Euclidean Hierarchy in Modal Logic

Johan van Benthem[1] Guram Bezhanishvili[2] and Mai Gehrke[2]

[1] Institute for Logic, Language and Computation
University of Amsterdam
Plantage Muidergracht 24, 1018 TV Amsterdam
E-mail: johan@science.uva.nl

[2] Department of Mathematical Sciences
New Mexico State University
Las Cruces, NM 88003-0001, USA
E-mail: {gbezhani,mgehrke}@nmsu.edu

Abstract

For an Euclidean space $\mathbb{R}^n$, let L_n denote the modal logic of chequered subsets of $\mathbb{R}^n$. For every $n \geq 1$, we characterize L_n using the more familiar Kripke semantics, thus implying that each L_n is a tabular logic over the well-known modal system **Grz** of Grzegorczyk. We show that the logics L_n form a decreasing chain converging to the logic L_∞ of chequered subsets of $\mathbb{R}^\infty$. As a result, we obtain that L_∞ is also a logic over **Grz**, and that L_∞ has the finite model property.

1 Introduction

The idea of interpreting the modal operators $\Box$ and $\Diamond$ as the interior and closure operators of a topological space first appeared in the pioneering work [7]. One of the main results of that paper is that the modal logic **S4** is complete with respect to the real line $\mathbb{R}$. This result was improved in [5] where it was shown that in order to refute a non-theorem of **S4** it is actually enough to interpret propositional variables as countable unions of convex subsets of $\mathbb{R}$. On the other hand, it follows from [4] that the modal logic of finite unions of convex subsets of $\mathbb{R}$ is the modal logic of the 2-fork Kripke frame, and thus is much stronger than **S4**. The aim of this paper is to expand on that result and obtain a characterization of modal logics of finite unions of products of convex subsets of $\mathbb{R}$. In other words, we characterize the modal logic of chequered subsets of an Euclidean space $\mathbb{R}^n$ for each $n \geq 1$, as well as the modal logic of chequered subsets of $\mathbb{R}^\infty$.

The paper is organized as follows. §2 has an auxiliary purpose. In it we recall the notion of a topo-bisimulation and an m-topo-bisimulation, and prove some basic facts about topo-bisimulations involving the standard topological operations such as taking open subspaces, open images, topological sums and topological products. In §3 we give a simpler proof of the fact that the modal logic L_1 of finite unions of convex subsets of $\mathbb{R}$ is the modal logic $L(\mathcal{F})$ of the 2-fork Kripke frame $\mathcal{F} = \langle W, R \rangle$. In §4 we show that for each $n > 1$, the modal logic L_n of chequered subsets of the Euclidean space $\mathbb{R}^n$ is the modal logic $L(\mathcal{F}^n)$, where $\mathcal{F}^n = \underbrace{\mathcal{F} \times \ldots \times \mathcal{F}}_{n-\text{times}}$ is the n-times Cartesian product of the 2-fork Kripke frame on itself. This implies that for each $n \geq 1$, L_n is a tabular logic over the modal logic **Grz** of Grzegorczyk. Therefore, each L_n is finitely axiomatizable and decidable. In §5 we characterize the modal logic L_∞ of chequered subsets of $\mathbb{R}^\infty$. We show that $L_\infty = \bigcap_{n=1}^\infty L_n = \bigcap_{n=1}^\infty L(\mathcal{F}^n)$. It follows that L_∞ is a modal logic over **Grz** of infinite depth and infinite width, and that L_∞ has the finite model property.

2 Topo-bisimulations

Recall that a *topological space* is a pair $\mathcal{X} = \langle X, \tau \rangle$ where X is a nonempty set and τ is a family of subsets of X containing $\varnothing$ and X, and closed under finite intersections and arbitrary unions. An element of τ is called an *open* subset of $\mathcal{X}$.

If $\mathcal{X}$ and $\mathcal{X}'$ are two topological spaces, then $\mathcal{X}'$ is called a *subspace* of $\mathcal{X}$ if $X' \subseteq X$ and $U' \in \tau'$ iff $U' = U \cap X'$ for some $U \in \tau$. If $X' \in \tau$, then $\mathcal{X}'$ is said to be an *open subspace* of $\mathcal{X}$.

If $\mathcal{X}$ and $\mathcal{X}'$ are two topological spaces, then a map $f : X \to X'$ is called *continuous* if $U \in \tau'$ implies $f^{-1}(U) \in \tau$. A continuous map f is called *open* if $U \in \tau$ implies $f(U) \in \tau'$. In other words, f is continuous if it reflects opens, and f is open if it both preserves and reflects opens. If f is surjective and open, then $\mathcal{X}'$ is called an *open image* of $\mathcal{X}$.

If $\{\mathcal{X}_i\}_{i \in I}$ is a family of pairwise disjoint topological spaces, then $\mathcal{X}$ is called the *topological sum* of $\{\mathcal{X}_i\}_{i \in I}$ if $X = \bigcup_{i \in I} X_i$ and $U \in \tau$ iff $U \cap X_i \in \tau_i$. If the members of the family $\{\mathcal{X}_i\}_{i \in I}$ are not pairwise disjoint, then the topological sum is defined using disjoint union instead of set-theoretical union.

If $\{\mathcal{X}_i\}_{i \in I}$ is a family of topological spaces, then $\mathcal{X}$ is called the *topological product* of $\{\mathcal{X}_i\}_{i \in I}$ if $X = \prod_{i \in I} X_i$ and a basis for τ is formed by the sets of the form $U = \prod_{i \in I} U_i$, where $U_i \in \tau_i$ and all but finitely many U_i coincide with X_i.

Also recall that a topological model M is a pair $\langle \mathcal{X}, \nu \rangle$ where $\mathcal{X}$ is a topological space and ν is a *valuation* on $\mathcal{X}$; that is $\nu : \mathbf{P} \to \mathcal{P}(X)$ is a function from the set $\mathbf{P}$ of propositional variables of our modal language to the powerset of X. For $x \in X$ and a formula φ we define what it means for φ to be *true* at x, written as $x \models \varphi$, by induction on the length of φ.

- if φ is a propositional variable p, then $x \models \varphi$ iff $x \in \nu(p)$;
- if $\varphi = \neg\psi$, then $x \models \varphi$ iff $x \not\models \psi$;
- if $\varphi = \psi \wedge \chi$, then $x \models \varphi$ iff $x \models \psi$ and $x \models \chi$;
- if $\varphi = \Box\psi$, then $x \models \varphi$ iff $(\exists U \in \tau)(x \in U$ and $(\forall y \in U)(y \models \psi))$.

Dualizing this last clause we obtain that

- if $\varphi = \Diamond\psi$, then $x \models \varphi$ iff $(\forall U \in \tau)(x \in U \Rightarrow (\exists y \in U)(y \models \psi))$.

We say that φ is *true* in $\mathcal{X}$ if $x \models \varphi$ for every $x \in X$; φ is said to be *valid* in $\mathcal{X}$ if φ is true in $\mathcal{X}$ for every valuation ν on $\mathcal{X}$. Let $L(\mathcal{X})$ denote the set of all valid formulas in $\mathcal{X}$. Then it is routine to check that $L(\mathcal{X})$ is a normal modal logic over **S4**. We call it *the modal logic of* $\mathcal{X}$.

We are in a position now to introduce our main technical tool.

Definition 2.1. *[2] Let two topological models $M = \langle \mathcal{X}, \nu \rangle$ and $M' = \langle \mathcal{X}', \nu' \rangle$ be given. A* topological bisimulation*, or simply a* topo-bisimulation*, between M and M' is a nonempty relation $T \subseteq X \times X'$ such that if xTx', then*

(base): $x \in \nu(p)$ *iff* $x' \in \nu'(p)$, *for any* $p \in \mathbf{P}$;

(forth condition): *if* $x \in U \in \tau$, *then* $(\exists U' \in \tau')(x' \in U' \ \& \ (\forall y' \in U')(\exists y \in U)(yTy'))$;

(back condition): *if* $x' \in U' \in \tau'$, *then* $(\exists U \in \tau)(x \in U \ \& \ (\forall y \in U)(\exists y' \in U')(yTy'))$.

If xTx', then we say that x is *topo-bisimilar* to x'. A topo-bisimulation T is said to be *total* if for every $x' \in X'$ there exists $x \in X$ topo-bisimilar to x', and conversely, for every $x \in X$ there exists $x' \in X'$ topo-bisimilar to x. If this is not the case, then we say that T is a *non-total* topo-bisimulation.

Below we give four examples of topo-bisimulations which will be important in subsequent sections.

Example 2.2. Suppose $\mathcal{X}'$ is an open subspace of $\mathcal{X}$. For every valuation ν on $\mathcal{X}$, define a valuation ν' on $\mathcal{X}'$ by putting $\nu'(p) = \nu(p) \cap X'$. Then it is routine to check that the identity map $i : X' \to X$ is a non-total topo-bisimulation between the topological models $M' = \langle \mathcal{X}', \nu' \rangle$ and $M = \langle \mathcal{X}, \nu \rangle$. Conversely, for every valuation ν' on $\mathcal{X}'$, define a valuation ν on $\mathcal{X}$ by putting $\nu(p) = \nu'(p)$. Then it is again routine to check that the identity map $i : X' \to X$ is a non-total topo-bisimulation between M' and M.

Example 2.3. Suppose $\mathcal{X}'$ is an open image of $\mathcal{X}$. Then there exists an open surjection $f : X \to X'$. For every valuation ν' on $\mathcal{X}'$, define a valuation ν on $\mathcal{X}$ by putting $\nu(p) = f^{-1}(\nu'(p))$. Then it is routine to check that f is a total topo-bisimulation between the topological models $M = \langle \mathcal{X}, \nu \rangle$ and $M' = \langle \mathcal{X}', \nu' \rangle$. Conversely, suppose ν is a valuation on $\mathcal{X}$ such that $x \in \nu(p)$ iff $y \in \nu(p)$ for every $x, y \in X$ with $f(x) = f(y)$ and $p \in \mathbf{P}$. We define a valuation ν' on $\mathcal{X}'$ by putting $\nu'(p) = f(\nu(p))$. Then it is again routine to check that f is a total topo-bisimulation between M and M'.

Example 2.4. Suppose $\mathcal{X}$ is the topological sum of $\{\mathcal{X}_i\}_{i \in I}$. For every valuation ν on $\mathcal{X}$, define a valuation ν_i on $\mathcal{X}_i$ by putting $\nu_i(p) = \nu(p) \cap X_i$. Similarly to Example 2.2 we have that the identity map $i : X_i \to X$ is a non-total topo-bisimulation between the topological models $M_i = \langle \mathcal{X}_i, \nu_i \rangle$ and $M = \langle \mathcal{X}, \nu \rangle$. Conversely, for every valuation ν_i on $\mathcal{X}_i$, define a valuation ν on $\mathcal{X}$ by putting $\nu(p) = \nu_i(p)$. Again similarly to Example 2.2 we have that the identity map $i : X_i \to X$ is a non-total topo-bisimulation between M_i and M.

Example 2.5. Suppose $\mathcal{X}$ is the topological product of $\{\mathcal{X}_i\}_{i \in I}$. Let $\pi_i : \prod_{i \in I} X_i \to X_i$ denote the i-th projection. It is well known that π_i is surjective and open. For every valuation ν_i on $\mathcal{X}_i$, define a valuation ν on $\mathcal{X}$ by putting $\nu(p) = \pi_i^{-1}(\nu_i(p))$. Similarly to Example 2.3 we have that π_i is a total topo-bisimulation between the topological models $M = \langle \mathcal{X}, \nu \rangle$ and $M_i = \langle \mathcal{X}_i, \nu_i \rangle$. Conversely, suppose ν is a valuation on $\mathcal{X}$ such that $x \in \nu(p)$ iff $y \in \nu(p)$ for every $x, y \in X$ with $\pi_i(x) = \pi_i(y)$ and $p \in \mathbf{P}$. We define a valuation ν_i on $\mathcal{X}_i$ by putting $\nu_i(p) = \pi_i(\nu(p))$. Again similarly to Example 2.3 we have that π_i is a total topo-bisimulation between M and M_i.

Suppose $M_i = \langle \mathcal{X}_i, \nu_i \rangle$ is a topological model for each $i \in I$, and $\mathcal{X}$ is the topological product of $\{\mathcal{X}_i\}_{i \in I}$. Define a valuation ν on $\mathcal{X}$ by putting $\nu(p) = \prod_{i \in I} \nu_i(p)$. Call ν the *product valuation* on $\mathcal{X}$.

Proposition 2.6. *(1) If I is finite and $M_i = \langle \mathcal{X}_i, \nu_i \rangle$ is topo-bisimilar to $M_i' = \langle \mathcal{X}_i', \nu_i' \rangle$ for each $i \in I$, then $M = \langle \mathcal{X}, \nu \rangle$ is topo-bisimilar to $M' = \langle \mathcal{X}', \nu' \rangle$, where $X' = \prod_{i \in I} X_i'$ and $\nu' = \prod_{i \in I} \nu_i'$.*

(2) If T_i is a total topo-bisimulation between $M_i = \langle \mathcal{X}_i, \nu_i \rangle$ and $M_i' = \langle \mathcal{X}_i', \nu_i' \rangle$ for each $i \in I$, then there exists a total topo-bisimulation between $M = \langle \mathcal{X}, \nu \rangle$ and $M' = \langle \mathcal{X}', \nu' \rangle$.

Proof. (1) Suppose I is finite and T_i is a topo-bisimulation between M_i and M_i' for each $i \in I$. Define $T \subseteq X \times X'$ by putting $T = \prod_{i \in I} T_i$. We want to show that T is a topo-bisimulation between M and M'. Suppose $f \in X$, $f' \in X'$ and fTf'. Then $f(i)T_i f'(i)$ for each $i \in I$. Therefore, $f(i) \in \nu_i(p)$ iff $f'(i) \in \nu_i'(p)$, and by the definition of the product valuation, $f \in \nu(p)$ iff $f' \in \nu'(p)$ for every $p \in \mathbf{P}$. Thus, T satisfies the base condition of the definition of topo-bisimulation. To check that T satisfies the forth condition, suppose U is a basic open of $\mathcal{X}$ and $f \in U$. Let $U_i = \pi_i(U)$. Then $f(i) \in U_i$ and U_i is open since π_i is open. Also since T_i is a topo-bisimulation, there exists $U_i' \in \tau_i'$ such that $f'(i) \in U_i'$ and for all $x' \in U_i'$ there exists $x \in U_i$ such that xT_ix'. Now let $U' = \prod_{i \in I} U_i'$. Since I is finite, it is obvious that U' is a basic open neighborhood of f'. Moreover, since every T_i is a topo-bisimulation and T is the product of T_is, for each $g' \in U'$ there exists $g \in U$ such that gTg'. So, T satisfies the forth condition of the definition of topo-bisimulation. That T also satisfies the back condition can be checked in a completely symmetric way. Therefore, we conclude that T is a topo-bisimulation between M and M'.

(2) is proved similarly to (1). The only difference is that in the forth condition of Definition 2.1, if I is infinite and $U = \prod_{i \in I} U_i$ is a basic open neighborhood of $f \in U$, then all but finitely many U_i are equal to X_i. Now

since T_i is a total topo-bisimulation, if $U_i = X_i$, then we can choose U_i' to be X_i'. Subsequently, $U' = \prod_{i \in I} U_i'$ will be the needed basic open neighborhood of f'. The same applies to the back condition of Definition 2.1. $\square$

Remark 2.7. We note that our proof of Proposition 2.6(1) does not work if I is infinite since there is no guarantee that the constructed $U' = \prod_{i \in I} U_i'$ is a basic open of $\mathcal{X}'$. Indeed, if $U_i = X_i$, we may not be able to choose U_i' to be the whole space $\mathcal{X}_i'$ since T_i is not a total topo-bisimulation. In fact, Proposition 2.6(1) is false if I is infinite as shows the following example. Let $X = \{0, 1\}$ be a two point discrete space and $X' = \{0\}$ be a one point subspace of X. Let also $\nu(p) = \{0\}$ and $\nu'(p) = \nu(p) \cap X' = \nu(p)$. Obviously the identity map $i : X' \to X$ is a non-total topo-bisimulation between $\langle \mathcal{X}', \nu' \rangle$ and $\langle \mathcal{X}, \nu \rangle$. On the other hand, the countable product $\mathcal{X}^\omega$ is homeomorphic to the Cantor space $\mathcal{C}$, while $(\mathcal{X}')^\omega$ is homeomorphic to $\mathcal{X}'$. So, $\langle \mathcal{X}^\omega, \nu^\omega \rangle$ is not topo-bisimilar to $\langle (\mathcal{X}')^\omega, (\nu')^\omega \rangle$. For example, $\Diamond p \to \Box p$ is falsified at $\langle 0, \ldots, 0, \ldots \rangle$ in $\langle \mathcal{X}^\omega, \nu^\omega \rangle$, but is true in $\langle (\mathcal{X}')^\omega, (\nu')^\omega \rangle$.

A crucial fact about topo-bisimulations is expressed in the following proposition.

Proposition 2.8. *[2] If T is a topo-bisimulation between two topological models $M = \langle \mathcal{X}, \nu \rangle$ and $M' = \langle \mathcal{X}', \nu' \rangle$, $x \in X$, $x' \in X'$ and xTx', then x and x' satisfy the same modal formulas.*

An immediate consequence of Proposition 2.8 and Examples 2.2–2.5 is the following proposition.

Proposition 2.9. *(1) If $\mathcal{X}'$ is an open subspace of $\mathcal{X}$, then $L(\mathcal{X}) \subseteq L(\mathcal{X}')$.*

(2) If $\mathcal{X}'$ is an open image of $\mathcal{X}$, then $L(\mathcal{X}) \subseteq L(\mathcal{X}')$.

(3) If $\mathcal{X}$ is the topological sum of the family $\{\mathcal{X}_i\}_{i \in I}$ of topological spaces, then $L(\mathcal{X}) = \bigcap_{i \in I} L(\mathcal{X}_i)$.

(4) If $\mathcal{X}$ is the topological product of the family $\{\mathcal{X}_i\}_{i \in I}$ of topological spaces, then $L(\mathcal{X}) \subseteq \bigcap_{i \in I} L(\mathcal{X}_i)$.

Proof. (1) Suppose $L(\mathcal{X}') \not\vdash \varphi$. Then there exists a valuation ν' on $\mathcal{X}'$ refuting φ. By Example 2.2 there exists a valuation ν on $\mathcal{X}$ such that $\langle \mathcal{X}', \nu' \rangle$ is topo-bisimilar to $\langle \mathcal{X}, \nu \rangle$. But then ν also refutes φ by Proposition 2.8. Therefore, $L(\mathcal{X}) \not\vdash \varphi$, and so $L(\mathcal{X}) \subseteq L(\mathcal{X}')$.

Statements (2)–(4) can be proved similarly to the way we have proved Statement (1) by using Examples 2.3–2.5 instead of Example 2.2. $\square$

In subsequent sections we will also need the notions of an *m-valuation*, a *topological m-model*, and an *m-topo-bisimulation*, where m is any natural number. The definitions are obtained from the definitions of a valuation, a topological model, and a topo-bisimulation, respectively, by replacing the set $\mathbf{P}$ of propositional variables by the set $\mathbf{P}_m$ of m-many propositional variables $p_1, \ldots, p_m$. Then the above propositions are also applicable to m-topo-bisimulations if we restrict ourselves to the formulas built from $\mathbf{P}_m$.

Finally, we recall that if $\langle W, R\rangle$ is a Kripke frame for **S4**, that is if $\langle W, R\rangle$ is a quasi-ordered set, then the *Alexandroff topology* τ_R associated with R is defined by putting

$$\tau_R = \{U \subseteq W : U \text{ is an upset of } W\},$$

where U is an *upset* of W if $w \in U$ and wRv imply $v \in U$. Then $\langle W, \tau_R\rangle$ is a topological space, and τ_R is characterized as a topology closed with respect to arbitrary intersections. As a result, the standard concept of bisimulation is a particular case of a more general concept of topo-bisimulation, and many known results on bisimulations can be obtained as particular cases of more general results on topo-bisimulations (see, e.g., [2, 3, 4, 6]).

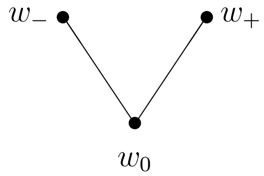
3 Logic of serial subsets of $\mathbb{R}$

Definition 3.1. *Call C a* convex *subset of $\mathbb{R}$ if $x, y \in C$ and $x \leq y$ imply $[x, y] \subseteq C$. Call S a* serial *subset of $\mathbb{R}$ if it is a finite union of convex subsets of $\mathbb{R}$.*

Let $\mathcal{C}(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$ denote the families of convex and serial subsets of $\mathbb{R}$, respectively. Note that $\mathcal{S}(\mathbb{R})$ forms a Boolean algebra closed with respect to the interior and closure operators of $\mathbb{R}$.

Definition 3.2. *Call a valuation ν on $\mathbb{R}$* serial *if $\nu(p) \in \mathcal{S}(\mathbb{R})$ for every $p \in \mathbf{P}$. Call a formula φ* 1-true *if it is true in $\mathbb{R}$ under a serial valuation. Call φ* 1-valid *if φ is 1-true for every serial valuation on $\mathbb{R}$.*

Let $L_1 = \{\varphi : \varphi \text{ is 1-valid}\}$. Since $\mathcal{S}(\mathbb{R})$ is a Boolean algebra closed with respect to the interior and closure operators of $\mathbb{R}$, it is obvious that L_1 is a normal modal logic over **S4**. We will refer to L_1 as *the logic of serial subsets of* $\mathbb{R}$. It was proved in [4] that $L_1 = L(\mathcal{F})$, where $L(\mathcal{F})$ is the logic of the 2-fork Kripke frame $\mathcal{F} = \langle W, R\rangle$. Here $W = \{w_0, w_-, w_+\}$ and $w_0Rw_0, w_-Rw_-, w_+Rw_+, w_0Rw_-, w_0Rw_+$:



Below we give a simpler proof of this fact. For this call a map f from $\mathbb{R}$ to a finite quasi-ordered set $\langle W, R\rangle$ *serial* if $f^{-1}(w) \in \mathcal{S}(\mathbb{R})$ for any $w \in W$. If f is onto, we call $\langle W, R\rangle$ a *serial image* of $\mathbb{R}$.

Lemma 3.3. *The 2-fork frame $\mathcal{F}$ is an open serial image of $\mathbb{R}$.*

Proof. Denote by τ_R the Alexandroff topology associated with R. Then $\tau_R = \{\emptyset, \{w_-\}, \{w_+\}, \{w_-, w_+\}, W\}$. Define $f : \mathbb{R} \to W$ by putting

$$f(x) = \begin{cases} w_0 & \text{for } x = 0, \\ w_- & \text{for } x < 0, \\ w_+ & \text{for } x > 0. \end{cases}$$

It is clear that

$f^{-1}(\emptyset) = \emptyset,$

$f^{-1}(\{w_-\}) = (-\infty, 0),$

$f^{-1}(\{w_+\}) = (0, +\infty),$

$f^{-1}(\{w_-, w_+\}) = (-\infty, 0) \cup (0, +\infty)$, and

$f^{-1}(W) = \mathbb{R}.$

So, f is continuous. Also since each of the sets $f^{-1}(\{w_0\}) = \{0\}$, $f^{-1}(\{w_-\}) = (-\infty, 0)$, and $f^{-1}(\{w_+\}) = (0, +\infty)$ is serial, so is f. Moreover, for any open subset U of $\mathbb{R}$, if $0 \in U$, then $f(U) = W$; and if $0 \notin U$, then $f(U) \subseteq \{w_-, w_+\}$, which is always open. Hence, f is open and serial. So, $\langle W, \tau_R\rangle$ is an open serial image of $\mathbb{R}$. □

Lemma 3.4. *If ν is a serial m-valuation on $\mathbb{R}$, then there is $\varepsilon > 0$, a serial m-valuation ν_ε on $(-\varepsilon, \varepsilon)$, and an m-valuation μ on the 2-fork frame $\mathcal{F}$ such that $\langle(-\varepsilon, \varepsilon), \nu_\varepsilon\rangle$ is m-topo-bisimilar to $\langle \mathcal{F}, \mu\rangle$.*

Proof. Let $\mathbf{P}_m = \{p_1, \ldots, p_m\}$ and pick $p_i \in \mathbf{P}_m$. Since $\nu(p_i)$ is a serial subset of $\mathbb{R}$, four cases are possible:

Case 1. There exists $\varepsilon_i > 0$ such that $((-\varepsilon_i, \varepsilon_i) - \{0\}) \cap \nu(p_i) = \varnothing$. Then put

$$\mu(p_i) = \begin{cases} \{w_0\} & \text{if } 0 \in \nu(p_i), \\ \varnothing & \text{otherwise.} \end{cases}$$

Case 2. There exists $\varepsilon_i > 0$ such that $(-\varepsilon_i, 0) \subseteq \nu(p_i)$ and $(0, \varepsilon_i) \cap \nu(p_i) = \varnothing$. Then put

$$\mu(p_i) = \begin{cases} \{w_0, w_-\} & \text{if } 0 \in \nu(p_i), \\ \{w_-\} & \text{otherwise.} \end{cases}$$

Case 3. There exists $\varepsilon_i > 0$ such that $(-\varepsilon_i, 0) \cap \nu(p_i) = \varnothing$ and $(0, \varepsilon_i) \subseteq \nu(p_i)$. Then put

$$\mu(p_i) = \begin{cases} \{w_0, w_+\} & \text{if } 0 \in \nu(p_i), \\ \{w_+\} & \text{otherwise.} \end{cases}$$

Case 4. There exists $\varepsilon_i > 0$ such that $(-\varepsilon_i, 0) \cup (0, \varepsilon_i) \subseteq \nu(p_i)$. Then put

$$\mu(p_i) = \begin{cases} W & \text{if } 0 \in \nu(p_i), \\ \{w_-, w_+\} & \text{otherwise.} \end{cases}$$

It is clear that μ is a well-defined m-valuation on the 2-fork frame $\mathcal{F}$, and that $\langle \mathcal{F}, \mu \rangle$ is an m-model. Let $\varepsilon = \min\{\varepsilon_i\}_{i=1}^m$ and consider the interval $(-\varepsilon, \varepsilon)$. Obviously $(-\varepsilon, \varepsilon) = \bigcap_{i=1}^m (-\varepsilon_i, \varepsilon_i)$. Also let $\nu_{(-\varepsilon,\varepsilon)}$ denote the restriction of ν to $(-\varepsilon, \varepsilon)$. Then $\nu_{(-\varepsilon,\varepsilon)}$ is a serial m-valuation on $(-\varepsilon, \varepsilon)$, and $\langle (-\varepsilon, \varepsilon), \nu_{(-\varepsilon,\varepsilon)} \rangle$ is an m-model. Let g denote the restriction of f to $(-\varepsilon, \varepsilon)$. Obviously g is an open map from $(-\varepsilon, \varepsilon)$ onto W. Also, by the definition of μ, we have

$$\mu(p_i) = g(\nu_{(-\varepsilon,\varepsilon)}(p_i)).$$

Now since every open map satisfying the base condition of the definition of m-topo-bisimulation is an m-topo-bisimulation, we obtain that g is an m-topo-bisimulation between $\langle (-\varepsilon, \varepsilon), \nu_{(-\varepsilon,\varepsilon)} \rangle$ and $\langle \mathcal{F}, \mu \rangle$. □

Theorem 3.5. $L_1 = L(\mathcal{F})$.

Proof. It follows immediately from Example 2.3 and Lemma 3.3 that for every valuation μ on the 2-fork frame $\mathcal{F}$ there exists a serial valuation ν on $\mathbb{R}$ such that f is a topo-bisimulation between $\langle \mathbb{R}, \nu \rangle$ and $\langle \mathcal{F}, \mu \rangle$. Now if φ is a non-theorem of $L(\mathcal{F})$, there exists a valuation μ on $\mathcal{F}$ such that $w_0 \not\models_\mu \varphi$.

Since f is a topo-bisimulation and $f(0) = w_0$, it follows from Proposition 2.8 that $0 \not\models_\nu \varphi$. So, φ is a non-theorem of L_1, and we have $L_1 \subseteq L(\mathcal{F})$.

Conversely, suppose $L_1 \not\vdash \varphi(p_1, \ldots, p_m)$. Then there exists a serial m-valuation ν on $\mathbb{R}$ refuting $\varphi(p_1, \ldots, p_m)$. Without loss of generality we may assume that $0 \not\models_\nu \varphi(p_1, \ldots, p_m)$.[1] By Lemma 3.4 there is $\varepsilon > 0$, a serial m-valuation ν_ε on $(-\varepsilon, \varepsilon)$, and an m-valuation μ on the 2-fork frame $\mathcal{F}$ such that $\langle(-\varepsilon, \varepsilon), \nu_\varepsilon\rangle$ is m-topo-bisimilar to $\langle \mathcal{F}, \mu\rangle$. Since $\nu_{(-\varepsilon,\varepsilon)}$ is the restriction of ν to $(-\varepsilon, \varepsilon)$, the identity map $i : (-\varepsilon, \varepsilon) \to \mathbb{R}$ is a non-total m-topo-bisimulation between $\langle(-\varepsilon, \varepsilon), \nu_{(-\varepsilon,\varepsilon)}\rangle$ and $\langle \mathbb{R}, \nu\rangle$. Hence, $0 \not\models_{(-\varepsilon,\varepsilon)} \varphi(p_1, \ldots, p_m)$. Since $0 \in (-\varepsilon, \varepsilon)$ is m-topo-bisimilar to $w_0 \in W$, we have $w_0 \not\models_\mu \varphi(p_1, \ldots, p_m)$. So $L(\mathcal{F}) \not\vdash \varphi(p_1, \ldots, p_m)$, and thus, $L(\mathcal{F}) \subseteq L_1$. Therefore, we have obtained that $L_1 = L(\mathcal{F})$.

□

4 Logic of chequered subsets of $\mathbb{R}^n$

Definition 4.1. *([8]) A set $C \subseteq \mathbb{R}^n$ is called a* hyper-rectangular convex *of $\mathbb{R}^n$ if $C = \prod_{i=1}^n C_i$, where $C_1, \ldots, C_n$ are convex subsets of $\mathbb{R}$. A set $S \subseteq \mathbb{R}^n$ is said to be* n-chequered *if it is a finite union of hyper-rectangular convex subsets of $\mathbb{R}^n$.*

Let $\mathcal{HR}(\mathbb{R}^n)$ and $\mathcal{CH}(\mathbb{R}^n)$ denote the families of hyper-rectangular convex and n-chequered subsets of $\mathbb{R}^n$, respectively. Note that similarly to $\mathcal{S}(\mathbb{R})$ we have that $\mathcal{CH}(\mathbb{R}^n)$ also forms a Boolean algebra closed with respect to the interior and closure operators of $\mathbb{R}^n$.

Definition 4.2. *Call a valuation ν on $\mathbb{R}^n$* n-chequered *if $\nu(p) \in \mathcal{CH}(\mathbb{R}^n)$ for every $p \in \mathbf{P}$. Call a formula φ* n-true *if it is true in $\mathbb{R}^n$ under an n-chequered valuation. Call φ* n-valid *if φ is n-true for any n-chequered valuation on $\mathbb{R}^n$.*

Let $L_n = \{\varphi : \varphi \text{ is } n\text{-valid}\}$. Since $\mathcal{CH}(\mathbb{R}^n)$ is a Boolean algebra closed with respect to the interior and closure operators of $\mathbb{R}^n$, it is obvious that L_n

[1]If $0 \models_\nu \varphi(p_1, \ldots, p_m)$, then there exists some $z \neq 0$ such that $z \not\models_\nu \varphi(p_1, \ldots, p_m)$, and we redefine f by putting

$$f(x) = \begin{cases} w_0 & \text{for } x = z, \\ w_- & \text{for } x < z, \\ w_+ & \text{for } x > z. \end{cases}$$

is a normal modal logic over **S4**. We refer to L_n as *the logic of n-chequered subsets of $\mathbb{R}^n$*.

Denote by $\mathcal{F}^n$ the n-times Cartesian product of the 2-fork frame $\mathcal{F}$ on itself. So, $\mathcal{F}^n = \langle W^n, R^n\rangle$, where $W^n = \underbrace{W \times \ldots \times W}_{n-\text{times}}$ and R^n is defined on W^n componentwise. Denote by τ_{R^n} the Alexandroff topology associated with R^n. It is obvious that τ_{R^n} is the product topology, that is

$$\tau_{R^n} = (\tau_R)^n.$$

Our goal is to characterize L_n by proving that $L_n = L(\mathcal{F}^n)$.

Lemma 4.3. *(1) For every formula $\varphi(p_1, \ldots, p_m)$ and every n-chequered m-valuation ν there exists a formula $\psi(q_1^1, \ldots, q_t^1, \ldots, q_1^m, \ldots, q_t^m)$ and a product t^m-valuation ν' such that $\nu(\varphi(p_1, \ldots, p_m)) = \nu'(\psi(q_1^1, \ldots, q_t^1, \ldots, q_1^m, \ldots, q_t^m))$.*

(2) For every formula $\varphi(p_1, \ldots, p_m)$ and every m-valuation ν on $\mathcal{F}^n$ there exists a formula $\psi(q_1^1, \ldots, q_t^1, \ldots, q_1^m, \ldots, q_t^m)$ and a product t^m-valuation ν' such that $\nu(\varphi(p_1, \ldots, p_m)) = \nu'(\psi(q_1^1, \ldots, q_t^1, \ldots, q_1^m, \ldots, q_t^m))$.

Proof. (1) Suppose $\varphi(p_1, \ldots, p_m)$ is a formula and ν is an n-chequered m-valuation. Then for each $i \leq m$ we have $\nu(p_i) = \bigcup_{k=1}^{t_i} C_k^i$, where each C_k^i is a hyper-rectangular convex of $\mathbb{R}^n$. Let $t = \max\{t_i\}_{i=1}^m$ and let $\psi(q_1^1, \ldots, q_t^1, \ldots, q_1^m, \ldots, q_t^m)$ be the formula obtained from $\varphi(p_1, \ldots, p_m)$ by substituting each propositional variable p_i by the disjunction $\bigvee_{k=1}^t q_k^i$, where each q_k^i is a fresh variable. Define a new t^m-valuation ν' on $\mathbb{R}^n$ by putting

$$\nu'(q_k^i) = \begin{cases} C_k^i & \text{if } k < t_i, \\ \emptyset & \text{otherwise.} \end{cases}$$

Since each C_k^i is a hyper-rectangular convex subset of $\mathbb{R}^n$, we obtain that ν' is a product t^m-valuation. Moreover, it directly follows from the definition that $\nu(\varphi(p_1, \ldots, p_m)) = \nu'(\psi(q_1^1, \ldots, q_t^1, \ldots, q_1^m, \ldots, q_t^m))$.

(2) Suppose $\varphi(p_1, \ldots, p_m)$ is a formula and ν is an m-valuation on $\mathcal{F}^n$. Since $\mathcal{F}$ is finite, for each $i \leq m$ we have $\nu(p_i) = \bigcup_{k=1}^{t_i} C_k^i$, where each C_k^i is the product of subsets of $\mathcal{F}$. The rest of the proof is identical with (1). □

Theorem 4.4. $L_n = L(\mathcal{F}^n)$ *for each $n \geq 1$.*

Proof. The $n = 1$ case follows from Theorem 3.5. Suppose $n > 1$ and $L_n \not\vdash \varphi(p_1, \ldots, p_m)$. Then there exists an n-chequered m-valuation ν on $\mathbb{R}^n$ refuting $\varphi(p_1, \ldots, p_m)$. By Lemma 4.3(1) we can transform $\varphi(p_1, \ldots, p_m)$ into $\psi(q_1^1, \ldots, q_t^1, \ldots, q_1^m, \ldots, q_t^m)$ and find a product t^m-valuation ν' refuting $\psi(q_1^1, \ldots, q_t^1, \ldots, q_1^m, \ldots, q_t^m)$. Without loss of generality we can assume that $\vec{0} \not\models' \psi(q_1^1, \ldots, q_t^1, \ldots, q_1^m, \ldots, q_t^m)$, where $\vec{0} = (\underbrace{0, \ldots, 0}_{n-\text{times}})$. Let $\pi_i : \mathbb{R}^n \to \mathbb{R}$ denote the i-th projection of $\mathbb{R}^n$ to $\mathbb{R}$. Then each $\pi_i \circ \nu'$ is a serial valuation on $\mathbb{R}$, and $\nu' = \prod_{i=1}^n (\pi_i \circ \nu')$. It follows from Lemma 3.4 that for each $\pi_i \circ \nu'$ there exist ε_i, a serial t^m-valuation $\nu_{(-\varepsilon_i, \varepsilon_i)}$, and a t^m-valuation μ_i on $\mathcal{F}$ such that $\langle(-\varepsilon_i, \varepsilon_i), \nu_{(-\varepsilon_i,\varepsilon_i)}\rangle$ is t^m-topo-bisimilar to $\langle\mathcal{F}, \mu_i\rangle$. From Proposition 2.6 it follows that $\langle(-\varepsilon_i, \varepsilon_i)^n, \prod_{i=1}^n \nu_{(-\varepsilon_i,\varepsilon_i)}\rangle$ is t^m-topo-bisimilar to $\langle\mathcal{F}^n, \mu\rangle$, where $\mu = \prod_{i=1}^n \mu_i$. On the other hand, $i : (-\varepsilon_i, \varepsilon_i)^n \to \mathbb{R}^n$ is a non-total topo-bisimulation between $\langle(-\varepsilon_i, \varepsilon_i)^n, \nu'_{\prod_{i=1}^n(-\varepsilon_i,\varepsilon_i)}\rangle$ and $\langle\mathbb{R}^n, \nu'\rangle$. Therefore, $\vec{0} \not\models'_{\prod_{i=1}^n(-\varepsilon_i,\varepsilon_i)} \psi(q_1^1, \ldots, q_t^1, \ldots, q_1^m, \ldots, q_t^m)$, and thus, $\vec{w_0} \not\models_\mu \psi(q_1^1, \ldots, q_t^1, \ldots, q_1^m, \ldots, q_t^m)$, where $\vec{w_0} = \langle\underbrace{w_0, \ldots, w_0}_{n-\text{times}}\rangle$. Now for each p_i define $\mu(p_i) = \bigcup_{k=1}^t \mu(q_k^i)$. Then it is obvious that $\mu(\varphi(p_1, \ldots, p_m)) = \mu(\psi(q_1^1, \ldots, q_t^1, \ldots, q_1^m, \ldots, q_t^m))$. Therefore, $\vec{w_0} \not\models_\mu \varphi(p_1, \ldots, p_m)$, which implies that $L(\mathcal{F}^n) \not\vdash \varphi(p_1, \ldots, p_m)$. Thus, $L(\mathcal{F}^n) \subseteq L_n$.

To prove the converse inclusion, recall from the proof of Theorem 3.5 that for every valuation μ on $\mathcal{F}$ there exists a serial valuation ν on $\mathbb{R}$ such that $\langle\mathcal{F}, \mu\rangle$ is topo-bisimilar to $\langle\mathbb{R}, \nu\rangle$. Therefore, by Proposition 2.6, for every product valuation μ^n on $\mathcal{F}^n$ there exists a product valuation ν^n on $\mathbb{R}^n$ such that $\langle\mathcal{F}^n, \mu^n\rangle$ is topo-bisimilar to $\langle\mathbb{R}^n, \nu^n\rangle$. Now Lemma 4.3(2) together with the same argument as above guarantee that product valuations 'generate' all other valuations on $\mathcal{F}^n$. But the pullbacks of product valuations on $\mathcal{F}^n$ are product valuations on $\mathbb{R}^n$. Therefore, $L(\mathcal{F}^n) \not\vdash \varphi$ implies $L_n \not\vdash \varphi$, and so $L_n \subseteq L(\mathcal{F}^n)$. Thus, we obtain that $L_n = L(\mathcal{F}^n)$. □

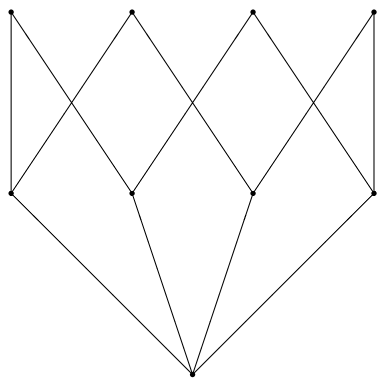
Remark 4.5. The main reason Theorem 4.4 holds true is that every serial valuation is 'generated' by the product valuations. Therefore, more general version of our theorem will also hold true. Namely, suppose the logics L_i and S_i of (special subsets of) topological spaces $\mathcal{X}_i$ and $\mathcal{Y}_i$ coincide for each $i \in I$, L denotes the logic of the subsets of $\prod_{i\in I} \mathcal{X}_i$ generated by the products of (special) subsets of $\mathcal{X}_i$, and S denotes the logic of the subsets of $\prod_{i\in I} \mathcal{Y}_i$ generated by the products of (special) subsets of $\mathcal{Y}_i$. Then the logics L and

S also coincide. In particular, this observation implies that if $L(\mathcal{X}_i) = L(\mathcal{Y}_i)$ for each $i \in I$, then $L(\prod_{i\in I} \mathcal{X}_i) = L(\prod_{i\in I} \mathcal{Y}_i)$, provided each $\mathcal{X}_i$, $\mathcal{Y}_i$ and I are finite. However, this fact will not hold true if we drop the finiteness condition from our assumptions.

Corollary 4.6. *Each L_n is a tabular logic over* **Grz**, *hence is finitely axiomatizable and decidable.*

Proof. Since each $L(\mathcal{F}^n)$ is a tabular logic, so is each L_n by Theorem 4.4. Since every $\mathcal{F}^n$ is a **Grz**-frame, every $L(\mathcal{F}^n)$, and hence every L_n, is a logic over **Grz**. Finally, it is well-known that every tabular logic over **K4**($\subseteq$ **Grz**) is finitely axiomatizable and decidable. □

A picture of $\mathcal{F} \times \mathcal{F}$ is shown below, for the readers convenience.



$\mathcal{F} \times \mathcal{F}$

5 Logic of chequered subsets of $\mathbb{R}^\infty$

Definition 5.1. *A set $C \subseteq \mathbb{R}^\infty$ is called a* ∞-rectangular convex *if $C = \prod_{i=1}^{\infty} C_i$, where each C_i is a convex subset of $\mathbb{R}$, and all but finitely many of C_i are equal to either $\mathbb{R}$ or $\varnothing$. A set $S \subseteq \mathbb{R}^\infty$ is said to be* ∞-chequered *if it is a finite union of ∞-rectangular convex sets.*

Let $\mathcal{RC}(\mathbb{R}^\infty)$ and $\mathcal{CH}(\mathbb{R}^\infty)$ denote the families of ∞-rectangular convex and ∞-chequered sets, respectively. Note that similarly to each $\mathcal{CH}(\mathbb{R}^n)$ we have that $\mathcal{CH}(\mathbb{R}^\infty)$ also forms a Boolean algebra closed with respect to the interior and closure operators of $\mathbb{R}^\infty$.

Definition 5.2. *Call a valuation ν on $\mathbb{R}^\infty$* ∞-chequered *if $\nu(p) \in \mathcal{CH}(\mathbb{R}^\infty)$ for every $p \in \mathbf{P}$. Call a formula φ* ∞-true *if it is true in $\mathbb{R}^\infty$ under a ∞-chequered valuation. Call φ* ∞-valid *if φ is ∞-true for any ∞-chequered valuation on $\mathbb{R}^\infty$.*

Let $L_\infty = \{\varphi : \varphi \text{ is } \infty\text{-valid}\}$. Since $\mathcal{CH}(\mathbb{R}^\infty)$ is a Boolean algebra closed with respect to the interior and closure operators of $\mathbb{R}^\infty$, it is obvious that L_∞ is a normal modal logic over **S4**. We refer to L_∞ as *the logic of ∞-chequered subsets of* $\mathbb{R}^\infty$.

Theorem 5.3. *$L_\infty = \bigcap_{n=1}^\infty L_n$.*

Proof. It is well-known that $\pi^n : \mathbb{R}^\infty \to \mathbb{R}^n$ defined by

$$\pi^n(\langle x_1, \ldots, x_n, \ldots\rangle) = \langle x_1, \ldots, x_n\rangle$$

is an onto open map. Moreover, the π^n-inverse image of any n-chequered set is ∞-chequered. Hence, for every n-chequered valuation ν on $\mathbb{R}^n$, the valuation ν_∞ defined by putting

$$\nu_\infty(p) = (\pi^n)^{-1}(\nu(p)), \text{ for every } p \in \mathbf{P},$$

is ∞-chequered. Thus, $\langle\mathbb{R}^\infty, \nu_\infty\rangle$ is topo-bisimilar to $\langle\mathbb{R}^n, \nu\rangle$, and so every non-theorem of L_n is a non-theorem of L_∞. Therefore, $L_\infty \subseteq L_n$ for each n, implying that $L_\infty \subseteq \bigcap_{n=1}^\infty L_n$.

Conversely, suppose $L_\infty \not\vdash \varphi(p_1, \ldots, p_m)$. Then there exists a ∞-chequered valuation ν_∞ refuting $\varphi(p_1, \ldots, p_m)$. Under ν_∞ each p_i corresponds to a ∞-chequered set $S_i = \bigcup_{k=1}^{t_i} C_k^i$, where each C_k^i is a ∞-rectangular convex of $\mathbb{R}^\infty$. Let n_k^i be the number of those C_k^i in the decomposition of S_i which are different from $\mathbb{R}$ and $\varnothing$. Put $n_i = \max\{n_k^i\}_{k=1}^{t_i}$ and $n = \max\{n_i\}_{i=1}^m$. Now consider $\langle\mathbb{R}^n, \nu\rangle$, where

$$\nu(p_i) = \pi^n(\nu_\infty(p_i)).$$

Then π^n is an m-topo-bisimulation between $\langle\mathbb{R}^\infty, \nu_\infty\rangle$ and $\langle\mathbb{R}^n, \nu\rangle$. Thus, $\langle\mathbb{R}^n, \nu\rangle$ also refutes $\varphi(p_1, \ldots, p_m)$, and so $\bigcap_{n=1}^\infty L_n \subseteq L_\infty$. Therefore, we obtain that $L_\infty = \bigcap_{n=1}^\infty L_n$.

□

As an immediate consequence of Theorems 4.4 and 5.3 we obtain the following:

Corollary 5.4. $L_\infty = \bigcap_{n=1}^{\infty} L(\mathcal{F}^n)$.

Since every L_n is a logic over **Grz**, it is obvious that so is L_∞. However, unlike each L_n, the logic L_∞ is not tabular. Moreover, both the depth and the width of L_∞ are infinite. Nevertheless, it follows from Corollary 5.4 that L_∞ has the finite model property.

6 Conclusions

In this paper we have characterized the logic L_1 of serial subsets (that is, finite unions of intervals) of the real line $\mathbb{R}$, as well as its natural generalizations – the logics L_n of sufficiently well-behaved n-chequered subsets of n-dimensional Euclidean spaces $\mathbb{R}^n$. Unlike the full modal logic of each Euclidean space $\mathbb{R}^n$ ($n \geq 1$), which coincides with **S4**, all logics L_n are different; forming a decreasing chain converging to the logic L_∞ of ∞-chequered subsets of $\mathbb{R}^\infty$. Thus, we have arrived at a new hierarchy in modal logic, which we call the *Euclidean hierarchy.* We conclude the paper by mentioning some interesting further questions related to this hierarchy.

We have shown that the logics L_n are tabular, and so finitely axiomatizable and decidable. It seems interesting to give a concrete axiomatization of these logics, like the one that we have for the one-dimensional case [4]. We have some work in progress on this concrete match between modal axioms and topological principles, together with Darko Sarenac. Also, one would like to know the complexity of this type of spatial reasoning, by determining the computational complexity of the validity problem for these logics. (It is easy to see an upper bound in exponential time; but how much lower can we get?) Also, though we have shown that L_∞ has the finite model property, the question whether L_∞ is finitely axiomatizable and/or decidable remains open.

Finally, it seems of interest to investigate enrichments of the basic modal language with further spatially inspired modalities. In particular, when describing useful patterns, one often wants *convexity* explicitly in the language, instead of leaving it implicit, as in our restriction to serial and chequered sets. More specifically, one can add a unary modal operator $C\varphi$ saying that the current point x lies in between two points satisfying φ, or a binary modality $C(\varphi, \psi)$ saying that x lies in between a φ-point and a ψ-point. This introduces some affine geometrical structure on top of topology. Aiello [1] has shown that various spaces have different modal logics with this addition. What can we say about their precise axiomatization?

References

[1] M. Aiello. *Spatial Reasoning: Theory and Practice.* PhD thesis, ILLC, University of Amsterdam, 2002.

[2] M. Aiello and J. van Benthem. Logical patterns in space. In D. Barker-Plummer, D. Beaver, J. van Benthem, and P. Scotto di Luzio, editors, *Words, Proofs, and Diagrams.* CSLI, Stanford, 2002. To appear. A longer version is available as Techinical Report PP-1999-19, University of Amsterdam".

[3] M. Aiello and J. van Benthem. A modal walk through space. *Journal of Applied Non-Classical Logics*, 2002. To appear. Also available as Techinical Report PP-2001-23, University of Amsterdam.

[4] M. Aiello, J. van Benthem, and G. Bezhanishvili. Reasoning about space: The modal way. Technical Report PP-2001-18, Univ. of Amsterdam, 2001.

[5] G. Bezhanishvili and M. Gehrke. A new proof of completeness of S4 with respect to the real line. Technical Report PP-2002-??, Univ. of Amsterdam, 2002.

[6] D. Gabelaia. Modal definability in topology. Master's thesis, ILLC, University of Amsterdam, 2001.

[7] J. McKinsey and A. Tarski. The algebra of topology. *Annals of Mathematics*, 45:141–191, 1944.

[8] J. van Benthem. *The Logic of Time*, volume 156 of *Synthese Library.* Reidel, Dordrecht, 1983. [Revised and expanded, Kluwer, 1991].