Bisimulation and path logic for sheaves: contextuality and beyond

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Abstract

In the setting of concurrency theory, Joyal, Winskel and Nielsen introduced a general notion of *path bisimulation* and showed that path bisimulations can be characterized as spans of open maps between presheaves. A modal logic for presheaves, called *path logic*, was shown to be expressive for such notion of bisimilarity. We consider the special case where the presheaves are defined over a topological space, and in particular where they are *sheaves*. We illustrate how natural properties of sheaves can be expressed in path logic and show how to encode the key concepts of the sheaf-theoretic treatment of contextuality [2]. We further investigate the associated notion of path bisimulation on sheaves, proving a characterization result for co-spans of open maps. Finally, we introduce a "hybrid path logic" and show that the notion of a sheaf itself can be captured by an axiom of this enriched logic.

1 Introduction

In computer science modal logic usually appears in *concurrency* theory, as a specification language for reasoning about concurrent systems. The most common approach here is to represent a concurrent system as a *labelled transition system*, and view these models as Kripke models for a multi-modal language, with different modalities corresponding to each label. Some standard modal logics used are linear time temporal logic, computation-tree logic or the modal μ-calculus. What drives this connection is the fact that modal logic is invariant

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for *bisimulations*, which are understood as behavioural equivalences of processes: models are bisimilar if they represent essentially the same process.

The so called "presheaf approach" to Concurrency Theory originated from the categorical outlook towards models of concurrency of [14]. In this paper many important models such as transition systems, synchronization trees and event structures are organized into categories and systematically related via adjunctions. Upon realization that each of these models was associated to a corresponding notion of path, in the seminal paper [8] Joyal, Winskel and Nielsen devised a representation of models of concurrency in terms of presheaves over suitable path categories, following the intuition that a model of concurrency consists of bundles of different paths glued together in a coherent way.

This perspective unveiled the possibility to define a general categorical notion of behavioural equivalence solely in terms of path preservation and path 'lifting'. While the former is usually inbuilt in the definition of morphism of the categories under examination, the latter had to be imposed, leading to the definition of open maps (which coincides on presheaves with the one in [7]). The desired general notion of bisimilarity was then at hand: two models of concurrency are deemed bisimilar if their presheaf representations are connected by a span of open maps. In a follow-up paper [15] it was observed that presheaves can themselves be regarded as transition system via the construction usually known as category of elements. A notion of bisimulation for these transition systems, called *path bisimulation*, was proved equivalent to the bisimulation in terms of span of open maps, and a modal logic called *path logic* was proposed and shown to be characteristic for path bisimulation. Path logic is a modal logic whose modalities are labelled by the morphisms of the path category. Given some conditions on the base category, presheaves can be thought of as generalized models of concurrency, with representables playing the role of path shapes. Path logic becomes then the natural choice of language for such models.

The main contribution of this paper is to bring the path logic developed for presheaves in concurrency theory, along with the closely related notion of a path bisimulation, into the context of topology. This is achieved by restricting our attention to the special case of presheaves over topological spaces, and in particular *sheaves*, a concept that is widely used in geometry and topology.[1]

The main test case for our approach is the recent sheaf-theoretic analysis of non-locality and contextuality pioneered by Abramsky and Brandenburger in [2]. Contextuality has proven to be a crucial feature of quantum phenomena and this line of research has since then been developed in a series of papers.[2] We will show how the pivotal concepts of that analysis can be captured by path logic; this in turn entails that such properties are invariant under path bisimulation. Such insights suggest that the study of Path Logic for sheaves could bear more fruits when developed in full generality. Moving the first steps in this direction, we prove some basic results concerning the apt notion of bisimulation in the context of sheaves, in particular a characterization result for co-spans of open maps, and suggest how to enrich Path Logic to capture the notion of sheaf itself.

[1] See [10] for a classic text.

[2] We will especially refer to [1, 2, 9].

2 Preliminaries

Suppose we are given a small category $\mathbb{C}$.[3] Along the lines of the presheaf approach surveyed in the previous sections, we assume $\mathbb{C}$ has an initial object, denoted with 0.

Definition 1. A presheaf $P : \mathbb{C}^{op} \to \mathbf{Set}$ is *rooted* if $P(0)$ is a singleton. The unique objects in $P(0)$ is called the *root*, and is usually denoted by r.

Note that due to the universal property of the initial objects all representable presheaves are rooted. Given a cardinal κ and a set Var of propositional variables, the syntax of *path logic* is defined by the grammar

$$\mathbf{PL}_\kappa(\mathbb{C}, \mathsf{Var}) \ni \varphi ::= e \mid \neg\varphi \mid \bigvee_{i \in I}\{\varphi_i\} \mid \langle f \rangle \varphi \mid \overline{\langle f \rangle}\varphi$$

where a ranges over Var, $f \in \mathbb{C}_1$ and the cardinality of I is less than κ. We define $\top = \bigwedge \emptyset$ and $\bot = \bigvee \emptyset$. The syntax of this logic is amenable for many interpretations, depending on the nature of the category $\mathbb{C}$; we will see in later sections how, when the base category is a poset, we can think of the modalities $\langle f \rangle \varphi$ and $\overline{\langle f \rangle}\varphi$ as extension and restriction of contexts.

In order to evaluate path logic on a presheaf, we first turn the presheaf into a labelled transition system:

Definition 2. Given a presheaf $P : \mathbb{C}^{op} \to \mathbf{Set}$, we can define a labelled transition system $\langle W, \{R_f\}_{f \in \mathbb{C}_1} \rangle$ via a variation of the category of elements, as described in [15]:

- $W := \{(x, C) | x \in P(C), C \in \mathbb{C}_0\}$
- $R_f := \{((x, C), (y, C')) | f : C \to C', P(f)(y) = x\}$

Definition 3. A *presheaf model M over* $\mathbb{C}$ is a presheaf P together with a valuation $V : \mathsf{Var} \to \mathcal{P}W$, where $\langle W, \{R_f\}_{f \in \mathbb{C}_1} \rangle$ is the LTS associated with P. The model M is said be rooted if P is a rooted presheaf.

We can now define the satisfaction relation for formulas of $\mathbf{PL}_\kappa(\mathbb{C}, \mathsf{Var})$ on a presheaf model $M = (P, V)$ over $\mathbb{C}$, essentially by doing standard Kripke semantics over the LTS associated with the presheaf P and treating $\overline{\langle f \rangle}$ as a backwards modality. For atomic propositions we have $M, (x, C) \vDash e$ iff $(x, C) \in V(e)$ and the clauses for connectives are as usual, while for the modalities put

- $M, (x, C) \vDash \langle f \rangle \varphi$ iff there is (y, C') such that $((x, C), (y, C')) \in R_f$ and $M, (y, C') \vDash \varphi$
- $M, (x, C) \vDash \overline{\langle f \rangle}\varphi$ iff there is (y, C') such that $((y, C'), (x, C)) \in R_f$ and $M, (y, C') \vDash \varphi$

The syntax and semantics of path logic were introduced in [8] to characterize the notion of strong path bisimulation:[4]

[3] We indicate with $\mathbb{C}_0$ and $\mathbb{C}_1$ the sets of objects and arrows respectively.

[4] We will always consider strong path bisimulation, hence we will drop the adjective strong henceforth.

Definition 4. A *path bisimulation* Z between rooted presheaf models $M_1 = (Q_1, V_1)$ and $M_2 = (Q_2, V_2)$ over $\mathbb{C}$ is a family $(Z_C)_{C \in \mathbb{C}_0}$ in which each Z_C is a set of pairs of objects (p, q) such that $p \in Q_1(C)$ and $q \in Q_2(C)$ satisfying the following conditions:

1. roots are related: $(r_1, r_2) \in Z_I$;
2. if $(p, q) \in Z_C$ then $p \in V_1(e)$ iff $q \in V_2(e)$
3. (forward) for $(p, q) \in Z_C$, if there is $p' \in Q_1(C')$ such that $Q_1(f)(p') = p$ for $f : C \to C'$ then there must be $q' \in Q_2(C')$ such that $Q_2(f)(q') = q$ and $(p', q') \in Z_{C'}$, and conversely reversing the role of the presheaves;[5]
4. (backward) if $(p, q) \in Z_C$ and $f : C' \to C$ then $(Q_1(f)(p), Q_2(f)(q)) \in Z_{C'}$.

In case $\mathsf{Var} = \emptyset$ we refer to Z simply as a path bisimulation between the presheaves Q_1, Q_2.

Path logic is expressive for path bisimulations:

Theorem 1 ([8]). *There is a path bisimulation between two rooted presheaf models M_1, M_2 over $\mathbb{C}$ iff the respective roots satisfy the same formulas of the path logic $\mathbf{PL}_{|\mathbb{C}_0|}(\mathbb{C}, \mathsf{Var})$.*

Definition 5. Given two presheaves $Q_1, Q_2 : \mathbb{C}^{op} \to \mathbf{Set}$, a natural transformation $\eta : Q_1 \to Q_2$ is an *open map* if, for every $f : C \to C'$ in $\mathbb{C}$, the following commuting square

$$\begin{array}{ccc} Q_1(C) & \xleftarrow{Q_1(f)} & Q_1(C') \\ {\scriptstyle \eta_C}\downarrow & & \downarrow{\scriptstyle \eta_{C'}} \\ Q_2(C) & \xleftarrow[Q_2(f)]{} & Q_2(C') \end{array}$$

is a quasi-pullback, that is, if $x \in Q_1(C)$ and $y \in Q_2(C')$ are such that $\eta_C(x) = Q_2(f)(y)$ then there exist $z \in Q_1(C')$ for which $\eta_{C'}(z) = y$ and $Q_1(f)(z) = x$.[6]

Theorem 2 ([8]). *A pair of rooted presheaves are path bisimilar if, and only if, they are related by a span of open maps.*

This concludes the review of the existing material; we now proceed to our contributions.

3 Path logic for sheaves

We now introduce path logic for sheaves and presheaves over a topological space and we study some properties of the associated notion of path bisimulation in the topological setting.

[5] The two directions of this condition are sometimes called "zig" and "zag" or "forth" and "back" conditions in Modal logic; here they are clustered together.

[6] This definition is equivalent to the one in term of path lifting, see [5].

From now on we shall only be interested in presheaves over a fixed topological space $\mathbb{X}$, that is, presheaves with base category the poset category of open sets $Open(\mathbb{X})$. In fact, we shall simply identify a topological space $\mathbb{X}$ with the associated poset category of open sets. Hence the path logic associated with a space $\mathbb{X}$ and a cardinal κ is just the logic $\mathbf{PL}_\kappa(\mathbb{X})$, where this notation is used just as before. Given a presheaf $P : \mathbb{X}^{op} \to \mathbf{Set}$, an inclusion $\iota : U \subseteq U'$ and an element $x \in P(U')$ we sometimes denote $P(\iota)(x) \in P(U)$ with $x|_U^P$; this is sometimes called the *restriction* of x to U. Elements of $P(U)$ for an open set U will be referred to as the *sections* of P over U. Elements of $P(X)$, where X refers to the whole space, are called *global sections*.

Definition 6. A *sheaf* P over $\mathbb{X}$ is a presheaf over $Open(\mathbb{X})$ satisfying the following conditions, for a given covering family $(U_i)_{i\in I}$ of an open U:

1. If $x, y \in P(U)$ are such that $x|_{U_i}^P = y|_{U_i}^P$ for all $i \in I$ then $x = y$, that is, if two elements agree on their restrictions to the members of the covering family then they must coincide. This condition is often called **locality**.
2. Suppose given a family $(x_i)_{i\in I}$ such that $x_i \in P(U_i)$ and $x_i|_{U_i\cap U_j}^P = x_j|_{U_i\cap U_j}^P$ for all $i, j \in I$ (the elements of the family 'agree on the intersections' of the covering family) then there exists a 'glueing' of such family, an element $x \in P(U)$ such that $x|_{U_i}^P = x_i$. This condition is known as **glueing**.

We denote by $\mathbf{Sh}(\mathbb{X})$ the category of sheaves over $\mathbb{X}$, and we let $\mathbf{PrSh}(\mathbb{X})$ denote the category of presheaves over $\mathbb{X}$. Note that every sheaf is a rooted presheaf.

Definition 7. A *sheaf model* is a presheaf model (P, V) such that P is a sheaf.

Since the morphisms in the base category are inclusion maps, and there is only one such inclusion for each pair of objects, in the corresponding path logic we denote with $\langle U, U'\rangle$ the modality associated to the inclusion $U \subseteq U'$ for U, U' opens in $\mathbb{X}$. A fairly natural way of interpreting the modalities in this setting, and more generally when the base category is a poset category, is in terms of *change of context*: the forward modality expresses the fact that a property holds when the context is extended from U to U', while the backward one handles the restriction from bigger to smaller contexts. This perspective on path logic is particularly apt for the sheaf-theoretic analysis of contextuality, as we will see in the next section.

For a formula φ of path logic and a rooted presheaf P, we may write $P \vDash \varphi$ to say that φ is true at the root. An example of a sheaf of immediate topological interest is the sheaf of sections of a covering map:

Definition 8. Let $\mathbb{X}$, and let $\pi : \mathbb{Y} \to \mathbb{X}$ be any continous map. Then π is called a *covering map* if, for every point u in $\mathbb{X}$, there is an open neighborhood U of u such that the inverse image $\pi^{-1}[U]$ is the union of disjoint sets $\{V_i\}_{i\in I}$ such that, for each $i \in I$, the restriction of π to V_i is a homeomorphism onto U. We say that U is *evenly covered*.

Fix a space $\mathbb{X}$. Then any covering map $\pi : \mathbb{Y} \to \mathbb{X}$ gives rise to a sheaf P, called the sheaf of sections of π. Given an open set U, the elements of $P(U)$ are continuous maps f mapping U into Y such that $\pi \circ f = \mathsf{Id}_U$. Restrictions of sections are given simply by function restriction.

3.1 Path logic and Contextuality

In this section we outline the framework put forward in [2] and describe how to encode the crucial notions of weak and strong contextuality. Suppose given a finite set X of variables, that in the quantum setting can be regarded as physical quantities, together with a set of possible outcomes O. Define a sheaf $\mathcal{E} : \wp(X)^{op} \to \mathbf{Set}$ mapping $U \subseteq X$ to O^U, the set of functions from U to O, while on arrows the function $\mathcal{E}(U \subseteq U')$ simply maps a function to the same function on the restricted domain. This functor is called the *sheaf of events*, as it associates to each set of variables all the possible assignments of outcomes to those variables.

Consider now a family of subsets of X, call it $\mathcal{M}$, such that the members of $\mathcal{M}$ form an antichain in $\wp(X)$ and $\bigcup \mathcal{M} = X$. Such family $\mathcal{M}$ is called *measurement cover* and represents the maximal sets of variables that can be tested together. For example the variables associated to position and momentum in a quantum system cannot be tested together: this and analogous constraints motivate this definition. Note that our inability to test two variables together does not preclude a priori the existence of a simultaneous assignment of values to both. Given a triple $\langle X, \mathcal{M}, O\rangle$, a subpresheaf $\mathcal{S}$ of $\mathcal{E}$ is called an *empirical model* if

1. $S(C) \neq \emptyset$ for all $C \in \mathcal{M}$: possible joint measurements give joint outcomes.
2. $S(U \subseteq U')$ is surjective if $U \subseteq U' \subseteq C$ for $C \in \mathcal{M}$: the model satisfies the no-signalling principle (a sheaf with this property is sometimes called *flasque* or *flabby*; here the condition is relative to $\mathcal{M}$).
3. For any family $\{s_C\}_{C\in\mathcal{M}}$ with $s_C \in \mathcal{S}(C)$ such that
$$\forall C, C' \in \mathcal{M} \quad s_C|_{C\cap C'} = s_{C'}|_{C\cap C'}$$
there exists a unique global section in $\mathcal{S}(X)$. This is the same as the glueing condition for sheaves, relativized to $\mathcal{M}$.

With respect to contextuality, the key properties of an empirical model are called *weak contextuality* and *strong contextuality*.[7] An empirical model is weakly contextual if there is a maximal context $C \in \mathcal{M}$ and a section $s \in \mathcal{S}(C)$ such that s cannot be extended to a global section in $\mathcal{S}(X)$. This means that there is a particular assignment of values to the measurements that cannot be reconciled with an assignment of values to all variables together.

We can express these properties in path logic in a natural way: given an empirical model $\mathcal{S}$, we turn the sheaf of events $\mathcal{E}$ into a sheaf model, introduce a propositional variable e and let the valuation $V_{\mathcal{S}}$ interpret e by

$$V_{\mathcal{S}}(e) = \{(s, C) \mid s \in \mathcal{S}(C)\}$$

The propositional variable a says about a value assignment s over the set of variables C that it represents a possible, real assignment according to the empirical model $\mathcal{S}$.[8] Representing an empirical model as a sheaf model in this way, we can now capture weak contextuality by the formula:

[7] Weak contextuality is called *logical* contextuality in [1].

[8] It is worth remarking that the notion of empirical model cannot itself be encoded in path logic. The first condition can be captured by $\bigwedge_{C\in\mathcal{M}}\langle\emptyset, C\rangle\top$, a formula stating that there

$$\bigvee_{C \in \mathcal{M}} \langle \emptyset, C \rangle (e \wedge [C, X] \neg e)$$

An empirical model is said to be strongly contextual if there is no global section: $\mathcal{S}(X) = \emptyset$. This condition states that there cannot be a simultaneous assignment of values to all variables; it can be encoded with the formula:

$$\neg \langle \emptyset, X \rangle e$$

A similar treatment of these notions, also casted in a modal language, was offered by Kishida in [9]. In said paper the labels for the modalities are measurements contexts, that is, compatible sets of measurements, and propositional variables are used to specify which outcome is associated to which measurement. The notion of weak and strong contextuality are captured via a formula *Det* expressing determinacy, namely a big disjunction encoding all the pairs measurements-outcomes and stating that one of them is the case.

We believe path logic constitutes an improvement over this line of work for three reasons.[9] First, the modalities of path logic contain all the identities of the objects of $\wp(X)$, thus we can encode a formula $[a]\phi$ from [9], meaning that ϕ will be the case whenever measurement a is performed, into the formula $[\emptyset, \{a\}]\phi$, stating that every section over a, hence any assignment of outcome to a brought about by measuring a, will satisfy ϕ. We can then use the propositional variables to associate measurements and outcomes to reproduce the formulas of Kishida within path logic. Second, path logic can express the *change* of measurement context: this allows for a characterization of contextuality in terms of the impossibility to extend to global sections, along the lines of the original paper [2]. Such characterization abstracts away from the particular specification of all the measurement-output pairs. Finally, path logic is not a logic designed specifically for this setting, but rather a very general language already studied in relation to concurrency. As known tools do, it not only solves the task at hand but also suggests connections with previous applications. In particular, since we know that path logic formulas are invariant under path bisimulations, expressing some of the central properties in the framework for contextuality in path logic leads to a simple but illuminating observation:

Corollary 1. *The properties of weak and strong contextuality are preserved by path bisimulations (and hence by spans of open maps).*

This connection, between contextuality and path bisimulation, seems to merit further investigation. In the setting of empirical models, the above corollary suggests that the properties of empirical models that are invariant for path bisimulation are typically properties concerning the degree and manner in which measurements are sensitive to context.

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is a section assigning outcomes to all the measurements in C, for all $C \in \mathcal{M}$. The second requirement is recorded by the formula $\bigwedge_{C \in \mathcal{M}} \bigwedge_{U \subseteq U' \subseteq C} [\emptyset, U]\langle U, U' \rangle \top$, expressing the fact that every section over U has a section over U' extending it. Note that we do not need infinitary logic to form these conjunctions, due to the finiteness of X. The third condition however cannot be rendered in path logic; this issue will be addressed in Section 5, where we will suggest how to express such properties with the use of nominals.

[9] These consideration address only the fragment without probabilities; we believe similar remarks can be made for the probabilistic case.

This raises some conceptual questions, in particular: is path bisimulation the "right" or most natural notion of equivalence for reasoning about such properties of empirical models? Related to this is the question if path logic is adequate as a language for reasoning about contextuality, i.e. can it express *all* properties of relevance to contextuality? We do not settle these questions here, but in the next section we shall see that there are indeed other possible candidates for notions of behavioural equivalence of sheaves.

3.2 Beyond Contextuality

The topological space that we considered in this section is a special one, the discrete topology over X. Nevertheless, the expressivity of path logic and the insight about the preservation of important properties via suitable bisimulations suggest that it might be fruitful to give a systematic treatment of path logic and path bisimulations in the wider context of sheaves. In fact, even the properties that we just discussed turn out to be of general significance.

In the case of the sheaf of sections associated with a covering map, strong contextuality describes how the space $\mathbb{Y}$ is related to $\mathbb{X}$ by the covering map π: the former looks locally like the latter, but not globally. An example is the covering map $\pi : \mathbf{R} \to \mathbf{S}_1$, where $\mathbf{R}$ is the real line and $\mathbf{S}_1$ is the unit circle. This covering map has no global section, although there are infinitely many sections over every smaller open set in $\mathbf{S}_1$. On the other hand, weak contextuality only forces the existence of some section that cannot be extended to a global one; an example of a covering map that is weak but not strongly contextual is $\pi : \mathbf{R} + \mathbf{R} \to \mathbf{R}$, the obvious projections of two disjoint copies of $\mathbf{R}$ into itself.

Another example of a general notion that can be encoded in path logic is *flabby sheaf*: a sheaf $P : \mathbb{X}^{op} \to \mathbf{Set}$ is said to be flabby if, for every inclusion $\iota : U \to X$, the restriction map $P\iota$ is surjective. It is not hard to see that the class of flabby sheaves over $\mathbb{X}$ is captured by the path logic formula $\bigwedge_{U \in Open(\mathbb{X})} [\emptyset, U]\langle U, X\rangle\top$. Flabby sheaves play a special role in homological algebra, see [4] for an overview.

This suggests that properties expressible in path logic may have more general importance in the context of topology. By previous results on path logic we can conclude that said properties are invariant under path bisimulation over presheaf models, giving a topological significance to path bisimulations.[10] We therefore study the connections between different candidates for bisimulations at the general level of sheaves over topological spaces.

4 Bisimulations for sheaves

In the general case path bisimulations could be neatly characterized in terms of spans of open maps in the category of presheaves over a fixed base category: path bisimulations correspond to spans of open maps, in the sense that rooted presheaves are path bisimilar if and only if they are related by a span of open

[10] A general study of the expressivity of path logic can be pursued along the lines of similar characterization results a là van Benthem. A straightforward modification of the usual argument shows that path logic is the fragment of many-sorted FOL (where the sorts are given by the objects of the base category) that is invariant for path bisimulation.

maps. Since open maps are special cases of coalgebra morphisms [11], spans of open maps correspond to what is known as an *Aczel-Mendler bisimulation*. Furthermore, it is not hard to show that two presheaves are related by a span of open maps iff they are related by a co-span of open maps (a pair of open maps from the respective presheaves with the same co-domain). Seeing open maps as coalgebra morphisms, this is in fact an instance of the coalgebraic concept of behavioural equivalence as a co-span in a category of coalgebras. So we have three equivalent descriptions of bisimilarity of presheaf models: path bisimilarity, spans of open maps and co-spans of open maps.

The situation for sheaves is less straightforward: the proofs of the equivalences mentioned above are not valid when we restrict attention to the category of sheaves over a space, i.e. when "span of open maps" means a span *in the category of sheaves* over the given space. So it seems we have three genuinely distinct candidates for behavioural equivalence of sheaves. It is easy to see that spans of open maps give rise to path bisimulations. Furthermore, co-spans of open maps give rise to spans of open maps (since the category of sheaves over a fixed space has pullbacks and open maps are stable under pullbacks). So we have:

$$\text{co-spans} \;\Rightarrow\; \text{spans} \;\Rightarrow\; \text{path bisimulations}$$

In the following sections, we shall look more closely at spans and co-spans of open maps, and relate them to special kinds of path bisimulations.

4.1 Path bisimulations and spans of open maps

We start by investigating the connection between path bisimulations and spans of open maps. It is certainly true that, for any pair of path bisimilar sheaves, we can construct a span of open maps connecting these sheaves. However, the presheaf at the "vertex" of this span may not be a sheaf, so the characterization of path bisimulations as spans of open maps is not internal to the category of sheaves over a given space. It turns out that a fairly natural condition on path bisimulations guarantees that the construction of a span of open maps from a bisimulation results in a proper sheaf:

Definition 9 (Locality axiom for path bisimulations)**.** Suppose we are given a covering $(U_i)_{i\in I}$ of an open set U, two sheaves $Q_1, Q_2 : Open(\mathbb{X})^{op} \to \mathbf{Set}$ and a path bisimulation Z between them. We say Z satisfies *Locality* if for all $p \in Q_1(U)$ and $q \in Q_2(U)$ such that $(p|^{Q_1}_{U_i}, q|^{Q_2}_{U_i}) \in Z_{U_i}$ for all $i \in I$, we have $(p, q) \in Z_U$.

The name of this property derives from the fact that it has the same shape of the locality condition on sheaves, with the difference that the equality is now replaced by the path bisimulation relation.

Proposition 1. *Suppose two sheaves on $\mathbb{X}$ are related by a path bisimulation satisfying the Locality axiom. Then these sheaves are also related by a span of open maps.*

In the context of empirical models, imposing conditions like Locality changes what sort of properties are invariants for bisimulations. So deciding what the "right" notion of bisimulation is in this setting amounts to making a decision regarding what sort of properties of empirical models we want to focus on. In particular the class of properties that are invariant for bisimulations with the Locality property will satisfy a corresponding locality constraint: in order to find what the invariant properties of a variable assignment v on a set of variables X is, given a cover $\mathcal{C}$ of X it suffices to investigate the invariant properties of each restriction of v to elements of the cover X.

At the present time we do not know whether the converse of Proposition 1 is true in general.[11]

4.2 Path bisimulations and co-spans of open maps

Another important notion of bisimulation in the coalgebra literature is given exactly by the dual of the span, i.e. a *co-span* of coalgebra morphisms. This is often called a *behavioural equivalence*. The same concept can be applied to sheaves, that is, we may consider co-spans of open maps rather than spans. It turns out that we can do better for co-spans than we did for spans: we can characterize the existence of a co-span of open maps precisely in terms of a concrete notion of path bisimulation. First, we introduce a Glueing property for path bisimulations, mimicking the corresponding axiom for sheaves in the same way as we defined the Locality property for bisimulations earlier:

Definition 10 (Gluing axiom for path bisimulations)**.** Suppose given a covering $(U_i)_{i\in I}$ of an open set U, two presheaves $Q_1, Q_2 : \mathit{Open}(\mathbb{X})^{op} \to \mathbf{Set}$ and a path bisimulation Z between them. The relation Z satisfies *Glueing* if the following is the case : whenever there are two families $(p_i)_{i\in I}$ and $(q_i)_{i\in I}$ such that $p_i \in Q_1(U_i)$ and $q_i \in Q_2(U_i)$ for all i and moreover for all i, j $(p_i|^{Q_1}_{U_i\cap U_j}, q_j|^{Q_2}_{U_i\cap U_j}) \in Z$ there exist two elements $p \in Q_1(U)$ and $q \in Q_2(U)$ such that $(p, q) \in Z$ and, for all i, $(p|^{Q_1}_{U_i}, q_i) \in Z$ and $(q|^{Q_2}_{U_i}, p_i) \in Z$.

Finally, we need a little technical side condition, that we borrow from [6]:

Definition 11. A path bisimulation Z is said to be *di-functional* if $(p, q) \in Z$, $(p', q) \in Z$ and $(p', q') \in Z$ entail $(p, q') \in Z$.

We can now state our characterization result:

Theorem 3. *Two sheaves Q_1 and Q_2 are related by a co-span of open maps*

$$Q_1 \to P \leftarrow Q_2$$

where P is a sheaf, if and only if they are related by a di-functional path bisimulation that satisfies the Gluing and Locality axioms.

Proof. From left to right, assume there are a sheaf P and two open maps $f : Q_1 \to P$ and $g : Q_2 \to P$. Define

$$Z_U = \{(p, q) \in Q_1(U) \times Q_2(U) | f_U(p) = g_U(q)\}$$

[11] The requirement of Locality can be lifted in some special circumstances, for example when the space satisfies the following property: given any open cover $(U_i)_{i\in I}$ of a non-empty open set U, and given any superset U' of U, there is a cover $(U'_i)_{i\in I}$ of U' such that $U'_i \cap U = U_i$ for all $i \in I$ and $U'_i \cap U'_j = U_i \cap U_j$ for all $i \neq j$. An example of a space with this property is natural numbers with the topology generated by all cofinite sets. This is however a very strong requirement, and typically many spaces of interest will not enjoy this property.

Clearly $Z = \bigcup_U Z_U$. We start with the forward condition of path bisimulation. Suppose $(p, q) \in Z_U$, $\iota : U \to U'$ and there is $p' \in U'$ such that $p'|^{Q_1}_U = p$. We need to show that there is $q' \in Q_2(U')$ such that $q'|^{Q_2}_U = q$ and $(p', q') \in Z_{U'}$, that is, $f_{U'}(p') = g_{U'}(q')$. By naturality we know that $f_{U'}(p')|^P_U = f_U(p'|^{Q_1}_U) = f_U(p) = g_U(q)$, so by weak pullback we obtain $q' \in Q_2(U')$ such that $q'|^{Q_2}_U = q$ and $f_{U'}(p') = g_{U'}(q')$.

For the backward condition suppose $(p, q) \in Z_U$ and $\iota : U' \to U$. We need to show that $p|^{Q_1}_{U'} = p'$ and $q|^{Q_2}_{U'} = q'$ are in relation: $(p', q') \in Z_{U'}$, that is, $f_{U'}(p') = g_{U'}(q')$. This follows immediately by the naturality of f and g and $f_U(p) = g_U(q)$. We proceed to check that the Locality property holds. Suppose given a covering $(U_i)_{i\in I}$ of U. Say there are $p \in Q_1(U)$ and $q \in Q_2(U)$ such that $(p|^{Q_1}_{U_i}, q|^{Q_2}_{U_i}) \in Z_{U_i}$ for all $i \in I$. We want to show that $(p, q) \in Z_U$, that is, that $f_U(p) = g_U(q)$. We have for every i that

$$
\begin{aligned}
f_U(p)|^P_{U_i} &= f_{U_i}(p|^{Q_1}_{U_i}) && (1)\\
&= g_{U_i}(q|^{Q_2}_{U_i}) && (2)\\
&= g_U(q)|^P_{U_i} && (3)
\end{aligned}
$$

where the first and last step are given by the naturality of f and g and the second is given by our assumption. Since $f_U(p)$and $g_U(q)$ agree on all the restrictions we can apply the locality property of the sheaf P to obtain $f_U(p) = g_U(q)$. Now for Gluing. Suppose given a covering $(U_i)_{i\in I}$ of U. Say there are two families $(p_i)_{i\in I}$ and $(q_i)_{i\in I}$ such that for all i $p_i \in Q_1(U_i)$ and $q_i \in Q_2(U_i)$ and moreover for all i, j $(p_i|^{Q_1}_{U_i\cap U_j}, q_j|^{Q_2}_{U_i\cap U_j}) \in Z$. We need to show that there are $p \in Q_1(U)$ and $q \in Q_2(U)$ such that $(p, q) \in Z$ and, for all i, $(p|^{Q_1}_{U_i}, q_i) \in Z$ and $(q|^{Q_2}_{U_i}, p_i) \in Z$. Our definition entails that $f_{U_i\cap U_j}(p_i|^{Q_1}_{U_i\cap U_j}) = g_{U_i\cap U_j}(q_j|^{Q_2}_{U_i\cap U_j})$. By naturality of f and g we get that $f_{U_i}(p_i)|^P_{U_i\cap U_j} = g_{U_j}(q_j)|^P_{U_i\cap U_j}$. When $i = j$ this means that $f_{U_i}(p_i) = g_{U_i}(q_i)$, hence we have a family of objects $(f_{U_i}(p_i) = g_{U_i}(q_i))_{i\in I}$ such that $f_{U_i}(p_i) = g_{U_i}(q_i) \in P(U_i)$ and the elements of this family agree at the intersections. Thus we can apply glueing in P to obtain $t \in P(U)$ such that $t|^P_{U_i} = f_{U_i}(p_i) = g_{U_i}(q_i)$ for all i.

Pick an index i; by the fact that f and g are open maps we obtain by weak pullback $p \in Q_1(U)$ and $q \in Q_2(U)$ such that $f_U(p) = t = g_U(q)$, which means $(p, q) \in Z_U$, and $p|^{Q_1}_{U_i} = p_i$ and $q|^{Q_2}_{U_i} = q_i$. This entails that $f_{U_i}(p|^{Q_1}_{U_i}) = t|^P_{U_i} = g_{U_i}(q_i)$ and $g_{U_i}(q|^{Q_2}_{U_i}) = t|^P_{U_i} = f_{U_i}(p_i)$. Now take $j \neq i$. We have that $f_{U_j}(p|^{Q_1}_{U_j}) = f_U(p)|^P_{U_j} = t|^P_{U_j} = g_{U_j}(q_j)$, where the first step is by naturality and the last is a consequence of gluing. Similarly we can show that $g_{U_j}(q|^{Q_1}_{U_j}) = f_{U_j}(p_j)$.

Difunctionality is immediate by the definition of the relation: if p, q, p' and q' are all sent to the same object then the relation will hold between p' and q'.

From right to left, suppose there is a difunctional path bisimulation between Q_1 and Q_2 satisfying Locality and Glueing. Take Eq_U to be the smallest equivalence relation containing Z_U. Define the sheaf P as follows

$$
\begin{aligned}
U &\mapsto Q_1(U) + Q_2(U)\backslash Eq_U\\
\iota : U \to U' &\mapsto P(\iota) : Q_1(U') + Q_2(U')\backslash Eq_{U'} \to Q_1(U) + Q_2(U)\backslash Eq_U
\end{aligned}
$$

where $P(\iota)([x]) = [Q_l(\iota)(x)]$ if $x \in Q_l(U')$, for $l \in \{1,2\}$. Notice that this definition automatically makes $[-] : Q_l \to P$ a natural transformation for $l = 1$ and $l = 2$. We sometimes omit the subscript when it is clear.

We first show that $P(\iota)$ is well defined. Suppose $x \neq y$ and $[x] = [y]$, we want to show that $P(\iota)([x]) = P(\iota)([y])$. Since $[x] = [y]$, only two scenarios can occur. Suppose $(x,y) \in Z_{U'}$. Then $P(\iota)([x]) = [Q_1(\iota)(x)]$ and $P(\iota)([y]) = [Q_2(\iota)(y)]$. By backward condition of path bisimulation we obtain from $(x,y) \in Z_{U'}$ that $(x|_U^{Q_1}, y|_U^{Q_2}) \in Z_{U'}$, so $[Q_1(\iota)(x)] = [Q_2(\iota)(y)]$ and we are done. Now suppose x and y are in relation because of a zig-zag of relations in $Z_{U'}$. Applying our previous argument to every pair in $Z_{U'}$ we get, by transitivity of equality, that $[Q_1(\iota)(x)] = [Q_2(\iota)(y)]$. We now show that $[-]$ is an open map. Suppose $\iota : U \to U'$ and say that there are $x_1 \in Q_1(U)$ and $[x_2] \in P(U')$ such that $[x_1] = P(\iota)([x_2])$. We know that $[x_1] = P(\iota)([x_2]) = [x_2|_U^{Q_l}]$, for some $l \in \{1,2\}$. Hence there is a zig-zag of Z_U edges between x_1 and $x_2|_U^{Q_l}$. Starting from x_2, we apply the forward condition to all the edges of the zig-zag (this is an argument by induction, similar to the one in [8]); in this way we obtain an element $x' \in Q_1(U')$ such that there is a zig-zag of $Z_{U'}$ edges between x_2 and x', hence $[x_2] = [x']$, and $x'|_U^{Q_1} = x_1$.

We proceed to show that P is a sheaf, beginning with locality. Suppose given a covering $(U_i)_{i \in I}$ of U. Consider $[x], [y] \in P(U)$ such that $[x]|_{U_i}^P = [y]|_{U_i}^P$ for all i. Because the bisimulation includes the roots, we can always assume that each equivalence class $[x]_U$ contains at least a member of $Q_1(U)$ and a member of $Q_2(U)$. So we can take $x \in Q_1(U)$ and $y \in Q_2(U)$. By $[x]|_{U_i}^P = [y]|_{U_i}^P$ we can infer that $[x|_{U_i}^{Q_1}] = [y|_{U_i}^{Q_2}]$ for every i. By difunctionality we can conclude that $(x|_{U_i}^{Q_1}, y|_{U_i}^{Q_2}) \in Z_{U_i}$ for all i. The Locality condition on Z allows us to infer that $(x,y) \in Z_U$, hence $[x] = [y]$. Finally we prove that P has the gluing property. Suppose given a covering $(U_i)_{i \in I}$ of U. Suppose there is a family $([x_i])_{i \in I}$ with $[x_i] \in P(U_i)$ such that, for all $i, j \in I$, $[x_i]|_{U_i \cap U_j}^P = [x_j]|_{U_i \cap U_j}^P$. We want to find $[x] \in P(U)$ such that $[x]|_{U_i}^P = [x_i]$ for all i. We know $[x]_U$ contains at least a member of $Q_1(U)$ and a member of $Q_2(U)$. So we can infer that there are two families $(p_i \in Q_1(U_i))_{i \in I}$ and $(q_i \in Q_2(U_i))_{i \in I}$ such that $[p_i] = [q_i] = [x_i]$. So from $[x_i]|_{U_i \cap U_j}^P = [x_j]|_{U_i \cap U_j}^P$ we infer that $[p_i|_{U_i \cap U_j}^{Q_1}] = [q_j|_{U_i \cap U_j}^{Q_2}]$. By difunctionality it must be that $(p_i|_{U_i \cap U_j}^{Q_1}, q_j|_{U_i \cap U_j}^{Q_2}) \in Z_{U_i \cap U_j}$, and this for all i and j. By the gluing property of the bisimulation we conclude that there are $p \in Q_1(U)$ and $q \in Q_2(U)$ such that $(p,q) \in Z_U$ and for all i $p|_{U_i}^{Q_1} = q_i$ and $q|_{U_i}^{Q_2} = p_i$. Take $[x] = [p] = [q]$: we have for all i that $[x]|_{U_i}^P = [p]|_{U_i}^P = [p|_{U_i}^{Q_1}] = [q_i] = [x_i]$. This concludes the proof. □

It is easy to see that every path bisimulation is contained in a difunctional path bisimulation, its "difunctional closure". But since we cannot assume that the difunctional closure operation preserves the Gluing and Locality axioms, we have to state difunctionality as an explicit premise of the previous theorem.

5 A hybrid path logic

While the path logic for presheaves has a natural interpretation on presheaves over a topological space, it is not obvious that this is the right modal language for reasoning about these structures. In fact, it is not hard to see that the basic property of being a *sheaf* over the base space $\mathbb{X}$ is not definable in the path logic corresponding to $\mathbb{X}$, since this property is easily seen not to be preserved by path bisimulations. Any example of a sheaf that is path bisimilar to a presheaf that is not a sheaf will show this, and such an example can easily be obtained using a presheaf that does not satisfy locality. We leave out the details.

So in order to remedy this, we need to extend the path logic with extra expressive power, but we want to do this as gently as possible. The suggestion that presents itself is to go to *hybrid logic*. We define the syntax of *hybrid path logic* for $\mathbb{X}$, a regular cardinal κ and over a set of nominals N, by the following grammar:

$$\mathbf{HPL}_\kappa(\mathbb{X}, N, \mathsf{Var}) \ni \varphi ::= e \mid i \mid @_i\varphi \mid \bigvee \Gamma \mid \neg\varphi \mid \langle U, V\rangle\varphi \mid \overline{\langle U, V\rangle}\varphi$$

Here i ranges over N, U and V range over open sets of $\mathbb{X}$, e ranges over Var and Γ ranges over sets of formulas of size $< \kappa$.

A *presheaf model* for this language is a rooted presheaf P over $\mathbb{X}$ together with a map $A : N \to \Sigma\{P(U) \mid U \in \mathit{Open}(\mathbb{X})\}$ and a valuation V for Var. Truth conditions of formulas in a model (P, A) at some $w \in P(U)$ are defined as before, with the added clauses:

- $(P, A, w) \vDash i$ if and only if $A(i) = w$
- $(P, A, w) \vDash @_i\varphi$ if and only if $(P, A, A(i)) \vDash \varphi$

Definition 12. We say that φ is true in (P, A, V), written $(P, A, V) \vDash \varphi$, if $(P, A, V, r) \vDash \varphi$ where r is the root of P. We say that φ is valid in P, written $P \vDash \varphi$, if $(P, A, V) \vDash \varphi$ for every A and every V.

Now, given a space $\mathbb{X}$, which we assume to be infinite, let κ be a regular cardinal greater than 2^ξ where ξ is the number of open sets of $\mathbb{X}$. Assuming the axiom of choice we can take this to be the successor of 2^ξ. Let N be a set of nominals with $2^\xi \leq |N| < \kappa$. Then consider the following formulas of $\mathbf{HPL}_\kappa(\mathbb{X}, N)$:

Loc: For any cover $\{U_i\}_{i \in I}$ of an open set U of $\mathbb{X}$, pick nominals $j, k, \{l_i\}_{i\in I}$ and construct the formula:

$$\bigwedge_{i\in I} @_{l_i}\langle U_i, U\rangle j \wedge \bigwedge_{i\in I} @_{l_i}\langle U_i, U\rangle k \to @_j k$$

Then we define Loc to be the conjunction of all these formulas, corresponding to all the covers of open sets in $\mathbb{X}$. The conjunction is well defined since there are at most 2^ξ covers to consider.

Glu: For any cover $\{U_i\}_{i\in I}$ of an open set U of $\mathbb{X}$, pick nominals $k, \{l_i\}_{i\in I}$ and construct the formula:

$$\bigwedge_{i,j\in I} @_{l_i}\overline{\langle U_i \cap U_j, U_i\rangle}\langle U_i \cap U_j, U_j\rangle l_j \to \langle \emptyset, U\rangle \bigwedge_{i\in I} \overline{\langle U_i, U\rangle} l_i$$

We take Glu to be the conjunction of all these formulas.

The proof of the following result is a simple check, but we list it as a theorem since we think it has some importance.

Theorem 4. *A rooted presheaf P is a sheaf if, and only if, $P \vDash \mathsf{Loc} \wedge \mathsf{Glu}$.*

It follows, of course, that validity of formulas in hybrid path logic is not preserved by path bisimulations. However, truth in a model is easily seen to be preserved by a natural extension of path bisimulations:

Definition 13. Let (P, A, V) and (P', A', V') be presheaf models. Then a *nominal path bisimulation* is a path bisimulation between (P, V) and (P', V') such that, for every nominal i, $A(i)$ is related to x by this path bisimulation if and only if $x = A'(i)$, and vice versa.

Proposition 2. *Formulas of* $\mathbf{HPL}_\kappa(\mathbb{X}, N, \mathsf{Var})$ *are invariant under nominal path bisimulations.*

6 Conclusions and further work

We started showcasing the expressing power of path logic by encoding the key concepts of the sheaf-theoretic approach to contextuality proposed in [2]. This encoding improves on the existing one and suggests that path logic might be the right logic to express in general the extension and restriction of contexts. Not only: the same formulas treated in the contextuality approach turned out to have a general significance for topology, for example encoding interesting properties of covering spaces. From previous results we know that such formulas will be invariant for path bisimulation on presheaves. Nevertheless,in the context of sheaves the notion of path bisimulation is more involved.

In this setting we give sufficient condition for the existence of a span of open maps and sufficient and necessary conditions for the existence of a co-span, therefore characterizing when two sheaves are behaviourally equivalent. It is also intriguing to note that said conditions, called here Locality and Gluing, resemble the sheaf properties of locality and gluing, when equality is replaced with the bisimulation relation. We outlined the relationship between spans and co-spans in presheaves and sheaves. Finally, we suggested how path logic has to be enriched in order to capture the key features of sheaves.

In future work we intend to address some of the questions left open and mentioned in the text; we list them here for easy reference:

- In light of the fact that sheaves are not definable in **FOL**, prove correspondence results over sheaf models.
- Find the conditions on path bisimulations that are equivalent to the existence of a span of open maps on sheaves.
- Find the conditions under which existence a span of open maps is equivalent to the existence of a co-span of open maps on sheaves.
- Axiomatize the class of transition systems arising from sheaves with hybrid path logic.
- Explore the connection with the usual topological semantics for modal languages [12, 13] and the sheaf semantics for first-order modal logic [3].

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