Geometry in Quantum Kripke Frames

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Abstract

Quantum Kripke frames and other related kinds of Kripke frames are introduced. The inner structures of these Kripke frames are studied in detail, and many of them turn out to form nice geometries. To be precise, geometric frames, which are more general than quantum Kripke frames, correspond to projective geometries with a pure polarity; and quantum Kripke frames correspond to irreducible Hilbertian geometries, which play an important role in foundations of quantum theory. Besides, many useful results of these structures are proved.

Contents

1 Definitions

In this section, I will define quantum Kripke frames and some related mathematical structures. All these structures are special cases of a class of very simple structures called Kripke frames defined as follows:

Definition 1.1. A *Kripke frame* is a tuple $\mathfrak{F} = (\Sigma, \to)$ in which Σ is a non-empty set and $\to \subseteq \Sigma \times \Sigma$.

Before defining other structures, I will introduce some important terminologies and notations in a Kripke frame $\mathfrak{F} = (\Sigma, \to)$:

- If $(s,t) \in \to$, call that *s and t are non-orthogonal* and write $s \to t$.
- If $(s,t) \notin \to$, call that *s and t are orthogonal* and write $s \not\to t$.
- $P \subseteq \Sigma$ is *orthogonal*, if $s \not\to t$ for any $s, t \in P$ satisfying $s \neq t$.
- The *orthocomplement of $P \subseteq \Sigma$ (with respect to $\to$)*, denoted by $\sim P$, is the set $\{s \in \Sigma \mid s \not\to t, \text{ for every } t \in P\}$.
- $P \subseteq \Sigma$ is *bi-orthogonally closed*, if $\sim\sim P = P$. $\mathcal{L}_{\mathfrak{F}}$ is used to denote the set $\{P \subseteq \Sigma \mid \sim\sim P = P\}$.
- $s, t \in \Sigma$ are *indistinguishable with respect to $P \subseteq \Sigma$*, denoted by $s \approx_P t$, if $s \to x \Leftrightarrow t \to x$ for every $x \in P$.
- $t \in \Sigma$ is *an approximation of $s \in \Sigma$ in P*, if $t \in P$ and $s \approx_P t$.

Remark 1.2. As it turns out, the notion of indistinguishability is very important. I list some basic properties of this relation, which are all easy to verified:

- for every $P \subseteq \Sigma$, $\approx_P$ is an equivalence relation on Σ;
- $\approx_\emptyset = \Sigma \times \Sigma$ and $Id_\Sigma \stackrel{\text{def}}{=} \{(s,s) \mid s \in \Sigma\} \subseteq \approx_\Sigma$[1];
- if $P \subseteq Q \subseteq \Sigma$, then $\approx_Q \subseteq \approx_P$.

Next, I mention several properties of a Kripke frame $\mathfrak{F} = (\Sigma, \to)$, some of which are common in the literature and the others may not:

- **Reflexivity:** $s \to s$, for every $s \in \Sigma$.

[1] Under Separation (introduced just after this remark), which is assumed for most of the time below, this inclusion becomes the identity.

- **Symmetry:** $s \to t$ implies that $t \to s$, for any $s, t \in \Sigma$.
- **Separation:** $s \neq t$ implies that there is a $w \in \Sigma$ such that $w \to s$ and $w \not\to t$.
- **Existence of Approximation for Lines (AL):**

  For any $s, t \in \Sigma$, if $w \in \Sigma \setminus {\sim}\{s,t\}$, then there is a $w' \in \Sigma$ which is an approximation of w in ${\sim\sim}\{s,t\}$, i.e. $w' \in P$ and $w \approx_{\sim\sim\{s,t\}} w'$.
- **Existence of Approximation for Hyperplanes (AH):**

  For each $s \in \Sigma$, if $w \in \Sigma \setminus {\sim\sim}\{s\}$, then there is a $w' \in \Sigma$ which is an approximation of w in ${\sim}\{s\}$, i.e. $w' \in {\sim}\{s\}$ and $w \approx_{\sim\{s\}} w'$.
- **Existence of Approximation (A):**

  For each $P \subseteq \Sigma$ with ${\sim\sim}P = P$, if $s \in \Sigma \setminus {\sim}P$, then there is an $s' \in \Sigma$ which is an approximation of s in P, i.e. $s' \in P$ and $s \approx_P s'$.
- **Superposition:** for any $s, t \in \Sigma$, there's a $w \in \Sigma$ such that $w \to s$ and $w \to t$.

In the above, I call a set of the form ${\sim\sim}\{s,t\}$ by 'line' and one of the form ${\sim}\{s\}$ by 'hyperplane'. The reason will be made clear below (Remark 3.8 and Proposition 3.9).

Now I'm ready to define the structures.

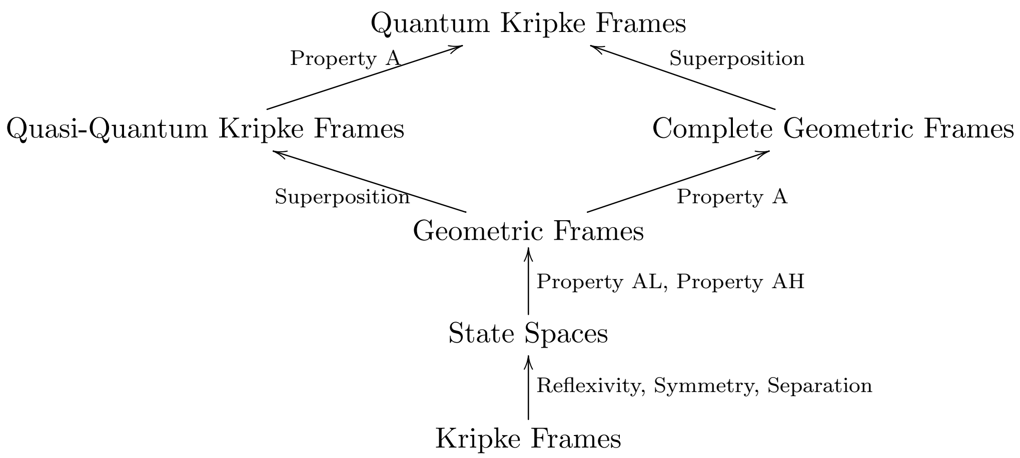
Definition 1.3. A *state space* is a Kripke frame satisfying Reflexivity, Symmetry and Separation.

The following structures are all special kinds of state spaces:

- A *geometric frame* is a state space satisfying Property AL and Property AH.
- A *complete geometric frame* is a state space satisfying Property A.
- A *quasi-quantum Kripke frame* is a state space satisfying Property AL, Property AH and Superposition.
- A *quantum Kripke frame* is a state space satisfying Property A and Superposition.

I emphasize that, according to the above definition, a quantum Kripke frame is a Kripke frame with five properties: Reflexivity, Symmetry, Separation, Property A and Superposition.

Below I will prove that Property A implies both Property AL and Property AH in state spaces (Proposition 2.4). It follows that a complete geometric frame is a geometric frame, and a quantum Kripke frame is a quasi-quantum Kripke frame (Corollary 2.5). Therefore, my way of naming these structures is justified. Given this, the relations among these structures can be summarized in the following picture:



In this picture, for example, the arrow from the node 'Quasi-Quantum Kripke Frames' to the node 'Quantum Kripke Frames' labelled by 'Property A' should be read as that quantum Kripke frames are quasi-quantum Kripke frames satisfying Property A.

Finally, I would like to point out that some of these structures are not new but have been proposed and studied in the literature (see e.g. [4]).

2 State Spaces

In this section, I'm going to investigate the structure inside a state space. I will start from some basic properties of the orthocomplement, and this will be followed by a study of the structure formed by bi-orthogonally closed subsets. This section will end with the relation among Property A, Property AH and Property AL.

First, I present some elementary properties of the orthocomplement.

Proposition 2.1. *In every state space* $\mathfrak{F} = (\Sigma, \to)$*,*

1. $\sim\Sigma = \emptyset$ *and* $\sim\emptyset = \Sigma$*, so both* Σ *and* $\emptyset$ *are bi-orthogonally closed;*

2. $P \subseteq Q$ *implies* $\sim Q \subseteq \sim P$*, for any* $P, Q \subseteq \Sigma$*;*

3. $P \subseteq \sim\sim P$*, for every* $P \subseteq \Sigma$*;*

4. $\sim P$ *is bi-orthogonally closed, for every* $P \subseteq \Sigma$*;*

5. $P \cap \sim P = \emptyset$*, for every* $P \subseteq \Sigma$*.*

Proof. *1*, *2*, *3* and *5* follow easily from the definition of state spaces.

For *4*, $P \subseteq \sim\sim P$ by *3*, so $\sim\sim\sim P \subseteq \sim P$ by *2*. Using *3* again, $\sim P \subseteq \sim\sim\sim P$. Hence $\sim P = \sim\sim\sim P$ and $\sim P$ is bi-orthogonally closed. $\square$

Remark 2.2. In a state space $\mathfrak{F} = (\Sigma, \rightarrow)$, from *2*, *3* and *4* of the above proposition, one can easily deduce that $\sim\sim(\cdot)$ is a *closure operator on* Σ in the sense that, for any $P, Q \subseteq \Sigma$,

- $P \subseteq \sim\sim P$;
- $P \subseteq Q$ implies that $\sim\sim P \subseteq \sim\sim Q$;
- $\sim\sim\sim\sim P = \sim\sim P$.

In the following, I may call $\sim\sim P$ *the bi-orthogonal closure of* $P \subseteq \Sigma$.

Second, I study the structure of the set of all bi-orthogonally closed subsets of a state space in some detail.

Proposition 2.3. *For every state space* $\mathfrak{F} = (\Sigma, \rightarrow)$*, the set* $\mathcal{L}_{\mathfrak{F}}$ *of all bi-orthogonally closed subsets of a state space* $\mathfrak{F} = (\Sigma, \rightarrow)$ *forms a complete atomistic orthocomplemented lattice with* $\subseteq$ *as the partial order and* $\sim(\cdot)$ *as the orthocomplementation. In particular,*

1. *for every* $\{P_i \mid i \in I\} \subseteq \mathcal{L}_{\mathfrak{F}}$*,* $\bigcap_{i \in I} P_i$ *is bi-orthogonally closed and is the greatest lower bound of* $\{P_i \mid i \in I\}$*;*

2. *for each* $s \in \Sigma$*,* $\{s\}$ *is bi-orthogonally closed, and thus is the atom of this lattice;*

3. *for every* $\{P_i \mid i \in I\} \subseteq \mathcal{L}_{\mathfrak{F}}$*,* $\bigvee_{i \in I} P_i \overset{\text{def}}{=} \bigcap \{Q \in \mathcal{L}_{\mathfrak{F}} \mid P_i \subseteq Q$ *for each* $i \in I\}$ *is bi-orthogonally closed and is the least upper bound of* $\{P_i \mid i \in I\}$*;*

4. $P = \bigvee \{\{s\} \in \mathcal{L}_{\mathfrak{F}} \mid \{s\} \subseteq P\}$*, for every* $P \in \mathcal{L}_{\mathfrak{F}}$*.*

5. $\sim\sim P = P$*, for each* $P \in \mathcal{L}_{\mathfrak{F}}$*;*

6. $P \subseteq Q$ *implies that* $\sim Q \subseteq \sim P$*, for any* $P, Q \in \mathcal{L}_{\mathfrak{F}}$*;*

7. $P \cap \sim P = \emptyset$ *and* $P \vee \sim P = \Sigma$*, for each* $P \in \mathcal{L}_{\mathfrak{F}}$*.*

8. *De Morgan's laws hold, i.e.* $\bigcap_{i\in I} {\sim}P_i = {\sim}\bigvee_{i\in I} P_i$ *and* $\bigvee_{i\in I} {\sim}P_i = {\sim}\bigcap_{i\in I} P_i$*, for every* $\{P_i \mid i \in I\} \subseteq \mathcal{L}_{\mathfrak{F}}$.

The lattice is complete in the sense of 1, and it's atomistic in the sense of 4. ${\sim}(\cdot)$ *is an orthocomplementation in the sense of 5, 6 and 7.*

Proof. *1* to *7* follow from Lemma 5.5 in [4] and *8* follows from *1* to *7* together with Lemma 4.1 in the same paper, although there the complement of $\rightarrow$, which is irreflexive, is taken as primitive. Direct proofs are not very hard. □

Moore in [4] shows that this proposition can be strengthened to a duality between a category with complete atomistic orthocomplemented lattices as objects and one with state spaces as objects.

Finally, I will use the above results to establish the relation among Property A, Property AL and Property AH.

Proposition 2.4. *In every state space* $\mathfrak{F} = (\Sigma, \rightarrow)$*, Property A implies both Property AL and Property AH.*

Proof. Assume that Property A holds.

For Property AL, assume that $w \in \Sigma \setminus {\sim}\{s, t\}$. By *4* of Proposition 2.1 ${\sim}{\sim}\{s, t\}$ is bi-orthogonally closed. *4* of Proposition 2.1 also implies that ${\sim}\{s, t\} = {\sim}{\sim}{\sim}\{s, t\}$, so $w \notin {\sim}{\sim}{\sim}\{s, t\}$. By Property A there is an approximation of w in ${\sim}{\sim}\{s, t\}$.

For Property AH, assume that $w \in \Sigma \setminus {\sim}{\sim}\{s\}$. By *4* of Proposition 2.1, ${\sim}\{s\}$ is bi-orthogonally closed. Since $w \notin {\sim}{\sim}\{s\}$, by Property A there is an approximation of w in ${\sim}\{s\}$. □

Corollary 2.5. *Every complete geometric frame is a geometric frame, and every quantum Kripke frame is a quasi-quantum Kripke frame.*

Proof. Straightforward from the definitions and the above proposition. □

This corollary justifies my way of naming the structures.

I'm going to end this section with a remark.

Remark 2.6. Observe that, in a reflexive and symmetric Kripke frame $\mathfrak{F} = (\Sigma, \rightarrow)$, Property AH and Separation together is equivalent to the following:

> **Strong Existence of Approximation for Hyperplanes (AH'):** For each $s \in \Sigma$, if $w \in \Sigma \setminus \{s\}$, then there is a $w' \in \Sigma$ which is an approximation of w in ${\sim}\{s\}$, i.e. $w' \in {\sim}\{s\}$ and $w \approx_{{\sim}\{s\}} w'$.

The reason is as follows: For one direction, assume that Property AH and Separation holds. Then $\mathfrak{F} = (\Sigma, \to)$ is a state space. According to *2* of Proposition 2.3, $w \in \Sigma \setminus \{s\}$ implies that $w \in \Sigma \setminus {\sim}{\sim}\{s\}$. Hence Property AH implies that there is an approximation of w in ${\sim}\{s\}$. For the other direction, assume that Property AH' holds. First I show that Property AH holds. Since $\{s\} \subseteq {\sim}{\sim}\{s\}$ holds in reflexive and symmetric Kripke frames, $w \in \Sigma \setminus {\sim}{\sim}\{s\}$ implies that $w \in \Sigma \setminus \{s\}$. Hence Property AH' implies that there is an approximation of w in ${\sim}\{s\}$. Second I show that Separation holds. Suppose that $s, t \in \Sigma$ are such that $s \neq t$. According to Property AH', there is an $s' \in \Sigma$ which is an approximation of s in ${\sim}\{t\}$, i.e. $s' \in {\sim}\{t\}$ and $s \approx_{\sim\{t\}} s'$. Then $s' \not\to t$ follows from $s' \in {\sim}\{t\}$. Moreover, $s' \to s$ follows from $s \approx_{\sim\{t\}} s'$, $s' \in {\sim}\{t\}$ and $s' \to s'$. Therefore, s' has the required property.

In the following, I will mainly discuss Kripke frames that are state spaces, and hence I may use Property AH' more often than Property AH, for the antecedent of Property AH' is simpler.

3 Geometric Frames

In this section, I'm going to investigate the structure inside a geometric frame. The main result is a representation theorem for projective geometries with a pure polarity using geometric frames. Definitions and results in projective geometry used in this report are reviewed in Appendix A.

3.1 From Geometric Frames to Projective Geometries

In this subsection, I will show that every geometric frame can be organised as a projective geometry with a pure polarity. For convenience, I will fix an arbitrary geometric frame $\mathfrak{F} = (\Sigma, \to)$ throughout this subsection.

Define a function $\star : \Sigma \times \Sigma \to \wp(\Sigma)$ such that $s \star t \stackrel{\text{def}}{=} {\sim}{\sim}\{s, t\}$ for every $s, t \in \Sigma$. Intuitively, one can think of $s \star t$ as the unique line passing through two points s and t, and, in the degenerate case when $s = t$, $s \star t$ is the point itself: this intuition will be justified formally below (Remark 3.8). Notice that $s \star t = t \star s$ for any $s, t \in \Sigma$, according to this definition. Also define a function $p : \Sigma \to \wp(\Sigma)$ such that $p(s) \stackrel{\text{def}}{=} {\sim}\{s\}$ for every $s \in \Sigma$. I will show that $(\Sigma, \star, p)$ is a projective geometry with a pure polarity by verifying the conditions in the definition one by one, and this will be presented as propositions. I will also prove some useful lemmas at the meantime.

As a start, notice that Σ is a non-empty set according to the definition of a geometric frame.

Proposition 3.1. $(\Sigma, \star, p)$ *satisfies (P1), i.e.* $s \star s = \{s\}$, *for every* $s \in \Sigma$.

Proof. By definition $s \star s = {\sim}{\sim}\{s, s\} = {\sim}{\sim}\{s\}$. By Proposition 2.3 ${\sim}{\sim}\{s\} = \{s\}$. Hence $s \star s = \{s\}$. $\square$

Proposition 3.2. $(\Sigma, \star, p)$ *satisfies (P2), i.e.* $s \in t \star s$, *for every* $s, t \in \Sigma$.

Proof. Notice that, for each $w \in {\sim}\{t, s\}$, $s \not\to w$ by definition. Hence $s \in {\sim}{\sim}\{t, s\} = t \star s$. $\square$

Before continuing to (P3), I will prove some very useful lemmas. The following lemma intuitively says that, for every $s \in \Sigma$, ${\sim}\{s\}$ intersects every line.

Lemma 3.3. *For any* $s, t, u \in \Sigma$, *if* $s \neq t$, *there is a* $v \in s \star t$ *such that* $u \not\to v$.

Proof. If $u \in {\sim}\{s, t\}$, taking v to be t will work, since $u \not\to t$ and $t \in s \star t$ by Proposition 3.2. In the following, I focus on the case when $u \notin {\sim}\{s, t\}$.

By Property AL there is a $u' \in {\sim}{\sim}\{s, t\}$ such that $u \approx_{{\sim}{\sim}\{s,t\}} u'$. Since $s \neq t$, $u' \neq s$ or $u' \neq t$. Without loss of generality, assume that $u' \neq s$. Then by Separation there is a $w \in \Sigma$ such that $w \to s$ and $w \not\to u'$. Since $w \to s$, $w \notin {\sim}\{s, t\}$. Hence by Property AL there is a $v \in {\sim}{\sim}\{s, t\}$ such that $w \approx_{{\sim}{\sim}\{s,t\}} v$.

I claim that this v has the required property. In fact, by the construction $v \in {\sim}{\sim}\{s, t\} = s \star t$. Moreover, $u' \in {\sim}{\sim}\{s, t\}$ and $v \approx_{{\sim}{\sim}\{s,t\}} w$ together with $w \not\to u'$ imply that $v \not\to u'$. Then, combining with $v \in {\sim}{\sim}\{s, t\}$ and $u \approx_{{\sim}{\sim}\{s,t\}} u'$, $v \not\to u$ is implied. Therefore, v has the required property. $\square$

The following lemma and its corollaries justify the intuition that $s \star t$ is a line when $s \neq t$ in the sense that $s \star t$ is determined by two distinct elements in Σ.

Lemma 3.4. *For any* $s, t, w \in \Sigma$, *if* $w \neq t$ *and* $w \in s \star t$, *then* $s \star t = w \star t$.

Proof. First, observe that $s \neq t$; otherwise, from $w \in s \star t = \{t\}$, $w = t$ can be derived, which contradicts $w \neq t$.

Second, I prove from $s \neq t$ that there is an $s' \in {\sim}\{t\}$ such that ${\sim}\{s, t\} = {\sim}\{s', t\}$. Since $s \neq t$, by Property AH' there is an $s' \in {\sim}\{t\}$ such that $s \approx_{{\sim}\{t\}} s'$. Then, for every $u \in \Sigma$,

$$\begin{aligned} & u \in {\sim}\{s, t\} \\ \Leftrightarrow\ & u \not\to s \text{ and } u \not\to t \\ \Leftrightarrow\ & u \not\to s' \text{ and } u \not\to t && \text{(by } s \approx_{{\sim}\{t\}} s') \end{aligned}$$

$$\Leftrightarrow u \in {\sim}\{s', t\}$$

Hence ${\sim}\{s, t\} = {\sim}\{s', t\}$.

Similarly, from $w \neq t$ one can find a $w' \in {\sim}\{t\}$ such that $w \approx_{\sim\{t\}} w'$, and thus ${\sim}\{w, t\} = {\sim}\{w', t\}$.

I claim that $w' = s'$. Suppose (towards a contradiction) that $w' \neq s'$. By Separation there is a $v \in \Sigma$ such that $v \to w'$ and $v \not\to s'$. Since $v \to w'$ and $w' \not\to t$, $v \neq t$. By Property AH' there is a $v' \in {\sim}\{t\}$ such that $v \approx_{\sim\{t\}} v'$. Now, on the one hand, since $w', s' \in {\sim}\{t\}$ and $v \approx_{\sim\{t\}} v'$, one can deduce that $v' \to w'$ and $v' \not\to s'$. On the other hand, since $v' \to w'$ and $v' \in {\sim}\{t\}$, by $w \approx_{\sim\{t\}} w'$ I get $v' \to w$. Since $w \in s \star t = {\sim\sim}\{s, t\}$, $v' \notin {\sim}\{s, t\}$, i.e. $v' \to s$ or $v' \to t$. Then $v' \to s$ follows, for $v' \not\to t$. Since again $v' \in {\sim}\{t\}$, by $s \approx_{\sim\{t\}} s'$ I get $v' \to s'$, contradicting that $v' \not\to s'$ which is proved just before. Therefore, $s' = w'$.

As a result, $s \star t = {\sim\sim}\{s, t\} = {\sim\sim}\{s', t\} = {\sim\sim}\{w', t\} = {\sim\sim}\{w, t\} = w \star t$. □

Corollary 3.5. *For any $s, t, w \in \Sigma$, if $w \neq t$ and $w \in s \star t$, then $s \in w \star t$.*

Proof. Two cases need to be considered.

Case 1: $s = t$. Then $s \in w \star s = w \star t$ by Proposition 3.2.

Case 2: $s \neq t$. Then by the above lemma $s \star t = w \star t$. Since $s \in t \star s = s \star t$, $s \in w \star t$. □

Corollary 3.6. *For any $s, t, u, v \in \Sigma$, if $u \neq v$ and $u, v \in s \star t$, then $u \star v = s \star t$.*

Proof. By the assumption $s \star t$ is not a singleton, and thus $s \neq t$ by Proposition 3.1. Then $u \neq s$ or $u \neq t$. Without loss of generality, assume that $u \neq s$. Then it follows from $u \neq s$ and $u \in s \star t$ that $u \star s = s \star t$, according to Lemma 3.4. Again by Lemma 3.4 it follows from $u \neq v$ and $v \in s \star t = u \star s$ that $u \star v = u \star s$. Therefore, $u \star v = u \star s = s \star t$. □

Proposition 3.7. $(\Sigma, \star, p)$ *satisfies (P3), i.e. $s \in t \star r$, $r \in x \star y$ and $s \neq x$ imply that $(s \star x) \cap (t \star y) \neq \emptyset$, for all $s, t, x, y, r \in \Sigma$.*

Proof. [2]Three cases need to be considered.

Case 1: $r = y$. In this case, $s \in t \star r = t \star y$. Hence $s \in (s \star x) \cap (t \star y)$, so $(s \star x) \cap (t \star y) \neq \emptyset$.

[2]This proof mimics the proof of the Theorem of Buekenhout and Parmentier in [1]. In my opinion, in principle, I can prove this proposition by introducing their terminologies and applying their theorem. However, since it's not very long, a direct proof may be more helpful in developing intuitions.

Case 2: $s \star x \subseteq t \star y$. In this case, $x \in s \star x = (s \star x) \cap (t \star y)$ and thus $(s \star x) \cap (t \star y) \neq \emptyset$.

Case 3: $r \neq y$ and $s \star x \not\subseteq t \star y$. In this case $\sim\{t, y\} \not\subseteq \sim\{s, x\}$ by Proposition 2.1. Then there is a $u \in \sim\{t, y\}$ such that $u \notin \sim\{s, x\}$. Since $s \neq x$, by Lemma 3.3 there is a $v \in s \star x$ such that $v \not\to u$. I claim that $v \in t \star y$. Under this claim, $v \in (s \star x) \cap (t \star y)$, and thus $(s \star x) \cap (t \star y) \neq \emptyset$.

It remains to prove the claim that $v \in t \star y$, i.e. $v \in \sim\sim\{t, y\}$.

Let $w \in \sim\{t, y\}$ be arbitrary. To establish the claim, it suffices to show that $v \not\to w$. If $w = u$, then one can deduce that $v \not\to w$ by the construction of v. Now it remains to deal with the case when $w \neq u$.

First observe that there is a $z \in w \star u$ such that $z \in \sim\{t, y, r\}$. Since $w \neq u$, by Lemma 3.3 there is a $z \in w \star u$ such that $z \not\to r$. Since $w, u \in \sim\{t, y\}$, it's not hard to show that $w \star u \subseteq \sim\{t, y\}$. Hence $z \in w \star u \subseteq \sim\{t, y\}$. Combining this with $z \not\to r$, $z \in \sim\{t, y\} \cap \sim\{r\} = \sim\{t, y, r\}$.

Second observe that $v \not\to z$. Since $s \in t \star r$, $z \not\to s$. Since $r \in x \star y$ and $r \neq y$, $x \in r \star y$ by Corollary 3.5. Hence $z \not\to x$. I have shown that $z \in \sim\{s, x\}$. Since $v \in s \star x$, $v \not\to z$.

Now it's ready for showing that $v \not\to w$. Since $z \in \sim\{s, x\}$ and $u \notin \sim\{s, x\}$, $z \neq u$. Since $z \in w \star u$, by Corollary 3.5 $w \in z \star u$. Remembering that $v \in \sim\{z, u\}$, $v \not\to w$. □

Remark 3.8. By now I have proved that $(\Sigma, \star)$ satisfies (P1), (P2) and (P3), so it is a projective geometry. Hence it's justified to think of $s \star t = \sim\sim\{s, t\}$ as the line passing through s and t whenever $s \neq t$.

Now I continue to show that the function p is a pure polarity on $(\Sigma, \star)$.

Proposition 3.9. *For every $s \in \Sigma$, $p(s) = \sim\{s\}$ is a hyperplane of $(\Sigma, \star)$.*

Proof. Assume that P is a subspace of $(\Sigma, \star)$ such that $\sim\{s\} \subseteq P$ and there is an $x \in P \setminus \sim\{s\}$. To prove that $\sim\{s\}$ is a hyperplane, by definition it suffices to show that $\Sigma \subseteq P$. Let $u \in \Sigma$ be arbitrary. If $u = x$, then $u \in P$ because $x \in P$. In the following, I focus on the case when $u \neq x$. By Lemma 3.3 there is a $v \in u \star x$ such that $v \not\to s$. Since $x \notin \sim\{s\}$, $x \neq v$. By Corollary 3.5 $u \in x \star v$. From $v \in \sim\{s\} \subseteq P$ and $x \in P$, one can deduce that $x \star v \subseteq P$ for P is a subspace. Therefore, $u \in P$. For u is arbitrary, $\Sigma \subseteq P$. □

Proposition 3.10.

- *$s \in p(t) \Leftrightarrow t \in p(s)$, for any $s, t \in \Sigma$.*
- *$s \notin p(s)$, for every $s \in \Sigma$.*

Proof. These follow from the definition of p, Reflexivity and Symmetry. □

Finally, I arrive at the conclusion of this subsection:

Theorem 3.11. $(\Sigma, \star, p)$ *defined as above from the geometric frame* $\mathfrak{F} = (\Sigma, \rightarrow)$ *is a projective geometry with a pure polarity.*

Proof. This follows immediately from the definition and all propositions proved in this subsection. □

In the following, I will call $(\Sigma, \star, p)$ *the projective geometry with a pure polarity corresponding to the geometric frame* $\mathfrak{F} = (\Sigma, \rightarrow)$.

3.2 From Projective Geometries to Geometric Frames

In this subsection, I will show that every projective geometry with a pure polarity can be organised as a geometric frame. For our discussion, I will fix an arbitrary projective geometry with a pure polarity $\mathcal{G} = (G, \star, p)$ throughout this subsection.

Define a relation $\rightarrow \subseteq G \times G$ such that, for any $a, b \in G$, $a \rightarrow b \Leftrightarrow a \notin p(b)$. Denote the orthocomplement operator with respect to $\rightarrow$ by $\sim(\cdot)$ again. I will show that $(G, \rightarrow)$ is a geometric frame by verifying the conditions in the definition one by one. (Given Remark 2.6, I will deal with Property AH' instead of Separation and Property AH.)

As a start, notice that G is non-empty by the definition of projective geometries.

Proposition 3.12. $(G, \rightarrow)$ *satisfies Reflexivity and Symmetry.*

Proof. Reflexivity holds because $a \notin p(a)$ for every $a \in G$. Symmetry holds because $a \notin p(b) \Leftrightarrow b \notin p(a)$ for any $a, b \in G$. □

I continue with a characterization of $\star$ and p in terms of $\sim(\cdot)$.

Lemma 3.13. *For any* $a, b \in G$, $p(a) = \sim\{a\}$ *and* $a \star b = \sim\sim\{a, b\}$.

Proof. $p(a) = \sim\{a\}$ is obvious from the definition of $\sim(\cdot)$.

For $a \star b = \sim\sim\{a, b\}$, observe that, for every $c \in G$,

$$\begin{aligned}
& c \in a \star b \\
\Leftrightarrow\ & p(a) \cap p(b) \subseteq p(c) && \text{(by Proposition A.17)} \\
\Leftrightarrow\ & x \in p(a) \text{ and } x \in p(b) \text{ imply that } x \in p(c), \text{ for every } x \in G \\
\Leftrightarrow\ & x \not\rightarrow a \text{ and } x \not\rightarrow b \text{ imply that } x \not\rightarrow c, \text{ for every } x \in G \\
\Leftrightarrow\ & x \in \sim\{a, b\} \text{ implies that } x \not\rightarrow c, \text{ for every } x \in G \\
\Leftrightarrow\ & c \in \sim\sim\{a, b\}
\end{aligned}$$

Therefore, $a \star b = \sim\sim\{a, b\}$. □

Proposition 3.14. $(G, \to)$ *satisfies Property AH'.*

Proof. Assume that $a, b \in G$ are such that $a \neq b$. It's required to show that there is a $c \in {\sim}\{b\}$ such that $a \approx_{\sim\{b\}} c$. By the above lemma, it suffices to find a $c \in p(b)$ such that $x \in p(a) \Leftrightarrow x \in p(c)$ for every $x \in p(b)$.

Observe that $(a \star b) \cap p(b)$ is a singleton. Since $a \neq b$, $(a \star b) \cap p(b)$ is either $a \star b$ or a singleton, according to Theorem A.5. Since p is pure, $b \notin p(b)$, so $(a \star b) \cap p(b) \neq a \star b$. Therefore, $(a \star b) \cap p(b)$ is a singleton. Denote by c the unique element in this singleton.

I claim that this c has the required property. On the one hand, by construction $c \in a \star b$, so by *1* of Proposition A.17

$$p(a) \cap p(b) \subseteq p(c) \tag{1}$$

On the other hand, since $c \in p(b)$ and $b \notin p(b)$, $b \neq c$. Since $c \in a \star b$, by (P4) of Lemma A.2 $a \in b \star c$. Then by *1* of Proposition A.17

$$p(c) \cap p(b) \subseteq p(a) \tag{2}$$

The required property of c follows easily from (1) and (2). □

Proposition 3.15. $(G, \to)$ *satisfies (AL).*

Proof. Assume that $c, a, b \in G$ are such that $c \notin {\sim}\{a, b\}$. It's required to show that there is a $d \in {\sim\sim}\{a, b\}$ such that $c \approx_{\sim\sim\{a,b\}} d$. By Lemma 3.13, it suffices to find a $d \in a \star b$ such that $c \in p(x) \Leftrightarrow d \in p(x)$ for every $x \in a \star b$.

As a start, notice that there is one easy case. When $a = b$, it's easy to see that a has the required property. In the following, I will focus on the case when $a \neq b$.

First, observe that $(a \star b) \cap p(c)$ is a singleton. Since $a \neq b$, $(a \star b) \cap p(c)$ is either $a \star b$ or a singleton, according to Theorem A.5. Since $c \notin {\sim}\{a, b\}$, $c \notin p(a)$ or $c \notin p(b)$, and thus $a \notin p(c)$ or $b \notin p(c)$. Hence $(a \star b) \cap p(c) \neq a \star b$. Therefore, $(a \star b) \cap p(c)$ is a singleton. Denote by s the unique element in this set.

Second, observe that $(a \star b) \cap p(s)$ is a singleton. Since $a \neq b$, $(a \star b) \cap p(s)$ is either $a \star b$ or a singleton, according to Theorem A.5. Since by construction $s \in a \star b$, $s \in {\sim\sim}\{a, b\}$ by Lemma 3.13. Since p is pure, it's not hard to see that $s \notin {\sim}\{a, b\}$. Hence $s \notin p(a)$ or $s \notin p(b)$, and thus $a \notin p(s)$ or $b \notin p(s)$. Hence $(a \star b) \cap p(s) \neq a \star b$. Therefore, $(a \star b) \cap p(s)$ is a singleton. Denote by d the unique element in this set.

Now I show that this d has the required property. Let $x \in a \star b$ be arbitrary. First assume that $c \in p(x)$. Then $x \in p(c)$, and thus $x \in (a \star b) \cap p(c)$. According to the construction of s, $x = s$. Hence $d \in p(s) = p(x)$.

Second assume that $d \in p(x)$. I claim that $x = s$. Suppose (towards a contradiction) that $s \neq x$. Since $a \neq b$, $s \neq a$ or $s \neq b$. Without loss of generality, assume that $s \neq a$. Since $s \neq a$, $s \in a \star b$, $x \neq s$ and $x \in a \star b$, applying (P7) in Lemma A.2 twice one can deduce that $a \star b = s \star a = s \star x$. It follows from $s, x \in p(d)$ that $a \star b = s \star x \subseteq p(d)$, since $p(d)$ is a hyperplane. Then $d \in a \star b \subseteq p(d)$, contradicting that p is pure. Therefore, $x = s$. Since $s \in p(c)$, $c \in p(s) = p(x)$. As a result, d has the required property. □

Finally, I come to the main conclusion of this subsection:

Theorem 3.16. *$(G, \to)$ defined as above from the projective geometry with a pure polarity $\mathcal{G} = (G, \star, p)$ is a geometric frame.*

Proof. This follows immediately from the definition and all propositions proved in this subsection. □

In the following, I will call $(G, \to)$ *the geometric frame corresponding to the projective geometry with a pure polarity* $\mathcal{G} = (G, \star, p)$.

3.3 Correspondence

In this subsection, I strengthen the results in the above two subsections to a correspondence between geometric frames and projective geometries with a pure polarity.

Theorem 3.17. *Every geometric frame $\mathfrak{F}$ is the corresponding geometric frame of the projective geometry with a pure polarity corresponding to $\mathfrak{F}$.*

Every projective geometry with a pure polarity $\mathcal{G}$ is the corresponding projective geometry with a pure polarity of the geometric frame corresponding to $\mathcal{G}$.

Proof. For the first part, let $\mathfrak{F} = (\Sigma, \to)$ be an arbitrary geometric frame. Take its corresponding projective geometry with a pure polarity $(\Sigma, \star, p)$ given in Theorem 3.11. Let $(\Sigma, \rightarrowtail)$ be the geometric frame corresponding to $(\Sigma, \star, p)$ given in Theorem 3.16. Then by the relevant definitions, for any $s, t \in \Sigma$,

$$s \rightarrowtail t \Leftrightarrow s \notin p(t) \Leftrightarrow s \notin {\sim}\{t\} \Leftrightarrow s \to t.$$

Therefore, $(\Sigma, \rightarrowtail)$ is identical to $\mathfrak{F} = (\Sigma, \to)$.

For the second part, let $\mathcal{G} = (G, \star, p)$ be an arbitrary projective geometry with a pure polarity. Take its corresponding geometric frame $(G, \to)$ given in Theorem 3.16, and let $(G, \circledast, p')$ be the projective geometry with a pure polarity corresponding to $(G, \to)$ given in Theorem 3.11. It's easy to see from relevant definitions and Lemma 3.13 that, for any $a, b \in G$,

$$a \circledast b = \sim\sim\{a, b\} = a \star b \text{ and } a \in p'(b) \Leftrightarrow a \not\to b \Leftrightarrow a \in p(b).$$

Therefore, $(G, \circledast, p')$ is identical to $\mathcal{G} = (G, \star, p)$. $\square$

Another correspondence can be drawn from the above theorem.

Corollary 3.18. *For every geometric frame $\mathfrak{F} = (\Sigma, \to)$,*

- *define a function $\star : \Sigma \times \Sigma \to \Sigma$ such that $s \star t \overset{\text{def}}{=} \sim\sim\{s, t\}$ for any $s, t \in \Sigma$;*
- *define $\perp \subseteq \Sigma \times \Sigma$ such that $s \perp t \Leftrightarrow s \not\to t$ for any $s, t \in \Sigma$;*

Then $(\Sigma, \star, \perp)$ is a pure orthogeometry, called the pure orthogeometry corresponding to $\mathfrak{F}$.

For every pure orthogeometry $\mathcal{G} = (G, \star, \perp)$, $(G, \not\perp)$ is a geometric frame, called the geometric frame corresponding to $\mathcal{G}$.

Every geometric frame $\mathfrak{F}$ is the corresponding geometric frame of the pure orthogeometry corresponding to $\mathfrak{F}$.

Every pure orthogeometry $\mathcal{G}$ is the corresponding pure orthogeometry of the geometric frame corresponding to $\mathcal{G}$.

Proof. This follows from Theorem 3.17 and Theorem A.20. $\square$

The theorem and its corollary mean that, on the one hand, projective geometries with a pure polarity and pure orthogeometries are Kripke frames in disguise, and on the other hand, geometric frames have nice geometric structures. Therefore, notions and results in projective geometry can be introduced into the study of geometric frames. The most useful and relevant ones are reviewed in Appendix A. Among them, I emphasize the following:

Definition 3.19. In a geometric frame $\mathfrak{F} = (\Sigma, \to)$,

- for any $s, t \in \Sigma$, the *line generated by s and t*, denoted by $s \star t$, is $\sim\sim\{s, t\}$;
- $P \subseteq \Sigma$ is a *subspace of* $\mathfrak{F}$, if $s \star t \subseteq P$ for any $s, t \in P$;
- $\mathcal{C}(P) \overset{\text{def}}{=} \bigcap\{Q \subseteq \Sigma \mid Q \text{ is a subspace of } \mathfrak{F} \text{ and } P \subseteq Q\}$ is called the *linear closure of* $P \subseteq \Sigma$.

According to Remark 2.2 and Lemma A.7, both the bi-orthogonal closure and the linear closure are closure operators on a geometric frame. In the remaining part of this subsection, I would collect some useful facts relating them.

Lemma 3.20. *In a geometric frame $\mathfrak{F} = (\Sigma, \rightarrow)$, if $P \subseteq \Sigma$ is bi-orthogonally closed, it is a subspace of $\mathfrak{F}$.*

Proof. It suffices to show that, if $s, t \in P$, then $\sim\sim\{s, t\} \subseteq P$. Assume that $s, t \in P$. Then $\{s, t\} \subseteq P$. Applying *2* of Proposition 2.1 twice, one can obtain $\sim\sim\{s, t\} \subseteq \sim\sim P$. Since P is bi-orthogonally closed, $\sim\sim\{s, t\} \subseteq \sim\sim P = P$. □

Lemma 3.21. *In a geometric frame $\mathfrak{F} = (\Sigma, \rightarrow)$, $\sim Q = \sim\mathcal{C}(Q)$, for every $Q \subseteq \Sigma$.*

Proof. By definition $Q \subseteq \mathcal{C}(Q)$, so $\sim\mathcal{C}(Q) \subseteq \sim Q$ by *2* of Proposition 2.1. It remains to show that $\sim Q \subseteq \sim\mathcal{C}(Q)$.

One can define a sequence of sets $\{Q_i\}_{i\in\mathbb{N}}$ in the same way as in Proposition A.8. Then by the proposition $\mathcal{C}(Q) = \bigcup_{i\in\mathbb{N}} Q_i$. It's easy to see from the definition that $\sim\mathcal{C}(Q) = \sim\bigcup_{i\in\mathbb{N}} Q_i = \bigcap_{i\in\mathbb{N}} \sim Q_i$. I prove $\sim Q \subseteq \bigcap_{i\in\mathbb{N}} \sim Q_i = \sim\mathcal{C}(Q)$ by showing that $\sim Q \subseteq \sim Q_i$, for every $i \in \mathbb{N}$. Use induction on i.

Base Step: $i = 0$. $\sim Q \subseteq \sim Q = \sim Q_0$ obviously holds.

Induction Step: $i = n + 1$. Let $s \in \sim Q$ and $t \in Q_{n+1}$ be arbitrary. By definition there are $u, v \in Q_n$ such that $t \in u \star v$. By Induction Hypothesis $s \in \sim Q \subseteq \sim Q_n$. Hence $s \not\rightarrow u$ and $s \not\rightarrow v$, i.e. $s \in \sim\{u, v\}$. Since $t \in u \star v$, $s \not\rightarrow t$. For t is arbitrary, $s \in \sim Q_{n+1}$. Therefore, $\sim Q \subseteq \sim Q_{n+1}$. □

The following lemma suggests a way to get a bigger bi-orthogonally closed set from a smaller one using linear closure.

Lemma 3.22. *In a geometric frame $\mathfrak{F} = (\Sigma, \rightarrow)$, let $s \in \Sigma$ and $P \subseteq \Sigma$ be bi-orthogonally closed. Then $\mathcal{C}(\{s\} \cup P)$ is bi-orthogonally closed.*

Proof. According to Corollary 3.18 $(\Sigma, \star, \not\rightarrow)$ is an orthogeometry. Notice that being bi-orthogonally closed in this report means the same as being a closed subspace in Proposition 14.2.5 of [2]. Since $P \subseteq \Sigma$ is bi-orthogonally closed, so is $\mathcal{C}(\{s\} \cup P)$ by this proposition. □

In the following proposition I show that in a geometric frame for finite sets linear closure coincides with bi-orthogonal closure.

Proposition 3.23. *In a geometric frame $\mathfrak{F} = (\Sigma, \rightarrow)$, for any $n \in \mathbb{N}$ and $s_1, ..., s_n \in \Sigma$, $\mathcal{C}(\{s_1, ..., s_n\}) = \sim\sim\{s_1, ..., s_n\}$.*

Proof. I prove by induction that, for every $n \in \mathbb{N}$, $\mathcal{C}(\{s_1, ..., s_n\})$ is bi-orthogonally closed.

Base Step: $n = 0$. By convention $\{s_1, ..., s_n\}$ is the empty set, which is bi-orthogonally closed by Proposition 2.1.

Induction Step: $n = k + 1$. By Induction Hypothesis $\mathcal{C}(\{s_1, ..., s_k\})$ is bi-orthogonally closed. Then by the above lemma $\mathcal{C}(\{s_{k+1}\} \cup \mathcal{C}(\{s_1, ..., s_k\}))$ is bi-orthogonally closed. By Corollary A.10 $\mathcal{C}(\{s_{k+1}\} \cup \mathcal{C}(\{s_1, ..., s_k\})) = \mathcal{C}(\{s_{k+1}\} \cup \{s_1, ..., s_k\}) = \mathcal{C}(\{s_1, ..., s_k, s_{k+1}\})$. Hence $\mathcal{C}(\{s_1, ..., s_k, s_{k+1}\})$ is bi-orthogonally closed. This finishes the proof by induction.

Now by Lemma 3.21 $\sim\sim\{s_1, ..., s_n\} = \sim\sim\mathcal{C}(\{s_1, ..., s_n\})$. As $\mathcal{C}(\{s_1, ..., s_n\})$ is bi-orthogonally closed, $\sim\sim\{s_1, ..., s_n\} = \mathcal{C}(\{s_1, ..., s_n\})$. $\square$

4 Complete Geometric Frames

In this section, I will start with introducing the important notion of saturated sets and study their properties. Then I will prove a correspondence between complete geometric frames and Hilbertian geometries. Finally I study the properties of subsets which are the bi-orthogonal closures of finite sets and the consequences of finite-dimensionality.

4.1 Saturated Sets in Geometric Frames

For convenience, I fix a geometric frame $\mathfrak{F} = (\Sigma, \to)$ throughout this subsection. I will start with the definition of saturated sets.

Definition 4.1. $P \subseteq \Sigma$ is *saturated*, if every $s \in \Sigma \backslash \sim P$ has an approximation in P, i.e. an $s' \in P$ satisfying $s \approx_P s'$.

Notice that in this terminology Property AL, Property AH and Property A say that every set of the form $\sim\sim\{s, t\}$, of the form $\sim\{s\}$ or being bi-orthogonally closed, respectively, is saturated.

The following proposition establishes that approximations in subspaces are unique, if they exist.

Proposition 4.2. *Let $s \in \Sigma$ and $P \subseteq \Sigma$ be a subspace. Approximation of s in P is unique, i.e. if $t, t' \in P$ are such that $s \approx_P t$ and $s \approx_P t'$, $t = t'$.*

Proof. Suppose (towards a contradiction) that $t \neq t'$.

Observe that there is a $w \in t{\star}t'$ such that $w \to t'$ and $w \not\to t$. Since $t' \neq t$, by Lemma 3.3 there is a $w \in t{\star}t'$ such that $w \not\to t$. Since $w \in t{\star}t' = \sim\sim\{t, t'\}$ and $w \to w$ by Reflexivity, $w \notin \sim\{t, t'\}$. Hence $w \to t'$.

Now since $t, t' \in P$ and P is a subspace, $t \star t' \subseteq P$. It follows from $w \in t \star t'$ that $w \in P$. On the one hand, since $w \to t'$ and $s \approx_P t'$, $s \to w$. On the other hand, since $w \not\to t$ and $s \approx_P t$, $s \not\to w$. Hence a contradiction is derived. Therefore, $t = t'$. $\square$

Next I introduce the notion of orthogonal decompositions, which generalizes a notion in the theory of Hilbert spaces with the same name. It will be very useful in studying saturated sets.

Definition 4.3. An *orthogonal decomposition of* $s \in \Sigma$ *with respect to* $P \subseteq \Sigma$ is a pair $(s_\parallel, s_\perp) \in P \times {\sim}P$ such that $s \in {\sim}{\sim}\{s_\parallel, s_\perp\}$.

$P \subseteq \Sigma$ *admits orthogonal decomposition*, if every $s \in \Sigma \setminus (P \cup {\sim}P)$ has an orthogonal decomposition with respect to P.

Remark 4.4. Notice that the two points in a pair forming an orthogonal decomposition are always orthogonal, and thus distinct.

The following proposition is a basic fact about the relation among bi-orthogonally closed sets, saturated sets and sets admitting orthogonal decomposition.

Proposition 4.5. *For every* $P \subseteq \Sigma$.

1. *if* P *admits orthogonal decomposition,* P *is bi-orthogonally closed;*
2. *if* P *is saturated,* P *admits orthogonal decomposition.*

Proof. **For 1:** Assume that P admits orthogonal decomposition. By Proposition 2.1 $P \subseteq {\sim}{\sim}P$, so it remains to show that ${\sim}{\sim}P \subseteq P$. I will prove the contraposition. Suppose that $s \notin P$. If $s \in {\sim}P$, $s \notin {\sim}{\sim}P$ since $s \to s$. If $s \notin {\sim}P$, by the assumption there is a pair $(s_\parallel, s_\perp) \in P \times {\sim}P$ such that $s \in {\sim}{\sim}\{s_\parallel, s_\perp\} = s_\parallel \star s_\perp$. Since $s \notin P$ and $s_\parallel \in P$, $s \neq s_\parallel$. It follows from $s \in s_\parallel \star s_\perp$ that $s_\perp \in s \star s_\parallel = {\sim}{\sim}\{s, s_\parallel\}$. Then $s_\perp \to s$ or $s_\perp \to s_\parallel$ follows from $s_\perp \to s_\perp$. Since $s_\perp \not\to s_\parallel$ by Remark 4.4, $s_\perp \to s$, and thus $s \to s_\perp$ by Symmetry. It follows from $s_\perp \in {\sim}P$ and $s \to s_\perp$ that $s \notin {\sim}{\sim}P$. As a result, P is bi-orthogonally closed.

For 2: Assume that P is saturated and $s \in \Sigma \setminus (P \cup {\sim}P)$. Then s has an approximation in P, i.e. there is an $s' \in P$ such that $s \approx_P s'$. Since $s \notin P$ and $s' \in P$, $s \neq s'$. Hence there is an $s_\perp \in s \star s'$ such that $s_\perp \not\to s'$ by Lemma 3.3.

Observe that $s_\perp \in {\sim}P$. Suppose (towards a contradiction) that $s_\perp \notin {\sim}P$. By the assumption $s_\perp$ has an approximation in P, i.e. there is an $s'_\perp \in P$ such that $s_\perp \approx_P s'_\perp$. On the one hand, it follows from $s'_\perp \to s'_\perp$ that $s_\perp \to s'_\perp$. On the other hand, since $s_\perp \not\to s'$ and $s' \in P$, one derives that $s'_\perp \not\to s'$.

Since $s \approx_P s'$ and $s'_\perp \in P$, $s'_\perp \not\to s$. Hence $s'_\perp \in \sim\{s, s'\}$. For $s_\perp \in s \star s' = \sim\sim\{s, s'\}$, $s_\perp \not\to s'_\perp$, contradicting $s_\perp \to s'_\perp$. Therefore, $s_\perp \in \sim P$.

Moreover, since $s' \in P$ and $s_\perp \in \sim P$, s' and $s_\perp$ are orthogonal and thus distinct. Hence it follows from $s_\perp \in s \star s'$ that $s \in s' \star s_\perp$.

I conclude that $(s', s_\perp) \in P \times \sim P$ satisfies $s \in s' \star s_\perp$, and thus is an orthogonal decomposition of s with respect to P. For s is arbitrary, P admits orthogonal decomposition. $\square$

It turns out that the converse of *2* in Proposition 4.5 also holds. Before proving this, a technical lemma need to be established.

Lemma 4.6. *If $Q \subseteq \Sigma$, $s \in \Sigma$, $t \in \sim Q$ and $s' \in (s \star t) \cap Q$, then $s \approx_Q s'$.*

Proof. Observe that $s \approx_{\sim\{t\}} s'$. Let $w \in \sim\{t\}$ be arbitrary. First assume that $w \not\to s$. Then $w \in \sim\{s, t\}$. Since $s' \in s \star t = \sim\sim\{s, t\}$, $w \not\to s'$. Second assume that $w \not\to s'$. Since $s' \in P$ and $t \in \sim P$, it follows easily that $s' \neq t$. Since $s' \in s \star t$, by Corollary 3.5 $s \in s' \star t = \sim\sim\{s', t\}$. By the assumption $w \in \sim\{s', t\}$, so $w \not\to s$.

Since $t \in \sim Q$, $Q \subseteq \sim\sim Q \subseteq \sim\{t\}$. Then from $s \approx_{\sim\{t\}} s'$ it follows that $s \approx_Q s'$ by Remark 1.2. $\square$

Now I show that, in geometric frames, saturated sets coincide with sets admitting orthogonal decomposition.

Theorem 4.7. *For every $P \subseteq \Sigma$, P is saturated, if and only if P admits orthogonal decomposition.*

Proof. **The 'Only If' Part:** This is *2* of Proposition 4.5.

The 'If' Part: Assume that P admits orthogonal decomposition and $s \notin \sim P$. If $s \in P$, it's easy to see by definition that s itself is an approximation of s in P. In the following, I focus on the case when $s \notin P$. Then by the assumption there is a pair $(s_\parallel, s_\perp) \in P \times \sim P$ such that $s \in s_\parallel \star s_\perp$. Since $s \notin \sim P$ and $s_\perp \in \sim P$, $s \neq s_\perp$. Hence by Corollary 3.5 $s_\parallel \in s \star s_\perp$, and thus $s_\parallel \in (s \star s_\perp) \cap P$. Since $s_\perp \in \sim P$, $s \approx_P s_\parallel$, according to Lemma 4.6. Therefore, $s_\parallel$ is an approximation of s in P. For s is arbitrary, P is saturated. $\square$

Two important corollaries can be drawn from this theorem. The first one is about properties of orthogonal decompositions.

Corollary 4.8. *Let $P \subseteq \Sigma$ admit orthogonal decomposition, $s \in \Sigma \setminus (P \cup \sim P)$ and $(s_\parallel, s_\perp) \in P \times \sim P$ be an orthogonal decomposition of $s \in \Sigma$ with respect to P, whose existence is guaranteed by the above theorem.*

1. *s, $s_\parallel$ and $s_\perp$ are three distinct points.*

2. *$s \approx_P s_\parallel$.*

3. *$s \approx_{\sim P} s_\perp$.*

4. *$s \to s_\parallel$.*

5. *$s \to s_\perp$.*

6. *If $(t_\parallel, t_\perp) \in P \times {\sim}P$ is an orthogonal decomposition of s with respect to P, then $(s_\parallel, s_\perp) = (t_\parallel, t_\perp)$.*

Proof. For *1*, the three disjoint sets $\Sigma \setminus (P \cup {\sim}P)$, P and ${\sim}P$ separate s, $s_\parallel$ and $s_\perp$, so these three points are all different.

For *2*, since $s \in s_\parallel \star s_\perp$, $s_\parallel \in s \star s_\perp$ by *1* and Corollary 3.5. Then $s_\parallel \in (s \star s_\perp) \cap P$. Since $s_\perp \in {\sim}P$, $s \approx_P s_\parallel$ by Lemma 4.6.

3 can be proved similar to *2*.

4 follows from *2*, $s_\parallel \in P$ and $s_\parallel \to s_\parallel$.

5 follows from *3*, $s_\perp \in {\sim}P$ and $s_\perp \to s_\perp$.

For *6*, assume that $(t_\parallel, t_\perp) \in P \times {\sim}P$ is an orthogonal decomposition of s with respect to P. By *2* and *3* $s \approx_P t_\parallel$ and $s \approx_{\sim P} t_\perp$. Notice that, according to the assumption and Proposition 4.5, P is bi-orthogonally closed, and thus it is a subspace by Lemma 3.20. Hence $s_\parallel = t_\parallel$ and $s_\perp = t_\perp$ follow from Proposition 4.2. Therefore, $(s_\parallel, s_\perp) = (t_\parallel, t_\perp)$. □

The second one is about the structure of saturated sets.

Corollary 4.9. *The set of all saturated sets in a geometric frame $\mathfrak{F} = (\Sigma, \to)$ forms an orthomodular poset. To be precise,*

- *the saturated sets are partially ordered by $\subseteq$ with $\emptyset$ as the least element and Σ the greatest element;*

- *${\sim}(\cdot)$ is an orthocomplementation on this poset in the sense of 5, 6 and 7 in Proposition 2.3;*

- *if $P \subseteq {\sim}Q$, $P \sqcup Q \stackrel{\text{def}}{=} {\sim}{\sim}(P \cup Q)$ is saturated and is the least upper bound of P and Q in the poset;*

- *for every $P, Q \subseteq \Sigma$, $P \subseteq Q$ implies that $P = Q \cap ({\sim}Q \sqcup P)$.*

Proof. This follows from Proposition 3.3 in [3], although there the irreflexive orthogonality relation is taken to be primitive. To be precise:

First, since $(\Sigma, \to)$ is a geometric frame, $(\Sigma, \not\to)$ is an orthogonality space satisfying (1°) and (3°) in the sense of [3]. Second, the set of all subspaces of $\mathfrak{F}$ forms a modular lattice, because $(\Sigma, \star)$ is a projective geometry by Remark 3.8 and the subspaces of a projective geometry form a modular lattice by Proposition 2.4.6 in [2]. Proposition 3.3 in [3] says that under such conditions the splitting sets form an orthomodular poset.

I claim that splitting sets in the sense of [3] are exactly sets admitting orthogonal decomposition in my terminology. First notice that both $\emptyset$ and Σ are splitting sets and sets admitting orthogonal decomposition by definition. Second notice that, for every non-empty proper subspace P of $\mathfrak{F}$, by Theorem A.9 and Lemma A.7

$$
\begin{aligned}
& \mathcal{C}(P \cup {\sim}P) = \Sigma \\
\Leftrightarrow\ & \text{there are } s_{\parallel} \in \mathcal{C}(P) \text{ and } s_{\perp} \in \mathcal{C}({\sim}P) \text{ such that } s \in s_{\parallel} \star s_{\perp}, \text{ for all } s \in \Sigma \\
\Leftrightarrow\ & \text{there are } s_{\parallel} \in P \text{ and } s_{\perp} \in {\sim}P \text{ such that } s \in s_{\parallel} \star s_{\perp}, \text{ for every } s \in \Sigma \\
\Leftrightarrow\ & \text{there are } s_{\parallel} \in P \text{ and } s_{\perp} \in {\sim}P \text{ such that } s \in s_{\parallel} \star s_{\perp}, \\
& \qquad \text{for every } s \in \Sigma \setminus (P \cup {\sim}P) \\
\Leftrightarrow\ & P \text{ admits orthogonal decomposition}
\end{aligned}
$$

Now, for a non-empty proper subset of Σ, on the one hand, if it's a splitting set, which is defined to be a subspace P satisfying $\mathcal{C}(P \cup {\sim}P) = \Sigma$, then it admits orthogonal decomposition by the above reasoning. On the other hand, if it admits orthogonal decomposition, then it is a subspace by Proposition 4.5 and Lemma 3.20, and thus it satisfies $\mathcal{C}(P \cup {\sim}P) = \Sigma$ by the above reasoning. Hence it is a splitting set.

As a result, saturated sets, which are exactly sets admitting orthogonal decomposition by the theorem and thus exactly splitting sets, form an orthomodular poset. $\square$

4.2 Complete Geometric Frames and Hilbertian Geometries

In this subsection, I'm going to do a survey on complete geometric frames, and to show a correspondence between them and Hilbertian geometries.

Based on the results of the previous subsection, one can prove a counterpart of Orthogonal Decomposition Theorem for Hilbert spaces.

Theorem 4.10. *In a complete geometric frame $\mathfrak{F} = (\Sigma, \to)$, for every $P \subseteq \Sigma$, the following are equivalent:*

(i) P is bi-orthogonally closed;

(ii) P is saturated;

(iii) P admits orthogonal decomposition.

Proof. The equivalence of (i) and (ii) follows from (A) and Proposition 4.5, and that of (ii) and (iii) follows from Theorem 4.7. $\square$

Now I continue to show a correspondence between complete geometric frames and Hilbertian geometries based on Theorem 3.17.

Theorem 4.11. *For every geometric frame, it is a complete geometric frame, if and only if its corresponding pure orthogeometry is a Hilbertian geometry.*

For every pure orthogeometry, it is a Hilbertian geometry, if and only if its corresponding geometric frame is a complete geometric frame.

Proof. For the first part, let $\mathfrak{F} = (\Sigma, \to)$ be an arbitrary geometric frame. Denote by $\mathcal{G}_{\mathfrak{F}} = (\Sigma, \star, \not\to)$ its corresponding pure orthogeometry given in Corollary 3.18. According to Lemma 3.20, every bi-orthogonally closed subsets of Σ is a subspace. Remember from the proof of Corollary 4.9 that, for every subspace $P \subseteq \Sigma$, $\mathcal{C}(P \cup {\sim}P) = \Sigma$, if and only if P admits orthogonal decomposition, and thus if and only if P is saturated by Theorem 4.7. Therefore, the following are equivalent:

(i) for every $P \subseteq \Sigma$ satisfying ${\sim}{\sim}P = P$, P is saturated;

(ii) for every $P \subseteq \Sigma$ satisfying ${\sim}{\sim}P = P$, $\mathcal{C}(P \cup {\sim}P) = \Sigma$.

According to the definitions, $\mathfrak{F}$ is a complete geometric frame if (i) holds, and $\mathcal{G}_{\mathfrak{F}}$ is a Hilbertian geometry if (ii) holds. As a result, $\mathfrak{F}$ is a complete geometric frame, if and only if its corresponding pure orthogeometry is a Hilbertian geometry.

The second part follows from the first part and Corollary 3.18, which says that every pure orthogeometry $\mathcal{G}$ is the corresponding pure orthogeometry of the geometric frame corresponding to $\mathcal{G}$. $\square$

4.3 Finite-Dimensionality

In this subsection, I'm going to study properties of subspaces generated by finite sets in geometric frames.

I start with a technical lemma, which gives a way to get a bigger saturated set from a smaller saturated set.

Lemma 4.12. *In a geometric frame $\mathfrak{F} = (\Sigma, \to)$, let $P \subseteq \Sigma$ be saturated and $t \in {\sim}P$. Then $\mathcal{C}(\{t\} \cup P)$ is also saturated.*

Proof. On the one hand, $t \in {\sim}P$ implies $\{t\} \subseteq {\sim}P$. Since P is saturated by the assumption and so is $\{t\}$ by the definition, ${\sim\sim}(\{t\} \cup P)$ is saturated by Corollary 4.9. According to Lemma 3.21, ${\sim\sim}\mathcal{C}(\{t\} \cup P) = {\sim\sim}(\{t\} \cup P)$ is saturated. On the other hand, since P is saturated, it's bi-orthogonally closed by Theorem 4.7. Then, according to Lemma 3.22, $\mathcal{C}(\{t\} \cup P)$ is bi-orthogonally closed and thus $\mathcal{C}(\{t\} \cup P) = {\sim\sim}\mathcal{C}(\{t\} \cup P)$. Therefore, $\mathcal{C}(\{t\} \cup P)$ is saturated. $\square$

The following proposition says that the bi-orthogonal closure of a finite orthogonal set is saturated.

Proposition 4.13. *In a geometric frame $\mathfrak{F} = (\Sigma, \to)$, for every $n \in \mathbb{N}$ and every orthogonal set $\{s_1, ..., s_n\} \subseteq \Sigma$, ${\sim\sim}\{s_1, ..., s_n\}$ is saturated.*

Proof. Use induction on n.

Base Step: $n = 0$. In this case, the set is the empty set. Since, for every $s \in \Sigma$, $s \in \Sigma = {\sim}\emptyset$, vacuously the set is saturated.

Induction Step: $n = k + 1$. By Induction Hypothesis ${\sim\sim}\{s_1, ..., s_k\}$ is saturated. Moreover, as $\{s_1, ..., s_k, s_{k+1}\}$ is orthogonal, $s_{k+1} \in {\sim}\{s_1, ..., s_k\} = {\sim\sim\sim}\{s_1, ..., s_k\}$. Then by Lemma 4.12 $\mathcal{C}(\{s_{k+1}\} \cup {\sim\sim}\{s_1, ..., s_k\})$ is saturated. According to Corollary A.10 and Proposition 3.23,

$$\begin{aligned}
&\mathcal{C}(\{s_{k+1}\} \cup {\sim\sim}\{s_1, ..., s_{k+1}\}) \\
=\ &\mathcal{C}(\{s_{k+1}\} \cup \mathcal{C}(\{s_1, ..., s_{k+1}\})) \\
=\ &\mathcal{C}(\{s_{k+1}\} \cup \{s_1, ..., s_{k+1}\}) \\
=\ &\mathcal{C}(\{s_1, ..., s_{k+1}\}) \\
=\ &{\sim\sim}\{s_1, ..., s_{k+1}\}
\end{aligned}$$

Therefore, ${\sim\sim}\{s_1, ..., s_k, s_{k+1}\}$ is saturated. $\square$

Next I prove the counterpart of the finite version of Gram-Schmidt Theorem for Hilbert spaces.

Theorem 4.14. *In a geometric frame $\mathfrak{F} = (\Sigma, \to)$, for any $n \in \mathbb{N}$ and any $s_1, ..., s_n \in \Sigma$, there is an $m \leq n$ and $t_1, ..., t_m \in \Sigma$ such that $\{t_1, ..., t_m\}$ is an orthogonal set and ${\sim\sim}\{s_1, ..., s_n\} = {\sim\sim}\{t_1, ..., t_m\}$.*

Proof. Use induction on n.

Base Step: $n = 0$. In this case the set is $\emptyset$. Since $\emptyset$ is orthogonal by definition, the result holds.

Induction Step: $n = k+1$. By Induction Hypothesis there is an $l \leq k$ and $t_1, ..., t_l \in \Sigma$ such that $\{t_1, ..., t_l\}$ is orthogonal and $\sim\sim\{s_1, ..., s_k\} = \sim\sim\{t_1, ..., t_l\}$. It follows easily that $\sim\{s_1, ..., s_k\} = \sim\{t_1, ..., t_l\}$. Three cases need to be considered.

Case 1: $s_{k+1} \in \sim\sim\{s_1, ..., s_k\}$. Then it's easy to see that $\sim\sim\{s_1, ..., s_k\} = \sim\sim\{s_1, ..., s_k, s_{k+1}\}$, and hence the above l and $t_1, ..., t_l$ suffice.

Case 2: $s_{k+1} \in \sim\{s_1, ..., s_k\}$. Define t_{l+1} to be s_{k+1}. Then it's not hard to see that $\{t_1, ..., t_l, t_{l+1}\}$ is orthogonal, $l+1 \leq k+1$ and

$$\begin{aligned}
&\sim\sim\{s_1, ..., s_k, s_{k+1}\} \\
&= \mathcal{C}(\{s_1, ..., s_k, s_{k+1}\}) \\
&= \mathcal{C}(\{s_{k+1}\} \cup \{s_1, ..., s_k\}) \\
&= \mathcal{C}(\{s_{k+1}\} \cup \mathcal{C}(\{s_1, ..., s_k\})) \\
&= \mathcal{C}(\{s_{k+1}\} \cup \sim\sim\{s_1, ..., s_k\}) \\
&= \mathcal{C}(\{t_{l+1}\} \cup \sim\sim\{t_1, ..., t_l\}) \\
&= \mathcal{C}(\{t_{l+1}\} \cup \mathcal{C}(\{t_1, ..., t_l\})) \\
&= \mathcal{C}(\{t_{l+1}\} \cup \{t_1, ..., t_l\}) \\
&= \mathcal{C}(\{t_1, ..., t_l, t_{l+1}\}) \\
&= \sim\sim\{t_1, ..., t_l, t_{l+1}\}
\end{aligned}$$

Case 3: $s_{k+1} \notin \sim\{s_1, ..., s_k\}$ and $s_{k+1} \notin \sim\sim\{s_1, ..., s_k\}$. It follows that $s_{k+1} \notin \sim\sim\sim\{t_1, ..., t_l\}$ and $s_{k+1} \notin \sim\sim\{t_1, ..., t_l\}$. Since, according to the above theorem, $\sim\sim\{t_1, ..., t_l\}$ is saturated, it admits orthogonal decomposition by Theorem 4.7. Hence s_{k+1} has an orthogonal decomposition $(t_\parallel, t_{l+1}) \in \sim\sim\{t_1, ..., t_l\} \times \sim\sim\sim\{t_1, ..., t_l\}$ with respect to $\sim\sim\{t_1, ..., t_l\}$. By Corollary 4.8 $s_{k+1} \approx_{\sim\sim\sim\{t_1,...,t_l\}} t_{l+1}$, i.e. $s_{k+1} \approx_{\sim\{t_1,...,t_l\}} t_{l+1}$. Notice that, since $t_{l+1} \in \sim\sim\sim\{t_1, ..., t_l\} = \sim\{t_1, ..., t_l\}$, $\{t_1, ..., t_l, t_{l+1}\}$ is orthogonal. Moreover, for every $x \in \Sigma$,

$$\begin{aligned}
&x \in \sim\{s_1, ..., s_k, s_{k+1}\} \\
\Leftrightarrow\ &x \in \sim\{s_1, ..., s_k\} \text{ and } s_{k+1} \not\to x \\
\Leftrightarrow\ &x \in \sim\{t_1, ..., t_l\} \text{ and } t_{l+1} \not\to x \\
\Leftrightarrow\ &x \in \sim\{t_1, ..., t_{l+1}\}
\end{aligned}$$

Therefore, $\sim\sim\{s_1, ..., s_k, s_{k+1}\} = \sim\sim\{t_1, ..., t_{l+1}\}$. Moreover, $l+1 \leq k+1$ follows from $l \leq k$. □

To draw an important corollary from the above results, I introduce the notion of finitely presentable sets.

Definition 4.15. In a Kripke frame $\mathfrak{F} = (\Sigma, \to)$, $P \subseteq \Sigma$ is *finitely presentable*, if there are $n \in \mathbb{N}$ and $s_1, ..., s_n \in \Sigma$ such that $P = {\sim}\{s_1, ..., s_n\}$ or $P = {\sim}{\sim}\{s_1, ..., s_n\}$.

Corollary 4.16. *In a geometric frame $\mathfrak{F} = (\Sigma, \to)$, every finitely presentable subsets of Σ is saturated.*

Proof. Assume that P is a finitely presentable subset of Σ.

First consider the case when there are $n \in \mathbb{N}$ and $s_1, ..., s_n \in \Sigma$ such that $P = {\sim}{\sim}\{s_1, ..., s_n\}$. By Theorem 4.14 $P = {\sim}{\sim}\{t_1, ..., t_m\}$ for some $m \leq n$ and some orthogonal set $\{t_1, ..., t_n\} \subseteq \Sigma$. Then, according to Proposition 4.13, P is saturated.

Second consider the case when there are $n \in \mathbb{N}$ and $s_1, ..., s_n \in \Sigma$ such that $P = {\sim}\{s_1, ..., s_n\}$. According to the above case, ${\sim}{\sim}\{s_1, ..., s_n\}$ is saturated. By Corollary 4.9 ${\sim}{\sim}{\sim}\{s_1, ..., s_n\}$ is saturated. Hence $P = {\sim}\{s_1, ..., s_n\} = {\sim}{\sim}{\sim}\{s_1, ..., s_n\}$ is saturated by Proposition 2.1. $\square$

Finally, I'm going to consider the special case when the whole space is finitely generated. I start from defining the notion of finite-dimensional geometric frames.

Definition 4.17. A geometric frame $\mathfrak{F} = (\Sigma, \to)$ is *finite-dimensional*, if there are $n \in \mathbb{N}$ and $s_1, ..., s_n \in \Sigma$ such that $\Sigma = {\sim}{\sim}\{s_1, ..., s_n\}$.

Next I prove that finite-dimensional geometric frames correspond to finite-dimensional projective geometries with a pure polarity, and also to finite-dimensional pure orthogeometries.

Proposition 4.18. *A geometric frame $\mathfrak{F} = (\Sigma, \to)$ is finite-dimensional, if and only if the corresponding projective geometry with a pure polarity is finite-dimensional, if and only if the corresponding pure orthogeometry is finite-dimensional.*

Proof. By definition a geometric frame $\mathfrak{F} = (\Sigma, \to)$ is finite-dimensional, if and only if there are $n \in \mathbb{N}$ and $s_1, ..., s_n \in \Sigma$ such that $\Sigma = {\sim}{\sim}\{s_1, ..., s_n\}$. This is equivalent to that there are $n \in \mathbb{N}$ and $s_1, ..., s_n \in \Sigma$ such that $\Sigma = \mathcal{C}(\{s_1, ..., s_n\})$, according to Proposition 3.23. Again this is equivalent to that the corresponding projective geometry with a pure polarity has a finite generating set $\{s_1, ..., s_n\}$ and thus is finite-dimensional. Finally, this is the case, if and only if the pure orthogeometry corresponding to $\mathfrak{F}$ is finite-dimensional, noticing that the projective geometry with a pure polarity corresponding to $\mathfrak{F}$ and the pure orthogeometry corresponding to $\mathfrak{F}$ are corresponding in the sense of Theorem A.20. $\square$

The following lemma is very useful.

Lemma 4.19. *In a geometric frame $\mathfrak{F} = (\Sigma, \to)$ that is finite-dimensional, for every $P \subseteq \Sigma$, the following are equivalent:*

(i) P is bi-orthogonally closed;

(ii) P is a subspace of $\mathfrak{F}$;

(iii) there are $n \in \mathbb{N}$ and $s_1, ..., s_n \in \Sigma$ such that $P = \sim\sim\{s_1, ..., s_n\}$.

Proof. **From (i) to (ii):** This is Lemma 3.20.

From (ii) to (iii): Since $\mathfrak{F}$ is finite-dimensional, by Proposition 4.18 the corresponding pure orthogeometry is finite-dimensional. By Theorem A.15 P has a finite generating set. This means that there are $n \in \mathbb{N}$ and $s_1, ..., s_n \in \Sigma$ such that $P = \mathcal{C}(\{s_1, ..., s_n\})$. Hence $P = \sim\sim\{s_1, ..., s_n\}$, according to Proposition 3.23.

From (iii) to (i): This follows from Proposition 2.1. □

Next I make a significant observation.

Proposition 4.20. *If a geometric frame $\mathfrak{F} = (\Sigma, \to)$ is finite-dimensional, then Property A holds.*

Proof. Let $P \subseteq \Sigma$ be bi-orthogonally closed. By the above lemma there are $n \in \mathbb{N}$ and $s_1, ..., s_n \in \Sigma$ such that $P = \sim\sim\{s_1, ..., s_n\}$. Hence P is saturated by Corollary 4.16. Since P is arbitrary, Property A holds. □

The significance of this proposition is that, if the condition of finite-dimensionality which is indeed second-order is added, then the second-order Property A in the definition of complete geometric frames and that of quantum Kripke frames can be replaced by the two first-order conditions Property AL and Property AH. I make this point clearer by the following theorem, which is the counterpart of the fact that every finite-dimensional pre-Hilbert space is a Hilbert space.

Theorem 4.21. *Every finite-dimensional geometric frame is a complete geometric frame. Every finite-dimensional quasi-quantum Kripke frame is a quantum Kripke frame.*

I end this subsection, as well as this section, by a discussion about bi-orthogonally closed hyperplanes in geometric frames. Remember that by definition hyperplanes are maximal proper subspaces. I show that bi-orthogonally closed hyperplanes in geometric frames all take a simple form.

Proposition 4.22. *In a geometric frame $\mathfrak{F} = (\Sigma, \to)$, every bi-orthogonally closed hyperplane is of the form $\sim\{s\}$ for some $s \in \Sigma$.*

Proof. Let P be a bi-orthogonally closed hyperplane of $\mathfrak{F}$. Since P is a hyperplane, $P \neq \Sigma$. By Proposition 2.1 $\sim P \neq \emptyset$.

Observe that $\sim P$ is a singleton. Suppose (towards a contradiction) that there are $u, v \in \sim P$ such that $u \neq v$. Since P is a hyperplane, by Theorem A.5 $(u \star v) \cap P$ is either $u \star v$ or a singleton. Since $u \in \sim P$, by Reflexivity $u \notin P$. Hence $(u \star v) \cap P$ can only be a singleton. Denote by w the unique element in this set. Since $\sim P$ is bi-orthogonally closed by Proposition 2.1, it is a subspace by Lemma 3.20. Since $u, v \in \sim P$, $w \in u \star v \subseteq \sim P$. Then $w \not\to w$ follows from $w \in P$, contradicting Reflexivity. Therefore, $\sim P$ must be a singleton. Denote by s the unique element in this singleton.

From $\sim P = \{s\}$ it follows that $P = \sim\sim P = \sim\{s\}$. □

A corollary of this proposition is that, for a finite-dimensional geometric frame, the pure polarity on the corresponding projective geometry is surjective.

Corollary 4.23. *Let $\mathfrak{F} = (\Sigma, \to)$ be a finite-dimensional geometric frame. The polarity in the corresponding projective geometry with a pure polarity is surjective.*

Proof. Let P be a hyperplane of the projective geometry with a pure polarity corresponding to $\mathfrak{F}$. Then P is a subspace. Since $\mathfrak{F}$ is finite-dimensional, by Lemma 4.19 P is bi-orthogonally closed. By the above proposition, $P = \sim\{s\}$ for some $s \in \Sigma$, and thus is in the image of the polarity. □

5 (Quasi-)Quantum Kripke Frames

In this section I investigate the relation between Superposition on geometric frames and irreducibility of projective geometries, and then obtain correspondence results for quasi-quantum Kripke frames and quantum Kripke frames.

Remember that a projective geometry is *irreducible*, if $a \star b$ contains (strictly) more than two points for any distinct points a and b.

Proposition 5.1. *Let $\mathfrak{F} = (\Sigma, \to)$ be a geometric frame. Then the following are equivalent:*

(i) Superposition holds;

(ii) for any $s, t \in \Sigma$ satisfying $s \neq t$, there is a $w \in s \star t$ such that $w \neq s$ and $w \neq t$.

Proof. **From (i) to (ii):** Let $s, t \in \Sigma$ satisfying $s \neq t$ be arbitrary. Two cases need to be considered.

Case 1: $s \to t$. Since $s \neq t$, by Lemma 3.3 there is a $w \in s \star t$ such that $s \not\to w$. Then $w \neq s$ by Reflexivity. Since $s \not\to w$ and $s \to t$, $w \neq t$.

Case 2: $s \not\to t$. According to (i), there is a $v \in \Sigma$ such that $v \to s$ and $v \to t$. It follows that $v \notin \sim\{s,t\}$. Since $s \neq t$ follows from Reflexivity and the assumption of this case, by Property AL there is a $w \in \sim\sim\{s,t\}$ such that $w \approx_{\sim\sim\{s,t\}} v$. This means that $w \in s \star t$ is such that $w \approx_{s\star t} v$. Notice that $w \neq s$; otherwise, from $w \approx_{s\star t} v$, $t \in s \star t$ and $s \not\to t$ one can deduce that $v \not\to t$, contradicting that $v \to t$. Similarly, one can show that $w \neq t$.

From (ii) to (i): Let $s, t \in \Sigma$ be arbitrary. Two cases need to be considered.

Case 1: $s \to t$. Then take w to be s. Hence $w \to t$. Moreover, by Reflexivity $w \to s$.

Case 2: $s \not\to t$. By Reflexivity $s \neq t$. By (ii) there is a $w \in s \star t$ such that $w \neq s$ and $w \neq t$. According to Lemma 3.4, $w \star t = s \star t$. Using Proposition 3.9 and Theorem A.5, one deduce that $(w \star t) \cap \sim\{s\}$ is either $w \star t$ or a singleton. Since $s \notin \sim\{s\}$ and $s \in s \star t = w \star t$, $(w \star t) \cap \sim\{s\}$ is a singleton. Since $t \in \sim\{s\}$, $(w \star t) \cap \sim\{s\} = \{t\}$. Therefore, $w \to s$ follows from $w \neq t$. Similarly one can deduce that $w \to t$. $\square$

Corollary 5.2.

- *For every geometric frame, it is a quasi-quantum Kripke frame, if and only if its corresponding projective geometry with a pure polarity is irreducible.*

  For every projective geometry with a pure polarity, it is irreducible, if and only if its corresponding geometric frame is a quasi-quantum Kripke frame.

- *For every geometric frame, it is a quasi-quantum Kripke frame, if and only if its corresponding pure orthogeometry is irreducible.*

  For every pure orthogeometry, it is irreducible, if and only if its corresponding geometric frame is a quasi-quantum Kripke frame.

- *For every geometric frame, it is a quantum Kripke frame, if and only if its corresponding pure orthogeometry is an irreducible Hilbertian geometry.*

  For every pure orthogeometry, it is an irreducible Hilbertian geometry, if and only if its corresponding geometric frame is a quantum Kripke frame.

Proof. This follows from Theorem 3.17, Corollary 3.18, Theorem 4.11 and Proposition 5.1. □

Remark 5.3. This corollary means that there is a one-to-one correspondence between quantum Kripke frames and irreducible Hilbertian geometries. In the literature, it's well known that irreducible Hilbertain geometries are closely related to Hilbert spaces, and thus play an important role in foundations of quantum theory. For the details, one may refer to Theorem 81 and Theorem 82 in [6][3], which is originally proved in [5]. Therefore, due to this correspondence, quantum Kripke frames will help in the study of foundations of quantum theory.

A Projective Geometry

In this appendix I will review some elements of projective geometry. If without explanation, the definitions and results are from [2].

A.1 Basic Notions in Projective Geometry

Definition A.1. A *projective geometry* is a tuple $\mathcal{G} = (G, \star)$, where G is a non-empty set and $\star : G \times G \to \wp(G)$ is a function such that:

(P1) $a \star a = \{a\}$, for every $a \in G$;

(P2) $a \in b \star a$, for any $a, b \in G$;

(P3) $a \in b \star r$, $r \in c \star d$ and $a \neq c$ imply that $(a \star c) \cap (b \star d) \neq \emptyset$, for all $a, b, c, d, r \in G$.

This definition is from [2]. Please refer to this book for the equivalence of this definition to the classical one, which is in terms of lines or a ternary collinearity relation. The following lemma collects some useful properties following from this definition:

Lemma A.2. *In a projective geometry $\mathcal{G} = (G, \star)$, the following holds, for any $a, b, c \in G$:*

(P4) if $a \in b \star c$ and $a \neq b$, then $c \in a \star b$;

(P5) if $a \in b \star c$, then $a \star b \subseteq b \star c$;

[3]Please be aware that in [6] Hilbertian geometries in this report are called Hilbert geometries.

(P6) $a \star b = b \star a$;

(P7) *if* $a \in b \star c$ *and* $a \neq b$, *then* $a \star b = b \star c$.

Proof. Please refer to the proof of Proposition 2.2.2 in [2]. $\square$

The following are some important notions in projective geometries.

Definition A.3. A *subspace* F of a projective geometry $\mathcal{G} = (G, \star)$ is a subset of G satisfying that $a, b \in F$ implies that $a \star b \subseteq F$, for any $a, b \in G$.

Definition A.4. A *hyperplane* H of a projective geometry $\mathcal{G} = (G, \star)$ is a subspace of $\mathcal{G}$ such that

- H is proper, i.e. $H \neq G$;
- H is maximal, i.e. $H \subseteq F$ implies that $F = H$ or $F = G$ for every subspace F of $\mathcal{G}$.

The following theorem is about an important property of hyperplanes.

Theorem A.5. *In a projective geometry* $\mathcal{G} = (G, \star)$, *for any hyperplane* H *and* $a, b \in G$ *satisfying* $a \neq b$, $(a \star b) \cap H$ *is* $a \star b$ *or a singleton.*

Proof. See Proposition 4.2.12 and Remark 4.2.13 (1°) of [2]. $\square$

A.2 Dimension in Projective Geometries

In this subsection, I'm going to recall the notion of dimension in projective geometries. For convenience, I fix a projective geometry $\mathcal{G} = (G, \star)$ throughout this subsection.

The notion of dimension in projective geometries is built on the notion of linear closures.

Definition A.6. Given $A \subseteq G$, the *linear closure*[4] of A, denoted by $\mathcal{C}(A)$, is defined as follows:

$$\mathcal{C}(A) := \bigcap \{E \in \wp(G) \mid A \subseteq E \text{ and } E \text{ is a subspace of } \mathcal{G}\}.$$

The following lemma collects some useful properties of linear closures.

Lemma A.7.

1. *For every subspace* E *of* $\mathcal{G}$, $\mathcal{C}(E) = E$.

[4] In [2] this notion is just called closure. In this report, to distinguish with the notion of bi-orthogonal closures, it's called linear closure.

2. $\mathcal{C}(A)$ *is a subspace, for each* $A \subseteq G$*.*

3. $\mathcal{C}(\cdot)$ *is a closure operator on* G*.*

Proof. *1* is obvious from the definition. *2* follows from the definition together with that any arbitrary intersection of subspaces of $\mathcal{G}$ is still a subspace (Proposition 2.3.3 in [2]). *3* is implied by Corollary 3.3.8 in [2]. $\square$

The following proposition suggests a recursive characterization of linear closures.

Proposition A.8. *For* $A \subseteq G$*, define a sequence* $\{A_i\}_{i\in\mathbb{N}}$ *of subsets of* G *as follows:*

- $A_0 = A$;
- $A_{n+1} = \bigcup\{a \star b \mid a, b \in A_n\}$.

Then $\mathcal{C}(A) = \bigcup_{i\in\mathbb{N}} A_i$*.*

Proof. First I prove by induction that $A_i \subseteq \mathcal{C}(A)$, for every $i \in \mathbb{N}$.

Base Step: $i = 0$. By the definition of linear closures, $A_0 = A \subseteq \mathcal{C}(A)$.

Induction Step: $i = n + 1$. Let $c \in A_{n+1}$ be arbitrary. By definition of A_{n+1}, there are $a, b \in A_n$ such that $c \in a \star b$. By Induction Hypothesis $a, b \in A_n \subseteq \mathcal{C}(A)$, so $c \in a \star b \subseteq \mathcal{C}(A)$ since $\mathcal{C}(A)$ is a subspace.

This finishes the proof by induction. Therefore, $\bigcup_{i\in\mathbb{N}} A_i \subseteq \mathcal{C}(A)$.

Second I prove that $\bigcup_{i\in\mathbb{N}} A_i$ is a subspace including A, and thus $\mathcal{C}(A) \subseteq \bigcup_{i\in\mathbb{N}} A_i$. By definition $A = A_0 \subseteq \bigcup_{i\in\mathbb{N}} A_i$. Now let $a, b \in \bigcup_{i\in\mathbb{N}} A_i$ be arbitrary. Then there are $n, n' \in \mathbb{N}$ such that $a \in A_n$ and $b \in A_{n'}$. Notice that by definition $A_i \subseteq A_{i+1}$, for every $i \in \mathbb{N}$. Hence $a, b \in A_m$, where $m = max\{n, n'\}$. Therefore, $a \star b \subseteq A_{m+1} \subseteq \bigcup_{i\in\mathbb{N}} A_i$. As a result, $\bigcup_{i\in\mathbb{N}} A_i$ is a subspace. $\square$

The following is a very important and useful result in projective geometry called *the projective law.*

Theorem A.9. *For any non-empty sets* $A, B \subseteq G$*,*

$$\mathcal{C}(A \cup B) = \bigcup\{a \star b \mid a \in \mathcal{C}(A),\ b \in \mathcal{C}(B)\}.$$

In particular, if E *is a non-empty subspace of* $\mathcal{G}$ *and* $a \in G$*, then*

$$\mathcal{C}(\{a\} \cup E) = \bigcup\{a \star b \mid b \in E\}.$$

Proof. Please refer to the proof of Corollary 2.4.5 in [2]. $\square$

Corollary A.10. *For any $a \in G$ and $A \subseteq G$, $\mathcal{C}(\{a\} \cup \mathcal{C}(A)) = \mathcal{C}(\{a\} \cup A)$.*

Proof. It's easy to see from the definition that $\{a\}$ is a subspace, and thus $\mathcal{C}(\{a\}) = \{a\}$. Using the projective law,

$$\begin{aligned}
&\mathcal{C}(\{a\} \cup A) \\
&= \bigcup \{c \star d \mid c \in \mathcal{C}(\{a\}),\ d \in \mathcal{C}(A)\} \\
&= \bigcup \{c \star d \mid c \in \{a\},\ d \in \mathcal{C}(A)\} \\
&= \bigcup \{a \star d \mid d \in \mathcal{C}(A)\} \\
&= \mathcal{C}(\{a\} \cup \mathcal{C}(A))
\end{aligned}$$

□

Based on the notion of linear closures, one can define the notions of independent sets, generating sets, bases and finite dimensionality.

Definition A.11.

- $A \subseteq G$ is *independent*, if $a \notin \mathcal{C}(A \setminus \{a\})$ for every $a \in A$.
- $A \subseteq G$ *generates a subspace*, or *is a generating set of a subspace*, E of $\mathcal{G}$, if $E = \mathcal{C}(A)$.
- A *basis* of a subspace E of $\mathcal{G}$ is a set $A \subseteq E$ which is independent and generates E.
- $\mathcal{G} = (G, \star)$ is called *finite-dimensional*, if G, as a subspace, has a finite generating set.

Next I cite some important theorems for defining the notion of dimension.

Theorem A.12. *Let E be a subspace of $\mathcal{G}$ and $A \subseteq D \subseteq E$ be such that A is independent and D generates E. Then there exists a basis B of E with $A \subseteq B \subseteq D$. In particular, every subspace of $\mathcal{G}$ has a basis.*

Proof. By Proposition 3.1.13 in [2] $\mathcal{G}$ is a geometry and thus a matroid. Then the conclusion is implied by Theorem 4.1.9 in [2]. □

Theorem A.13. *Let E be a subspace of $\mathcal{G}$, $A \uplus B_1$ and $A \uplus B_2$ be two bases of E. Then B_1 and B_2 are of the same cardinality. In particular, any two bases of a subspace are of the same cardinality.*

Proof. This is similar to the proof of the above theorem, but Theorem 4.2.2 in [2] is applied instead of Theorem 4.1.9. □

Now it's time to define the notion of rank.

Definition A.14. The *rank* of a subspace E of $\mathcal{G}$, denoted by $r(E)$, is the cardinality of one (and thus any) basis of E.

I cite an important property of ranks.

Theorem A.15. *For two subspaces E and F of $\mathcal{G}$, if $E \subseteq F$, then $r(E) \leq r(F)$.*

Proof. Please refer to the proof of Proposition 4.3.1 in [2]. □

Intuitively, in projective geometries, the rank of a subspace is the smallest number of independent points needed to generate the subspace. Unfortunately, this natural notion doesn't match the ordinary conception of dimension, which intuitively is the (cardinal) number of degrees of freedom. This mismatch is the reason why the new term 'rank' is needed. For example, a line in a projective geometry is generated by two distinct points, and thus is of rank 2; but on a line one only has one degree of freedom, and thus normally a line is said to be of dimension 1. In general, if a subspace is of finite rank $n \geq 1$, it is said to be of dimension $n - 1$.

A.3 Projective Geometries with Additional Structures

In this subsection, I will discuss three kinds of projective geometries with additional structures: projective geometries with a polarity, orthogeomtries and Hibertian geometries.

I start from projective geometries with a polarity.

Definition A.16. A *projective geometry with a polarity* is a tuple $\mathcal{G} = (G, \star, p)$, where $(G, \star)$ is a projective geometry and p is a function from G to the set of all hyperplanes of $(G, \star)$, called a *polarity* on $(G, \star)$, such that $a \in p(b) \Leftrightarrow b \in p(a)$, for any $a, b \in G$.

Moreover, a polarity p is *pure*, if $a \notin p(a)$, for every $a \in G$.

Sometimes a polarity is required to be surjective (e.g. [7]), but I don't make this requirement in this report.

The following proposition collects some useful results about projective geometry with a polarity.

Proposition A.17. *Let $\mathcal{G} = (G, \star, p)$ be a projective geometry with a polarity.*

1. *For any $a, b, c \in G$, $c \in a \star b \Rightarrow p(a) \cap p(b) \subseteq p(c)$.*

2. p is injective.

3. For any $a, b, c \in G$, $c \in a \star b \Leftarrow p(a) \cap p(b) \subseteq p(c)$.

Proof. [5] For *1*, assume that $c \in a \star b$. Let $x \in p(a) \cap p(b)$ be arbitrary. For $x \in p(a)$, $a \in p(x)$ by the definition of a polarity. Similarly one can deduce $b \in p(x)$ from $x \in p(b)$. Since $p(x)$ is a hyperplane, $a \star b \subseteq p(x)$. Hence $c \in a \star b \subseteq p(x)$. By the definition of a polarity, $x \in p(c)$. Therefore, $p(a) \cap p(b) \subseteq p(c)$.

For *2*, suppose (towards a contradiction) that there are $a, a' \in G$ such that $a \neq a'$ but $p(a) = p(a')$. Since $p(a)$ is a hyperplane, one can find a $b \in G$ such that $b \notin p(a)$. Since $a \neq a'$, by Theorem A.5 $(a \star a') \cap p(b)$ is non-empty. Take $x \in (a \star a') \cap p(b)$. Since $x \in a \star a'$, by *1* and supposition we have $p(a) = p(a) \cap p(a') \subseteq p(x)$. Since both $p(a)$ and $p(x)$ are hyperplanes, by definition $p(a) = p(x)$. Now it follows from $x \in p(b)$ that $b \in p(x) = p(a)$, contradicting that $b \notin p(a)$. Therefore, p is injective.

For *3*, assume that $p(a) \cap p(b) \subseteq p(c)$. As a start, notice that there're two easy cases: when $c \in \{a, b\}$ the conclusion follows easily from the definition; and when $a = b$ the conclusion follows easily with the help of *2*. In the following, I focus on the case when a, b, c are pairwise distinct. Then by *2* $p(a)$ and $p(b)$ are different hyperplanes of $(G, \star)$. Hence one can find an $x \in G$ such that $x \in p(b)$ and $x \notin p(a)$.

Observe that $(a \star c) \cap p(x)$ is a singleton. Since $p(x)$ is a hyperplane and $a \neq c$, by Theorem A.5 $(a \star c) \cap p(x)$ is either $a \star c$ or a singleton. Since $x \notin p(a)$, $a \notin p(x)$ and thus $(a \star c) \cap p(x) \neq a \star c$. Therefore, $(a \star c) \cap p(x)$ is a singleton. Denote by y the unique element in it.

For this y, observe that $p(b) \subseteq p(y)$. Let $r \in p(b)$ be arbitrary. If $r = x$, then $r = x \in p(y)$ since $y \in p(x)$. If $r \neq x$, one can show from $x \notin p(a)$ that $(x \star r) \cap p(a)$ is a singleton. Denote by u the unique element in it. Since $x, r \in p(b)$, $u \in x \star r \subseteq p(b)$. Hence $u \in p(a) \cap p(b) \subseteq p(c)$ by assumption. It follows that $u \in p(a) \cap p(c)$. Since $y \in a \star c$, $u \in p(a) \cap p(c) \subseteq p(y)$ by *1*. Since $x \notin p(a)$ and $u \in p(a)$, $u \neq x$. Then $r \in x \star u$ follows from $u \in x \star r$ and (P4). Since $x, u \in p(y)$, $r \in x \star u \subseteq p(y)$. For r is arbitrary, $p(b) \subseteq p(y)$.

Now it's time to show that $c \in a \star b$. Since both $p(b)$ and $p(y)$ are hyperplanes, $p(b) = p(y)$. By *2* $b = y$. Then $b \in a \star c$ follows from $y \in a \star c$. Since $a \neq b$, $c \in a \star b$ by (P4). $\square$

[5] The proof of *2* is inspired by that of Proposition 11.3.3 in [2], and that of *3* by that of 14.2.5 in the same book. One could arrive at the same conclusions by introducing some terminologies from this book and applying these propositions. However, since they're not very long, direct proofs may be more helpful in developing intuitions.

Remark A.18. The above proposition shows that a polarity p on a projective geometry $\mathcal{G} = (G, \star)$ is an injection with the property that $c \in a \star b \Leftrightarrow p(a) \cap p(b) \subseteq p(c)$, for any $a, b, c \in G$. One can define the *dual geometry* $\mathcal{G}^*$ of $\mathcal{G}$ to be the tuple $(G^*, *)$, where G^* is the set of all hyperplanes of $\mathcal{G}$ and $H \in E * F \Leftrightarrow E \cap F \subseteq H$ for any $E, F, H \in G^*$, which can be proved to be a projective geometry (Proposition 11.2.3 of [2]). Then the above proposition means that p is an embedding of $\mathcal{G}$ into its dual $\mathcal{G}^*$.

Another kind of special projective geometries, called orthogeometries, is closely related to projective geometries with a polarity.

Definition A.19. An *orthogeometry* is a tuple $\mathcal{G} = (G, \star, \perp)$, where $(G, \star)$ is a projective geometry and $\perp \subseteq G \times G$, called *the orthogonality relation*, satisfies the following properties:

(O1) $a \perp b$ implies that $b \perp a$, for any $a, b \in G$;

(O2) if $a \perp q$, $b \perp q$ and $c \in a \star b$, then $c \perp q$, for any $a, b, c, q \in G$;

(O3) if $a, b, c \in G$ and $b \neq c$, then there is a $q \in b \star c$ such that $q \perp a$;

(O4) for every $a \in G$, there is a $b \in G$ such that $a \not\perp b$.

The orthogonality relation is *pure*, if $a \not\perp a$ for every $a \in G$. An orthogeometry is *pure*, if the orthogonality relation in it is pure.

The close relation between projective geometries with a polarity and orthogeometries is expressed in the following theorem:

Theorem A.20. *For every projective geometry $\mathcal{G} = (G, \star)$, there is a canonical bijection from the set of all polarities on $\mathcal{G}$ to the set of all orthogonality relations on $\mathcal{G}$. To be precise, a polarity $p : G \to \wp(G)$ on $\mathcal{G}$ is mapped by this bijection to the orthogonality relation $\perp \subseteq G \times G$ satisfying that $a \perp b \Leftrightarrow a \in p(b)$, for any $a, b \in G$.*

Moreover, a polarity p on $\mathcal{G}$ is pure, if and only if its image under this bijection is a pure orthogonality relation.

Proof. This is implied by Proposition 14.1.3 in [2]. $\square$

Remark A.21. On a projective geometry $\mathcal{G} = (G, \star)$, if a polarity p is mapped by the canonical bijection to the orthogonality relation $\perp$, $(G, \star, p)$ will be called *the projective geometry with a polarity corresponding to* $(G, \star, \perp)$, and $(G, \star, \perp)$ *the orthogeometry corresponding to* $(G, \star, p)$.

Finally I discuss Hilbertian geometries.

Definition A.22. A *Hilbertian geometry* is an orthogeometry $\mathcal{G} = (G, \star, \perp)$ satisfying the following condition:

$$\text{for every } E \subseteq G \text{ satisfying } (E^\perp)^\perp = E,\ \mathcal{C}(E \cup E^\perp) = G,$$

where $E^\perp = \{a \in G \mid a \perp b, \text{ for every } b \in E\}$.

I will take a closer look of the orthogonality relations in Hilbertian geometries.

Lemma A.23. *Every Hilbertian geometry is a pure orthogeometry.*

Proof. Let $\mathcal{G} = (G, \star, \perp)$ be a Hilbertian geometry. By definition it suffices to show that $\perp$ is pure. Notice that by Theorem A.20 $\{\cdot\}^\perp : G \to \wp(G)$ is a polarity on $(G, \star)$.

As a preparation, I show that $(\{a\}^\perp)^\perp = \{a\}$, for every $a \in G$. First I show that $\{a\} \subseteq (\{a\}^\perp)^\perp$. For every $b \in \{a\}^\perp$, $b \perp a$ by definition, so $a \perp b$ by (O1). Hence $\{a\} \subseteq (\{a\}^\perp)^\perp$. Second I show that $(\{a\}^\perp)^\perp \subseteq \{a\}$. Suppose (towards a contradiction) that $b \in (\{a\}^\perp)^\perp$ and $a \neq b$. For $\{\cdot\}^\perp$ is a polarity on $(G, \star)$, by *2* of Proposition A.17 $\{a\}^\perp$ and $\{b\}^\perp$ are different hyperplanes of $(G, \star)$. Then there is a $c \in \{a\}^\perp$ such that $c \notin \{b\}^\perp$, i.e. $c \in \{a\}^\perp$ but $b \not\perp c$, contradicting that $b \in (\{a\}^\perp)^\perp$. Therefore, $(\{a\}^\perp)^\perp \subseteq \{a\}$, and thus $(\{a\}^\perp)^\perp = \{a\}$.

Now suppose (towards a contradiction) that $\perp$ is not pure, i.e. $a \perp a$ for some $a \in G$. Since $(\{a\}^\perp)^\perp = \{a\}$, by definition of Hilbertian geometries $G = \mathcal{C}(\{a\} \cup \{a\}^\perp)$. Since $a \in \{a\}^\perp$, $\{a\} \cup \{a\}^\perp = \{a\}^\perp$. Hence $G = \mathcal{C}(\{a\}^\perp) = \{a\}^\perp$, contradicting that $\{a\}^\perp$ is a hyperplane. Therefore, $\perp$ is pure. □

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