Successor Large Cardinals in Symmetric Extensions*

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Abstract

We give an exposition in modern language (and using partial orders) of Jech's method for obtaining models where successor cardinals have large cardinal properties. In such models, the axiom of choice must necessarily fail. In particular, we show how, given any regular cardinal and a large cardinal of the requisite type above it, there is a symmetric extension of the universe in which the axiom of choice fails, the smaller cardinal is preserved, and its successor cardinal is measurable, strongly compact or supercompact, depending on what we started with. The main novelty of the exposition is a slightly more general form of the Lévy-Solovay Theorem, as well as a proof that fine measures generate fine measures in generic extensions obtained by small forcing.

1 Introduction

In set theory, large cardinals serve as barometers of consistency strength, allowing us to compare the consistency strength of theories extending the basic set theory by comparing the amount of large cardinals that the theories are equiconsistent with. As the name suggests, in the ZFC context, these cardinals are usually quite large. At the very least, they are closure points of the von Neumann hierarchy, and hence the standard ZFC techniques for getting larger sets of taking powersets and unions are bounded below them. But often they are much larger, implying the existence of unboundedly many smaller large cardinals below them.

In the ZF context, however, not only can large cardinals be quite small, they can in many cases even be the first uncountable cardinal. The first results of this type were independently obtained by Jech [Jec68] and Takeuti [Tak70], who showed that "ZFC + there is a measurable cardinal" is equiconsistent with "$\mathsf{ZF} + \mathsf{DC} + \aleph_1$ is a measurable cardinal" (Takeuti also showed how to obtain models with $\aleph_1$ being strongly compact and supercompact). It is the method of Jech that we shall focus on, and we shall show how, in the context of a large cardinal defined by means of an ultrafilter, it can be used to make any successor of a regular cardinal have this large cardinal property. We note that a proof that the Jech construction preserves supercompactness is also contained in [AH86], although it is couched in some extra notation. Our treatment follows that of [Dim11], which contains an exposition of Jech's method in modern language. In particular, Dimitriou adapted the argument to the context of partial orders, and isolated some key notions which allow us to extend her argument (which was for measurable cardinals) to strongly compact and supercompact cardinals. We note that our exposition owes a lot to [Dim11], as well as to the notes from an ILLC Research Project taught by Dimitriou on "Singularizing Successive Cardinals" in Amsterdam in June 2009.

Before we start, some clarifications should be made. Over ZFC, there are often various equivalent ways of capturing that κ has some large cardinal property. For example, the existence of a κ-complete

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ultrafilter on κ is equivalent to κ being the critical point of a non-trivial elementary embedding of the universe into a transitive class. Over ZF, however, they are not equivalent. $\aleph_1$ can never satisfy the latter, whereas as we shall show below, it can satisfy the former. This is not just the case with definitions involving elementary embeddings (which technically cannot be expressed in ZFC as they require quantification over classes), but even ZF-expressible, ZFC-equivalent notions of large cardinals need not be ZF-equivalent. A detailed example can be found in [Kie06], where the relations between various ZFC-equivalent and ZF-inequivalent notions related to strong compactness are studied. To prevent confusion, before we start our discussion of large cardinals, we shall make clear which definitions we use. Another thing to note is that in some cases, the consistency strength of $\mathsf{ZFC} + \mathsf{L}$ is more than $\mathsf{ZF} + \mathsf{L}$. An example is when L is the statement "There a cardinal κ such that there is a normal measure on $\mathcal{P}_\kappa(\kappa^+)$". For details, see [Bol02].

In this report, we shall show how the Jech construction preserves the following large cardinal properties: measurable, γ-compact, γ-supercompact. The last two also show that the construction preserves strong compactness and supercompactness. We shall try to keep the proofs as modular as possible, so that the reader may be convinced that this technique can be used to show the preservation of any (ZFC-)large cardinal notion which is defined by way of the existence of an ultrafilter. We assume some background in forcing as can be found in [Jec78] or [Kun80].

2 Basic Forcing

We shall work with partial orders as opposed to Boolean algebras. In what follows, let $\langle \mathbb{P}, \leq, 1\rangle$ be a forcing poset, with 1 being the top element. For $p, q \in \mathbb{P}$, say that p is stronger than q if $p \leq q$.

Definition 1. The class $V^{\mathbb{P}}$ of all $\mathbb{P}$-*names* is obtained by recursion as follows:
For τ a set, $\tau \in V^{\mathbb{P}}$ iff $\tau \subseteq V^{\mathbb{P}} \times \mathbb{P}$.

Definition 2. The *canonical names* for elements of V are defined recursively as follows (for $x \in V$):

$$\check{x} \triangleq \{(\check{y}, 1) \mid y \in x\}.$$

The canonical name for the generic filter is defined as $\Gamma \triangleq \{(\check{p}, p) \mid p \in \mathbb{P}\}$.

From now on, $\check{x}$ is always assumed to be a canonical name for an element, x, of the ground model. Consequently, if $\check{x}$ is a name for x, it is implicitly assumed that x is in V.

Definition 3. For G a $\mathbb{P}$-generic filter and τ a $\mathbb{P}$-name, τ^G, the *interpretation* of τ by G, is defined recursively as:

$$\tau^G \triangleq \{\sigma^G \mid \exists p \in G((\sigma, p) \in \tau)\}$$

Correspondingly, the generic extension is defined as $V[G] \triangleq \{\tau^G \mid \tau \in V^{\mathbb{P}}\}$.

It can be shown that $V[G]$ is the smallest transitive model of ZFC containing both V and G. The canonical names defined above are crucial for the proof. From now on, if $\dot{x}$ is a name, then we write x for its interpretation in a forcing extension. That is, we write x instead of $(\dot{x})^G$ where G is the generic filter. Also, for M a class of interpretations of names, when we say $V \subseteq M$ we mean that for every $x \in V$, $(\check{x})^G \in M$. Also, we denote the class $\{\check{x} \mid x \in V\}$ by $\check{V}$. Other similar shortcuts with names and their interpretations will also be taken without mentioning. Hopefully, this will not contribute to the reader's confusion.

3 Symmetric Extensions

In this section, we introduce symmetric extensions and prove some basic results about them.

Definition 4. Let $\langle \mathbb{P}, \leq, 1 \rangle$ be a partial order. An *automorphism* a of $\mathbb{P}$ is a bijection of $\mathbb{P}$ such that a and its inverse both preserve $\leq$ and 1. Given such an automorphism a, it can be seen to act on $V^{\mathbb{P}}$ by recursion in the following way (we denote the corresponding action by a_* here for clarity):

$$a_*(\tau) \triangleq \{(a_*(\sigma), a(p)) \mid (\sigma, p) \in \tau\}$$

From now on, given an automorphism a of $\mathbb{P}$, we denote its induced action on $V^{\mathbb{P}}$ by a as well.

The following lemma is called the symmetry lemma. The proof is an easy simultaneous induction on formula complexity.

Lemma 5 (Symmetry Lemma). *Let $\mathbb{P}$ be a partial order and $\mathcal{G}$ an automorphism group of $\mathbb{P}$. Let φ be a formula of the forcing language with n free variables and let $\sigma_1, \ldots, \sigma_n \in V^{\mathbb{P}}$ be names. If $a \in \mathcal{G}$ then*

$$p \Vdash \varphi(\sigma_1, \ldots, \sigma_n) \Leftrightarrow a(p) \Vdash \varphi(a(\sigma_1), \ldots, a(\sigma_n))$$

□

Definition 6. Let $\mathbb{P}$ be a partial order, $\mathcal{G}$ an automorphism group of $\mathbb{P}$. A set $\mathcal{F} \subseteq \mathcal{P}(\mathcal{G})$ of subgroups of $\mathcal{G}$ is called a *filter* if for all subgroups H, K of $\mathcal{G}$,

(i) If $H \in \mathcal{F}$ and $H \subset K$, then $K \in \mathcal{G}$.

(ii) If $H, K \in \mathcal{F}$ then $H \cap K \in \mathcal{F}$.

Further, it is *normal* if for every $F \in \mathcal{F}$ and every $g \in \mathcal{G}$, $gFg^{-1} \in \mathcal{F}$.

For the rest of this section, we fix a partial order $\mathbb{P}$, a group $\mathcal{G}$ of automorphisms of $\mathbb{P}$ and a normal filter $\mathcal{F}$ over $\mathcal{G}$.

Definition 7. For each $\tau \in V^{\mathbb{P}}$, its symmetry group with respect to $\mathcal{G}$ is defined as:

$$\mathsf{sym}^{\mathcal{G}}\tau \triangleq \{g \in \mathcal{G} \mid g\tau = \tau\}$$

That is, if $\mathcal{G}$ is seen as acting on $V^{\mathbb{P}}$, then $\mathsf{sym}^{\mathcal{G}}\tau$ is the stabiliser group of τ. Call τ symmetric if $\mathsf{sym}^{\mathcal{G}}\tau \in \mathcal{F}$. We denote the class of all hereditarily symmetric names by $\mathsf{HS}^{\mathcal{F}}$. That is, recursively, for $\tau \in V^{\mathbb{P}}$,

$$\tau \in \mathsf{HS}^{\mathcal{F}} \text{ iff } \mathsf{sym}^{\mathcal{G}}\tau \in \mathcal{F} \text{ and } \forall \sigma \in \mathrm{dom}(\tau), \sigma \in \mathsf{HS}^{\mathcal{F}}$$

Often, we shall not mention $\mathcal{G}$ and $\mathcal{F}$ when they are clear from the context, and refer to the above as sym and HS respectively.

An easy proof by induction gives us that for any automorphism a of $\mathbb{P}$, and any canonical name $\check{x} \in V^{\mathbb{P}}$, $a(\check{x}) = \check{x}$. Consequently, every canonical name is hereditarily symmetric. That is, $\check{V} \subseteq \mathsf{HS} \subseteq V^{\mathbb{P}}$.

Definition 8. We define the *symmetric submodel* of $V[G]$ with respect to $\mathcal{F}$ as

$$V(G)^{\mathcal{F}} \triangleq \{\tau^G \mid \tau \in \mathsf{HS}^{\mathcal{F}}\}$$

Here as well, we will not mention $\mathcal{F}$ if it is clear from context.

Definition 9. For a formula φ with n free variables and names $\sigma_1, \dots \sigma_n \in \mathsf{HS}^{\mathcal{F}}$ we define the *symmetric forcing relation* $\Vdash_{\mathsf{HS}}$ by

$$p \Vdash_{\mathsf{HS}} \varphi(\sigma_1, \dots \sigma_n) \overset{\Delta}{\Leftrightarrow} \text{for every } \mathbb{P}\text{-generic filter } G \text{ over } V \text{ such that } p \in G,\ V(G) \vDash \varphi(\sigma_1, \dots \sigma_n).$$

This semantic forcing relation $\Vdash_{\mathsf{HS}}$ can be defined in the ground model in a syntactic way analogous to the usual forcing relation $\Vdash$, and it has properties similar to those of the latter, except that the quantifiers are relativized to the hereditarily symmetric names. The proofs of these properties are similar to the case for $\Vdash$.

Lemma 10. *Let φ and ψ be arbitrary sentences in the forcing language for $\mathbb{P}$. Then the following hold:*

(i) *If $p \Vdash_{\mathsf{HS}} \varphi$ and $q \leq p$ then $q \Vdash_{\mathsf{HS}} \varphi$.*

(ii) *There is no p such that $p \Vdash_{\mathsf{HS}} \varphi$ and $p \Vdash_{\mathsf{HS}} \neg\varphi$.*

(iii) *For every p there is $q \leq p$ such that $q \Vdash_{\mathsf{HS}} \varphi$ or $q \Vdash_{\mathsf{HS}} \neg\varphi$.*

(iv) $p \Vdash_{\mathsf{HS}} \neg\varphi \iff$ *for every $q \leq p$ it is not the case that $q \Vdash_{\mathsf{HS}} \varphi$.*

(v) $p \Vdash_{\mathsf{HS}} \varphi \wedge \psi \iff p \Vdash_{\mathsf{HS}} \varphi$ *and* $p \Vdash_{\mathsf{HS}} \psi$.

(vi) $p \Vdash_{\mathsf{HS}} \forall x\varphi \iff$ *for every $\sigma \in \mathsf{HS}$, $p \Vdash_{\mathsf{HS}} \varphi(\sigma)$.*

(vii) $p \Vdash_{\mathsf{HS}} \varphi \vee \psi \iff$ *for every $q \leq p$ there is $r \leq q$ such that $r \Vdash_{\mathsf{HS}} \varphi$ or $r \Vdash_{\mathsf{HS}} \psi$.*

(viii) $p \Vdash_{\mathsf{HS}} \exists x\varphi \iff$ *for every $q \leq p$ there is $r \leq q$ and $\sigma \in \mathsf{HS}$ such that $r \Vdash_{\mathsf{HS}} \varphi(\sigma)$.*

□

Also, corresponding to the Forcing Theorem, a similar result can be proven for $\Vdash_{\mathsf{HS}}$:

Lemma 11. *Let φ be a sentence in the forcing language of $\mathbb{P}$. If $V(G) \vDash \varphi$, then there is $q \in G$ such that $q \Vdash_{\mathsf{HS}} \varphi$.*

Proof. For each $p \in \mathbb{P}$, there is a $p' \leq p$ such that p' decides φ. That is, such that $p' \Vdash_{\mathsf{HS}} \varphi$ or $p' \Vdash_{\mathsf{HS}} \neg\varphi$. Hence, the set

$$D \triangleq \{p' \in \mathbb{P} \mid p' \text{ decides } \varphi\}$$

is a dense open set of $\mathbb{P}$, and hence there is a $q \in G \cap D$. Now, if $q \Vdash_{\mathsf{HS}} \neg\varphi$, then $V(G) \vDash \neg\varphi$. Therefore, $q \Vdash_{\mathsf{HS}} \varphi$, and we are done. □

These two lemmas in conjunction allow us to argue about $\Vdash_{\mathsf{HS}}$ in much the same way as $\Vdash$, as can be seen from the proof of the next theorem.

Theorem 12. *$V(G)$ has the following properties:*

(i) $V \subseteq V(G) \subseteq V[G]$.

(ii) $V(G)$ *is a transitive model of* ZF.

Proof. (i) This follows from the fact that $\check{V} \subseteq \mathsf{HS} \subseteq V^{\mathbb{P}}$.

(ii) Transitivity follows from the class HS being hereditary. Extensionality, Foundation, Empty Set and Infinity follow because $\check{V} \subseteq V(G)$ and $V(G)$ is transitive. The proof of Pairing is easy, and Union and Powerset have similar proofs assuming Separation. So we prove Separation, Union and Replacement:

For Separation, let φ be a formula and let $x, y \in V(G)$ with names $\dot{x}, \dot{y} \in \mathsf{HS}$. We would like to show that the set $w \triangleq \{z \in x \mid V(G) \vDash \varphi(y, z)\}$ is in $V(G)$, so we need to find a name $\dot{w} \in \mathsf{HS}$ for it. Towards this end, consider the name

$$\dot{w} \triangleq \{(\sigma, p) \mid \sigma \in \mathrm{dom}(x) \text{ and } p \Vdash_{\mathsf{HS}} \varphi(y, \sigma)\}.$$

It is clear that $\forall \sigma \in \mathrm{dom}(\dot{w})$, $\sigma \in \mathsf{HS}$. So we only need to show that $\mathsf{sym}(\dot{w}) \in \mathcal{F}$. But $\mathsf{sym}(\dot{x}) \cap \mathsf{sym}(\dot{y}) \subseteq \mathsf{sym}(\dot{w})$. To see this, let $a \in \mathsf{sym}(\dot{x}) \cap \mathsf{sym}(\dot{y})$. Then

$$\begin{aligned} a(\dot{w}) &= \{(a(\sigma), a(p)) \mid \sigma \in \mathrm{dom}(\dot{x}) \text{ and } a(p) \Vdash_{\mathsf{HS}} \varphi(y, a(\sigma))\} \\ &= \{(\tau, q) \mid a^{-1}(\tau) \in \mathrm{dom}(\dot{x}) \text{ and } a^{-1}(q) \Vdash_{\mathsf{HS}} \varphi(y, a^{-1}(\tau))\} \\ &= \{(\tau, q) \mid \tau \in \mathrm{dom}(\dot{x}) \text{ and } q \Vdash_{\mathsf{HS}} \varphi(y, \tau)\} \\ &= \dot{w}. \end{aligned}$$

The first step here uses the Symmetry Lemma, and the third step the observation that $a^{-1} \in \mathsf{sym}(\dot{x}) \cap \mathsf{sym}(\dot{y})$. Hence, $\dot{w} \in \mathsf{HS}$, and we are done.

For Union, let $x \in V(G)$ and $\dot{x} \in \mathsf{HS}$ a name for x. Consider the name

$$\tau \triangleq \{(\sigma, 1) \mid \exists \pi \in \mathrm{dom}(\dot{x})(\sigma \in \mathrm{dom}(\pi))\}.$$

This name is hereditarily symmetric because for any $a \in \mathsf{sym}(\dot{x})$,

$$\mathrm{dom}(\dot{x}) = \{a(\pi) \mid \pi \in \mathrm{dom}(\dot{x})\},$$

hence $\mathsf{sym}(\dot{x}) \subseteq \mathsf{sym}(\tau)$. It is also clear that $\bigcup x \subseteq \tau^G$, and Union follows by using Separation.

For Replacement, let $x \in V(G)$ and $\dot{x} \in \mathsf{HS}$ a name for it and φ a formula such that

$$V(G) \vDash \forall z \in x \exists! y \varphi(z, y).$$

We would then like to exhibit a set $w \in V(G)$ and a name $\dot{w} \in \mathsf{HS}$ for it such that

$$V(G) \vDash \forall z \in x \exists y \in w \varphi(z, y).$$

Towards this, for each $\dot{z} \in \mathrm{dom}(\dot{x})$, consider the set

$$S_{\dot{z}} \triangleq \{\dot{y} \in \mathsf{HS} \mid \exists p \in \mathbb{P}(p \Vdash_{\mathsf{HS}} \varphi(\dot{z}, \dot{y}))\}$$

and define the name

$$\tau \triangleq \{(\sigma, 1) \mid \exists \dot{z} \in \mathrm{dom}(\dot{x})(\sigma \in S_{\dot{z}})\}.$$

It is clear that if for some $z \in x$, $y \in V(G)$ is such that

$$V(G) \vDash \varphi(z, y),$$

then $y \in \tau^G$, and to see that $\mathsf{sym}(\dot{x}) \subseteq \mathsf{sym}(\tau)$ (and consequently, $\mathsf{sym}(\tau) \in \mathcal{F}$ and $\tau \in \mathsf{HS}$), it suffices to observe that for any $a \in \mathsf{sym}(\dot{x})$,

$$\begin{aligned} \mathrm{dom}(\dot{x}) &= \mathrm{dom}(a(\dot{x})) \\ &= \{a(\pi) \mid \pi \in \mathrm{dom}(\dot{x})\}, \end{aligned}$$

so if

$$\exists \pi \in \mathrm{dom}(\dot{x})(\sigma \in \mathrm{dom}(\pi)),$$

then

$$\exists \pi \in \mathrm{dom}(\dot{x})(a(\sigma) \in \mathrm{dom}(\pi)).$$

Hence, by the Symmetry Lemma, if there is a $p \in \mathbb{P}$ and $\dot{z} \in \mathrm{dom}(\dot{x})$ such that $p \Vdash_{\mathsf{HS}} \varphi(\dot{z}, \sigma)$, then $a(p) \Vdash_{\mathsf{HS}} \varphi(a(\dot{z}), a(\sigma))$. In particular, since $a(\dot{z}) \in \mathrm{dom}(\dot{x})$, if $(\sigma, 1) \in \tau$, then $(a(\sigma), 1) \in \tau$. And so Replacement follows by using Separation.

□

Definition 13. For $E \subseteq \mathbb{P}$, define its *pointwise stabiliser group* to be

$$\mathsf{fix}_{\mathcal{G}} E \triangleq \{g \in \mathcal{G} \mid \forall p \in E, g(p) = p\}$$

that is, it is the set of automorphisms which pointwise fix E. Again, we will usally omit subscripts.

Definition 14. Let $\mathbb{P}$ be a partial order, $\mathcal{G}$ an automorphism group of $\mathbb{P}$. A subset $I \subseteq \mathcal{P}(\mathbb{P})$ is called a *$\mathcal{G}$-symmetry generator* if it consists of up-sets, is closed under unions, and if for all $g \in \mathcal{G}$ and $E \in I$, there is an $E' \in I$ s.t. $g(\mathsf{fix}E)g^{-1} \supseteq \mathsf{fix}E'$.

As we shall see below, $\mathcal{G}$-symmetry generators generate a normal filter over $\mathcal{G}$. They also help us identify a property which shall be crucial in the report, that of being *projectable*. We shall see this in the next section. We have also added a clause of them containing only up-sets. This allows us to conclude that if $E \in I$, then E itself is a forcing poset (with the same top element as $\mathbb{P}$). We shall use this property while proving the Approximation Lemma.

Lemma 15. *If I is a $\mathcal{G}$-symmetry generator, then the set $\{\mathsf{fix}E \mid E \in I\}$ generates a normal filter over $\mathcal{G}$.*

Proof. Let $\mathcal{F}_I$ be the set obtained from $\{\mathsf{fix}E \mid E \in I\}$ by closing it under supergroups. We now need to show that $\mathcal{F}_I$ is closed under intersections, and that it is normal. Towards closure under intersections, let $K_1, K_2 \in \mathcal{F}_I$. Then, by the construction of $\mathcal{F}_I$, there must be $E_1, E_2 \in I$ such that $\mathsf{fix}E_i \subseteq K_i$. It follows that

$$\mathsf{fix}(E_1 \cup E_2) \subseteq \mathsf{fix}E_1 \cap \mathsf{fix}E_2 \subseteq K_1 \cap K_2.$$

But $\{\mathsf{fix}E \mid E \in I\}$ generates $\mathcal{F}_I$, and I is closed under unions, so it follows that $\mathcal{F}_I$ is a filter. For normality, let $a \in \mathcal{G}$ and $K \in \mathcal{F}_I$. Then there is an $E \in I$ such that $\mathsf{fix}E \subseteq K$. Therefore, $a(\mathsf{fix}E)a^{-1} \subseteq aKa^{-1}$. As I is a $\mathcal{G}$-symmetry generator, there is a $E' \in I$ such that $\mathsf{fix}E' \subseteq a(\mathsf{fix}E)a^{-1} \subseteq aKa^{-1}$. Consequently, $aKa^{-1} \in \mathcal{F}_I$, and so the filter is normal. □

We see now that $\mathcal{G}$-symmetry generators generate a normal filter, and consequently, if a name σ is in HS, then its symmetry group must be in this normal filter generated by the $\mathcal{G}$-symmetry generator. Consequently, with each name $\sigma \in \mathsf{HS}$ we can asssociate an element of the $\mathcal{G}$-symmetry generator which is responsible for the name being symmetric. The next definition makes this notion precise.

Definition 16. If I is a $\mathcal{G}$-symmetry generator, a set $E \in I$ *supports* a name $\sigma \in \mathsf{HS}$ if $\mathsf{sym}\sigma \supseteq \mathsf{fix}E$.

4 Approximation Lemma

In this section, we prove what we call the Approximation Lemma which will be crucial in our proofs. It essentially allows us to reduce the question about whether some property holds in the symmetric extension to one of verifing this property in an intermediate extension. These intermediate extensions are models of ZFC, thus simplifying the task of verifing the property.

Definition 17. Let $\mathbb{P}$ be a partial order, $\mathcal{G}$ an automorphism group, and I a symmetry generator. We say that I is *projectable* for the pair $(\mathbb{P}, \mathcal{G})$ if for every $p \in \mathbb{P}$ and every $E \in I$, there is a $p^* \in E$ that is minimal in the partial order such that $p^* \geq p$. We shall call $p \restriction E \stackrel{\Delta}{=} p^*$ the *projection* of p to E.

The importance of a projectable symmetry generator is that for any $E \in I$, and any forcing condition p, we can approximate p in the best possible way by an element of E. This property is useful because I generates the normal filter with respect to which our symmetric extension is defined. The next definition christens the situation when this approximation is tight.

Definition 18. Let $\mathbb{P}$ be a partial order, $\mathcal{G}$ an automorphism group, and I a projectable symmetry generator for $(\mathbb{P}, \mathcal{G})$. We say that the triple $(\mathbb{P}, \mathcal{G}, I)$ has the *approximation property* if for any formula φ with n free variables, and names $\sigma_1, \ldots, \sigma_n \in \mathsf{HS}$ all with support $E \in I$, and any $p \in \mathbb{P}$,

$$p \Vdash \varphi(\sigma_1, \ldots, \sigma_n) \text{ implies that } p \restriction E \Vdash \varphi(\sigma_1, \ldots, \sigma_n)$$

The importance of the approximation property is clear: it lets us compute the truth of formulas whose parameters all have the same set as a support *by an element of this support set.*

All of the partial orders $\mathbb{P}$, automorphism groups $\mathcal{G}$ and symmetry generators I that we will use in this report to construct symmetric extensions will have the following property, which will imply that $(\mathbb{P}, \mathcal{G}, I)$ has the approximation property:

Definition 19. Let $\mathbb{P}$ be a partial order, $\mathcal{G}$ an automorphism group, and I a projectable symmetry generator for $(\mathbb{P}, \mathcal{G})$. We say that $\mathbb{P}$ is $(\mathcal{G}, I)$-*homogenous* if for every $E \in I$, every $p \in \mathbb{P}$ and every $q \leq p \restriction E$ there is an automorphism $a \in \mathsf{fix}E$ s.t. $a(p) \parallel q$.

Lemma 20. *If $\mathbb{P}$ is $(\mathcal{G}, I)$-homogenous, then $(\mathbb{P}, \mathcal{G}, I)$ has the approximation property.*

Proof. Fix a formula φ and names $\sigma_1, \ldots, \sigma_n \in \mathsf{HS}$ with support $E \in I$, and a $p \in \mathbb{P}$. Suppose $p \Vdash \varphi(\sigma_1, \ldots, \sigma_n)$ and $p \restriction E \nVdash \varphi(\sigma_1, \ldots, \sigma_n)$. Then, there is a $q \leq p \restriction E$ such that $q \Vdash \neg\varphi(\sigma_1, \ldots \sigma_n)$. But then, for every automorphism $a \in \mathsf{fix}E$, $a(p) \Vdash \varphi(a(\sigma_1), \ldots, a(\sigma_n))$, and hence $a(p) \Vdash \varphi(\sigma_1, \ldots, \sigma_n)$. Consequently, $a(p) \perp q$, and we have a contradiction to $(\mathcal{G}, I)$-homogeniety. □

And finally, the main result of this section falls out quite easily from our definitions.

Lemma 21 (Approximation Lemma)**.** *If $\mathbb{P}$ is a partial order, $\mathcal{G}$ is an automorphism group, and I a projectable symmetry generator such that the triple $(\mathbb{P}, \mathcal{G}, I)$ has the approximation property, then for any set $X \in V(G)^{\mathcal{F}_{\mathcal{I}}}$ such that $X \subseteq V$, there is an $E \in I$ and an E-name for X. That is, we can say that $X \in V[G \cap E]$. Consequently, if $Y \in V(G)$, there there is an $E \in I$ such that $Y \cap V \in V[G \cap E]$.*

Proof. Let $\dot{X} \in \mathsf{HS}$ be a name for X with support $E \in I$. Let

$$\dot{X}_1 \triangleq \{(\dot{y}, p{\restriction}E) \mid p \Vdash y \in X\}.$$

Let

$$\dot{X}_2 \triangleq \{(\check{x}, p) \mid \exists y \in \mathsf{HS} \text{ s.t. } (y, p) \in \dot{X}_1 \text{ and } p \Vdash (x = y) \wedge (y \in X)\}$$

. Clearly, $V(G) \vDash X = X_1 = X_2$, and $\dot{X}_2$ is an E-name. □

Note that above, we defined $\dot{X}_2$ after we had defined $\dot{X}_1$ because we had no way of ensuring that $\dot{X}_1$ is an E-name. The problem there is that $\dot{Y}$ is an E-name iff for every $(\sigma, p) \in \dot{Y}$, σ is an E-name and $p \in E$. We circumvented this problem in the proof by choosing $\dot{X}_2$ such that for every $(x, p) \in \dot{X}_2$, x is actually a *canonical name*, and hence an E-name for sure. Here we are using that if $E \in I$, then E is an up-set, and hence we can say that any canonical name is also an E-name.

Now, before we go any further in this report, there is some clarification that needs to be done. We have so far introduced two different forcing relations, $\Vdash$ and $\Vdash_{\mathsf{HS}}$. In this section we used the former, whereas in the previous section we used the latter. Throughout the report, thanks to our techniques, we will in fact never use the latter. An explanation of this phenomenon might be found by looking at a sketch of the kind of arguments that we employ in the rest of the report.

If $\dot{X} \in \mathsf{HS}$ is a name for $X \in V(G)$, then, as HS is a hereditary class, the interpretation of $\dot{X}$ is the same in $V[G]$ as in $V(G)$. Even more importantly for us, if $E \in I$, our symmetry generator, and $X \in V(G)$ is such that it has an E-name, then $X \in V[G \cap E] \subset V(G)$. Hence, if we prove that X has a property in $V[G \cap E]$ which is upwards-absolute over transitive models of ZF (for example, if this property is Δ_0 in some parameters in V), then X has this property in $V(G)$ too. The important point here is that we only use the $\Vdash$ relation (relativised to the partial order E, which as we mentioned in the discussion after Definition 14 is itself a forcing poset) to prove that X has this property in $V[G \cap E]$, and we have in fact not used the $\Vdash_{\mathsf{HS}}$ relation in this 'proof' at all.

5 Lévy-Solovay Theorems

In this section we prove some Lévy-Solovay theorems for some large cardinals [LS67]. These are theorems which say that if κ has some large cardinal property L in V, $\mathbb{P}$ is a partial order of size less than κ in V, and G is a V-generic filter over $\mathbb{P}$, then in $V[G]$, κ still has the property L. As a start, we give precise definitions of the large cardinal properties we shall consider.

Definition 22. (i) An uncountable cardinal κ is *measurable* if there is a nonprincipal κ-complete ultrafilter $\mathcal{U}$ on κ. If $\mathcal{U}$ is such an ultrafilter on κ, we say that it is *normal* if whenever $f : \kappa \to \kappa$ is such that $f(\lambda) < \lambda$ for a set in $\mathcal{U}$, then f is constant on a set in $\mathcal{U}$.

(ii) Let κ be an infinite cardinal. Let A be a set of size greater than or equal to κ. Then

$$\mathcal{P}_\kappa(A) \triangleq \{B \subseteq A \mid |B| < \kappa\}.$$

We note that in practice we shall only be talking about $\mathcal{P}_\kappa(A)$ for A an ordinal, and as all subsets of ordinals can always be well-ordered (and hence, have a size that can be compared with κ), $\mathcal{P}_\kappa(A)$ will have the same meaning in the ZF context as in the ZFC context. For each $x \in \mathcal{P}_\kappa(A)$, let

$$\hat{x} \triangleq \{y \in \mathcal{P}_\kappa(A) \mid x \subseteq y\}.$$

Let F be the filter generated by the sets $\hat{x}$ for all $x \in \mathcal{P}_\kappa(A)$. That is:

$$F \triangleq \{X \subseteq \mathcal{P}_\kappa(A) \mid X \supseteq \hat{x} \text{ for some } x \in \mathcal{P}_\kappa(A)\}.$$

If κ is a regular cardinal, this filter is κ-complete. Call $\mathcal{U}$ a *fine measure* on $\mathcal{P}_\kappa(A)$ if $\mathcal{U}$ is a κ-complete ultrafilter extending F. Call $\mathcal{U}$ a *normal measure* on $\mathcal{P}_\kappa(A)$ if it is fine, and further, if whenever $f : \mathcal{P}_\kappa(A) \to A$ is such that $f(X) \in X$ for a set in $\mathcal{U}$ (such an f is called a *choice function*), then f is constant on a set in $\mathcal{U}$.

(iii) Let $\kappa \leq \gamma$ be uncountable cardinals. An uncountable cardinal κ is *γ-compact* if there is a fine measure on $\mathcal{P}_\kappa(\gamma)$. It is *strongly compact* if it is γ-compact for all $\kappa \leq \gamma$.

(iv) Let $\kappa \leq \gamma$ be uncountable cardinals. An uncountable cardinal κ is *γ-supercompact* if there is normal measure on $\mathcal{P}_\kappa(\gamma)$. It is *supercompact* if it is γ-supercompact for all $\kappa \leq \gamma$.

We now state three lemmas that will come in handy in the later sections. The first two can be proved over ZF whereas the latter is proved over ZFC. The first one shows that measurable cardinals are 'large':

Lemma 23. *Let κ be a measurable cardinal with $\mathcal{U}$ a nonprincipal κ-complete ultrafilter on it. Then:*

(i) κ is regular.

(ii) For every cardinal $\lambda < \kappa$, there is no injective function $f : \kappa \to \mathcal{P}(\lambda)$.

Proof. (i) Follows easily from $\mathcal{U}$ being κ-complete and non-principal.

(ii) Towards a contradiction, let $\lambda < \kappa$ be a cardinal such that such a function f exist. Let $\text{ran}(f) = S$, and let $\mathcal{W}$ be the non-principal κ-complete ultrafilter on S which is the image of $\mathcal{U}$ under f. We construct a sequence $\langle S_\alpha \mid \alpha < \lambda \rangle$ of sets in $\mathcal{W}$ as follows:

$$S_0 \triangleq S,$$

$$S_\alpha \triangleq \bigcap_{\beta<\alpha} S_\beta \text{ for } \alpha \text{ a limit ordinal},$$

$$S_{\alpha+1} \triangleq \text{either } \{X \in S_\alpha \mid \alpha \in X\} \text{ or } \{X \in S_\alpha \mid \alpha \notin X\}, \text{ whichever of them is in } \mathcal{W}.$$

By κ-completeness, the intersection $\bigcap_{\alpha<\lambda} S_\alpha$ is in $\mathcal{W}$ as well, but it can contain at most one element, a contradiction to $\mathcal{W}$ being non-principal.

□

The next lemma, whose proof we omit, shows that normal measures generalise normal filters.

Lemma 24. *(i) If $\mathcal{U}$ is a normal measure on κ then $\mathcal{W} \triangleq \{X \subseteq \mathcal{P}_\kappa(\kappa) \mid X \cap \kappa \in \mathcal{U}\}$ is a normal filter on $\mathcal{P}_\kappa(\kappa)$.*

(ii) If $\mathcal{W}$ is a normal measure on $\mathcal{P}_\kappa(\kappa)$ then $\mathcal{U} \triangleq \mathcal{W} \cap \mathcal{P}(\kappa)$ is a normal filter on κ.

It can also be shown that over ZFC, κ is measurable if and only if there is a *normal* measure on κ. Hence, over ZFC, κ is measurable iff κ is κ-compact iff κ is κ-supercompact. Consequently, given two uncountable cardinals $\kappa \leq \gamma$, κ being measurable, γ-compact and γ-supercompact are increasingly stronger hypotheses. While it is known [Men75] that strong compactness and supercompactness are not equivalent, the following is a long-standing open problem in set theory (see [Kan08] for a discussion):

Question 25. *Are the following theories equiconsistent?*

(i) $\mathsf{ZFC} + \exists\kappa(\kappa$ *is strongly compact*).

(ii) $\mathsf{ZFC} + \exists\kappa(\kappa$ *is supercompact*).

Over ZF however, there can be measurable cardinals without any normal measures. A first step towards this was taken by Spector [Spe80] by assuming the consistency of much more than a measurable cardinal, where, forcing over a model of AD, he obtained a model with a measurable cardinal κ carrying a measure μ such that κ^κ/μ is not well-founded, and hence μ cannot be normalised in the standard way. And recently, Bilinsky and Gitik have shown [BG12] that starting from a model of ZFC+GCH+"there is a measurable cardinal κ", one can obtain a symmetric submodel of a generic extension which satisfies ZF+"κ is a measurable cardinal but there is no normal measure on κ".

The lemma that follows relates $\mathcal{P}_\kappa(\gamma)$ in the ground model to $\mathcal{P}_\kappa(\gamma)$ in the generic extension obtained by *small* forcing. We will use it to prove that γ-compactness is preserved in such generic extensions, and in particular, to show that fine measures generate fine measures in generic extensions by small forcing.

Lemma 26. *Let κ be a regular cardinal. Let $\gamma \geq \kappa$. Let $\mathbb{P}$ be a partial order of size less than κ. For every $C \in \mathcal{P}_\kappa(\gamma)^{V[G]}$, there is a $D \in \mathcal{P}_\kappa(\gamma)^V$ such that in $V[G]$, $C \subseteq D$. Consequently, in $V[G]$, $\hat{C} \supseteq \hat{D}$.*

Proof. Let $\dot{C}$ be a name and $p \in \mathbb{P}$ a condition such that $p \Vdash \dot{C} \in \mathcal{P}_\kappa(\gamma)$. For each $q \leq p$, define

$$D_q \triangleq \{\alpha \mid q \Vdash \alpha \in \dot{C}\}.$$

Clearly, $|D_q| < \kappa$. Let $D = \bigcup_{q \leq p} D_q$. Then, as κ is regular and $|\mathbb{P}| < \kappa$, it follows that $|D| < \kappa$, and in particular, $D \in \mathcal{P}_\kappa(\gamma)^V$. It is also clear that $p \Vdash \dot{C} \subseteq \check{D}$. Hence, $p \Vdash \exists D \in \mathcal{P}_\kappa(\gamma)^V(\dot{C} \subseteq D)$, and we are done.

□

We shall call the next lemma the Lévy-Solovay Lemma. The Lévy-Solovay and Approximation lemmas are the most important tools that we shall use in our main results. In conjunction, they allow us to approximate functions with certain domains in our symmetric extension $V(G)$ by functions in V which are "almost everywhere" the same. The use of the quotation marks shall be demystified in the last section.

Lemma 27 (Lévy-Solovay Lemma)**.** *In V, let κ be a regular cardinal, D be a set and $\mathcal{U}$ a κ-complete ultrafilter on D. Let $\mathbb{P}$ be a poset of size less than κ and G a V-generic filter on $\mathbb{P}$. Suppose $V[G] \vDash f : D \to V$. Then there is an $S \in \mathcal{U}$ and a $g : S \to V$ in V s.t. $V[G] \vDash f \restriction S = g$.*

Proof. Let $p \in \mathbb{P}$ and $\dot{f}$ a name for f be such that $p \Vdash \dot{f} : \check{D} \to \check{V}$. We shall show that there are $q \leq p$, $Z \in \mathcal{U}$, $f : \check{Z} \to \check{V}$ s.t. $q \Vdash \dot{f} \restriction \check{Z} = \check{f}$. For $r \leq p$, let

$$X_r \triangleq \{s \in D \mid \exists x(r \Vdash \dot{f}(\check{s}) = \check{x})\}.$$

Now, $p \Vdash \text{dom}(\dot{f}) = \check{D}$, so $\bigcup_{r \leq p} X_r = D$, and as $|\mathbb{P}| < \kappa$, κ-completeness gives us a $q \leq p$ such that $X_q \in \mathcal{U}$ and a function g with domain X_q in V defined by $g(s) = x$ where x is such that $q \Vdash \dot{f}(\check{s}) = \check{x}$. Clearly then, $q \Vdash \dot{f} \restriction \check{X}_q = \check{g}$. Hence, by standard density arguments, we are done. □

Once we have the above lemma, we can show that κ-complete ultrafilters in the ground model generate κ-complete ultrafilters in the generic extension:

Lemma 28. *In V, let κ be a regular cardinal, $\mathbb{P}$ a poset of size less than κ, D a set and $\mathcal{U}$ a κ-complete ultrafilter on D. Let G be a V-generic filter on $\mathbb{P}$. Let $\mathcal{W}$ be the filter on D generated by $\mathcal{U}$ in $V[G]$. Then $\mathcal{W}$ is a κ-complete ultrafilter.*

Proof. We use the above lemma repeatedly.

(i) **Ultrafilter.** In $V[G]$, let $X \subseteq D$. Let $(f : D \to 2)^{V[G]}$ be the indicator function of X. Let $Y \in \mathcal{U}$ and $(g : Y \to 2)^V$ be such that $V[G] \vDash f \restriction Y = g$. As $\mathcal{U}$ is an ultrafilter, it follows that $g^{-1}(i) \in \mathcal{U}$ for exactly one $i \in 2$. Depending on which, we can conclude that there is a $Z \in \mathcal{U}$ such that $V[G] \vDash Z \subseteq X$ or $V[G] \vDash Z \subseteq (D \setminus X)$. Accordingly, exactly one of X and $D \setminus X$ is in $\mathcal{W}$.

(ii) **κ-complete.** Let $\gamma < \kappa$ and $(f : D \to \gamma)^{V[G]}$. Using the Lévy-Solovay Lemma, obtain $Y \in \mathcal{U}$ and $(g : Y \to \gamma)^V$ such that $V[G] \vDash f \restriction Y = g$. By κ-completeness it follows that $g^{-1}(\alpha) \in \mathcal{U}$ for some $\alpha \in \gamma$. Hence we get $f^{-1}(\alpha) \in \mathcal{W}$ for some $\alpha \in \gamma$.

□

In fact, the above lemma could have been sharpened further depending on the specific properties that the set D and the ultrafilter $\mathcal{U}$ have. The next theorem, which we call the Lévy-Solovay Theorem, does this for the specific properties that define some large cardinal properties:

Theorem 29. *In V, let κ be a measurable or γ-compact or γ-supercompact cardinal. Let $\mathbb{P}$ be a poset of size less than κ, and let G be a V-generic filter over $\mathbb{P}$. Then κ continues to be measurable, or γ-compact or γ-supercompact in $V[G]$. Further, the required ultrafilters in $V[G]$ are the ones generated by the corresponding ultrafilters in V. Consequently, if κ is strongly compact or supercompact in V, then it stays so in $V[G]$.*

Proof. (i) **Measurable.** This follows immediately from the lemma above, by setting D to be κ and $\mathcal{U}$ to be some κ-complete nonprincipal ultrafilter on it. Note that a non-principal filter on κ in V will generate a non-principal filter on κ in $V[G]$. We only show that if $\mathcal{U}$ is normal, then so is $\mathcal{W}$. So, suppose $f \in V[G]$ is such that

$$((f : \kappa \to \kappa) \wedge (f \text{ is decreasing}))^{V[G]}.$$

Using the Lévy-Solovay lemma, obtain $Y \in \mathcal{U}$ and $g \in V$ such that

$$((g : Y \to \kappa) \wedge (g \text{ is decreasing}))^V$$

such that $V[G] \vDash f \restriction Y = g$. Clearly then, g is decreasing on a set in $\mathcal{U}$, and if $\mathcal{U}$ is normal in V, it follows that there is some $Y \in \mathcal{U}$ s.t. g is constant on Y. Consequently, we get that in $V[G]$, f is constant on a set in $\mathcal{U}$, and hence on a set in $\mathcal{W}$ as well.

(ii) γ-**compact.** In V, let $\mathcal{U}$ be a fine measure on $\mathcal{P}_\kappa(\gamma)$. Let $\mathcal{V}$ be the κ-complete measure it generates on $\mathcal{P}_\kappa(\gamma)^V$ in $V[G]$. Let $\mathcal{W}$ be the filter generated by $\mathcal{V}$ on $\mathcal{P}_\kappa(\gamma)$ in $V[G]$. As $\mathcal{W}$ is generated by a κ-complete ultrafilter on $\mathcal{P}_\kappa(\gamma)^V \subseteq \mathcal{P}_\kappa(\gamma)$, we get that $\mathcal{W}$ is a κ-complete ultrafilter itself. All that is left then is to show that it is fine. This follows from Lemma 26: for any $X \in \mathcal{P}_\kappa(\gamma)^{V[G]}$, there is a $Y \in \mathcal{P}_\kappa(\gamma)^V$ such that $V[G] \vDash \hat{X} \supseteq \hat{Y} \supseteq \hat{Y}^V$. And clearly, $\hat{Y}^V \in \mathcal{U}$, so $\hat{X} \in \mathcal{W}$.

(iii) γ-**supercompact.** Using what we have proved so far, all that is left to show is that if $\mathcal{U}$ is a normal measure on $\mathcal{P}_\kappa(\gamma)$ in V, then $\mathcal{W}$ is normal on $\mathcal{P}_\kappa(\gamma)$ in $V[G]$. So, let $f \in V[G]$ be such that

$$((f : \mathcal{P}_\kappa(\gamma) \to \gamma) \wedge (\forall X \in \mathcal{P}_\kappa(\gamma)[f(X) \in X]))^{V[G]}.$$

Let h be the restriction of f to $\mathcal{P}_\kappa(\gamma)^V$. Clearly then,

$$((h : \mathcal{P}_\kappa(\gamma)^V \to \gamma) \wedge (\forall X \in \mathcal{P}_\kappa(\gamma)^V[h(X) \in X]))^{V[G]}.$$

Using the Lévy-Solovay Lemma obtain $Y \in \mathcal{U}$ and $g \in V$ such that $(g : Y \to \gamma)^V$ and $V[G] \vDash h \restriction Y = g$. Clearly then, g is a choice function in V on a set in $\mathcal{U}$, and by the normality of $\mathcal{U}$ in V we get that g is constant on a set in $\mathcal{U}$, and so h is constant on a set in $\mathcal{U}$, and consequently f is constant on a set in $\mathcal{W}$ in $V[G]$.

□

It should be mentioned that the Lévy-Solovay theorem can also be generalised to other sorts of large cardinals which are defined not just in terms of ultrapowers, but by directed systems of ultrapowers such as Woodin cardinals as well [HW00]. It can also be generalised in a different direction by allowing larger partial orders, whose size may even be larger than the size of specific large cardinal itself, but which enjoy certain other properties [Ham01]. In another direction, it can also be shown to hold true for large cardinal notions very close to the Kunen inconsistency, where the elementary embeddings are defined not for the entire universe, but for specific levels of the von Neumann hierarchy [Lav07].

6 The Forcing Notion and its Properties

For the rest of this report we shall (for some fixed regular cardinal η and *large* cardinal $\kappa > \eta$) work with a specific triple $(\mathbb{P}, \mathcal{G}, I)$:

- $\mathbb{P} \triangleq \{p : \eta \rightharpoonup \kappa \mid |p| < \eta \text{ and } p \text{ is injective}\}$. The ordering is reverse inclusion.
- $\mathcal{G} \triangleq \mathcal{S}_\kappa$ i.e. the full permutation group of κ. We extend it to the partial order in the usual way.

- $I \triangleq \{E_\alpha \mid \eta < \alpha < \kappa\}$ where $E_\alpha \triangleq \{p \cap (\eta \times \alpha) \mid p \in \mathbb{P}\}$.

The specific notion of large cardinal will be mentioned whenever some specific property needs to be used. For the most part, we shall leave it unspecified and prove general results. The important thing to notice is that κ is a large cardinal in a model of ZFC, hence it is atleast an inaccessible cardinal. Consequently, for any α, $\eta < \alpha < \kappa$, $\kappa > |\mathbb{P} \cap E_\alpha| (= |E_\alpha|)$. Other observations which we shall use:

(i) $\mathbb{P}$ has the κ^+-cc. Therefore, all cardinals above κ are preserved. Further, η is regular, and hence $\mathbb{P}$ is η-closed and so all cardinals $\leq \eta$ are preserved as well.

(ii) I is a projectable symmetry generator, and given $p \in \mathbb{P}$ and $E_\alpha \in I$, $p \restriction E_\alpha = p \cap \eta \times \alpha$.

(iii) $V[G \cap E_\alpha]$ is an E_α-generic extension of V. As E_α is itself a forcing poset (and has size less than κ), the Lévy-Solovay theorem is applicable in $V[G \cap E_\alpha]$, and we can conclude that κ is *large* in $V[G \cap E_\alpha]$ as well. As a most special case, κ is not collapsed in $V[G \cap E_\alpha]$.

Lemma 30. *$\mathbb{P}$ is $(\mathcal{G}, I)$-homogenous, and hence $(\mathbb{P}, \mathcal{G}, I)$ has the approximation property.*

Proof. Let $\alpha \in (\eta, \kappa)$, $p \in \mathbb{P}$ and $q \in \mathbb{P}$ be s.t. $q \leq p \restriction E_\alpha$. We have to find an automorphism $a \in \mathsf{fix} E_\alpha$ such that $a(p) \parallel q$. For this, consider any automorphism $a \in \mathsf{fix} E_\alpha$ such that

$$(\mathrm{dom}(a(p)) \setminus (\mathrm{dom}(p^*))) \cap (\mathrm{dom}(q) \setminus (\mathrm{dom}(p^*))) = \emptyset.$$

Or, more intelligibly, pick an automorphism $a \in \mathsf{fix} E_\alpha$ so that the 'extra domain' of $a(p)$ is disjoint from the 'extra domain' of q. By 'extra domain' we mean the domain which is not included in the domain of p^*. Since η is regular and p and q are partial functions of size less than η, such an automorphism can be found. □

Lemma 31. *In $V(G)$, κ is the successor of η.*

Proof. Let $\eta < \gamma < \kappa$. Consider the name $\dot{\tau}$ given by:

$$\dot{\tau} \triangleq \{(\check{p}, p \restriction E_\gamma) \mid p \in \mathbb{P}\}.$$

Clearly, $\dot{\tau} \in \mathsf{HS}$ (with support E_γ) and τ^G is a bijection from η to γ. Now, suppose that in $V(G)$, there is a bijection $f : \eta \to \kappa$. By the approximation lemma, there is an E_α such that $f \in V[G \cap E_\alpha]$. But this is a contradiction as κ is not collapsed in $V[G \cap E_\alpha]$. □

The next lemma will, once we've shown that measurable cardinals are preserved by the Jech construction, give us a handle on how much of the axiom of choice is valid in the resulting model.

Lemma 32. *In $V(G)$, the powerset of η, $\mathcal{P}(\eta)$, is a κ-long union of sets of size less than κ. Also, for all $X \in V[G \cap E_\alpha]$ for some α, X is wellorderable.*

Proof. For the first part, notice that for any $X \in \mathcal{P}(\eta)$, the Approximation Lemma implies that there is an α such that $X \in \mathcal{P}(\eta)^{V[G \cap E_\alpha]}$. Therefore, if

$$C_\alpha \triangleq \{x \subseteq \eta \mid x \in V[G \cap E_\alpha]\},$$

Then for each $\eta < \alpha < \kappa$,

$$C_\alpha = \mathcal{P}(\eta)^{V[G\cap E_\alpha]} \in V[G \cap E_\alpha] \subset V(G),$$

and in $V(G)$, $\mathcal{P}(\eta) = \bigcup_{\eta<\alpha\kappa} C_\alpha$. What remains then is to show that each of the C_α have size less than κ. For this, notice that in $V[G \cap E_\alpha]$, κ remains a large cardinal by the Lévy-Solovay theorem, and in particular, remains an inaccessible cardinal. Hence, in $V[G\cap E_\alpha]$, there is a bijection between $\mathcal{P}(\eta)$ and some ordinal smaller than κ. This bijection is therefore also present in $V(G)$, and as κ remains a cardinal in $V(G)$, we can conclude that the C_α have are well-ordered and have size less than κ in $V(G)$. For the second part, if $X \in V[G\cap E_\alpha]$, then as $V[G\cap E_\alpha]$ is a model of ZFC, there is an ordinal γ and a bijection $f : \gamma \to X$ in $V[G \cap E_\alpha]$. As $V[G \cap E_\alpha] \subseteq V(G)$, $f \in V(G)$ as well, and so X can be well-ordered. □

7 Successor Large Cardinals

By now, most of the hard work has already been done. All the proofs that follow are essentially the same, where the Approximation lemma and the theorems developed in Section 5 are applied one after the other.

To explain our general strategy, we will refer to filters which have certain closure properties such as κ-completeness, fineness, normality etc. as *large cardinal filters*. In this terminology, an ordinal is called a large cardinal if certain large cardinal filters exist. Now, our aim is to show that if such a filter exists 'on' a cardinal κ in V, then it exists 'on' the same cardinal in $V(G)$ as well.

Our method for showing this is the following: using the filters that we know exist in V, we will generate filters in $V(G)$ on the corresponding domains. Then, to show that these filters have certain closure properties, we use the Approximation Lemma to conclude that the sets in $V(G)$ which are in the domain of the filter are already in some $V[G \cap E_\alpha]$. Then, we use the Lévy-Solovay Lemma to approximate these sets in V. We then use the fact that the generating filters in V already had these closure properties to reach our conclusion.

In the case of γ-compactness and γ-supercompactness, because the corresponding domains actually change in each of the universes, there will be an extra step involved in showing that these generated filters are fine filters, and there we will use Lemma 26, in particular the fact that $V \subset V[G \cap E_\alpha] \subset V(G)$, and that the fine filters in $V[G \cap E_\alpha]$ which the Lévy-Solovay theorems give us are also those generated by the fine filters in V on the corresponding domains. Hence, there is some coherence in the filters on $\mathcal{P}_\kappa(\gamma)$ which exist in all of these universes, which we shall exploit in the proofs.

Lemma 33. *Let D be a set and $\mathcal{U}$ a κ-complete ultrafilter on D in V. Suppose $V(G) \vDash f : D \to V$. Then there is $S \in \mathcal{U}$ and $g : S \to V$ in V s.t. $V(G) \vDash f\restriction S = g$.*

Proof. First we apply the Approximation Lemma to obtain an α such that $f \in V[G \cap E_\alpha]$. Next, we use the Lévy-Solovay lemma to obtain $S \in \mathcal{U}$ and $g : S \to V$ such that $V[G \cap E_\alpha] \vDash f\restriction S = g$. Clearly then, $V(G) \vDash f\restriction S = g$ as well. □

Lemma 34. *In V, let D be a set and $\mathcal{U}$ a κ-complete ultrafilter on D. Let $\mathcal{W}$ be the filter on D generated by $\mathcal{U}$ in $V(G)$. Then $\mathcal{W}$ is a κ-complete ultrafilter.*

Proof. The proof is exactly the same as that of Lemma 28, using the previous lemma instead of the Lévy-Solovay Lemma.

(i) **Ultrafilter.** In $V(G)$, let $X \subseteq D$. Let $(f : D \to 2)^{V(G)}$ be the indicator function of X. Let $Y \in \mathcal{U}$ and $(g : Y \to 2)^V$ be such that $V(G) \vDash f \restriction Y = g$. As $\mathcal{U}$ is an ultrafilter, it follows that $g^{-1}(i) \in \mathcal{U}$ for exactly one $i \in 2$. Depending on which, we can conclude that there is a $Z \in \mathcal{U}$ such that $V(G) \vDash Z \subseteq X$ or $V(G) \vDash Z \subseteq (D \setminus X)$. Accordingly, exactly one of X and $D \setminus X$ is in $\mathcal{W}$.

(ii) **κ-complete.** Let $\gamma < \kappa$ and $(f : D \to \gamma)^{V(G)}$. Using the previous lemma, obtain $Y \in \mathcal{U}$ and $(g : Y \to \gamma)^V$ such that $V(G) \vDash f \restriction Y = g$. By κ-completeness it follows that $g^{-1}(\alpha) \in \mathcal{U}$ for some $\alpha \in \gamma$. Hence we get $f^{-1}(\alpha) \in \mathcal{W}$ for some $\alpha \in \gamma$.

□

Theorem 35. *In V, let $\kappa \leq \gamma$. If κ is a measurable, γ-compact, γ-supercompact cardinal, then it stays so in $V(G)$. Further, the required ultrafilters in $V(G)$ are the ones generated by the corresponding ultrafilters in V. Consequently, if κ is strongly compact or supercompact in V, it stays so in $V(G)$.*

Proof. (i) **Measurable.** This follows easily from the above, noting that if $\mathcal{U}$ is a non-principal filter on κ in V, then the filter $\mathcal{W}$ it generates on $V[G]$ is non-principal as well. We only show that if $\mathcal{U}$ is normal, then $\mathcal{W}$ is normal as well. So, let $f \in V(G)$ be such that

$$((f : \kappa \to \kappa) \wedge (f \text{ is decreasing}))^{V(G)}.$$

Using Lemma 33, obtain $Y \in \mathcal{U}$ and $g \in V$ such that

$$((g : Y \to \kappa) \wedge (g \text{ is decreasing}))^V$$

and $V(G) \vDash f \restriction Y = g$. Clearly then, g is decreasing on a set in $\mathcal{U}$, and if $\mathcal{U}$ is normal in V, it follows that there is some $Y \in \mathcal{U}$ such that g is constant on Y. Consequently, we get that in $V(G)$, f is constant on a set in $\mathcal{U}$, and hence on a set in $\mathcal{W}$.

(ii) **γ-compact.** Let $\mathcal{U}$ be a fine measure on $\mathcal{P}_\kappa(\gamma)$ in V. Let $\mathcal{V}$ be the κ-complete measure it generates on $\mathcal{P}_\kappa(\gamma)^V$ in $V(G)$, and let $\mathcal{W}$ be the filter generated by $\mathcal{V}$ on $\mathcal{P}_\kappa(\gamma)$ in $V(G)$. As $\mathcal{W}$ is generated by a κ-complete ultrafilter on $\mathcal{P}_\kappa(\gamma)^V \subseteq \mathcal{P}_\kappa(\gamma)$, we get that $\mathcal{W}$ is a κ-complete ultrafilter itself. For fineness, let $X \in \mathcal{P}_\kappa(\gamma)^{V(G)}$. By the Approximation lemma, there is an α such that $X \in V[G \cap \alpha]$, and as κ is not collapsed when going from V to $V[G \cap E_\alpha]$, $X \in \mathcal{P}_\kappa(\gamma)^{V[G \cap E_\alpha]}$. Hence, by the Lévy-Solovay theorem, $\hat{X} \in \mathcal{V}'$, where $\mathcal{V}'$ is the fine measure that $\mathcal{U}$ generates on $\mathcal{P}_\kappa(\gamma)^{V[G \cap E_\alpha]}$. But $\mathcal{P}_\kappa(\gamma)^{V[G \cap E_\alpha]} \subseteq \mathcal{P}_\kappa(\gamma)^{V(G)}$, and so $\mathcal{U} \subseteq \mathcal{V}' \subseteq \mathcal{W}$. So $\mathcal{W}$ is fine as well.

(iii) **γ-supercompact.** Using what we have proved so far, all that is left to show is that if $\mathcal{U}$ is a normal measure on $\mathcal{P}_\kappa(\gamma)$ in V, then $\mathcal{W}$ is normal on $\mathcal{P}_\kappa(\gamma)$ in $V(G)$. So, let

$$((f : \mathcal{P}_\kappa(\gamma) \to \gamma) \wedge (\forall X \in \mathcal{P}_\kappa(\gamma)[f(X) \in X]))^{V(G)}.$$

Let h be the restriction of f to $\mathcal{P}_\kappa(\gamma)^V$. Using the Approximation Lemma, obtain α such that $h \in V[G \cap E_\alpha]$. Using the Lévy-Solovay Lemma obtain $Y \in \mathcal{U}$ and $(g : Y \to \gamma)^V$ such that $V[G \cap E_\alpha] \vDash h \restriction Y = g$. Clearly, g is a choice function on a set in $\mathcal{U}$ in V, and by the normality of $\mathcal{U}$ in V, we get that g is constant on a set in $\mathcal{U}$, and so h is constant on a set in $\mathcal{U}$, and consequently f is constant on a set in $\mathcal{W}$ in $V[G]$.

□

Note that the measurability of κ in $V(G)$ in combination with Lemma 23, which we recall was proven in ZF, shows that $\mathcal{P}(\eta)$ is not well-orderable in $V(G)$. This, in combination with Lemma 32 shows that AC_κ fails in $V(G)$.

Corollary 36. *Let η be a regular cardinal, and $\kappa > \eta$ be a measurable cardinal or a γ-compact cardinal or a γ-supercompact cardinal. Then there is a symmetric submodel of a generic extension of the universe which is a model of $\mathsf{ZF} + \neg\mathsf{AC}_\kappa +$"$\eta$ is a cardinal, $\kappa = \eta^+$ and κ is a measurable or γ-compact or γ-supercompact cardinal". Consequently, the same is true for κ being strongly compact or supercompact.* □

8 Conclusion

In this report, we have shown how models of set theory where the axiom of choice fails can have very counterintuitive properties. For all of the large cardinal properties that we chose, the methods were very uniform, in that only minor modifications had to be made depending on the choice of the particular largr cardinal notion in order to make our two basic lemmas, the Approximation Lemma and the Lévy-Solovay Lemma applicable. It would be interesting to see whether models for various other counterintuitive statements which are consistent with ZF and the negation of AC can be found by similar techniques. As our investigation was inspired by statements that are true in models of determinacy, this biases the following list:

(i) A model with consecutive measurable, strongly compact and supercompact cardinals. It should be mentioned that such a model is obtained in [AH86].

(ii) A model where $\aleph_1$ is measurable and the club filter is an ultrafilter [Mit02].

(iii) A model where there is a cardinal with the strong partition property (cf. [Kle77]).

(iv) A model with an arbitrarily large initial segment such that each cardinal in this initial segment has countable cofinality [Git80].

(v) A model with a measurable cardinal with no normal measure on it [BG12].

It should be pointed out that as the references suggest, models as mentioned above have been shown to exist already. What would be of interest would be to see if similar 'elementary' methods as in this report can be used to obtain them.

Another avenue would be to investigate other large cardinal properties which successor cardinals can be made to satisfy. In the case of properties defined by way of (finite exponent) partition relations, this can be easily done. For the case of weakly compact cardinals for example, the relevant lemmas can be found in [Jec08], or, in our terminology, in [Dim11]. For other types of large cardinal properties which can be meaningfully expressed in ZF alone, it would be interesting to see if they can also be satisfied by successor cardinals in models where the axiom of choice fails. Another interesting direction would be to construct models of $\mathsf{ZF} + \mathsf{L}$ starting from a model of a theory which has consistency strength less than $\mathsf{ZFC} + \mathsf{L}$, for those large cardinal properties L for whom this is possible. We leave such questions for future investigations.

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