Individual Choice Sequences in the Work of L.E.J.Brouwer

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Abstract

Choice sequences are sequences not completely determined by a law. We state that the introduction of particular choice sequences by Brouwer in the late twenties was not recognised as such. We claim that their later use in the method of the creative subject was not traced back to this original use of them and has been misinterpreted. We show where these particular choice sequences appear in the work of Brouwer and we show how they should be handled.

Key words: intuitionism, choice sequences, the method of the creative subject.

I. Introduction

Brouwer made his first steps in the foundations of mathematics in his thesis of 1907 ([Br 07]). In the decade after this his main concern was topology. He returned to the foundations in 1918 when he presented with [Br 18] a reconstruction of mathematics along the lines set out in his thesis. The striking feature of this reconstruction is the introduction of the real numbers by infinitely proceeding sequences with terms chosen more or less freely from objects already constructed: *choice sequences*. He kept publishing on the subject until 1930.

After a lapse of more than fifteen years he started to publish again on the subject in 1948. Characteristic for his papers in this second period of intuitionistic activity is a technique for deriving counterexamples against consequences of the law of the excluded middle. The technique has generally been considered to be radically new; in this period it is known as the method of the creative subject.

Extensive research has been done on choice sequences. The standard text on the subject is A.S.Troelstra's monograph [Tr 77]. This work is based on ideas originating with G.Kreisel. It contains a considerable amount of technical work on formal systems of classes of choice sequences. For these systems Troelstra proves elimination theorems: a sentence with quantification over choice sequences can be translated into an equivalent sentence without choice sequences. So, what can be proved with choice sequences, can also be proved without them.[i]

The formal systems of [Tr 77] are not relevant for the method of the creative subject. In order to reconstruct this method, Kreisel and Troelstra developed the theory of the idealised mathematician. From seemingly straightforward assumptions concerning the properties of the idealised mathematician, Troelstra derived a paradox, which could not be resolved satisfactorily (see [TD 88], pp. 842-846 and our analysis in [Ni 87]).

The subject of this paper is Brouwers use of *individual* choice sequences. An individual choice sequence (our terminology) is a particular sequence with an incomplete description for the construction of its terms. They have not been an object of study after Brouwer; the research was concentrated on global properties of choice sequences. They do not occur in [Tr 77], nor in [Tr 82], a survey article on Brouwer's use of choice sequences. Our aim is twofold. Firstly we want to show where individual choice sequences appear in the work of Brouwer, secondly we want to show how they should be treated.

In Section 3 we give the first example known to us that Brouwer deliberately uses an individual choice sequence. It is from a lecture in 1927, but not published until 1991 ([Br 91]). We also give the first in print, which is in [Br 30]. Further we give a fragment from [Br 48], which we think shows without doubt, that Brouwer is exploiting individual choice sequences, as introduced in 1927, in the method of the creative subject.

All three cited fragments of choice sequences deal with the same result: the non-equivalence of the apartness relation and inequality. Only in [Br 48] does he give a proof, which we shall analyse in the Section 4. In our view the properties of a choice sequence are determined during the construction, which is the future. So reasoning about a choice sequence should involve principles of the logic of time. We shall use a very obvious one in our reconstruction. We do not use an idealised mathematician, so neither its paradoxical features.

From the current reconstructions of the method of the creative subject Kripke's Scheme (KS) has been derived. KS is widely accepted in intuitionistic research as a reasonable principle. In Section 5 we show that their is no basis for KS in Brouwers creative subject arguments.

We start in Section 2 with the introduction of the basic intuitionistic notions. The most important one is that of a *spread*. Its discovery made the intuitionistic reconstruction possible, since it yields the construction of uncountably many objects.[ii] Brouwer's use of words is confusing. For a spread he used the word "Menge" which is German for set. Closer to the classical notion of set is Brouwers notion of *species*, a property applicable to mathematical objects. Real numbers are introduced as a species of species of elements of some spread. We describe the introduction with one particular spread, and mention another, with which one can understand the original texts we give in Section 3.

II. Definitions

In Brouwers intuitionism mathematics consists of mental constructions of the human individual. The prime material for these constructions is the sequence of natural numbers. Of this sequence the first element is given and every next one is constructable from its predecessors. They have their origin in our perception of the move of time.

From the natural numbers the integers and rationals can be constructed in a standard way. The resulting mathematics up to this point is contained in classical mathematics. It is the introduction of real numbers by means of elements of a spread that gives intuitionism its special character.

A *spread* is a *law* that regulates the construction of infinite sequences a_{n_0}, a_{n_1}, a_{n_2} … . It consists of two parts. The first decides whether a natural number m is admitted as v+1-th index n_v in a row of admitted choices n_0, n_1, …, n_{v-1} The second part assigns an already constructed mathematical object a_{n_v} to an admitted choice n_v. For each row of admitted indices n_0, n_1, …, n_{v-1} their exists at least one natural number m such that m is admitted as v+1-th index n_v. Within the limitations of the spread law the choice of indices is free, but this freedom can be further restricted at any moment of the construction of a sequence, even completely, so that a law determines the remainder of the sequence. A sequence constructed according to a spread law is called an *element of the spread*. We reserve the name *lawlike sequences* for those spread elements that are determined by a law from their first term onwards. All other sequences we call *choice sequences*. In the original fragments of Brouwer below we shall see examples of choice sequences.

A *species* is a property that can be possessed by a mathematical entity. If the entity has the property, the entity is called an *element of the species*. A spread is a species, but a species need not be a spread. Real numbers are introduced as a species of species of elements of a certain spread.

Let (q_n) be an enumeration of the rational numbers. The spread S is such that in a row of indices n_0, n_1, n_2,... each natural number m is admitted as choice for n_0; the assignment a_{n_0} to n_0 is q_{n_0}. Further, m is admitted as choice for n_{v+1} as successor of n_v in the row of indices iff $| q_{n_v} - q_m | < 2^{-v-1}$; the assignment to n_{v+1} for that choice is q_m.

Elements of S are convergent sequences of rationals, which we shall denote by (a_n), (b_n), etc. Between two elements (a_n) and (b_n) of S we define the relation R by $(a_n)\ R\ (b_n)$ iff $\forall k\ \exists n\ \forall m{>}n\ | a_m - b_m | < 2^{-k}$. R is an equivalence relation. The real numbers are the elements of the species of equivalence classes of this relation. They form the *full continuum*, also called the *continuum*. The *reduced continuum* is the species of equivalence classes of R restricted to the lawlike elements of S. We shall denote real numbers by a, b, …x, y,… . Arithmetical operations and relations can be defined on real numbers via the representatives, as in the following definitions.

$a < b$ iff for representatives (a_n) and (b_n) $\forall k\ \exists n\ \forall m\ (b_{n+m} - a_{n+m}) > 2^{-k}$,
$a > b$ iff $b < a$,
$a \# b$ iff $a < b$ or $b < a$.

In intuitionism one should distinguish between the apartness relation a # b and the inequality relation $a \neq b$. The latter expression means $\neg\, a = b$, i.e. the supposition of a=b is contradictory. It is in the cited fragment with choice sequences below, that Brouwer proves the non-equivalence of these relations.

An alternative way to introduce the real numbers is by a spread of which the elements are infinite sequences of nested intervals of rational numbers. The n-th term of such an element is a $\lambda^{(n)}$-interval which is encompassed by its predecessor. A $\lambda^{(n)}$-interval has the form $[a.2^{-n-1}, (a + 2).2^{-n-1}]$, with *a* an integer, so the length of the terms converges to 0. The real numbers are defined as the species of the species of "coinciding" elements. It is this definition that Brouwer uses in the first examples we cite below. There is no essential difference between the two definitions of real numbers.

III. Individual choice sequences – where they appear

Brouwer introduced and started to exploit the distinction reduced versus full continuum in his Berlin Lectures of 1927. In the following way he proved that the above defined relation < is not a "full" order on the reduced continuum.

> "[…] Weiter bezeichnen wir mit k_1 die kleinste natürliche Zahl n mit der Eigenschaft, daß die n-te bis (n+9)-te Ziffer der Dezimalbruchenentwicklung von π eine Sequenz 0123456789 bilden und dazu definieren wir wie folgt den Punkt r des reduzierten Kontinuums: Das n-te λ-Intervall λ_n ist ein symmetrisch um den Nullpunkt gelegenes $\lambda^{(n-1)}$-Intervall, solange $n<k_1$; für $n \geq k_1$ aber ist λ_n das symmetrisch um den Punkt $(-2)^{k1}$ gelegene $\lambda^{(n)}$-Intervall."
>
> "[…] Weiter ist der zu r gehörende Punktkern des reduziertes Kontinuums, solange die Existenz von k_1 weder bewiesen noch noch ad absurdum geführt ist, weder =0, noch >0, noch <0. Bis zum stattfinden einer dieser beiden Entdeckungen ist also das reduzierte Kontinuum nicht vollständig geordnet." ([Br 91], pp.31-32)

Given an algorithm calculating π the sequence generating r in the passage above is lawlike, in Brouwers words a *sharp point*. So r belongs to the reduced continuum. The technique applied is not new. Already in [Br1908] Brouwer used unknown properties of the decimal expansion of π to demonstrate the unreliability of the law of the excluded third. The technique is generalised in [Br 29], the text of the first of two lectures he gave in Vienna in 1928. The property of a natural number being equal to k_1 above, is an example of a *fleeing property*: for each natural number it is decidable whether it possesses it or not, no natural number possessing the property is known, the assumption of the existence of a number possessing the property is not known to be contradictory. The critical number λ_f (in German "Lösungszahl") of a fleeing property f is the smallest natural number possessing f. For a fleeing property and its λ_f we can define a sequence (a_n) as above for k_1. The German name for the thus generated real number is "duale Pendelzahl".

Let us return to [Br 91]. That the full continuum is not ordered had been shown in the following way:

> "[...] Dazu betrachten wir eine mathematische Entität oder Species S, eine Eigenschaft E, und definieren wie folgt den Punkt s des Kontinuums: Das n-te λ-Intervall λ_n ist eine symmetrisch um den Nullpunkte gelegenes $\lambda^{(n-1)}$-Intervall, so lange man die Gültigheit noch die Absurdität von E für S kennt, dagegen ist es ein symmetrisch um den Punkt 2^{-m}, bzw. um den Punkt -2^{-m} gelegenes $\lambda^{(n)}$-Intervall ,

wenn n≥m und zwischen der Wahl des (m-1)-ten und der Wahl des m-ten Intervalles ein Beweis der Gültigkeit bzw. der Absurdität von E für S gefunden worden ist."

" [...] Alsdann ist der zu s gehörende Punktkern des Kontinuums ≠0, aber solange man weder die Absurdität noch die Absurdität der Absurdität von E für S kennt, weder >0 noch <0. Bis zum stattfinden einer dieser beide Entdeckungen kann also das Kontinuum nicht geordnet sein." ([Br 91], pp.31-32)

The sequence generating s above depends on whether or not some proof will be found, so it is not lawlike. It is the first time that Brouwer deliberately gives an example of a choice sequence, in his own words an "unfertiges Element".[iii] He does not give further argument here for the result. We delay our first comment till the next choice sequence, which is in [Br 30], the text of his second Vienna lecture of 1928.

In [Br 30] Brouwer examines the intuitionistic continuum with regards to seven properties, all valid for the classical continuum. Each time he distinguishes between the reduced and the full continuum. He uses a lawlike sequence when it is sufficient for his purpose, like in the first example below, where he applies the apparatus introduced in [Br 29].

"Daß das Kontinuum (und ebenso das reduzierte Kontinuum) **nicht diskret ist**, folgt z. B. daraus, daß die Zahl $1/2 + p_f$, wo p_f die duale pendelzahl der fliehenden Eigenschaft *f* vorstellt, weder gleich 1/2 noch von 1/2 verschieden ist." ([Br 30], CW p.435).

But if necessary he does use a choice sequence:

"Daß das Kontinuum durch die der Anschauung entnommene Reihenfolge ihrer Elemente **nicht geordnet ist**, erweist sich am Elemente *p*, für dessen bestimmte konvergente Folge $c_1, c_2, \ldots$ c_1 im Nullpunkt und jedes $c_{v+1} = c_v$ gewählt wird, mit der einzige Ausnahme, daß ich, sobald von einer bestimmten fliehenden Eigenschaft *f* mir eine Lösungszahl λ_f bekannt wird, das nächste c_v gleich -2^{-v-1} wähle, und daß ich, sobald mir eine Beweis der Absurdität dieser Lösungszahl bekannt wird, das nächste c_v gleich 2^{-v-1} wähle. Dieses Element *p* ist von Null verschieden, ist aber trotzdem weder kleiner als Null noch größer als Null." ([Br 30], CW p.436).

Note the difference between these two sequences. If for some natural number m it is proved that $m=\lambda_f$, then the number defined in the first fragment becomes $1/2 + 2^{-m}$. Such a relation does not follow from the definition in the second case.

Since the Berlin lecture notes were not published until 1991, the sequence used in the second fragment is the first choice sequence of Brouwer in print. It is a peculiar fact that this sequence has never been recognised as something special.[iv] Whether the sequence is the same as in his Berlin lecture, depends on whether one may conclude from the definition of a fleeing property that it is *non-tested* (neither ¬A nor ¬¬A is known). An indication that Brouwer intended to give the same example is that in [Br 48], which we treat below, he gave the fleeing property used in the definition of r in [Br 91] as an example of a non-tested proposition. As in [Br 91], there is no further proof of the above result, or of the results obtained with other choice sequences.[v]

As any infinite sequence in intuitionism, a choice sequence is given by a description to construct its terms. But the description does not determine the sequence algorithmically. In the examples above the values of the terms are made to depend on future mathematical experiences of the one who constructs the sequence. In the examples above Brouwer denotes with *I* or *we* the one who constructs it.

It is remarkable that Brouwer fell into inactivity after the introduction of these choice sequences. After [Br 30] he hardly published anything for more then fifteen years.

After the Second World War he became active again. His papers of that period are characterised by what has been considered as a new method, which is known as *the method of the creative subject*. We treat the example from [Br 48]. His proof of the non-equivalence of the apartness and inequality relation starts with the following definition:

> "Let α be a mathematical assertion that cannot be tested, i.e. for which no method is known to prove either its absurdity or the absurdity of its absurdity.
>
> Then the creating subject can, in connection with this assertion α, create an infinitely proceeding sequence a_1, a_2, a_3, ... according to the following direction: As long as, in the course of choosing the a_n, the creating subject has experienced neither the truth, nor the absurdity of α, a_n is chosen equal to 0.
>
> However, as soon as between the choice of a_{r-1} and a_r the creating subject has obtained a proof of the truth of α, a_r as well as a_{r+v} for every natural number v is chosen equal to 2^{-r}. And as soon as between the choice of a_{s-1} and a_s the creating subject has experienced the absurdity of α, a_s, as well as a_{s+v} for every natural number v is chosen equal to -2^{-s}.
>
> This infinitely proceeding sequence a_1, a_2, a_3, ... is positively convergent, so it defines a real number ρ." ([Br 48], CW p.478)

There has been discussion about what the expression *creating subject* could mean. Remark that in Brouwers view mathematics consists of mental constructions, created by the human individual. A definition, e.g. as above, cannot be else then a description of a construction, to be carried out by that individual. As it seems to us, Brouwer denotes with the expression *creating subject* the individual who can carry out the construction, where he used *we* or *I* before. Interpreted this way, the definition in [Br 48] is the same as in the cited fragments of choice sequences above.

Brouwer himself remarks in the introduction of [Br 48] that he uses this example in his lectures from 1927, and there is no other candidate in these lectures.

We conclude that Brouwer applies individual choice sequences, as introduced in [Br 91], in the method of the creative subject.

IV. Individual choice sequences – how to treat them

Contrary to [Br 91] and [Br 30], Brouwer gives in [Br 48] a detailed proof of the stated result, which we shall analyse in this section. We shall take the expression *the creating subject* to denote *any mathematician*, which could be ourselves. Therefore we shall use *we* in our reconstruction, as Brouwer did in his early use of choice sequences.

Thus the sequence defining ρ in the definition given in [Br 48] above is a sequence we *can* construct. We reason about this sequence before the construction actually has started, we just use the definition. Since the terms of the sequence depend on our future mathematical experience, we need principles for reasoning about the future. We start with the formalisation of such principles.

Suppose the future to be divided into ω discrete stages. For a mathematical assertion φ and a natural number *n*

$$\Box_n \varphi$$

is defined as: on the n-th stage from now we shall have a proof of φ. A proof of $\Box_n\varphi$ may depend on information coming free before stage n, but it may also be the case that $\Box_n\varphi$ holds because we already have a proof of φ now, since we suppose that a proof remains valid in the move of time. This is expressed by adopting for all natural numbers *m* and *n*, and for any mathematical assertion φ

(1) $\Box_n \varphi \Rightarrow \Box_{n+m}\varphi$.

Let the present be stage 0, which is expressed by

(2) $\varphi \Leftrightarrow \Box_0 \varphi$.

Then obviously, for any mathematical assertion

(3) $\varphi \Rightarrow \exists n \Box_n \varphi$ is valid.

From (3) immediately follows

(4) $\neg \exists n \Box_n\varphi \Rightarrow \neg \varphi$.

The principles above are general principles for reasoning about future mathematical activity. They will be used in the reconstruction below; the basic step is (4).

We repeat the definition of ρ in [Br 48] with *we* instead of *creating subject* ; A is a mathematical proposition which is not tested, i.e. we have neither a proof of ¬A nor of ¬¬A *now*.

As long as, while choosing values for (a_n), we neither have attained a proof of A nor of ¬A, we take a_n=0. If we find a proof of A between the choice of a_{n-1} and a_n, we take $a_{n+m}= 2^{-n}$ for all *m*. If we find a proof of ¬A between the choice a_{n-1} and a_n we take $a_{n+m}= -2^{-n}$ for all *m*.

The sequence (a_n) is convergent, so it defines a real number, say ρ.

We connect the definition of (a_n) with the introduced stages by taking the division of stages such, that a_n is chosen at stage n. Given this division, if for some *k*, $\Box_k$ A holds, then $\rho \geq 2^{-k}$ holds as well. So we have

(5) $\exists n \Box_n A \Rightarrow \rho>0$, and analogously

(6) $\exists n \Box_n \neg A \Rightarrow \rho<0$.

Note that the newly introduced term $\Box_n A$ can not be used in the definition of (a_n), e.g. by defining $a_n=0$ iff neither $\Box_n A$ nor $\Box_n \neg A$ holds. The fact that $\Box_n A$ is not valid now, does not exclude that we will find a proof of A before stage n.

We are now ready for the proof. After the definition of ρ in [Br 48] we gave in Section 3, Brouwer continues with:

> "If for this real number ρ the relation ρ>0 were to hold, then ρ<0 would be impossible, so it would be certain α could never be proved to be absurd, so the absurdity of the absurdity of α would be known, so α would be tested, which it is not. Thus the relation ρ>0 does not hold.
>
> Further, if for the real number ρ the relation ρ<0 were to hold, then ρ>0 would be impossible, so it would be certain that α could never be proved to be true, so the absurdity of α would be known, so again α would be tested, which it is not. Thus the relation ρ<0 does not hold.
>
> Finally let us suppose that the relation ρ=0 holds. In this case neither ρ<0 nor ρ>0 could ever be proved, so neither the absurdity nor the truth of α could ever be proved, so the absurdity as well as the absurdity of the absurdity of α would be known. This is a contradiction, so the relation ρ=0 is absurd, in other words the real numbers ρ and 0 are different ." ([Br 48], CW pp 478-479)

We rewrite the first paragraph of this proof as follows:

If ρ>0 holds, then $\neg\rho<0$ holds, so $\neg\exists n \Box_n \neg A$ holds, so $\neg\neg A$ holds, and A would be tested. Since A is not tested, ρ>0 does not hold.

Crucial in this rewriting is that "it would be certain that α could never be proved to be absurd" is expressed by $\neg\exists n \Box_n \neg A$. The reasoning in the rewritten paragraph is valid because of the following implications:

1. $\rho>0 \Rightarrow \neg\rho<0$
2. $\neg\rho<0 \Rightarrow \neg\exists n \Box_n \neg A$ because of (6)
3. $\neg\exists n \Box_n \neg A \Rightarrow \neg\neg A$ because of (4).

Analogously, if ρ<0 were to hold, ¬ A, and so the fact that A is tested, would follow from

1. $\rho<0 \Rightarrow \neg\rho>0$
2. $\neg\rho>0 \Rightarrow \neg\exists n \Box_n A$ because of (5)
3. $\neg\exists n \Box_n A \Rightarrow \neg A$ because of (4).

So ρ>0 does not hold either.

However, if ρ=0 were to hold, it would follow from

1. $\rho=0 \Rightarrow \neg\rho<0$
2. $\neg\rho<0 \Rightarrow \neg\exists n\Box_n\neg A$
3. $\neg\exists n\Box_n\neg A \Rightarrow \neg\neg A$

and from

4. $\rho=0 \Rightarrow \neg\rho>0$
5. $\neg\rho>0 \Rightarrow\neg\exists n\Box_n A$
6. $\neg\exists n\Box_n A \Rightarrow \neg A$

that $\neg A$ and $\neg\neg A$ were to hold, which is a contradiction, i.e. $\rho\neq 0$ does hold.

5. The idealised mathematician

We took the notation $\Box_n\varphi$ from the existing reconstructions but we gave it another meaning. In these reconstructions, for example Troelstra's in [TD 88] pp. 842-846, the expression *creating* subject is changed in *creative* subject and interpreted as *the idealised mathematician* (for short IM). The ω discrete stages do not cover the future before us, as in our version, but they cover all the mathematical activity of the IM; $\Box_n\varphi$ is defined as: the IM has a proof of φ at stage n. In these reconstructions

(7) $\exists n\Box_n\varphi \Rightarrow \varphi$

is supposed to be obvious. In our interpretation of the basic term $\Box_n\varphi$, (7) is not plausible at all, because it eliminates the distinction we want to make with $\Box_n\varphi$. We want to draw attention to the fact that Brouwer avoids the use of (7).

If he had wanted to use it he could have simplified his argument by using

1. $\rho>0 \Rightarrow \exists n\Box_n A$
2. $\exists n\Box_n A \Rightarrow A$

and

1. $\rho<0 \Rightarrow \exists n\Box_n\neg A$
2. $\exists n\Box_n\neg A \Rightarrow \neg A$.

In that case he would not have to resort to an untested proposition, an undecided one (A nor $\neg A$ is known) would have been sufficient. From (7) Kripke's Scheme (KS) is derived:

KS $\quad \exists\alpha(\exists x\alpha(x)=1 \leftrightarrow A)$, for α a sequence with values 0 or 1, A any formula.

KS is often accepted in intuitionism as a reasonable principle. For a recent application see [Da 99]. We think to have shown that there is no basis for KS in Brouwers creating subject arguments. On the contrary, he deliberately avoids using it.

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Notes

i In two recent lectures Troelstra maintains the view that choice sequences have no mathematical use because of the elimination theorems; to him their interest is philosophical ([Tr 01A] p.227 and [Tr 01B] p.19).

ii In a forthcoming paper we shall follow Brouwer's development from 1907 until the culmination in the spread definition of 1918, and we shall follow his elaboration of the concept during the rest of his career. A lot of the modifications in the definition concern the kind of restrictions that can be imposed on choice sequences. Our version of the definition comes closest to the one Brouwer used around 1927.

iii In his proof of the negative continuity theorem of [Br 27] Brouwer may seem to use an individual choice sequence. This precedes the famous continuity theorem. According to Brouwer the theorem is, contrary to the continuity theorem, an immediate consequence of basic intuitionistic principles, and its proof appears in his lectures from 1918 (contrary to the continuity theorem). However, the proof is not at all clear. There are, among others, reconstructions in [Po 76], [Ma 85] and [Tr 82], all different.
Brouwer continuously developed his ideas on choice sequences during his entire career, see also note ii). In our opinion the notion of choice sequence was not sufficiently crystallised out when he used it in [Br 27]. At least he never used it again in the same manner. In [Br 81] p.81 Brouwer proves the negative continuity theorem again. Whereas he was fully exploiting choice sequences as introduced in [Br 91] in his method of the creating subject, he now uses a lawlike sequence in his proof.
The second proof is not taken into consideration in the reconstructions mentioned.

iv See also note iv. The choice sequence of [Br 91] has for the first time been recognised as such and cited in [Da 99]. No reconstruction is given. The fact that Brouwer uses choice sequences in [Br 30] is mentioned there; they are not shown. The expression *individual choice sequence* is for the first time used in the reconstruction of the creating subject arguments in our [Ni 87].

[v] There are two other choice sequences in [Br 30]. We give them both. The first is:

> "De (auf virtuele Ordnung erweiterte) **Insichdichtheit** im obige Sinne besteht für das intuitionistische Kontinuum **nicht**, und zwar weil die obige Charakterisierung der Elemente desselben als Hauptelemente versagt. Denken wir nämlich eine Charakterisierung des Elementes 1/2 als Hauptelement auf Grund einer konvergenten Folge $a_1 < a_2 ... < 1/2$. Alsdan konstruieren wir in folgender Weise eine Folge d_1, d_2, ...: Wir bestimmen der Reihe nach $d_1=a_1$, $d_2=a_2$, ..., und setzen in dieser Weise $d_v=a_v$, so lange uns von einer bestimmten fliehenden Eigenschaft weder eine Lösungszahl, noch die Absurdität einer solchen bekannt geworden ist; wenn aber zwischen der Bestimmung von d_v und d_{v+1} eines dieser beiden Ergebnisse eintritt, so setzen wir $d_u=d_v=a_v$ für u>v. Das zu dieser Folge d_1, d_2, ... gehörende Element des Kontinuums d ist , 1/2; trotzdem kann kein solches a_v angegeben werden, daß a_v>d ist. Für das volle intuitionistische Kontinuum ist mithin die Insichdichtheit nach der obige Definition mindestens hoffnungslos; für das reduzierte Kontinuum läßt sie das gleiche zeigen." ([Br 30], CW. p.437)

The second:

> "Von der (auf virtuelle Ordnung erweiterten) **Separabilität in sich** stellt sich für das intuitionistische Kontinuum wie folgt die Unhaltbarkeit heraus.: Es sei F die diskrete und geordnete Fundamentahlreihe, auf welcher die Separabilität in sich des Kontinuums K beruhen soll. Es sei p_1 das erste Element von F. Wir dürfen annehmen, daß $p_1>2^{-n}$ für eine passende natürliche Zahl n. Es sei p_2 das erste in F auf p_1 folgende Element von F, das zwischen p_1 und dem Nullpunkt gelegen ist, p_3 das erste in F auf p_2 folgende Element von F, das zwischen p_1 und p_2 gelegen ist, p_4 das erste in F auf p_3 folgende Element von F, das zwischen p_1 und p_3 gelegen ist, usw. Wir konstruieren in folgender Weise eine konvergente Folge m_1, m_2, ... von Elementen von F: Wir setzen $m_v=p_v$, so lange uns von einer bestimmten fliehenden Eigenschaft weder eine Lösungszahl noch die Absurdität einer solchen bekannt geworden ist; wird aber zwischen der Bestimmung von m_k und m_{k+1} eine Lösungszahl gefunden oder die Absurdität einer solchen bewiesen, so setzen wir $m_v=p_k$ für v>k. Das zu dieser konvergenten gehörende Element p von K ist verschieden von p_1; trozdem kann kein zwischen p und p_1 gelegenes Element von K angegeben werden." ([Br 30], CW p.438)

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