Name (to appear in publication): Johan van Benthem

Position (to appear in publication): University Professor of Logic and Professor of Philosophy

Affiliation (to appear in publication): University of Amsterdam & Stanford University

The 5 Questions

1. Why were you initially drawn to the foundations of mathematics and/or the philosophy of mathematics?

I started out as a student of physics, hard-working, interested, but alas, not 'in love' with my subject. Then logic struck, and having become interested in this subject for various reasons – including the fascinating personality of my first teacher –, I switched after my candidate's program, to take two master's degrees, in mathematics and in philosophy. The beauty of mathematics was clear to me at once, with the amazing power, surprising twists, and indeed the music, of abstract arguments. As our professor of Analysis wrote at the time in our study guide "Mathematics is about the delight in the purity of trains of thought", and old-fashioned though this phrasing sounded in the revolutionary 1960s, it did resonate with me. Then I had the privilege of being taught set-theoretic topology by a group of brilliant students around De Groot, our leading expert around the time, who worked with Moore's method of discovering a subject for oneself. Topology unfolded from a few definitions and examples to real theorems that we had to prove ourselves – and the take-home exam took sleepless nights, as it included proving some results from scratch which came from a recent dissertation (as it turned out later). Only at the very end did De Groot appear, to give one lecture on Tychonoff's Theorem where an application was made of the Axiom of Choice, a sacral act only to be performed by tenured full professors.

At the same time, I was learning how mathematical logic and mathematics formed a natural unity, through my evening reading of Nagel & Newman's little book *Gödel's Proof* (1957), which reinforced this mystery that the deepest insights are accessible through mathematical clarity. Also later on, I have often been influenced by books bought on recommendation from other students, not as course requirements from our teachers – such as Benacerraf & Putnam's *Philosophy of Mathematics*, van Heijenoort's *From Frege to Gödel*, or the little gem edited by Jaakko Hintikka called *The Philosophy of Mathematics*, replete with logic classics. And then formal courses backed this up. I have been taught by intuitionists like Heyting and

Troelstra, by formalists like Curry, and I eventually wrote my Ph.D. thesis in mathematics with Martin Löb, a very purist proof theorist, and a gentleman from the Grand Old German School in mathematical logic and the foundations of mathematics. Of course – these were the revolutionary early seventies – I immediately rejected all intellectual views of all my professors out of hand. I turned to modal logic, semantics, and model theory. Löb was disappointed. Himself a Jewish refugee from Germany, his view was that proof theory was about deep structure, elegance and style, qualities he attributed to German and Japanese culture, whereas model theory was about fast images and muddling through, which he associated with Anglo-Saxon and French culture. (In his highly original view of history, this explained the core alliances in the Second World War.) My own desire for independence was helped to some extent by these and other curious features. For instance, Curry really believed in formalism and formula games, and did not interact with his audience. I once was the only student in the room, but he still lectured at a distance as if there were a crowd. The exam consisted in two hours of formal deductions in his office – he collected streams of computer print-out paper for this purpose – in pure implicational propositional calculus, with no appeals to the Deduction Theorem allowed. Halfway through the exam, a silver-haired lady knocked on the door, and watched me struggle with formulas and paper. She said: "Good afternoon. I am Mrs. Curry, and I want you to know that my husband is a kind man..." That gentle gesture did more for me than the Deduction Theorem could ever have. I will save my student observations of the other professors for my Memoirs, and move ahead now.

I find it hard to separate thinking about mathematics itself, mathematical logic, 'foundations', and the philosophy of mathematics. The sensibilities needed for one may be close to those needed for the other. I definitely found what I was looking for in the delights of proving significant results in the meta-theory of modal 'intensional' logic, and through developing 'modal correspondence theory', seeing patterns and connections with classical 'extensional' logic that were below the surface of received opinion at the time, only brought to light by mathematical analysis. But I also had the transcendent moments of despair one needs to experience, when thinking about layer on layer of meta-analysis in set theory, or the reversals of perspective when translating one theory or logic into another, and then back again. Nowhere a piece of solid ground! It was like the way I often feel when lying on a beach and looking at the blue sky overhead: an immediate stomach-twisting sense of falling off into the Universe. Where was the safe Archimedean Standpoint which would prevent one from falling into bottomless intellectual pits? I would now see these experiences as necessary intellectual initiation rites – and they were far more mind-blowing than the drugs also available in my student environment at that time. What finally attracted me was the esotherical sense of joining a sort of secret society of logicians who could develop formal theories and then emerge in broad daylight in any discipline: mathematics, computer science, philosophy, or theology. This fitted my general intellectual inclinations, then and now.

Of course, there is also a sociological aspect of initiation. Years later, when I was appointed in Groningen, I had to speak at a no-nonsense mathematics colloquium which was famous for intimidating newcomers and outsiders. I decided at once that I would not do what most people did, and try to impress them with cramming as much technical mathematics into my talk as possible. Instead, without any pre-amble, I started by telling them that I would prove the Existence of God, and began writing exotic modal formulas for that without much explanation. This surprise opening went down extremely well. Of course, after that, I had also prepared a lot of heavy modal logic stuff with ultraproducts and model theory, but the main point was made. After the talk, I was taken to the office of the department head, and offered some hard liquor from an illegal stock he kept there. I knew that I had passed the test.

2. What examples from your work (or the work of others) illustrate the use of mathematics for philosophy?

There are so many great names right at the interface, and so many well-documented cross-currents between the two fields, that I have little to add to that. In some ways, Plato's inscription 'Mèdeis Ageometrètos Eisito" on the Gate of the Academy said it all, and 'the rest is history'.

Apart from the great names, I myself have always admired the gentle but persuasive influence of my predecessor Evert Willem Beth, the first occupant of the Amsterdam logic chair in the philosophy department, a companion to Brouwer's in mathematics. For Beth, insights into the history of mathematics and then contemporary work on its foundations had immediate repercussions for philosophy, and beyond that, for modern society. His magnum opus *The Foundations of Mathematics* was not a purely technical book, but it also contains his views on what happened when our views of science were transformed through the abstract postulational revolution in 19th century mathematics, which took us from one unique Space to a multiplicity of 'spaces', and from one true theory wearing its intended interpretation on its sleeves to wide-scope axiom systems. According to Beth, this came with profound changes in general intellectual methodology, making traditional philosophical "*What is X*" questions largely redundant – an anti-essentialist view which still seems worth expounding today. Another example which still seems highly relevant to me is his analysis of the 'Problem of Locke-Berkeley', putting metaphysical talk of arbitrary triangles to rest in a clear understanding of the role of variable and quantifiers in mathematical deduction.

And as a final example of entanglement, I would mention Beth's masterful analysis of the methods of analysis and synthesis in mathematical history since Antiquity, their connection with Kant's notoriously difficult distinction between 'analytic' and 'synthetic' truths, and eventually, their harmonic coexistence in Beth's *semantic tableaux* – still a ubiquitous method in the field, which search for proofs and counter-examples at the same time. Beth's paper developing the latter has a lot of history (he likens them to the Tree of Porphyry) which modern referees would cut out at once as non-mathematical ballast and suspicious erudition. But for him, doing mathematical logic and doing philosophy was a natural unity. And this unity also worked the other way around: he did his mathematics driven by philosophical 'taste', rather than displays of cleverness per se. His famous Definability Theorem is an answer to a real issue about a whole field: how logic can be a systematic theory of *definition* as well as *deduction*. And likewise, his 'Beth models' for intuitionistic logic, the first plausible semantics for what Brouwer meant, high-lighted the informational processes behind constructive mathematics, a philosophical insight as much as a mathematical one. People with Beth's intellectual span still exist, though seeking mathematical respectability as such may have become more of a dominant norm.

Now to my own answers! How one thinks that mathematics influences, or should influence philosophy, depends first and foremost on what one considers essential *to mathematics itself*. And in this respect, I have my doubts about (or let us say, a lack of affinity with) many of the themes usually taught in courses in the philosophy of mathematics. Is the nature of *mathematical objects* important: are they Platonic entities, free constructions of the mind, nominalist fictions? I think it matters not at all to mathematical practice, and what is worse, concentrating on 'What is *X*' questions like this misrepresents what mathematics really is, and what makes it so appealing. Likewise, is *mathematical knowledge* a web of formal theories, connected by various embeddings and translations? Again, I think this view, despite its endorsement by non-logicians like Bourbaki, is a highly loaded description coming out of foundational research, rather than an unbiased account of what mathematics consists of. In particular, it cuts the field into disjoint territories like 'algebra', geometry', and so on, which often do not represent natural boundaries. Indeed, both earlier views put the cart before the horse. Mathematics is first and foremost a *human activity*, a family of ways of thinking (geometrical, combinatorial, and so on), a way of phrasing things abstractly, and a set of methods, aided by a store of accumulated results. Of course, this activity has given us a constant stream of intellectual *products*: definitions, theorems, theories, and so on – but the heartbeat of the field is in those activities, and in the creative patterns which they exhibit. The know-how is inextricable from the knowledge that. In the light of all this, my own analysis of how mathematics can or should influence philosophy would start from the mathematician's modus operandi, and then look at its repercussions in philosophical methodology, because. Moreover, – and any reader worth her salt must have seen this coming – I would make the same unconventional *"Activities over Products"* claim about philosophy in general. It is about intellectual activities: styles of analysis, defining, and reasoning, rather than any warehouse of special propositional insights.

Now to the Great Themes which I see as crucial to the flow of ideas from mathematics to philosophy. I will just give a few examples. Beth emphasized the philosophical, and indeed the general cultural importance of the 19th century move toward pluralism and creation of abstracts models for given postulates: spaces, algebras, groups. I would emphasize one that I think even more important, also from Beth's own interest in the 'definability' aspect of logic – namely, the theme of *structural invariance and the genesis of language*. When Helmholtz made his great proposal that geometrical language arises because of existing invariances in space, he set into motion what is arguably one of the most powerful mathematical perspectives. As set forth in Klein's "Erlanger Programm" (much more influential than "Hilbert's Program") any mathematical theory arises by defining a class of structures plus an equivalence relation over these, often given via a group of transformations. And the 'natural properties' of structures which arise in that way are the invariants of these transformations, which then give rise to languages defining these at the right level of expressive power. In particular, one mathematical structure, say Euclidean Space, can be analyzed at various grain levels in this way, from finer (as in geometry) to coarser (as in topology). This I think is absolutely crucial, and indeed, it fits some philosophers' ontological interest in a careful analysis of 'criteria of identity'. The invariance-to-language perspective still shows its power all the time, and across many disciplines: from cognitive psychology to image processing, and from new parts of mathematics to process theories in computer science. Defining abstract structures is just half the job, studying their interrelations at the right level, and then looking at the matching languages is the other.

A special case of this perspective has made it into philosophy over the last decade, viz. the study of 'logical constants' as the invariants under the broadest class of structural transformations, say all permutations, or even all homomorphisms from objects to objects. This powerful idea has come up independently in computer science, formal semantics, and the philosophy of logic, and, though I myself find the invariance criterion necessary but clearly insufficient, I think this mathematical style of thinking has lifted philosophical discussions about logical constants to a much higher level. Even so, I must admit to a sense of bewilderment when my colleagues in this area then go on to use these wonderful definitions in their quest for an Eldorado: the location of the 'exact border-line between Logic and Mathematics': a non-issue if I ever saw one. Who cares, as long as there is lively intellectual smuggling and traffic?

Another grand theme in mathematics which I see as crucial to philosophy is, of course, the notion of deductive argument and *proof* as a way of organizing compelling thought. I will not pursue this in detail, except to note that thinking about the structure and variety of exact inference has been a constant source of philosophical inspiration. Bolzano's "Wissenschaftslehre" (again a 19th century achievement, I really love that period!) comes to mind as a masterful account of the variety of intellectual reasoning styles, acknowledged and expanded in the work of Peirce, and finding its way eventually into the modern study of varieties of common sense reasoning by McCarthy and others in AI. In the latter area, it was clearly mathematical notions and results, rather than 'natural language philosophy' or 'common sense talk', which has made for genuine new insights. – "But" you might object, "this is AI, not philosophy!" Well, the way I see it, Von Clausewitz is right, here as ever: "AI is philosophy continued by other means". – In the same stream of mathematical modeling for inference, I see the influence of mathematical proof-theoretic paradigms, classical, constructive, linear logic-based, or more general sub-structural, on theories of meaning, knowledge, and nowadays even interaction – but others in this Volume will no doubt make this case much more eloquently.

Still, it seems fair to say that there is much more serious structure to mathematical proof than has been brought to light in the areas I have cited, including strategies for definition, ways of revising arguments in response to counter-examples, and finding pivotal structures for generalization. This has been pointed out by Lakatos and many other free-thinkers at some distance from modern logic, and these aspects too, seem worthy of serious philosophical reflection. I have formulated some thoughts on this in my 2006 *Topoi* paper "Where is Logic Going, and Should It?", which I will not repeat here.

Next, what about meta-mathematics? Is not that the greatest mathematical influence on philosophy of all? Here too, I will defer to other authors in this volume. Let me just say this on my own behalf. The great body of meta-mathematical results about the logical structure of mathematical and scientific theories, the families they form and their inter-theory relations, seems highly relevant far beyond the foundations of mathematics. It pertains to any serious philosophical discussion of the structure of knowledge, and issues like reduction, explanation, or other key themes in the philosophy of science. Perhaps not surprisingly, Beth's work, like that of Carnap, Peirce, or Bolzano, was an inextricable mixture of logic and philosophy of science. And let's not forget, even Tarski's first textbook was called "Logic and the Methodology of the

Deductive Sciences". I feel this role of pure logic has been neglected, with philosophers of science going their own way – leaving logicians just the philosophy of mathematics as their private preserve.

But there is much more than standard meta-mathematics. From my own work on modal logic (clearly a branch of mathematics in its theoretical aspects), I would mention the role of epistemic, dynamic and temporal logics in epistemology, belief revision theory, or the philosophy of action, and a whole range of philosophical topics besides. Moreover, these formal ideas link up naturally with influences on philosophy coming from other areas of mathematics, such as learning theory (witness Kevin Kelly's work) or Information Theory (witness the work of Fred Dretske). Of course, I am not saying that every mathematical issue or method means something deep philosophically. But is seems good strategy to see if it does. For instance, in contemporary modal logic, one of the most delicate issues is the following additional dimension to the earlier-mentioned language – structural invariance duality. There is also the *Golden Rule of Balance*: the greater the expressive power of a logical language (tied to a finer structural equivalence), the greater its *computational complexity*. And once the language can say a whole lot of things, its logic will have become undecidable. This trade-off between expressive power and complexity is a deep phenomenon, whose true laws are yet poorly understood. What it adds to the research agenda is awareness of the *algorithmic* aspects of any semantic structures and informational tasks we choose to study. While I see this further dimension as absolutely central, algorithmics and complexity theory have failed to make any significant impact in philosophy so far that I can see.

Let me conclude with something closer to my home turf. Based on the influences that *have* occurred over the last decades, I would personally make the case that what is called 'philosophical logic' has been largely an interface area developing a mixture of mathematical methodology and philosophical sensitivity: a 'buffer state', if you wish, between mathematics and philosophy. I have chronicled the history of some relevant themes at the interface of Logic and Philosophy over the last century in my article in the *Handbook of the Philosophy of Logic*. There is a surprising amount of these, far beyond what one finds reflected in standard texts in philosophy of logic or mathematics, and together they show that one can have surprising, sometimes tortured, but often highly productive relationships here.

Of course, my list of influences is slanted toward logic – and there are many further areas of contemporary mathematics with proven philosophical impact, such as Probability Theory, Decision Theory, Game Theory, or Information Theory. Hence, I would be in favour of philosophy curricula offering a broad non-parochial band-width of formal methods. Philosophers should not just be able to appreciate logical deduction, but also probabilistic, game-theoretic, complexity-theoretic, and other mathematical reasoning. But who am I to make this plea? Logic evidently has no monopoly on the mathematics–philosophy interface, but let the other mathematicians speak for themselves.

3. What is the proper role of philosophy of mathematics in relation to logic, foundations of mathematics, the traditional core areas of mathematics, and science?

As I said already, I find it hard to keep some of these categories apart. Sometimes, I even fear that courses in Philosophy of Mathematics are just courses in mathematical logic made palatable to philosophy students by drawing all of its teeth. In any case, what I do find improper is the selling of in-house features of mathematical systems as philosophical issues. For instance, I am always amazed by the way some logicians have managed to get philosophers excited about issues like 'existence as the range of quantification', about 'meaning as what you get in a natural deduction proof', or about the 'border-line (another boundary, it's really an obsession!) between first-order and higher-order logic' – as if these were philosophical issues, rather than mainly internal technical system concerns for logicians, of no great outside significance that I can see – neither to mathematicians nor to philosophers.

4. What do you consider the most neglected topics and/or contributions in late 20th century philosophy of mathematics?

Here is what I miss most, in the form of three laments.

First of all, some sustained reflection on the *received agenda in the foundations of mathematics*. The way the history has gone 'After the Fall' by Gödel's Theorems has been an impressive sequence of insights into formalized theories, and the development of rich sub-disciplines of Recursion Theory, Model Theory, and Proof Theory. Sure, all this is truly admirable – and a more culturally rewarding 'catastrophe' than the demise of Hilbert's Program is hard to imagine. But still, there are also other roads that could have been taken, and that are still there for us to walk.

For instance, looking back at the foundational era, I have always been amazed by the almost pathological fears of inconsistency. Frege says somewhere that, if a single contradiction were to be discovered in mathematics, "the whole building would collapse like a House of Cards". Please, why? This claim seems largely an artefact of the wrong metaphor. Mathematics is not a house with foundations which have to bear the whole weight. It is rather a *planetary system* of different theories entering into various relationships, and happily spinning together in logical space. Damage one, and the system will continue, maybe with some debris orbiting here and there. Or, here is another metaphor for mathematics, equally attractive, due to Chaim Perelman: it is a wonderful *tapestry* of many strands woven together by the great mathematicians. Pull out one strand, and the tapestry may be weaker by an epsilon, but tears can be mended. And this brings me to my most central objection: we know from the history of mathematics and the sciences that contradictions are never the end of a story. To the contrary, one of the most striking ability of scientists is not to create infallible theories, but rather, having creative ways of coping with problems once they arise. In a wonderful, little-known study in the early 1960s, the Czech philosopher of law Ota Weinberger pointed out the persistent strategies for removing inconsistencies that can be found both in common sense reasoning (removing disagreements in conversation) and in science. Most of them go back to medieval logic and beyond. Trivially, one can give up some assumptions, the way Zermelo-Fraenkel Set Theory gave up Cantor's Full Comprehension for the Separation Axiom. Other strategies include making distinctions between kinds of objects that had been identified before, such as 'sets' versus 'classes' in NBG Set Theory. Another powerful strategy is the introduction of 'hidden variables', such as contextual arguments: I am tall for a human, but not tall for an animal. The history of science is replete with these, and much more inventive strategies. Where is the problem of foundations then, which we teach our students to gasp at in religious awe? I say that, on the basis of all this experience, that, if an inconsistency were to be discovered in, say, Peano Arithmetic this very evening, as I am writing this line, a marvelous analysis would be made within a short time, and an incomparably more subtle New Arithmetic would be found. Indeed, I often worry that *not enough* paradoxes and inconsistencies are being discovered these days to keep our most innovative mathematicians on edge… In this light, logic should also have studied patterns of repair, and ways of changing theories. But more on that below.

A second conspicuous lack is the closedness of the agenda in other directions. To name the most spectacular of these, to me, one of the most spectacular new areas of mathematics in the 20th century has been the emergence of modern computer science (or *informatics*, as we say in Europe), with its fast-growing set of mathematical insights into computation, algorithmics, complexity theory, data structures, process theories, and so on. Of course, philosophers do discuss insights from recursion theory, and the like – but the fact that computer science has transformed our whole fundamental thinking about information, computation, 'information dynamics' (the term used by Milner and other protagonists), and highly sophisticated general theories of sequential and parallel action, far beyond the era of Turing and Church, seems to have bypassed the philosophy of mathematics – and general philosophy – altogether.

The only claim one sometimes hears is that this is really about 'constructive mathematics', accompanied by wholly false claims that modern computer science is deeply dominated by intuitionistic logic, type theory and the like. Nothing like that is true! Indeed, the reverse should happen: insights from the study of information and computation are already informing logic, and they should also come to inform philosophy. This point is now slowly emerging in current programs for a 'Philosophy of Information', on which a separate *Five Questions* Volume is being published.

This closed nature of the agenda is also being reinforced by the current wave of historical research into the Golden Age of the foundations of mathematics. While this may seem a harmless pursuit aimed at increasing our understanding how we got to be here, I also sense another, much more problematic undercurrent in this whole historicizing trend. 'The only issues of real value are those from the past, and the variations we can find for them'. In this nostalgic way, Frege, Gödel and Turing are still made into our teachers, rather than the innovative thinkers of today changing the agenda of logic.

Finally, now that I am in complaining mode, let me step on the accelerator full throttle. The philosophy and foundations of mathematics, it seems to me, show a curiously defensive attitude, ill-fitting their status and achievements. Friendly proposals to extend the classical agenda (we cannot keep singing hymns to Gödel and Tarski *forever* – unless we are already in Heaven) are perceived as threats, to be received with suspicion and sometimes even personal attacks. And innovative 'agents provocateurs' like Lakatos in his mind-opener "Proofs and Refutations" were subjected to torrents of abuse, rather than the lively interest in new ideas one would expect from a vigorous community. As Samson Abramsky once put it, the established order in foundations reminds him of what was said about the Bourbon Dynasty, when they re-entered France again after the Revolution: "They had learnt nothing, – and they had forgotten nothing". Indeed! I myself am often struck by the analogies between modern attitudes in foundational research and Islamic fundamentalism. Both insist on a mythical purity of the past, sanitizing our founding fathers of their (Heaven knows) many ambiguities and weaknesses, and both claim that if we can only stick to the past, ignoring the key ideas transforming the modern world outside, all evils will be cured. To be sure, I see myself as a logician, and a great lover of mathematical technique, but I have never been able to understand this defense of the status quo. In particular, I have always found many outspoken critics of modern logic (Blanshard, Perelman, Toulmin, Lakatos) extremely interesting and well-worth reading, and a useful reminder of the many doors our Founding Fathers have closed historically – doors that could be opened again now, *precisely because* we can feel confident in what has already been achieved.

5. What are the most important open problems in the philosophy of mathematics and what are the prospects for progress?

Here is what I take to be the main challenges for the future in the philosophy of mathematics, and indeed foundational research. I think we are still far from a true understanding of mathematics and what it can mean for philosophy. I have to formulate the following points somewhat apodictically, for lack of space – but I have some publications backing them up, for instance, 'Logic and Reasoning: Do the Facts Matter?' in a forthcoming issue of *Studia Logica* edited by Hannes Leitgeb, and 'Intelligent Interaction: Logic as a Theory of Rational Agency', to appear in the *Proceedings of the 2007 DLMPS Conference* held in Beijing (an organization founded by Beth and Tarski, by the way).

First, mathematics is not some isolated faculty of the human mind which needs to be approached with special reverence. It is rather an *in vitro* version of one crucial faculty of rational agents in general, and I think we need to embed our understanding of mathematics in more encompassing theories of rational agency. Thus, philosophy of mathematics and epistemology should go hand in hand, but both with suitably broad agendas. I myself see a fluid transition all the way from common sense reasoning to mathematical proof, and from knowledge structures in daily life to mathematical theories. Accordingly, we should arrive at an integrated understanding of that whole spectrum. Moreover, but I guess this point is too obvious to need much elaboration, we should not take deduction as our only model here, since intelligent handling of information also involves precise observation and communication.

Next, in line with what I said about 'foundationalism', the true measure of intelligent human behaviour, in my view, is not to be always right, and to have infallible procedures keeping us on that track forever. We see human intelligence at its finest when we *correct ourselves*, learn from mistakes, and create something new and better out of broken dreams and refuted expectations. These processes are studied nowadays under the heading of belief revision and similar parts of logic, but they apply equally well to a real understanding of mathematical practice. In medical terms, the foundationalists wanted to make the whole world of mathematics free from disease, by killing off every last unclarity and inconsistency. I say, in contrast, that we should study the ubiquitous 'sanitary' processes of mathematical *precisation* and *revision*, which swing into action every time a new problem is discovered. Mathematics is also about processes of correction, invention of language, precisation when clarity is to be seved, but also less formal paraphrase when an overview is needed of a proof or a notion. Accordingly, in my favourite slogan, logic is not just the guardian of being right, its true role is much better described as follows: *logic is the immune system of the mind!*

My third agenda item is this. Mathematics is a social activity, just as much as any other branch of science. The true locus of mathematical progress is the *seminar room*, just as much as the attic with a Rodinesque thinker. Thus, we are talking multi-agent interaction, including human subjects, and not just one, but many of them. Indeed, this theme has been hidden under the floor boards ever since the Golden Thirties. Carnap stressed the 'intersubjective nature' of scientific insight, but how to explain that when our logical theories have no explicit subjects accounted for? Likewise, even Turing, when discussing his famous Test whether a machine could mimic a human, emphasized the essentially social nature of learning and scientific progress. I expect that a true understanding of mathematics must be embedded eventually, not just in a theory of self-correcting rational agents, but in a theory of intelligent interaction.

My fourth point is the role of the empirical facts. As we are rapidly learning more about the actual functioning of mathematical abilities in the human brain, *cognitive science* enters the picture. We are fast learning about the delicate interplay of visual, linguistic, and planning components of the brain in meaningful reasoning activities, as well as counting and measuring. I think a modern philosophy of mathematics must deal with this new source of information, which replaces centuries of speculation about intuition and the like, in just the same way as other areas of philosophy, and indeed, logic itself.

But let me end with something simple, and almost pedestrian. If we are going to do philosophy of mathematics today, it might be good to *just take a simple look at what mathematics today really is*, in the hands of its best practitioners – and decide on its major features only afterwards. Some years ago, I decided to teach a public evening course on modern mathematics with my distinguished Amsterdam colleague Robbert Dijkgraaf, our leading expert in string theory and our best-known mathematician nationally. The reason was that we had had a conversation in which we both felt that the usual division of the field into areas like Algebra, Geometry, etc. gives no true picture of the intellectual geography of the field. We both thought the essence lay elsewhere, in ubiquitous notions and patterns. We decided on an experiment. We both drew up a list of what we considered key aspect of mathematics, and if those lists would overlap enough, we had a basis for a joint course. Indeed, the two lists, one by a mathematical logician and one by a mathematical physicists, were strikingly similar. And here are the topics that we taught: (a) Symmetry, Invariance, and Language, (b) Counting with Numbers and with Language, (c) Order, (d) Proof, (e) Computation and Complexity, (f) Paradoxes and Meta-theorems, (g) Probability, (h)

Games and Information Dynamics, and (i) Prediction and Dynamical Systems. Each evening, we would show how these topics arose in Nature (with an emphasis on physics), and then in Cognition (with an emphasis on logic), and the third hour would be for discussion with our audience. I have seldom felt the vibrancy and cultural impact of mathematics like in this evening course. I wish the Philosophy of Mathematics could achieve equal vibrancy, doing justice to the true attractions of its subject.

The Philosophy of Mathematics: 5 Questions website is located at

http://www.phil-math.org