Definability in Quantum Kripke Frames

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Abstract

I characterize the first-order definable, bi-orthogonally closed subsets of a quasi-quantum Kripke frame satisfying a reasonable assumption. The techniques are generalization of those in Goldblatt's paper published in 1984. Combining these techniques with Goldblatt's idea, I prove that quantum Kripke frames are not first-order definable in the class of quasi-quantum Kripke frames.

1 Preliminaries

In [10], two kinds of Kripke frames, quasi-quantum Kripke frames and quantum Kripke frames, are introduced and shown to have nice connections with projective geometry and foundations of quantum physics.

I review the basic definitions here. For many useful notions and results about these structures, please refer to [10].

Definition 1.1. A *Kripke frame* is a tuple $\mathfrak{F} = (\Sigma, \to)$ in which Σ is a non-empty set and $\to \subseteq \Sigma \times \Sigma$.

Definition 1.2. In a Kripke frame $\mathfrak{F} = (\Sigma, \to)$:

- If $(s, t) \in \to$, call that *s and t are non-orthogonal* and write $s \to t$.
- If $(s, t) \notin \to$, call that *s and t are orthogonal* and write $s \not\to t$.
- $P \subseteq \Sigma$ is *orthogonal*, if $s \not\to t$ for any $s, t \in P$ satisfying $s \neq t$.
- The *orthocomplement of $P \subseteq \Sigma$ (with respect to $\to$)*, denoted by $\sim P$, is the set $\{s \in \Sigma \mid s \not\to t$, for every $t \in P\}$.
- $P \subseteq \Sigma$ is a *subspace*, if $\sim\sim\{s, t\} \subseteq P$ for any $s, t \in P$.
- $P \subseteq \Sigma$ is *bi-orthogonally closed*, if $\sim\sim P = P$.
- $s, t \in \Sigma$ are *indistinguishable with respect to $P \subseteq \Sigma$*, denoted by $s \approx_P t$, if $s \to x \Leftrightarrow t \to x$ for every $x \in P$.
- $t \in \Sigma$ is *an approximation of $s \in \Sigma$ in P*, if $t \in P$ and $s \approx_P t$.

Definition 1.3. The following is a list of conditions which a Kripke frame $\mathfrak{F} = (\Sigma, \to)$ may satisfy.

- **Reflexivity:** $s \to s$, for every $s \in \Sigma$.

- **Symmetry:** $s \to t$ implies that $t \to s$, for any $s, t \in \Sigma$.
- **Separation:** $s \neq t$ implies that there is a $w \in \Sigma$ such that $w \to s$ and $w \not\to t$.
- **Existence of Approximation for Lines (AL):**

  For any $s, t \in \Sigma$, if $w \in \Sigma \setminus {\sim}\{s, t\}$, then there is a $w' \in \Sigma$ which is an approximation of w in ${\sim\sim}\{s, t\}$, i.e. $w' \in {\sim\sim}\{s, t\}$ and $w \approx_{\sim\sim\{s,t\}} w'$.
- **Existence of Approximation for Hyperplanes (AH):**

  For each $s \in \Sigma$, if $w \in \Sigma \setminus {\sim\sim}\{s\}$, then there is a $w' \in \Sigma$ which is an approximation of w in ${\sim}\{s\}$, i.e. $w' \in {\sim}\{s\}$ and $w \approx_{\sim\{s\}} w'$.
- **Existence of Approximation (A):**

  For each $P \subseteq \Sigma$ with ${\sim\sim}P = P$, if $s \in \Sigma \setminus {\sim}P$, then there is an $s' \in \Sigma$ which is an approximation of s in P, i.e. $s' \in P$ and $s \approx_P s'$.
- **Superposition:** for any $s, t \in \Sigma$, there's a $w \in \Sigma$ such that $w \to s$ and $w \to t$.

Definition 1.4. A *quasi-quantum Kripke frame* is a Kripke frame satisfying Reflexivity, Symmetry, Separation, Property AL, Property AH and Superposition.

A *quantum Kripke frame* is a Kripke frame satisfying Reflexivity, Symmetry, Separation, Property A and Superposition.

Remark 1.5. According to Proposition 2.4 in [10], Property A implies Property AL and Property AH with the help of Reflexivity, Symmetry and Separation. Hence Property AL and Property AH hold in every quantum Kripke frame. In other words, every quantum Kripke frame is a quasi-quantum Kripke frame.

It's natural to talk about Kripke frames in a formal language with one binary relation symbol. In this report, I hope to investigate the first-order definable (with parameters), bi-orthogonally closed subsets of quasi-quantum Kripke frames.

Let me start from the special case of finite-dimensional quasi-quantum Kripke frames.

Definition 1.6. A Kripke frame $\mathfrak{F} = (\Sigma, \to)$ is *finite-dimensional*, if there are $n \in \mathbb{N}$ and $s_1, ..., s_n \in \Sigma$ such that $\Sigma = {\sim\sim}\{s_1, ..., s_n\}$; otherwise, it is *infinite-dimensional.*

Proposition 1.7. *In a finite-dimensional quasi-quantum Kripke frame $\mathfrak{F} = (\Sigma, \to)$, for every $P \subseteq \Sigma$, the following are equivalent:*

(i) P is first-order definable and bi-orthogonally closed;

(ii) there are $n \in \mathbb{N}$ and $s_1, ..., s_n \in \Sigma$ such that $P = {\sim\sim}\{s_1, ..., s_n\}$.

Proof. **From (i) to (ii):** Assume that P is bi-orthogonally closed. Since $\mathfrak{F}$ is a quasi-quantum Kripke frame, it's a geometric frame in the sense of Definition 1.3 in [10]. Then, according to Lemma 4.19 in the same report, there are $n \in \mathbb{N}$ and $s_1, ..., s_n \in \Sigma$ such that $P = {\sim\sim}\{s_1, ..., s_n\}$.

From (ii) to (i): Assume that there are $n \in \mathbb{N}$ and $s_1, ..., s_n \in \Sigma$ such that $P = {\sim\sim}\{s_1, ..., s_n\}$. By Proposition 2.1 in [10] $P = {\sim\sim}\{s_1, ..., s_n\}$ is bi-orthogonally closed. Moreover, P is definable by the following first-order formula, where R is interpreted by $\to$ and $x_1, ..., x_n$ by $s_1, ..., s_n$, respectively:

$$\forall y(xRy \to yRx_1 \vee ... \vee yRx_n)$$

Therefore, P is first-order definable and bi-orthogonally closed. □

One may wonder whether this also holds in the infinite-dimensional case. The bad news is that the answer is no, but the good news is that there is something similar. As it turns out, for the infinite-dimensional case, two things are needed. First, (ii) in the above proposition need to be generalized to the notion of finitely presentable subsets in Definition 4.15 in [10], recalled in the following:

Definition 1.8. Let $\mathfrak{F} = (\Sigma, \to)$ be a Kripke frame.

$P \subseteq \Sigma$ is *finite-dimensional*, if there are $n \in \mathbb{N}$ and $s_1, ..., s_n \in \Sigma$ such that $P = \sim\sim\{s_1, ..., s_n\}$; otherwise, it is *infinite-dimensional*.

$P \subseteq \Sigma$ is *finite-codimensional*, if there are $n \in \mathbb{N}$ and $s_1, ..., s_n \in \Sigma$ such that $P = \sim\{s_1, ..., s_n\}$; otherwise, it is *infinite-codimensional*.

$P \subseteq \Sigma$ is *finitely presentable*, if it is finite-dimensional or finite-codimensional.

Remark 1.9. I use the convention that, when $n = 0$, $\{s_1, ..., s_n\}$ denotes the empty set. By Proposition 2.1 in [10] $\sim\emptyset = \Sigma$ and $\sim\sim\emptyset = \emptyset$. Therefore, both $\emptyset$ and Σ are finitely presentable.

Moreover, a Kripke frame $\mathfrak{F} = (\Sigma, \to)$ is finite-dimensional in the sense of Definition 1.6, if and only if Σ is finite-dimensional in the sense of Definition 1.8.

Second, an additional assumption is needed. The remaining part of this report will be devoted to the details of generalizing the above proposition to the infinite-dimensional case and the applications of this generalization.

The organization is as follows: Section 2 is devoted to a study of properties of infinite-dimensional or infinite-codimensional subsets. Automorphisms, which play an important role in determining first-order definable sets, will be studied in Section 3 with the help of the additional assumption. The main theorem will be stated and proved in Section 4. In Section 5, as an application of the techniques developed in the previous sections, quantum Kripke frames will be shown not to be first-order definable following the idea of Goldblatt's argument in [4].

2 Infinite-Dimensional or -Codimensional Sets

In this section, I study properties of infinite-dimensional or infinite-codimensional subsets in quasi-quantum Kripke frames. I will prove three technical lemmas.

Lemma 2.1. *Let $\mathfrak{F} = (\Sigma, \to)$ be a quasi-quantum Kripke frame and $P \subseteq \Sigma$ be an infinite-dimensional subspace.*

1. *If Q is a finite orthogonal subset of P, then there is an $s \in P$ such that $Q \cup \{s\}$ is also an orthogonal subset of P.*
2. *For every $n \in \mathbb{N}$, there is an orthogonal subset of P of cardinality n.*
3. *P has an infinite orthogonal subset.*

Proof. **For 1:** Let Q be an arbitrary finite orthogonal subset of P. Then $\sim\sim Q = \mathcal{C}(Q) \subseteq P$ by Proposition 3.23 in [10]. Since P is infinite-dimensional, the above inclusion is proper, so there is a $w \in P$ such that $w \notin \sim\sim Q$. If $w \in \sim Q$, then define s to be w, and thus it's obvious that $Q \cup \{s\}$ is an orthogonal subset of P. If $w \notin \sim Q$, since $\mathfrak{F}$ is a geometric frame, by Theorem 4.7 and Corollary 4.16 in [10] there are $w_{\parallel} \in \sim\sim Q$ and $w_{\perp} \in \sim\sim\sim Q = \sim Q$ such that $w \in \sim\sim\{w_{\parallel}, w_{\perp}\}$. Since $w \notin \sim\sim Q$ and $w_{\parallel} \in \sim\sim Q$,

$w \neq w_{\parallel}$, and thus $w_{\perp} \in {\sim}{\sim}\{w, w_{\parallel}\}$ by Lemma 3.5 in [10]. Since $w, w_{\parallel} \in P$ and P is a subspace, ${\sim}{\sim}\{w, w_{\parallel}\} \subseteq P$, so $w_{\perp} \in P$. Define s to be $w_{\perp}$. From $w_{\perp} \in P$ and $w_{\perp} \in {\sim}Q$, it's easy to conclude that $Q \cup \{s\}$ is an orthogonal subset of P.

For *2*: Observe that $\emptyset$ is an orthogonal subset of P by definition. Using *1* an easy induction shows that one can build a sequence $\{Q_n\}_{n\in\mathbb{N}}$ of orthogonal subsets of P such that Q_n is of cardinality n for each $n \in \mathbb{N}$ and $Q_i \subseteq Q_j$ for any $i, j \in \mathbb{N}$ with $i \leq j$.

For *3*: Take $\bigcup_{n\in\mathbb{N}} Q_n$ of the above chain $\{Q_n\}_{n\in\mathbb{N}}$ of orthogonal subsets of P. It's easy to verify by definition that $\bigcup_{n\in\mathbb{N}} Q_n$ is an infinite orthogonal subset of P. □

Lemma 2.2. *Let $\mathfrak{F} = (\Sigma, \to)$ be a quasi-quantum Kripke frame and $P \subseteq \Sigma$ be bi-orthogonally closed and infinite-codimensional. For any $n \in \mathbb{N}$ and $s_1, ..., s_n \in \Sigma$, there is a $v \in {\sim}\{s_1, ..., s_n\}$ such that $v \notin P$.*

Proof. Suppose (towards a contradiction) that ${\sim}\{s_1, ..., s_n\} \subseteq P$. It follows that ${\sim}P \subseteq {\sim}{\sim}\{s_1, ..., s_n\}$. According to Proposition 3.23 in [10], ${\sim}P \subseteq \mathcal{C}(\{s_1, ..., s_n\})$. It follows from Theorem A.15 in [10] that there are $m \leq n$ and $w_1, ..., w_m \in \Sigma$ such that ${\sim}P = \mathcal{C}(\{w_1, ..., w_m\})$. By Proposition 3.23 in [10] again ${\sim}P = {\sim}{\sim}\{w_1, ..., w_m\}$. Then $P = {\sim}{\sim}P = {\sim}{\sim}{\sim}\{w_1, ..., w_m\} = {\sim}\{w_1, ..., w_m\}$, contradicting that P is infinite-codimensional. Therefore, ${\sim}\{s_1, ..., s_n\} \nsubseteq P$.

It follows that there is a $v \in {\sim}\{s_1, ..., s_n\}$ such that $v \notin P$. □

Lemma 2.3. *Let $\mathfrak{F} = (\Sigma, \to)$ be a quasi-quantum Kripke frame, $P \subseteq \Sigma$ be an infinite-dimensional subspace, $n \in \mathbb{N}$ and $s_1, ..., s_n \in \Sigma$. If $v \in {\sim}\{s_1, ..., s_n\}$ and $v \notin P$, then there is a $u \in {\sim}\{s_1, ..., s_n\}$ such that $u \in P$ and $u \not\to v$.*

Proof. Take an orthogonal subset $\{t_1, ..., t_{n+2}\}$ of P, whose existence is granted by Lemma 2.1. Without loss of generality, assume that there is a $k \in \{0, 1, ..., n\}$ such that $s_k \in {\sim}\{t_1, ..., t_{n+2}\}$ if and only if $i > k$. Then write $s'_1, ..., s'_k$ for the approximations of $s_1, ..., s_k$ in ${\sim}{\sim}\{t_1, ..., t_{n+2}\}$, respectively, given by Corollary 4.16 in [10]. Now two cases need to be considered.

- **Case 1:** $v \notin {\sim}\{t_1, ..., t_{n+2}\}$.

  Then v has an approximation v' in ${\sim}{\sim}\{t_1, ..., t_{n+2}\}$ given by Corollary 4.16 in [10]. By definition $\{s'_1, ..., s'_k, v'\} \subseteq {\sim}{\sim}\{t_1, ..., t_{n+2}\}$, and thus ${\sim}{\sim}\{s'_1, ..., s'_k, v'\} \subseteq {\sim}{\sim}\{t_1, ..., t_{n+2}\}$. By Proposition 3.23 in [10] $\mathcal{C}(\{s'_1, ..., s'_k, v'\}) \subseteq \mathcal{C}(\{t_1, ..., t_{n+2}\})$. Since $\{t_1, ..., t_{n+2}\}$ is an orthogonal subset larger than $\{s'_1, ..., s'_k, v'\}$, the above inclusion must be proper, according to Theorem A.15 in [10]. Hence there is an $x \in \mathcal{C}(\{t_1, ..., t_{n+2}\})$ such that $x \notin \mathcal{C}(\{s'_1, ..., s'_k, v'\})$. By Proposition 3.23 in [10] $x \in {\sim}{\sim}\{t_1, ..., t_{n+2}\}$ and $x \notin {\sim}{\sim}\{s'_1, ..., s'_k, v'\}$.

  If $x \in {\sim}\{s'_1, ..., s'_k, v'\}$, define u to be x.

  If $x \notin {\sim}\{s'_1, ..., s'_n, v'\}$, then by Theorem 4.7 and Corollary 4.16 in [10] there're $x' \in {\sim}{\sim}\{s'_1, ..., s'_k, v'\}$ and $x_{\perp} \in {\sim}{\sim}{\sim}\{s'_1, ..., s'_k, v'\} = {\sim}\{s'_1, ..., s'_k, v'\}$ such that $x \in {\sim}{\sim}\{x', x_{\perp}\}$. Notice that $x \in {\sim}{\sim}\{t_1, ..., t_{n+2}\}$ and $x' \in {\sim}{\sim}\{s'_1, ..., s'_k, v'\} \subseteq {\sim}{\sim}\{t_1, ..., t_{n+2}\}$. Hence $x, x' \in {\sim}{\sim}\{t_1, ..., t_{n+2}\}$, and ${\sim}{\sim}\{x, x'\} \subseteq {\sim}{\sim}\{t_1, ..., t_{n+2}\}$. Since $x \neq x'$, $x_{\perp} \in {\sim}{\sim}\{x, x'\}$ follows from $x \in {\sim}{\sim}\{x', x_{\perp}\}$ and Corollary 3.5 in [10]. Hence $x_{\perp} \in {\sim}{\sim}\{t_1, ..., t_{n+2}\}$. Define u to be $x_{\perp}$.

  In both cases, the chosen u is such that $u \in {\sim}{\sim}\{t_1, ..., t_{n+2}\}$ and $u \in {\sim}\{s'_1, ..., s'_k, v'\}$. Since $\{t_1, ..., t_{n+2}\} \subseteq P$, $u \in {\sim}{\sim}\{t_1, ..., t_{n+2}\} = \mathcal{C}(\{t_1, ..., t_{n+2}\}) \subseteq P$. Moreover, by definition of v' and s'_i for $i \in \{1, ..., k\}$, $u \not\to v$ and $u \not\to s_i$ for $i \in \{1, ..., k\}$. For every $k < i \leq n$, since $s_i \in \sim\{t_1, ..., t_{n+2}\}$ and $u \in \sim\sim\{t_1, ..., t_{n+2}\}$, $u \not\to s_i$. As a result, $u \in P$, $u \in \sim\{s_1, ..., s_n\}$ and $u \not\to v$.

- **Case 2:** $v \in \sim\{t_1, ..., t_{n+2}\}$.

  In this case, the argument is simpler, for I don't need to care about v. Apply an argument similar to the above to the set $\{s'_1, ..., s'_k\}$ instead of $\{s'_1, ..., s'_k, v'\}$. Then one gets a u such that $u \in \sim\sim\{t_1, ..., t_{n+2}\} \subseteq P$ and $u \in \sim\{s'_1, ..., s'_k\}$. Since $v \in \sim\{t_1, ..., t_{n+2}\}$ and $u \in \sim\sim\{t_1, ..., t_{n+2}\}$, $u \not\to v$. Moreover, one can prove that $u \in \sim\{s_1, ..., s_n\}$ by the same argument as the above.

In both cases, I find a $u \in \sim\{s_1, ..., s_n\}$ such that $u \in P$ and $u \not\to v$. □

3 Automorphisms of a Quasi-Quantum Kripke Frame

In this section, I need to introduce the following condition on a quasi-quantum Kripke frame $\mathfrak{F} = (\Sigma, \to)$, the idea of which is from Holland in [6]:

> **Assumption:** For any $s, t \in \Sigma$ with $s \not\to t$, there is a $w \in \sim\sim\{s, t\}$ such that every harmonic conjugate of w with respect to s and t is orthogonal to w.

The notion of harmonic conjugates is as follows:

Definition 3.1. In a quasi-quantum Kripke frame $\mathfrak{F} = (\Sigma, \to)$, for any $s, t, u \in \Sigma$ such that $s \neq t$ and $u \in \sim\sim\{s, t\}$, $v \in \Sigma$ is *a harmonic conjugate of u with respect to s and t*, if there are $a, b, c, d \in \Sigma$ satisfying the following:

1. $w_1 \notin \sim\sim\{w_2, w_3\}$, for any $w_1, w_2, w_3 \in \{a, b, c, d\}$ such that $w_1 \neq w_2 \neq w_3 \neq w_1$;
2. $s \in \sim\sim\{a, b\} \cap \sim\sim\{c, d\}$;
3. $t \in \sim\sim\{a, c\} \cap \sim\sim\{b, d\}$;
4. $u \in \sim\sim\{a, d\} \cap \sim\sim\{s, t\}$;
5. $v \in \sim\sim\{b, c\} \cap \sim\sim\{s, t\}$.

Remark 3.2. Please observe that in the formal language with one binary relation symbol there is a *first-order* formula $\varphi(x_1, x_2, y_1, y_2)$ with four free variables saying that (the denotation of) x_1 is a harmonic conjugate of x_2 with respect to y_1 and y_2. Hence the Assumption is first-order in this language.

Moreover, using Lemma C.2 it can be shown that, in a quasi-quantum Kripke frame $\mathfrak{F} = (\Sigma, \to)$, if there are $s, t, u \in \Sigma$ such that $s \neq t$ and $u \in \sim\sim\{s, t\}$ has a harmonic conjugate with respect to s and t, then $\mathfrak{F}$ has an orthogonal set of cardinality 3.

In fact, this notion is employed from projective geometry. For a brief review, please refer to Appendix C.

Next I reveal the analytic content of the Assumption.

Lemma 3.3. *Let V be a vector space of dimension at least 3 over some division ring $\mathcal{K}$ equipped with an anisotropic Hermitian form Φ, whose accompanying involution on $\mathcal{K}$ is $(\cdot)^*$. For any $\mathbf{u}, \mathbf{v} \in V \setminus \{\mathbf{0}\}$, define that $\langle \mathbf{u} \rangle \to_V \langle \mathbf{v} \rangle$, if $\Phi(\mathbf{u}, \mathbf{v}) \neq 0$. Then the following are equivalent:*

(i) $(\Sigma(V), \to_V)$ *is a quasi-quantum Kripke frame that satisfies the Assumption;*

(ii) for any $\mathbf{s}, \mathbf{t} \in V \setminus \{\mathbf{0}\}$*,* $\Phi(\mathbf{s}, \mathbf{t}) = 0$ *implies that there is an* x *in* $\mathcal{K}$ *such that* $\Phi(\mathbf{s}, \mathbf{s}) = \Phi(x\mathbf{t}, x\mathbf{t})$*.*

(iii) for any $\mathbf{s}, \mathbf{t} \in V \setminus \{\mathbf{0}\}$*, there is an* x *in* $\mathcal{K}$ *such that* $\Phi(\mathbf{s}, \mathbf{s}) = \Phi(x\mathbf{t}, x\mathbf{t})$*.*

Proof. **From (i) to (ii):** Let $\mathbf{s}, \mathbf{t} \in V \setminus \{\mathbf{0}\}$ be arbitrary such that $\Phi(\mathbf{s}, \mathbf{t}) = 0$. Then $\langle \mathbf{s} \rangle \not\to_V \langle \mathbf{t} \rangle$. By (i) there is a $\mathbf{w} \in V \setminus \{\mathbf{0}\}$ such that $\langle \mathbf{w} \rangle \in {\sim\sim}\{\langle \mathbf{s} \rangle, \langle \mathbf{t} \rangle\}$ and every harmonic conjugate of $\langle \mathbf{w} \rangle$ with respect to $\langle \mathbf{s} \rangle$ and $\langle \mathbf{t} \rangle$ is orthogonal to $\langle \mathbf{w} \rangle$. Since $\langle \mathbf{w} \rangle \in {\sim\sim}\{\langle \mathbf{s} \rangle, \langle \mathbf{t} \rangle\}$, it's not hard to see that there must be y and z in $\mathcal{K}$ such that $\mathbf{w} = y\mathbf{s} + z\mathbf{t}$. Since V is of dimension at least 3, by Lemma C.3 a harmonic conjugate of $\langle \mathbf{w} \rangle$ with respect to $\langle \mathbf{s} \rangle$ and $\langle \mathbf{t} \rangle$ is $\langle y\mathbf{s} - z\mathbf{t} \rangle$. Since it's orthogonal to $\langle \mathbf{w} \rangle$, $\Phi(y\mathbf{s} + z\mathbf{t}, y\mathbf{s} - z\mathbf{t}) = 0$. It follows that $y \cdot \Phi(\mathbf{s}, \mathbf{s}) \cdot y^* - y \cdot \Phi(\mathbf{s}, \mathbf{t}) \cdot z^* + z \cdot \Phi(\mathbf{t}, \mathbf{s}) \cdot y^* - z \cdot \Phi(\mathbf{t}, \mathbf{t}) \cdot z^* = 0$. Since $\Phi(\mathbf{s}, \mathbf{t}) = 0$, $y \cdot \Phi(\mathbf{s}, \mathbf{s}) \cdot y^* - z \cdot \Phi(\mathbf{t}, \mathbf{t}) \cdot z^* = 0$. Take x to be $y^{-1} \cdot z$. Then

$$\begin{aligned}
&\Phi(x\mathbf{t}, x\mathbf{t}) \\
&= \Phi\left((y^{-1} \cdot z)\mathbf{t}, (y^{-1} \cdot z)\mathbf{t}\right) \\
&= y^{-1} \cdot z \cdot \Phi(\mathbf{t}, \mathbf{t}) \cdot z^* \cdot (y^{-1})^* \\
&= y^{-1} \cdot y \cdot \Phi(\mathbf{s}, \mathbf{s}) \cdot y^* \cdot (y^{-1})^* \\
&= \Phi(\mathbf{s}, \mathbf{s})
\end{aligned}$$

From (ii) to (iii): Let $\mathbf{s}, \mathbf{t} \in V \setminus \{\mathbf{0}\}$ be arbitrary. Since V is of dimension at least 3, it's not hard to find $\mathbf{v} \in V \setminus \{\mathbf{0}\}$ such that $\Phi(\mathbf{s}, \mathbf{v}) = \Phi(\mathbf{v}, \mathbf{t}) = 0$. By (ii) there are y and z in $\mathcal{K}$ such that $\Phi(\mathbf{s}, \mathbf{s}) = \Phi(y\mathbf{v}, y\mathbf{v})$ and $\Phi(\mathbf{v}, \mathbf{v}) = \Phi(z\mathbf{t}, z\mathbf{t})$. Take $x = y \cdot z$. Then

$$\begin{aligned}
&\Phi(x\mathbf{t}, x\mathbf{t}) \\
&= \Phi((y \cdot z)\mathbf{t}, (y \cdot z)\mathbf{t}) \\
&= y \cdot z \cdot \Phi(\mathbf{t}, \mathbf{t}) \cdot z^* \cdot y* \\
&= y \cdot \Phi(z\mathbf{t}, z\mathbf{t}) \cdot y^* \\
&= y \cdot \Phi(\mathbf{v}, \mathbf{v}) \cdot y^* \\
&= \Phi(y\mathbf{v}, y\mathbf{v}) \\
&= \Phi(\mathbf{s}, \mathbf{s})
\end{aligned}$$

From (iii) to (i): By Proposition B.6 $(\Sigma(V), \to)$ is a quasi-quantum Kripke frame. To prove that the Assumption holds, let $s, t \in \Sigma(V)$ such that $s \not\to_V t$. Let $\mathbf{s}, \mathbf{t} \in V \setminus \{\mathbf{0}\}$ such that $s = \langle \mathbf{s} \rangle$ and $t = \langle \mathbf{t} \rangle$. Then $\Phi(\mathbf{s}, \mathbf{t}) = 0$. By (iii) there is an x in $\mathcal{K}$ such that $\Phi(\mathbf{s}, \mathbf{s}) = \Phi(x\mathbf{t}, x\mathbf{t})$. Since $\mathbf{s}, \mathbf{t} \in V \setminus \{\mathbf{0}\}$, $\Phi(\mathbf{s}, \mathbf{t}) = 0$ and Φ is anisotropic, $\mathbf{s}$ and $\mathbf{t}$ must be linearly independent, and thus neither $\mathbf{s} + x\mathbf{t}$ nor $\mathbf{s} - x\mathbf{t}$ is $\mathbf{0}$. Consider $\langle \mathbf{s} + x\mathbf{t} \rangle$. Let $\langle \mathbf{v} \rangle$ be an arbitrary harmonic conjugate of $\langle \mathbf{s} + x\mathbf{t} \rangle$ with respect to $\langle \mathbf{s} \rangle$ and $\langle \mathbf{t} \rangle$, where $\mathbf{v} \in V \setminus \{\mathbf{0}\}$. According to Lemma C.3, $\langle \mathbf{v} \rangle = \langle \mathbf{s} - x\mathbf{t} \rangle$. Moreover,

$$\Phi(\mathbf{s} + x\mathbf{t}, \mathbf{s} - x\mathbf{t}) = \Phi(\mathbf{s}, \mathbf{s}) - \Phi(\mathbf{s}, \mathbf{t}) \cdot x^* + x \cdot \Phi(\mathbf{t}, \mathbf{s}) - \Phi(x\mathbf{t}, x\mathbf{t}) = 0$$

Hence $\langle \mathbf{s} + x\mathbf{t} \rangle \not\to_V \langle \mathbf{s} - x\mathbf{t} \rangle$, i.e. $\langle \mathbf{s} + x\mathbf{t} \rangle \not\to_V \langle \mathbf{v} \rangle$. □

Remark 3.4. According to the above lemma, a quasi-quantum Kripke frame $(\Sigma(V), \to)$, rising from some vector space V of dimension at least 3 and equipped with an anisotropic

Hermitian form, satisfies the Assumption, if and only if, for every non-zero vector $\mathbf{s}$, there is always a vector of the same 'length' of $\mathbf{s}$ in every one-dimensional subspace of V.

In my opinion, the Assumption is reasonable. It's not hard to verify that, for every pre-Hilbert space over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, the Assumption holds on the induced quasi-quantum Kripke frame. Therefore, no vector space used in quantum mechanics is excluded by the Assumption.

However, it's fair to say that the Assumption is restrictive. The reason is that, when only *infinite-dimensional quantum Kripke frames* are the concern, even a condition weaker than the Assumption is enough to exclude those induced by vector spaces over division rings other than $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. This is a consequence of the famous Solèr's Theorem. For the details, please refer to Section 3, especially Theorem 3.7, in [6].

I proceed to show that the Assumption is consistent with and independent of the conditions in the definition of quantum Kripke frames.

Proposition 3.5. *There is a quantum Kripke frame satisfying the Assumption, and there is a quantum Kripke frame which doesn't satisfy the Assumption.*

Proof. For the first part, it's not hard to see that the vector space $\mathbb{R}^3$ equipped with the inner product defined as usual induces a quantum Kripke frame satisfying the Assumption with the help of the above lemma, Proposition B.6 and Corollary B.7.

For the second part, I consider the infinite-dimensional vector space V over a division ring $\mathcal{K}$, different from $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, equipped with an orthomodular Hermitian form Φ defined in [8]. It's not hard to verify that $(\Sigma(V), \to_V)$ is a quantum Kripke frame with the help of Proposition B.6 and Corollary B.7. Suppose (towards a contradiction) that the quantum Kripke frame $(\Sigma(V), \to)$ satisfies the Assumption. On the one hand, since V is of dimension at least 3, by the above lemma (iii) holds in V. Since V is infinite-dimensional, it's not hard to see that it has an infinite orthogonal sequence. (For example, one may derive that $(\Sigma(V), \to_V)$ is infinite-dimensional and take a sequence of vectors, each of which generates an element in the infinite orthogonal subset given in *3* of Lemma 2.1.) Without loss of generality, assume that one vector in this sequence, $\mathbf{v}$, satisfies $\Phi(\mathbf{v}, \mathbf{v}) = 1$. Then an infinite orthonormal sequence can be obtained by normalization, which is made possible by (iii). On the other hand, it can be proved that V doesn't have an infinite orthonormal sequence. (For example, this follows from Solèr's Theorem.) Therefore, a contradiction is derived. As a result, the quantum Kripke frame $(\Sigma(V), \to)$ doesn't satisfy the Assumption. □

Finally, a useful result about existence of automorphisms can be obtained.

Proposition 3.6. *If a quasi-quantum Kripke frame $\mathfrak{F} = (\Sigma, \to)$ has an orthogonal set of cardinality* 4 *and satisfies the Assumption, then for any $s, t \in \Sigma$ with $s \not\to t$ there is an automorphism F of $\mathfrak{F}$ such that $F(s) = t$, $F(t) = s$ and F restricted to $\sim\{s, t\}$ is the identity.*

Proof. I use the analytic method. Since $\mathfrak{F}$ has an orthogonal set of cardinality 4, by Proposition B.6 there is a vector space V over some division ring $\mathcal{K}$ equipped with an anisotropic Hermitian form Φ such that $(\Sigma(V), \to_V)$ is isomorphic to $\mathfrak{F}$. For simplicity, I identify $(\Sigma(V), \to_V)$ with $\mathfrak{F}$.

Let $\langle \mathbf{s} \rangle, \langle \mathbf{t} \rangle \in \Sigma(V)$ be such that $\langle \mathbf{s} \rangle \not\to_V \langle \mathbf{t} \rangle$, where $\mathbf{s}, \mathbf{t} \in V \setminus \{\mathbf{0}\}$. Then $\Phi(\mathbf{s}, \mathbf{t}) = 0$. Since $\mathfrak{F}$ has an orthogonal set of cardinality at least 4, it's easy to see that V is of dimension at least 4. Since $\mathfrak{F}$ satisfies the Assumption, by the above lemma I assume that

$\Phi(\mathbf{s},\mathbf{s}) = \Phi(\mathbf{t},\mathbf{t})$ without loss of generality. Using the idea from Gram-Schmidt Theorem, for every $\mathbf{u} \in V$, it's not hard to see that $\mathbf{u}$ can be uniquely decomposed as follows:

$$\begin{aligned}\mathbf{u} = \quad & \left(\Phi(\mathbf{u},\mathbf{s}) \cdot \Phi(\mathbf{s},\mathbf{s})^{-1}\right)\mathbf{s} \\ & + \left(\Phi(\mathbf{u},\mathbf{t}) \cdot \Phi(\mathbf{t},\mathbf{t})^{-1}\right)\mathbf{t} \\ & +(\mathbf{u} - \left(\Phi(\mathbf{u},\mathbf{s}) \cdot \Phi(\mathbf{s},\mathbf{s})^{-1}\right)\mathbf{s} - \left(\Phi(\mathbf{u},\mathbf{t}) \cdot \Phi(\mathbf{t},\mathbf{t})^{-1}\right)\mathbf{t})\end{aligned}$$

with the last component $\mathbf{u} - (\Phi(\mathbf{u},\mathbf{s}) \cdot \Phi(\mathbf{s},\mathbf{s})^{-1})\,\mathbf{s} - (\Phi(\mathbf{u},\mathbf{t}) \cdot \Phi(\mathbf{t},\mathbf{t})^{-1})\,\mathbf{t}$ in the orthocomplement of $\{\mathbf{s},\mathbf{t}\}$. In the following, for simplicity, I denote $\Phi(\mathbf{u},\mathbf{s}) \cdot \Phi(\mathbf{s},\mathbf{s})^{-1}$ by u_s, $\Phi(\mathbf{u},\mathbf{t}) \cdot \Phi(\mathbf{t},\mathbf{t})^{-1}$ by u_t and $\mathbf{u} - (\Phi(\mathbf{u},\mathbf{s}) \cdot \Phi(\mathbf{s},\mathbf{s})^{-1})\,\mathbf{s} - (\Phi(\mathbf{u},\mathbf{t}) \cdot \Phi(\mathbf{t},\mathbf{t})^{-1})\,\mathbf{t}$ by $\mathbf{u}_\perp$. Hence under this notation $\mathbf{u} = u_s\mathbf{s} + u_t\mathbf{t} + \mathbf{u}_\perp$ with $\mathbf{u}_\perp$ in the orthocomplement of $\{\mathbf{s},\mathbf{t}\}$.

Define a map $f : V \to V$ such that every $\mathbf{u} = u_s\mathbf{s} + u_t\mathbf{t} + \mathbf{u}_\perp$ is mapped to $f(\mathbf{u}) = u_s\mathbf{t} + u_t\mathbf{s} + \mathbf{u}_\perp$. It follows from existence and uniqueness of the decomposition that f is a bijection, and linearity of f can be verified easily as follows: let $\mathbf{u} = u_s\mathbf{s} + u_t\mathbf{t} + \mathbf{u}_\perp$ and $\mathbf{v} = v_s\mathbf{s} + v_t\mathbf{t} + \mathbf{v}_\perp$ be arbitrary, then

$$\begin{aligned}& f(a\mathbf{u} + b\mathbf{v}) \\ = \ & f\Big(a(u_s\mathbf{s} + u_t\mathbf{t} + \mathbf{u}_\perp) + b(v_s\mathbf{s} + v_t\mathbf{t} + \mathbf{v}_\perp)\Big) \\ = \ & f\Big((au_s + bv_s)\mathbf{s} + (au_t + bv_t)\mathbf{t} + (a\mathbf{u}_\perp + b\mathbf{v}_\perp)\Big) \\ = \ & (au_s + bv_s)\mathbf{t} + (au_t + bv_t)\mathbf{s} + (a\mathbf{u}_\perp + b\mathbf{v}_\perp) \\ = \ & a(u_s\mathbf{t} + u_t\mathbf{s} + \mathbf{u}_\perp) + b(v_s\mathbf{t} + v_t\mathbf{s} + \mathbf{v}_\perp) \\ = \ & af(\mathbf{u}) + bf(\mathbf{v})\end{aligned}$$

Moreover, f preserves Φ, for

$$\begin{aligned}& \Phi(f(\mathbf{u}), f(\mathbf{v})) \\ = \ & \Phi(u_s\mathbf{t} + u_t\mathbf{s} + \mathbf{u}_\perp, v_s\mathbf{t} + v_t\mathbf{s} + \mathbf{v}_\perp) \\ = \ & \quad u_s\Phi(\mathbf{t},\mathbf{t})v_s^* + u_s\Phi(\mathbf{t},\mathbf{s})v_t^* + u_s\Phi(\mathbf{t},\mathbf{v}_\perp) \\ & + u_t\Phi(\mathbf{s},\mathbf{t})v_s^* + u_t\Phi(\mathbf{s},\mathbf{s})v_t^* + u_t\Phi(\mathbf{s},\mathbf{v}_\perp) \\ & + \Phi(\mathbf{u}_\perp,\mathbf{t})v_s^* + \Phi(\mathbf{u}_\perp,\mathbf{s})v_t^* + \Phi(\mathbf{u}_\perp,\mathbf{v}_\perp) \\ = \ & \quad u_s\Phi(\mathbf{s},\mathbf{s})v_s^* + u_s\Phi(\mathbf{s},\mathbf{t})v_t^* + u_s\Phi(\mathbf{s},\mathbf{v}_\perp) \\ & + u_t\Phi(\mathbf{t},\mathbf{s})v_s^* + u_t\Phi(\mathbf{t},\mathbf{t})v_t^* + u_t\Phi(\mathbf{t},\mathbf{v}_\perp) \\ & + \Phi(\mathbf{u}_\perp,\mathbf{s})v_s^* + \Phi(\mathbf{u}_\perp,\mathbf{t})v_t^* + \Phi(\mathbf{u}_\perp,\mathbf{v}_\perp) \\ = \ & \Phi(u_s\mathbf{s} + u_t\mathbf{t} + \mathbf{u}_\perp, v_s\mathbf{s} + v_t\mathbf{t} + \mathbf{v}_\perp) \\ = \ & \Phi(\mathbf{u},\mathbf{v})\end{aligned}$$

Therefore, it's not hard to verify that the map $F : \Sigma(V) \to \Sigma(V)$ defined by $F(\langle\mathbf{u}\rangle) = \langle f(\mathbf{u})\rangle$ for every $\mathbf{u} \in V \setminus \{\mathbf{0}\}$ is an automorphism of $\mathfrak{F}$. Moreover, from the definition,

$$\begin{aligned}& F(\langle\mathbf{s}\rangle) = \langle f(\mathbf{s})\rangle = \langle\mathbf{t}\rangle \\ & F(\langle\mathbf{t}\rangle) = \langle f(\mathbf{t})\rangle = \langle\mathbf{s}\rangle \\ & F(\langle\mathbf{v}\rangle) = \langle f(\mathbf{v})\rangle = \langle\mathbf{v}\rangle\text{, for every } \mathbf{v} \neq \mathbf{0} \text{ in the orthocomplement of } \{\mathbf{s},\mathbf{t}\}\end{aligned}$$

Therefore, F is an automorphism of $\mathfrak{F}$ with the required property. □

4 The Main Theorem

Now I state and prove the main result.

Theorem 4.1. *In a quasi-quantum Kripke frame $\mathfrak{F} = (\Sigma, \to)$ satisfying the Assumption, for every $P \subseteq \Sigma$, the following are equivalent:*

(i) P is first-order definable and bi-orthogonally closed;

(ii) P is finitely presentable.

Proof. If $\mathfrak{F} = (\Sigma, \to)$ is finite-dimensional, then the conclusion follows from Proposition 1.7. In the following, I focus on the case when $\mathfrak{F}$ is infinite-dimensional. Let $P \subseteq \Sigma$ be arbitrary.

From (i) to (ii): I prove the contrapositive. Assume that P is bi-orthogonally closed but not finitely presentable. I need to show that P is not first-order definable. Let $n \in \mathbb{N}$ and $s_1, ..., s_n \in \Sigma$ be arbitrary. Since P is bi-orthogonally closed, it is a subspace by Lemma 3.20 in [10]. Since P is not finitely presentable, by Lemma 2.2 there is a $v \in {\sim}\{s_1, ..., s_n\}$ such that $v \notin P$. By Lemma 2.3 there is a $u \in {\sim}\{s_1, ..., s_n\}$ such that $u \in P$ and $u \not\to v$. Since $\mathfrak{F}$ is infinite-dimensional, it has an orthogonal set of cardinality 4 by Lemma 2.1. Then by Proposition 3.6 there is an automorphism F of $\mathfrak{F}$ such that $F(u) = v$, $F(v) = u$ and F restricted to ${\sim}\{u, v\}$ is the identity. It follows that, for every $i \in \{1, ..., n\}$, $F(s_i) = s_i$, because $s_i \in {\sim}\{u, v\}$. This means that F fixes the set $\{s_1, ..., s_n\}$ pointwise. However, F doesn't fix P setwise, for $u \in P$ and $F(u) = v \notin P$. Hence P is not definable by any first-order formula with parameters from $\{s_1, ..., s_n\}$, according to a well-known result in model theory (Lemma 2.1.1 in [5], Proposition 1.3.5 in [9]). Since $s_1, ..., s_n$ are arbitrary, P is not first-order definable.

From (ii) to (i): Assume that P is finitely presentable. By definition there are $n \in \mathbb{N}$ and $s_1, ..., s_n \in \Sigma$ such that $P = {\sim}\{s_1, ..., s_n\}$ or $P = {\sim}{\sim}\{s_1, ..., s_n\}$. According to Proposition 2.1 in [10], P is bi-orthogonally closed. To see that P is first-order definable, two cases need to be considered. If $P = {\sim}\{s_1, ..., s_n\}$, then P can be defined by the following first-order formula with R interpreted by $\to$ and $x_1, ..., x_n$ by $s_1, ..., s_n$, respectively:

$$\neg\,(xRx_1 \vee ... \vee xRx_n)$$

If $P = {\sim}{\sim}\{s_1, ..., s_n\}$, then P can be defined by the following first-order formula with R interpreted by $\to$ and $x_1, ..., x_n$ by $s_1, ..., s_n$, respectively:

$$\forall y(xRy \to yRx_1 \vee ... \vee yRx_n)$$

Therefore, P is first-order definable. □

Remark 4.2. In a pre-Hilbert space over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, two vectors $\mathbf{u}, \mathbf{v}$ are orthogonal, if their inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 0$. The idea of proving the above theorem can be used to show that a set of vectors is a first-order definable (with parameters and in terms of orthogonality only), closed linear subspaces, if and only if it is the orthocomplement of a finite set of vectors or the orthocomplement of the orthocomplement of a finite set of vectors. This is a characterization of closed linear subspaces which are first-order definable in terms of orthogonality.

To show an interesting consequence of this theorem, I introduce the notion of first-order quantum Kripke frames and review the notion of saturated sets in Definition 4.1 in [10].

Definition 4.3. A *first-order quantum Kripke frame* is a Kripke frame $\mathfrak{F} = (\Sigma, \to)$ such that $\mathfrak{F} \models T$, where T is a first-order theory in a formal language with exactly one binary relation symbol R consisting of the following formulas:

- **Reflexivity**
$$\forall x(xRx)$$
- **Symmetry**
$$\forall x \forall y(xRy \to yRx)$$
- **Separation**
$$\forall x \forall y\big(x \neq y \to \exists z(zRx \wedge \neg zRy)\big)$$
- **Superposition**
$$\forall x \forall y \exists z(zRx \wedge zRy)$$
- **Schema for Existence of Approximation**
For every first-order formula $\varphi(y, \overline{w})$,
$$\forall \overline{w}\Big[BOC(\varphi(y,\overline{w})) \to \forall x\Big(\exists y(xRy \wedge \varphi(y,\overline{w})) \\ \to \exists z\big(\varphi(z,\overline{w}) \wedge \forall u(\varphi(u,\overline{w}) \to (uRz \leftrightarrow uRx))\big)\Big)\Big],$$
where $\overline{w}$ is a finite tuple of variables and $BOC(\varphi(y, \overline{w}))$ is a formula defined as follows, saying that $\varphi(y, \overline{w})$ defines a bi-orthogonally closed set:
$$\forall x\Big[\forall z\big(xRz \to \exists u(zRu \wedge \varphi(u,\overline{w}))\big) \to \varphi(x,\overline{w})\Big]$$

Definition 4.4. In a Kripke frame $\mathfrak{F} = (\Sigma, \to)$, $P \subseteq \Sigma$ is *saturated*, if every $s \in \Sigma \setminus {\sim}P$ has an approximation in P, i.e. an $s' \in P$ satisfying $s \approx_P s'$.

Intuitively, Schema for Existence of Approximation says that every first-order definable, bi-orthogonally closed subset is saturated, while Property A says that every bi-orthogonally closed subset is saturated. It's not hard to see that a quantum Kripke frame is a first-order quantum Kripke frame. In fact, the definition of first-order quantum Kripke frames is obtained by just replacing Property A, which is second-order, in the definition of quantum Kripke frames by a first-order schema. The relation between first-order quantum Kripke frame and quantum Kripke frame is similar to that between first-order arithmetic and Peano arithmetic.

The following corollary can be drawn from the theorem:

Corollary 4.5. *For a Kripke frame $\mathfrak{F} = (\Sigma, \to)$ satisfying the Assumption, the following are equivalent:*

(i) $\mathfrak{F}$ is a quasi-quantum Kripke frame;

(ii) $\mathfrak{F}$ is a first-order quantum Kripke frame.

Proof. **From (i) to (ii):** Assume that $\mathfrak{F}$ is a quasi-quantum Kripke frame. By Corollary 4.16 in [10] every finitely presentable subsets of Σ is saturated. Since $\mathfrak{F}$ is quasi-quantum Kripke frame satisfying the Assumption, by the theorem every first-order definable, bi-orthogonally closed subset is finitely presentable. Hence every first-order definable, bi-orthogonally closed subset is saturated. Therefore, $\mathfrak{F}$ is a first-order quantum Kripke frame.

From (ii) to (i): Assume that $\mathfrak{F}$ is a first-order quantum Kripke frame. It suffices to derive Property AL and Property AH from the schema. Notice that the set $\sim\sim\{s,t\}$ for $s,t \in \Sigma$ can be defined by the following formula with R interpreted by $\to$ and x_1, x_2 by s,t, respectively:

$$\forall y\,(xRy \to (yRx_1 \vee yRx_2))\,,$$

the set $\sim\{s\}$ for $s \in \Sigma$ can be defined by the following formula with R interpreted by $\to$ and y by s, respectively:

$$\neg xRy$$

Then it's not hard to see that Property AL and Property AH follow from the schema. Therefore, $\mathfrak{F}$ is a quasi-quantum Kripke frame. □

Remark 4.6. According to this corollary, first-order quantum Kripke frames satisfying the Assumption are finitely first-order axiomatizable.

5 Application

In this section, as an application of the techniques developed above, I will show that quantum Kripke frames are not first-order definable, employing Goldblatt's idea in [4].

The following proposition generalizes Theorem 1 in [4] in two aspects. First, this proposition is about quantum Kripke frames satisfying the Assumption, while Theorem 1 in [4] is only about Hilbert spaces, which correspond to a very specific kind of quantum Kripke frames. Second, this proposition is about quantum Kripke frames of arbitrary infinite dimension, while Theorem 1 in [4] is only about Hilbert spaces being separable, i.e. countably infinite-dimensional.

Proposition 5.1. *Let $\mathfrak{F} = (\Sigma, \to)$ be a quantum Kripke frame satisfying the Assumption and $P \subseteq \Sigma$ be an infinte-dimensional subspace. Then if $s_1, ..., s_n \in P$ and $t \in \Sigma$, there exists an automorphism F of $\mathfrak{F}$ such that $F(s_i) = s_i$ for $i = 1, ..., n$ and $F(t) \in P$.*

Proof. Assume that $s_1, ..., s_n \in P$ and $t \in \Sigma$. Then $\sim\sim\{s_1, ..., s_n\} = \mathcal{C}(\{s_1, ..., s_n\}) \subseteq P$. Moreover, since $\mathfrak{F}$ has an infinite-dimensional subspace, it follows from Lemma 2.1 that it has an orthogonal subset of cardinality 4. Now three cases need to be considered.

Case 1: $t \in P$. Then the identity map on Σ is an automorphism of $\mathfrak{F}$ with the required property.

Case 2: $t \notin P$ and $t \in \sim\{s_1, ..., s_n\}$. By Lemma 2.3 there is an $s \in \sim\{s_1, ..., s_n\}$ such that $s \in P$ and $t \not\to s$. Hence $s, t \in \sim\{s_1, ..., s_n\}$ and $t \not\to s$. By Proposition 3.6 there is an automorphism F of $\mathfrak{F}$ such that $F(t) = s$, $F(s) = t$ and F restricted to $\sim\{s,t\}$ is the identity. Since $\{s_1, ..., s_n\} \subseteq \sim\sim\{s_1, ..., s_n\} \subseteq \sim\{s,t\}$, $F(s_i) = s_i$ for $i = 1, ..., n$. Moreover, $F(t) = s \in P$. Therefore, F is an automorphism of $\mathfrak{F}$ with the required property.

Case 3: $t \notin P$ and $t \notin \sim\{s_1, ..., s_n\}$. Then $t \notin \sim\sim\sim\{s_1, ..., s_n\}$. Moreover, since $\sim\sim\{s_1, ..., s_n\} \subseteq P$, $t \notin \sim\sim\{s_1, ..., s_n\}$. Hence by Proposition 4.5 and Corollary 4.16 in

[10] there are $t_\parallel \in \sim\sim\{s_1, ..., s_n\}$ and $t_\perp \in \sim\sim\sim\{s_1, ..., s_n\} = \sim\{s_1, ..., s_n\}$ such that $t \in \sim\sim\{t_\parallel, t_\perp\}$. Notice that $t_\perp \notin P$; otherwise, since P is a subspace, $t_\parallel, t_\perp \in P$ implies that $t \in \sim\sim\{t_\parallel, t_\perp\} \subseteq P$, contradicting that $t \notin P$. Since $t_\perp \notin P$ and $t_\perp \in \sim\{s_1, ..., s_n, t_\parallel\}$, by Lemma 2.3 there is an $s \in \sim\{s_1, ..., s_n, t_\parallel\}$ such that $s \in P$ and $s \not\to t_\perp$. It follows that $s, t_\perp \in \sim\{s_1, ..., s_n, t_\parallel\}$ and $t_\perp \not\to s$. By Proposition 3.6 there is an automorphism F of $\mathfrak{F}$ such that $F(t_\perp) = s$, $F(s) = t_\perp$ and F restricted to $\sim\{s, t_\perp\}$ is the identity. Since $\{s_1, ..., s_n, t_\parallel\} \subseteq \sim\sim\{s_1, ..., s_n, t_\parallel\} \subseteq \sim\{s, t_\perp\}$, $F(s_i) = s_i$ for $i = 1, ..., n$ and $F(t_\parallel) = t_\parallel$. Moreover, from $t \in \sim\sim\{t_\parallel, t_\perp\}$ it's not hard to verify that $F(t) \in \sim\sim\{F(t_\parallel), F(t_\perp)\}$. Since $F(t_\parallel) = t_\parallel \in P$, $F(t_\perp) = s \in P$ and P is a subspace, $F(t) \in \sim\sim\{F(t_\parallel), F(t_\perp)\} \subseteq P$. Therefore, F is an automorphism of $\mathfrak{F}$ with the required property. □

Corollary 5.2. *Let $\mathfrak{F} = (\Sigma, \to)$ be a quantum Kripke frame satisfying the Assumption and $P \subseteq \Sigma$ be an infinite-dimensional subspace. Then $(P, \to)$ is an elementary substructure of $\mathfrak{F}$, where I use the same symbol for the relation $\to$ on Σ and its restriction to P.*

Proof. I use the Tarski-Vaught Test. Let $n \in \mathbb{N}$ be arbitrary and φ an arbitrary first-order formula with exactly one binary relation symbol and at most $n+1$ free variables. Assume that $\mathfrak{F} \models \varphi[s_1, ..., s_n, t]$ for some $s_1, ..., s_n \in P$ and $t \in \Sigma$. By the above proposition, there exists an automorphism F of $\mathfrak{F}$ such that $F(s_i) = s_i$ for $i = 1, ..., n$ and $F(t) \in P$. Hence $\mathfrak{F} \models \varphi[F(s_1), ..., F(s_n), F(t)]$, and thus $\mathfrak{F} \models \varphi[s_1, ..., s_n, F(t)]$. It follows from the Tarski-Vaught Theorem that $(P, \to)$ is an elementary substructure of $(\Sigma, \to)$. □

Theorem 5.3. *Quantum Kripke frames are not first-order definable in the class of quasi-quantum Kripke frames.*

Proof. I use the same counterexample as in [4]. Let l_2 be the separable Hilbert space of absolutely squared summable sequences of complex numbers, and $l_2^{(0)}$ the pre-Hilbert space of finitely non-zero sequences of complex numbers. It can be proved that $l_2^{(0)}$ is a linear subspace of l_2, which is infinite-dimensional but not closed. (Please refer to, for example, Example 2.1.13 in [7].) Based on the above, the following can be derived:

1. $(\Sigma(l_2), \to_{l_2})$ is a quantum Kripke frame, using Proposition B.6 and Corollary B.7.
2. $\Sigma(l_2^{(0)})$ is an infinite-dimensional subspace of $(\Sigma(l_2), \to_{l_2})$.
3. $(\Sigma(l_2^{(0)}), \to_{l_2})$ is a quasi-quantum Kripke frame, according to Proposition B.6; but it's not a quantum Kripke frame, according to the Piron-Amemiya-Araki Theorem in [1].
4. $(\Sigma(l_2), \to_{l_2})$ satisfies the Assumption, according to Remark 3.4.

Now, on the one hand, by 1 and 3 $(\Sigma(l_2), \to_{l_2})$ is a quantum Kripke frame, and $(\Sigma(l_2^{(0)}), \to_{l_2})$ is a quasi-quantum Kripke frame but not a quantum Kripke frame. On the other hand, by 2, 4 and the above corollary $(\Sigma(l_2^{(0)}), \to_{l_2})$ is an elementary substructure of $(\Sigma(l_2), \to_{l_2})$. Hence no first-order formula can distinguish between them. □

A Linear Algebra

In this appendix, I would like to review some relevant elements of linear algebra. The point of this review lies in its generality: vector spaces over division rings are investigated, instead of those over just $\mathbb{R}$ or $\mathbb{C}$. If without explanation, the definitions are from [3].

First I recall the notion of division rings.

Definition A.1. A *division ring* $\mathcal{K}$ is a tuple $(K, +, \cdot, 0, 1)$ where K is a non-empty set, $+, \cdot : K \times K \to K$ are two functions and $0, 1 \in K$ such that:

- $(K, +, 0)$ is an Abelian group;
- $(K \setminus \{0\}, \cdot, 1)$ is a group;
- $\cdot$ distributes over $+$, i.e. for any $x, y, z \in K$,

$$x \cdot (y + z) = x \cdot y + x \cdot z,$$
$$(x + y) \cdot z = x \cdot z + y \cdot z.$$

If $(K \setminus \{0\}, \cdot, 1)$ forms an Abelian group, then $\mathcal{K}$ is called a *field*.

Please notice that in [3] the word 'field' actually means division rings. However, I stick to the common usage of this word in this report.

Next, I introduce the notion of vector spaces over a division ring.

Definition A.2. A *(left) vector space over a division ring* $\mathcal{K} = (K, +, \cdot, 0, 1)$ is a set V equipped with functions $+ : V \times V \to V$, called *addition*, and $x(\cdot) : V \to V$, called *multiplication by scalar* x, for every $x \in K$, as well as $\mathbf{0} \in V$[1]such that:

- $(V, +, \mathbf{0})$ is an Abelian group;
- $x(\mathbf{v} + \mathbf{w}) = x\mathbf{v} + x\mathbf{w}$, for any $x \in K$ and $\mathbf{v}, \mathbf{w} \in V$;
- $(x + y)\mathbf{v} = x\mathbf{v} + y\mathbf{v}$, for any $x, y \in K$ and $\mathbf{v} \in V$;
- $(x \cdot y)\mathbf{v} = x(y\mathbf{v})$, for any $x, y \in K$ and $\mathbf{v} \in V$;
- $1\mathbf{v} = \mathbf{v}$, for every $\mathbf{v} \in V$.

I proceed to the notion of Hermitian forms on a vector space over some division ring.

Definition A.3. An *Hermitian form* on a vector space V over a division ring $\mathcal{K} = (K, +, \cdot, 0, 1)$ is a function $\Phi : V \times V \to K$ such that there is a function $\mu : K \to K$ satisfying:

- μ is an involution on $\mathcal{K}$, i.e.
  - μ is bijective;
  - $\mu(x + y) = \mu(x) + \mu(y)$ and $\mu(x \cdot y) = \mu(y) \cdot \mu(x)$, for any $x, y \in K$;
  - $\mu \circ \mu(x) = x$, for every $x \in K$;
- $\Phi(\mathbf{u} + \mathbf{v}, \mathbf{w}) = \Phi(\mathbf{u}, \mathbf{w}) + \Phi(\mathbf{v}, \mathbf{w})$, for any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$;
- $\Phi(x\mathbf{v}, \mathbf{w}) = x \cdot \Phi(\mathbf{v}, \mathbf{w})$, for any $\mathbf{v}, \mathbf{w} \in V$;
- $\Phi(\mathbf{v}, \mathbf{w}) = \mu(\Phi(\mathbf{w}, \mathbf{v}))$, for any $\mathbf{v}, \mathbf{w} \in V$;

[1]In this report, when discussing about vector spaces, I'm not going to present them in a rigorous set-theoretic way, which is tedious and unusual. Instead, I will just follow the usual way how mathematicians talk about them.

μ is called *the accompanying involution* of Φ.

An Hermitian form is *non-singular*, if $\Phi(\mathbf{v}, \mathbf{w}) = 0$ for all $\mathbf{w} \in V$ implies $\mathbf{v} = \mathbf{0}$.

An Hermitian form is *anisotropic*, if $\Phi(\mathbf{v}, \mathbf{v}) = 0$ implies $\mathbf{v} = \mathbf{0}$, for every $\mathbf{v} \in V$.

Remark A.4. First, notice that Hermitian forms and inner products on Hilbert spaces are very much alike. For example, they're both additive in each arguments while the other is fixed. However, an important difference is that, an Hermitian form is linear in the first argument, while an inner product is linear in the second argument.

Second, from the above definition, for any $x, y \in K$ and $\mathbf{v}, \mathbf{w} \in V$,

$$\Phi(x\mathbf{u} + y\mathbf{v}, \mathbf{w}) = x \cdot \Phi(\mathbf{u}, \mathbf{w}) + y \cdot \Phi(\mathbf{v}, \mathbf{w})$$
$$\Phi(\mathbf{w}, x\mathbf{u} + y\mathbf{v}) = \Phi(\mathbf{w}, \mathbf{u}) \cdot \mu(x) + \Phi(\mathbf{w}, \mathbf{v}) \cdot \mu(y)$$

Finally, I consider a special kind of Hermitian forms.

Definition A.5. An Hermitian form Φ on a vector space V over some division ring $\mathcal{K}$ is *orthomodular*, if $E = (E^\perp)^\perp$ implies that $E \oplus E^\perp = V$, for every $E \subseteq V$, where $E^\perp \overset{\text{def}}{=} \{\mathbf{v} \in V \mid \Phi(\mathbf{v}, \mathbf{u}) = 0, \text{ for every } \mathbf{u} \in E\}$, is called *the orthocomplement of* E.

Lemma A.6. *Every orthomodular Hermitian form is anisotropic. On finite-dimensional vector spaces, every anisotropic Hermitian form is orthomodular.*

Proof. Please refer to the last but one paragraph on the second page of [6]. □

B Geometry and Algebra

In this appendix, I review the intimate relation between projective geometry and linear algebra. If without explanation, the definitions are from [3]. For elements in projective geometry, please refer to the appendix in [10] or [3].

I start from the important observation that every vector space over a division ring gives rise to a projective geometry in a canonical way.

Theorem B.1. *Let V be a vector space over a division ring $\mathcal{K} = (K, +, \cdot, 0, 1)$. Denote by $\Sigma(V)$ the set $\{\langle\mathbf{v}\rangle \mid \mathbf{v} \in V \setminus \{\mathbf{0}\}\}$, where $\langle\mathbf{v}\rangle = \{x\mathbf{v} \mid x \in K\}$. Define a function $* : \Sigma(V) \times \Sigma(V) \to \wp(\Sigma(V))$ such that, for any $\mathbf{u}, \mathbf{v} \in V \setminus \{\mathbf{0}\}$,*

$$\langle\mathbf{u}\rangle * \langle\mathbf{v}\rangle = \{\langle\mathbf{w}\rangle \mid \mathbf{w} \in V \setminus \{\mathbf{0}\} \text{ and } \mathbf{w} = x\mathbf{u} + y\mathbf{v} \text{ for some } x, y \in K\}.$$

Then $(\Sigma(V), *)$ is a projective geometry, called the projective geometry of V *and denoted by $\mathcal{P}(V)$.*

Proof. This follows from Proposition 2.1.6 and Proposition 2.2.3 in [3]. □

Now I consider about the converse of this. As it turns out, it's not always possible to get a vector space over a division ring from a projective geometry. To be able to do so, the geometry needs to satisfy a special property defined as follows.

Definition B.2. $\mathcal{G} = (G, \star)$ satisfies the *Desargues' property*, if, for any six distinct points $a, b, c, a', b', c' \in G$ such that $c \notin a \star b$, $c' \notin a' \star b'$, if the lines $a \star a'$, $b \star b'$ and $c \star c'$ intersect at one point, then the three points $(a \star b) \cap (a' \star b')$, $(b \star c) \cap (b' \star c')$ and $(c \star a) \cap (c' \star a')$ are collinear[2].

[2]Three points a, b, c are collinear, if $w_1 \in w_2 \star w_3$ for some w_1, w_2, w_3 such that $\{w_1, w_2, w_3\} = \{a, b, c\}$.

$\mathcal{G} = (G, \star)$ is *irreducible*, if $a \star b$ contains at least three points for any $a, b \in G$ with $a \neq b$.

$\mathcal{G} = (G, \star)$ is *arguesian*, if $\mathcal{G}$ is irreducible, satisfies the Desargues' property and has three non-collinear points.

The importance of being arguesian is manifested in the following theorem:

Theorem B.3. *For a projective geometry $\mathcal{G} = (G, \star)$, the following are equivalent:*

(i) $\mathcal{G}$ is arguesian;

(ii) there is a division ring $\mathcal{K}$ and a (left) vector space V of dimension at least 3 over $\mathcal{K}$ such that $\mathcal{P}(V) \cong \mathcal{G}$.

Both $\mathcal{K}$ and V are unique up to isomorphism, when they exist.

Proof. This follows from Proposition 9.2.6 and Theorem 9.4.4 in [3]. □

If i is an isomorphism from a projective geometry $(G, \star)$ to a projective geometry $\mathcal{P}(V)$, then the element $i(a)$ in $\Sigma(V)$ is called the *homogeneous coordinate of the point $a \in G$ with respect to i.*

Next I turn to a similar relation between arguesian pure orthogeometries and vector spaces with an anisotropic Hermitian form.

Theorem B.4. *For every vector space V over some division ring $\mathcal{K}$ equipped with an anisotropic Hermitian form Φ, $(\Sigma(V), *, \perp)$ is an irreducible pure orthogeometry, where $\perp \subseteq \Sigma(V) \times \Sigma(V)$ is defined so that $\langle \mathbf{v} \rangle \perp \langle \mathbf{w} \rangle \Leftrightarrow \Phi(\mathbf{v}, \mathbf{w}) = 0$, for any $\mathbf{v}, \mathbf{w} \in V \setminus \{\mathbf{0}\}$.*

Moreover, every arguesian pure orthogeometry $\mathcal{G} = (G, \star, \perp)$ is isomorphic to one of this form.

Proof. For the first part, by Proposition 14.1.6 in [3] $(\Sigma(V), *, \perp)$ is an orthogeometry. Irreducibility is obvious, and being pure follows easily from Φ being anisotropic.

For the second part, since $(G, \star)$ is arguesian, by Theorem B.3 there is a division ring $\mathcal{K}$ and a (left) vector space V over $\mathcal{K}$ such that $i : \mathcal{P}(V) \cong (G, \star)$. Then notice that a pure orthogeometry is non-null in the sense of Definition 14.1.7 in [3]. Hence Theorem 14.1.8 in [3] implies that there is a non-singular Hermitian form Φ on V such that $i(\langle \mathbf{v} \rangle) \perp i(\langle \mathbf{w} \rangle) \Leftrightarrow \Phi(\mathbf{v}, \mathbf{w}) = 0$, for any $\mathbf{v}, \mathbf{w} \in V \setminus \{\mathbf{0}\}$. It remains to show that Φ is anisotropic.

Suppose (towards a contradiction) that Φ is not anisotropic. Then there is a $\mathbf{v} \in V$ such that $\Phi(\mathbf{v}, \mathbf{v}) = 0$ but $\mathbf{v} \neq \mathbf{0}$. By the above equivalence $i(\langle \mathbf{v} \rangle) \perp i(\langle \mathbf{v} \rangle)$, contradicting that the orthogeometry is pure. As a result, Φ is anisotropic. □

Desargues' property has very nice and important consequences, but its statement is very complicated. However, there is a very simple sufficient condition for it to hold.

Theorem B.5. *An irreducible projective geometry satisfies Desargues' property, and thus is arguesian, if it is of rank at least 4.*

Proof. Please refer to the proof of Theorem 8.4.6 in [3]. □

Next, I prove a useful connection between linear algebra and quasi-quantum Kripke frames.

Proposition B.6. *For every vector space V over a division ring $\mathcal{K}$ equipped with an Hermitian form Φ, define $\rightarrow_V \subseteq \Sigma(V) \times \Sigma(V)$ such that, for any $\mathbf{u}, \mathbf{v} \in V \setminus \{\mathbf{0}\}$, $\langle \mathbf{u} \rangle \rightarrow_V \langle \mathbf{v} \rangle$ if and only if $\Phi(\mathbf{u}, \mathbf{v}) \neq 0$, then $(\Sigma(V), \rightarrow_V)$ is a quasi-quantum Kripke frame.*

Every quasi-quantum Kripke frame, which has an orthogonal set of cardinality 4, is isomorphic to one of the above form.

Proof. For the first part, let V be a vector space over a division ring $\mathcal{K}$ equipped with an anisotropic Hermitian form Φ. According to Theorem B.4, $(\Sigma(V), *, \perp)$ is an irreducible pure orthogeometry, where $\perp \subseteq \Sigma(V) \times \Sigma(V)$ is defined so that $\langle \mathbf{v} \rangle \perp \langle \mathbf{w} \rangle \Leftrightarrow \Phi(\mathbf{v}, \mathbf{w}) = 0$, for any $\mathbf{v}, \mathbf{w} \in V \setminus \{\mathbf{0}\}$. Notice that $\langle \mathbf{u} \rangle \rightarrow_V \langle \mathbf{v} \rangle$, if and only if $\langle \mathbf{u} \rangle \not\perp \langle \mathbf{v} \rangle$, for any $\mathbf{u}, \mathbf{v} \in V \setminus \{\mathbf{0}\}$. According to Corollary 3.18 and Corollary 5.2 in [10], $(\Sigma(V), \rightarrow_V)$ is the geometric frame corresponding to the irreducible pure orthogeometry $(\Sigma(V), *, \perp)$, and thus is a quasi-quantum Kripke frame.

For the second part, let $\mathfrak{F} = (\Sigma, \rightarrow)$ be a quasi-quantum Kripke frame, which has an orthogonal set of cardinality 4. Then it's a geometric frame in the sense of Definition 1.3 in [10]. Let $(\Sigma, \star, \not\rightarrow)$ be its corresponding pure orthogeometry given in Corollary 3.18 in [10]. Then by Corollary 5.2 in [10] $(\Sigma, \star, \not\rightarrow)$ is an irreducible pure orthogeometry. Moreover, since $\mathfrak{F}$ has an orthogonal set of cardinality 4, it's not hard to see that this set is an independent set of cardinality 4 in $(\Sigma, \star, \not\rightarrow)$, so this orthogeometry is of rank at least 4. Then by the above theorem $(\Sigma, \star)$ is arguesian. Hence by Theorem B.4 there is a vector space V over some division ring $\mathcal{K}$ equipped with an anisotropic Hermitian form Φ such that $i : (\Sigma(V), *, \perp) \cong (\Sigma, \star, \not\rightarrow)$. Notice that, for any $\mathbf{u}, \mathbf{v} \in V \setminus \{\mathbf{0}\}$,

$$\begin{aligned}
&\langle \mathbf{u} \rangle \rightarrow_V \langle \mathbf{v} \rangle \\
\Leftrightarrow\ & \Phi(\mathbf{u}, \mathbf{v}) \neq 0 \\
\Leftrightarrow\ & \text{not } \Phi(\mathbf{u}, \mathbf{v}) = 0 \\
\Leftrightarrow\ & \text{not } \langle \mathbf{u} \rangle \perp \langle \mathbf{v} \rangle \\
\Leftrightarrow\ & \text{not } i(\langle \mathbf{u} \rangle) \not\rightarrow i(\langle \mathbf{v} \rangle) \\
\Leftrightarrow\ & i(\langle \mathbf{u} \rangle) \rightarrow i(\langle \mathbf{v} \rangle)
\end{aligned}$$

As a result, $\mathfrak{F}$ is isomorphic to $(\Sigma(V), \rightarrow_V)$. □

A corollary involving quantum Kripke frames can be drawn from this proposition.

Corollary B.7. *Let V be a vector space over a division ring $\mathcal{K}$ equipped with an anisotropic Hermitian form Φ, whose accompanying involution is μ. Then, for the quasi-quantum Kripke frame $(\Sigma(V), \rightarrow_V)$, the following are equivalent:*

(i) *$(\Sigma(V), \rightarrow_V)$ is a quantum Kripke frame;*

(ii) *Φ is orthomodular.*

Proof. Please notice three things. First, for a subspace E of V, $E = (E^{\perp})^{\perp}$, if and only if $\Sigma(E) \subseteq \Sigma(V)$ is bi-orthogonally closed. Second, for $P \subseteq \Sigma(V)$, $\bigcup P$ is a subspace of V; and P is bi-orthogonally closed, if and only if $\bigcup P = ((\bigcup P)^{\perp})^{\perp}$. Third, $\Phi(\mathbf{t}, \mathbf{t})^{-1}$ is well-defined, for every non-zero vector $\mathbf{t}$.

From (i) to (ii): Assume that E is a subspace of V such that $E = (E^{\perp})^{\perp}$ and $\mathbf{v} \in V$. If $\mathbf{v} \in E$, then $\mathbf{v} = \mathbf{v} + \mathbf{0}$ and $\mathbf{0} \in E^{\perp}$, so $\mathbf{v} \in E \oplus E^{\perp}$. If $\mathbf{v} \in E^{\perp}$, then $\mathbf{v} \in E \oplus E^{\perp}$ can be proved similarly. In the following, I focus on the case when $\mathbf{v} \notin E \cup E^{\perp}$. Then $\mathbf{v} \neq \mathbf{0}$. Since $\mathbf{v} \notin E^{\perp}$, it's easy to see that $\langle \mathbf{v} \rangle \notin \Sigma(E^{\perp}) = {\sim}\Sigma(E)$. Since $E = (E^{\perp})^{\perp}$,

$\Sigma(E)$ is bi-orthogonally closed. Then by (i) there is a $u \in \Sigma(E)$ such that $u \approx_{\Sigma(E)} \langle \mathbf{v} \rangle$. Take $\mathbf{u} \in V \setminus \{\mathbf{0}\}$ such that $u = \langle \mathbf{u} \rangle$. It follows that $\mathbf{u} \in E$. Consider the vector $\mathbf{w} = \mathbf{v} - (\Phi(\mathbf{v},\mathbf{u}) \cdot \Phi(\mathbf{u},\mathbf{u})^{-1})\,\mathbf{u}$. Since $\mathbf{u} \in E$, $\mathbf{v} \notin E$ and E is a subspace, $\mathbf{u}$ and $\mathbf{v}$ are linearly independent. Hence $\mathbf{w} \neq \mathbf{0}$.

I claim that $\mathbf{w} \in E^{\perp}$. Let $\mathbf{x} \in E$ be arbitrary. Two cases need to be considered.

Case 1: $\mathbf{x}$ and $\mathbf{u}$ are linearly dependent. Then $\mathbf{x} = k\mathbf{u}$, for some k in $\mathcal{K}$. Hence

$$\begin{aligned}
&\Phi(\mathbf{w},\mathbf{x}) \\
&= \Phi(\mathbf{v} - \left(\Phi(\mathbf{v},\mathbf{u}) \cdot \Phi(\mathbf{u},\mathbf{u})^{-1}\right)\mathbf{u}, k\mathbf{u}) \\
&= \Phi(\mathbf{v},\mathbf{u}) \cdot \mu(k) - \Phi(\mathbf{v},\mathbf{u}) \cdot \Phi(\mathbf{u},\mathbf{u})^{-1} \cdot \Phi(\mathbf{u},\mathbf{u}) \cdot \mu(k) \\
&= 0
\end{aligned}$$

Case 2: $\mathbf{x}$ and $\mathbf{u}$ are linearly independent. Let $\mathbf{y} = \mathbf{x} - (\Phi(\mathbf{x},\mathbf{u}) \cdot \Phi(\mathbf{u},\mathbf{u})^{-1})\,\mathbf{u}$, which can not be $\mathbf{0}$ in this case. Notice that $\mathbf{y} \in E$ and

$$\begin{aligned}
&\Phi(\mathbf{y},\mathbf{u}) \\
&= \Phi(\mathbf{x} - \left(\Phi(\mathbf{x},\mathbf{u}) \cdot \Phi(\mathbf{u},\mathbf{u})^{-1}\right)\mathbf{u}, \mathbf{u}) \\
&= \Phi(\mathbf{x},\mathbf{u}) - \Phi(\mathbf{x},\mathbf{u}) \cdot \Phi(\mathbf{u},\mathbf{u})^{-1} \cdot \Phi(\mathbf{u},\mathbf{u}) \\
&= 0
\end{aligned}$$

Then $\langle \mathbf{y} \rangle \not\to_V \langle \mathbf{u} \rangle$. Since $\langle \mathbf{y} \rangle \in \Sigma(E)$ and $\langle \mathbf{u} \rangle \approx_{\Sigma(E)} \langle \mathbf{v} \rangle$, $\langle \mathbf{y} \rangle \not\to_V \langle \mathbf{v} \rangle$, so $\Phi(\mathbf{v},\mathbf{y}) = 0$. Hence

$$\begin{aligned}
&\Phi(\mathbf{w},\mathbf{x}) \\
&= \Phi(\mathbf{v} - \left(\Phi(\mathbf{v},\mathbf{u}) \cdot \Phi(\mathbf{u},\mathbf{u})^{-1}\right)\mathbf{u}, \mathbf{y} + \left(\Phi(\mathbf{x},\mathbf{u}) \cdot \Phi(\mathbf{u},\mathbf{u})^{-1}\right)\mathbf{u}) \\
&= \Phi(\mathbf{v},\mathbf{y}) + \Phi(\mathbf{v},\mathbf{u}) \cdot \mu\left(\Phi(\mathbf{x},\mathbf{u}) \cdot \Phi(\mathbf{u},\mathbf{u})^{-1}\right) - \Phi(\mathbf{v},\mathbf{u}) \cdot \Phi(\mathbf{u},\mathbf{u})^{-1} \cdot \Phi(\mathbf{u},\mathbf{y}) \\
&\quad -\Phi(\mathbf{v},\mathbf{u}) \cdot \Phi(\mathbf{u},\mathbf{u})^{-1} \cdot \Phi(\mathbf{u},\mathbf{u}) \cdot \mu\left(\Phi(\mathbf{x},\mathbf{u}) \cdot \Phi(\mathbf{u},\mathbf{u})^{-1}\right) \\
&= 0
\end{aligned}$$

In both cases, $\Phi(\mathbf{w},\mathbf{x}) = 0$. Since $\mathbf{x}$ is arbitrary, $\mathbf{w} \in E^{\perp}$.

Therefore, $\mathbf{v} = (\Phi(\mathbf{v},\mathbf{u}) \cdot \Phi(\mathbf{u},\mathbf{u})^{-1})\,\mathbf{u} + \mathbf{w}$, $(\Phi(\mathbf{v},\mathbf{u}) \cdot \Phi(\mathbf{u},\mathbf{u})^{-1})\,\mathbf{u} \in E$ and $\mathbf{w} \in E^{\perp}$. As a result, $\mathbf{v} \in E \oplus E^{\perp}$.

From (ii) to (i): By the above proposition it suffices to show that Property A holds in $(\Sigma(V), \to_V)$. Let $P \subseteq \Sigma(V)$ and $s \in \Sigma(V)$ be arbitrary such that $\sim\sim P = P$ and $s \notin \sim P$. Since P is bi-orthogonally closed, $\bigcup P$ is a subspace of V and $\bigcup P = ((\bigcup P)^{\perp})^{\perp}$. Take $\mathbf{s} \in V \setminus \{\mathbf{0}\}$ such that $s = \langle \mathbf{s} \rangle$. By (ii) there are $\mathbf{s}_{\parallel} \in \bigcup P$ and $\mathbf{s}_{\perp} \in (\bigcup P)^{\perp}$ such that $\mathbf{s} = \mathbf{s}_{\parallel} + \mathbf{s}_{\perp}$. Since $s \notin \sim P$, $\mathbf{s}_{\parallel} \neq \mathbf{0}$, so $\langle \mathbf{s}_{\parallel} \rangle$ is an element in $\Sigma(\bigcup P) = P$. Moreover, for each $t \in P$, if $\mathbf{t} \in V \setminus \{\mathbf{0}\}$ is such that $\langle \mathbf{t} \rangle = t$, then $\Phi(\mathbf{t},\mathbf{s}) = \Phi(\mathbf{t},\mathbf{s}_{\parallel}) + \Phi(\mathbf{t},\mathbf{s}_{\perp}) = \Phi(\mathbf{t},\mathbf{s}_{\parallel}) + 0 = \Phi(\mathbf{t},\mathbf{s}_{\parallel})$. It follows that $t \to_V s$ if and only if $t \to_V \langle \mathbf{s}_{\parallel} \rangle$. Therefore, $\langle \mathbf{s}_{\parallel} \rangle$ is an approximation of s in P. $\square$

C Harmonic Conjugate

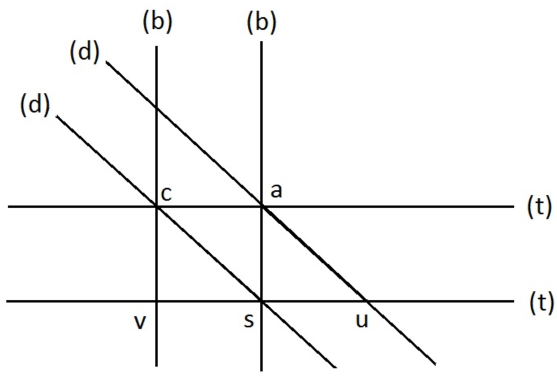
In this section, I would like to review the notion of harmonic conjugates, which is very important in projective geometry.

I use the definition of harmonic conjugates in [2] instead of [3].

Definition C.1. In a projective geometry $\mathcal{G} = (G, \star)$, for any $s, t, u \in G$ such that $s \neq t$ and $u \in s \star t$, $v \in G$ is *a harmonic conjugate of u with respect to s and t*, if there are $a, b, c, d \in G$ satisfying the following:

1. no three of a, b, c, d are collinear;
2. $s \in a \star b \cap c \star d$;
3. $t \in a \star c \cap b \star d$;
4. $u \in a \star d \cap s \star t$;
5. $v \in b \star c \cap s \star t$.

The definition of harmonic conjugates seems very complicated. The following picture of the analogue in an affine plane may help to make sense of it: (in the picture '(t)' means that t is not a point in the affine plane; instead, it's an imaginary point at infinity where parallel lines intersect.)



The following lemma reveals that existence of harmonic conjugate actually has a consequence in the dimension of a projective geometry.

Lemma C.2. *Let $\mathcal{G} = (G, \star)$ be a projective geometry and $s, t, u, v \in G$ be such that $s \neq t$, $u \in s \star t$ and v is a harmonic conjugate of u with respect to s and t. Moreover, let $a, b, c, d \in G$ witness this, i.e. 1 to 5 in the above definition are satisfied. Then $b \notin s \star t$.*

In particular, $\mathcal{G}$ is of rank at least 3.

Proof. I point out that, according to Proposition 2.2.5 in [3],

(P8) for any $x, y, p, q \in G$, $x, y \in p \star q$ and $x \neq y$ imply that $x \star y = p \star q$.

Observe that $b \neq s$; otherwise, $b = s \in c \star d$, contradicting that b, c, d are not collinear. Also observe that $b \neq t$; otherwise, $b = t \in a \star c$, contradicting that a, b, c are not collinear.

Now suppose (towards a contradiction) that $b \in s \star t$. Then $b, s \in (a \star b) \cap (s \star t)$. Since $s \neq b$, by (P8) $s \star t = s \star b = a \star b$. Moreover, by the supposition $b, t \in (b \star d) \cap (s \star t)$. Since $t \neq b$, by (P8) $s \star t = t \star b = b \star d$. Therefore, $a \in a \star b = s \star t = b \star d$, contradicting that a, b, d are not collinear. As a result, $b \notin s \star t$.

Since $s \neq t$ and $b \notin s \star t$, it's not hard to show that $\{b, s, t\}$ is an independent set in $\mathcal{G}$, so $\mathcal{G}$ is of rank at least 3. $\square$

The following is a characterization of the notion of harmonic conjugates in projective geometries of the form $\mathcal{P}(V)$ for some vector space V over some division ring. In the literature, its proof is referred to as a direct computation (e.g. [6]). However, due to the existential definition that I adopt, it may not be obvious how to compute. Hence I provide a detailed proof below.

Lemma C.3. *Let V be a vector space, of dimension at least 3, over some division ring, $\mathbf{s}, \mathbf{t} \in V \setminus \{\mathbf{0}\}$ be linearly independent and $\mathbf{v} \in V \setminus \{\mathbf{0}\}$. In the projective geometry $\mathcal{P}(V)$, $\langle \mathbf{v} \rangle$ is a harmonic conjugate of $\langle \mathbf{s} + \mathbf{t} \rangle$ with respect to $\langle \mathbf{s} \rangle$ and $\langle \mathbf{t} \rangle$, if and only if $\langle \mathbf{v} \rangle = \langle \mathbf{s} - \mathbf{t} \rangle$.*

Proof. Since $\mathbf{s}$ and $\mathbf{t}$ are linearly independent, $\langle \mathbf{s} \rangle$, $\langle \mathbf{t} \rangle$ and $\langle \mathbf{s} + \mathbf{t} \rangle$ are distinct.

The 'Only If' part: Assume that $\langle \mathbf{v} \rangle$ is a harmonic conjugate of $\langle \mathbf{s} + \mathbf{t} \rangle$ with respect to $\langle \mathbf{s} \rangle$ and $\langle \mathbf{t} \rangle$. By definition there are $a, b, c, d \in \Sigma(V)$ such that

1. no three of a, b, c, d are collinear;
2. $\langle \mathbf{s} \rangle \in (a * b) \cap (c * d)$;
3. $\langle \mathbf{t} \rangle \in (a * c) \cap (b * d)$;
4. $\langle \mathbf{s} + \mathbf{t} \rangle \in (a * d) \cap (\langle \mathbf{s} \rangle * \langle \mathbf{t} \rangle)$;
5. $\langle \mathbf{v} \rangle \in (b * c) \cap (\langle \mathbf{s} \rangle * \langle \mathbf{t} \rangle)$.

By the above lemma $b \notin \langle \mathbf{s} \rangle * \langle \mathbf{t} \rangle$, and thus $b \neq \langle \mathbf{s} \rangle$ and $b \neq \langle \mathbf{t} \rangle$.

Let $\mathbf{b} \in V \setminus \{\mathbf{0}\}$ be such that $\langle \mathbf{b} \rangle = b$. Since $b \notin \langle \mathbf{s} \rangle * \langle \mathbf{t} \rangle$, $\mathbf{b}, \mathbf{s}, \mathbf{t}$ are linearly independent.

I find the homogeneous coordinate of a. Since $\langle \mathbf{s} \rangle \in a * b$ and $\langle \mathbf{s} \rangle \neq b$, by (P4) of Lemma A.2 in [10] $a \in \langle \mathbf{s} \rangle * b$. Then there are x, y in $\mathcal{K}$ such that $a = \langle x\mathbf{s} + y\mathbf{b} \rangle$. Without loss of generality, I assume that $x = y = 1$, so $a = \langle \mathbf{s} + \mathbf{b} \rangle$.

I find the homogeneous coordinate of d. Since $\langle \mathbf{t} \rangle \in b * d$ and $\langle \mathbf{t} \rangle \neq b$, by (P4) of Lemma A.2 in [10] $d \in \langle \mathbf{t} \rangle * b$. Since $a = \langle \mathbf{s} + \mathbf{b} \rangle \neq \langle \mathbf{s} + \mathbf{t} \rangle$ and $\langle \mathbf{s} + \mathbf{t} \rangle \in a * d$, by (P4) $d \in a * \langle \mathbf{s} + \mathbf{t} \rangle$. Hence $d \in (\langle \mathbf{t} \rangle * b) \cap (a * \langle \mathbf{s} + \mathbf{t} \rangle)$. It follows that $d = \langle \mathbf{b} - \mathbf{t} \rangle$.

I find the homogeneous coordinate of c. Since $a = \langle \mathbf{s} + \mathbf{b} \rangle \neq \langle \mathbf{t} \rangle$ and $\langle \mathbf{t} \rangle \in a * c$, by (P4) $c \in a * \langle \mathbf{t} \rangle$. Since $d = \langle \mathbf{b} - \mathbf{t} \rangle \neq \langle \mathbf{s} \rangle$ and $\langle \mathbf{s} \rangle \in c * d$, by (P4) $c \in d * \langle \mathbf{s} \rangle$. Hence $c \in (a * \langle \mathbf{t} \rangle) \cap (d * \langle \mathbf{s} \rangle)$. It follows that $c = \langle \mathbf{b} + \mathbf{s} - \mathbf{t} \rangle$.

Finally, I find the homogeneous coordinate of $\langle \mathbf{v} \rangle$. Since $\langle \mathbf{v} \rangle \in (b * c) \cap (\langle \mathbf{s} \rangle * \langle \mathbf{t} \rangle) = (\langle \mathbf{b} \rangle * \langle \mathbf{b} + \mathbf{s} - \mathbf{t} \rangle) \cap (\langle \mathbf{s} \rangle * \langle \mathbf{t} \rangle)$, $\langle \mathbf{v} \rangle = \langle \mathbf{s} - \mathbf{t} \rangle$.

The 'If' Part: Assume that $\langle \mathbf{v} \rangle = \langle \mathbf{s} - \mathbf{t} \rangle$. Since V is of dimension at least 3, one can find $\mathbf{b} \in V$ such that $\mathbf{b}, \mathbf{s}, \mathbf{t}$ are linearly independent. Let $a = \langle \mathbf{s} + \mathbf{b} \rangle$, $b = \langle \mathbf{b} \rangle$, $c = \langle \mathbf{b} + \mathbf{s} - \mathbf{t} \rangle$ and $d = \langle \mathbf{b} - \mathbf{t} \rangle$. Then it's easy to check that 1 to 5 in the definition of harmonic conjugates hold. Therefore, $\langle \mathbf{v} \rangle = \langle \mathbf{s} - \mathbf{t} \rangle$ is a harmonic conjugate of $\langle \mathbf{s} + \mathbf{t} \rangle$ with respect to $\mathbf{s}$ and $\mathbf{t}$. □

Remark C.4. From this lemma and Theorem B.3 it's clear that in arguesian projective geometries, for every three points s, t and w, w has at most one harmonic conjugate with respect to s and t.

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