A_j)$ and q_{j+1} have the same truth value everywhere in our new model, $a \models \tau(A_j) \to q_{j+1}$. Analogously, since s_{j+1} has the same truth value as $(p_{j+1} \to q_{j+1}) \vee (\neg p_{j+1} \to q_{j+1})$ everywhere in the model, $a \models ((p_{j+1} \to q_{j+1}) \vee (\neg p_{j+1} \to q_{j+1})) \to s_{j+1}$. Now, it follows that $a \Vdash (\neg p_{j+1} \to q_{j+1}) \to s_{j+1}$ and $a \Vdash (p_{j+1} \to q_{j+1}) \to s_{j+1}$. Thus, $a \Vdash (\tau(A_j) \to q_{j+1}) \wedge (((\neg p_{j+1} \to q_{j+1}) \to s_{j+1}) \wedge ((p_{j+1} \to q_{j+1}) \to s_{j+1}))$.
We have that $a \nVdash \neg p_{j+1} \to q_{j+1}$ and $a \nVdash p_{j+1} \to q_{j+1}$ because $a_0 \Vdash \neg p_{j+1}, a_0 \nVdash q_{j+1}, a_1 \Vdash p_{j+1}$ and $a_1 \nVdash q_{j+1}$. Thus, $a \nVdash (\neg p_{j+1} \to q_{j+1}) \vee (p_{j+1} \to q_{j+1})$. So, $a \nVdash s_{j+1}$.

Therefore, $a \nVdash [((\tau(A_j) \to q_{j+1}) \wedge ((\neg p_{j+1} \to q_{j+1}) \to s_{j+1}) \wedge ((p_{j+1} \to q_{j+1}) \to s_{j+1}))] \to s_{j+1}$ and thus $a \nVdash \tau(A_{j+1})$.

($\Leftarrow$)

Let K be a model with root a where everywhere in the model p_i for $i > j+1$ is evaluated according to V, and let $a \nVdash \tau(A_{j+1})$ for the root a. So, $a \nVdash [((\tau(A_j) \to q_{j+1}) \wedge ((\neg p_{j+1} \to q_{j+1}) \to s_{j+1}) \wedge ((p_{j+1} \to q_{j+1}) \to s_{j+1}))] \to s_{j+1}$. Thus, there is a state $a' \geq a$ such that $a' \Vdash (\tau(A_j) \to q_{j+1}) \wedge ((\neg p_{j+1} \to q_{j+1}) \to s_{j+1}) \wedge ((p_{j+1} \to q_{j+1}) \to s_{j+1})$ and $a' \nVdash s_{j+1}$.

Since $a \Vdash (p_{j+1} \to q_{j+1}) \to s_{j+1}$ and $a' \nVdash s_{j+1}$, it must be that $a' \nVdash p_{j+1} \to q_{j+1}$. Thus, $\exists a_1 \geq a'$ such that $a_1 \Vdash p_{j+1}$ and $a_1 \nVdash q_{j+1}$. Since $a_1 \geq a'$, we also have that $a_1 \Vdash (\tau(A_j) \to q_{j+1}) \wedge ((\neg p_{j+1} \to q_{j+1}) \to s_{j+1}) \wedge ((p_{j+1} \to q_{j+1}) \to s_{j+1})$ and therefore $a_1 \Vdash \tau(A_{j+1}) \to q_{j+1}$. Since $a_1 \nVdash q_{j+1}$, $a_1 \nVdash \tau(A_{j+1})$.

Now, consider the submodel generated by a_1. In this model, p_{j+2}, ..., p_m are evaluated according to V and p_{j+1} is everywhere true. Furthermore, it is a counterexample for $\tau(A_j)$. Thus, by I.H. it follows that $V_1 \nvDash Q_j p_j \dots Q_1 p_1 B(p)$ for the valuation V_1 as it is defined in the above definition.

Analogously, since $a \Vdash (\neg p_{j+1} \to q_{j+1}) \to s_{j+1}$ and $a' \nVdash s_{j+1}$, it holds that $a' \nVdash \neg p_{j+1} \to q_{j+1}$. Thus, $\exists a_0 \geq a'$ such that $a_0 \Vdash \neg p_{j+1}$ and $a_0 \nVdash q_{j+1}$. Since $a_0 \geq a'$, we also have that $a_0 \Vdash (\tau(A_j) \to q_{j+1}) \wedge ((\neg p_{j+1} \to q_{j+1}) \to s_{j+1}) \wedge ((p_{j+1} \to q_{j+1}) \to s_{j+1})$ and therefore $a_0 \Vdash \tau(A_{j+1}) \to q_{j+1}$. Since $a_0 \nVdash q_{j+1}$, $a_0 \nVdash \tau(A_{j+1})$.
Now, consider the submodel generated by a_0. In this model p_{j+2}, ..., p_m are evaluated according to V and p_{j+1} is everywhere false. Furthermore, it is a counterexample for $\tau(A_j)$. Thus, by I.H. it follows for the valuation V_0, as defined in the above definition, $V_0 \nvDash Q_j p_j \dots Q_1 p_1 B(p)$.

Hence $V_0 \nvDash Q_j p_j \dots Q_1 p_1 B(p)$ and $V_1 \nvDash Q_j p_j \dots Q_1 p_1 B(p)$.

Thus, it follows from the definition of V_0 and V_1 that

$$V \nvDash \exists p_{j+1} Q_j p_j \dots Q_1 p_1 B(p).$$

$Q_{j+1} = \forall$

Recall that in this case
$\tau(A_{j+1}) =$
$[(\tau(A_j) \to q_{j+1}) \wedge ((p_{j+1} \to q_{j+1}) \wedge (\neg p_{j+1} \to q_{j+1}) \to q_{j+1})] \to q_{j+1}$.

($\Rightarrow$) Let V be an evaluation of the atoms $p_{j+2}, \dots, p_m$ and assume that $V \nvDash \forall p_{j+1} Q_j p_j \dots Q_1 p_1 B(p)$. Then by the definition of the quantifiers and as by the definition of V_0 and V_1 as above , it follows that

(1) $V_0 \nvDash Q_j p_j \dots Q_1 p_1 B(p)$
or
(2) $V_1 \nvDash Q_j p_j \dots Q_1 p_1 B(p)$.

Case 1 $\overset{I.H.}{\Rightarrow}$ $\exists$ a model K_0 where $p_{j+2}, \dots, p_m$ are evaluated according to V and where p_{j+1} is false everywhere in the model. Furthermore, for the root a it holds that $a \nVdash \tau(A_j)$. Since p_{j+1} is false everywhere in the model , $a \Vdash \neg p_{j+1}$. Next we have to build a countermodel for $\tau(A_{j+1})$. We will build it by extending the model K_0 in the following way:
Everywhere in the model, the truth value of q_{j+1} is set to the same as the one of $\tau(A_j)$.

Then we have that $a \Vdash \tau(A_j) \to p_{j+1}$.

Next, let $a' \geq a$ with $a' \Vdash (p_{j+1} \to q_{j+1}) \wedge (\neg p_{j+1} \to q_{j+1})$. Then $a' \Vdash \neg p_{j+1} \to q_{j+1}$. Because $a \Vdash \neg p_{j+1}$, also $a' \Vdash \neg p_{j+1}$ and since $a' \Vdash \neg p_{j+1} \to q_{j+1}$, $a' \Vdash q_{j+1}$.

Thus, $a \Vdash (\tau(A_j) \to p_{j+1}) \wedge (((p_{j+1} \to q_{j+1}) \wedge (\neg p_{j+1} \to q_{j+1})) \to q_{j+1})$. But since q_{j+1} has everywhere the same truth value as $\tau(A_j)$, $a \nVdash q_{j+1}$ and thus the model K_0 with additional atomic proposition q_{j+1} is a counterexample for $\tau(A_{j+1})$, where $p_{j+2}, \ldots, p_m$ are evaluated according to V.

Case 2 $\overset{I.H.}{\Rightarrow}$ $\exists$ a model K_1 where $p_{j+2}, \ldots, p_m$ are evaluated according to V and where p_{j+1} is true everywhere in the model. Furthermore, for the root a it holds that $a \nVdash \tau(A_j)$. Since p_{j+1} is everywhere in the model true, $a \Vdash p_{j+1}$. Next we have to build a countermodel for $\tau(A_{j+1})$.

We will build the model by extending the model K_0 in the following way:

The truth value of q_{j+1} is everywhere set to the same as the one of $\tau(A_j)$.

Then we have that $a \Vdash \tau(A_j) \to p_{j+1}$.

Next, let $a' \geq a$ with $a' \Vdash (p_{j+1} \to q_{j+1}) \wedge (\neg p_{j+1} \to q_{j+1})$. Then $a' \Vdash p_{j+1} \to q_{j+1}$. Because $a \Vdash p_{j+1}$, also $a' \Vdash p_{j+1}$ and since $a' \Vdash p_{j+1} \to q_{j+1}$, $a' \Vdash q_{j+1}$.

Thus, $a \Vdash ((p_{j+1} \to q_{j+1}) \wedge (\neg p_{j+1} \to q_{j+1})) \to q_{j+1}$. But since q_{j+1} has everywhere the same truth value as $\tau(A_j)$, $a \nVdash q_{j+1}$ and thus the model K_0 with additional atomic proposition q_{j+1} is a counterexample for $\tau(A_{j+1})$, where $p_{j+2}, \ldots, p_m$ are evaluated according to V.

($\Leftarrow$) Let V be an evaluation of atoms $p_{j+2}, \ldots, p_m$ and let K be a model where $p_{j+2}, \ldots, p_m$ are everywhere evaluated according to V and where for the root a it holds that $a \nVdash \tau(A_{j+1})$, i.e. $a \nVdash [(\tau(A_j) \to q_{j+1}) \wedge ((p_{j+1} \to q_{j+1}) \wedge (\neg p_{j+1} \to q_{j+1}) \to q_{j+1})] \to q_{j+1}$. Then there is a state $a' \geq a$ such that $a' \Vdash (\tau(A_j) \to q_{j+1}) \wedge ((p_{j+1} \to q_{j+1}) \wedge (\neg p_{j+1} \to q_{j+1}) \to q_{j+1})$ and $a' \nVdash q_{j+1}$. So, since $a' \Vdash (p_{j+1} \to q_{j+1}) \wedge (\neg p_{j+1} \to q_{j+1}) \to q_{j+1}$, it holds that $a' \nVdash (p_{j+1} \to q_{j+1}) \wedge (\neg p_{j+1} \to q_{j+1})$.

Thus,

(1) $a' \nVdash p_{j+1} \to q_{j+1}$

or

(2) $a' \nVdash \neg p_{j+1} \to q_{j+1}$.

Case 1 $\Rightarrow \exists a_1 \geq a'$ such that $a_1 \Vdash p_{j+1}$ and $a_1 \nVdash q_{j+1}$. Since $a' \Vdash \tau(A_{j+1}) \to q_{j+1}$, $a_1 \nVdash \tau(A_{j+1})$. So the submodel generated by a_1 is a counterexample for $\tau(A_{j+1})$. Furthermore, in this model $p_{j+2}, \ldots, p_m$ are everywhere evaluated according to V.

$\overset{I.H.}{\Rightarrow} V_1 \nvDash Q_j p_j, \ldots, Q_1 p_1 B(p)$.
Thus, $V \nvDash \forall q_{j+1} Q_j p_j, \ldots, Q_1 p_1 B(p)$.

Case 2 $\Rightarrow \exists a_0 \geq a'$ such that $a_0 \Vdash \neg p_{j+1}$ and $a_0 \nVdash q_{j+1}$. Since $a' \Vdash \tau(A_{j+1}) \to q_{j+1}$, $a_0 \nVdash \tau(A_{j+1})$. So the submodel generated by a_0 is a counterexample for $\tau(A_{j+1})$. Furthermore, in this model $p_{j+2}, \ldots, p_m$ are everywhere evaluated according to V.
$\overset{I.H.}{\Rightarrow} V_0 \nvDash Q_j p_j, \ldots, Q_1 p_1 B(p)$.
Thus, $V \nvDash \forall q_{j+1} Q_j p_j, \ldots, Q_1 p_1 B(p)$.

So, $V \nvDash \forall q_{j+1} Q_j p_j, \ldots, Q_1 p_1 B(p)$.

Hence, given an evaluation V of the atoms $p_{j+1}, \ldots, p_m$, $V \nvDash Q_j p_j \ldots Q_1 p_1 B(p)$ iff there is a model where each atom $p_i, i > j$ is evaluated according to V and where for the root a, $a \nVdash \tau(A_j)$.

Now, the main theorem follows immediately:

Theorem 26. *The validity problem for* ***IPC*** *is* PSPACE*-complete.*

Proof. For $j = m$ the previous lemma says that the quantified Boolean formula $Q_m p_m \ldots Q_1 p_1 B(p)$ is false if and only if $\tau(A_m)$ has a Kripke-counterexample. So, the function that maps A to $\tau(A) = \tau(A_m)$ is a reduction from the validity problem of **QBF** to the validity problem of **IPC**. The reduction can be computed in polynomial time because there is one step in the recursion for every quantifier Q_j in the original formula and in each of the steps a new formula $\tau(A_j)$ is constructed, whose length is clearly bound by a polynomial. So, the validity problem of **IPC** is PSPACE-hard. Since it is also in PSPACE, it is PSPACE-complete.

Note that in the proof of Lemma 25, we did not use any specific property of **IPC** except the disjunction property. Thus, the proof for *PSPACE*-hardness would also go through for other intermediate logics with disjunction property.

Corollary 27. *Any intermediate logic that has the disjunction property is* PSPACE*-hard.*

5 Complexity of Intermediate Logics

5.1 Every Intermediate Logic is co-NP-hard

Let **L** be a consistent intermediate logic. To show co-NP-hardness we have to show that **L** is at least as hard as all the problems in NP. This is done by showing that the validity problem of **L** is at least as hard as the validity problem of **CPC**, which is known to be co-NP-hard.

Recall Glivenko's Theorem, which says that for every formula

$$\varphi, \varphi \in \mathbf{CPC} \text{ iff } \neg\neg\varphi \in \mathbf{IPC}.$$

We can use this result in constructing the reduction,τ, from **CPC** to **L** by defining

$$\tau(\varphi) = \neg\neg\varphi.$$

Lemma 28. *Given a consistent intermediate logic* $\boldsymbol{L}$*, for all formulas* $\varphi, \varphi \in$ $\mathbf{CPC}$ *iff* $\tau(\varphi) \in \mathbf{L}$.

Proof. Let $\varphi \in \mathbf{CPC}$. Then by Glivenko's theorem, $\tau(\varphi) \in \mathbf{IPC}$ and since $\mathbf{IPC} \subseteq \mathbf{L}$, $\tau(\varphi) \in \mathbf{L}$.

Next, assume that $\varphi \notin \mathbf{CPC}$. Then $\neg\varphi \in \mathbf{CPC}$ and by Glivenko's Theorem $\tau(\neg\varphi) = \neg\neg(\neg\varphi) \in \mathbf{IPC}$. Then also $\neg\neg(\neg\varphi) \in \mathbf{L}$ and since $\mathbf{LC}$ is consistent, $\neg\neg\varphi \notin \mathbf{L}$ because otherwise we would have that $\neg\neg\varphi \in \mathbf{L}$ and $\neg(\neg\neg\varphi) \in \mathbf{L}$. Hence, $\varphi \in \mathbf{CPC}$ iff $\tau(\varphi) \in \mathbf{L}$.

Theorem 29. *Every intermediate logic is co-*NP*-hard.*

Proof. From the previous lemma it follows that the validity problem of classical propositional logic can be reduced to the one of every intermediate logic with the reduction $\tau(\varphi) = \neg\neg\varphi$. Since this reductiton consists only of a double negation of the original formula, it can be computed in polynomial time. Then, since the validity problem of $\mathbf{CPC}$ is co-NP-hard, it follows directly that the validity problems of intermediate logics are also co-NP-hard.

5.2 LC

One of the best known intermediate logics is the logic $\mathbf{LC}$, also known as *Dummett logic*, which is defined as follows:

$$\mathbf{LC} = \mathbf{IPC} + (p \to q) \vee (q \to p)$$

Semantically, $\mathbf{LC}$ is the logic of the class of finite linearly ordered frames.

The non-validity problem of LC is in NP To show that the non-validity problem of $\mathbf{LC}$ is in NP it is sufficient to show the following [1]:

- Every formula that is not in the logic $\mathbf{LC}$ can be refuted in an $\mathbf{LC}$ frame of size at most polynomial in the number of subformulas of the formula.
- Given a frame, one can decide in polynomial time whether it is an $\mathbf{LC}$ frame.

Lemma 30. *For all formulas* φ*, if* $\varphi \notin \mathbf{LC}$*, then there is an* $\mathbf{LC}$ *frame* $\mathfrak{F}$ *such that* $\mathfrak{F} \not\Vdash \varphi$ *and* $|\mathfrak{F}| \leq |Sub\varphi| + 1$.

Proof. Let $\varphi \notin \mathbf{LC}$. Then there is a finite linearly ordered frame $\mathfrak{F} = \langle W, R\rangle$ such that $\mathfrak{F} \not\Vdash \varphi$, i.e. there is a model $M = \langle \mathfrak{F}, V\rangle$ such that $M \not\Vdash \varphi$. Now we have to show that we can build a new $\mathbf{LC}$ model out of this model whose size is linear in the number of subformulas of φ in such a way that the new model still refutes φ.

Definition 31. Given a formula φ and an $\mathbf{LC}$ model $M = \langle W, R, V\rangle$ with maximal element m, define the following set $X \subseteq W$:

$$X = \{m\} \cup \max\{x | x \not\Vdash p, p \in Var\varphi\}.$$

Then define the model M' as follows: $M' = \langle X, R', V'\rangle$, where R' and V' are the restrictions of R and V to X.

Then $\mathfrak{F}' = \langle X, R' \rangle$ is also an **LC** frame because it is finite and R' imposes a linear ordering on the elements of X. Furthermore, $|X| \leq n+1$, where n is the number of subformulas in φ. By a simple induction the following fact can be shown:

Fact 32. *For all $\psi \in Sub\varphi$ and for all $x \in X$:*

$$M', x \Vdash \psi \textit{ iff } M, x \Vdash \psi$$

Next, we also have to show that for all formulas $\psi \in Sub\varphi$ it holds that if ψ is refuted in M, then it is also refuted in the new model M'.

More formally: For all $\psi \in Sub\varphi$ and for all $x \in W$, if $M, x \nVdash \psi$, then there is a $y \in X$ such that $y \geq x$ and $M', y \nVdash \psi$. This is shown using a proof by induction on the complexity of ψ:

Base Case $\psi = p$
Let $M, x \nVdash p$. Then take $y = \max\{x | x \nVdash p\}$. We know that $y \in X$ because $p \in VAR\varphi$. So, $y \geq x$ and $y \nVdash p$.

I.H Assume that for a $\psi \in Sub\varphi$ it holds that for all $x \in W$, if $M, x \nVdash \psi$, then there is a $y \geq x$ such that $y \in X$ and $M', y \nVdash \psi$.

Inductive Step $\psi = \psi_1 \wedge \psi_2$
Let $M, x \nVdash \psi_1 \wedge \psi_2$. Then $M, x \nVdash \psi_1$ or $M, x \nVdash \psi_2$. In the first case, it follows from I.H. that there is a , $y \in X$ such that $y \geq x$ and $M', y \nVdash \psi_1$, and in the second case that there is a $y' \in X$ such that $y' \geq x$ and $M', y' \nVdash \psi_2$. Thus, in either case we have that there is some $y \in X$ such that $y \geq x$ and $M', y \nVdash \psi_1 \wedge \psi_2$.

$\psi = \psi_1 \vee \psi_2$
Let $M, x \nVdash \psi_1 \vee \psi_2$. Then $M, x \nVdash \psi_1$ and $M, x \nVdash \psi_2$. From I.H. it follows that there is a $y_1 \in X, y_1 \geq x$ such that $M', y_1 \nVdash \psi_1$ and that there is a $y_2 \in X, y_2 \geq x$ such that $M', y_2 \nVdash \psi_2$. Now, let $y = min\{y_1, y_2\}$. Then $y \in X, y \geq x$ and furthermore, $M', y \nVdash \psi_1$ and $M', y \nVdash \psi_2$. Thus, $M', y \nVdash \psi_1 \vee \psi_2$.

$\psi = \psi_1 \rightarrow \psi_2$
Let $M, x \nVdash \psi_1 \rightarrow \psi_2$. Then there is a $x' \in W, x' \geq x$ such that $M, x' \Vdash \psi_1$ and $M, x' \nVdash \psi_2$. From I.H. it follows that there is a $y \in X, y \geq x'$ such that $M', y \nVdash \psi_2$. Since $y \geq x$, it must be the case that $M, y \Vdash \psi_1$ and then it follows from Fact 32 that $M', y \Vdash \psi_1$. Thus, $M', y \nVdash \psi_1 \rightarrow \psi_2$.

$\psi = \neg\psi'$
Let $M, x \nVdash \neg\psi'$. This means that it is not the case that for all $x' \geq x$ it holds that $M, x' \nVdash \neg\psi'$. Thus, there is a $x'' \geq x$ such that $M, x'' \Vdash \psi'$. Then also $M, m \Vdash \psi'$ for the maximal element m of the frame $\mathfrak{F}$. Recall that $m \in X$. Then it follows from Fact 32 that $M', m \Vdash \psi'$ and therefore $M', m \nVdash \neg\psi'$.

Hence, for all formulas $\psi \in VAR\varphi$ it holds that for all $x \in W$, if $M, x \nVdash \psi$, then there is a $y \in X$ such that $y \geq x$ and $M', y \nVdash \psi$.

Therefore, for all formulas φ, if $\varphi \notin$ **LC**, then there is an **LC** frame $\mathfrak{F}$ such that $\mathfrak{F} \nVdash \varphi$ and $|\mathfrak{F}| \leq |Sub\varphi| + 1$.

Lemma 33. *Given a frame $\mathfrak{F} = \langle W, R\rangle$, we can decide in polynomial time whether $\mathfrak{F}$ is an **LC** frame or not.*

Proof. Given a frame $\mathfrak{F} = \langle W, R\rangle$, we only have to check whether it is finite and linearly ordered – i.e. we look at every state x and check whether xRx and whether for all states $y \neq x$ it is the case that either xRy or yRx. Furthermore we should check if the accessibility relation is transitive. This can be done in polynomial time.

Theorem 34. *The validity problem of **LC** is co-NP-complete.*

Proof. Given a formula φ, we can nondeterministically choose a frame $\mathfrak{F}$ polynomial in number of subformulas of φ and then check whether there is a model based on $\mathfrak{F}$ that refutes φ. If $\varphi \notin$ **LC**, then we can find such a frame (Lemma 30). Next, it has to be checked whether the frame we picked is an **LC** frame. By Lemma 33, this can be done in polynomial time. Thus, it can be decided in polynomial time whether a formula φ is not valid in **LC** and thus the non-validity problem of **LC** is in NP. Hence, we have that the validity problem of **LC** is in co-NP. Since we know that it is also co-NP-hard (Theorem 29), it follows that the validity problem of **LC** is co-NP-complete.

5.3 Tabular Logics

Tabular logics are among the simplest intermediate logics to characterize since a tabular logic, **TL**, is defined to be a logic of a single finite frame, i.e., **TL** $=$ $Log(\mathfrak{F})$ where $\mathfrak{F}$ is a finite intuitionistic Kripke frame.

The non-validity problem of any Tabular Logic is in NP

Theorem 35. *The validity problem of any tabular logic, **TL**$=$ $Log(\mathfrak{F})$, is co-NP-complete.*

Proof. Given a formula φ, we can nondeterministically choose a valuation V on the frame $\mathfrak{F}$ and then check whether $\langle\mathfrak{F}, V\rangle$ refutes φ. Since checking whether a formula φ is not valid in $\langle\mathfrak{F}, V\rangle$ can be decided in polynomial time, we know that the validity problem of **TL** is in co-NP. Since we know that it is also co-NP-hard (Theorem 29), it follows that the validity problem of **TL** is co-NP-complete.

6 Conclusions

In the last section, we investigated the complexity of some simple intermediate logics that are NP-complete. It is one thing to show NP-completeness, however, as demonstrated by the highly complicated proofs that **IPC** is PSPACE-complete in Sections 3 and 4, it is quite another to show that a logic is PSPACE-complete. At the end of Section 4, we mention a corollary of the PSPACE-hardness of **IPC** that does provide us with a way to avoid the tricky reductions – namely, that any intermediate logic that has the disjunction property is PSPACE-hard. This corollary provides us with an avenue to investigate the complexity of the class of intermediate logics with the disjunction property. Another known result, due to Chagrov, is that the intermediate logic **KC** (i.e., **IPC**+ $\neg p \vee \neg\neg p$) is PSPACE-complete. This result shows that there are intermediate logics with the finite model property and without the disjunction property, that are PSPACE-complete. Since we have these completeness results, we can also attempt to reduce other intermediate logics to **IPC**, **KC** or **LC** for instance.

There are many open problems related to the results presented in this paper that are waiting to be investigated. We have identified two priority avenues for future investigation as briefly mentioned in [7] and [1]:

1. The complexity of variable fragments of intermediate logics. Rybakov [10] has recently shown that the two-variable fragment of IPC is PSPACE-complete.
2. Given that all normal modal logics extending **S4.3** have an NP-complete satisfiability problem, it would be interesting to investigate the possibility of an analogous result for intermediate logics.

Finally, we conclude by motivating reasons for pursuing further research in this area:

1. It has the potential to give insights into the NP=PSPACE problem as suggested in [6].
2. It will aid in the classification of intermediate logics.

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