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Modal definability in topology

Master's thesis

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Modal definability in topology

Abstract

This thesis is mainly concerned with the definability issue when modal logic is interpreted on topological spaces. A topological analogon of Goldblatt-Thomason theorem is proved. Some new topological constructions are introduced. One of them, namely the notion of compact extension, is a generalization of the concept of Stone-Čech compactification known in topology. The preservation of modal validity with respect to suitable topological operations is presented.

Keywords: Modal Logic, semantics, topology, definability.

Introduction

This work deals with the topological semantics of modal logic. We consider the interpretation of the modal diamond as a closure operator. Several interpretations of the modal diamond as an operator over a topological space are possible. Two of them, namely diamond-as-closure-operator and diamond-as-limit-operator, have been pioneering in the semantics of modal logic as far back as in 1944, in the celebrated paper of McKinsey and Tarski [McKinsey and Tarski 1944]. The topological semantics has proved rather useful, providing nice completeness results for such well-known modal system(s) as (extensions of) $S4$. However, after the discovery of what is now known as the Kripke semantics for modal logic, the latter has become the heaviestly used modal semantical tool, leaving the topological interpretations in the shadow. This is probably due to the more intuitive and illustrative nature of the Kripke semantics. Frames are easier to picture and operate upon, while topological spaces can be quite mysterious even on the basic level. Nevertheless, spatial intuitions behind the topological structures provide a quite startling view on modal logic, and the latter appears a good medium for reasoning about spaces. This summarizes two possible approaches. One approach is originating from modal logic. The abstract syntactic entities, formal theories like modal logic ask for the meaning, for the realization. Collection of symbols and the rules of operating on them seek semantics. The Glass Bead Game needs a board to be played on after all. Topology is one such board for modal logic. One among others, but first and unique. The spatial intuitions granted to humans by recurrent experiencing of the space and time gave birth to topology. Topology gave a meaning to modal logic. The abstract syntactic objects come to denote some spatial phenomena, and trigger our intuition to draw conclusions which can be expressed in the same modal symbolism. Thus the first approach is roughly topological decoration of modal logic. The second approach originates in spatial reasoning. What is the suitable language for reasoning about spaces? We would like to have a rich enough stock of resources to code at least the most relevant topological properties. We would like to find out what kind of logical rules govern the space. We need a formal system to reason. At the same time the resulting system should be transparent, decidable, finitary accessible, in short - feasible in one or another way. Where is the golden middle? In attempts of accomplishing these tasks, among others, modal logic comes into play. We may want to reason about regions and their interrelations, or about metric spaces, Euclidean spaces, connected spaces, etc. Modal logic offers a clear, accessible paradigm of formal reasoning and a great flexibility to fine-tune the expressive power of the language+interpretation to meet the particular spatial needs. The second approach in general terms takes modal logic as a reasoning tool for topological spaces. These two streams interplay greatly and, it is our fine belief, that is were the beauty shines through. There is of course no strict distinction, but every paper where modal logic meets topology can be classified to belong in one of these categories, if not in both. For example, works like [McKinsey and Tarski 1944], [Shehtman 1999], [Mints 1998], [Bezhanishvili and Gehrke 2001] fall more in the first category, while the papers [Goldblatt 1980], [Shehtman 1983], [Aiello and van Benthem 2001] belong more to the second. We leave it to the reader to classify this paper.

The main contribution of this thesis is the topological Goldblatt-Thomason theorem. It answers the question: "which classes of topological spaces can be defined by a modal formula in the basic modal language?". In order to approach this issue, we developed the construction called the Alexandroff extension - the topological analogon of the ultrafilter extension for a frame. Another interesting construction - the compact extension is introduced as well. These two constructions appear to be tightly connected. At the same time, the former is the crucial in the main result of the thesis, while the latter is the generalization of the well-known topological notion of the Stone-Čech compactification. In conclusion, invariance of modally expressible topological properties under certain topological transformations is summarizing the results obtained in the thesis.

The material is organized as follows. In the first part we bring in necessary definitions both from logic and topology, introduce the semantics interpreting the modal diamond as a closure operator, prove soundness and strong completeness of $S4$ with respect to the topological semantics and build a bridge between Kripke frames and Alexandroff topologies; in the second part we address the definability issue, develop four basic operations on topological spaces and prepare the algebraic duality to prove the main result of this part - a topological Goldblatt-Thomason theorem; in the third part several enriched languages are discussed, the generalization of the notion of Stone-Čech compactification is presented and the results of the earlier parts are summarized in terms of topological invariance.

1 Topological semantics and completeness

Most of the results presented in this part have already been established in the works of various authors either implicitly or explicitly. The main aim for the present chapter is to carefully lead the reader into the modal world of topological spaces, or rather, the topological world of modal logic. The necessary topological definitions have been taken from [Engelking 1977] mostly, while for modal logic we used [Blackburn et al. 2001] as the dominating reference.

1.1 Preliminaries

We will define the topological semantics for modal logic and present the soundness result in this section.
First of all, let us specify what kind of structure is a topological space.

Definition 1.1.1 A Topological Space $\mathcal{T} = (X, \Omega)$ *is a nonempty set* X *with a collection of its subsets* Ω *satisfying the following:*

T_1 $\emptyset \in \Omega$ *and* $X \in \Omega$,

T_2 *If* $O_1 \in \Omega$ *and* $O_2 \in \Omega$ *then* $O_1 \cap O_2 \in \Omega$,

T_3 *If* $O_i \in \Omega$ *for all* $i \in I$, *with* I *some index set, then* $\bigcup_{i \in I} O_i \in \Omega$.

The elements of Ω are called *opens*. So, the empty set and the universe are opens, opens are closed under finite intersections and under arbitrary unions. If $x \in O \in \Omega$ then we say that O is an open neighbourhood of x. Informally, open neighbourhoods of the given point tell us which points are "close" to the chosen one. It is known that the set $O \subseteq X$ is open iff every point in O has an open neighbourhood included in O. The universe is the (biggest) open neighbourhood of all of its points. The complements of opens are called *closed* and with the de Morgan rules in mind, it does not take long to see that arbitrary intersections, as well as finite unions of closed sets are closed, as are the universe and the empty set. There are many equivalent means of defining topological structure on a set (fixing the family of closed sets is one of them) and we will see some further on in this paper, but what we said so far is enough to define the topological semantics for the modal language with the single box operator.

Definition 1.1.2 *The basic modal language consists of countable stock of proposition letters* $p, p_1, p_2, ..$ *the constant truth* $\top$, *boolean connectives* $\wedge, \neg$

and the modal operator $\Box$. Modal formulas are denoted by Greek letters and are built in the following way:

$$\phi \ ::= \ p_i \mid \top \mid \phi \wedge \psi \mid \neg\phi \mid \Box\phi.$$

Now we are ready to define topological models for the basic modal language. The valuation on the topological space will be the function assigning a subset of the topological space to each propositional letter.

Definition 1.1.3 A topological model $\mathcal{M}$ *is a triple* (X, Ω, ν) *where* $\mathcal{T} = (X, \Omega)$ *is a topological space and the valuation* ν *sends propositional letters to subsets of* X*. The notation* $\mathcal{M}, x \models \phi$ *(or simply* $x \models \phi$*) will read "the point* x *of the model* $\mathcal{M}$ *makes the formula* ϕ *true"; if* A *is a subset of* X*,* $A \models \phi$ *will mean that* $x \models \phi$ *for all* x *in* A*. The definition of truth proceeds like this:*

$$\begin{array}{rll} x \models p_i & \text{iff} & x \in \nu(p_i), \\ x \models \top & & \text{always}, \\ x \models \phi \wedge \psi & \text{iff} & x \models \phi \quad \text{and} \quad x \models \psi, \\ x \models \neg\phi & \text{iff} & x \not\models \phi, \\ x \models \Box\phi & \text{iff} & \exists O \in \Omega : \ x \in O \models \phi. \end{array}$$

The definitions of the model validity and the space validity are standard. In words the above definition says that if we associate formulas with the set of points making this formula true, then the abstract operations $\neg$ and $\wedge$ become set-theoretic complement and intersection operations respectively. The last part of the truth definition is not that explicit - it says that formula $\Box\phi$ is true at x, just when ϕ is true at all the nearby points - everywhere in some open neighbourhood of x, that is. Taking a closer look, $\nu(\Box\phi)$, defined as the set of points making the formula $\Box\phi$ true, consists of all the points having an open neighbourhood $O \subseteq \nu(\phi)$. This is nothing else but the interior part of $\nu(\phi)$ - a topologist will say. More formally, here comes another definition from general topology:

Definition 1.1.4 *For* A *subset of the topological space, the* interior *of* A*, denoted* $\mathbf{I}(A)$ *is the union of all open sets included in* A*, or, equivalently, the biggest open subset of* A*.*

To make the picture complete, we recall the following proposition from [Engelking 1977]:

Proposition 1.1.5 *The point x belongs to $\mathbf{I}(A)$ iff there is an open neighbourhood O of x such that $O \subseteq A$.*

In the light of the latter definition and proposition, it should be clear that $\nu(\Box\phi) = \mathbf{I}(\nu(\phi))$. So the abstract $\Box$ operator becomes the interior operator over the subsets of the topological space in our interpretation. We would like to mention the topological equivalents for other logical connectives definable in our language. It is rather obvious that the boolean connectives obtain their traditional set-theoretic meaning, but what is the topological interpretation of the modal $\Diamond$? A simple derivation shows that $\nu(\Diamond\phi) = -\mathbf{I}-(\nu(\phi))$ where $-$ stands for set-theoretic complement operation. Recalling that $-\mathbf{I}(-A)$ with A subset of the topological space is nothing else than the *closure set* of A, we conclude $\nu(\Diamond\phi) = \mathbf{C}(A)$. Here $\mathbf{C}$ denotes the topological closure operator, assigning to each subset A of the topological space the smallest closed set containing this subset - the intersection of all closed sets which extend A, in other words.

Now, we know more about the $\mathbf{I}$ and $\mathbf{C}$ from general topology - they satisfy certain conditions. In fact, given the set equipped with the operator working on its subsets, imposing the interior- or closure- conditions on this operator is another way of defining the topological structure on the set. Let us have a look at these conditions and see what they oblige our $\Box$ to be like.

Proposition 1.1.6 *Let (X, Ω) be any topological space, $\mathbf{I}$ and $\mathbf{C}$ the interior and closure operators defined by topology, then the following holds for any subsets of X:*

(I1)	$\mathbf{I}(X) = X$	*(C1)*	$\mathbf{C}(\emptyset) = \emptyset$
(I2)	$\mathbf{I}(A) \subseteq A$	*(C2)*	$A \subseteq \mathbf{C}(A)$
(I3)	$\mathbf{I}(A \cap B) = \mathbf{I}(A) \cap \mathbf{I}(B)$	*(C3)*	$\mathbf{C}(A \cup B) = \mathbf{C}(A) \cup \mathbf{C}(B)$
(I4)	$\mathbf{I}(\mathbf{I}(A)) = \mathbf{I}(A)$	*(C4)*	$\mathbf{C}(\mathbf{C}(A)) = \mathbf{C}(A)$

It is not difficult to notice that the conditions (I1),..,(I4) when read with $\Box$ instead of $\mathbf{I}$ look like the axioms for the modal system $S4$! Dually, conditions (C1),..,(C4) look like $S4$ axioms written with $\Diamond$. In fact, as we will see shortly, any topological model makes the modal logic $S4$ valid, turning $S4$ into the smallest normal modal logic of topological spaces. Let us put all these in precise mathematical form.

Definition 1.1.7 *The modal logic $S4$ is the smallest set of modal formulas which contains all the classical tautologies, the following axioms:*

(N) $\Box\top$

(T) $\Box p \rightarrow p$

(R) $\Box(p \wedge q) \leftrightarrow \Box p \wedge \Box q$

(4) $\Box p \rightarrow \Box\Box p$

and is closed under modus ponens, substitution and monotonicity (from $\phi \rightarrow \psi$ *derive* $\Box\phi \rightarrow \Box\psi$*).*

The above axioms are not exactly the standard way of defining $S4$, but equivalent to it and the best for our present needs they are. We claim that every topological space validates all the modal formulas in $S4$ thus proving the soundness of $S4$ with respect to the topological semantics.

Theorem 1.1.8 *The theorems of* $S4$ *are valid on every topological space.*

Proof: That the axioms (N), (T), (R) and (4) are valid on any topological space is an easy observation after comparing to the conditions (I1), (I2), (I3), (I4). That the derivation rules preserve the validity is a trivial exercise. $\dashv$

McKinsey and Tarski prove even more in [McKinsey and Tarski 1944], namely that $S4$ is the complete logic of topological spaces. We will approach this matter in the next section via the Kripke completeness for $S4$.

1.2 Frames and Completeness

In this section we will build the connections between the topological and the Kripke semantics. We will consider $S4$-frames and see that they come in one-to-one correspondence with the special class of topological spaces, called Alexandroff spaces. This correspondence often facilitates some completeness proofs, showing that for the modal logics above $S4$ the topological semantics is a generalization of the frame semantics. Concluding this section, we will consider the modal logics $S4.1$ and $S4.2$ and see what class of topological spaces they characterize.
We have seen in the previous section that every topological model is a model for $S4$. This is not the case in the Kripke semantics. The frame is validating $S4$ iff the relation on the frame is reflexive and transitive.

Definition 1.2.1 *The structure* $\mathcal{F} = (W, R)$ *is called a qo-set (quasi-ordered set) if* W *is a nonempty set and* R *is a reflexive and transitive relation on* W*.*

Such frames are also enough to refute any non-theorem of $S4$, as it is well-known.

Proposition 1.2.2 *The modal logic $S4$ is sound and strongly complete with respect to the class of all qo-sets.*

To illustrate the connection with the topological semantics, let us examine upward closed subsets of a qo-frame $\mathcal{F} = (W, R)$. The subset $A \subseteq W$ is upward closed, if together with any point w it contains all of its successors: $w \in A$ & $Rwv \Rightarrow v \in A$. It is not difficult to check that an arbitrary union or intersection of upward closed sets is again upward closed. This is sufficient for upward closed sets to form a topological structure of open sets on W. Of course, the topology obtained in this way is of a special kind - opens become closed under *arbitrary*, rather then *finite*, intersection. This sort of topological spaces are known from general topology:

Definition 1.2.3 *A topological space is called Alexandroff space (A-space for short) if either of the following equivalent conditions hold:*
1. Arbitrary intersections of opens are open.
2. Every point has a least open neighbourhood.

The second condition in the above definition opens the way from A-spaces to qo-sets. To define a reflexive-transitive relation on an A-space, we simply say that Rxy holds iff y is included in the least open neighbourhood of x[1]. Reflexivity of such a relation is immediate, for transitivity note that if y is a member of the least open neighbourhood O_x of x, then O_x becomes an open neighbourhood of y as well and thus includes in itself all the members of the least open neighbourhood of y. It is worth mentioning that if we perform the A-space-qo-set transformation both ways starting from either one, we will get the initial structure back.

Well then, we have seen how any qo-set becomes an A-space, but why treat upward closed sets as opens, why not closed sets? They will happily satisfy the conditions imposed on the collection of closed sets for topology, will they not? The reason lies behind our definition of the interpretation of

[1]One could define a quasi-order on *any* topological space by saying that xRy holds iff y is included in every open neighbourhood of x or, equivalently, x is included in the closure of the singleton $\{y\}$. This is called a *specialization order* in the literature and in the case of A-spaces is clearly identical to the one defined above.

modal formulas. Indeed, the structures of a qo-set and the corresponding A-space are very much the same - just seen through different mathematical glasses, but what about the modal models built on them? Are they the same in any sense? The following theorem sheds light on these questions.

Theorem 1.2.4 *Let $\mathcal{F} = (W, R)$ be a qo-set and $\mathcal{T} = (W, \Omega)$ corresponding A-space. For any valuation ν on W, for any point $w \in W$ and for any modal formula ϕ, we have:*

$$\mathcal{F}, \nu, w \models \phi \text{ iff } \mathcal{T}, \nu, w \models \phi.$$

Proof: The proof goes by induction on the length of ϕ. The propositional case and the case for the boolean connectives are trivial. Recalling that the R-successors of any point constitute precisely the least open neighbourhood of this point; and that in A-space $\Box\phi$ is true at a point iff ϕ is true in the least open neighbourhood of this point, we observe that $\mathcal{F}, \nu, w \models \Box\phi$ iff $\mathcal{T}, \nu, w \models \Box\phi$, which finishes the proof. $\dashv$

With the help of the latter theorem we can now prove the topological completeness of $S4$.

Theorem 1.2.5 *$S4$ is strongly complete with respect to the class of all topological spaces.*

Proof: Take any $S4$-consistent set of formulas Σ. We know that Σ can be satisfied in a model of which the underlying frame is a qo-set. Take the A-space corresponding to this qo-set, keeping the valuation. This will be a topological model where Σ is satisfied. $\dashv$

All we have done so far in this section shows that for $S4$ and its extensions the topological semantics is more general then the Kripke semantics. If the modal logic containing $S4$ is (strongly) complete with respect to some class of frames (qo-sets, in fact), then this logic is (strongly) complete with respect to the corresponding class of A-spaces. Certainly the most interesting topological spaces fail to have Alexandroff structure (although note that all finite topological spaces are clearly A-spaces), but we can still exploit A-spaces for completeness results. Witness, for example, the following picture: take any $S4$-consistent set of formulas Γ; surely Γ defines some class K of topological spaces which make all the formulas in Γ valid. If, in addition, $S4 + \Gamma$ is

Kripke (strongly) complete, then it will be topologically (strongly) complete with respect to K, or, indeed, with respect to any subclass of K containing enough A-spaces to refute non-theorems of $S4+\Gamma$ - thanks to theorem 1.2.4. This is quite a general and vague statement, but we are going to illustrate it in the following section.

Before proceeding to the next section though, we would like to address one interesting question concerning the topological semantics. Namely, is there a Kripke-incomplete extension of $S4$, which is topologically complete? In other words, is the topological semantics really more general than the frame semantics, from the completeness point of view? The answer is affirmative. In [Gerson 1975] the extension of $S4$ is presented which is complete with respect to the neighbourhood semantics, but incomplete with respect to the Kripke semantics. We can prove that for $S4$ and its extensions, the neighbourhood semantics is equivalent to the topological semantics (cf. [Shehtman 1998]). It follows that the example given by Gerson will also work for proving that the topological semantics is strictly more general then the Kripke semantics.

1.3 Some more topological completeness

If the topological semantics is more general than Kripke semantics, we could just take any of the extensions of $S4$ known to be Kripke complete and automatically get their topological completeness, as outlined at the end of the previous section. In case we have an extension of $S4$ axiomatizable with Sahlqvist formulas, we would even get strong completeness! More challenging seems the question which topological property will then the logic in question define. We start with investigating the modal system $S4.2$.

Definition 1.3.1 *The modal logic $S4.2$ is the extension of $S4$ with the axiom (.2) $\Diamond\Box p \to \Box\Diamond p$.*

It is known that the frames for $S4.2$ are strongly directed, i.e. any two worlds having a common predecessor must share a successor, too. The formula (.2) being in a Sahlqvist form, gives strong Kripke completeness, and therefore strong topological completeness - with respect to the class of topological spaces characterized by (.2). To describe this class in topological terms, we will need the following definition:

Definition 1.3.2 *The topological space is* extremally disconnected *if any of the following two equivalent conditions hold:*

1. The closure of any open is open.
2. The closures of any two disjoint opens are disjoint.

Extremally disconnected spaces were defined in [Stone 1937] and have been around in general topology ever since. As it appears, (.2) characterizes the class of extremally disconnected topological spaces. This statement seems to be known to scholars working in the field. However we could not find the exact reference who first established it. Here is the proof of this fact:

Theorem 1.3.3 $S4.2$ *defines the class of extremally disconnected topological spaces.*

Proof: That $S4$ is valid on any topological space is already known, let us check that the formula (.2) is valid on a topological space $\mathcal{T} = (X, \Omega)$ iff $\mathcal{T}$ is extremally disconnected.
It is an easy exercise to prove that in $S4$ the axiom (.2) can be written in an equivalent way as $\Diamond\Box p \leftrightarrow \Box\Diamond\Box p$. Now just replace the modal operators with their topological interpretations in this equivalent form of (.2) and get $\mathbf{CI}(P) = \mathbf{ICI}(P)$ with P an arbitrary subset of X. This is exactly to say that the closure of every open is open (note that if P denotes an arbitrary subset, $\mathbf{I}(P)$ will denote an arbitrary open subset). Thus to say $\mathcal{T}$ is extremally disconnected is to say $\mathcal{T}$ validates (.2). $\dashv$

As an easy consequence we obtain that extremally disconnected A-spaces are in one-to-one correspondence with strongly directed Kripke frames!
Summarizing all about $S4.2$ we have accomplished here, we get the following:

Theorem 1.3.4 $S4.2$ *is sound and strongly complete with respect to the class of extremally disconnected topological spaces.*

Proof: Soundness is the consequence of the previous theorem, for completeness we use the Kripke completeness of $S4.2$. Indeed, any Kripke frame validating $S4.2$ gives rise to an A-space which validates $S4.2$ by theorem 1.2.4 and is, thus, extremally disconnected by the previous theorem. Then any consistent set of modal formulas Σ can be satisfied in the A-space corresponding to the frame underlying the $S4.2$-Kripke-model for Σ. $\dashv$

We were quite lucky with the formula (.2) - it defined a well-known, topologically valuable class. Our next example shows that finding the topologically transparent property characterizing the modally defined class of spaces is not always possible. There are plethora of consistent extensions of $S4$ around (uncountably many, even) and no wonder some of them define rather bizzare topological properties. Nevertheless, if not entirely transparent and immediate to the intuition, some of the modal formulas still define interesting classes of spaces. A good example is the McKinsey formula $\Box\Diamond\phi \to \Diamond\Box\phi$. This formula first appeared in [McKinsey 1945] where the author baptized the corresponding extension of $S4$ as $S4.1$.

Definition 1.3.5 *The modal logic $S4.1$ is the extension of $S4$ with the axiom (.1) $\Box\Diamond p \to \Diamond\Box p$.*

It is known that in the presence of transitivity the McKinsey formula means atomicity for frames, i.e. every point sees a reflexive maximum: $\forall x \exists y (Rxy \mathbin{\&} \forall z (Ryz \to y{=}z))$. The topological counterpart of this property can be defined in various equivalent ways, giving different intuitive grasp of what (.1) really says spatially. The following definition is inspired by the notion used in [Esakia 1979] for the analysis of the system $S4.1$.

Definition 1.3.6 *Let $\mathcal{T} = (X, \Omega)$ be a topological space and $A \subseteq X$ its any subset. Define the* frontier *set for A to be the set $\mathbf{C}(A) \cap \mathbf{C}(-A)$. Call the operator $\mathbf{Fr}$ defined by the equation $\mathbf{Fr}(A) = \mathbf{C}(A) \cap \mathbf{C}(-A)$ the* frontier *operator.*
Call the topological space atomic *if the frontier of any subset has an empty interior, i.e. $\mathbf{IFr}(A) = \emptyset$ for any A.*

To say but a bit more, in atomic spaces no open set can be included in the frontier of some other set. We will demonstrate that (.1) characterizes the class of atomic spaces.

Theorem 1.3.7 *$S4.1$ is the logic of atomic spaces.*

Proof: We have to prove that for any $\mathcal{T} = (X, \Omega)$, $\mathcal{T} \models \Box\Diamond p \to \Diamond\Box p$ iff $\mathcal{T}$ atomic.
Let us again give the interpretation to modal connectives. $\mathcal{T}$ validates (.1) means for any subset P we have $\mathbf{IC}(P) \subseteq \mathbf{CI}(P) \Leftrightarrow \mathbf{IC}(P) \cap -\mathbf{CI}(P) = \emptyset \Leftrightarrow \mathbf{IC}(P) \cap --\mathbf{I}--\mathbf{C}(-P) = \emptyset \Leftrightarrow \mathbf{IC}(P) \cap \mathbf{IC}(-P) = \emptyset \Leftrightarrow \mathbf{I}(\mathbf{C}(P) \cap \mathbf{C}(-P)) = \emptyset \Leftrightarrow \mathbf{IFr}(P) = \emptyset$. The last equation in this chain of equivalent statements is expressing that $\mathcal{T}$ is an atomic space. $\dashv$

For topological strong completeness witness the following:

Theorem 1.3.8 $S4.1$ *is sound and strongly complete with respect to all atomic spaces.*

Proof: Soundness follows from the previous theorem, for strong completeness we use the analogous result with respect to Kripke frames for $S4.1$ proved in [Lemmon and Scott 1966]. $\dashv$

While (.2) defined the essential topological property, spaces characterized by (.1) seem to receive minor attention in the topological literature. As we already mentioned, this is likely to happen with the most topological properties defined by modal formulas in the basic language. This is only one side of the coin though - taking modal formulas and tackling with the corresponding topological property to make some spatial sense of it. Another approach would be to take a topologically sensible class and see which modal formulas may describe it. Or indeed, when is a class of topological spaces *definable* at all? What are the conditions to impose on the class of topological spaces to be sure that this class can be characterized by a modal formula? We address this issue in the next part of this work.

2 Definable spaces

This part deals with the question of space definability. When is a class of topological spaces modally definable? In the case of the Kripke semantics the Goldblatt-Thomason theorem has an answer. Something similar for topological spaces would have been useful. Let us recall this theorem:

Theorem (Goldblatt-Thomason) *A class of frames, closed under taking ultrafilter extensions, is modally definable, if and only if it is closed under the formation of bounded morphic images, generated subframes, and disjoint unions, and reflects ultrafilter extensions.*

In the first part we have defined topological models, translated frame structure into the topological one, but we did not mention which topological operations preserve space validity. The analogs for bounded morphism, generated subframe and disjoint union will be of a major importance in space definability; the first section of this chapter is devoted to developing this kind of topological tools. In the second section we will prepare some algebraic apparatus and work out algebraic duality to approach the definability; all of the latter will allow us to prove the topological Goldblatt-Thomason theorem in the third section, with the help of the new notion of Alexandroff extension, the topological equivalent of ultrafilter extensions for frames.

2.1 Validity preserving operations

We have built the bridge between Kripke frames and Alexandroff spaces in the first chapter. Certain operations on frames preserve the validity of modal formulas. Such operations - bounded morphisms, generated subframes and disjoint unions, become crucial when dealing with the frame definability. No wonder we need similar notions for the topological semantics and that is what we are about to bring in this section.

While we need to rethink in topological terms the notions of bounded morphism and generated subframe, the formation of disjoint union appears to have the straightforward topological equivalent. Here is the definition:

Definition 2.1.1 *For a family of disjoint topological spaces $\mathcal{T}_i = (X_i, \Omega_i)$, $i \in I$, their* topological sum *is the topological space $\mathcal{T} = (X, \Omega)$ with $X = \bigcup_{i\in I} X_i$ and $\Omega = \{O \in X | \forall i \in I : O \cap X_i \in \Omega_i\}$.*

It is easy to observe that when restricted to A-spaces, this operation will yield the structure corresponding to the disjoint union of respective frames. This means we are on the right track, and the following theorem, proving that the topological sum of topological spaces preserves the validity of modal formulas, justifies our choice.

Theorem 2.1.2 *Let $(\mathcal{T}_i)_{i\in I}$ be a family of disjoint topological spaces and let ϕ be a modal formula such that $\mathcal{T}_i \models \phi$ holds for each $i \in I$. Then for $\mathcal{T} = \bigcup_{i\in I} \mathcal{T}_i$ we have $\mathcal{T} \models \phi$.*

Proof: Suppose $\mathcal{T} \not\models \phi$ for the sake of contradiction. This means there is a valuation ν and a point $x \in X$ such that $\mathcal{T}, \nu, x \not\models \phi$. Obviously, $x \in X_i$ for some $i \in I$. Define the valuation ν_i as described: $\nu_i(p) = \nu(p) \cap X_i$. An easy induction argument shows that:

$$\mathcal{T}_i, \nu_i, x_i \models \psi \text{ iff } \mathcal{T}, \nu, x_i \models \psi,$$

for all $x_i \in X_i$ and all modal formulas ψ. This gives the desired contradiction, because we get $\mathcal{T}_i, \nu_i, x \not\models \phi$. $\dashv$

NB In fact an easy argument shows that the converse holds as well - each member of a family of spaces validates some modal formula, if their topological sum does so.

This already allows us to show that certain topologically interesting properties (or rather, their corresponding classes) are not definable by (sets of) modal formula(s) in the basic modal language. Our examples are connectedness and compactness.

Definition 2.1.3 *For a topological space $\mathcal{T} = (X, \Omega)$ we say that:*
(i) $\mathcal{T}$ is connected *if X can not be presented as the union of two disjoint open subsets.*
(ii) $\mathcal{T}$ is compact *if any family $(F_i)_{i\in I}$ of closed subsets of X with the finite intersection property has a non-empty intersection.*

Connected and compact topological spaces play the starring role in general topology. Unfortunately, neither of these properties can be defined in our language. Witness the following:

Theorem 2.1.4 *The class of connected topological spaces and the class of compact topological spaces are not modally definable.*

Proof: The following example serves for both reasons: consider countably many disjoint sets $X_i = \{i\}$ with the corresponding topologies $\Omega_i = \{\emptyset, X_i\}$, $i \in I = \{1, 2, ..\}$. These are one-point sets with the only possible topology. All of them are compact (any finite topological space clearly is), and connected. If connectedness (compactness) were definable by a modal formula ϕ, all $\mathcal{T}_i = (X_i, \Omega_i)$ would validate ϕ and, by the preservation of modal validity under the formation of topological sum, so would $\mathcal{T} = \bigcup_{i \in I} \mathcal{T}_i$. This would imply that $\mathcal{T}$ is connected (compact), which it is not. To observe why, note that by the definition of topology on the topological sum of spaces, $\mathcal{T}$ is just a countable set equipped with the discrete topology[2]. In particular, every cofinite set is closed. The cofinite sets form a family of closed sets with the finite intersection property, but their intersection is empty. This shows $\mathcal{T}$ is not compact. Any cofinite set different from the universe, together with its complement, form a disjoint couple of opens giving the universe in union. Thus connectedness of $\mathcal{T}$ holds wrong as well. $\dashv$

This is rather unfortunate. We will see how the enriching of the modal language with the global diamond will help us to define connectedness in the last chapter of this work, but compactness (being essentially higher-order property) will still remain wild in this sense. We leave this issue till later and go on to see the topological equivalents for other basic frame operations.

Next comes the notion of generated subframe in our agenda. Recall that a generated subframe is based on a subset of a frame which together with any of its members contains all of its successors. But this is exactly what we called "upward closed sets" in the first chapter and we saw that the topological notion of open subset is the equivalent for that. At least this was the case with A-spaces - upward closed sets constituted the topology when forming the A-space from a qo-set. The following theorem shows that open subspaces of topological spaces inherit the validity of modal formulas thus proving that open subspaces are the right topological substitute for the notion of generated subframe.

Theorem 2.1.5 *Let U be the open subset of the topological space $\mathcal{T} = (X, \Omega)$ and ϕ be a modal formula. If $\mathcal{T} \models \phi$ and $\mathcal{T}_U = (U, \Omega_U)$ is the topological space with $\Omega_U = \{O \subseteq U | O \in \Omega\}$, then $\mathcal{T}_U \models \phi$.*

[2]Discrete is the topology where every subset is open (and closed).

Proof: For any valuation ν on U let ν' be any extension of ν to all of X, agreeing with ν on U. We will prove by induction on the length of modal formulas that for any point $u \in U$, any modal formula ψ:

$$\mathcal{T}, \nu', u \models \psi \text{ iff } U, \nu, u \models \psi,$$

which readily implies the claim of the theorem.
Since the propositional letters and boolean connectives need not any special care, consider the case $\psi = \Box\psi_1$.
$\boxed{\Rightarrow}$ $\mathcal{T}, \nu', u \models \Box\psi_1$ means there is an open neighbourhood O of u, such that $\mathcal{T}, \nu', O \models \psi_1$. By the definiton of topology on U and the induction hypothesis, we get $U, \nu, O \cap U \models \psi_1$ with $O \cap U$ the open neighbourhood of u in Ω_U. This means $U, \nu, u \models \Box\psi_1$.
$\boxed{\Leftarrow}$ $U, \nu, u \models \Box\psi_1$ means there is an open neighbourhood O_U of u in U such that $U, \nu, O_U \models \psi_1$. From the definition of Ω_U it follows that $O_U \in \Omega$. By induction hypothesis then $\mathcal{T}, \nu', O_U \models \psi_1$ with O_U open neighbourhood of u in $\mathcal{T}$, which yields $\mathcal{T}, \nu', u \models \Box\psi_1$. $\dashv$

The class of disconnected topological spaces is that of complementing the class of connected topological spaces. We have shown the latter to be modally undefinable. The former appears to be undefinable as well. Just consider the two-point space with discrete topology - it is disconnected, but the one-point open subspace of it is clearly connected. As open subspaces preserve validity, disconnectedness can not be modally defined.

So open subspaces are the topological twins of generated subframes. The difference should be mentioned though - in frames, if A is any subset of a frame, there always exists the least upward closed set containing A - the A-generated subframe. This is the case with Alexandroff spaces as well, but not generally. In an arbitrary topological space we can always take *some* open subspace surrounding the given subset A, but one can not always find the smallest open of this kind.

With this correspondence in hand, it is not hard to find out what the topological notion of bounded morphism should be. Look what a bounded morphism does with upward closed sets - the forth condition in the definition of a bounded morphism is just a monotonicity requirement saying that bounded-morphic preimages of the upward closed sets are upward closed. In topological terms this would mean that preimages of opens are open. Such maps are called *continuous* in general topology:

Definition 2.1.6 *The map $f : \mathcal{T}_1 \to \mathcal{T}_2$ between topological spaces $\mathcal{T}_1 = (X_1, \Omega_1)$ and $\mathcal{T}_2 = (X_2, \Omega_2)$ is said to be* continuous *if whenever $O_2 \subseteq X_2$ is open in $\mathcal{T}_2$, the set $f^{-1}(O_2)$ is open in $\mathcal{T}_1$, i.e.:*

$$O_2 \in \Omega_2 \text{ implies } f^{-1}(O_2) \in \Omega_1.$$

The back condition in the definition of bounded morphism, when slightly rephrased, is claiming that the images of upward closed sets are upward closed. This corresponds to the definition of *open* maps in topology.

Definition 2.1.7 *The map $f : \mathcal{T}_1 \to \mathcal{T}_2$ between topological spaces $\mathcal{T}_1 = (X_1, \Omega_1)$ and $\mathcal{T}_2 = (X_2, \Omega_2)$ is called* open *if it sends open sets to open sets, i.e.:*

$$O_1 \in \Omega \text{ implies } f(O_1) \in \Omega_2.$$

A map is called an interior map, *if it is both open and continuous.*

The latter definition and our earlier remarks suggest interior maps play the same role in the topological semantics as bounded morphisms do in the Kripke semantics. The following theorem proves this suggestion correct.

Theorem 2.1.8 *Let $\mathcal{T}_1 = (X_1, \Omega_1)$ be a topological space and f an interior map from $\mathcal{T}_1$ onto $\mathcal{T}_2 = (X_2, \Omega_2)$. For any modal formula ϕ, $\mathcal{T}_1 \models \phi$ implies $\mathcal{T}_2 \models \phi$.*

Proof: Again, for any valuation ν_2 on $\mathcal{T}_2$ define the valuation ν_1 as follows: $\nu_1(p) \equiv f^{-1}(\nu(p))$. We claim for any modal formula ψ and any $x \in X_1$ the following holds:

$$\mathcal{T}_1, \nu_1, x \models \psi \text{ iff } \mathcal{T}_2, \nu_2, f(x) \models \psi.$$

This will then yield the desired validity preservation result, exploiting the surjectiveness of f. The proof proceeds by induction, with propositional and boolean cases straightforward. Assume $\psi = \Box\psi_1$:

$\boxed{\Rightarrow}$ From $\mathcal{T}_1, \nu_1, x \models \Box\psi_1$ it follows that $\mathcal{T}_1, \nu_1, U_x \models \psi_1$ for some open neighbourhood U_x of x. It is by openness of f that we get $f(U_x)$ is an open neighbourhood of $f(x)$. By induction hypothesis, $\mathcal{T}_2, \nu_2, f(U_x) \models \psi_1$ and we arrive at $\mathcal{T}_2, \nu_2, f(x) \models \Box\psi_1$.

$\boxed{\Leftarrow}$ Same as above, with f^{-1} instead of f. The continuity of f ensures $f^{-1}(O_2)$ to be open in $\mathcal{T}_1$ provided O_2 is open in $\mathcal{T}_2$. $\dashv$

We are almost done with finding the topological equivalents for basic validity-preserving operations on frames. The only one left is the formation of ultrafilter extension and we will postpone the definition of the analogous topological construction till the last section of this chapter. Before that, let us put the constructions we have already defined above to work. In particular, our immediate aim is to show that the separation axioms are not definable by the modal formula. The definitions of higher separation axioms employ some involved topological terminology, which we would like to avoid here. The axioms T_0 and T_1 are defined in the last part of the work. For our present purposes it is sufficient to know (cf. [Engelking 1977]) that the real line with its usual topology obeys the separation axioms $T_0, T_{\frac{1}{2}}, T_1, T_2, T_3, T_{3\frac{1}{2}}, T_4, T_5, T_6$. Unlucky as it is, none of these separation axioms can be defined by the modal formula, not being preserved under interior maps. Here is the evidence:

Theorem 2.1.9 *The separation axioms T_i with $i \leq 6$ are not definable in the basic modal language.*

Proof: Consider the interior map from the reals with their standard topology to $X = \{1, 2\}$ equipped with antidiscrete topology[3], sending rationals to 1 and irrationals to 2. It is easy to verify that the reals obey all separation axioms, while X obeys none. As interior maps preserve the validity, none of the separation axioms could be defined by a formula in the basic modal language. $\dashv$

Note that this example also exhibits non-compactness to be undefinable. Again, we can gain a little by enriching the language we are working with, and we address this issue in the third part of the present work, where we will see how nominals can help to define some lower separation axioms.
We would like to make a few remarks on the constructions developed in this section before going on to the next one.

Other topological constructions: In some cases well-known topological constructions give rise to natural interior maps. The validity preservation automatically transfers to these constructions. As an example we briefly discuss the topological product of spaces and the quotient of the topological space.

[3]Antidiscrete topology (also called trivial topology in the literature) consists solely of empty set and the universe.

The product of topological spaces can be defined in two ways. We would like to avoid bringing in the respective definitions, but in both cases the natural projections are interior maps. This means that if the product of topological spaces validates some modal formula, then the components must validate this formula as well. For the quotient of a topological space note that any equivalence relation defined on the universe of the space determines a quotient space with the natural topology on it (so that the natural quotient map becomes continuous). Under the (necessary and sufficient) condition that for any open set O the union of all the equivalence classes intersecting with O is open, the natural quotient map becomes an interior map. This would mean that whenever the above condition is satisfied for a space and an equivalence relation on it, and the space validates the modal formula ϕ, the quotient space validates ϕ as well. For the exact definitions of these topological constructions we refer to [Engelking 1977].

Topo-bisimulations: We would like to mention that the three validity-preserving operations are manifestations of the notion of *topo-bisimulation*, first introduced in [Aiello and van Benthem 2001] as a generalization of the notion of bisimulation between Kripke models to topological models. Topo-bisimulations are a powerful tool for a modal reasoning about topological spaces. Each of the constructions - topological sum, open subspace and interior image give rise to natural topo-bisimulations between the corresponding models. The validity preservation results presented above can be proved through a more general theorem from [Aiello and van Benthem 2001] claiming that topo-bisimulation implies modal equivalence. The interested reader is referred to [Aiello and van Benthem 2001] and [Aiello et al. 2001] for results concerning topo-bisimulations.

Topological invariance: Say P is a topological property which our modal language is powerful enough to express. It is worth mentioning that the results of this section imply P is invariant for the formation of topological sums, open subspaces and interior images. This obvious consequence of the earlier theorems bears topologically interesting insight. Consider, for example, atomicity. It is straightforward from the modal point of view that being an atomic space is a topological property which is invariant under topological sums, open subspaces and interior images. This is easy to establish by direct topological proof, but still not entirely obvious. Describing invariance of properties under various topological transformations is the major task of general topology. For modally definable properties and three basic validity-preserving topological operations the invariance is automatic, witness the earlier theorems of this section. The same holds for extremal disconnectedness. We will say a bit more about extremal disconnectedness in the last chapter and get back to the topological invariance in the concluding remark of the thesis.

In the meantime, we are going to proceed to the next section of this chapter. We choose to approach the desired definability result via algebraic machinery and the next section is devoted to developing such techniques. Let us conclude this section with the remark that when restricted to A-spaces, the topological notions of topological sum, open subspace and interior map coincide with the frame-theoretic notions of disjoint union, generated subframe and bounded morphism respectively, applied on corresponding qo-sets.

2.2 Interior algebras

First introduced by McKinsey and Tarski, Interior Algebras are BAOs with the topological operator. Actually, the pioneering completeness proof for modal logic has been carried out with the help of Interior Algebras and their topological interpretations. Here we repeat some of the basic definitions and prepare the firm grounds to base our definability result upon. An Abstract Interior Algebra is just a boolean algebra with $S4$-operator, as formally stated below:

Definition 2.2.1 *An Interior Algebra is a pair $(B, \Box)$ where B is a boolean algebra, $\Box$ is an operator, assigning to each element a of B element $\Box a \in B$, such that the following holds:*

(I_1) $\Box(a \wedge b) = \Box a \wedge \Box b$,

(I_2) $\Box a \leq a$,

(I_3) $\Box\Box a = \Box a$,

(I_4) $\Box\top = \top$.

An obvious example of such an algebra would be the full set algebra over any topological space, with **I** operator. Indeed, any set algebra over the topological space, closed under **I**, will do. Following the route of the celebrated

Stone Representation Theorem, we can actually embed any abstract Interior Algebra into the power set algebra over some topological space. The carrier of the topology, as in the case of boolean algebras, is the set of all ultrafilters of the algebra. The topology on this set can be defined in various ways, giving rise to different structures. Our aim favours the definition where the representation space has an Alexandroff topology. As we will see shortly, this can be done in a natural way. Namely, we declare the set of ultrafilters over an abstract Interior Algebra to be open, if it consists of all the ultrafilters extending some open filter.

Definition 2.2.2 *We call a filter F of an interior algebra $(B, \Box)$ an* open filter *if for any $a \in B$ we have that $a \in F$ implies $\Box a \in F$.*
We call an element a of an interior algebra open*, if $a = \Box a$.*

Before checking this definition of the topology to be correct, it should be mentioned ahead that the representation space for an interior algebra $\mathcal{A}$ we are about to bring turns out to be the A-space obtained from the ultrafilter frame of $\mathcal{A}$ by declaring upward-closed sets open. This means our next theorem, although topologically flavoured, is simply building up the ultrafilter frame - in topological terms.

Theorem 2.2.3 *Any Interior Algebra $\mathcal{A} = (A, \Box)$ is isomorphic to subalgebra of all subsets of some A-space.*

Proof: Consider the set $A^* \equiv \mathit{Uf}(A)$ of all ultrafilters of A. For arbitrary filter F of the algebra by F^* we will denote the set of all ultrafilters extending F. In case F is generated by an element a of the algebra, $F = \{b \in A | b \geq a\}$, it is clear that F^* consists of all ultrafilters having a as an element. In this case we may also use the notation $a^* \equiv \{u \in A^* | a \in u\}$.
Let O be any open filter, consider the set O^* of all ultrafilters which extend O. An easy argument shows that the collection of all O^* will qualify for the base of topology. Denote the resulting topological space $\mathcal{A}^* = (A^*, \Omega^*)$ and call it the *Alexandroff extension* of A.
To show that Ω^* is an Alexandroff topology, it suffices to prove that any ultrafilter has the smallest open neighbourhood. Let u be an arbitrarily chosen ultrafilter of A, consider the set of all open elements from this ultrafilter $O_u^1 \equiv \{o \in u | o = \Box o\}$, which is closed under finite meets according to (I_1) and thus generates the filter $O_u \equiv \{a \in A | a \geq o \in O_u^1\}$ which obviously is

an open filter. From the fact that u is an extension of O_u it follows that O_u^* is an open neighbourhood of u, it is also the smallest such, as all the open filters contained in u are also contained in O_u.
Now consider the sets $a^* = \{u \in A^* | a \in u\}$ for each $a \in A$. By Stone's representation theorem, family of subsets $(a^*)_{a \in A}$ of A^* is a boolean algebra isomorphic to A. Now, showing that $\mathcal{A}$ is a subalgebra of $(\wp A^*, I^*)$, where I^* stands for the interior operator defined by Ω^*, boils down to showing that $(\Box a)^* = I^* a^*$ for each $a \in A$:
First observe that $(\Box a)^*$ is an open set in Ω^*. Indeed, as we already mentioned, $(\Box a)^*$ coincides with the set of all ultrafilters extending the filter generated by $\Box a$. This filter is open because $\Box a \leq b$ implies $\Box a \leq \Box b$ by (I_1) and (I_3). The derivation $u \in (\Box a)^* \Rightarrow \Box a \in u \Rightarrow a \in u \Rightarrow u \in a^*$ shows that $(\Box a)^* \subseteq a^*$. We thus proved that $(\Box a)^*$ is an open set included in a^*. To complete the proof we will show that $(\Box a)^*$ is the biggest open set included in a^*.
Indeed, take any open set $O \subseteq a^*$, from the definition of Ω^* we retrieve $O = \bigcup_{i \in I} O_i^*$ with O_i^* an open filter for each $i \in I$. It is straightforward that $O_i^* \subseteq a^*$ for each $i \in I$. This means that an ultrafilter extending O_i must contain a and by ultrafilter theorem this is precisely to say $a \in O_i$. Thus $\Box a \in O_i$ by openness of O_i for each $i \in I$, whence $O_i^* \subseteq (\Box a)^*$, for each $i \in I$. But then $O = \bigcup_{i \in I} O_i^* \subseteq (\Box a)^*$. Since O was the arbitrary open set included in a^*, we conclude that $(\Box a)^*$ is the greatest open contained in a^*. So $(\Box a)^* = I^* a^*$. $\dashv$

To get a grasp of what the above construction really does, consider the following.

Definition 2.2.4 *For an Interior Algebra $\mathcal{A} = (A, \Box)$ define the relation R^* on A^* as follows:*

$$R^* uv \quad \text{iff} \quad v \in O_u.$$

Where O_u denotes the smallest open neighbourhood of u in Ω^*.

We will show now that R^* coincides with the relation R_+ of the ultrafilter frame for $\mathcal{A}$, thus linking the Alexadroff space we have just built with the ultrafilter frame of the respective algebra.

Theorem 2.2.5 *Let $\mathcal{A} = (A, \Box)$ be any Interior Algebra. The frame (A^*, R^*) coincides with the ultrafilter frame $\mathcal{A}_+$ of $\mathcal{A}$.*

Proof: Let us first turn $\mathcal{A}$ into BAO by means of defining the operator $\Diamond \equiv -\Box-$ on the boolean algebra A. This is merely another way of looking at $\mathcal{A}$. Now of two arbitrary ultrafilters u and v the following holds:

$$R^*uv \text{ iff } \Diamond a \in u \text{ for all } a \in v.$$

This is to say that the relation R^* coincides with the relation R_+ of the ultrafilter frame of $\mathcal{A}$. The proof is as follows:
$\boxed{\Rightarrow}$ Assume R^*uv and for some $a \in v$, $\Diamond a \notin u$. Since u is an ultrafilter $-\Diamond a \in u \Rightarrow --\Box-a \in u \Rightarrow \Box-a \in u$. $\Box-a$ is an open element, recall form the proof of the previous theorem that v is some extension of the open filter of all open elements from u, therefore $\Box-a \in v$ and as far as $\Box-a \leq -a$ and v is an ultrafilter, $-a \in v$ which contradicts the assumption $a \in v$.
$\boxed{\Leftarrow}$ Now let $\Diamond a \in u$ hold for all $a \in v$ and assume R^*uv fails; then v must lack at least one open element from u, or, putting it more precisely, there exists $o \in u$ such that $o = \Box o$ and $o \notin v$. By the properties of ultrafilters, then, $-o \in v$ and by our assumption $\Diamond-o \in u$. An easy verification shows that $\Diamond-o = -\Box--o = -\Box o = -o$, so $-o \in u$ and again, we find a contradiction with $o \in u$. $\dashv$

To make the picture complete, we just need some duality results concerned with homomorphisms and products. In the light of the last theorem we could just mention that the homomorphism between two interior algebras gives rise to the bounded morphism between respective ultrafilter frames and thus, the interior map between their Alexandroff extensions (the correspondence established in the first chapter). We supply the proof of this fact in the terms of theorem 2.2.3 for interested reader. First let us define how to lift the homomorphism between algebras to the map between corresponding Alexandroff extensions.

Definition 2.2.6 *Let $h : \mathcal{A}_1 \to \mathcal{A}_2$ be a homomorphism of Interior Algebras. Define $h^* : \mathcal{A}_2^* \to \mathcal{A}_1^*$ as follows:*

$$h^*(u_2) = \{a_1 \in A_1 | h(a_1) \in u_2\}.$$

A simple verification shows that h^* is defined correctly (it maps ultrafilters to ultrafilters). Topologically, it also appears an interior map. We prove this fact below.

Theorem 2.2.7 *If $h : \mathcal{A}_1 \to \mathcal{A}_2$ is a homomorphism of Interior Algebras, then $h^* : \mathcal{A}_2^* \to \mathcal{A}_1^*$ is an interior map between topological spaces. Additionally, if h is injective (surjective) then h^* is surjective (injective).*

Proof: h^* can be easily checked to be sending ultrafilters to ultrafilters, surjectively if h is injective and injectively if h is surjective. The only non-trivial part to prove remains h^* being interior (that is, open and continuous). This suffices to be checked just on the elements of the respective bases (provided we first establish continuity and then openness), as it is known from general topology (cf. [Engelking 1977], section 1.4). Let us proceed towards this aim. To prove the continuity of h^* consider $O_1^* \in \Omega_1^*$ element of the base generated by the open filter O_1 and take $h^{*-1}O_1^*$ which consists of all such ultrafilters of A_2 h^*-images of which contain O_1 as a subset, or, in other words, all ultrafilters of A_2 which contain the set $h(O_1)$. Although $h(O_1)$ may not be a filter, it is closed under finite meets because O_1 was closed under finite meets and h respects the boolean operations. Hence the set of all ultrafilters containing $h(O_1)$ will coincide with the set of all ultrafilters containing the smallest filter generated by $h(O_1)$. This filter is an open filter, because O_1 was an open filter and h respects the $\Box$. So we obtain $h^{*-1}O_1^* = (h(O_1))^* \in \Omega_2^*$.
For openness of h^* we take an element of the base of Ω_2 denoted O_2^* and consider the set $h^*(O_2^*)$. A typical representative of this set has the form $h^*(u_2)$ where u_2 is such that $O_2 \subseteq u_2$. Since $h^*(u_2) = \{a_1 \in A_1 | h(a_1) \in u_2\}$, it is immediate that $h^{-1}(O_2) \subseteq h^*(u_2)$. For h is a homomorphism and O_2 is an open filter, it follows that $h^{-1}(O_2)$ is an open filter of A_1 and this means nothing but $h^*(O_2^*) = (h^{-1}(O_2))^* \in \Omega_1^*$. $\dashv$

Concluding this section, we show how to deal with the products of the set interior algebras.

Proposition 2.2.8 *Let $\mathcal{T}_i$ be the collection of topological spaces indexed with the set I. Denote by $(\mathcal{T}_i)_*$ the power set algebra over $\mathcal{T}_i$ with respective interior operator. $(\mathcal{T}_i)_*$ are clearly interior algebras, and the following holds:*

$$\prod_{i \in I} (\mathcal{T}_i)_* \cong (\bigcup_{i \in I} \mathcal{T}_i)_* \ .$$

With $\cong$ standing for IA*-isomorphism and $\bigcup$ for the topological sum of topological spaces.*

Proof: Define the map $f : \prod_{i\in I}(\mathcal{T}_i)_* \to (\bigcup_{i\in I}\mathcal{T}_i)_*$ as follows:
For $(a_i)_{i\in I} \in \prod_{i\in I}(\mathcal{T}_i)_*$ note that each $a_i \in (\mathcal{T}_i)_*$ corresponds to the subset $A_i \subseteq \mathcal{T}_i$. Then define $f((a_i)_{i\in I}) = (\bigcup_{i\in I} A_i)_*$. In other words, take the topological sum of all A_i and send $f((a_i)_{i\in I})$ to corresponding element of Interior Algebra of all subsets of $\bigcup_{i\in I}\mathcal{T}_i$. It is a subject of a routine check that f is an isomorphism of interior algebras. To clarify why f commutes with the interior operator, note that on the product of algebras interior operator is defined componentwise; on the topological sum of topological spaces interior operator also works componentwise. $\dashv$

2.3 Topological Goldblatt-Thomason theorem

We have exhibited the topological equivalents for basic frame operations of disjoint union, generated subframe and bounded morphic image. The fourth crucial operation on frames is formation of the ultrafilter extension. We still lack the topological equivalent for that. Since A-spaces are in one-to-one correspondence with $S4$-frames, such an equivalent should yield the ultrafilter extension of corresponding frame when applied to an A-space. But then we already have one implicit candidate that will do this job! Just take the interior algebra of all subsets of topological space and consider the Alexandroff extension of it - when starting from an A-space, this procedure will result in the ultrafilter extension for this frame. This entire section is the justification of this choice, with application to topological version of Goldblatt-Thomason theorem.

Definition 2.3.1 *For a given topological space $\mathcal{T} = (X, \Omega)$ define its* Alexandroff extension *to be the A-space $\mathcal{T}^* \equiv (\wp(X)^*, \Omega^*)$ of the interior algebra of all subsets of X.*

In other words, just take the upward-closed set topology on the ultrafilter frame for the power set algebra of $\mathcal{T}$ and define the standard valuation on it as follows:

Definition 2.3.2 *If $\mathcal{M} = (\mathcal{T}, \nu)$ is a topological model, define the standard valuation ν^* on the $\mathcal{T}^*$ as follows:*

$$u \in \nu^*(p) \text{ iff } \nu(p) \in u.$$

This definition will help us to show that Alexandroff extension anti-preserves the modal validity. We could use the relational, rather than topological, structure on $\mathcal{T}^*$ in the proof, reproducing the analogous result about ultrafilter extension of a frame, but the presented proof is shorter and fits better in the framework of this chapter.

Theorem 2.3.3 *Let $\mathcal{T} = (X, \Omega)$ be a topological space, $\mathcal{M} = (\mathcal{T}, \nu)$ a topological model based on it, $\mathcal{T}^*$ the Alexandroff extension of $\mathcal{T}$ and ν^* the standard valuation, then the following holds for any modal formula ϕ:*

$$u \in \nu^*(\phi) \ \text{ iff } \ \nu(\phi) \in u.$$

Proof: We proceed by induction on the length of ϕ. The propositional case is taken care of by the definition of ν^*, the cases of the boolean connectives are rather obvious, so we only address the modality case. Consider a formula of $\Box\phi$ form:

$\boxed{\Rightarrow}$ By definition $u \in \nu^*(\Box\phi)$ means that u has an open neighbourhood (restrict to the element of the base without loss of generality) O_u^* such that $v \models \phi$ holds for all $v \in O_u^*$. In other words, $O_u \subseteq v$ implies $v \models \phi$. By the induction hypothesis this can be rephrased as $O_u \subseteq v$ implies $\nu(\phi) \in v$ for all v. This indicates that $\nu(\phi)$ is an element of O_u, for if $\nu(\phi) \notin O_u$ were the case, by ultrafilter theorem we could find an ultrafilter extending O_u and still leaving $\nu(\phi)$ aside. So $\nu(\phi) \in O_u$. Hence, O_u being an open filter yields $I(\nu(\phi)) \in O_u$ and since u extends O_u, we arrive at $I(\nu(\phi)) \in u$.

$\boxed{\Leftarrow}$ Suppose $\nu(\Box\phi) \in u$, this means $I(\nu(\phi)) \in u$. Consider any ultrafilter v from the smallest open neighbourhood of u. Any such ultrafilter contains all the open elements from u. Thus $I(\nu(\phi)) \in v$ because $I(\nu(\phi))$ is an open element of u. Then $I(\nu(\phi)) \subseteq \nu(\phi)$ implies $\nu(\phi) \in v$. By the induction hypothesis $v \in \nu^*(\phi)$. As v was arbitrarily chosen from the smallest open neighbourhood of u, we get $u \in \nu^*(\Box\phi)$. $\dashv$

We are finally in the position of presenting the main result of this chapter: a topological version of Goldblatt-Thomason theorem about definable classes.

Theorem 2.3.4 (Topological Goldblatt-Thomason) *The class K of topological spaces which is closed under formation of Alexandroff extensions is modally definable iff it is closed under taking opens subspaces, interior images, topological sums and it reflects Alexandroff extensions.*

Proof: We only deal with the right to left direction, as the other direction appears trivial given the earlier theorems 2.1.2, 2.1.5, 2.1.8 and 2.3.3.
Assume that K is any class of spaces satisfying the conditions of the theorem and let $\mathcal{T} = (X, \Omega)$ be a space validating the modal theory of K, then the Interior Algebra of all subsets of X, denoted $\mathcal{T}_*$, is validating the equational theory of algebras corresponding to the spaces from K. Thus by Birkhoff's theorem, $\mathcal{T}_*$ is isomorphic to **HSP** of such algebras. So there are algebras $\mathcal{H}$, $\mathcal{S}$ and mappings h, s such that:

(1) $h : \mathcal{H} \to \mathcal{T}_*$ is onto IA-homomorphism.

(2) $s : \mathcal{H} \to S$ is injective IA-homomorphism.

(3) $\mathcal{S} = \prod_{i \in I} (\mathcal{P}_i)_*$ where $\mathcal{P}_i \in K$ and $(\mathcal{P}_i)_*$ is the Interior Algebra of all subsets of $\mathcal{P}_i$ for each $i \in I$.

From (3), definition of Alexandroff extension and the proposition 2.2.8 we obtain that $\mathcal{S} = (\bigcup_{i \in I} \mathcal{P}_i)_*$ and denoting $\mathcal{P} = \bigcup_{i \in I} \mathcal{P}_i$ we get $\mathcal{P} \in K$ by the closure under the formation of topological sum. So $\mathcal{S} = \mathcal{P}_*$.
Now we exploit (2) lifting up s to Alexandroff extensions, and as far as s is injective IA-homomorphism, we find that $s^* : (\mathcal{P}_*)^* \to H^*$ is an onto interior map. Note that $(\mathcal{P}_*)^*$ is nothing else but the Alexandroff extension of $\mathcal{P}$ and thus belongs to K by the conditions we imposed on this class; but then so is H^* being the interior image of the space from K. So $H^* \in K$.
Finally we treat (1), again lifting up h to Alexandroff extensions and obtaining $h^* : (\mathcal{T}_*)^* \to H^*$ where h^* in now an injective interior map, so $h^*((\mathcal{T}_*)^*)$ is an open subset of H^* and, by closure conditions, $h^*((\mathcal{T}_*)^*) \in K$. Note that $h^* : (\mathcal{T}_*)^* \to h^*((\mathcal{T}_*)^*)$ is bijective interior map (homeomorphism). This implies $(\mathcal{T}_*)^* \in K$. We just have to mention that $(\mathcal{T}_*)^*$ is simply an Alexandroff extension of $\mathcal{T}$, so, by K reflecting Alexandroff extensions, $(\mathcal{T}_*)^* \in K$ readily implies desired $T \in K$. $\dashv$

As an immediate consequence, we get that the class of Alexandroff spaces, obviousely closed under the formation of Alexandroff extensions, is not modally definable - it clearly does not *reflect* Alexandroff extensions!

Unlike the three basic topological validity-preserving operations, the Alexandroff extension is a somewhat new concept to topology. Topological sums, open subspaces and interior maps are part of the general topology folklore. Alexandroff extensions, although having a natural topological structure, are brought into life for logical reasons. They helped us to prove the topological Goldblatt-Thomason theorem, but if we are to put this theorem to work, we need a better topological understanding of the Alexandroff extension. One possible way to describe the structure of a new construction is to connect it somehow with already known and well-investigated constructions. In the case of the Alexandroff extension, it turns out that the topological notion of the *Stone-Čech compactification* is destined to play this role. We will present in the next chapter the construction which we call the *compact extension.* Compact extension is a generalization of the well-known notion of Stone-Čech compactification. We will show that Alexandroff extension is the smallest Alexandroff topology containing the topology of the compact extension.

3 Further directions and conclusions

3.1 Enriching the language

As promised in the earlier chapters, we will show in this section how extending the modal language helps us to define some of otherwise undefinable topological properties/classes. We will consider adding the global modality, the difference operator and the nominals. The case with global modality has already been considered in [Shehtman 1999] where connectedness is expressed as well. Let us consider all these options in turn.

Global Modality: Add to the language with topological box operator the global modality. We could even view the global box (written A) as a special kind of topological box with the fixed *antidiscrete* topology interpretation! The global modality combined with the topological one gives rise to the logic $S4 * S5$ and the connectedness becomes expressible by the modal formula $A(\Diamond\phi \to \Box\phi) \to (A\phi \vee A\neg\phi)$, as shown in [Shehtman 1999]. We would like to add that the example displayed in the theorem 2.1.9 still shows non of the separation axioms can be expressed even in this richer language - interior maps preserve validity here too.

What happens with definability when we add the global diamond to our language? In the case of the Kripke semantics we know that two of the closure conditions in the Goldblatt-Thomason theorem drop out. Namely the closure under disjoint unions and generated subframes, for those no longer preserve the validity of modal formulas in the enriched language. This of course implies that the topological sum and the open subspace do not preserve the modal validity in the language with additional global modalities. However, interior images and Alexandroff extensions still do. We sketch the proofs below.

Theorem 3.1.1 *Let $\mathcal{T}_1 = (X_1, \Omega_1)$ be a topological space and f an interior map from $\mathcal{T}_1$ onto $\mathcal{T}_2 = (X_2, \Omega_2)$. For any modal formula ϕ in the language with additional global modality, $\mathcal{T}_1 \models \phi$ implies $\mathcal{T}_2 \models \phi$.*

Proof sketch: Note that the mere fact that f is a surjective mapping ensures that f is still an interior map when we equip X_1 and X_2 with the antidiscrete topologies. Use the theorem 2.1.8 thinking of A as a topological box with the fixed antidiscrete interpretation. $\dashv$

Theorem 3.1.2 *Let $\mathcal{T} = (X, \Omega)$ be a topological space, $\mathcal{M} = (\mathcal{T}, \nu)$ a topological model based on it, $\mathcal{T}^*$ the Alexandroff extension of $\mathcal{T}$ and ν^* the standard valuation, then the following holds for any modal formula ϕ in the modal language with additional global modality:*

$$u \in \nu^*(\phi) \text{ iff } \nu(\phi) \in u.$$

Proof sketch: Proving by induction on the length of formulas, witness the following chain of equivalent statements: $u \in \nu^*(A\phi) \Leftrightarrow \forall v(v \in \nu^*(\phi)) \Leftrightarrow \forall v(\nu(\phi) \in v) \Leftrightarrow \nu(\phi) = X \Leftrightarrow \nu(A\phi) = X \in u$. $\dashv$.

These prove the easy direction of the following suggestion:

Conjecture 3.1.3 *A class K of topological spaces which is closed under taking Alexandroff extensions is modally definable in the language with additional global diamond iff K is closed under taking the interior images and it reflects Alexandroff extensions.*

Next in our agenda is the difference operator.

The difference operator: Enrich the basic modal language with the new diamond D saying something is true somewhere else. The global diamond is definable in this language by the formula $E\phi \leftrightarrow \phi \vee D\phi$. It follows readily that connectedness is expressed as well. We can do even better - the separation axioms T_0 and T_1 become modally definable. Let us first give the topological definition of these axioms.

Definition 3.1.4 *We say that a topological space $\mathcal{T}$ obeys the T_0 separation axiom if for any two distinct points x and y, one of them has an open neighbourhood not containing the other.*
We say that a topological space $\mathcal{T}$ obeys the T_1 separation axiom if for any two distinct points x and y, each of them has an open neighbourhood not containing the other.

The following is how the separation axioms T_0 and T_1 can be defined in the modal language enriched with the difference operator.

Theorem 3.1.5 *Consider the basic modal language enriched with the difference operator D and with the diamond interpreted as a closure operator. The following two formulas define the T_0 and T_1 separation axioms respectively:*

$$t_0 = Up \wedge DUq \rightarrow \Box\neg q \vee D(q \wedge \Box\neg p),$$

$$t_1 = Up \rightarrow A(p \leftrightarrow \Diamond p).$$

Where Up is defined as $p \wedge \neg Dp$ and Ap is defined as $p \wedge \neg D\neg p$.

Proof: First recall that Up is true at a world x in a model iff p is true at x and only there.

$\boxed{T_0 \Rightarrow t_0}$ Take a T_0 topological space $\mathcal{T} = (X, \Omega)$ and any valuation ν on it. If some point x_1 makes the antecedent of t_0 true, this proves $\nu(p) = \{x_1\}$ and $\nu(q) = \{x_2\}$ for some $x_2 \neq x_1$. At least one of the points x_1 or x_2 has an open neighbourhood not containing the other one, which is exactly what the consequent of t_0 needs to be true.

$\boxed{t_0 \Rightarrow T_0}$ If $\mathcal{T} \models t_0$ then it does so for any valuation and any substitution instance of t_0. For any two distinct points $x_1, x_2 \in X$ consider the valuation $\nu(p) = \{x_1\}$, $\nu(q) = \{x_2\}$ and employ $x_1 \models Up \wedge DUq \rightarrow \Box\neg q \vee D(q \wedge \Box\neg p)$. Antecedent is true (we designed the valuation for it) and the consequent must be true as well, which means either $x_1 \models \Box\neg q$ or $x_1 \models D(q \wedge \Box\neg p)$. If $x_1 \models \Box\neg q$ is true, then x_1 obtains the open neighbourhood with $\neg q$ true throughout. Thus none of the points in this neighbourhood can be x_2. If $x_1 \models D(q \wedge \Box\neg p)$ is the case, for some $x_3 \in X$ we have $x_3 \models q \wedge \Box\neg p$. Since q is only true at x_2, from $x_3 \models q$ we get $x_3 = x_2$. Thus $x_2 \models \Box\neg p$, which means x_2 has an open neighbourhood making $\neg p$ true. Hence x_2 has an open neighbourhood disjoint from x_1.

$\boxed{T_1 \Rightarrow t_1}$ Recall that T_1 spaces are precisely those where singletons are closed, or equivalently, where the closure of every point is this point itself. If $\mathcal{T}$ is such a space and ν any valuation on it, consider any point x making the antecedent of t_1 true. This means $\nu(p) = \{x\}$ and by T_1 also $\nu(\Diamond p) = \{x\}$ implying that the consequent is true as well.

$\boxed{t_1 \Rightarrow T_1}$ If $\mathcal{T} \models t_1$ and $x \in X$ is any point, take the valuation $\nu(p) = \{x\}$ and exploit $x \models Up \rightarrow A(p \leftrightarrow \Diamond p)$ bearing in mind that $x \models Up$ by the definition of ν. We get $\nu(\Diamond p) = \{x\} = \mathbf{C}\{x\}$ which is to say $\mathcal{T}$ is T_1-space given x was arbitrary. $\dashv$

See [de Rijke 1992] for a thorough survey of the modal languages with difference operator. Some aspects are discussed in [Blackburn et al. 2001] also, where a hybrid languages are considered as well.

Hybrid language: Consider the basic modal language equipped with countably many additional propositional letters denoted $i, j, ..$ and meant to be sent to exactly one point with the valuation, topologically interpreted $\Box$ and the satisfaction operators $@_i$ for each nominal with the interpretation

$$x \models @_i\phi \text{ iff } \nu(i) \models \phi.$$

Again, we can express T_0 and T_1 separation axioms as stated below:

Theorem 3.1.6 *Consider the following formulas in the hybrid modal language with @ operator:*

$$t_0 = @_i\neg j \to @_j\Box\neg i \vee @_i\Box\neg j,$$
$$t_1 = i \leftrightarrow \Diamond i$$

(i) The topological space $\mathcal{T} = (X, \Omega)$ is a T_0-space iff $\mathcal{T} \models t_0$.

(ii) The topological space $\mathcal{T} = (X, \Omega)$ is a T_1-space iff $\mathcal{T} \models t_1$.

Proof: (i) Recall that the T_0 separation axiom requires one of the any two distinct points to have an open neighbourhood not containing the other.
$\boxed{t_0 \Rightarrow T_0}$ If $\mathcal{T} \models t_0$, then for any two distinct points $x, y \in X$ with valuation ν such that $\nu(i) = \{x\}$, $\nu(j) = \{y\}$, $\mathcal{T}, \nu, x \models @_i\neg j$. Then $\mathcal{T}, \nu, x \models @_j\Box\neg i \vee @_i\Box\neg j$, which means that either $\mathcal{T}, \nu, x \models \Box\neg j$ or $\mathcal{T}, \nu, y \models \Box\neg i$. This is precisely to say that either x has an open neighbourhood not containing y or vice versa.
$\boxed{T_0 \Rightarrow t_0}$ If $\mathcal{T}$ is not a T_0-space, then it has two distinct points x, y such that they are inseparable by an open set. Take any valuation which sends i to x and j to y, then $\mathcal{T}, \nu, x \models @_i\neg j$ and at the same time, as every open neighbourhood of x (y) contains y (x), we have $X, x \models \Diamond j$ and $X, y \models \Diamond i$. This means $X, x \not\models \Box\neg j$ and $X, y \not\models \Box\neg i$. In other words, $X, x \not\models @_j\Box\neg i \vee @_i\Box\neg j$ and $X \not\models t_0$.
(ii) The formula t_1 says every singleton is closed (the diamond is interpreted as the closure operator), which is equivalent way of expressing $\mathcal{T}$ obeys the T_1 separation axiom. $\dashv$

This hybrid topological language seems quite promising in expressing some topological properties the closure-diamond alone could not afford.

3.2 Compactness and compactifications

In this section we will discuss the compactness phenomenon in topology. We have already mentioned that defining this property in modal terms is not something we should expect. The equivalent definition of compactness requires any open cover of a space to have a finite subcover. This is a higher order statement and unless we allow infinite disjunctions in our formulas, it is unlikely to allow itself being modally captured. Nevertheless, definability is not the only way of looking at things after all. We will show how to embed any Interior Algebra in the full set algebra over some compact topological space. This construction will allow us to generalize the topological notion of Stone-Čech compactification. We will also show that the new notion of compact extension anti-preserves modal validity. The Alexandroff extension will be characterized in terms of topologically more intuitive compact extension.

To start with, let us exhibit what does compactness mean for Kripke frames when we consider them topologically. In other words, how does compactness reflect on Alexandroff spaces? Recall that Alexandroff spaces are those topological spaces where every point has the smallest open neighbourhood. Alexandroff spaces are in one-to-one correspondence with quasi ordered Kripke frames. We will call a Kripke frame $\mathcal{F} = (W, R)$ *finitely rooted* if the finite set F of worlds can be found such that every other world is accessible from one of the members of F. The corresponding Alexandroff spaces will have the similar property spelled out in the next theorem.

Theorem 3.2.1 *Alexandroff space* (X, Ω) *is compact iff it is "finitely rooted", i.e. if it can be covered by the smallest open neighbourhoods of finite number of its points.*

Proof: Denote by O_x the smallest open neighbourhood of a point x.
$\boxed{\Rightarrow}$ Suppose (X, Ω) is a compact A-space. Consider the open cover $(O_x)_{x \in X}$. By compactness, there exists a finite subcover $O_{x_1}, .., O_{x_n}$, which is to say that (X, Ω) is finitely rooted.
$\boxed{\Leftarrow}$ Take a finitely rooted A-space (X, Ω). Let $x_1, .., x_n$ be the points smallest open neighbourhoods of which cover X. Any open cover $(O_s)_{s \in S}$ of X has to capture each of the points $x_1, .., x_n$, so for some $s_1, .., s_n$ we will have $x_1 \in O_{s_1}, .., x_n \in O_{s_n}$. As each of O_{s_i} is bigger than the smallest open neighbourhood of the corresponding x_i, the collection of opens $O_{s_1}, .., O_{s_n}$ is a cover of X. It follows that X is compact. $\dashv$

So qo-sets corresponding to the compact Alexandroff spaces are finitely rooted. In case a qo-set is of a finite depth, compactness becomes equivalent to having a finite minimum. Let us now turn to interior algebras and their connections to compact topological spaces.

That every Interior Algebra is isomorphic to a set algebra over some compact topological space has already been established in [Aiello et al. 2001]. The authors use the topological canonical model techniques to approach the matter. They prove a rather pretty result that the topology of their topological canonical model is an intersection of the Stone and Kripke topologies naturally arising on the same set. This implies compactness, as Stone spaces are known to be compact and subtopologies inherit this property. Here we present yet another proof of this fact, using the machinery very similar to the one presented in the second chapter of this work where we embedded Interior Algebras into set algebras over Alexandroff spaces. As both proofs follow the same reasoning thread, we omit some details below for the sake of readability.

Theorem 3.2.2 *An abstract Interior Algebra $\mathcal{A} = (A, \Box)$ is isomorphic to a subalgebra of all subsets of some compact topological space.*

Proof: Desired topological space is a modification of Stone's construction. More concretely, take the collection of all ultrafilters of A, denoted $U\!f(A)$ and for each $o \in A$ such that $o = \Box o$ (we called such elements *open*), define the set $o^{\#} = \{u \in U\!f(A) | o \in u\}$. Straightforward verification shows that the collection $(o^{\#})_{\Box o = o \in A}$ will form a base for topology. Call the resulting topology $\Omega^{\#}$ and the resulting topological space $\mathcal{A}^{\#} = (U\!f(A), \Omega^{\#})$ the *Compact Extension* of $\mathcal{A}$. We claim that $\mathcal{A}^{\#}$ is a compact topological space and the interior algebra of all subsets of $\mathcal{A}^{\#}$ contains $\mathcal{A}$ as its subalgebra. Let's first prove compactness:

Take any family $(F_i)_{i \in I}$ of closed subsets of $U\!f(A)$ which has the finite intersection property and consider $F \equiv \bigcap_{i \in I} F_i$. For each $i \in I$ we have $F_i = U\!f(A) \backslash O_i$ where O_i is open. This means $O_i = \bigcup_{j \in J_i} o_j^{\#}$, by de Morgan, $F_i = \bigcap_{j \in J_i} U\!f(A) \backslash o_j^{\#}$. Now, by definition, $f_j^{\#} \equiv U\!f(A) \backslash o_j^{\#} = \{u \in U\!f(A) | o_j \notin u\}$. Using the properties of ultrafilters, denoting $f_j \equiv -o_j$, we obtain $f_j^{\#} = \{u \in U\!f(A) | f_j \in u\}$. Taking $J \equiv \bigcup_{i \in I} J_i$ allows us to obtain $F = \bigcap_{j \in J} f_j^{\#}$. Clearly, family $(f_j^{\#})_{j \in J}$ still has the finite intersection property which now

also means that $(f_j)_{j \in J}$, collection of elements from A, has a finite meet property and thus is included in some ultrafilter of A; call this ultrafilter f. It is obvious that $f \in F$, which confirms the compactness of $\mathcal{A}^{\#}$.
All what is left to show is that the interior operator $I^{\#}$ induced by $\Omega^{\#}$ coincides with $\Box$ when sets $a^{\#} \equiv \{u \in \mathit{Uf}(A) | a \in u\}$ are considered for each $a \in A$. This is just as in theorem 2.2.3. More formally, we have to prove that $I^{\#} a^{\#} = (\Box a)^{\#}$ for arbitrary $a \in A$. This proceeds as follows:
First observe that $(\Box a)^{\#}$ is an open set. The derivation $u \in (\Box a)^{\#} \Rightarrow \Box a \in u \Rightarrow a \in u \Rightarrow u \in a^{\#}$ shows that $(\Box a)^{\#} \subseteq a^{\#}$. Now take any open $O \subseteq a^{\#}$, from the definition of $\Omega^{\#}$ we retrieve $O = \bigcup_{i \in I} o_i^{\#}$ with $\Box o_i = o_i \in A$ for each $i \in I$. Note that properties of the ultrafilters ensure that $o_i \leq a$ for each $i \in I$. Take now any ultrafilter $u \in O \subseteq a^{\#}$, for some $i \in I$ we have then $u \in o_i^{\#}$, or, in other words $o_i \in u$, as $o_i \leq a$ and $\Box o_i = o_i$, we get (using (I_1)) $\Box o_i \leq \Box a$, thus $\Box a \in u$, therefore $u \in (\Box a)^{\#}$. This shows that $(\Box a)^{\#}$ is the biggest open contained in $a^{\#}$, completing our proof. $\dashv$

The reader might have noticed that the topology $\Omega^{\#}$ is contained in the topology Ω^* of the theorem 2.2.3. Indeed, in the base for $\Omega^{\#}$ we included all the ultrafilters sharing the same open *element* from the algebra, while Ω^* was based on the sets of ultrafilters sharing the same open *filter*. Moreover, as we will show shortly, Ω^* is the coarsest Alexandroff topology containing $\Omega^{\#}$. In general $\Omega^{\#}$ is not an Alexandroff topology, but if we would like to add open sets to $\Omega^{\#}$ so that it would become closed under arbitrary intersections, we would have to take all of Ω^* in. The next result is a formalization of this statement:

Theorem 3.2.3 *Let $\mathcal{A}$ be an interior algebra, $\mathcal{A}^{\#} = (\mathit{Uf}(A), \Omega^{\#})$ its compact extension and Ω an Alexandroff topology on $\mathit{Uf}(A)$ such that $\Omega^{\#} \subseteq \Omega$; then $\Omega^* \subseteq \Omega$ holds, with Ω^* standing for the topology of the Alexandroff extension of $\mathcal{A}$.*

Proof: It is sufficient to show that for any ultrafilter $u \in \mathit{Uf}(A)$, the smallest open neighbourhood O_u of u in Ω^* is an element of Ω. Recall that O_u consists of all the ultrafilters having all open elements from u. This means $O_u = \bigcap_{\Box o = o \in u} o^*$. Recall from the theorem 2.2.3 that o^* can be viewed in two equivalent ways. One is that o^* consists of all the ultrafilters extending the open filter generated by the element o. Another one is that o^* includes all the ultrafilters having o as an element. The latter means that o^* denotes the same set as $o^\#$. Then we have $O_u = \bigcap_{\Box o = o \in u} o^\#$. Each $o^\#$ is open in Ω by the assumptions and Ω is closed under arbitrary intersections, so $O_u \in \Omega$. $\dashv$

The above theorem connects the two notions: the Alexandroff extension and the compact extension. We will continue to explore the topological nature of the compact extension to get a better insight of both structures. If we exploit the construction of compact extension in the same way as the Alexandroff extensions were used in the section 2.3, we will obtain a compact topological space from an arbitrary one. Here is the corresponding definition:

Definition 3.2.4 *For a given topological space $\mathcal{T} = (X, \Omega)$ define its* Compact Extension *to be the compact extension $\mathcal{T}^\# \equiv (Uf(\wp X), \Omega^\#)$ of the interior algebra over all subsets of X.*

The similarities between two constructions go further, unleashing the modal nature of the compact extension. We can show that the compact extensions, like Alexandroff extensions, anti-preserve modal validity.

Let us consider a topological model $(\mathcal{T}, \nu)$. We can naturally lift up the valuation to the Compact Extension as follows:

$$u \in \nu^\#(p) \text{ iff } \nu(p) \in u.$$

We can prove that this correspondence enlarges to all formulas, just like in the earlier theorem 2.3.3, as stated below.

Theorem 3.2.5 *For arbitrary modal formula ϕ, the following takes place:*

$$u \in \nu^\#(\phi) \text{ iff } \nu(\phi) \in u.$$

Proof: We proceed by induction on the length of ϕ. Propositional case is taken care of by the definition of $\nu^\#$, boolean connectives employ apparent properties of ultrafilters and the modality step goes as follows (I and $I^\#$ stand for the interior operators in respective spaces):
$\nu(\Box\phi) \in u \Rightarrow I(\nu(\phi)) \in u \Rightarrow u \in (I(\nu(\phi)))^\# \in \Omega^\#$. Take now any $v \in (I(\nu(\phi)))c \Rightarrow I(\nu(\phi)) \in v$, $I(\nu(\phi)) \subseteq \nu(\phi) \Rightarrow \nu(\phi) \in v$ and from here by induction hypothesis $u \models \phi$ and as v was arbitrarily chosen from the open neighbourhood of u, we get to $u \models \Box\phi$.
For the other direction, let $u \models \Box\phi$, this means for some open neighbourhood

of u (we may take an element of the base), $O^{\#} \models \phi$, by induction hypothesis $O \models \phi$, as O is open in X, this implies $O \subseteq I(\nu(\phi))$, now recall $u \in O^{\#}$, which means, by definition, $O \in u$ and thus, $I(\nu(\phi)) \in u$. $\dashv$

It follows that the Compact Extensions anti-preserve modal validity. In words, if the Compact Extension of a space validates some modal formula in the basic modal language, then so does the space itself. To tie the ends up, we exhibit that our Compact Extension is a case of the well-known topological construction called Stone-Čech compactification. We bring in the notion of compactification generalizing that of accepted in general topology. Note that a *homeomorphism* between topological spaces is the interior bijection, or in other words, the one-to-one onto map which is continuous together with its inverse. Homeomorphic embedding of one space into another is the injective map which turns out to be a homeomorphism when restricted to the map from domain to the image of domain in codomain. The subset A of a topological space is said to be *dense* in it, if the closure of A gives the universe.

Definition 3.2.6 *The pair $(\mathcal{T}_1, c)$ is called a compactification of the topological space $\mathcal{T}$ if $\mathcal{T}_1$ is a compact space, $c : \mathcal{T} \to \mathcal{T}_1$ is a homeomorphic embedding and $c(\mathcal{T})$ is dense in $\mathcal{T}_1$.*

In general topology compactifications are considered only when the space $\mathcal{T}_1$ is a T_2-space in addition, i.e. when any two distinct points from $\mathcal{T}_1$ have the respective disjoint open neighbourhoods. It is a fact in general topology that only spaces having such compactifications are the ones satisfying the $T_{3\frac{1}{2}}$ separation axiom. Such spaces are also referred as Tychonoff spaces. When Tychonoff spaces are considered, all compactifications of a given space can be naturally ordered and there is always the biggest, called the Stone-Čech compactification. The Stone-Čech compactification has a certain universality property, which we reproduce here for the further reference.

Proposition 3.2.7 *If $\mathcal{T}$ is a Tychonoff space, $(\mathcal{T}_1, c)$ is its compactification with $\mathcal{T}_1$ a T_2-space and any continuous map from $\mathcal{T}$ to arbitrary compact T_2-space $\mathcal{T}_2$ can be continuously extended to the map from $\mathcal{T}_1$ to $\mathcal{T}_2$, then $(\mathcal{T}_1, c)$ is equivalent to the Stone-Čech compactification of $\mathcal{T}$.*

This is one of the central properties of the Stone-Čech compactification. The word "equivalent" in the above proposition has the natural topological meaning, which we would like to omit in this section, already being overloaded by numerous topological terms. What matters to our present purposes is that the definition of the compact extension introduced in this section will turn out to be a compactification, equivalent to the Stone-Čech compactification when applied to a Tychonoff space. That is precisely what we meant by saying that the notion of the compact extension is generalizing that of the Stone-Čech compactification.

Theorem 3.2.8 *Let $\mathcal{T}$ be a topological space. The pair $(\pi, \mathcal{T}^{\#})$ where π is a natural map sending points to the corresponding principal ultrafilters, is a compactification. If additionally $\mathcal{T}$ is a Tychonoff space, then the space $\mathcal{T}^{\#}$ is equivalent to the Stone-Čech compactification of $\mathcal{T}$.*

Proof: First we prove that the compact extension of the space $\mathcal{T} = (X, \Omega)$ is a compactification. We already know that $\mathcal{T}^{\#}$ is a compact space. We have to show that the map π is a homeomorphic embedding and that $\pi(X)$ is dense in $\mathcal{T}^{\#}$. Recall that π sends points to corresponding principal ultrafilters, i.e. $\pi(x) = \{A \subseteq X | x \in A\}$. Consider the subspace of all principal ultrafilters $F(X)$ in $\mathcal{T}^{\#}$. Every basic open in $\Omega^{\#}$ is of the form $O^{\#}$ where $O \in \Omega$. Of course $O^{\#}$ contains all the principal ultrafilters corresponding to the points $x \in O$. This proves the density of $F(X)$ in $\mathcal{T}^{\#}$, as every open set in $\Omega^{\#}$ contains a basic open and with it, an ultrafilter from $F(X)$. To prove that $F(X)$ with induced topology is homeomorphic to $\mathcal{T}$ observe that the only principal ultrafilters contained in arbitrary basic open $O^{\#}$ are those generated by the elements $x \in O \subseteq X$.
Now we show that for any compact T_2-space $\mathcal{T}_1 = (Y, \Omega_1)$ and any continuous map $f : \mathcal{T} \to \mathcal{T}_1$, f can be extended to continuous $f^{\#} : \mathcal{T}^{\#} \to \mathcal{T}_1$. This will prove that when $\mathcal{T}$ happens to be a Tychonoff space, $\mathcal{T}^{\#}$ will coincide with the Stone-Čech compactification of $\mathcal{T}$ by the proposition 3.2.7. The map $f^{\#}$ is defined as follows:

$$f^{\#}(u) = \bigcap_{A \in u} \mathbf{C}(f(A))$$

Where $\mathbf{C}$ is the closure operator of $\mathcal{T}_1$.
First, $f^{\#}$ is well-defined. Indeed, the family $(\mathbf{C}(f(A)))_{A \in u}$ is a family of closed sets having the finite meet property. This is because the collection of all sets from any ultrafilter has the finite intersection property and the f-image of this collection will clearly inherit it, while the closure of a set is always bigger then the set itself. Since $\mathcal{T}_1$ is compact, the intersection will be non-empty. We have to show that the intersection is a singleton. Assume

$y_1, y_2 \in f^{\#}(u)$. As $\mathcal{T}_1$ is a T_2-space, any two distinct points have the disjoint open neighbourhoods. Let O_1 and O_2 be such neighbourhoods of y_1 and y_2 respectively. Consider the sets $f^{-1}(O_1)$ and $f^{-1}(O_2)$. They are clearly disjoint. Then at least one of the sets $-f^{-1}(O_1)$ or $-f^{-1}(O_2)$ belongs to the ultrafilter u, because otherwise two disjoint sets $f^{-1}(O_1)$ and $f^{-1}(O_2)$ will have to fall in u, contradicting the fact that u is an ultrafilter. Say $-f^{-1}(O_1) \in u$. Since $y_1 \in f^{\#}(u)$, we derive that $y_1 \in \mathbf{C}(f(-f^{-1}(O_1)))$. This means that any open neighbourhood of y_1 intersects with the set $f(-f^{-1}(O_1))$. O_1 is an open neighbourhood of y_1, so $O_1 \cap f(-f^{-1}(O_1)) \neq \emptyset$. This is a contradiction. Hence $f^{\#}(u)$ consists of one and only one point for each ultrafilter u. This implies that $f^{\#}$ is a well-defined mapping. We claim that it is continuous as well. Indeed, take any open set $O \in \Omega_1$. Consider the ultrafilter $u \in f^{\# \ -1}(O)$. This means $f^{\#}(u) \in O$. Since $\mathcal{T}_1$ is a compact T_2-space, we can claim that $f^{\#}(u)$ has an open neighbourhood O_1 such that $\mathbf{C}(O_1) \subseteq O$. Consider the set $f^{-1}(O_1)$. This set is open in Ω by continuity of f. Then $(f^{-1}(O_1))^{\#}$ is an open neighbourhood of u. We claim that every ultrafilter from this open set will be mapped inside O by $f^{\#}$. This will prove the continuity of $f^{\#}$. Indeed, take $v \in (f^{-1}(O_1))^{\#}$. This means $f^{-1}(O_1) \in v$. Then $f^{\#}(v) \in \mathbf{C}(f(f^{-1}(O_1))) = \mathbf{C}(O_1) \subseteq O$, which completes our proof. $\dashv$

Now we can say more about the Alexandroff extensions. In the light of this section, Alexandroff extension of a Tychonoff space is the coarsest Alexandroff topology containing the Stone-Čech compactification of this space. Stone-Čech compactifications date back to 1937 and the problems involving this construction are known to be among the most interesting topological questions. Stone-Čech compactifications of discrete spaces are closely related to set theory and present an independent logical interest. All this suggests that Alexandroff extensions are not as alien to topology as they might seem. Arising from logical purposes, the notion of Alexandroff extension has a clear topological meaning in terms of the well-established notion of the Stone-Čech compactification.
Some more topological flavour is hidden here. Consider the Stone-Čech compactification $\beta\mathcal{T}$ of the topological space $\mathcal{T}$. If $\beta\mathcal{T}$ validates the formula (.2), then by the anti-preservation result, $\mathcal{T} \models$ (.2) holds as well. This means that if the Stone-Čech compactification of a space is extremally disconnected, then the space itself is extremally disconnected. This is a well-known topological fact. We brought the almost trivial modal proof to illustrate our next point.

Proposition 3.2.9 *Let P be a topological property expressible in the basic modal language, then:*
P is invariant with respect to the formation of topological sums, open subspaces and interior images.
P is inverse-invariant with respect to the formation of Alexandroff extensions, compact extensions and StoneČech compactifications.

Proof: Trivial given the earlier (anti-)preservation results. $\dashv$

This proposition is just a straightforward consequence of what we have displayed in this work. Nevertheless, it provides a nice perspective on how modal logic can help to obtain automatic invariance results of some class of spatial properties with respect to certain transformations.

References

[Aiello and van Benthem 2001] M. Aiello and J. van Benthem, "Logical patterns in space", D. Barker-Plummer and Etchemendy, J. and Scotto di Luzio, P. and van Benthem, J. (editors), *Logic Unleashed: Language, Diagrams, and Computation,* Forthcoming, CSLI Publications, Stanford, 2001.

[Aiello et al. 2001] M. Aiello, J. van Benthem, G. Bezhanishvili, *Reasoning about Space: the Modal Way,* Manuscript, Univ. Amsterdam and New Mexico, 2001.

[Bezhanishvili and Gehrke 2001] G. Bezhanishvili and M. Gehrke, *A New Proof of Completeness of S4 with respect to the Real Line,* submitted, 2001.

[Blackburn et al. 2001] P. Blackburn, M. de Rijke and Y. Venema, *Modal Logic,* Cambridge University Press, 2001.

[Engelking 1977] R. Engelking, *General Topology,* Polish Scientific Publishers, Warszawa, 1977.

[Esakia 1979] L. Esakia "On the theory of modal and superintuitionistic systems", in *Logical Inference,* Moscow, Nauka, 1979, pp. 147-171 (In Russian).

[Gerson 1975] M. Gerson, "An extension of $S4$ complete for the neighbourhood semantics but incomplete for the relational semantics", *Studia Logica,* 34(1975), pp. 333-342.

[Goldblatt 1980] R. Goldblatt, "Diodorean Modality in Minkowski Spacetime", *Studia Logica,* 39(1980), pp. 219-236.

[Lemmon and Scott 1966] E. Lemmon and D. Scott, *Intentional Logics,* Stanford, 1966.

[McKinsey and Tarski 1944] J.C.C. McKinsey and A. Tarski, "The algebra of topology", *Annals of Mathematics,* 45(1944), pp. 141-191.

[McKinsey 1945] J.C.C. McKinsey, "On the syntactic construction of systems of modal logic", *Journal of Symbolic Logic,* 10(1945), pp. 83-94.

[Mints 1998] G. Mints, "A completeness Proof for Propositional S4 in Cantor Space", In E. Orlowska (editor), *Logic at Work: Essays Dedicated to the Memory of Helena Rasiowa,* Physica-Verlag, Heidelberg, 1998.

[Kripke 1959] S. Kripke, "A completeness theorem in modal logic", *Journal of Symbolic Logic,* 24(1959), pp. 1-14.

[Rasiowa and Sikorski 1963] H. Rasiowa and R. Sikorski, *Metamathematics of Mathematics,* Polish Scientific Publishers, Warszawa, 1963.

[de Rijke 1992] M. de Rijke, "The Modal Logic of Inequality", *Journal of Symbolic Logic,* 57(1992), pp. 566-584.

[Shehtman 1983] V. Shehtman, "Modal Logics of Domains on the real plane", *Studia Logica,* 42(1980), pp. 63-80.

[Shehtman 1998] V. Shehtman, "On Strong Neighbourhood Completeness of Modal and Intermediate Propositional Logics (Part I)", in Marcus Kracht, Maarten de Rijke, Heinrich Wansing and Michael Zakharyaschev (editors), *Advances in Modal Logic*, Vol. 1, CSLI Publications, Stanford, 1998, pp. 209-222.

[Shehtman 1999] V. Shehtman "'Everywhere' and 'Here'", *Journal of Appplied Non-Classical Logic,* 9(1999), pp. 369-380.

[Stone 1937] M. H. Stone, "Algebraic characterizations of special Boolean rings", *Fundamenta Mathematicae,* 29(1937), pp. 223-302.

[Venema 1993] Y. Venema, "Derivation rules as anti-axioms in modal logic", *Journal of Symbolic Logic,* 58(1993), pp. 1003-1054.