FINITE PROJECTIVE FORMULAS

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1 Introduction

The notion of projective formula was introduced by Ghilardi [8] in 1999. Let us denote by $\mathcal{P}_n$ a set of fixed $p_1, \ldots, p_n$ propositional variables and by Φ_n – all equivalence classes of intuitionistic formulas with variables in $\mathcal{P}_n$. Consider a substitution $\sigma : \mathcal{P}_n \to \Phi_n$ and extend it to all of Φ_n by $\sigma(\phi(p_1, \ldots, p_n)) = \phi(\sigma(p_1), \ldots, \sigma(p_n))$. Now, a formula $\phi \in \Phi_n$ is called *projective* if there exists a substitution $\sigma : \Phi_n \to \Phi_n$ such that $\vdash \sigma(\phi)$ and $\phi \vdash \psi \leftrightarrow \sigma(\psi)$, for all $\psi \in \Phi_n$. In this paper we study projective formulas from the relational and algebraic semantical point of view.

We show a close connection between projective formulas and projective Heyting algebras (for definition see Section 4). Namely, to each finitely generated projective Heyting algebra there corresponds a projective formula; to non-isomorphic finitely generated projective algebras there correspond non-equivalent projective formulas, but there can be non-equvalent projective formulas which correspond to isomorphic projective algebras. To a fixed n-generated projective Heyting algebra H there correspond as many projective formulas as there are different retractions between H and Φ_n (Φ_n being the free n-generated Heyting algebra and H its retract). We have a one-to-one correspondence between projective formulas and (H, ir) couples, where H is a projective algebra, and i, r the retractions.

Ghilardi (together with the results of Dick de Jongh and Albert Visser) has shown in [8] that projective formulas are (the same as) the exact and extendible formulas introduced earlier by de Jongh and Visser. A formula $\phi \in \Phi_n$ is called *extendible* if for all finite, rooted models $\mathcal{M}_1, \ldots, \mathcal{M}_k$ which force ϕ, there can be defined a valuation on the model $\mathcal{M}$ – obtained by adding a point as a new root to the disjoint union of the $\mathcal{M}_i$'s – in such a way that $\mathcal{M} \models \phi$.

A subset A of the n-universal model $\mathcal{M}_n$ (see Section 3) is called *admissible* if there exists a formula $\phi \in \Phi_n$ such that $A = \{w \in \mathcal{M}_n : w \models \phi\}$. A set A is called *extendible* if for any anti-chain in it, A contains at least one element totally covered by this anti-chain. Now, a formula is extendible iff its corresponding admissible set is extendible. Then we proceed with characterizing the admissible extendible subsets of the n-universal model. Here the problem of the characterization of infinite admissible extendible sets seems rather complicated for n greater than one; we just give an alternative proof of the fact that there is an infinite number of them.

However, it is easier to approach the finite admissible extendible subsets of the n-universal model which correspond to the so-called *finite* projective formulas. All finite subsets of the n-universal model are admissible (Grigolia [11], de Jongh [5],[6]) and a necessary condition for them to be extendible is that their widths should be less than or equal to two and their depths less than or equal to $n+1$. Using this fact, we write out all the finite extendible subsets of the 2-universal model – there are 26 such – and present their corresponding finite projective formulas.

In addition, we give a combinatorial formula which is a counter of the number of finite projective formulas of n variables. A computer program which realizes this combinatorial formula is attached to the thesis. We have conducted calculations for n ranging from 1 to 6. The finite formulas of one variable were found by Dick de Jongh (see [4]), there are four of them. As we mentioned above, there are 26 finite formulas of two variables; for $n = 3$ there are 256 finite formulas; for $n = 4$: 3386; for $n = 5$: 55984 and for $n = 6$: 1110506.

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2 Preliminaries

In this section we recall some basic facts on Intuitionistic Propositional Calculus, which we subsequently denote by **IPC**, and its (relational and algebraic) semantics.

The language of **IPC** consists, as usual, of the propositional variables $p, q, r, \ldots$ and the connectives $\wedge, \vee, \rightarrow$ and $\perp$ ($\top$, $\neg$ and $\leftrightarrow$ are introduced by their usual abbreviations). The formulas will be denoted by $\phi, \psi, \chi, \ldots$.

We choose the following formulas:

1. $\phi \rightarrow (\psi \rightarrow \phi)$
2. $\phi \rightarrow (\psi \rightarrow \chi) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi))$
3. $(\phi \wedge \psi) \rightarrow \phi$; $(\phi \wedge \psi) \rightarrow \psi$
4. $\phi \rightarrow (\phi \vee \psi)$; $\psi \rightarrow (\phi \vee \psi)$
5. $\phi \rightarrow (\psi \rightarrow (\phi \wedge \psi))$
6. $(\phi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow ((\phi \vee \psi) \rightarrow \chi))$
7. $\perp\rightarrow \phi$

as the only axiom-schemes for **IPC**.

Modus Ponens:

$$\frac{\phi, \phi \rightarrow \psi}{\psi}$$

will be the only rule of derivation for **IPC**.

As usual, **IPC** $\vdash \phi$ will mean that ϕ is a theorem of **IPC**.

2.1 Kripke semantics

A *Kripke frame* for **IPC** is a couple $\mathcal{F} = (W, R)$, where W is a nonempty set, and R is a reflexive partial order on W.

$A \subseteq W$ is said to be *upward closed*, if $w \in A$ and wRv imply $v \in A$. Denote by $ConW$ the set of all upward closed subsets of W. $R(w) = \{v \in W : wRv\}$ is the *upward closed set generated by* w. We also denote $R(A) = \bigcup_{w \in A} R(w)$.

A *Kripke model* for **IPC** is a couple $\mathcal{M} = (\mathcal{F}, \models)$, where $\mathcal{F}$ is a Kripke frame, and $\models$ a binary relation on $W \times \mathcal{P}$ ($\mathcal{P}$ the set of propositional variables) such that

$w \models p$ and wRv imply $v \models p$

$\models$ is then extended to all formulas by the following usual clauses:

$w \models \phi \wedge \psi$ iff $w \models \phi$ and $w \models \psi$;
$w \models \phi \vee \psi$ iff $w \models \phi$ or $w \models \psi$;
$w \models \phi \rightarrow \psi$ iff $\forall v \in R(w)(v \models \phi \Rightarrow v \models \psi)$;
$w \not\models \bot$.

It follows immediately that

$w \models \neg\phi$ iff $\forall v \in R(w)(v \not\models \phi)$,
$w \models \phi \leftrightarrow \psi$ iff $\forall v \in R(w)(v \models \phi \Leftrightarrow v \models \psi)$
and
$w \models \top$.

A formula ϕ is *forced* in w, if $w \models \phi$. ϕ is *forced* in a Kripke model $\mathcal{M}$, if ϕ is forced by every world of the underlying set of $\mathcal{M}$. ϕ is *valid* in a Kripke frame $\mathcal{F}$, if ϕ is forced in every Kripke model based on $\mathcal{F}$. Finally, ϕ is *valid* in a class $\mathcal{C}$ of Kripke frames, if ϕ is valid in every $\mathcal{F} \in \mathcal{C}$.

Now we have that **IPC** is sound and complete with respect to the class $\mathcal{KF}$ of all Kripke frames, that is, **IPC** $\vdash \phi$ iff ϕ is valid in $\mathcal{KF}$, and that, in addition, **IPC** enjoys the finite model property (f.m.p., for short), that is, **IPC** is sound and complete with respect to the class $\mathcal{KF}_{FIN}$ of all finite Kripke frames (see e.g., Fitting [7]).

2.2 Operations on Kripke frames

Given two Kripke frames $\mathcal{F} = (W, R)$ and $\mathcal{F}' = (W', R')$, $\mathcal{F}'$ is called a *generated subframe* of $\mathcal{F}$, if W' is an upward closed subset of W and R' is the restriction of R to W'. $\mathcal{F}'$ is called a *p-morphic image* of $\mathcal{F}$, if there exists a surjection $f : W \rightarrow W'$ such that

(i) wRv implies $f(w)R'f(v)$;

(ii) $f(w)R'f(v)$ implies that there is a $u \in W$ such that wRu and $f(u) = f(v)$.

$\mathcal{F}$ and $\mathcal{F}'$ are called *isomorphic* if there exists a bijection $f : W \rightarrow W'$ such that wRv iff $f(w)R'f(v)$.

Given a family $\{\mathcal{F}_i\}_{i \in I}$ of Kripke frames, the disjoint union $\coprod_{i \in I} \mathcal{F}_i$ is the frame (W, R), where W is the disjoint union of the sets W_i, and $(w, i)R(v, j)$ iff $i = j$ and wR_iv, for any $(w, i), (v, j) \in W$.

For any Kripke frame $\mathcal{F}$, denote by $Th(\mathcal{F})$ the set of all formulas (of the language of **IPC**) which are valid in $\mathcal{F}$. It is well-known (see e.g. Chagrov and Zakharyaschev [3]) that $\mathbf{IPC} \subseteq Th(\mathcal{F})$, and that $Th(\mathcal{F})$ is closed w.r.t. Modus Ponens. For every family $M = \{\mathcal{F}_i\}_{i \in I}$ of Kripke frames, let $Th(M) = \bigcap_{i \in I} Th(\mathcal{F}_i)$. Now we have that

1) If $\mathcal{F}'$ is a generated subframe of $\mathcal{F}$, then $Th(\mathcal{F}) \subseteq Th(\mathcal{F}')$;
2) If $\mathcal{F}'$ is a p-morphic image of $\mathcal{F}$, then $Th(\mathcal{F}) \subseteq Th(\mathcal{F}')$;
3) If $\mathcal{F}$ is the disjoint union of the family M of Kripke frames, then $Th(\mathcal{F}) = Th(M)$ (again consult [3] for a proof);
4) If $\mathcal{F}$ and $\mathcal{F}'$ are isomorphic, then $Th(\mathcal{F}) = Th(\mathcal{F}')$.

2.3 Operations on Kripke Models

Given two Kripke models $\mathcal{M} = (W, R, \models)$ and $\mathcal{M}' = (W', R', \models')$, $\mathcal{M}'$ is called a *generated submodel* of $\mathcal{M}$, if the frame (W', R') is a generated subframe of the frame (W, R) and $\models'$ coincides with $\models$ on the set W'. $\mathcal{M}'$ is called a *p-morphic image* of $\mathcal{M}$, if there exists a p-morphism f from (W, R) onto (W', R') such that for all $w \in W$ and every propositional variable p, $w \models p$ iff $f(w) \models' p$. $\mathcal{M}$ and $\mathcal{M}'$ are called *isomorphic* if there exists an isomorphism f between the frames (W, R) and (W', R') and for all $w \in W$ and every propositional variable p, $w \models p$ iff $f(w) \models' p$.

Given a family $\{\mathcal{M}_i\}_{i \in I}$ of Kripke models, the disjoint union $\coprod_{i \in I} \mathcal{M}_i$ is the model $(W, R, \models)$, where (W, R) is the disjoint union of the frames (W_i, R_i), and $(w, i) \models p$ iff $w \models_i p$.

For any model $\mathcal{M}$, denote by $Th(\mathcal{M})$ the set of all formulas which are forced in $\mathcal{M}$. Now we have that

1) If $\mathcal{M}'$ is a generated submodel of $\mathcal{M}$, then $Th(\mathcal{M}) \subseteq Th(\mathcal{M}')$;
2) If $\mathcal{M}'$ is a p-morphic image of $\mathcal{M}$, then $Th(\mathcal{M}) = Th(\mathcal{M}')$;
3) If $\mathcal{M}$ is the disjoint union of the family $\{\mathcal{M}_i\}_{i \in I}$ of Kripke models, then $Th(\mathcal{M}) = Th(\coprod_{i \in I} \mathcal{M}_i) = \bigcap_{i \in I} Th(\mathcal{M}_i)$;
4) If $\mathcal{M}$ and $\mathcal{M}'$ are isomorphic, then $Th(\mathcal{M}) = Th(\mathcal{M}')$.

2.4 Algebraic semantics

A *Heyting algebra* $\mathcal{H} = (H, \wedge, \vee, \to, 0)$ is a distributive lattice $(H, \wedge, \vee, 0)$ with an additional binary operation $\to$ which satisfies the following condition: $x \leq a \to b$ iff $a \wedge x \leq b$ for any $a, b \in H$. (If $a, b \in H$, then $\neg a$ and $a \leftrightarrow b$ are defined as $a \to 0$ and $(a \to b) \wedge (b \to a)$, and 1 is $\neg 0$.) We denote the class of all Heyting algebras by $\mathcal{HA}$.

A function $v : \mathcal{P} \to H$ is called a *valuation* (of the set of propositional variables $\mathcal{P}$ in a Heyting algebra $\mathcal{H}$). v is then extended to all formulas by the following usual clauses:

$v(\phi \wedge \psi) = v(\phi) \wedge v(\psi)$;
$v(\phi \vee \psi) = v(\phi) \vee v(\psi)$;
$v(\phi \to \psi) = v(\phi) \to v(\psi)$;
$(v(\neg\phi) = \neg v(\phi))$;
$(v(\phi \leftrightarrow \psi) = v(\phi) \leftrightarrow v(\psi))$;
$v(\bot) = 0$;
$(v(\top) = 1)$.

For a given $\mathcal{H}$ and a given valuation v in $\mathcal{H}$, a formula ϕ is *true* in $\mathcal{H}$, if $v(\phi) = 1$. ϕ is *valid* in $\mathcal{H}$, if ϕ is true in $\mathcal{H}$ for every valuation in $\mathcal{H}$. Finally, ϕ is *valid* in a class $\mathcal{C} \subseteq \mathcal{HA}$, if ϕ is valid in every $\mathcal{H} \in \mathcal{C}$.

Here too, we have that algebraic semantics is adequate for **IPC**, that is, **IPC** $\vdash \phi$ iff ϕ is valid in $\mathcal{HA}$ (and even $\mathcal{HA}_{FIN}$, where $\mathcal{HA}_{FIN}$ denotes the class of all finite Heyting algebras), see e.g., Fitting [7].

2.5 Connection between Kripke frames and Heyting algebras

There is a close correspondence between Kripke frames and Heyting algebras. Indeed, with every Kripke frame $\mathcal{F}$ is associated the Heyting algebra $\mathcal{F}^+ = (ConW, \cap, \cup, \to, \emptyset)$, where $ConW$ denotes the set of all upward closed subsets of W and $A \to B = \{w \in W : \forall v \in R(w)(v \in A \Rightarrow v \in B)\}$. Now we have that a formula ϕ is valid in $\mathcal{F}$ iff ϕ is valid in $\mathcal{F}^+$.

Conversely, with every Heyting algebra $\mathcal{H}$ is associated the Kripke frame $\mathcal{H}_+ = (W, R)$, where W is the set of all prime filters of H, and wRv iff $w \subseteq v$. Now we have that, if ϕ is valid in $\mathcal{H}_+$, then ϕ is valid in $\mathcal{H}$, but the converse is not true in general (it is true though, if $\mathcal{H}$ is finite).

Further, $\mathcal{F}$ is p-morphically embedded in ${\mathcal{F}^+}_+$, $\mathcal{H}$ is homomorphically embedded in ${\mathcal{H}_+}^+$, and $\mathcal{F} \cong {\mathcal{F}^+}_+$, if $\mathcal{F}$ is finite, and $\mathcal{H} \cong {\mathcal{H}_+}^+$, if $\mathcal{H}$ is finite (see e.g., Fitting [7]).

Furthermore, if $\mathcal{H}_2$ is a subalgebra of $\mathcal{H}_1$, then $\mathcal{H}_{2+}$ is a p-morphic image of $\mathcal{H}_{1+}$, and if $\mathcal{F}_2$ is a p-morphic image of $\mathcal{F}_1$, then $\mathcal{F}_2^+$ is a subalgebra of $\mathcal{F}_1^+$.

In addition, if $\mathcal{H}_2$ is a homomorphic image of $\mathcal{H}_1$, then $\mathcal{H}_{2+}$ is a generated subframe of $\mathcal{H}_{1+}$, and conversely, if $\mathcal{F}_2$ is a generated subframe of $\mathcal{F}_1$, then $\mathcal{F}_2^+$ is a homomorphic image of $\mathcal{F}_1^+$.

We will need one more notion in the sequel, namely the notion of a principal filter. Given a Heyting algebra H and $a \in H$, call the filter $[a) = \{b \in H : a \leq b\}$ the *principal filter* (generated by a).

3 Free algebras and universal models

Recall that for any variety $\mathbf{V}$, $A \in \mathbf{V}$ and $X \subseteq A$, A is said to be the *X-generated free algebra over* $\mathbf{V}$, if for every $B \in \mathbf{V}$, any map $h : X \to B$ can be extended to a homomorphism $\hat{h} : A \to B$. If this is the case, then we denote A by $F(X)$.

If $|X| = \omega$, then $F(X)$ is called the *ω-generated free* algebra over $\mathbf{V}$, and is denoted by $F(\omega)$, and if $|X| = n$, then $F(X)$ is said to be the *n-generated free* algebra over $\mathbf{V}$, and is denoted by $F(n)$.

In the case of $\mathcal{HA}$ we have that the ω-generated free Heyting algebra $F(\omega)$ is (isomorphic to) the algebra $\Phi/_\equiv$ – where Φ denotes the set of all formulas of the language of **IPC**, and $\phi \equiv \psi$ iff $\vdash \phi \leftrightarrow \psi$ – and the n-generated free Heyting algebra $F(n)$ is (isomorphic to) the algebra $\Phi_n/_\equiv$ – where Φ_n denotes the set of all formulas in n fixed variables. Subsequently we do not distinguish between the formulas and their equivalence classes. Hence we simply write Φ for $F(\omega)$, and Φ_n for $F(n)$. Since Φ is a lattice, we have an order $\leq$ on Φ. It follows from the definition of $\to$ that for all $\phi, \psi \in \Phi$, $\phi \leq \psi$ iff $\vdash \phi \to \psi$.

As follows from the above, for any $n \leq \omega$, Φ_n is embedded into ${(\Phi_n)_+}^+$. The description of $(\Phi_n)_+$ can be found in Urquhart [15], Grigolia [11], Shehtman [14], Rybakov [13] and Bellissima [2]. Here we will recall the description of the upper part of $(\Phi_n)_+$, which is usually called *the n-universal model*, and which keeps all the information about Φ_n. We will follow the so-called *colouring technique* of Grigolia [11], closely related to the methods of de Jongh [1968,1970].

Suppose a frame (W, R) is given. Let $x < y$ mean xRy and $x \neq y$. For $w, v \in W$, v is said to *cover* w, if $w < v$ and there does not exist u such that $w < u < v$. $A \subseteq W$ is called an *anti-chain* of W, if for all $w, v \in A$ we have $w \neq v$ implies $(w, v) \notin R$ and $(v, w) \notin R$. We say that $A \subseteq W$ *totally covers* $w \in W$, written as $w \prec A$, if A coincides with the set of all elements which cover w. If $A = \{v\}$ then v is said to *totally cover* w (written as $w \prec v$). $w \in W$ is said to have the *depth* m, if the length of the maximal $<$-chain with the root w equals m.

A colour is defined as any subset of the set $\{1, \ldots, n\}$. So, for fixed n, we have 2^n colours. Now the n-universal model $\mathcal{M}_n$ is constructed recurrently by levels: the mth level contains the points of depth m, to each of which we assign a colour. Since we have fixed n, we have n fixed variables $p_1, \ldots, p_n$; by assigning a colour to a point w ($Col(w)$) we set a valuation on w (this will be defined in the sequel) in the following sense: p_i is true in w whenever $i \in Col(w)$. If an anti-chain totally covers two points, we want those two points to have different colours. Furthermore, if wRv, then we want $Col(w) \subseteq Col(v)$, and if $w \prec v$, then $Col(w) \subset Col(v)$.

Now the elements of the first level are $w_1, \ldots, w_{2^n}$, coloured in 2^n different colours. Let W_n^1 denote the set of points of the first level, that is, $W_n^1 = \{w_1, \ldots, w_{2^n}\}$. The elements of the first level are going to be the maximal elements of the model.

Further, each element w of the first level totally covers $2^{|Col(w)|}-1$ elements of the second level with the colours $Col(v) \subset Col(w)$ ($2^{|Col(w)|} - 1$ is the number of proper subsets of $Col(w)$). Furthermore, let $A = \{w_{i_1} \ldots, w_{i_k}\}$, $k \geq 2$, be a subset of W_n^1. Then A totally covers $2^{|\bigcap_{j=1}^k Col(w_{i_j})|}$ elements of the second level with $Col(v) \subseteq \bigcap_{j=1}^k Col(w_{i_j})$ ($2^{|\bigcap_{j=1}^k Col(w_{i_j})|}$ the number of all subsets of the set $\bigcap_{j=1}^k Col(w_{i_j})$) and two distinct elements have different colours. These and only these are the elements of the second level, which we denote by W_n^2.

Now suppose we have constructed W_n^m, $m \geq 2$. Each element w of the level m totally covers $2^{|Col(w)|} - 1$ elements of the level $m+1$ with the colours $Col(v) \subset Col(w)$. Let $A = \{w_{i_1} \ldots, w_{i_k}\} \subseteq W_n^1 \cup \ldots \cup W_n^m$, $k \geq 2$, be an anti-chain of which at least one element is of level m. Then A totally covers $2^{|\bigcap_{j=1}^k Col(w_{i_j})|}$ elements of the level $m + 1$ with $Col(v) \subseteq \bigcap_{j=1}^k Col(w_{i_j})$ and two distinct elements have different colours. These and only these are the elements of the level $m+1$, which we denote by W_n^{m+1}.

Finally, the n-universal model $\mathcal{M}_n = (W_n, R_n, \models_n)$ is defined as follows: $W_n = \bigcup_{m=1}^{\infty} W_n^m$; wR_nv iff $w = v$ or there exists a finite sequence $(x_1, \ldots, x_m)$ of points of W_n such that $w = x_1$, $v = x_m$ and x_i is covered by x_{i+1}; for fixed n variables $p_1, \ldots, p_n$, $w \models_n p_i$ iff $i \in Col(w)$, $i = 1, \ldots, n$.

The underlying frame (W_n, R_n) of the n-universal model $\mathcal{M}_n$ is usually called *the n-universal frame*. It should be clear from the construction that every level W_n^m contains only finitely many points.

In order to demonstrate how the construction works we consider the case of $n = 1$. Then we only have two colours $\emptyset$ and $\{1\}$. Hence W_1^1 consists of two elements of different colours. Further the element of W_1^1 with the empty colour cannot totally cover anything, while the other element of W_1^1 with the colour $\{1\}$ totally covers just one element of W_1^2 with the empty colour. Furthermore, W_1^1 (as an anti-chain) totally covers another element of W_1^2 also with the empty colour. Now there are two new anti-chains in $W_1^1 \cup W_1^2$, and each of them totally covers an element with the colour $\emptyset$ of W_1^3, and so on. The resulting model, the so-called *Rieger-Nishimura ladder*, is shown in Fig.1 below.

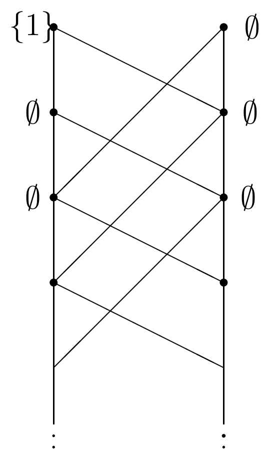

Fig.1

In the case of $n = 2$ there are four different colours. In Fig.2 a part of $\mathcal{M}_2$ is shown. We have omitted the points covered by the anti-chains with more than two elements, since in the characterization of finite projective formulas of two variables we will not need them. As it will turn out in Section 6, all the information on finite projective formulas of two variables is already coded in the part of $\mathcal{M}_2$ shown in Fig.2 below.

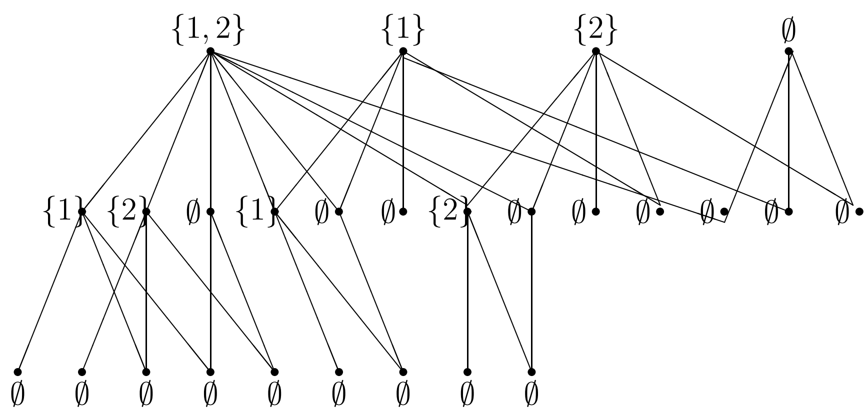

Fig.2

Now, after getting acquainted with the construction of the n-universal model, let us show that they give representations of the n-generated free Heyting algebras.

Define $h : \Phi_n \to ConW_n$ by putting $h(\phi) = \{w \in W_n : w \models_n \phi\}$. Obviously h is defined correctly. Moreover, $h(p_i) = \{w \in W_n : i \in Col(w)\}$, $i = 1, \ldots, n$.

Call a set $A \in ConW_n$ *admissible* if $A = h(\phi)$ for some $\phi \in \Phi_n$. Denote the set of all admissible upward closed subsets of W_n by $AdmW_n$. We obviously have $AdmW_n \subseteq ConW_n$. Consult Grigolia [11], de Jongh [5],[6] for the fact that $AdmW_n \subset ConW_n$ and also for the following fact:

Proposition 1 *If $A \in ConW_n$ is finite, then it is admissible.*

However, there do exist infinite admissible sets, for example W_n itself ($W_n = h(\top)$).

Suppose there is given a Kripke frame $\mathcal{F} = (W, R)$. Call a valuation $\models$ on $\mathcal{F}$ an *n-valuation* if $\models$ is defined only for the propositional variables $p_1, \ldots, p_n$, i.e. $\models$ is a binary relation on $W \times \mathcal{P}_n$ ($\mathcal{P}_n = \{p_1, \ldots, p_n\}$). An n-valuation $\models$ gives us a colouring on the model $(\mathcal{F}, \models)$ in the following way: for each $w \in W$, define $Col(w) = \{i : w \models p_i\}$. Obviously $Col(w) \subseteq \{1, \ldots, n\}$, for all $w \in W$. Call $(W, R, \models)$ an *n-model* if $\models$ is an n-valuation.

Definition 2 *Call a finite n-model $(W, R, \models)$* n-generated *if for all distinct $w, v \in W$, from $w \prec v$ it follows that $Col(w) \neq Col(v)$ and for any finite anti-chain $A \subset W$, $Col(w) \neq Col(v)$ whenever $w, v \prec A$.*

Lemma 3 *For any finite, rooted, n-model $\mathcal{M} = (W, R, \models)$, there exists a unique rooted, p-morphic image $\mathcal{M}' = (W', R', \models')$ which is a generated submodel of the n-universal model $\mathcal{M}_n$. Moreover, if $\mathcal{M}$ is n-generated, then there exists a rooted generated submodel of the n-universal model isomorphic to $\mathcal{M}$.*

Proof. We will prove the lemma for finite, rooted n-models using induction on the depths of the roots of the models.

Suppose there is given a finite, rooted n-model $\mathcal{M} = (W, R, \models)$ with the root w of the depth one. Thus $W = \{w\}$. Since $\mathcal{M}$ is an n-model, there exists $w' \in W_n^1$ such that $Col(w') = Col(w)$. Define a model $\mathcal{M}'$ to be the one element model $\{w'\}$. $\mathcal{M}'$ is obviously isomorphic to $\mathcal{M}$ and it is a rooted generated submodel of the n-universal model $\mathcal{M}_n$.

Now suppose that for each finite, rooted n-model $\mathcal{M}$ with the root w of the depth less than m, we have constructed a rooted generated submodel of the n-universal model which is a p-morphic image of $\mathcal{M}$ (if $\mathcal{M}$ is n-generated, then suppose we have constructed its isomorphic model). To show the induction step we prove the following

Claim. Given $f_i : \mathcal{M}_i \to \mathcal{M}'_i$, $i = 1, \ldots, k$, p-morphisms (isomorphisms), where $\mathcal{M}_1, \ldots, \mathcal{M}_k$ are finite, rooted n-models (such that if $\mathcal{M}_i$ has common points with $\mathcal{M}_j$ ($i \neq j$), then their intersection is a generated submodel of both $\mathcal{M}_i$ and $\mathcal{M}_j$) and $\mathcal{M}'_1, \ldots, \mathcal{M}'_k$ are generated submodels of the n-universal model. Then $\bigcup_{i=1}^k f_i$ is a p-morphism (isomorphism) from $\bigcup_{i=1}^k \mathcal{M}_i$ onto $\bigcup_{i=1}^k \mathcal{M}'_i$.

Proof. We will use induction on the depths of the roots of $\mathcal{M}_1, \ldots, \mathcal{M}_k$. The basis follows from the fact that the maximal points of the n-universal model have different colours. Now suppose that the conditions of the claim hold for models with the depths of the roots less than or equal to m. Consider all the immediate successors of each w_i – the root of $\mathcal{M}_i$ – and the models generated by them. The restrictions of f_i's to these models (appropriate for each i) are also p-morphisms (isomorphisms), and since the induction hypothesis holds for them, we will easily obtain that $\bigcup_{i=1}^k f_i : \bigcup_{i=1}^k \mathcal{M}_i \to \bigcup_{i=1}^k \mathcal{M'}_i$ is again a p-morphism (an isomorphism). $\square$

Now we continue with our lemma. Consider a finite n-model $\mathcal{M} = (W, R, \models)$ with the root w of depth m. Denote by A the set of all immediate successors of w. By the induction hypothesis, for each $v \in A$, the model $R(v)$ has its p-morphic (isomorphic) image $\mathcal{M}_v$ in $\mathcal{M}_n$. Then, by our claim, there exists a p-morphism (an isomorphism) from $R(A)$ onto $\mathcal{M}_A = \bigcup_{v \in A} \mathcal{M}_v$. Denote by A' the bottom of $\mathcal{M}_A$ and let us distinguish two cases: A' contains more than one point or A' consists of one point. In the first case, consider a point w' in the n-universal model such that $w' \prec A'$ and $Col(w') = Col(w)$. Add w' to $\mathcal{M}_A$ as the new root and denote the resulting model by $\mathcal{M}'$. In the second case, let $\mathcal{M}'$ be $\mathcal{M}_A$.

Obviously (in both cases) a new model $\mathcal{M}'$ is the p-morphic (isomorphic) image of $\mathcal{M}$ and it is a rooted generated submodel of the n-universal model. Thus we have shown the induction step. $\square$

Now we have the following representation of Φ_n:

Theorem 4 *(Grigolia [11]) Φ_n is isomorphic to $AdmW_n$.*

Proof. The surjection $h : \Phi_n \to AdmW_n$ preserves the algebraic operations. Indeed, $w \in h(\phi \wedge \psi)$ iff $w \models_n \phi \wedge \psi$ iff $w \models_n \phi$ and $w \models_n \psi$ iff $w \in h(\phi)$ and $w \in h(\psi)$ iff $w \in h(\phi) \cap h(\psi)$. Hence $h(\phi \wedge \psi) = h(\phi) \cap h(\psi)$. Analogously for disjunction, $h(\phi \vee \psi) = h(\phi) \cup h(\psi)$. Let us show now that $h(\phi \to \psi) = h(\phi) \to h(\psi)$. $w \in h(\phi \to \psi)$ iff $w \models_n \phi \to \psi$ iff $\forall v \in R_n(w)(v \models_n \phi \Rightarrow v \models_n \psi)$ iff $\forall v \in R_n(w)(v \in h(\phi) \Rightarrow v \in h(\psi))$ iff $w \in h(\phi) \to h(\psi)$.

To show that h is an injection, assume $\phi \neq \psi$ in Φ_n. Then $\not\vdash \phi \leftrightarrow \psi$. By the finite model property of **IPC**, there exists a finite, rooted model $\mathcal{M} = (W, R, \models)$ and a $w \in W$ such that $w \not\models \phi \leftrightarrow \psi$. Without loss of generality, assume $w \not\models \phi \to \psi$. Then there exists $v \in W$ such that wRv, $v \models \phi$ and $v \not\models \psi$. According to Lemma 3, there exists a model $(W', R', \models')$ in the n-universal model $\mathcal{M}_n$ which is a p-morphic image of $\mathcal{M}$ and makes the same formulas true. Then there exists a point $v' \in W'$ ($v' \in W_n$), which corresponds to v, such that $v' \models \phi$ and $v' \not\models \psi$. By the definition of h, $v' \in h(\phi)$ and $v' \notin h(\psi)$. So we have that $h(\phi) \neq h(\psi)$ and hence h is an injection. $\square$

4 Projective algebras and projective formulas

Recall that $A \in \mathbf{V}$ is said to be *projective*, if for any $B, C \in \mathbf{V}$, any surjective homomorphism $f : B \to C$ and any homomorphism $g : A \to C$, there is a homomorphism $h : A \to B$ such that $fh = g$.

$$\begin{tikzcd} & A \arrow[dl, "h"'] \arrow[dr, "g"] & \\ B \arrow[rr, two heads, "f"] & & C \end{tikzcd}$$

Fig.3

It is well-known (consult e.g. Grätzer [10]) that an algebra is projective if and only if it is a retract of a free algebra. Recall that B is a *retract* of A, if there exists an injective homomorphism $g : B \to A$ and a surjective homomorphism $f : A \to B$ such that $fg = id_B$ (see Fig.4 below).

$$\begin{tikzcd} B \arrow[r, shift left, "g"] & A \arrow[l, shift left, "f"] \end{tikzcd}$$

Fig.4

Now we recall the notion of projective formulas, originally introduced by Ghilardi [8], and show a close connection between them and projective algebras. Let $\mathcal{P}_n = \{p_1, \ldots, p_n\}$. A substitution $\sigma : \mathcal{P}_n \to \Phi_n$ is a function assigning to each propositional variable p_i a formula of the variables $p_1, \ldots, p_n$. Then σ can be extended to an endomorphism $\sigma : \Phi_n \to \Phi_n$ by putting

$$\sigma(\phi(p_1, \ldots, p_n)) = \phi(\sigma(p_1), \ldots, \sigma(p_n)).$$

Definition 5 *A formula $\phi \in \Phi_n$ is called* projective *if there exists a substitution $\sigma : \Phi_n \to \Phi_n$ such that $\vdash \sigma(\phi)$ and $\phi \vdash \psi \leftrightarrow \sigma(\psi)$, for all $\psi \in \Phi_n$.*

The following lemmas show the close connection between projective formulas and projective algebras.

Lemma 6 *Suppose H is an n-generated projective Heyting algebra. Then to H there corresponds a projective formula ϕ of n-variables, such that H is (isomorphic to) $\Phi_n/_{[\phi)}$.*

Proof. Suppose H is an n-generated projective Heyting algebra with the generators $a_1, \ldots, a_n$. Then H is a retract of Φ_n, and there exist an injective homomorphism $i : H \to \Phi_n$ and a surjective homomorphism $r : \Phi_n \to H$ such that $ri = id_H$. Then it is obvious that $ir : \Phi_n \to \Phi_n$ is an endomorphism of Φ_n. We will show now that $\phi = \bigwedge_{j=1}^n (p_j \leftrightarrow ir(p_j))$ is a projective formula, namely, $\vdash ir(\phi)$ and $\phi \vdash \psi \leftrightarrow ir(\psi)$ for any $\psi \in \Phi_n$.

Indeed, $ir(\bigwedge_{j=1}^n (p_j \leftrightarrow ir(p_j))) = \bigwedge_{j=1}^n (ir(p_j) \leftrightarrow irir(p_j))$, and since $ri = id_H$, we have $ir(\bigwedge_{j=1}^n (p_j \leftrightarrow ir(p_j))) = \bigwedge_{j=1}^n (ir(p_j) \leftrightarrow ir(p_j))$. Thus $\vdash ir(\phi)$. Further, for any $\psi \in \Phi_n$, $ir(\psi(p_1, \ldots, p_n)) = \psi(ir(p_1), \ldots, ir(p_n))$, and since $\phi \vdash p_j \leftrightarrow ir(p_j)$, $j = 1, \ldots, n$, we have $\phi \vdash \psi \leftrightarrow ir(\psi)$.

Now we will prove that H is isomorphic to $\Phi_n/_{[\phi)}$. For this recall that $\Phi_n/_{[\phi)}$ is isomorphic to the ϕ-relativized algebra $\Phi_n^\phi = \{\psi \in \Phi_n : \psi \leq \phi\}$, while $(\phi \wedge \cdot) : \Phi_n \to \Phi_n^\phi$ corresponds to the natural homomorphism $h : \Phi_n \to \Phi_n/_{[\phi)}$. What we want to show is that $(\phi \wedge \cdot)i : H \to \Phi_n^\phi$ is an isomorphism. Indeed, it is obvious that it is a homomorphism. Further, since $\phi \vdash \psi \leftrightarrow ir(\psi)$, for any $\psi \in \Phi_n$, $\phi \wedge \psi = \phi \wedge ir(\psi)$, $(\phi \wedge \cdot)i(r\psi) = \phi \wedge \psi$, and $(\phi \wedge \cdot)i$ is surjective. Furthermore, for any $a, b \in H$, if $(\phi \wedge \cdot)i(a) = (\phi \wedge \cdot)i(b)$, $\phi \wedge i(a) = \phi \wedge i(b)$. Hence $\phi \vdash i(a) \leftrightarrow i(b)$, $ir(\phi) \vdash i(a) \leftrightarrow i(b)$, and since $\vdash ir(\phi)$, we have that $\vdash i(a) \leftrightarrow i(b)$. Thus $i(a) = i(b)$, and since i is an injection, we have that $a = b$. Thus $(\phi \wedge \cdot)i$ is injective as well. $\square$

Now we will recall the definition of a finitely presented algebra. We will show that every projective algebra is finitely presented. See e.g. Ghilardi and Zawadowski [9]):

An algebra H is called *finitely presented* if
1) H is finitely generated (with the generators $a_1, \ldots, a_n \in H$);
2) there exists a finite number of equations

$$\begin{gathered} P_1(a_1, \ldots, a_n) = Q_1(a_1, \ldots, a_n) \\ \ldots \\ P_m(a_1, \ldots, a_n) = Q_m(a_1, \ldots, a_n) \end{gathered}$$

holding in H such that any other equation $P(a_1, \ldots, a_n) = Q(a_1, \ldots, a_n)$ which holds in H follows from them. (In other words, if Σ denotes the formula $\bigwedge_{i=1}^{m}(P_i(p_1, \ldots, p_n) \leftrightarrow Q_i(p_1, \ldots, p_n))$ and σ – the formula $P(p_1, \ldots, p_n) \leftrightarrow Q(p_1, \ldots, p_n)$, then $\Sigma \vdash \sigma$.)

The following lemma which characterizes finitely presented algebras is well-known:

Lemma 7 *An n-generated Heyting algebra H is finitely presented iff there exists a principal filter F of Φ_n such that H is isomorphic to $\Phi_n/_F$.*

Proof. Suppose H is finitely presented and n-generated, with the generators $a_1, \ldots, a_n \in H$. Since H is n-generated, it is a homomorphic image of Φ_n and hence there exists a filter F of Φ_n such that H is (isomorphic to) $\Phi_n/_F$. Let a be $\bigwedge_{i=1}^{m}(P_i(p_1, \ldots, p_n) \leftrightarrow Q_i(p_1, \ldots, p_n))$ and show that $F = [a)$. Indeed, since the equations $P_i(a_1, \ldots, a_n) = Q_i(a_1, \ldots, a_n)$, $i = 1, \ldots, m$, hold in H, we have that $P_i(p_1, \ldots, p_n) \leftrightarrow Q_i(p_1, \ldots, p_n) \in F$. Hence $a \in F$. On the other hand, for any $b \in F$, the equality $b(a_1, \ldots, a_n) = \top$ holds in H, and hence the formula $b(p_1, \ldots, p_n)$ follows from Σ. Therefore $a \leq b$, and $F = [a)$. Thus F is principal.

Conversely, suppose H is (isomorphic to) Φ_n/F, where $F = [a)$ is a principal filter. Then let $\Sigma = \{a(p_1, \ldots, p_n)\}$. Obviously an equality $\Sigma = \top$ holds in $\Phi_n/_F$, and for any equality $P(a_1, \ldots, a_n) = Q(a_1, \ldots, a_n)$ holding in H, $P(p_1, \ldots, p_n) \leftrightarrow Q(p_1, \ldots, p_n) \in F$. Hence $\Sigma \vdash P(p_1, \ldots, p_n) \leftrightarrow Q(p_1, \ldots, p_n)$ and $\Phi_n/_F$ is finitely presented. $\square$

It follows from the above that every finitely generated projective algebra is finitely presented. Now we will show that projective formulas also give rise to projective algebras.

Lemma 8 *If ϕ is a projective formula of n variables, then $\Phi_n/_{[\phi)}$ is a projective algebra.*

Proof. Suppose ϕ is projective. Then there exists a substitution $\sigma : \Phi_n \to \Phi_n$ such that $\vdash \sigma(\phi)$ and $\phi \vdash \psi \leftrightarrow \sigma(\psi)$ for all $\psi \in \Phi_n$. Since σ is an endomorphism of Φ_n, $\sigma(\Phi_n)$ is a subalgebra of Φ_n. We will prove now that $\sigma(\Phi_n)$ is a retract of Φ_n by showing that $\sigma^2 = \sigma$. Indeed, since ϕ is projective, $\sigma(\phi) = 1_{\Phi_n}$, and $\phi \leq \psi \leftrightarrow \sigma(\psi)$. But then $\sigma(\phi) \leq \sigma(\psi) \leftrightarrow \sigma^2(\psi)$, $\sigma(\psi) \leftrightarrow \sigma^2(\psi) = 1_{\Phi_n}$, $\sigma(\psi) = \sigma^2(\psi)$, and $\sigma^2 = \sigma$. Hence $\sigma(\Phi_n)$ is a retract of Φ_n.

Now we will show that $\sigma(\Phi_n)$ is isomorphic to $\Phi_n/_{[\phi)}$. For this we need to show that the restriction of the natural homomorphism $h : \Phi_n \to \Phi_n/_{[\phi)}$ to $\sigma(\Phi_n)$ is an isomorphism. Indeed, if $\sigma(\psi) = \sigma(\chi)$, then $\vdash \sigma(\psi) \leftrightarrow \sigma(\chi)$. Since $\phi \vdash \psi \leftrightarrow \sigma(\psi), \chi \leftrightarrow \sigma(\chi)$, we have $\phi \vdash \psi \leftrightarrow \chi$, and hence $h(\psi) = h(\chi)$. Conversely, if $h(\psi) = h(\chi)$, then $\phi \vdash \psi \leftrightarrow \chi$, $\sigma(\phi) \vdash \sigma(\psi) \leftrightarrow \sigma(\chi)$, and from $\vdash \sigma(\phi)$ it follows that $\vdash \sigma(\psi) \leftrightarrow \sigma(\chi)$. Hence $\sigma(\psi) = \sigma(\chi)$. Thus, $\sigma(\psi) = \sigma(\chi)$ iff $h(\psi) = h(\chi)$, $\sigma(\Phi_n)$ is isomorphic to $\Phi_n/_{[\phi)}$, $\Phi_n/_{[\phi)}$ is a retract of Φ_n, and hence is a projective algebra. $\square$

Thus we have the following correspondence between projective formulas and projective algebras: to each finitely generated projective algebra there corresponds a projective formula, to two non-isomorphic finitely generated projective algebras there correspond non-equivalent projective formulas, but there can be non-equivalent projective formulas which correspond to isomorphic projective algebras. Actually, to a fixed n-generated projective algebra H there correspond as many projective formulas as there are different retractions between H and Φ_n. Therefore, we arrive at the following

Corollary 9 *There exists a one-to-one correspondence between projective formulas and (H, ir) couples, where H is a projective algebra, and $i : H \to \Phi_n$, $r : \Phi_n \to H$ are retractions.*

Dually principal filters of Φ_n correspond to admissible upward closed subsets of the n-universal frame W_n. In order to give the dual characterization of those principal filters of Φ_n which give rise to projective algebras, we need the additional definition of an extendible subset of W_n, which is the subject of next section.

5 Exact and extendible formulas

Recall from de Jongh [4] that a formula $\phi \in \Phi_n$ is called *exact* if there exist $\chi_1, \dots, \chi_n \in \Phi_n$ such that $\vdash \phi(\chi_1, \dots, \chi_n)$ and, for any $\psi \in \Phi_n$, if $\vdash \psi(\chi_1, \dots, \chi_n)$, then $\vdash \phi \to \psi$.

Actually, de Jongh just asked for formulas $\chi_1, \dots, \chi_n$ with any number of variables, but the results of Ghilardi (see e.g. Theorem 10) and others show that the definition can be restricted to formulas in Φ_n.

Theorem 10 *If a formula is projective, then it is exact.*

Proof. Suppose ϕ is projective. Then there exists a substitution $\sigma : \Phi_n \to \Phi_n$, such that $\vdash \sigma(\phi)$ and $\phi \vdash \psi \leftrightarrow \sigma(\psi)$, for all $\psi \in \Phi_n$. To prove that ϕ is exact we need to construct $\chi_1, \dots, \chi_n$ such that $\vdash \phi(\chi_1, \dots, \chi_n)$ and, for any $\psi \in \Phi_n$, if $\vdash \psi(\chi_1, \dots, \chi_n)$, then $\vdash \phi \to \psi$.

Let $\chi_i = \sigma(p_i),\ i = 1, \dots, n$. Then $\phi(\chi_1, \dots, \chi_n) = \phi(\sigma(p_1), \dots, \sigma(p_n)) = \sigma(\phi)$, and hence $\vdash \phi(\chi_1, \dots, \chi_n)$. Suppose $\psi \in \Phi_n$ is such that $\vdash \psi(\chi_1, \dots, \chi_n)$. Then $\vdash \sigma(\psi)$. Now, since $\phi \vdash \psi \leftrightarrow \sigma(\psi)$, we have that $\phi \vdash \psi$ and hence ϕ is exact. $\square$

Now we recall the notion of an extendible formula. Suppose there is given a finite family of finite rooted models $\mathcal{M}_1 = (W_1, R_1, \models_1), \dots, \mathcal{M}_k = (W_k, R_k, \models_k)$ with the roots $w_1, \dots, w_k$, and ϕ is forced in $\mathcal{M}_1, \dots, \mathcal{M}_k$. Now consider the disjoint union of W_i's and add to it a new point w as a new root. Denote the resulting frame by (W, R). Call ϕ *extendible* if always in such a case there exists a valuation $\models$ on (W, R) such that $\models$ agrees with $\models_1, \dots, \models_k$ on $W_1, \dots, W_k$ and $w \models \phi$.

Assume that $\mathcal{M}_1, \dots, \mathcal{M}_k$ are pairwise nonisomorphic and n-models.

In order to describe those admissible subsets of the n-universal model which correspond to extendible formulas we need the following:

Definition 11 *An upward closed subset of the n-universal model, $A \in ConW_n$, is called* extendible *if for every finite anti-chain $B \subseteq A$, there exists an element $w \in A$ such that $w \prec B$.*

Denote the set of all extendible subsets of W_n by $ExtW_n$.

Theorem 12 *A formula $\phi \in \Phi_n$ is extendible iff its corresponding admissible set is extendible.*

Proof. Suppose a formula $\phi \in \Phi_n$ is extendible. Consider the admissible subset A_ϕ of the n-universal model $\mathcal{M}_n = (W_n, R_n, \models_n)$ corresponding to ϕ and a finite anti-chain $B = \{w_1, \dots, w_k\} \subseteq A_\phi$. We want to show that there exists $w \in A_\phi$ such that $w \prec B$. Indeed, consider a family of rooted models $\mathcal{M}_1 = (R_n(w_1), R_{\mathcal{M}_1}, \models_1), \dots, \mathcal{M}_k = (R_n(w_k), R_{\mathcal{M}_k}, \models_k)$ (where $R_{\mathcal{M}_i}$ and $\models_i$ are the restrictions of R_n and $\models_n$ to $R_n(w_i)$, $i = 1, \dots, k$). Obviously the models $\mathcal{M}_1, \dots, \mathcal{M}_k$ are finite and they force ϕ. But then ϕ is forced in the disjoint union of $\mathcal{M}_i$, $i = 1, \dots, k$. Add a new point v as a new root to the disjoint union of the models $\mathcal{M}_i$, $i = 1, \dots, k$. Since ϕ is extendible, there can be defined a valuation $\models$ on v in such a way that $v \models \phi$. Denote the resulting model by $\mathcal{M} = (W, R, \models)$. According to the construction of the n-universal model, for $B = \{w_1, \dots, w_k\}$, there exists a point $w \in W_n$ such that $w \prec B$ and $Col(w) = Col(v)$. Define a model $\mathcal{M}' = (W', R'_n, \models'_n)$, where $W' = \bigcup_{i=1}^{k} R_n(w_i) \cup \{w\}$ and R'_n, $\models'_n$ are the restrictions of R_n, $\models_n$ to W'. Since there exists a natural p-morphism h from the disjoint union of $\mathcal{M}_1, \dots, \mathcal{M}_k$ onto the union of $\mathcal{M}_1, \dots, \mathcal{M}_k$, we also have a p-morphism from $\mathcal{M}$ onto $\mathcal{M}'$ defined by adding the couple (v, w) to h. $\mathcal{M} \models \phi$, and hence $\mathcal{M}' \models'_n \phi$. Then $w \models'_n \phi$ and therefore $w \in A_\phi$. Thus A_ϕ is extendible.

For the other direction, assume $A_\phi \in ExtW_n$. Consider any family of finite, rooted models $\mathcal{M}_1 = (W_1, R_1, \models_1), \dots, \mathcal{M}_k = (W_k, R_k, \models_k)$ such that ϕ is forced in $\mathcal{M}_1, \dots, \mathcal{M}_k$. By Lemma 3, for each $\mathcal{M}_i$, $i = 1, \dots, k$, there exists its p-morphic image $\mathcal{M}'_i$ which is a generated submodel of the n-universal model. Since $\mathcal{M}'_i \models \phi$, $i = 1, \dots, k$, $W'_i \subset A_\phi$, $i = 1, \dots, k$. Denote the roots of $\mathcal{M}'_1 \dots, \mathcal{M}'_k$ by $w_1, \dots, w_k$. If the set $\{w_1, \dots, w_k\}$ is not an anti-chain in A_ϕ, then form an anti-chain $B \subseteq \{w_1, \dots, w_k\}$ by taking B to be the set $\{w_{i_1}, \dots, w_{i_m}\}$ of R_n-minimal elements from $\{w_1, \dots, w_k\}$. We distinguish two cases, $m > 1$ and $m = 1$. First, we cover the case that $m > 1$. Then, from the extendibility of A_ϕ, we have that there exists a $v \in A_\phi$ such that $v \prec \{w_{i_1}, \dots, w_{i_m}\}$. Add a new point w with $Col(w) = Col(v)$ as a new root to the disjoint union of the models $\mathcal{M}_1, \dots, \mathcal{M}_k$. Such w exists since, for all $u \in \bigcup_{i=1}^{k} W'_i$, $Col(v) \subseteq Col(u)$ and, for all $i = 1, \dots, k$, $\mathcal{M}'_i$ is a p-morphic image of $\mathcal{M}_i$. Denote the new model obtained by adding w as a new root to the disjoint union of the models $\mathcal{M}_1, \dots, \mathcal{M}_k$ by $\mathcal{M}$. Denote by $\mathcal{M}'$ the model obtained by adding v as a new root to the models $\mathcal{M}'_1 \dots, \mathcal{M}'_k$. Since there exists a natural p-morphism h from the disjoint union of $\mathcal{M}_1, \ldots, \mathcal{M}_k$ onto the union of $\mathcal{M}'_1, \ldots, \mathcal{M}'_k$, we also have a p-morphism from $\mathcal{M}$ onto $\mathcal{M}'$ defined by adding the couple (w, v) to h. Hence $w \models \phi$ and thus ϕ is an extendible formula. Next we consider the case that $m = 1$. Then, we have $B = \{w_{i_1}\}$. Add a new point w with $Col(w) = Col(w_{i_1})$ as a new root to the disjoint union of the models $\mathcal{M}_1, \ldots, \mathcal{M}_k$. Such w exists since, for all $u \in \bigcup_{i=1}^{k} W'_i$, $Col(w_{i_1}) \subseteq Col(u)$ and, for all $i = 1, \ldots, k$, $\mathcal{M}'_i$ is a p-morphic image of $\mathcal{M}_i$. As before, denote the new model obtained by adding w as a new root to the disjoint union of the models $\mathcal{M}_1, \ldots, \mathcal{M}_k$ by $\mathcal{M}$. Denote by $\mathcal{M}'$ the union of $\mathcal{M}'_1 \ldots, \mathcal{M}'_k$. Since there exists a natural p-morphism h from the disjoint union of $\mathcal{M}_1, \ldots, \mathcal{M}_k$ onto $\mathcal{M}'$, we also have a p-morphism from $\mathcal{M}$ onto $\mathcal{M}'$ defined by adding the couple (w, w_{i_1}) to h. Hence $w \models \phi$ and, again, ϕ is an extendible formula. $\square$

The following result is from Visser [16]. His proof was rather complicated. Even more complicated is the proof sketched by Ghilardi [8]. We will give a simple proof that directly uses our knowledge of the n-universal model.

Theorem 13 *If a formula is exact then it is extendible.*

Proof. Let ϕ be exact by $\chi_1, \ldots, \chi_n$. By theorem 12 it is sufficient to prove the corresponding admissible set A_ϕ to be extendible. Consider a finite anti-chain $B = \{w_1, \ldots, w_k\} \subseteq A_\phi$. We want to show that there exists $v \in A_\phi$ such that $v \prec B$.

Consider the family of rooted models $\mathcal{M}_1 = (R_n(w_1), R_{\mathcal{M}_1}, \models_1), \ldots, \mathcal{M}_k = (R_n(w_k), R_{\mathcal{M}_k}, \models_k)$ (where $R_{\mathcal{M}_i}$ and $\models_i$ are the restrictions of R_n and $\models_n$ to $R_n(w_i)$, $i = 1, \ldots, k$). Since finite sets in $ConW_n$ are admissible (Proposition 1), $R_n(w_1), \ldots, R_n(w_k)$ are admissible, say by the formulas $\psi_1, ..., \psi_k$. Note that ψ_i implies ϕ in **IPC**, $i = 1, \ldots, k$. Indeed, we have $R_n(w_i) \subset A_\phi$.

Consider now the direct successors $w_{i_1}, \ldots, w_{i_{k_i}}$ of the w_i. The sets $R_n(w_{i_j})$ ($i = 1, \ldots, k$, $j = 1, \ldots, k_i$) are also admissible, say by the formulas $\psi_{i_1}, \ldots, \psi_{i_{k_i}}$. Now, for each i, $\psi_i \to \psi_{i_1} \vee \ldots \vee \psi_{i_{k_i}}$ is not derivable in **IPC**, since this formula is false in w_i. With $\psi_i \vdash \phi$, the formula $\phi \to (\psi_i \to \psi_{i_1} \vee \ldots \vee \psi_{i_{k_i}})$ is equivalent to $\psi_i \to \psi_{i_1} \vee \ldots \vee \psi_{i_{k_i}}$ and hence is not derivable in **IPC** either. Then, because ϕ is exact for $\chi_1, \ldots, \chi_n$, neither is derivable $\psi_i \to \psi_{i_1} \vee \ldots \vee \psi_{i_{k_i}}$ with $\chi_1, \ldots, \chi_n$ substituted for $p_1, \ldots, p_n$, which we denote by $\psi^*_i \to \psi^*_{i_1} \vee \ldots \vee \psi^*_{i_{k_i}}$ ($i = 1, \ldots, k$).

That means that, for each i, a Kripke-model $\mathcal{M}_i^*$ exists such that in its root ψ_i^* is forced but $\psi_{i_1}^*, \ldots, \psi_{i_{k_i}}^*$ are not. Consider the models $\mathcal{M}'_1, \ldots, \mathcal{M}'_k$ where $\mathcal{M}'_i$ has the domain of $\mathcal{M}_i^*$ and valuation defined as follows: $w \models p_i$ iff $w \models \chi_i$ in $\mathcal{M}_i^*$. Thus we have that, for each i, in the root of $\mathcal{M}'_i$ ψ_i is forced but $\psi_{i_1}, \ldots, \psi_{i_{k_i}}$ are not. According to Lemma 3, to each $\mathcal{M}'_i$, there corresponds a unique rooted, p-morphic image $\mathcal{M}''_i$ which is a generated submodel of the n-universal model. Let us show that $\mathcal{M}''_i$ coincides with $\mathcal{M}_i$. Indeed, since $\mathcal{M}''_i \models \psi_i$, it is a submodel of $\mathcal{M}_i$. But since ψ_{i_j} is not forced in the root of $\mathcal{M}''_i$, $j = 1, \ldots, k_i$, it must be that the root of $\mathcal{M}''_i$ coincides with the root of $\mathcal{M}_i$ and thus $\mathcal{M}''_i = \mathcal{M}_i$. So we have that from each $\mathcal{M}'_i$ there exists a p-morphism f_i onto $\mathcal{M}_i$.

Add a new root w^* to the disjoint union of the $\mathcal{M}_i^*$'s with some arbitrary forcing defined on it and denote the resulting model by $\mathcal{M}^*$. Do the same to $\mathcal{M}'_i$'s (with the new root w') and denote the resulting model by $\mathcal{M}'$. We have that $\phi(\chi_1, \ldots, \chi_k)$ is forced in w^* (since it is provable by the exactness of ϕ) and hence ϕ is forced in w'. Now extend the disjoint union of $\mathcal{M}_i$'s by a new root w putting $Col(w) = Col(w')$ thus obtaining a new model $\mathcal{M}$. Now join the p-morphisms $f_1, \ldots, f_k$, add a couple (w', w) and thus form a new p-morphism f from $\mathcal{M}'$ onto $\mathcal{M}$. This makes clear that ϕ is forced in w. Now, as in the proof of Theorem 12 (the end of the first direction), we get that there exists a new point $v \in A_\phi$ such that $v \prec B$. Hence A_ϕ is extendible and ϕ is an extendible formula. $\square$

In order to establish a correspondence between projective, exact, extendible formulas, admissible extendible sets and projective algebras we need the following result of Revaz Grigolia from [11].

Theorem 14 *An algebra H is projective if and only if H_+ is admissible and extendible.*

So we have the following correspondence between projective, exact, extendible formulas, admissible extendible sets and projective algebras: If $\phi \in \Phi_n$ is projective then it is exact, hence it is extendible. Moreover, A_ϕ is an extendible (admissible) set and according to Theorem 14, an algebra $\Phi_n/_{[\phi)}$ is projective. Then we have the retractions $i : \Phi_n/_{[\phi)} \to \Phi_n,\ r : \Phi_n \to \Phi_n/_{[\phi)}$ and hence, the couple $(\Phi_n/_{[\phi)}, ri)$ gives rise to a (projective formula) ϕ. Thus, we can state the following corollary:

Corollary 15 *There is a one-to-one correspondence between projective, exact, extendible formulas, admissible extendible sets and* (H, ri) *couples, where* H *is a projective algebra, and* $i : H \to \Phi_n,\ r : \Phi_n \to H$ *are retractions.*

6 Finite projective formulas

Definition 16 *(First in de Jongh 1968,1970) A formula* $\phi \in \Phi_n$ *is called* finite *if its corresponding admissible set in the* n*-universal model is finite.*

The main purpose of the current section is a characterization of the finite projective formulas of n variables. Recall from Section 3 that every finite upward closed subset of the n-universal model is admissible. Thus, according to Theorem 12, we have to characterize the finite extendible subsets of the n-universal model. First we will show that there is only a finite number of finite extendible sets, then we will construct a combinatorial formula which counts the number of the finite extendible sets in the n-universal model and finally, we will list the finite projective formulas of one and two variables.

6.1 The number of the finite projective formulas of n variables

In order to characterize finite projective formulas we need some definitions. Suppose $A \in ConW_n$, $A_m = W_n^m \cap A = \{w \in A : depth(w) = m\}$.

Definition 17 *A has the* depth m *($depth(A) = m$) if* $\exists w \in A(depth(w) = m)$ *and* $\forall v \in A(depth(v) \leq m)$.

Definition 18 *The width of A is* $max_{1 \leq m \leq depth(A)} |A_m|$.

Definition 19 *The rank of the colour of* $w \in W_n$ *is* $|Col(w)|$ *and we denote it by* $rank(Col(w))$.

For example, $rank(\emptyset) = 0$, $rank(\{1\}) = 1$, $rank(\{1, \ldots, n\}) = n$. So we have $n + 1$ ranks for 2^n colours.

Definition 20 The rank of the level m *in* A, $rank(A_m) = max\{rank(Col(w)) : w \in A_m\}$.

Lemma 21 *Suppose $A \in ExtW_n$ and A is finite, then for no three (distinct) points $w, v, u \in A$, the sets $\{w, v\}$ and $\{w, u\}$ can both be anti-chains.*

Proof. Suppose there exist distinct $w, v, u \in A$ such that the sets $\{w, v\}$ and $\{w, u\}$ are both anti-chains. Since A is extendible, $\{w, v\}$, $\{w, u\}$ must totally cover two points v', u' respectively in A, i.e. $v' \prec \{w, v\}$ and $u' \prec \{w, u\}$. Let us show that v' and u' are R-incomparable and at least one of the sets $\{u', v\}$,$\{v', u\}$ is an anti-chain of A. Indeed, let us show first that $(u', v') \notin R$ (the proof $(v', u') \notin R$ is analogous). Since u' is totally covered by $\{w, u\}$, $u'Rv'$ implies wRv' or uRv'. wRv' is a contradiction since we have $v'Rw$ and $v' \neq w$. If uRv', then from $v'Rw$ it follows that uRw which contradicts the assumption that w and u are incomparable. To show that at least one of the sets $\{u', v\}$,$\{v', u\}$ is an anti-chain of A consider two possible cases: first, u and v are incomparable and second, u and v are not incomparable. Suppose u and v are incomparable. Since u' is totally covered by $\{w, u\}$, $u'Rv$ implies wRv or uRv and both are not the case by our assumption. Hence $(u', v) \notin R$. But $(v, u') \notin R$ as well since, if vRu', then from $u'Rw$ we would get vRw which is a contradiction. Thus in the first case $\{u', v\}$ is an anti-chain. Consider the second case, assume vRu. Let us show that $\{u', v\}$ is an anti-chain. $(u', v) \notin R$, since $u \prec \{w, u\}$ and $(w, v) \notin R$ and $(u, v) \notin R$. It is left to show that $(v, u') \notin R$, but if vRu', then from $u'Rw$ it follows vRw which is a contradiction. Analogously for the second case with the assumption of uRv: we will get that $\{v', u\}$ is an anti-chain.

Hence the conditions of the lemma imply that we get at least two new anti-chains in A with higher level points. The new anti-chains must totally cover two points in A. Then again we get two anti-chains in A and so forth. This contradicts the assumption that A is finite. $\square$

Corollary 22 *Suppose there is given a finite $A \in ExtW_n$ and $w, v, u \in A$. Then if v and u are incomparable and wRv, then we have wRu.*

Proof. Suppose otherwise. Let us show that $(u, w) \notin R$. $(w, u) \notin R$ would get us that w and u are incomparable contradicting the previous lemma. Indeed, if uRw, then from wRv we get uRv contradicting the assumption that $\{v, u\}$ is an anti-chain. $\square$

Corollary 23 *Suppose there is given a finite $A \in ExtW_n$ such that $width(A) \leq 2$. Then each node in A_{m+1} is covered only by node(s) from A_m.*

Proof. Consider $w \in A_{m+1}$ and assume w is covered by some $u \in A_l$, $l < m$. Then, according to the construction, u belongs to some anti-chain $A' \succ w$ containing some point $v \in A_m$; (A is upward closed). But this is a contradiction since u and v are not incomparable; namely we have vRu. Indeed, note that, since A is upward closed, according to the construction of the n-universal model, each node in A_{k+1} is covered by at least one node from A_k. Then if A_k contains two nodes, then, by the previous corollary, both cover each node in A_{k+1}. It follows from here that there exists a chain $v < u_1 \ldots < u_{m-l+1} < u$ and hence vRu. $\square$

Now we can state a proposition which characterizes the finite extendible subsets of the n-universal model. It follows from the proposition that there is a finite number of finite extendible subsets in the n-universal model and hence there is a finite number of finite projective formulas of n variables.

Proposition 24 *(For an algebraic version see R. Balbes and A. Horn [1]). Suppose $A \in ExtW_n$ and A is finite, then $depth(A) \leq n+1$ and $width(A) \leq 2$.*

Proof. A is finite, hence it has finite depth. Say $depth(A) = k$. Note that, since A is extendible, there is only one point $w_k \in A_k$.

If $width(A) \geq 3$ then there exist $w, v, u \in A_m$ ($depth(w) = depth(v) = depth(u) = m$). But then from the fact that A is extendible it follows that there are at least four points of the depth $m+1$ in A totally covered by the four anti-chains, $\{w, v, u\}$, $\{w, v\}$, $\{w, u\}$ and $\{v, u\}$, produced by the set $\{w, v, u\}$. Then it is obvious that the number of the elements of the levels will increase with the growth of the levels. This contradicts the assumption that A is finite.

To prove that $depth(A) \leq n+1$ we will show by induction on the depth of A that the ranks of the levels in A (with $width(A) \leq 2$) decrease (according to the construction of the n-universal model) with the growth of the levels, i.e. $rank(A_{m+1}) < rank(A_m)$, for all $1 \leq m < k$. Since there are $n+1$ ranks we will get that the depth of A may not exceed $n+1$. We start with a

Claim It follows from Corollary 23 that if $w_1, w_2 \in A_m$, then $Col(w_1) \neq Col(w_2)$, $m = 1, \dots, k-1$.

Proof. Indeed, if $w_1, w_2 \in A_1$, then they have different colours. Now suppose we have shown that if $w_1, w_2 \in A_l$, $l \leq m$, then $Col(w_1) \neq Col(w_2)$. Then if there is one point $w \in A_m$ which totally covers two points $v_1, v_2 \in A_{m+1}$ then v_1 and v_2 have different colours and if there are two points $w_1, w_2 \in A_m$ such that the anti-chain $\{w_1, w_2\}$ totally covers two points $v_1, v_2 \in A_{m+1}$ then, again, the colours of v_1 and v_2 must be different. Thus we have the induction step. $\square$

Suppose now that there is one point $w \in A_m$. According to Corollary 23, there are two cases, either w totally covers one point $v \in A_{m+1}$ or w totally covers two points $v_1, v_2 \in A_{m+1}$. In the first case $Col(v) \subset Col(w)$ and hence $rank(A_{m+1}) < rank(A_m)$. In the second case at least one of $Col(v_1)$, $Col(v_2)$ is a proper subset of $Col(w)$ and hence $rank(A_{m+1}) < rank(A_m)$.

Next suppose that there are two points $w_1, w_2 \in A_m$. Again there are two cases: first, the anti-chain $\{w_1, w_2\}$ totally covers one point $v \in A_{m+1}$; second, $\{w_1, w_2\}$ totally covers two points $v_1, v_2 \in A_{m+1}$. In the first case $Col(v) \subseteq Col(w_1) \cap Col(w_2)$, but since $Col(w_1) \neq Col(w_2)$, we have that $Col(w_1) \cap Col(w_2) \subset Col(w_1), Col(w_2)$ and hence $rank(A_{m+1}) < rank(A_m)$. In the second case $Col(v_1), Col(v_2) \subseteq Col(w_1) \cap Col(w_2)$, but since $Col(w_1) \neq Col(w_2)$, we have $Col(w_1) \cap Col(w_2) \subset Col(w_1), Col(w_2)$, and hence $rank(A_{m+1}) < rank(A_m)$. Thus the proposition is proved. $\square$

Corollary 25 *The number of finite projective formulas of n variables is finite.*

Proof. Since each W_n^m is finite, from Proposition 24 it follows that the number of the finite extendible subsets of W_n is finite. Hence there are finitely many finite projective formulas of n variables. $\square$

6.2 Combinatorial formulas.

Using Proposition 24 we can count the number of finite extendible subsets of the n-universal model and thus the number of finite projective formulas of n variables.

We will proceed with the construction of a combinatorial formula which is a counter of the finite projective formulas of n variables for the input n.

First let us count the number of finite extendible subframes of the n-universal frame.

Denote by $S_F(n)$ the number of finite extendible subframes of $\mathcal{F}_n$. Then

$$S_F(n) = \sum_{k=1}^{n+1} F(k),$$

where $F(k)$ is the number of the finite extendible subframes of the depth k. According to Proposition 24, k can take the values from 1 to $n+1$. Now,

$$F(k) = 2^{k-1}.$$

Indeed, according to Proposition 24, in each level $m = 1, \dots, k-1$ of the finite extendible set of the depth k there may be one or two points and there must be one point of level k. Denote the number of points in the level m by i_m. It follows from Corollary 22 (see the proof of Corollary 23) that each point in the level m covers each point in the level $m+1$. In addition, by Corollary 23, nodes in A_{m+1} are covered only by nodes from A_m. Thus, each of the vectors $(i_1, \dots, i_k)$, $i_1, \dots, i_{k-1} = 1, 2$ and $i_k = 1$ determines one frame. Now, the number of different vectors $(i_1, \dots, i_k)$, $i_1, \dots, i_{k-1} = 1, 2$ and $i_k = 1$, is 2^{k-1}. Thus,

$$S_F(n) = \sum_{k=1}^{n+1} 2^{k-1} = 2^{n+1} - 1.$$

We will proceed with the construction of the combinatorial formula which is a counter of the finite projective formulas of n variables for the input n.

As this subsection uses a strict theoretical approach to the computation, we recommend to consider it along with the appendix which describes the same calculations in more practical style.

Denote by $S_M(n)$ the number of finite extendible sets in $\mathcal{M}_n$ (the number of submodels).

$$S_M(n) = \sum_{k=1}^{n+1} M(k),$$

where $M(k)$ is the number of admissible extendible sets with the depth k in $\mathcal{M}_n$. According to Proposition 24, k can take the values $1, \ldots, n+1$.

$$M(k) = \sum_{i_1=1, i_2, \ldots, i_k=1,2} F(i_1, \ldots, i_k),$$

where $F(i_1, \ldots, i_k)$ is the number of such finite extendible subsets (submodels) of $\mathcal{M}_n$ which have i_j nodes ($i_j = 1$ or 2) on the level $k - j + 1$ ($j = 1, \ldots, k$). From the definition of the extendible set it follows that $i_1 = 1$ and, according to Proposition 24, each i_j, $j = 2, \ldots, k$, can have only the values 1 or 2.

Fix a frame of depth k. So, we have all values for the parameters $i_1, i_2, \ldots, i_k$, ($i_1 = 1$). The construction of the n-universal model limits the number of acceptable colours that a node w_1 of the depth k can have. Denote this number by f^{i_1} (f^1). If we fix some (acceptable) colour for w_1, then we will have a number of possible colours for the nodes of the depth $k-1$ which totally cover w_1. Denote this number by $f^{i_2}_{i_1}$, (i_2 – the number of nodes that totally cover w_1). Analogously, let $f^{i_j}_{i_{j-1}}$ be the number of all possible (colours of) nodes of the depth $k - j + 1$ which correspond (totally cover) to already fixed nodes of the depth $k - j + 2$.

We will see in the sequel, that each of the expressions $f^{i_j}_{i_{j-1}}$, $j = 2, \ldots, k$, is a sum which uses fixed parameters (the number of nodes in the previous level and the ranks of their colours) from the sum $f^{i_{j-1}}_{i_{j-2}}$. Then $F(i_1, \ldots, i_k)$ can be obtained by multiplying the (nested) sums $f^{i_1}, f^{i_2}_{i_1}, f^{i_3}_{i_2}, \ldots, f^{i_k}_{i_{k-1}}$, which we write in the following way:

$$F(i_1, \ldots, i_k) = f^{i_1} f^{i_2}_{i_1} f^{i_3}_{i_2} \ldots f^{i_k}_{i_{k-1}}.$$

Let us agree on the following notation: if there is one node of the depth $k - j + 1$, then denote it by w_j (so e.g. w_1 is the unique node of depth k) and $l_j = rank(Col(w_j))$. If there are two nodes of the depth $k - j + 1$, then denote them by w^1_j and w^2_j, $l^0_j = rank(Col(w^1_j) \cap Col(w^2_j))$ and $l^1_j = rank(Col(w^1_j))$, $l^2_j = rank(Col(w^2_j))$. We denote by C^n_k the choose of k over n. Then

$$f^1 = \sum_{l_1=0}^{n-k+1} C^n_{l_1},$$

Indeed, the rank of the colour of a node w_1 of the depth k can vary from

0 to $n-k+1$ and to each possible rank $l_1 = 0, \ldots, n-k+1$ there correspond $C^n_{l_1}$ different variants (colours).

There are the following four different cases for $f^{i_j}_{i_{j-1}}$:

1. $i_{j-1} = 1, i_j = 1$;
2. $i_{j-1} = 2, i_j = 1$;
3. $i_{j-1} = 1, i_j = 2$;
4. $i_{j-1} = 2, i_j = 2$.

We will consider each case separately.

1. This is the case when on the each of the levels $k-j+2$ and $k-j+1$ there is one node. In order for w_j to totally cover w_{j-1} it must be that $Col(w_{j-1}) \subset Col(w_j)$. This means $l_j > l_{j-1}$. At the same time, $l_j \leq n-k+j$. In $Col(w_j)$ l_{j-1} elements are fixed and there are free $l_j - l_{j-1}$ elements. The number of elements from which we can choose elements to fill the free places in $Col(w_j)$ is $n - l_{j-1}$. It follows from the above that:

$$f^1_1(j) = \sum_{l_j = l_{j-1}+1}^{n-k+j} C^{n-l_{j-1}}_{l_j - l_{j-1}}.$$

2. This is the case when one node of the depth $k-j+1$ totally covers two nodes of the depth $k-j+2$. In order for the node w_j to totally cover the nodes w^1_{j-1}, w^2_{j-1}, it must be that $Col(w^1_{j-1}) \cup Col(w^2_{j-1}) \subset Col(w_j)$ (*), hence l_j can take the values from $l^1_{j-1} + l^2_{j-1} - l^0_{j-1} + s$ to $n-k+j$, where $s = \begin{cases} 1, & \text{if } l^1_{j-1} = l^0_{j-1} \text{ or } l^2_{j-1} = l^0_{j-1} \\ 0, & \text{otherwise.} \end{cases}$

The value of s forces the strict set theoretical inclusion in (*). For each $\{w^1_{j-1}, w^2_{j-1}\}$, we have $l^1_{j-1} + l^2_{j-1} - l^0_{j-1}$ fixed elements and $l_j - (l^1_{j-1} + l^2_{j-1} - l^0_{j-1})$ free elements in $Col(w_j)$. The number of elements from which we can choose elements to fill the free spaces in $Col(w_j)$ is $n - (l^1_{j-1} + l^2_{j-1} - l^0_{j-1})$. Thus:

$$f^1_2(j) = \sum_{l_j = l^1_{j-1} + l^2_{j-1} - l^0_{j-1} + s}^{n-k+j} C^{n-(l^1_{j-1}+l^2_{j-1}-l^0_{j-1})}_{l_j-(l^1_{j-1}+l^2_{j-1}-l^0_{j-1})}; s = \begin{cases} 1, & \text{if } l^1_{j-1} = l^0_{j-1} \text{ or } l^2_{j-1} = l^0_{j-1} \\ 0, & \text{otherwise.} \end{cases}$$

3. This is the case when there is one element of the depth $k-j+2$ totally covered by two elements of the depth $k-j+1$. In this case we will consider two sums. First will be the number of anti-chains $\{w^1_j, w^2_j\}$ which totally cover w_{j-1} with $l^1_j < l^2_j$. Second will be the number of anti-chains $\{w^1_j, w^2_j\}$ which totally cover w_{j-1} with $l^1_j = l^2_j$.

In the first sum, l_j^0 can take values from l_{j-1} to $n-k+j-1$. Indeed, in order for the anti-chain $\{w_j^1, w_j^2\}$ to totally cover node w_{j-1} it must be that $Col(w_{j-1}) \subseteq Col(w_j^1) \cap Col(w_j^2)$. Here $n-k+j-1$ is an upper bound instead of $n-k+j$ because the colours of w_j^1 and w_j^2 may not coincide. In $Col(w_j^1) \cap Col(w_j^2)$ there are fixed l_{j-1} elements and free $l_j^0 - l_{j-1}$ elements. The number of elements from which we can choose elements to fill the free spaces in $Col(w_j^1) \cap Col(w_j^2)$ is $n-l_{j-1}$, hence there are $C_{l_j^0-l_{j-1}}^{n-l_{j-1}}$ such variants (ranks of possible intersections) for each w_{j-1}. After this, for each possible w_j^1, in $Col(w_j^1)$ there are fixed l_j^0 elements and free $l_j^1 - l_j^0$ elements. The number of elements from which we can choose elements to fill the free spaces in $Col(w_j^1)$ is $n-l_j^0$, hence for each l_j^1, which can take the values from l_j^0 to $n-k+j-1$ (not $n-k+j$ because this maximal value will be taken by rank of w_j^2), the number of possible w_j^1's for each w_{j-1} is $C_{l_j^0-l_{j-1}}^{n-l_{j-1}} C_{l_j^1-l_j^0}^{n-l_j^0}$. After the node w_j^1 is fixed, in $Col(w_j^2)$ we get l_j^0 fixed elements and $l_j^2 - l_j^0$ free elements. We can fill the free spaces in $Col(w_j^2)$ only using those elements which do not belong to $Col(w_j^1)$. The number of such elements is $n-l_j^1$. Hence there are $C_{l_j^2-l_j^0}^{n-l_j^1}$ possible variants of w_j^2's for the fixed w_j^1. Since we are considering our first sum, we must have $l_j^2 > l_j^1$ and thus the first sum is:

$$\sum_{l_j^0=l_{j-1}}^{n-k+j-1} \sum_{l_j^1=l_j^0}^{n-k+j-1} C_{l_j^0-l_{j-1}}^{n-l_{j-1}} C_{l_j^1-l_j^0}^{n-l_j^0} \sum_{l_j^2=l_j^1+1}^{n-k+j} C_{l_j^2-l_j^0}^{n-l_j^1}. \quad (1)$$

In the second sum, as in the first, l_j^0 can take values from l_{j-1} to $n-k+j-1$ and the number of ranks of possible intersections is $C_{l_j^0-l_{j-1}}^{n-l_{j-1}}$ (for each w_{j-1}). Since we are considering the case $l_j^1 = l_j^2$, we cannot have $l_j^1 = l_j^0$ (if so, we would get $Col(w_j^1) = Col(w_j^2)$); so here l_j^1 ranges from l_j^0+1 to $n-k+j$. For each w_j^1, there are l_j^0 elements fixed and $n-l_j^0$ elements free in $Col(w_j^2)$. We can fill the free spaces in $Col(w_j^2)$ only using those elements which do not belong to $Col(w_j^1)$. The number of such elements is $n-l_j^1$. Hence there are $C_{l_j^1-l_j^0}^{n-l_j^1}$ possible variants of w_j^2's for the fixed w_j^1. But now we cannot multiply the number of possible w_j^1's (which is $C_{l_j^1-l_j^0}^{n-l_j^0}$) by the number of corresponding w_j^2's, because, in this way we would get repetitions, i.e. we would *count* two anti-chains $\{w_j^1, w_j^2\}$ and $\{w_j^2, w_j^1\}$ instead of one. In fact, this is the case when there is given a set of n_1 elements and each element forms (*unordered*) pairs with n_2 ($n_2 < n_1$) other elements of the same set. The number of all pairs then will be equal to $(n_1 n_2)/2$; if the product $n_1 n_2$ is an odd number, then it means that, for given numbers n_1, n_2, such pairs cannot be formed (for example: $n_1 = 3$, $n_2 = 1$). Since in our case we are preserved from the odd values of $C^{n-l0}_{l1-l0} C^{n-l1}_{l1-l0}$, we take our second sum to be:

$$\sum_{l_j^0 = l_{j-1}}^{n-k+j-1} \sum_{l_j^1 = l_j^2 = l_j^0 + 1}^{n-k+j} C^{n-l_{j-1}}_{l_j^0 - l_{j-1}} (C^{n-l_j^0}_{l_j^1 - l_j^0} C^{n-l_j^1}_{l_j^1 - l_j^0})/2, \quad (2).$$

Finally:

$$f_1^2(j) = (1) + (2).$$

4. This is the case when there are two nodes of the depth $k - j + 1$ which form an anti-chain which totally covers each of the two nodes of the depth $k - j + 2$. This case is analogous to the previous case. Here we consider $Col(w^1_{j-1}) \cup Col(w^2_{j-1})$ instead of $Col(w_{j-1})$ as in the previous case. Thus we have:

$$f_2^2(j) = \sum_{l_j^0 = l^1_{j-1} + l^2_{j-1} - l^0_{j-1}}^{n-k+j-1} \sum_{l_j^1 = l_j^0}^{n-k+j-1} C^{n-(l^1_{j-1} + l^2_{j-1} - l^0_{j-1})}_{l_j^0 - (l^1_{j-1} + l^2_{j-1} - l^0_{j-1})} C^{n-l_j^0}_{l_j^1 - l_j^0} \sum_{l_j^2 = l_j^1 + 1}^{n-k+j} C^{n-l_j^1}_{l_j^2 - l_j^0} +$$

$$+ \sum_{l_j^0 = l^1_{j-1} + l^2_{j-1} - l^0_{j-1}}^{n-k+j-1} \sum_{l_j^1 = l_j^2 = l_j^0 + 1}^{n-k+j} C^{n-(l^1_{j-1} + l^2_{j-1} - l^0_{j-1})}_{l_j^0 - (l^1_{j-1} + l^2_{j-1} - l^0_{j-1})} (C^{n-l_j^0}_{l_j^1 - l_j^0} C^{n-l_j^1}_{l_j^1 - l_j^0})/2.$$

Below there is presented the formula $S_{\mathcal{M}}(n)$ which gives the number of the finite projective formulas of n variables in **IPC**. The computer program realizing this computation can be found in the appendix.

$$S_M(n) = \sum_{k=1}^{n+1} M(k),$$

where

$$M(k) = \sum_{i_1 = 1, i_2, \ldots, i_k = 1,2} F(i_1, \ldots, i_k),$$

where

$$F(i_1,\ldots,i_k) = f^{i_1} f^{i_2}_{i_1} f^{i_3}_{i_2} \ldots f^{i_k}_{i_{k-1}},$$

where

$$f^1 = \sum_{l_1=0}^{n-k+1} C^n_{l_1},$$

$$f^1_1(j) = \sum_{l_j=l_{j-1}+1}^{n-k+j} C^{n-l_{j-1}}_{l_j-l_{j-1}},$$

$$f^1_2(j) = \sum_{l_j=l^1_{j-1}+l^2_{j-1}-l^0_{j-1}+s}^{n-k+j} C^{n-(l^1_{j-1}+l^2_{j-1}-l^0_{j-1})}_{l_j-(l^1_{j-1}+l^2_{j-1}-l^0_{j-1})}; s = \begin{cases} 1, & \text{if } l^1_{j-1} = l^0_{j-1} \text{ or } l^2_{j-1} = l^0_{j-1} \\ 0, & \text{otherwise.} \end{cases},$$

$$f^2_1(j) = \sum_{l^0_j=l_{j-1}}^{n-k+j-1} \sum_{l^1_j=l^0_j}^{n-k+j-1} C^{n-l_{j-1}}_{l^0_j-l_{j-1}} C^{n-l^0_j}_{l^1_j-l^0_j} \sum_{l^2_j=l^1_j+1}^{n-k+j} C^{n-l^1_j+l^0_j}_{l^2_j-l^0_j} +$$

$$+ \sum_{l^0_j=l_{j-1}}^{n-k+j-1} \sum_{l^1_j=l^2_j=l^0_j}^{n-k+j} C^{n-l_{j-1}}_{l^0_j-l_{j-1}} (C^{n-l^0_j}_{l^1_j-l^0_j} C^{n-l^1_j}_{l^1_j-l^0_j})/2,$$

$$f^2_2(j) = \sum_{l^0_j=l^1_{j-1}+l^2_{j-1}-l^0_{j-1}}^{n-k+j-1} \sum_{l^1_j=l^0_j}^{n-k+j-1} C^{n-(l^1_{j-1}+l^2_{j-1}-l^0_{j-1})}_{l^0_j-(l^1_{j-1}+l^2_{j-1}-l^0_{j-1})} C^{n-l^0_j}_{l^1_j-l^0_j} \sum_{l^2_j=l^1_j+1}^{n-k+j} C^{n-l^1_j+l^0_j}_{l^2_j-l^0_j} +$$

$$+ \sum_{l^0_j=l^1_{j-1}+l^2_{j-1}-l^0_{j-1}}^{n-k+j-1} \sum_{l^1_j=l^2_j=l^0_j}^{n-k+j} C^{n-(l^1_{j-1}+l^2_{j-1}-l^0_{j-1})}_{l^0_j-(l^1_{j-1}+l^2_{j-1}-l^0_{j-1})} (C^{n-l^0_j}_{l^1_j-l^0_j} C^{n-l^1_j}_{l^1_j-l^0_j})/2,$$

We have conducted calculations for $n = 1, 2, 3, 4, 5$ and 6. As it was known, there are four finite formulas of one variable. There are 26 finite formulas of two variables; for $n = 3$ there are 256 finite formulas; for $n = 4$: 3386; for $n = 5$: 55984 and for $n = 6$: 1110506.

6.3 Finite projective formulas of one variable

It was shown by D. de Jongh (see [4]) that there are five projective formulas of one variable in **IPC**, four of which are finite. Now we will give an easier proof of this fact using the colouring technique.

So, we restrict ourselves to the case of $n = 1$. Proposition 24, makes it possible to write out all of the finite extendible subsets of W_1:

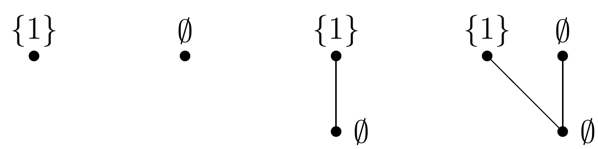

Fig.5

W_1 is shown in Fig.1. To each of the finite extendible sets in Fig.5. there corresponds a finite projective formula of one variable. Now we will introduce a technique by which we can find the projective formulas which correspond to the given finite extendible sets.

Let us recall an equivalent definition of the algebraic operation $\to$: For $A, B \in ConW$, $A \to B = -R^{-1}(A \setminus B)$, where $R^{-1}(w) = \{v : vRw\}$ and $R^{-1}(A) = \bigcup_{w \in A} R^{-1}(w)$. Thus $\neg A = A \to \emptyset = -R^{-1}(A)$ and $A \leftrightarrow B = -R^{-1}(A \setminus B) \cap -R^{-1}(B \setminus A)$.

The free algebra Φ_n is generated by $p_1, \ldots, p_n$. Hence its isomorphic algebra $AdmW_n$ is generated by $G_i = h(p_i) = \{w \in W_n : w \models_n p_i\}$, $i = 1, \ldots, n$. Thus, any admissible extendible subset of W_n can be obtained by applying the algebraic operations $\cap, \cup, \to, \emptyset$ to the sets $G_1, \ldots, G_n$. In the case $n = 1$, we have $G = \{w \in W_n : Col(w) = \{1\}\}$. Denote by p the propositional variable (free generator of the algebra Φ_1). Thus G is the subset of W_1 in every point of which p is forced.

Now we give the finite projective formulas which correspond to the finite extendible sets shown in Fig.5

$\phi_1 = p$; first in Fig.5, which is G itself.
$\phi_2 = \neg p$; second in Fig.5, which is $-R^{-1}(G)$.

$\phi_3 = \neg\phi_2$; third in Fig.5, which is $-R^{-1}(\phi_2)$.
$\phi_4 = \phi_3 \rightarrow p$; fourth in Fig.5, which is $-R^{-1}(\phi_3 \setminus G)$.

6.4 Finite projective formulas of two variables

Let us restrict ourselves to the case of $n = 2$. Proposition 24 makes it possible to write out all of the finite extendible subsets of W_2; first we will see the finite extendible frames (Fig.6) and then all existing models on those frames (Fig.7):

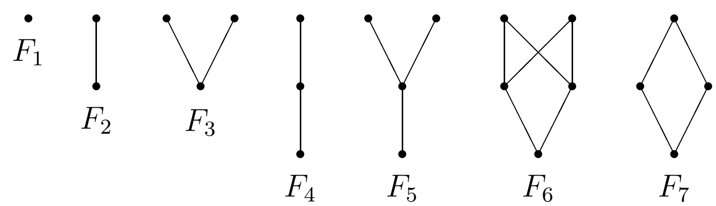

Fig.6

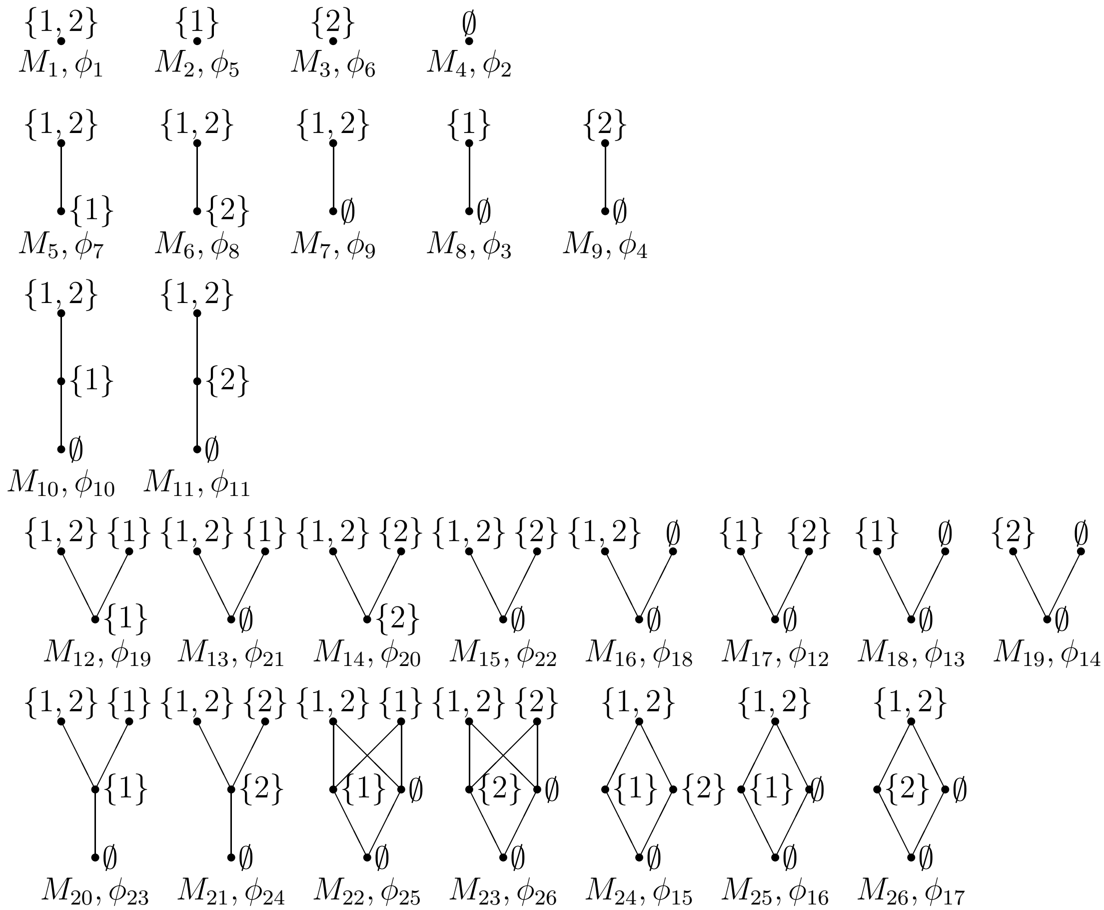

Fig.7

There are 26 finite extendible subsets of W_2 and to each of them there corresponds a finite projective formula of two variables.

In the case $n = 2$ we have, $G_1 = \{w \in W_n : Col(w) = \{1\}$ or $Col(w) = \{1,2\}\}$ and $G_2 = \{w \in W_n : Col(w) = \{2\}$ or $Col(w) = \{1,2\}\}$, Denote by p, q two propositional variables (the free generators of the algebra Φ_2). Thus G_1 contains one point in which both p and q are forced and the rest of the points of G_1 make p true and q false. Analogously G_2 contains one point in which both p and q are forced and the rest of the points of G_2 make q true and p false.

Now we give the finite projective formulas which correspond to the finite extendible sets shown in Fig.7.

$\phi_1 = p \wedge q$; M_1, which is $G_1 \cap G_2$;
$\phi_2 = \neg(p \vee q)$; M_4, which is $-R^{-1}(G_1 \cup G_2)$;
$\phi_3 = \neg(q \vee \phi_2)$; M_8, $-R^{-1}(G_2 \cup \phi_2)$;
$\phi_4 = \neg(p \vee \phi_2)$; M_9, $-R^{-1}(G_1 \cup \phi_2)$;
$\phi_5 = \phi_3 \wedge p$; M_2, $\phi_3 \cap G_1$;
$\phi_6 = \phi_4 \wedge q$; M_3, $\phi_4 \cap G_2$;

Define $U = \neg(\phi_2 \vee \phi_3 \vee \phi_4)$. In $U_1 = U \cap W_n^1 = \{w \in U : depth(w) = 1\}$ there is one point u with the colour $\{1, 2\}$, in U_2 there are three points which are totally covered by u and on each next level of U there are only the points totally covered by the points or anti-chains only from the previous levels of U. Then,

$\phi_7 = U \wedge p$; M_5, $U \cap G_1$;
$\phi_8 = U \wedge q$; M_6, $U \cap G_2$;
$\phi_9 = \neg(\phi_7 \vee \phi_8) \wedge U \vee \phi_1$; M_7, $-R^{-1}(\phi_7 \cup \phi_8) \cap U \cup \phi_1$;
$\phi_{10} = ((\phi_8 \vee \phi_9) \to \phi_1) \wedge U$; M_{10}, $-R^{-1}((\phi_8 \cup \phi_9) \setminus \phi_1) \cap U$;
$\phi_{11} = ((\phi_7 \vee \phi_9) \to \phi_1) \wedge U$; M_{11}, $-R^{-1}((\phi_7 \cup \phi_9) \setminus \phi_1) \cap U$;
$\phi_{12} = (\phi_3 \to \phi_5) \wedge (\phi_4 \to \phi_6) \wedge \neg(\phi_1 \vee \phi_2)$; M_{17}, $-R^{-1}(\phi_3 \setminus \phi_5) \cap -R^{-1}(\phi_4 \setminus \phi_6) \cap -R^{-1}(\phi_1 \cup \phi_2)$;
$\phi_{13} = \neg(\phi_1 \vee \phi_6) \wedge (\phi_3 \to \phi_5)$; M_{18}, $-R^{-1}(\phi_1 \cup \phi_6) \cap -R^{-1}(\phi_3 \setminus \phi_5)$;
$\phi_{14} = \neg(\phi_1 \vee \phi_5) \wedge (\phi_4 \to \phi_6)$; M_{19}, $-R^{-1}(\phi_1 \cup \phi_5) \cap -R^{-1}(\phi_4 \setminus \phi_6)$;
$\phi_{15} = (\phi_{10} \to \phi_7) \wedge (\phi_{11} \to \phi_8) \wedge (\phi_9 \to \phi_1) \wedge U$; M_{24}, $-R^{-1}(\phi_{10} \setminus \phi_7) \cap -R^{-1}(\phi_{11} \setminus \phi_8) \cap -R^{-1}(\phi_9 \setminus \phi_1) \cap U$;
$\phi_{16} = (\phi_{10} \to \phi_7) \wedge (\phi_8 \to \phi_1) \wedge U$; M_{25}, $-R^{-1}(\phi_{10} \setminus \phi_7) \cap -R^{-1}(\phi_8 \setminus \phi_1) \wedge U$;
$\phi_{17} = (\phi_{11} \to \phi_8) \wedge (\phi_7 \to \phi_1) \wedge U$; M_{26}, $-R^{-1}(\phi_{11} \setminus \phi_8) \cap -R^{-1}(\phi_7 \setminus \phi_1) \wedge U$;

Define $\bar{U} = (\phi_7 \to \phi_1) \vee (\phi_8 \to \phi_1) \vee (\phi_9 \to \phi_1)$. $\bar{U} = W_2 \setminus U \cup \{u\}$, ($U$ and u were defined above). Then,

$\phi_{18} = \neg(\phi_5 \to \phi_6) \wedge \bar{U}$; M_{16}, $-R^{-1}(\phi_5 \setminus \phi_6) \cap \bar{U}$;
$\phi_{19} = \neg(\phi_6 \vee \phi_2) \wedge \bar{U} \wedge p$; M_{12}, $-R^{-1}(\phi_6 \cup \phi_2) \cap \bar{U} \cap p$;
$\phi_{20} = \neg(\phi_5 \vee \phi_2) \wedge \bar{U} \wedge q$; M_{14}, $-R^{-1}(\phi_5 \cup \phi_2) \cap \bar{U} \cap p$;
$\phi_{21} = \neg(\phi_6 \vee \phi_2) \wedge \bar{U} \wedge (\phi_{19} \to (\phi_1 \vee \phi_5)) \wedge (\phi_3 \to \phi_5)$; M_{13}, $-R^{-1}(\phi_6 \cup \phi_2) \cap \bar{U} \cap -R^{-1}(\phi_{19} \setminus (\phi_1 \cup \phi_5)) \cap -R^{-1}(\phi_3 \setminus \phi_5)$;

$\phi_{22} = \neg(\phi_5 \vee \phi_2) \wedge \bar{U} \wedge (\phi_{20} \rightarrow (\phi_1 \vee \phi_6)) \wedge (\phi_4 \rightarrow \phi_6)$; M_{15}, $-R^{-1}(\phi_5 \cup \phi_2) \cap \bar{U} \cap -R^{-1}(\phi_{20} \setminus (\phi_1 \cup \phi_6)) \cap -R^{-1}(\phi_4 \setminus \phi_6)$;

$\phi_{23} = (\neg(\phi_2 \vee \phi_6) \wedge \bar{U}) \wedge (\phi_{21} \rightarrow (\phi_1 \vee \phi_5)) \wedge (\phi_3 \rightarrow \phi_5)$; M_{20}, $-R^{-1}(\phi_2 \cup \phi_6) \cap \bar{U} \cap -R^{-1}(\phi_{21} \setminus (\phi_1 \cup \phi_5)) \cap -R^{-1}(\phi_3 \setminus \phi_5)$;

$\phi_{24} = (\neg(\phi_2 \vee \phi_5) \wedge \bar{U}) \wedge (\phi_{22} \rightarrow (\phi_1 \vee \phi_6)) \wedge (\phi_4 \rightarrow \phi_6)$; M_{21}, $-R^{-1}(\phi_2 \cup \phi_5) \cap \bar{U} \cap -R^{-1}(\phi_{22} \setminus (\phi_1 \cup \phi_6)) \cap -R^{-1}(\phi_4 \setminus \phi_6)$;

$\phi_{25} = (\neg(\phi_2 \vee \phi_6) \wedge \bar{U}) \wedge (\phi_3 \rightarrow \phi_5) \wedge (\phi_{23} \rightarrow \phi_{19})$; M_{22}, $-R^{-1}(\phi_2 \cup \phi_6) \cap \bar{U} \cap -R^{-1}(\phi_3 \cup \phi_5) \cap -R^{-1}(\phi_{23} \setminus \phi_{19})$;

$\phi_{26} = (\neg(\phi_2 \vee \phi_5) \wedge \bar{U}) \wedge (\phi_4 \rightarrow \phi_6) \wedge (\phi_{24} \rightarrow \phi_{20})$; M_{23}, $-R^{-1}(\phi_2 \cup \phi_5) \cap \bar{U} \cap (\phi_4 \setminus \phi_6) \cap -R^{-1}(\phi_{24} \setminus \phi_{20})$.

The formulas, $\phi_1, \ldots, \phi_{26}$ were obtained using Fig.2 and Fig.7 and the set theoretical operations ($\cup, \cap, \setminus$ and $-$) on the sets G_1, G_2 which correspond dually to the propositional variables p, q. Already obtained finite formulas were used for finding new finite formulas so most of the formulas look complicated and can be simplified. The final, simplified, list of the finite projective formulas of two variables is the following:

$\phi_1 = p \wedge q$;
$\phi_2 = \neg p \wedge \neg q$;
$\phi_3 = \neg\neg p \wedge \neg q$;
$\phi_4 = \neg\neg q \wedge \neg p$;
$\phi_5 = \neg q \wedge p$;
$\phi_6 = \neg p \wedge q$;
$\phi_7 = p \wedge \neg\neg q$;
$\phi_8 = q \wedge \neg\neg p$;
$\phi_9 = (p \leftrightarrow q) \wedge \neg\neg p$;
$\phi_{10} = \neg\neg q \wedge ((p \rightarrow q) \rightarrow p)$;
$\phi_{11} = \neg\neg p \wedge ((q \rightarrow p) \rightarrow q)$;
$\phi_{12} = (p \leftrightarrow \neg q) \wedge (q \leftrightarrow \neg p)$;
$\phi_{13} = \neg q \wedge (\neg\neg p \rightarrow p)$;
$\phi_{14} = \neg p \wedge (\neg\neg q \rightarrow q)$;
$\phi_{15} = \neg\neg p \wedge \neg\neg q \wedge ((p \rightarrow q) \rightarrow q) \wedge ((q \rightarrow p) \rightarrow p)$;
$\phi_{16} = \neg\neg q \wedge (q \rightarrow p)$;
$\phi_{17} = \neg\neg p \wedge (p \rightarrow q)$;
$\phi_{18} = (p \leftrightarrow q) \wedge (\neg\neg p \rightarrow p)$;
$\phi_{19} = p \wedge (\neg\neg q \rightarrow q)$;
$\phi_{20} = q \wedge (\neg\neg p \rightarrow p)$;

$\phi_{21} = (\neg\neg q \to q) \wedge (p \leftrightarrow (q \vee \neg q))$;
$\phi_{22} = (\neg\neg p \to p) \wedge (q \leftrightarrow (p \vee \neg p))$;
$\phi_{23} = ((p \to q \vee \neg q) \to p) \wedge (\neg\neg q \to q)$;
$\phi_{24} = ((q \to p \vee \neg p) \to q) \wedge (\neg\neg p \to p)$;
$\phi_{25} = \neg\neg p \wedge (\neg\neg q \to q) \wedge (((p \to q \vee \neg q) \to p) \to p)$;
$\phi_{26} = \neg\neg q \wedge (\neg\neg p \to p) \wedge (((q \to p \vee \neg p) \to q) \to q)$.

6.5 Number of all projective formulas

Now we proceed with the calculation of the number of all projective formulas of n variables in **IPC**. We will construct an infinite number of admissible extendible sets in the n-universal model from which it follows that there is a countably infinite number of projective formulas of n variables.

We need the following observation in order to proceed with the construction of the needed admissible extendible sets:

Proposition 26 *Consider $w \in W_n$ and $A \subseteq W_n$ a finite anti-chain such that $w \prec A$ and there exists $v \in W_n$ such that $v \neq w$ and $v \prec A$. Then the set $-R^{-1}(w)$ is extendible.*

Proof. Assume $-R^{-1}(w)$ is not extendible. Then there exists a finite anti-chain $B \subseteq -R^{-1}(w)$ such that, for all $u \in W_n$, from $u \prec B$ it follows that $u \notin -R^{-1}(w)$. Suppose $u \prec B$, then $u \in R^{-1}(w)$. Then uRw. Since $u \prec B$ and $w \notin B$ it must be that $u = w$. But then $B = A$. Note that since $w \prec A$ and $v \prec A$, $v \in -R^{-1}(w)$ which contradicts the assumption that all the points that are totally covered by B belong to $R^{-1}(w)$. $\square$

Theorem 27 *There is an infinite number of projective formulas of n variables, $n \geq 2$.*

Proof. We will prove this theorem by constructing countable number of admissible extendible subsets of W_n. Since there is a one-to-one correspondence between the elements of Φ_n and admissible sets, the theorem will be proved.

Consider the Rieger Nishimura ladder type infinite subset of W_n ($n \geq 2$) shown on Fig.8.

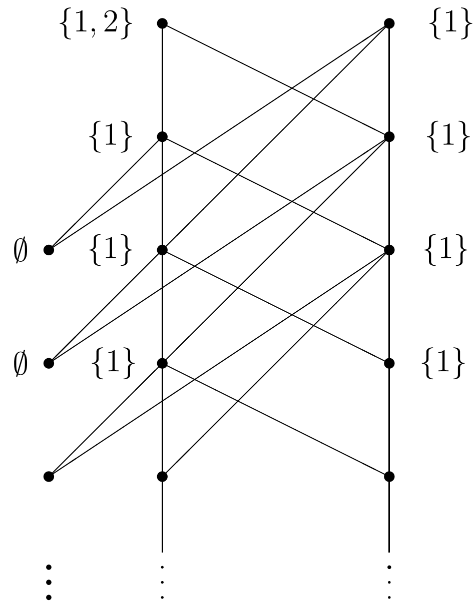

Fig.8

This set is generated by two elements of the first level of W_n with the colours $\{1,2\}$ and $\{1\}$. The element with the colour $\{1,2\}$ totally covers an element with the colour $\{1\}$. Then two elements of the first level totally cover an element of the second level with the colour $\{1\}$. Now, there are two anti-chains and each of them totally covers an element with the colour $\{1\}$. Each of these anti-chains also covers an element with the colour $\emptyset$. In Fig.8 we show only one such element totally covered by the one of these anti-chains. If we proceed, we will get the picture shown in Fig.8. Now, the elements with the colour $\emptyset$ in Fig.8 satisfy the conditions of the previous proposition and hence every such element w gives rise to a unique extendible set $-R^{-1}(w)$. There is a countable number of such elements. Thus we have constructed infinitely many infinite extendible subsets of W_n. Consult Grigolia [11] for the fact that for all $w \in W_n$, $-R^{-1}(w)$ is admissible. Thus we have constructed countably many infinite admissible extendible sets in the n-universal model.
□

Theorem 28 *If $A \in ConW_n$, then $-R^{-1}(A) \in ExtW_n$.*

Proof. The proof is analogous of the proof of Proposition 26. Assume $-R^{-1}(A)$ is not extendible. Then there exists a finite anti-chain $B \subseteq -R^{-1}(A)$ such that, for all $w \in W_n$, from $w \prec B$ it follows that $w \notin -R^{-1}(A)$. Suppose $w \prec B$, then $w \in R^{-1}(A)$. Then wRv for some $v \in A$, which is a contradiction since B totally covers w and $B \cap A = \emptyset$. □

For any $A \in ConW_n$, $-R^{-1}(A) = -R^{-1}(A_1)$ (where $A_1 = W_n^1 \cap A$). Hence, each formula of the type $\neg\phi$ is projective. The exception is $\perp$ ($\emptyset = -R^{-1}(W_n)$) which we do not consider as a projective formula. Since there are 2^n elements in W_n^1, we get that there are 2^{2^n} different subsets of W_n^1 and hence $2^{2^n} - 1$ nonequivalent negated projective formulas of n variables.

It also follows from the above that in **IPC** there are 2^{2^n} nonequivalent negated formulas of n variables, but this is of course not surprising, since it is well-known that in **IPC** negated formulas behave exactly as in classical logic.

Appendix. Computation of finite extendible subsets of the n-universal model

Here we give a description of the computation of finite extendible sets in the n-universal model and a computer program realizing these computations.

We refer to Section 6 (subsection 6.2) for more theoretical details.

If we can count the number $M(k)$ of all possible finite extendible models of depth k, then we can easily take the sum:

$$S_M(n) = \sum_{k=1}^{n+1} M(k)$$

to be the number of all finite extendible models (see subsection 6.2).

For a given depth k, different vectors $(i_1, \ldots, i_k)$ $(i_1, \ldots, i_{k-1} = 1, 2$ and $i_k = 1)$ represent the possible (finite) extendible frames which have i_k nodes on the first level, i_{k-1} nodes on the second level and so on (with $i_1 = 1$ node as the root). So

$$M(k) = \sum_{i_1=1, i_2, \ldots, i_k=1,2} F(i_1, \ldots, i_k),$$

where $F(i_1, \ldots, i_k)$ is the number of such finite extendible sets which have i_j points ($i_j = 1$ or 2) in the level $k - j + 1$ ($j = 1, \ldots, k$).

To calculate the function F for given parameters $i_1, \ldots, i_k$, fix a frame of depth k. The construction of the n-universal model limits the number of acceptable colours that a node w of the depth k can have. If we fix some (acceptable) colour for w, then we will have a number of possible colours for the nodes of the depth $k-1$ which totally cover w. Then for the fixed (colours of) nodes of the depth $k-1$ we have a number of possible (colours for) nodes of the depth $k-2$ and so on. This approach brings us to a computational tree starting from the root and going up which considers all the possible cases of 'moving' from already fixed colours of the given level to the possible colours of nodes of the next level.

Let us recall the notation introduced in Subsection 6.2. If there is one node of the depth $k - j + 1$, then denote by l_j the rank of its colour. If there are two nodes of the depth $k - j + 1$, then denote by l_j^1 the rank of the colour of the first node, by l_j^2 the rank of the colour of the second one and denote by l_j^0 the rank of the intersection of their colours. We denote by C_k^n the choose of k over n. Then the number of possible colours for the root of the fixed frame of the depth k is:

$$\sum_{l_1=0}^{n-k+1} C_{l_1}^n.$$

See Subsection 6.2 for the proof. In the comments made in program below we denote this sum by f1.

Then there are four different cases for computation of the possible (colours of) nodes which cover the given nodes of the previous level.

1. There is one node on the each of the levels $k - j + 2$ and $k - j + 1$; (f11)

2. There are two nodes on the level $k - j + 2$ covered by one node on the level $k - j + 1$; (f21)

3. There is one node on the level $k - j + 2$ covered by two nodes on the level $k - j + 1$; (f12)

4. There are two nodes on the level $k - j + 2$ covered by two nodes on the level $k - j + 1$; (f22)

See Section 6.2 for the sums that correspond to each of the above cases.

Our computational tree mentioned above consists of nested sums - starting with the sum f1 and having a corresponding sum from the sums f11, f21, f12 and f22 on each of its levels. Here the word 'corresponding' is used in the following sense: since a level in our computational tree means the depth of the nodes (in a fixed frame) to be coloured (with acceptable valuations), it depends which case out of our four cases we are dealing with; if, for instance, a program is about to colour two nodes which cover one node, then it means that it is dealing with the third case and hence it should use the sum f12 for its calculations.

The parameter j in the program indicates the level in the computational tree.

The comments about the computer program can be found in the text of the program enclosed in {,} brackets.

```
procedure TMainForm.StartBitBtnClick(Sender: TObject);
var n, k, j, Sn, S, memlines : Integer;
    fact : array[0..7] of Integer;
    i : array[1..7] of Integer;
    C : array[0..7,0..7] of Integer;

{Initializing C matrix, so that computer will not have
 to calculate binomials (C[n,m]) every time}
procedure InitC;
var n1, n2 : Integer;
begin
  for n1 := 0 to 7 do
    for n2 := 0 to 7 do
      if n1 < n2 then C[n1,n2] := 0  else
        if n1 = n2 then C[n1,n2] := 1  else
          C[n1,n2] := fact[n1] div (fact[n2]*fact[n1-n2]);
end;
```

```
{The 'heart' of the program - recursive procedure}

procedure Rec(l0j,l1j,l2j, P : Integer);
var l0,l1,l2: Integer;
begin
  {j indicates the depth of points to be coloured}
  Inc(j);
  if j = k+1 then
  {if we have coloured points of all needed depths then...}
    begin
      {S is the sum and P is the product (of binomials)}
      S := S + P;
      Dec(j);
      Exit; {Return to the state from where the procedure was called}
    end;
  {Decision procedure for which a sum (f1, f11, f12, f21 or f22)
   must be calculated}
  {Here the case of j=1 (the sum f1) should be considered separately,
   while the other four cases exclude each other}
  if j=1 then
    for  l1 := l1j to n-k+1 do Rec(l1,l1,l1, P*C[n,l1])
    {P becomes a product of binomials}
  else begin
  if (i[j-1]=1) and (i[j]=1) then
    for l1 := l1j+1 to n-k+j do Rec(l1,l1,l1, P*C[n-l1j,l1-l1j]);
  if (i[j-1]=1) and (i[j]=2) then
  begin
    for l0 := l1j to n-k+j-1 do
      for l1 := l0 to n-k+j-1 do
        for l2 := l1+1 to n-k+j do
          Rec(l0,l1,l2, P*C[n-l1j,l0-l1j]*C[n-l0,l1-l0]*C[n-l1,l2-l0]);
    for l0 := l1j to n-k+j-1 do
      for l1 := l0+1 to n-k+j do
        Rec(l0,l1,l1, P*C[n-l1j,l0-l1j]
                     *((C[n-l0,l1-l0]*C[n-l1,l1-l0])div 2));
  end;
  if (i[j-1]=2) and (i[j]=1) then
```

```
    if (l1j=l0j) or (l2j=l0j) then for l1 := l1j+l2j-l0j+1 to n-k+j do
      Rec(l1,l1,l1, P*C[n-(l1j+l2j-l0j),l1-(l1j+l2j-l0j)])
    else  for l1 := l1j+l2j-l0j to n-k+j do
      Rec(l1,l1,l1, P*C[n-(l1j+l2j-l0j),l1-(l1j+l2j-l0j)]);
  if (i[j-1]=2) and (i[j]=2) then
  begin
    for l0 := l1j+l2j-l0j to n-k+j-1 do
      for l1 := l0 to n-k+j-1 do
        for l2 := l1+1 to n-k+j do
          Rec(l0,l1,l2, P*C[n-(l1j+l2j-l0j),l0-(l1j+l2j-l0j)]*C[n-l0,l1-l0]
                        *C[n-l1,l2-l0]);
    for l0 := l1j+l2j-l0j to n-k+j-1 do
      for l1 := l0+1 to n-k+j do
        Rec(l0,l1,l1, P*C[n-(l1j+l2j-l0j),l0-(l1j+l2j-l0j)]
                      *((C[n-l0,l1-l0]*C[n-l1,l1-l0])div 2));
  end;
  end;
  Dec(j);
end;

{A counter of models for a fixed frame}
procedure F;
var memcount : Integer;
begin
  S := 0; {S - the number of models for a fixed frame}
  j := 0;
  Rec(0,0,0, 1);
  {Sn - the global sum}
  Sn := Sn + S;

  {Taking care of output}
  for memcount := 1 to n+1 do
    Memo.Lines[memlines] := Memo.Lines[memlines] + IntToStr(i[memcount])+',';
  Memo.Lines[memlines] :=
    Memo.Lines[memlines] + '                  '+IntToStr(S);
  Inc(memlines);
end;
```

```
Begin
  {Initializing the variable n}
  n := StrToInt(nEdit.Text);
  if n = 0 then
  begin
    ResultLabel.Caption := '0';
    for memlines := 0 to 200 do Memo.Lines[memlines] := '';
    Exit;
  end;
  {Taking care of output}
  for memlines := 0 to 200 do Memo.Lines[memlines] := '';
  Memo.Lines[0] := 'Frame            Number of models';
  memlines := 2;
  {Initializing factorials from 1 to 7}
  fact[0] := 1; fact[1] := 1; fact[2] := 2; fact[3] := 6;
  fact[4] := 24; fact[5] := 120; fact[6] := 720; fact[7] := 5040;
  {Initializing binomials}
  InitC;
  {Initializing the matrix i}
  i[1] := 1; i[2] := 0; i[3] := 0; i[4] := 0; i[5] := 0; i[6] := 0; i[7] := 0;
  {Initializing Sn}
  Sn := 0;
  {"Creating frames".
   The direct way was prefered to the use of another recursion
   which would make the program almost unreadable}
  for k := 1 to n+1 do
    Case k of
    1: F;
    {Here begins the manual construction of frames
     (without using recursion which would be necessary since k is varying)}
    2: begin
         i[2]:=1; F;
         i[2]:=2; F;
       end;
    3: begin
         i[2]:=1; i[3]:=1; F;
```

```
              i[3]:=2; F;
         i[2]:=2; i[3]:=1; F;
              i[3]:=2; F;
      end;
   4: begin
         i[2]:=1; i[3]:=1; i[4]:=1; F;
                       i[4]:=2; F;
              i[3]:=2; i[4]:=1; F;
                       i[4]:=2; F;
         i[2]:=2; i[3]:=1; i[4]:=1; F;
                       i[4]:=2; F;
              i[3]:=2; i[4]:=1; F;
                       i[4]:=2; F;
      end;
   5: begin
         i[2]:=1; i[3]:=1; i[4]:=1; i[5]:=1; F;
                                i[5]:=2; F;
                       i[4]:=2; i[5]:=1; F;
                                i[5]:=2; F;
              i[3]:=2; i[4]:=1; i[5]:=1; F;
                                i[5]:=2; F;
                       i[4]:=2; i[5]:=1; F;
                                i[5]:=2; F;
         i[2]:=2; i[3]:=1; i[4]:=1; i[5]:=1; F;
                                i[5]:=2; F;
                       i[4]:=2; i[5]:=1; F;
                                i[5]:=2; F;
              i[3]:=2; i[4]:=1; i[5]:=1; F;
                                i[5]:=2; F;
                       i[4]:=2; i[5]:=1; F;
                                i[5]:=2; F;
      end;
   6: begin
         i[2]:=1; i[3]:=1; i[4]:=1; i[5]:=1; i[6]:=1; F;
                                         i[6]:=2; F;
                                i[5]:=2; i[6]:=1; F;
                                         i[6]:=2; F;
```

```
                    i[4]:=2; i[5]:=1; i[6]:=1; F;
                                      i[6]:=2; F;
                             i[5]:=2; i[6]:=1; F;
                                      i[6]:=2; F;
           i[3]:=2; i[4]:=1; i[5]:=1; i[6]:=1; F;
                                      i[6]:=2; F;
                             i[5]:=2; i[6]:=1; F;
                                      i[6]:=2; F;
                    i[4]:=2; i[5]:=1; i[6]:=1; F;
                                      i[6]:=2; F;
                             i[5]:=2; i[6]:=1; F;
                                      i[6]:=2; F;
  i[2]:=2; i[3]:=1; i[4]:=1; i[5]:=1; i[6]:=1; F;
                                      i[6]:=2; F;
                             i[5]:=2; i[6]:=1; F;
                                      i[6]:=2; F;
                    i[4]:=2; i[5]:=1; i[6]:=1; F;
                                      i[6]:=2; F;
                             i[5]:=2; i[6]:=1; F;
                                      i[6]:=2; F;
           i[3]:=2; i[4]:=1; i[5]:=1; i[6]:=1; F;
                                      i[6]:=2; F;
                             i[5]:=2; i[6]:=1; F;
                                      i[6]:=2; F;
                    i[4]:=2; i[5]:=1; i[6]:=1; F;
                                      i[6]:=2; F;
                             i[5]:=2; i[6]:=1; F;
                                      i[6]:=2; F;
end;
7: begin
  i[2]:=1; i[3]:=1; i[4]:=1; i[5]:=1; i[6]:=1; i[7]:=1; F;
                                               i[7]:=2; F;
                                      i[6]:=2; i[7]:=1; F;
                                               i[7]:=2; F;
                             i[5]:=2; i[6]:=1; i[7]:=1; F;
                                               i[7]:=2; F;
                                      i[6]:=2; i[7]:=1; F;
```

```
                                             i[7]:=2; F;
                  i[4]:=2; i[5]:=1; i[6]:=1; i[7]:=1; F;
                                             i[7]:=2; F;
                                    i[6]:=2; i[7]:=1; F;
                                             i[7]:=2; F;
                           i[5]:=2; i[6]:=1; i[7]:=1; F;
                                             i[7]:=2; F;
                                    i[6]:=2; i[7]:=1; F;
                                             i[7]:=2; F;
         i[3]:=2; i[4]:=1; i[5]:=1; i[6]:=1; i[7]:=1; F;
                                             i[7]:=2; F;
                                    i[6]:=2; i[7]:=1; F;
                                             i[7]:=2; F;
                           i[5]:=2; i[6]:=1; i[7]:=1; F;
                                             i[7]:=2; F;
                                    i[6]:=2; i[7]:=1; F;
                                             i[7]:=2; F;
                  i[4]:=2; i[5]:=1; i[6]:=1; i[7]:=1; F;
                                             i[7]:=2; F;
                                    i[6]:=2; i[7]:=1; F;
                                             i[7]:=2; F;
                           i[5]:=2; i[6]:=1; i[7]:=1; F;
                                             i[7]:=2; F;
                                    i[6]:=2; i[7]:=1; F;
                                             i[7]:=2; F;
i[2]:=2; i[3]:=1; i[4]:=1; i[5]:=1; i[6]:=1; i[7]:=1; F;
                                             i[7]:=2; F;
                                    i[6]:=2; i[7]:=1; F;
                                             i[7]:=2; F;
                           i[5]:=2; i[6]:=1; i[7]:=1; F;
                                             i[7]:=2; F;
                                    i[6]:=2; i[7]:=1; F;
                                             i[7]:=2; F;
                  i[4]:=2; i[5]:=1; i[6]:=1; i[7]:=1; F;
                                             i[7]:=2; F;
                                    i[6]:=2; i[7]:=1; F;
                                             i[7]:=2; F;
```

```
                                    i[5]:=2; i[6]:=1; i[7]:=1; F;
                                                      i[7]:=2; F;
                                             i[6]:=2; i[7]:=1; F;
                                                      i[7]:=2; F;
                  i[3]:=2; i[4]:=1; i[5]:=1; i[6]:=1; i[7]:=1; F;
                                                      i[7]:=2; F;
                                             i[6]:=2; i[7]:=1; F;
                                                      i[7]:=2; F;
                                    i[5]:=2; i[6]:=1; i[7]:=1; F;
                                                      i[7]:=2; F;
                                             i[6]:=2; i[7]:=1; F;
                                                      i[7]:=2; F;
                           i[4]:=2; i[5]:=1; i[6]:=1; i[7]:=1; F;
                                                      i[7]:=2; F;
                                             i[6]:=2; i[7]:=1; F;
                                                      i[7]:=2; F;
                                    i[5]:=2; i[6]:=1; i[7]:=1; F;
                                                      i[7]:=2; F;
                                             i[6]:=2; i[7]:=1; F;
                                                      i[7]:=2; F;
       end;



    end;

  {Outputting the global sum}
  ResultLabel.Caption := IntToStr(Sn);

End;
```

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