Duality for Distributive Modal Algebras

with an application on subdirect irreducibility

Be Birchall

June 25, 2004

Acknowledgments

Thanks first to the Beth foudation for the generous grant making it possible for me to study at the ILLC. I feel fortunate to have been introduced through the Master of Logic programme to the friendly, supportive and collaborative environment here.

Two people especially contributed to this environment, making it an engaging one where I could learn a lot. Yde Venema, whose inspiring and clear lectures first introduced me to algebraic modal logic, was an ideal thesis supervisor: he provided structure for the research, as well as invaluable (prompt and extensive) feedback, discussions and advice. His support and encouragement were just as important as these things.

Dick de Jongh, my mentor in the programme, always helped me feel more at home at the ILLC. I am very glad to have been a student while he was the director of the Master of Logic programme.

I am also grateful to the people on the committee that read my thesis for their helpful questions and comments: to Dick and Yde, as well as Johan van Benthem and Nick Bezhanishvili. Thanks also to Nick for helpful discussions along the way.

Contents

Chapter 0

Introduction

This work is concerned with developing some dualities for distributive algebras with additional modal operators, and investigating the conditions on their duals under which the algebras are subdirectly irreducible. The notions playing key roles in this are introduced below.

0.1 Distributive Modal Algebras

As modal algebras are based on Boolean algebras, distributive modal algebras (*DMA*'s) are based on distributive algebras. Thus *DMA*'s include function symbols for conjunction, disjunction, true and false, but not for negation. The four modal operators $\Diamond$, $\Box$, $\rhd$ and $\lhd$ added to the underlying distributive algebra, though, enable the representation of various forms of weak negation. The modal operator $\Diamond$ preserves disjunction and false, while $\Box$ preserves conjunction and true. Thus the diamond in a *DMA* behaves as a diamond $\Diamond$ in a modal algebra, while the box behaves as $\neg\Diamond\neg$ in such an algebra. The other two modal operators of a *DMA* reverse conjunction and disjunction and also reverse true and false, as follows. The operator $\rhd$ turns disjunctions into conjunctions and false into true, while $\lhd$ turns conjunctions into disjunctions and true into false. Thus $\rhd$ behaves as $\neg\Diamond$ in a modal algebra, while $\lhd$ behaves as $\Diamond\neg$ in such an algebra. Since negation is not present, $\Box$ and $\Diamond$ are not interdefinable, as would be the case with a modal algebra. This explains the need for modal operators $\Box$, $\rhd$ and $\lhd$ in addition to $\Diamond$.

Various special classes of *DMA*'s are often considered in the literature. For example, Positive Modal Logic, as developed in [Dun95] and further studied in [CJ99], concerns distributive modal algebras with only the two operators $\Diamond$ and $\Box$ that are constrained to satisfy certain axioms. Cignoli, in [Cig91], generalizes a discussion (in [Hal62]) of quantifiers for Boolean algebras to the distributive setting by considering a special class of *DMA*'s with only the modal operator $\diamond$, again constrained to satisfy certain conditions. Ockham algebras, as in [], involve only the modal operators $\rhd$ and $\lhd$. An additional axiom, ensuring that $\rhd a = \lhd a$ for elements a of the algebra, allows the representation of a certain weak negation known as "Ockham negation". Distributive modal algebras are the concern of [GNV02], where further special cases are discussed.

0.2 Dualities for *DMA*'s

A frame is a partially ordered set together with four additional binary relations, one corresponding to each modal operator of a *DMA*. Frames play the same role in representing *DMA*'s as Kripke frames play in the representation of modal algebras. If we restrict attention to only the finite objects of each type, we find that frames and *DMA*'s are indeed dual to one another. However, for arbitrary frames and arbitrary *DMA*'s such a duality does not obtain. This, then, gives rise to two dualities: one in which arbitrary frames are represented, and one representing arbitrary *DMA*'s.

A special case of *DMA*'s, perfect distributive modal algebras (DMA^+'s), are the duals of frames. This duality is mentioned in [GNV02], where it is partially developed. The duality is analogous in the Boolean case to the duality (developed in [Tho75]) between Kripke frames and complete atomic and completely additive modal algebras, a special case of modal algebras.

The class of Kripke frames is not adequate for the representation of modal algebras; instead, finite Kripke frames are generalized to descriptive general frames, structures involving a topology, to yield the duals of modal algebras. Similarly, the dual of an arbitrary *DMA* is a structure involving topology that in the finite case coincides with a frame. Thus, duals of *DMA*'s consist of an underlying set with four binary relations together with both an order relation and a topology.

This duality for *DMA*'s extends Priestley duality, in which distributive algebras without modal operators are represented as Priestley spaces. Other generalizations of Priestley duality exist in the literature, establishing dualities between various extensions of Priestley spaces and distributive algebras with certain additional operators. Goldblatt, for example provides (in

[Gol89]) a duality between an extension of Priestley spaces and a slight generalization (in which modal operators are not constrained to be only unary) of *DMA*'s with only the order-preserving modal operators $\Diamond$ and $\Box$. Goldblatt's duality is further generalized in [SS00]. In [Cig91], Priestley spaces are enriched with the addition of an equivalence relation to represent those *DMA*'s mentioned above having only the modal operator $\Diamond$. And in [CJ99], a generalized Priestley duality is established for the modal algebras of Positive Modal Logic.

0.3 Subdirect Irreducibility

The *direct product* of a collection of algebras is the usual product for algebras, generalizing that based on the cartesian product of the underlying sets of a finite collection of algebras. We might wonder if every algebra can be expressed as the direct product of algebras that cannot themselves be further reduced to a direct product of other algebras. For finite algebras this is indeed the case. That is, where a *directly indecomposable* algebra is one that is not isomorphic to a direct product of two nontrivial algebras, every finite algebra is isomorphic to a direct product of directly indecomposable algebras.

This does not, though, obtain for infinite algebras in general. However, something similar does hold for a weaker notion of product. An algebra A is a *subdirect product* of a collection of algebras if it is a subalgebra of the direct product of that collection such that the natural projection map of A to each algebra in the collection is surjective. If one such projection map is an isomorphism for every collection of algebras of which A is a subdirect product, then A is *subdirectly irreducible.* That is, informally, an algebra is subdirectly irreducible iff it cannot be reduced to other algebras via subdirect product. A theorem of Birkhoff states that every algebra is isomorphic to a subdirect product of subdirectly irreducible algebras. In this sense, subdirectly irreducible algebras are the basic building blocks of algebras.

An alternative characterization of subdirect irreducibility, in terms of the congruence lattice of an algebra, follows from the definition of subdirectly irreducible. According to this characterization, an algebra is subdirectly irreducible iff it is trivial or has a smallest nontrivial congruence.

A precise presentation of the above can be found in §II.8 of [BS81]. The alternative characterization just mentioned is the one we will work with here. For a more detailed discussion of this characterization, consult appendix B.

0.4 Subdirect Irreducibility in Dual Perspective

A duality between two categories provides a new perspective on concepts important for one category by enabling the translation of those concepts into a new setting. Here we are concerned with finding the dual notion of subdirect irreducibility. For Kripke frames, there is (see [Kra99] p.174) a simple characterization according to which a Kripke frame is rooted iff its dual is subdirectly irreducible.

Next, we can consider how this generalizes to arbitrary modal algebras with their topological duals. In this setting we do not have the simple characterization that a modal algebra is subdirectly irreducible iff its dual is rooted (see [Sam99] for counterexamples). Sambin does, however, provide a generalization for the special case of $K4$ algebras, whose duals have a transitive accessibility relation. According to this characterization, a $K4$ algebra is subdirectly irreducible iff the set of roots of its dual is open and nonempty. In [Ven02], Venema provides a generalization of this to arbitrary modal algebras. This is via a generalized notion of rootedness, topo-rootedness, that involves the topology as well as the accessibility relation of the algebra's dual.

Here the concern is with seeing how such a characterization generalizes to the distributive setting. As for modal algebras, this involves generalizing the notion of rootedness to take into account aspects of the dual structure. For frame duality, the appropriate notion of rootedness, order-rootedness, involves both the ordering as well as the relations corresponding to the modal operators. We find that a frame has an order-root iff its dual is subdirectly irreducible. For *DMA* duality, the notion of rootedness, again called topo-rootedness here, involves the topology, ordering and the relations corresponding to the modal operators. We find that for this notion of topo-root, a *DMA* is subdirectly irreducible iff the set of topo-roots of its dual is open and nonempty.

0.5 Overview

The first half of this thesis concerns the duality for frames. The relevant categories are introduced, and shown to be dually equivalent in chapter 1. In chapter 2 this duality is applied to characterize the subdirectly irreducible duals of frames.

The duality for *DMA*'s is developed in chapter 3. Finally, chapter 4 contains an application of this duality to obtain a dual characterization of subdirect irreducibility for *DMA*'s. Some steps are made towards a more transparent characterization than that initially given, and results are obtained for *DMA*'s that satisfy a certain simplifying condition. The chapter closes with a characterization of subdirect irreducibility for a special class of *DMA*'s that do satisfy that simplifying condition.

Chapter 1

Representing Frames

In this chapter the category $\mathcal{FR}$ of frames with order-preserving bounded morphisms is shown to be dually equivalent to the category $\mathcal{DMA}^+$ of perfect distributive modal algebras (*DMA*$^+$'s) with *DMA*$^+$-homomorphisms. As discussed in appendix B, there is an"object part" and a "morphism part" to be established in proving such a duality to hold. The "object part" is established in the first section, where frames and *DMA*$^+$'s are defined; the "morphism part" is established in the second section, where order-preserving bounded morphisms and *DMA*$^+$-homomorphisms are defined. This duality is mentioned in [GNV02], where the "object part" is provided.

1.1 Perfect Distributive Modal Algebras

The class of perfect distributive modal algebras (*DMA*$^+$'s) is introduced in this section. These are distributive modal algebras (*DMA*'s), also introduced here, satisfying certain additional conditions. We see that frames, to be defined below, can be represented as *DMA*$^+$'s. The dual $\mathbb{F}^+$ of a frame $\mathbb{F}$ is defined and shown to be a *DMA*$^+$, and the dual $\mathbb{A}_+$ of a *DMA*$^+$ $\mathbb{A}$ is defined and shown to be a frame. We then see that $\mathbb{F} \cong (\mathbb{F}^+)_+$ for a frame $\mathbb{F}$, and $\mathbb{A} \cong (\mathbb{A}_+)^+$ for a *DMA*$^+$ $\mathbb{A}$. That is, the "object part" of the duality of this chapter is established.

A *frame* $\mathbb{F} = (F, \leqslant, R_\Diamond, R_\Box, R_\rhd, R_\lhd)$ is a tuple such that $\leqslant$ is a partial order on F and $R_\Diamond, R_\Box, R_\rhd, R_\lhd \subseteq F \times F$ satisfy the relation conditions, stated below.

Definition 1.1 Where $R_\Diamond, R_\Box, R_\rhd$ and $R_\lhd$ are binary relations on a set with partial order $\leqslant$, the *relation conditions* are:

$(R1)$ $\leqslant \circ R_\Diamond \circ \leqslant\ \subseteq R_\Diamond$,

$(R2)\ \geqslant \circ R_{\Box} \circ \geqslant\ \subseteq R_{\Box}$,

$(R3)\ \geqslant \circ R_{\rhd} \circ \leqslant\ \subseteq R_{\rhd}$,

$(R4)\ \leqslant \circ R_{\lhd} \circ \geqslant\ \subseteq R_{\lhd}$.

Here $\circ$ denotes the composition of relations, so that for example $(R1)$ holds just in case $tR_{\Diamond}w$ for all $t, u, v, w \in F$ for which $t \leqslant u$, $uR_{\Diamond}v$ and $v \leqslant w$. Notice that Kripke frames are a special case of these frames. Kripke frames coincide with those frames $\mathbb{F} = (F, \leqslant, R_{\Diamond}, R_{\Box}, R_{\rhd}, R_{\lhd})$ with the discrete ordering so that $u \leqslant v$ iff $u = v$ for $u, v \in F$ and with empty relations $R_{\Box}$, $R_{\rhd}$ and $R_{\lhd}$.

Definition 1.2 A *distributive modal algebra (DMA)* $\mathbb{A} = (A, \vee, \wedge, 0, 1, \Diamond, \Box, \rhd, \lhd)$ is an algebra with $(A, \vee, \wedge, 0, 1)$ bounded distributive lattice such that the *modal operators* $\Diamond, \Box, \rhd, \lhd$ satisfy:

$$\begin{aligned}
\Diamond(a \vee b) &= \Diamond a \vee \Diamond b\ , \quad & \Diamond 0 &= 0,\\
\Box(a \wedge b) &= \Box a \wedge \Box b\ , \quad & \Box 1 &= 1,\\
\rhd(a \vee b) &= \rhd a \wedge \rhd b\ , \quad & \rhd 0 &= 1,\\
\lhd(a \wedge b) &= \lhd a \vee \lhd b\ , \quad & \lhd 1 &= 0.
\end{aligned}$$

Definition 1.3 A *DMA* $\mathbb{A}$ is a *perfect distributive modal algebra (DMA$^+$)* if in addition $(A, \vee, \wedge, 0, 1)$ is complete, completely distributive, join generated by $J^{\infty}(\mathbb{A})$ (as well as meet generated by $M^{\infty}(\mathbb{A})$) and such that

$$\begin{aligned}
\Diamond(\bigvee X) &= \bigvee(\Diamond X),\\
\Box(\bigwedge X) &= \bigwedge(\Box X),\\
\rhd(\bigvee X) &= \bigwedge(\rhd X),\\
\lhd(\bigwedge X) &= \bigvee(\lhd X).
\end{aligned}$$

Here $J^{\infty}(\mathbb{A})$ is the set of all completely join irreducible elements of $\mathbb{A}$, where $a \in A$ is such an element iff $a = \bigvee X$ implies $a \in X$ for all $X \subseteq A$. Analogous remarks apply to $M^{\infty}(\mathbb{A})$ and completely meet irreducible elements of $\mathbb{A}$. The reader can consult [BD74] for definitions of bounded distributive lattice, completely distributive, complete, and join (meet) generated.

Where $\mathbb{A}$ is clear from context and $X \subseteq A$, the notation $J^\infty(X)$ is used to denote the set $J^\infty(\mathbb{A}) \cap X$ of completely join irreducible elements of $\mathbb{A}$ in X. The condition that $\mathbb{A}$ is join generated by $J^\infty(\mathbb{A})$ (and meet generated by $M^\infty(\mathbb{A})$) will be put to use via the following:

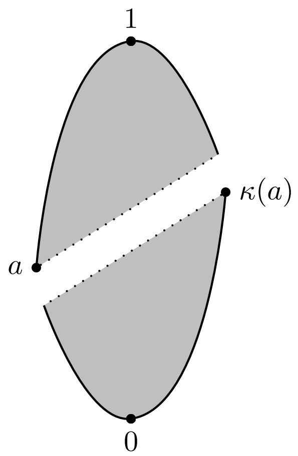
Proposition 1.4 For a *DMA*$^+$ $\mathbb{A}$ with $a \in A$,

$$\bigvee J^\infty(\downarrow a) = a = \bigwedge M^\infty(\uparrow a).$$

Proof. Since $\mathbb{A}$ is perfect, A is join-generated by $J^\infty(\mathbb{A})$. So $a = \bigvee u$ for some $u \subseteq J^\infty(\mathbb{A})$. Then for every $b \in u$, $b \leqslant a$. So $u \subseteq J^\infty(\downarrow a)$. Thus $a = \bigvee u \leqslant \bigvee J^\infty(\downarrow a)$. And since $a \geqslant b$ for every $b \in J^\infty(\downarrow a)$, also $a \geqslant \bigvee J^\infty(\downarrow a)$.

The second equality holds by analogous reasoning. $\square$

Consider a *DMA*$^+$ $\mathbb{A} = (A, \vee, \wedge, 0, 1, \Diamond, \Box, \rhd, \lhd)$. Soon we will make use of the map $\kappa : J^\infty(\mathbb{A}) \to A$ defined by $\kappa(a) = \bigvee(A \setminus \uparrow a)$. The following proposition tells us that $\kappa(a) \in M^\infty(\mathbb{A})$ for $a \in J^\infty(\mathbb{A})$ and that for all $b \in M^\infty(\mathbb{A})$, $b = \kappa(a)$ for some unique $a \in J^\infty(\mathbb{A})$. For $a, b \in A$, (a, b) *splits* $\mathbb{A}$ if $\uparrow a \cup \downarrow b = A$ and $\uparrow a \cap \downarrow b = \emptyset$. (See the diagram below.) Notice that the proof of this next proposition uses $\mathbb{A}$'s completeness and appeals to the infinite distributivity of $\mathbb{A}$, a property weaker than complete distributivity.



$(a, \kappa(a))$ splits the lattice

Proposition 1.5 Let $\mathbb{A} = (A, \vee, \wedge, 0, 1, \diamond, \square, \rhd, \lhd)$ be a *DMA*$^+$, with $a, b \in A$. Then $a \in J^\infty(\mathbb{A})$ iff a splits $\mathbb{A}$; $b \in M^\infty(\mathbb{A})$ iff there is some a such that (a, b) splits $\mathbb{A}$. Moreover, if (a, b) splits $\mathbb{A}$ then $b = \kappa(a)$.

Proof. Only the first half of the first claim will be proved; the second half follows by analogous reasoning.

First observe that for any $a \in A$, there is at most one $b \in A$ such that (a, b) splits $\mathbb{A}$. For suppose (a, b) and (a, b') each split $\mathbb{A}$. Then since $\uparrow a \bigcap \downarrow b' = \emptyset$, $b' \notin \uparrow a$. So since $\uparrow a \bigcup \downarrow b = A$, we have $b' \in \downarrow b$ and thus $b' \leqslant b$. But then interchanging the roles of b and b' yields also $b \leqslant b'$, and thus $b = b'$.

Consider $a \in A$ and let $b = \bigvee(A \backslash \uparrow a)$. It suffices to show that if a splits $\mathbb{A}$ then a is completely join irreducible, and that if a is completely join irreducible then (a, b) splits $\mathbb{A}$.

Suppose (a, b') splits $\mathbb{A}$ for some $b' \in A$. Then certainly $a \neq 0$, for we must have $0 \in \downarrow b'$ and $\uparrow a \bigcap \downarrow b = \emptyset$. Suppose $a = \bigvee C$ for some $C \subseteq A$. If $a \notin C$, then $c < a$ for all $c \in C$ and thus $C \bigcap \uparrow a = \emptyset$. So $C \subseteq \downarrow b'$. But then $a = \bigvee C \leqslant b'$ and hence $a \in \downarrow b'$, which is impossible as $\uparrow a \bigcap \downarrow b' = \emptyset$. So $a \in C$. Thus a is completely join irreducible.

Finally, suppose that a is completely join irreducible. To show that (a, b) splits $\mathbb{A}$, we must show that $\uparrow a \bigcup \downarrow b = A$ and that $\uparrow a \bigcap \downarrow b = \emptyset$. The first of these claims is immediate from the definition $b = \bigvee(A \backslash \uparrow a)$. To see that $\uparrow a \bigcap \downarrow b = \emptyset$, suppose otherwise assuming that $a \leqslant c$ and $c \leqslant b$ for some $c \in A$. Then $a \leqslant b$, and so

$$a = a \wedge b = a \wedge \bigvee(A \backslash \uparrow a) = \bigvee\{a \wedge b' : b' \in A \backslash \uparrow a\},$$

where this last equality follows from the complete distributivity of $\mathbb{A}$. Now observe that $\{a \wedge b' : b' \in A \backslash \uparrow a\} = \{c \in A : c < a\}$. For if $c < a$, then $a \wedge c = c \in A \backslash \uparrow a$; and if $c = a \wedge b'$ for $b' \ngeqslant a$ then $c < a$. Thus we obtain $a = \bigvee\{c \in A : c < a\}$. But this contradicts a's being completely join irreducible, as certainly $a \notin \{c \in A : c < a\}$. $\square$

The following states a useful consequence of this last proposition:

Proposition 1.6 Let $\mathbb{A} = (A, \vee, \wedge, 0, 1, \diamond, \square, \rhd, \lhd)$ be a *DMA*$^+$. For $a \in J^\infty(\mathbb{A}), b \in A$,

$$a \leqslant b \text{ iff } b \nleqslant \kappa(a).$$

Proof. We have

$$\begin{aligned} a \leqslant b &\text{ iff } b \in {\uparrow} a, \\ &\text{ iff } b \notin {\downarrow}\kappa(a), \\ &\text{ iff } b \nleqslant \kappa(a), \end{aligned}$$

where the second biconditional follows from $(a, \kappa(a))$'s splitting $\mathbb{A}$. □

Notice that the following proposition is a consequence of this definition and the conditions on the modal operators.

Proposition 1.7 Let $\mathbb{A}$ be a *DMA*$^+$ with carrier A and order $\leqslant$ on A; suppose $a \leqslant b$ for $a, b \in A$. Then:

(*i*) $\Diamond a \leqslant \Diamond b$,

(*ii*) $\Box a \leqslant \Box b$,

(*iii*) $\rhd b \leqslant \rhd a$,

(*iv*) $\lhd b \leqslant \lhd a$,

(*v*) $\kappa(a) \leqslant \kappa(b)$, if $a, b \in J^\infty(\mathbb{A})$.

The κ function will play a role in the definition of the dual of a *DMA*$^+$. One more definition is needed before defining the dual of a frame $\mathbb{F}$:

Definition 1.8 Where $R \subseteq F \times F$, the operations $\langle R\rangle, [R], [R\rangle$ and $\langle R]$ on $\mathcal{P}(F)$ are defined by

$\langle R\rangle X = \{u : \exists v(uRv \text{ and } v \in X)\}$

$[R]X = \{u : \forall v(uRv \to v \in X\}$

$[R\rangle X = \{u : \forall v(uRv \to v \notin X)\}$

$\langle R]X = \{u : \exists v(uRv \text{ and } v \notin X)\}$

The dual $\mathbb{F}^+$ of a frame $\mathbb{F} = (F, \leqslant, R_\Diamond, R_\Box, R_\triangleright, R_\triangleleft)$ is defined by

$$\mathbb{F}^+ = (\mathcal{D}(\mathbb{F}), \cup, \cap, \emptyset, F, \langle R_\Diamond\rangle, [R_\Box], [R_\triangleright\rangle, \langle R_\triangleleft]).$$

Here $\mathcal{D}(\mathbb{F})$ is the collection of down-sets of $\mathbb{F}$, where $a \subseteq F$ is a *down-set* of $\mathbb{F}$ if for all $u, v \in F$ we have $u \in a, v \leqslant u$ implies $v \in a$.

We must check that $\mathbb{F}^+$ is indeed a perfect distributive modal algebra. Certainly $(\mathcal{D}(\mathbb{F}), \cup, \cap, \emptyset, F)$ is a complete, completely distributive lattice. To see that this lattice is join generated by $J^\infty(\mathcal{D}(\mathbb{F}))$, consider the following proposition which will also be useful later:

Proposition 1.9 Where F is the underlying set of a frame $\mathbb{F}$,

$$\{{\downarrow}w : w \in F\} = J^\infty(\mathcal{D}(\mathbb{F})),$$

and the lattice reduct of $\mathbb{F}$ is join generated by $J^\infty(\mathcal{D}(\mathbb{F}))$.

Proof. Consider $w \in F$ and suppose ${\downarrow}w = \bigcup U$ for some $U \subseteq \mathcal{D}(\mathbb{F})$. Then $w \in a$ for some $a \in U$. So as $a \subseteq \bigcup U = {\downarrow}w$, $a \subseteq {\downarrow}w$. And as a is a down-set containing w, also ${\downarrow}w \subseteq a$. So ${\downarrow}w = a$ for some $a \in U$. Thus ${\downarrow}w \in J^\infty(\mathcal{D}(\mathbb{F}))$.

Next consider $a \in \mathcal{D}(\mathbb{F})$. Then $a = \bigcup\{{\downarrow}w : w \in a\}$. So if $a \in J^\infty(\mathcal{D}(\mathbb{F}))$ then $a = {\downarrow}w$ for some $w \in a \subseteq F$.

Now it is immediate that the lattice is join generated by $J^\infty(\mathcal{D}(\mathbb{F}))$, as $a = \bigcup\{{\downarrow}w : w \in a\}$ for $a \in \mathcal{D}(\mathbb{F})$. $\Box$

So far we have seen that $(\mathcal{D}(\mathbb{F}), \cup, \cap, \emptyset, F)$ is a complete infinitely distributive bounded lattice join generated by $J^\infty(\mathcal{D}(\mathbb{F}))$. Similarly, the lattice is meet generated by $M^\infty(\mathcal{D}(\mathbb{F}))$. Next observe that the relation conditions of definition 1.1 guarantee that $\mathcal{D}(\mathbb{F})$ is closed under $\langle R_\Diamond\rangle, [R_\Box], [R_\triangleright\rangle$ and $\langle R_\triangleleft]$. Finally, notice that the definitions of $\langle R_\Diamond\rangle, [R_\Box], [R_\triangleright\rangle$ and $\langle R_\triangleleft]$ ensure that

$$\begin{array}{lcl}
\langle R_\Diamond\rangle\emptyset = \emptyset & \text{and} & \langle R_\Diamond\rangle \bigcup U = \bigcup \langle R_\Diamond\rangle U \text{ for } U \subseteq \mathcal{D}(\mathbb{F}), \\
[R_\Box]F = F & \text{and} & [R_\Box] \bigcap U = \bigcap [R_\Box]U \text{ for } U \subseteq \mathcal{D}(\mathbb{F}), \\
[R_\triangleright\rangle\emptyset = F & \text{and} & [R_\triangleright\rangle \bigcup U = \bigcap [R_\triangleright\rangle U \text{ for } U \subseteq \mathcal{D}(\mathbb{F}), \\
\langle R_\triangleright]F = \emptyset & \text{and} & \langle R_\triangleright] \bigcap U = \bigcup \langle R_\triangleright]U \text{ for } U \subseteq \mathcal{D}(\mathbb{F}).
\end{array}$$

This establishes the following.

Proposition 1.10 If $\mathbb{F}$ is a frame, $\mathbb{F}^+$ is a *DMA*$^+$.

The dual of a *DMA*$^+$ $\mathbb{A} = (A, \vee, \wedge, 0, 1, \diamond, \Box, \rhd, \lhd)$ is defined by

$$\mathbb{A}_+ = (J^\infty(\mathbb{A}), \leqslant, S_\diamond, S_\Box, S_\rhd, S_\lhd)$$

where $\leqslant$ is the order on A restricted to $J^\infty(\mathbb{A})$ and $S_\diamond, S_\Box, S_\rhd, S_\lhd \subseteq J^\infty(\mathbb{A}) \times J^\infty(\mathbb{A})$ are defined by

$uS_\diamond v$ iff $u \leqslant \diamond v$,

$uS_\Box v$ iff $\kappa(u) \geqslant \Box\kappa(v)$,

$uS_\rhd v$ iff $\kappa(u) \geqslant \rhd v$,

$uS_\lhd v$ iff $u \leqslant \lhd\kappa(v)$.

To ensure that $\mathbb{A}_+$ is a frame, we need only ensure that the relation conditions of definition 1.1 are satisfied. These are immediate consequences of proposition 1.7 above. For example, to check that $(R1)$ holds, suppose that $uS_\diamond v, t \leqslant u$ and $v \leqslant w$. We must check that $tS_\diamond w$. Since $uS_\diamond v$, we have $u \leqslant \diamond v$. So using proposition 1.7 (i) we have $t \leqslant u \leqslant \diamond v \leqslant \diamond w$. Thus $tS_\diamond w$ as required. The other cases are similar, making use of the relevant clauses of proposition 1.7. Thus we arrive at:

Proposition 1.11 Where $\mathbb{A}$ is a *DMA*$^+$, $\mathbb{A}_+$ is a frame.

To establish the "object part" of the duality between frames and distributive modal algebras, it remains only to prove that $\mathbb{F} \cong (\mathbb{F}^+)_+$ and $\mathbb{A} \cong (\mathbb{A}_+)^+$ for a frame $\mathbb{F}$ and *DMA*$^+$ $\mathbb{A}$.

Proposition 1.12 Where $\mathbb{F}$ is a frame, $\mathbb{F} \cong (\mathbb{F}^+)_+$.

Proof. Let $\mathbb{F} = (F, \leqslant, R_\diamond, R_\Box, R_\rhd, R_\lhd)$ so that $(\mathbb{F}^+)_+ = (J^\infty(\mathcal{D}(\mathbb{F})), \subseteq, S_{\langle R_\diamond \rangle}, S_{[R_\Box]}, S_{[R_\rhd \rangle}, S_{\langle R_\lhd]})$. It will be shown that $f : u \mapsto {\downarrow} u$ is an isomorphism from $\mathbb{F}$ onto $(\mathbb{F}^+)_+$.

First observe that by proposition 1.9 $f(u) \in J^\infty(\mathcal{D}(\mathbb{F}))$ for $u \in F$. And as ${\downarrow} u, {\downarrow} v \in J^\infty(\mathcal{D}(\mathbb{F}))$ for $u, v \in F$ we have $u \leqslant v$ iff ${\downarrow} u \cap J^\infty(\mathcal{D}(F)) \subseteq {\downarrow} v \cap J^\infty(\mathcal{D}(\mathbb{F}))$. That is, $u \leqslant v$ in $\mathbb{F}$ iff ${\downarrow} u \leqslant {\downarrow} v$ in $(\mathbb{F}^+)_+$.

So far we have seen that $(F, \leqslant) \cong (J^\infty(\mathcal{D}(\mathbb{F})), \subseteq)$ via f. It remains to establish that for $u, v \in F$, (*i*) $uR_\Diamond v$ iff $f(u)S_{\langle R_\Diamond \rangle} f(v)$,

(*ii*) $uR_\Box v$ iff $f(u)S_{[R_\Box]} f(v)$,

(*iii*) $uR_\rhd v$ iff $f(u)S_{[R_\rhd \rangle} f(v)$,

(*iv*) $uR_\lhd v$ iff $f(u)S_{\langle R_\lhd]} f(v)$.

Only (*i*) and (*ii*) will be shown; the remaining cases are similar. We have $f(u)S_{\langle R_\Diamond \rangle} f(v)$ iff $f(u) \subseteq \langle R_\Diamond \rangle f(v)$, iff for all $u' \in F$,

$$\text{if } u' \leqslant u \text{ then } \exists v'(u' R_\Diamond v' \text{ and } v' \leqslant v).$$

If $uR_\Diamond v$ and $u' \leqslant u$ then the relation conditions guarantee that also $u' R_\Diamond v$. So $uR_\Diamond v$ implies $f(u)S_{\langle R_\Diamond \rangle} f(v)$. Now suppose $f(u)S_{\langle R_\Diamond \rangle} f(v)$. Then since $u \leqslant u$ there is some v' for which $uR_\Diamond v'$ and $v' \leqslant v$. But then by the relation conditions $uR_\Diamond v$, as required.

For (*ii*), we consider $\kappa(\downarrow u)$ for $\downarrow u \in J^\infty(\mathcal{D}(\mathbb{F}))$. Notice that proposition 1.6 yields $\kappa(\downarrow u) \supseteq b$ iff $\downarrow u \not\supseteq b$ for $\downarrow u \in J^\infty(\mathcal{D}(\mathbb{F}))$ and $b \in \mathcal{D}(\mathbb{F})$. Thus we have

$$\begin{aligned} f(u)S_{[R_\Box]} f(v) &\text{ iff } \kappa(\downarrow u) \supseteq [R_\Box]\kappa(\downarrow v), \\ &\text{ iff } \downarrow u \not\subseteq [R_\Box]\kappa(\downarrow v), \\ &\text{ iff } \downarrow u \not\subseteq \{u' : \forall v'(u' R_\Box v' \to v' \in \kappa(\downarrow v)\}, \\ &\text{ iff } \downarrow u \not\subseteq \{u' : \forall v'(u' R_\Box v' \to v \not\leqslant v')\}, \\ &\text{ iff } \exists u' \text{ such that } u' \leqslant u \text{ and } \exists v'(u' R_\Box v' \text{ and } v \leqslant v'). \end{aligned}$$

The penultimate biconditional here makes use of $v' \in \kappa(\downarrow v)$ iff $v \not\leqslant v'$, which follows from an application of proposition 1.6. To see this, notice that $v' \in \kappa(\downarrow v)$ iff $\downarrow v' \subseteq \kappa(\downarrow v)$, iff (by proposition 1.6) $\downarrow v \not\subseteq \downarrow v'$, iff $v \not\leqslant v'$.

Suppose $uR_\Box v$. Then where $u' := u$ and $v' := v$, we have $u' \leqslant u$ and $(u' R_\Box v'$ and $v \leqslant v')$. Thus $f(u)S_{[R_\Box]} f(v)$. Now for the converse, suppose that $f(u)S_{[R_\Box]} f(v)$. Thus for some u' we have $u' \leqslant u$ and $\exists v'(u' R_\Box v'$ and $v \leqslant v')$. Then the relation conditions guarantee that also $uR_\Box v$. $\Box$

Proposition 1.13 Where $\mathbb{A}$ is a perfect distributive modal algebra, $\mathbb{A} \cong (\mathbb{A}_+)^+$.

Proof. Let $\mathbb{A} = (A, \vee, \wedge, 0, 1, \Diamond, \Box, \rhd, \lhd)$ so that $(\mathbb{A}_+)^+ = (\mathcal{D}(J^\infty(\mathbb{A})), \cup, \cap, \emptyset, A, \langle R_\Diamond\rangle, [R_\Box], [R_\rhd\rangle, \langle R_\lhd])$. It will be shown that $g : a \mapsto J^\infty(\downarrow a)$ is an isomorphism from $\mathbb{A}$ onto $(\mathbb{A}_+)^+$.

Clearly $g(a) \in \mathcal{D}(J^\infty(\mathbb{A}))$ for $a \in A$. Consider $u \in \mathcal{D}(J^\infty(\mathbb{A}))$. Showing that $u = J^\infty(\downarrow(\bigvee u))$ establishes that g is surjective. Well, $u \subseteq J^\infty(\mathbb{A})$ and for $a \in u$ we have $a \leqslant \bigvee u$. Thus $u \subseteq J^\infty(\downarrow(\bigvee u))$. Now suppose $a \in J^\infty(\downarrow(\bigvee u))$. Since $a \in \downarrow(\bigvee u)$, $a \leqslant \bigvee u$. Thus, since $a \in J^\infty(\mathbb{A})$ and hence is join prime, $a \leqslant b$ for some $b \in u$. And thus $a \in u$ as u is a down-set. So $u = J^\infty(\downarrow(\bigvee u))$.
Next observe that for $a, b \in A$ we have $a \leqslant b$ iff $g(a) \subseteq g(b)$. The left to right direction is immediate, and the other direction is a consequence of proposition 1.4 as $J^\infty(\downarrow a) \subseteq J^\infty(\downarrow b)$ implies $\bigvee J^\infty(\downarrow a) \leqslant \bigvee J^\infty(\downarrow b)$. From this it not only follows that G is bijective, but also that g is an order isomorphism. And hence g is an isomorphism.

So far we have seen that g is an isomorphism between $(A, \vee, \wedge, 0, 1)$ and $(\mathcal{D}(J^\infty(\mathbb{A})), \cup, \cap, \emptyset, A)$. It remains to establish that for $a \in A$,

(*i*) $\langle S_\Diamond\rangle g(a) = g(\Diamond a)$,

(*ii*) $[S_\Box]g(a) = g(\Box a)$,

(*iii*) $[S_\rhd\rangle g(a) = g(\rhd a)$, and

(*iv*) $\langle S_\lhd]g(a) = g(\lhd a)$.

Again, proofs for only the first two claims will be given. We have

$$\langle S_\Diamond\rangle g(a) = \langle S_\Diamond\rangle J^\infty(\downarrow a) = \{u : \exists v(uS_\Diamond v \text{ and } v \in J^\infty(\downarrow a))\},$$

and

$$g(\Diamond a) = J^\infty(\downarrow \Diamond a).$$

So to establish the claim we must show that for all $u \in J^\infty(\mathbb{A})$,

$$u \leqslant \Diamond a \text{ iff there is a } v \in J^\infty(\mathbb{A}) \text{ with } u \leqslant \Diamond v \text{ and } v \leqslant a.$$

Consider $u \in J^\infty(\mathbb{A})$ and suppose first that the right hand side is satisfied for some v. Then as $v \leqslant a$, we have $\Diamond v \leqslant \Diamond a$. So $u \leqslant \Diamond a$, as required. For the other direction observe that

$$\Diamond a = \Diamond \bigvee J^\infty(\downarrow a) = \bigvee \Diamond[J^\infty(\downarrow a)],$$

where the first equality follows from proposition 1.4 and the second from $\mathbb{A}$'s being perfect. So supposing $u \leqslant \Diamond a$ and using the complete join primeness of u we have $u \leqslant \Diamond v$ for some $v \in J^\infty(\mathbb{A})$ with $v \leqslant a$.

Turning attention to (ii), we have

$$[S_\Box]g(a) = [S_\Box]J^\infty(\downarrow a) = \{u : \forall v(uS_\Box v \to v \in J^\infty(\downarrow a))\}$$

and

$$g(\Box a) = J^\infty(\downarrow \Box a).$$

So it suffices to show that for all $u \in J^\infty(\mathbb{A})$,

$$u \nleqslant \Box a \text{ iff there is a } v \in J^\infty(\mathbb{A}) \text{ with } \kappa(u) \geqslant \Box\kappa(v) \text{ and } v \nleqslant a.$$

Suppose first that v satisfies the condition on the right hand side. Then $v \nleqslant a$ ensures that $a \leqslant \kappa(v)$. So $\Box a \leqslant \Box\kappa(v) \leqslant \kappa(u)$. Thus $\Box a \leqslant \kappa(u)$ and hence $u \nleqslant \Box a$. For the converse suppose $u \nleqslant \Box a$ so that

$$\kappa(u) \geqslant \Box a = \Box \bigwedge(M^\infty(\mathbb{A}) \cap \uparrow a) = \bigwedge \Box(M^\infty(\mathbb{A}) \cap \uparrow a).$$

Then as $\kappa(u)$ is meet prime, $\kappa(u) \geqslant \Box m$ for some $m \in M^\infty(\mathbb{A})$ with $m \geqslant a$. Thus where $v \in J^\infty(\mathbb{A})$ is such that $m = \kappa(v)$, we have $\kappa(u) \geqslant \Box\kappa(v)$ and $\kappa(v) \geqslant a$. So since $\kappa(v) \geqslant a$ implies $a \ngeqslant v$, v satisfies the required conditions. $\Box$

1.2 Morphisms

The morphisms for *DMA*$^+$'s, *DMA*$^+$-homomorphisms, are introduced in this section. In the course of this, *DMA*-homomorphisms, the morphisms for *DMA*'s, are defined. We see that a *DMA*$^+$-homomorphism is a *DMA*-homomorphism that satisfies certain additional conditions. The morphisms for frames are order-preserving bounded morphisms, and are also defined below. This section contains the "morphism part" of the duality between the category of *DMA*$^+$'s with *DMA*$^+$-homomorphisms and the category of frames with order-preserving bounded morphisms.

Definition 1.14 Let $\mathbb{A} = (A, \vee, \wedge, 0, 1, \Diamond, \Box, \rhd, \lhd)$ and $\mathbb{A}' = (A', \vee, \wedge, 0, 1, \Diamond, \Box, \rhd, \lhd)$ be *DMA*'s. A *DMA-homomorphism from* $\mathbb{A}$ *to* $\mathbb{A}'$ is a function $\eta : A \to A'$ satisfying

$$\eta(0) = 0' \text{ and } \eta(1) = 1',$$

$$\eta(a \vee b) = \eta(a) \vee \eta(b) \text{ and } \eta(a \wedge b) = \eta(a) \wedge \eta(b) \text{ for } a, b \in A$$

and

$$\eta(\triangle a) = \triangle'(\eta a) \text{ for } a \in A$$

for each modal operator $\triangle$ or $\mathbb{A}$ with corresponding operator $\triangle'$ of $\mathbb{A}'$.

If $\mathbb{A}$ is a *DMA*$^+$ with $\bigvee X$ ($\bigwedge X$) the join (meet) of $X \subseteq A$ (and similarly for $\mathbb{A}', \bigvee'$ and $\bigwedge'$), then a *DMA*$^+$*-homomorphism* is a *DMA*-homomorphism such that

$$\eta(\bigvee X) = \bigvee' \eta X \text{ and } \eta(\bigwedge X) = \bigwedge' \eta X \text{ for } X \subseteq A.$$

For the remainder of this section, let $\mathbb{A} = (A, \vee, \wedge, 0, 1, \diamondsuit, \Box, \rhd, \lhd)$ and $\mathbb{A}' = (A', \vee', \wedge', 0', 1', \diamondsuit', \Box', \rhd', \lhd')$ be *DMA*$^+$'s; let $\mathbb{F} = (F, \leqslant, R_\diamondsuit, R_\Box, R_\rhd, R_\lhd)$ and $\mathbb{F}' = (F', \leqslant', R'_\diamondsuit, R'_\Box, R'_\rhd, R'_\lhd)$ be frames.

Definition 1.15 An *order-preserving bounded morphism* is a function $\chi : F \to F'$ satisfying

(Bi) $u \leqslant v$ implies $\chi(u) \leqslant \chi(v)$ for $u, v \in F$,

(Bii) $uR_\triangle v$ implies $\chi(u)R'_\triangle\chi(v)$ for $u, v \in F$ and $\triangle \in \{\diamondsuit, \Box, \rhd, \lhd\}$,

$(Biii)(a)$ if $\chi(u)R'_\diamondsuit v'$ for $u \in F, v' \in F'$ then $\exists v \in F$ with $uR_\diamondsuit v$ and $\chi(v) \leqslant' v'$,

$(Biii)(b)$ if $\chi(u)R'_\Box v'$ for $u \in F, v' \in F'$ then $\exists v \in F$ with $uR_\Box v$ and $\chi(v) \geqslant' v'$,

$(Biii)(c)$ if $\chi(u)R'_\rhd v'$ for $u \in F, v' \in F'$ then $\exists v \in F$ with $uR_\rhd v$ and $\chi(v) \leqslant' v'$,

$(Biii)(d)$ if $\chi(u)R'_\lhd v'$ for $u \in F, v' \in F'$ then $\exists v \in F$ with $uR_\lhd v$ and $\chi(v) \geqslant' v'$.

Next the dual χ^+ of an order-preserving bounded morphism χ and the dual η_+ of a *DMA*$^+$-homomorphism are defined. We must then ensure that χ^+ is a *DMA*$^+$-homomorphism and that η_+ is an order-preserving bounded morphism. For a function $\chi : F \to F'$, define the dual $\chi^+ : \mathcal{D}(\mathbb{F}') \to \mathcal{P}(F)$ by

$$\chi^+ : a' \mapsto \{u \in F : \chi(u) \in a'\}.$$

And for a function $\eta : A \to A'$, let $\eta_+ : A' \to A$ be defined by

$$\eta_+ : a' \mapsto \bigwedge\{x \in J^\infty(\mathbb{A}) : \eta(x) \geqslant' a'\}.$$

Notice that η_+ is defined on all of A'. The restriction of η_+ to $J^\infty(\mathbb{A}')$ is the dual of η, and is also denoted by η_+.

Proposition 1.16 If χ is an order preserving bounded morphism between $\mathbb{F}$ and $\mathbb{F}'$ then χ^+ is a *DMA*-homomorphism from $\mathbb{F}'^+$ to $\mathbb{F}^+$.

Proof. To check that the image of $\mathcal{D}(\mathbb{F}')$ under χ^+ is indeed a subset of $\mathcal{D}(\mathbb{F})$, consider $a' \in \mathcal{D}(\mathbb{F}')$ with $u \in \chi^+(a')$ and $v \leqslant u$. To show that $\chi^+(a')$ is a down-set we must ensure that $v \in \chi^+(a')$. Well, $\chi(u) \in a'$ and so as χ is order preserving also $\chi(v) \leqslant \chi(u) \in a'$. So since a' is a down-set, $\chi(v) \in a'$ and hence $v \in \chi^+(a')$.

χ^+ is a homomorphism between the lattice reducts of $\mathbb{F}'^+$ and $\mathbb{F}^+$ if

(*i*) $\chi^+(\emptyset) = \emptyset$,

(*ii*) $\chi^+(F') = F$,

(*iii*) $\chi^+(\bigcup U) = \bigcup \chi^+[U]$ for $U \subseteq \mathcal{D}(\mathbb{F})$, and

(*iv*) $\chi^+(\bigcap U) = \bigcap \chi^+[U]$ for $U \subseteq \mathcal{D}(\mathbb{F})$.

To see that the first two obtain, observe that $\chi^+(\emptyset) = \{u \in F : \chi(u) \in \emptyset\} = \emptyset$ and $\chi^+(F') = \{u \in F : \chi(u) \in F'\} = F$. For the third claim observe that

$$\chi^+(\bigcap U) = \{u \in F : \chi(u) \in \bigcap U\} = \bigcap_{a \in U} \{u \in F : \chi(u) \in a\} = \bigcap \chi^+[U].$$

Claim (*iv*) follows analogously.

Finally, we must show that $\chi^+(\triangle' a') = \triangle \chi^+(a')$ for each modal operator $\triangle$ of $\mathbb{F}^+$ and $a' \in \mathcal{D}(\mathbb{F}')$. For $\triangle = \langle R_\diamond \rangle$ this is the claim that for $a' \in \mathcal{D}(\mathbb{F}')$,

$$\chi^+(\langle R'_\diamond \rangle a') = \langle R_\diamond \rangle \chi^+(a').$$

We have

$$\chi^+(\langle R'_\diamond \rangle a') = \{u \in F : \exists v' \in F'(\chi(u) R'_\diamond v' \text{ and } v' \in a')\}$$

and

$$\langle R_\diamond \rangle \chi^+(a') = \{u \in F : \exists v \in F(u R_\diamond v \text{ and } \chi(v) \in a'\}.$$

Observe that the conditions (Bii) and $(Biii)(a)$ in the definition of bounded morphism are just what we need to show that the right hand sides of these two equations are equal.

The arguments for the other modal operators in $\mathbb{F}$ are similar. $\square$

The following result is useful in checking that the dual η_+ of a *DMA*$^+$-homomorphism $\eta : A \to A'$ is an order-preserving bounded morphism. Recall that η_+ is defined on all of A'.

Proposition 1.17 Where $\eta : A \to A'$ is a *DMA*$^+$-homomorphism from $\mathbb{A}$ to $\mathbb{A}'$,

(i) $\eta_+(\eta(a)) \leqslant a$ for $a \in J^\infty(\mathbb{A})$,

(ii) $\eta(\eta_+(a')) \geqslant' a'$ for $a' \in A'$,

(iii) $\eta_+(b') \geqslant \eta_+(a')$ for $a', b' \in A'$ with $b' \geqslant' a'$.

Proof. For (i), consider $a \in J^\infty(\mathbb{A})$. Then $a \in \{b \in J^\infty(\mathbb{A}) : \eta(b) \geqslant' \eta(a)\}$, and so

$$a \geqslant \bigwedge\{b \in J^\infty(\mathbb{A}) : \eta(b) \geqslant' \eta(a)\} = \eta_+(\eta(a)).$$

And for (ii), consider $a' \in A'$. Let

$$X := \{a \in J^\infty(\mathbb{A}) : \eta(a) \geqslant' a'\},$$

so that $a' \leqslant' \eta(a)$ for all $a \in X$. Then $a' \leqslant' \bigwedge' \eta[X] = \eta(\bigwedge X) = \eta(\eta_+(a'))$.

Finally for (iii) suppose $a' \geqslant' b'$ for $a', b' \in A'$. Then $\{x \in J^\infty(\mathbb{A}) : \eta(x) \geqslant' b'\} \subseteq \{x \in J^\infty(\mathbb{A}) : \eta(x) \geqslant' a'\}$ and thus

$$\eta_+(b) = \bigwedge\{x \in J^\infty(\mathbb{A}) : \eta(x) \geqslant' b'\} \geqslant \bigwedge\{x \in J^\infty(\mathbb{A}) : \eta(x) \geqslant' a'\} = \eta_+(a').$$

$\square$

Proposition 1.18 If η is a *DMA*$^+$-homomorphism from $\mathbb{A}$ to $\mathbb{A}'$ then η_+ is an order-preserving bounded morphism between $\mathbb{A}'_+$ and $\mathbb{A}_+$.

Proof. Consider $a' \in J^\infty(\mathbb{A}')$. First we check that $\eta_+(a') \in J^\infty(\mathbb{A})$. Let $X = \{x \in J^\infty(\mathbb{A}) : \eta(x) \geqslant' a'\}$ so that $\eta_+(a') = \bigwedge X$, and observe that

$$a' \leqslant' \eta(\bigwedge X).$$

For since $a' \leqslant' \eta(x)$ for all $x \in X$ it follows that $a' \leqslant' \bigwedge' \eta[X] = \eta(\bigwedge X)$. Now by proposition 1.4, $\bigwedge X = \bigvee J^\infty(\downarrow \bigwedge X)$. Thus $\eta(\bigwedge X) = \bigvee' \eta[J^\infty(\downarrow \bigwedge X)]$, whence from $a' \leqslant' \eta(\bigwedge X)$ we obtain $a' \leqslant' \bigvee' \eta[J^\infty \downarrow \bigwedge X)]$. So since a' is join irreducible, and so join prime, $a' \leqslant' \eta(b)$ for some $b \in J^\infty(\mathbb{A})$ with $b \leqslant \bigwedge X$. As $b \in J^\infty(\mathbb{A})$ and $a' \leqslant' \eta(b)$ we have $b \in X$; so $b \geqslant \bigwedge X$. Thus since also $b \leqslant \bigwedge X$, $\bigwedge X = b \in X \subseteq J^\infty(\mathbb{A})$, and hence $\eta_+(a') = \bigwedge X \in J^\infty(\mathbb{A})$.

Proposition 1.17 (*iii*) ensures that η_+ is order-preserving.

It remains to show that

(*i*) $a' S'_\diamond b'$ implies $\eta_+(a') S_\diamond \eta_+(b')$ for $a', b' \in J^\infty(\mathbb{A}')$, and

(*ii*) $\eta_+(a') S_\diamond b$ implies $\exists b' \in J^\infty(\mathbb{A}')$ such that $a' S'_\diamond b'$ and $\eta_+(b') \leqslant b$, for $a' \in J^\infty(\mathbb{A})$, $b \in J^\infty(\mathbb{A})$,

and similarly for the other modal operators. For (*i*) suppose that $a' S'_\diamond b'$ for $a', b' \in J^\infty(\mathbb{A}')$. By proposition 1.17 (*ii*), we have $b' \leqslant' \eta(\eta_+(b'))$. Thus, using proposition 1.7(*i*), also $\diamond' b' \leqslant' \diamond' \eta(\eta_+(b') = \eta(\diamond \eta_+(b'))$. So since $a' \leqslant' \diamond' b'$, making use of proposition 1.4 again we have $a' \leqslant' \eta(\diamond \eta_+(b')) = \eta(\bigvee J^\infty(\downarrow \diamond \eta_+(b'))) = \bigvee' \eta[J^\infty(\downarrow \diamond \eta_+(b'))]$. So as a' is completely join prime, $a' \leqslant' \eta(c)$ for some $c \in J^\infty(\mathbb{A})$ with $c \leqslant \diamond \eta_+(b')$. Thus

$$\eta_+(a') = \bigwedge \{x \in J^\infty(\mathbb{A}) : \eta(x) \geqslant' a'\} \leqslant c \leqslant \diamond \eta_+(b').$$

So $\eta_+(a') \leqslant \diamond \eta_+(b')$; that is, $\eta_+(a') S_\diamond \eta_+(b')$.

And for (*ii*) suppose that $\eta_+(a') S_\diamond b$ for $a' \in J^\infty(\mathbb{A}')$ and $b \in J^\infty(\mathbb{A})$. From $\eta_+(a') S_\diamond b$ we have $\eta_+(a') \leqslant \diamond b$ from which it follows that $\eta(\eta_+(a')) \leqslant \eta(\diamond b)$. Thus by proposition 1.17 (*ii*), $a' \leqslant \eta(\diamond b)$. So since

$$\eta(\diamond b) = \diamond' \eta(b) = \diamond' \bigvee{}' \{x' \in J^\infty(\mathbb{A}') : x' \leqslant' \eta(b)\} = \bigvee{}' \{\diamond' x' : x' \in J^\infty(\mathbb{A}') \text{ and } x' \leqslant' \eta(b)\},$$

the complete join primeness of a' guarantees the existence of some $b' \in J^\infty(\mathbb{A}')$ such that $a' \leqslant' \diamond' b'$ and $b' \leqslant' \eta(b)$. Since $b' \leqslant' \eta(b)$, from proposition 1.17 (*i*) and (*iii*) we obtain

$\eta_+(b') \leqslant b$. Thus b' is as required.

The analogues of (i) and (ii) for the other modal operators follow by similar considerations. $\Box$

So far we have seen that the dual χ^+ of an order-preserving bounded morphism χ is a *DMA*$^+$-homomorphism, and that the dual η_+ of a *DMA*$^+$-homomorphism is an order-preserving bounded morphism. Thus, following the remarks in appendix B, proving the following propositions is what remains to establish the "morphism part" of the duality.

Proposition 1.19 Where $f_{\mathbb{F}}$ and $f_{\mathbb{F}'}$ are bijections as in proposition 1.12 from $\mathbb{F}$ to $(\mathbb{F}^+)_+$ and from $\mathbb{F}'$ to $(\mathbb{F}'^+)_+$, respectively, and χ is an order-preserving bounded morphism between $\mathbb{F}$ and $\mathbb{F}'$,

$$f_{\mathbb{F}'}(\chi(w)) = (\chi^+)_+(f_{\mathbb{F}}(w)) \text{ for all } w \in F.$$

Proof. For $w \in F$,

$$\begin{aligned}
(\chi^+)_+(f_{\mathbb{F}}(w)) &= (\chi^+)_+(\downarrow w) \\
&= \bigwedge\{x \in J^\infty(\mathcal{D}(\mathbb{F})) : \chi^+(x) \geqslant \downarrow w\} \\
&= \bigwedge\{x \in J^\infty(\mathcal{D}(\mathbb{F}')) : \downarrow w \subseteq \{u \in F : \chi(u) \in x\}\} \\
&= \bigcap\{\downarrow u' : u' \in F' \text{ such that } \chi(u) \leqslant u' \text{ for all } u \leqslant w\},
\end{aligned}$$

where this last equality follows from proposition 1.9. Thus since $f_{\mathbb{F}'}(\chi(w)) = \downarrow\chi(w)$, to prove the proposition it suffices to show that for all $v' \in F'$,

$$v' \leqslant \chi(w) \text{ iff } v' \leqslant u' \text{ for every } u' \in F' \text{ satisfying } \forall u \in F(u \leqslant w \to \chi(u) \leqslant u').$$

Consider $w \in F$ and $v' \in F'$. For the left to right direction, suppose $v' \leqslant \chi(w)$ and $\forall u \in F(u \leqslant w \to \chi(u) \leqslant u')$; we must show that $v' \leqslant u'$. Since $w \leqslant w$, the supposition ensures $\chi(w) \leqslant u'$. Thus $v' \leqslant \chi(w) \leqslant u'$. For the other direction observe that for $u \in F$, $u \leqslant w$ implies $\chi(u) \leqslant \chi(w)$. Thus, supposing the right hand side to hold, $v' \leqslant \chi(w)$. $\Box$

Proposition 1.20 Where $g_{\mathbb{A}}$ and $g_{\mathbb{A}'}$ are the bijections as in proposition 1.13 from $\mathbb{A}$ to $(\mathbb{A}_+)^+$ and from $\mathbb{A}'$ to $(\mathbb{A}'_+)^+$ respectively and η is a *DMA*$^+$-homomorphism from $\mathbb{A}$ to $\mathbb{A}'$,

$$g_{\mathbb{A}'}(\eta(a)) = (\eta_+)^+(g_{\mathbb{A}}(a)), \text{ for all } a \in A.$$

Proof. For $a \in A$, $g_{\mathbb{A}}(a) = J^\infty(\downarrow a)$ so

$$\begin{aligned}(\eta_+)^+(g_{\mathbb{A}}(a)) &= (\eta_+)^+ J^\infty(\downarrow a)\\ &= \{x \in J^\infty(\mathbb{A}') : \eta_+(x) \in J^\infty(\downarrow a)\}\\ &= \{x \in J^\infty(\mathbb{A}') : \eta_+(x) \leqslant a\}\\ &= \{x \in J^\infty(\mathbb{A}') : \bigwedge\{y \in J^\infty(\mathbb{A}) : \eta(y) \geqslant x\} \leqslant a\}.\end{aligned}$$

The third equality here follows from proposition 1.18, which ensures that $\eta_+(x) \in J^\infty(\mathbb{A})$. Thus since $g_{\mathbb{A}'}(\eta(a)) = J^\infty(\downarrow \eta(a))$, to prove the proposition it is enough to establish that for all $x \in J^\infty(\mathbb{A}')$ we have

$(*)$ $x \leqslant \eta(a)$ iff $\bigwedge\{y \in J^\infty(\mathbb{A}) : \eta(y) \geqslant x\} \leqslant a$.

Consider $x \in J^\infty(\mathbb{A}')$ and $a \in A$. For the right to left direction of $(*)$ observe that if $\bigwedge\{y \in J^\infty(\mathbb{A}) : \eta(y) \geqslant x\} \leqslant a$ then $\eta(\bigwedge\{y \in J^\infty(\mathbb{A}) : \eta(y) \geqslant x\} \leqslant \eta(a))$. So

$$x \leqslant \bigwedge\{\eta(y) : x \leqslant \eta(y) \text{ and } y \in J^\infty(\mathbb{A})\} = \eta(\bigwedge\{y \in J^\infty(\mathbb{A}) : \eta(y) \geqslant x\}) \leqslant \eta(a).$$

For the other direction, suppose $x \leqslant \eta(a)$. By proposition 1.4, $\eta(a) = \eta(\bigvee J^\infty(\downarrow a)) = \bigvee \eta[J^\infty(\downarrow a)]$. So $x \leqslant \bigvee \eta[J^\infty(\downarrow a)$. Thus since x is completely join prime, there is a $y \in J^\infty(\downarrow a)$ such that $x \leqslant \eta(y)$. So $\bigwedge\{y \in J^\infty(\mathbb{A}) : \eta(y) \geqslant x\} \leqslant a$. This establishes $(*)$ and completes the proof. $\square$

We now have all the results in place required to establish:

Theorem 1.21 The category $\mathcal{FR}$ of frames with order-preserving bounded morphisms, and the category DMA^+ of *DMA*$^+$'s with *DMA*$^+$-homomorphisms are dually equivalent.

Chapter 2

Subdirect Irreducibility of Perfect Distributive Modal Algebras

A characterization of subdirect irreducibility of *DMA*$^+$'s is provided (in §3). This characterization generalizes the slogan "a frame is rooted iff its dual is subdirectly irreducible" for Kripke frames. Some terminology is introduced in the first two sections.

For the remainder of this chapter, let $\mathbb{F} = (F, \leqslant, R_\Diamond, R_\Box, R_\rhd, R_\lhd)$ be a frame with dual $\mathbb{F}^+ = (\mathcal{D}(\mathbb{F}), \cup, \cap, \emptyset, F, \langle R_\Diamond \rangle, [R_\Box], [R_\rhd\rangle, \langle R_\lhd])$.

2.1 Order-Heredity

In this section the notions of order-heredity and order-root, which play a crucial role in §3, are introduced. We see also how these notions are natural analogues of the notions of order-heredity and root for Kripke frames.

Definition 2.1 A subset $c \subseteq F$ is an *order-hereditary* subset of $\mathbb{W}$ if for all $u, v \in F$,

(*i*) if $u \in c$ and $uR_\Diamond v$ then there is a $w \in c$ such that $uR_\Diamond w$ and $w \leqslant v$,

(*ii*) if $u \in c$ and $uR_\Box v$ then there is a $w \in c$ such that $uR_\Box w$ and $w \geqslant v$,

(*iii*) if $u \in c$ and $uR_\rhd v$ then there is a $w \in c$ such that $uR_\rhd w$ and $w \leqslant v$, and

(*iv*) if $u \in c$ and $uR_\lhd v$ then there is a $w \in c$ such that $uR_\lhd w$ and $w \geqslant v$.

A *proper order-hereditary subset* of $\mathbb{F}$ is an order-hereditary subset of $\mathbb{F}$ that is also a proper subset of F. The set of all order-hereditary subsets of $\mathbb{F}$ is denoted $o\mathcal{H}(\mathbb{F})$.

Definition 2.2 An *order-root* of $\mathbb{F}$ is an element of F not contained in any proper order-hereditary subset of $\mathbb{F}$.

The remainder of this section may make the notion of order-root seem more natural to readers familiar with the usual notion of root for Kripke frames. The notions of root and bounded-morphism for Kripke frames are as in [BdRV01] pp.138-9. A subset $c \subseteq W$ of the underlying set of a Kripke frame $\mathbb{W} = (W, R)$ is a *hereditary* subset of $\mathbb{W}$ if it is closed under R. That is, $c \subseteq W$ is hereditary iff $v \in c$ for all $u \in c$ and $v \in W$ with uRv. So a root of $\mathbb{W}$ is an element of W not contained in any proper hereditary subset of $\mathbb{W}$.

Thus heredity plays the same role in the notion of root for Kripke frames as order-heredity plays in the notion of order-root for frames. The following two propositions show that order-heredity can be characterized in terms of order-preserving bounded morphisms, and that an analogous characterization of heredity exists in terms of bounded morphisms for Kripke frames.

Proposition 2.3 For $c \subseteq F$, c is order-hereditary iff there is some order-preserving bounded morphism $\chi : F' \to F$ from a frame $\mathbb{F}' = (F', \leqslant', R'_\Diamond, R'_\Box, R'_\rhd, R'_\lhd)$ to $\mathbb{F}$ such that $\chi[F'] = c$.

Proof. First suppose $\chi : F' \to F$ is an order-preserving bounded morphism and $\chi[F'] = c$. To see that $\chi[F']$ satisfies (i) in the definition of order-hereditary, consider $u \in \chi[F']$ and $v \in F$ for which $uR_\Diamond v$; we must ensure that there is some $w \in \chi[F']$ such that $uR_\Diamond w$ and $w \leqslant v$. Well, since $uR_\Diamond v$ and $u \in \chi[F']$ there is some $u' \in F'$ such that $\chi(u')R_\Diamond v$. Thus $(Biii)(a)$ guarantees the existence of some $v' \in F'$ such that $u'R'_\Diamond v'$ and $\chi(v') \leqslant v$. Making use of (Bii) from $u'R'_\Diamond v'$ we obtain $\chi(u')R_\Diamond \chi(v')$. Thus $w := \chi(v')$ is as required. Analogous use of $(Biii)(b)$, (c) and (d) respectively establishes that $\chi[F'] = c$ satisfies (ii), (iii) and (iv) in the definition of order-hereditary.

For the converse, suppose c is order-hereditary. Let $\mathbb{F}_c = (c, \leqslant_c, R_{\Diamond c}, R_{\Box c}, R_{\rhd c}, R_{\lhd c})$ be the frame with underlying set c and order and relations the respective restrictions to c of the order and relations of $\mathbb{F}$. Thus, for example, $R_{\Diamond c} = R_\Diamond \cap c^2$. Observe that $\mathbb{F}_c$ is indeed a frame. To see that $\mathbb{F}_c$ satisfies $(R1)$, consider $t, u, v, w \in c$ for which $t \leqslant_c uR_{\Diamond c}v \leqslant w$. Then also $t \leqslant uR_\Diamond v \leqslant w$ and thus since $\mathbb{F}$ satisfies $(R1)$ we have $tR_\Diamond w$. But then since $t, w \in c$ also $tR_{\Diamond c}w$, as required to ensure that $\mathbb{F}_c$ satisfies $(R1)$. Checking that $\mathbb{F}_c$ satisfies the other relation conditions is similarly straightforward.

Now define $\chi_c : c \to F$ by $\chi_c : w \mapsto w$, and observe that χ_c is an order-preserving bounded morphism from $\mathbb{F}_c$ to $\mathbb{F}$. Certainly χ_c is order-preserving and satisfies (Bii). To see that χ_c satisfies $(Biii)(a)$, suppose $\chi_c(u)R_\diamond v$ for $u \in c$ and $v \in F$. Then $uR_\diamond v$. So since c is order-hereditary, there is some $w \in c$ for which $uR_\diamond w$ and $w \leqslant v$. But then $uR_{\diamond c}w$ and $\chi(w) \leqslant v$. Analogous use of the clause (ii), (iii) and (iv) of the definition of order-hereditary respectively establishes that χ_c satisfies each of $(Biii)(b)$, (c) and (d). And certainly $\chi(c) = c$. $\square$

This last proposition, as well as the definition of χ_c occurring in its proof, will be useful later.

Proposition 2.4 Consider a Kripke frame $\mathbb{W} = (W, R)$. For $c \subseteq W$, c is a hereditary subset of $\mathbb{W}$ iff there is some bounded morphism $\chi : W' \to W$ from a Kripke frame $\mathbb{W}' = (W', R')$ to $\mathbb{W}$ such that $\chi[W'] = c$.

Proof. This is a special case of the previous proposition, where the ordering on frames is taken to be discrete and the relations $R_\Box$, $R_\triangleright$ and $R_\triangleleft$ are taken to be empty. $\square$

2.2 Complete Congruences

As explained in the final section of appendix B, for each category $\mathcal{C}$ to which an algebra belongs there corresponds a notion of subdirect irreducibility (with respect to $\mathcal{C}$). We will be concerned with the notion of subdirect irreducibility corresponding to the category $\mathcal{DMA}^+$ of *DMA*$^+$'s with *DMA*$^+$-homomorphism; this notion will be referred to by "subdirect irreducibility (with respect to $\mathcal{DMA}^+$)".

A *complete congruence* of $\mathbb{A}$ is the kernel of a *DMA*$^+$-homomorphism. The set $Con^+(\mathbb{A})$ of complete congruences of $\mathbb{A}$ forms a lattice under $\subseteq$ (see [BS81]). Notice that the trivial congruence $\{(a, a) : a \in A\}$ is the bottom element of the lattice $(Con^+(\mathbb{A}), \leqslant)$.

The following, then, characterizes the relevant notion of subdirect irreducibility of a nontrivial *DMA*$^+$:

Proposition 2.5 A nontrivial *DMA*$^+$ $\mathbb{A}$ is subdirectly irreducible (with respect to *DMA*$^+$'s) iff there is a smallest nontrivial element of $(Con^+(\mathbb{A}), \leqslant)$.

2.3 Subdirect Irreducibility Results

In this section it will be shown, via a correspondence between order-hereditary subsets of $\mathbb{F}$ and complete congruences of $\mathbb{F}^+$, that $\mathbb{F}$ has an order-root iff $\mathbb{F}^+$ is subdirectly irreducible (with respect to DMA^+'s). The following observation is used in establishing this correspondence:

Proposition 2.6 Let $\chi : F' \to F$ be an order-preserving bounded morphism from a frame with underlying set F' to $\mathbb{F}$. Then for all $a, b \in \mathcal{D}(\mathbb{F})$,

$$(a, b) \in \mathsf{ker}\chi^+ \text{ iff } \chi(F') \cap a = \chi(F') \cap b.$$

Proof. We have

$$\begin{aligned}(a, b) \in \mathsf{ker}\chi^+ \ &\text{iff } \chi^+(a) = \chi^+(b),\\ &\text{iff } \{w' \in F' : \chi(w') \in a\} = \{w' \in F' : \chi(w') \in b\},\\ &\text{iff } \chi(F') \cap a = \chi(F') \cap b.\end{aligned}$$

□

Proposition 2.7 The ordered set $(o\mathcal{H}(\mathbb{F}), \supseteq)$ is isomorphic to the congruence lattice $(Con^+(\mathbb{F}^+), \leqslant)$ of $\mathbb{F}^+$.

Proof. An isomorphism $\varepsilon : o\mathcal{H}(\mathbb{F}) \to Con^+(\mathbb{F}^+)$ will be defined. Consider an order-hereditary subset c of $\mathbb{F}$. Where $\chi_c : c \to F$ is the order-preserving bounded morphism as in the proof of proposition 2.3, the duality results of the previous chapter ensure that $\chi_c^+ : \mathbb{F}^+ \to \mathbb{F}_c^+$ is a DMA^+-homomorphism. And hence $\mathsf{ker}\chi_c^+ \in Con^+(\mathbb{F}^+)$. This licenses the definition of $\varepsilon : o\mathcal{H}(\mathbb{F}) \to Con^+(\mathbb{F}^+)$ by

$$\varepsilon : c \mapsto \ker \chi_c^+.$$

To prove the proposition it now suffices to show that ε is a surjection satisfying $c \subseteq d$ iff $\varepsilon(d) \leqslant \varepsilon(c)$ for all $c, d \in o\mathcal{H}(\mathbb{F})$.

To see that ε is surjective, consider $\theta \in Con^+(\mathbb{F}^+)$. The duality results guarantee the existence of an order-preserving bounded morphism $\chi : F' \to F$ between a frame $\mathbb{F}' = (F', \leqslant' R'_\Diamond, R'_\Box, R'_\triangleright, R'_\triangleleft)$ and $\mathbb{F}$ such that $\mathsf{ker}\chi^+ = \theta$. Let $c \subseteq F$ be the image $\chi[F']$ of χ. Then proposition 2.3 ensures that c is order-hereditary. Now making use of proposition 2.6, for $a, b \in \mathcal{D}(\mathbb{F})$ we have $(a, b) \in \theta = \mathsf{ker}\chi^+$, iff $c \cap a = c \cap b$, iff $(a, b) \in \mathsf{ker}\chi_c^+$. Thus $\theta = \mathsf{ker}\chi_c^+ = \varepsilon(c)$, and so

ε is surjective.

Next suppose $c \subseteq d$ for $c, d \in o\mathcal{H}(\mathbb{F})$; we must show that $\mathsf{ker}\chi_d^+ \leqslant \mathsf{ker}\chi_c^+$. For this suppose $(a, b) \in \mathsf{ker}\chi_d^+$, where $a, b \in \mathcal{D}(\mathbb{F})$. Then by proposition 2.6, $d \cap a = d \cap b$. But then since $c \subseteq d$ also $c \cap a = c \cap b$. And thus, again using proposition 2.6, $(a, b) \in \mathsf{ker}\chi_c^+$.

Finally suppose $c \nsubseteq d$ for $c, b \in o\mathcal{H}(\mathbb{F})$, considering $w \in c \setminus d$. Then $d \cap {\downarrow}w = d \cap ({\downarrow}w \setminus \{w\})$, but $c \cap {\downarrow}w \neq c \cap ({\downarrow}w \setminus \{w\})$. So by proposition 2.6, $({\downarrow}w, {\downarrow}w \setminus \{w\}) \in \mathsf{ker}\chi_d^+$ but $({\downarrow}w, {\downarrow}w \setminus \{w\}) \notin \mathsf{ker}\chi_c^+$. Thus $\mathsf{ker}\chi_d^+ \nleqslant \mathsf{ker}\chi_c^+$. $\square$

Notice that from this last proposition it follows that $o\mathcal{H}(\mathbb{F})$ forms a lattice under $\supseteq$; this lattice has top element $\emptyset$ and bottom element F. Thus we have so far seen how the lattice of order-hereditary subsets of a frame corresponds to the congruence lattice of its dual. From this correspondence it follows that a *DMA*$^+$ has a smallest nontrivial complete congruence iff its dual has a greatest proper order-hereditary subset. Thus we can arrive at a dual characterization of subdirect irreducibility for *DMA*$^+$'s by investigating the conditions under which a frame has a greatest proper order-hereditary subset.

Proposition 2.8 $\mathbb{F}$ has a greatest proper order-hereditary subset iff $\mathbb{F}$ has an order-root.

Proof. First suppose $a \subseteq F$ is the greatest proper order-hereditary subset of $\mathbb{F}$. Since a is proper, there is some $w \in F \setminus a$. Consider any order-hereditary subset b of $\mathbb{F}$ such that $w \in b$. Since a is the greatest proper order-hereditary subset of $\mathbb{F}$ but $b \nsubseteq a$, b is not a proper subset of F. Thus w is an order-root of $\mathbb{F}$.

Suppose for the converse that $\mathbb{F}$ has an order-root, so that the set a of elements of F which are not order-roots is a proper subset of F. To see that a satisfies clause (i) of the definition of order-hereditary, consider $w, v \in F$ such that $w \in a$ and $wR_\Diamond v$. We must ensure that there is some $u \in a$ for which $wR_\Diamond u$ and $u \leqslant v$. Since w is not an order-root, there is some proper order-hereditary subset b containing w. Then using $w \in b$ and $wR_\Diamond v$, the order-heredity of b guarantees the existence of some $u \in b$ such that $wR_\Diamond u$ and $u \leqslant v$. Since b is a proper order-hereditary subset, u is not an order-root and hence $u \in a$. Establishing that a satisfies the remaining conditions for being order-hereditary proceeds by analogous reasoning. Now if $c \nsubseteq a$ for $c \subseteq F$ then c must contain some order-root; so if c is order-hereditary then $c = F$. This establishes that a is the greatest proper order-hereditary subset of $\mathbb{F}$. $\square$

Thus we arrive at:

Theorem 2.9 $\mathbb{F}$ has an order-root iff $\mathbb{F}^+$ is subdirectly irreducible (with respect to $\mathcal{DMA}^+$).

Proof. $\mathbb{F}^+$ is subdirectly irreducible (with respect to $\mathcal{DMA}^+$) iff $\mathbb{F}^+$ has a least nontrivial complete congruence. And $\mathbb{F}^+$ has a least nontrivial complete congruence iff (by proposition 2.7) $\mathbb{F}$ has a greatest proper order-hereditary subset, iff (by proposition 2.8) $\mathbb{F}$ has an order-root. $\square$

2.4 Duals of Kripke Frames

The aim of this section is to show that the above characterization generalizes the familiar characterization according to which a Kripke frame is rooted iff its dual is subdirectly irreducible. For this we must more carefully state that familiar characterization, specifying the relevant notion of subdirect irreducibility.

Recall that, as explained in appendix B, there are different notions of subdirect irreducibility for an algebra corresponding to the different categories to which the algebra belongs. Modal algebras are a special case of *DMA*'s, and the homomorphisms for modal algebras are simply *DMA*-homomorphisms between modal algebras. That is, a homomorphism η between two modal algebras $\mathbb{B}$ and $\mathbb{B}'$ with underlying sets B and B' and modal operators $\Box$ and $\Box'$ respectively is a homomorphism between the lattice reducts of $\mathbb{B}$ and $\mathbb{B}'$ such that $\eta(\Box b) = \Box'\eta(b)$ for $b \in B$. The category of modal algebras with these homomorphisms is denoted $\mathcal{MA}$. This is the category corresponding to the notion of subdirect irreducibility figuring in the familiar characterization mentioned above.

Let $\mathbb{W} = (W, R)$ be a Kripke frame. Its dual $\mathbb{W}^\#$ is defined by

$$\mathbb{W}^\# = (\mathcal{P}(W), \cup, \cap, \setminus, \emptyset, W, [R]),$$

where $\mathcal{P}(W)$ is the power set of W and $[R]$ is as in definition 1.8. Complete atomic and completely additive modal algebras, or *CAMA*'s, are those modal algebras $\mathbb{B} = (B, \vee, \wedge, \neg, 0, 1, \Box)$ isomorphic to the dual of some Kripke frame (see [Tho75] for a proof of this and definition of *CAMA*). Now we are ready to state the characterization:

Proposition 2.10 $\mathbb{W}^\#$ is subdirectly irreducible (with respect to $\mathcal{MA}$) iff $\mathbb{W}$ has a root.

Proofs of this (as in [Kra99] or [Sam99]) follow a different route than the proof of theorem 2.9.

A homomorphism η between *CAMA*'s $\mathbb{B}$ (with underlying set B) and $\mathbb{B}'$ with infinite meets $\bigwedge$ and $\bigwedge'$ respectively is a *complete homomorphism* if

$$\eta(\bigwedge X) = \bigwedge{}' \eta[X], \text{ for } X \subseteq B.$$

Notice that complete homomorphisms do coincide with *DMA*$^+$-homomorphisms between *CAMA*'s. The category of *CAMA*'s with complete homomorphisms is denoted $\mathcal{CAMA}$. Thus theorem 2.9 does generalize the proposition:

Proposition 2.11 $\mathbb{W}^{\#}$ is subdirectly irreducible (with respect to $\mathcal{CAMA}$) iff $\mathbb{W}$ has a root.

So to show that theorem 2.9 generalizes the characterization 2.10, it is enough to show that the two notions of subdirect irreducibility introduced here for *CAMA*'s coincide. For this we consider the characterization of subdirect irreducibility stated in terms of congruences.

A *congruence* of a *CAMA* $\mathbb{B}$ is a kernel of a homomorphism from $\mathbb{B}$. A *complete congruence* θ of a *CAMA* $\mathbb{B}$ with underlying set B and infinite meet $\bigwedge$ is a congruence such that for $\{a_i, b_i : i \in I\} \subseteq B$,

$$a_i \theta b_i \text{ for all } i \in I \text{ implies } \bigwedge\{a_i : i \in I\} \ \theta \ \bigwedge\{b_i : i \in I\}.$$

Observe that the complete congruences of a *CAMA* $\mathbb{B}$ are the kernels of complete homomorphisms from $\mathbb{B}$. Thus, returning to the dual $\mathbb{W}^{\#}$ of $\mathbb{W}$, we have:

Proposition 2.12

- (i) $\mathbb{W}^{\#}$ is subdirectly irreducible (with respect to $\mathcal{MA}$) iff $\mathbb{W}^{\#}$ has a smallest nontrivial congruence.
- (ii) $\mathbb{W}^{\#}$ is subdirectly irreducible (with respect to $\mathcal{CAMA}$) iff $\mathbb{W}^{\#}$ has a smallest nontrivial complete congruence.

Thomason (see [Tho75]) provides a duality between $\mathcal{CAMA}$ and the category of Kripke frames with bounded morphisms. Via this duality and proposition 2.4 we can obtain the analogues of propositions 2.7, 2.8 and 2.9:

Proposition 2.13 The lattice $(\mathcal{H}(\mathbb{W}), \supseteq)$ of hereditary subsets of $\mathbb{W}$ is isomorphic to the lattice $(Con^{+}(\mathbb{W}^{\#}), \leqslant)$ of complete congruences of $\mathcal{W}^{\#}$.

Proposition 2.14 $\mathbb{W}$ has a greatest proper hereditary subset iff $\mathbb{W}$ has a root.

Proposition 2.15 $\mathbb{W}$ has a root iff $\mathbb{W}^{\#}$ is subdirectly irreducible (with respect to $\mathcal{CAMA}$).

These propositions follow by similar reasoning as that used to establish their analogues in the previous section. The proofs will not be provided; the aim of this section is to show that the analogue (proposition 2.15) of the subdirect irreducibility characterization (proposition 2.9) in the previous section is equivalent to the familiar characterization (proposition 2.10) of subdirect irreducibility for the duals of Kripke frames. That is, we will see that:

Proposition 2.16 $\mathbb{W}^{\#}$ is subdirectly irreducible iff $\mathbb{W}^{\#}$ is subdirectly irreducible (with respect to $\mathcal{CAMA}$).

Proof. This follows from proposition 2.23 below together with proposition 2.12. $\square$

To prove this proposition, we will see that although there are congruences which are not complete, $\mathbb{W}^{\#}$ has a least nontrivial congruence iff $\mathbb{W}^{\#}$ has a least nontrivial complete congruence. This is established via a correspondence between complete congruences and certain filters of $\mathbb{W}^{\#}$.

A *modal filter* $F \subseteq \mathcal{P}(W)$ of $\mathbb{W}^{\#}$ is a filter such that $a \in F$ implies $[R]a \in F$ for all $a \in \mathcal{P}(W)$. A *complete filter* on $\mathbb{W}^{\#}$ is a filter F of $\mathbb{W}^{\#}$ such that $\bigcap X \in F$ for all $X \subseteq F$.

Where F is a modal filter on $\mathbb{W}^{\#}$, let $\theta_F \subseteq \mathcal{P}(W) \times \mathcal{P}(W)$ be defined by $a\theta_F b$ iff $(W \setminus ((a \cup b) \setminus (a \cap b)) \in F$. And where θ is a congruence on $\mathbb{W}^{\#}$, let $F_\theta = \{a \in \mathcal{P}(W) : a\theta W\}$. Then (see [SV88] p.280 for a proof) the following correspondence obtains:

Proposition 2.17 The map $F \mapsto \theta_F$ is an isomorphism between the lattice of modal filters of $\mathbb{W}^{\#}$ and the congruence lattice of $\mathbb{W}^{\#}$, and has inverse $\theta \mapsto F_\theta$.

Notice that the top element $\{(a, b) : a, b \in \mathcal{P}(W)\}$ of the lattice of congruences corresponds to the top element $\mathcal{P}(W)$ of the lattice of filters, and that the trivial congruence $\{(a, a) : a \in \mathcal{P}(W)\}$ corresponds to the trivial filter $\{W\}$.

Proposition 2.18 Let F be a filter of $\mathbb{W}^\#$. Then F is a complete filter iff θ_F is a complete congruence.

Proof. For the left to right direction, suppose F is a complete modal filter and consider $\{a_i, b_i : i \in I\} \subseteq \mathcal{P}(W)$ such that $a_i \theta_F b_i$ for all $i \in I$. Then $W \setminus ((a_i \cup b_i) \setminus (a_i \cap b_i)) \in F$ for all $i \in I$, and thus the completeness of F guarantees that also $\bigcap_{i \in I} W \setminus ((a_i \cup b_i) \setminus (a_i \cap b_i)) \in F$. Observe that

$$\bigcap_{i \in I} W \setminus ((a_i \cup b_i) \setminus (a_i \cap b_i)) \;\subseteq\; W \setminus ((\bigcap_{i \in I} a_i \cup \bigcap_{i \in I} b_i) \setminus (\bigcap_{i \in I} a_i \cap \bigcap_{i \in I} b_i)).$$

To see this, consider $w \in \bigcap_{i \in I} W \setminus ((a_i \cup b_i) \setminus (a_i \cap b_i))$. Then for all $i \in I$, $w \in (a_i \cup b_i)$ implies $w \in (a_i \cap b_i)$. Thus $w \in a_i \cup b_i$ for all $i \in I$ implies $w \in (a_i \cap b_i)$ for all $i \in I$. And hence $w \in \bigcap a_i \cup \bigcap b_i$ implies $w \in \bigcap a_i \cap \bigcap b_i$. But then $w \in W \setminus ((\bigcap_{i \in I} a_i \cup \bigcap_{i \in I} b_i) \setminus (\bigcap_{i \in I} a_i \cap \bigcap_{i \in I} b_i))$. This establishes the displayed inclusion; from it together with $\bigcap_{i \in I} W \setminus ((a_i \cup b_i) \setminus (a_i \cap b_i)) \in F$, we obtain $W \setminus ((\bigcap_{i \in I} a_i \cup \bigcap_{i \in I} b_i) \setminus (\bigcap_{i \in I} a_i \cap \bigcap_{i \in I} b_i)) \in F$. Hence $\bigcap_{i \in I} a_i \theta_F \bigcap_{i \in I} b_i$.

And for the converse, suppose θ_F is a complete congruence. Consider $A \subseteq F$. By the previous proposition, $F = F_{\theta_F}$, and thus $a \theta_F W$ for all $a \in A$. So by the completeness of θ_F, also $\bigcap A$ θ_F W; hence $\bigcap A \in F_{\theta_F} = F$. Thus F is a complete modal filter. $\square$

It follows from the preceding two propositions that:

Proposition 2.19 $\mathbb{W}^\#$ has a least nontrivial congruence iff $\mathbb{W}^\#$ has a least nontrivial modal filter, and $\mathbb{W}^\#$ has a least nontrivial complete congruence iff $\mathbb{W}^\#$ has a least nontrivial complete modal filter.

So to show that $\mathbb{W}^\#$ has a least nontrivial congruence iff $\mathbb{W}^\#$ has a least nontrivial complete congruence, we need only ensure that $\mathbb{W}^\#$ has a least nontrivial modal filter iff $\mathbb{W}^\#$ has a least nontrivial complete modal filter. This is made easier by the following two observations:

Proposition 2.20 For $a \in \mathcal{P}(W)$, $\uparrow a$ is a modal filter iff $a \subseteq [R]a$.

Proof. Certainly $\uparrow a$ is a filter, and if $\uparrow a$ is a modal filter then $a \subseteq [R]a$. Suppose $a \subseteq [R]a$. Then for any $b \in \uparrow a$, $a \subseteq b$ and thus $[R]a \subseteq [R]b$. So $a \subseteq [R]b$, and hence $[R]b \in \uparrow a$. $\square$

Proposition 2.21 Where F is a filter on $\mathbb{W}^\#$, F is complete iff F is principal.

Proof. First suppose F is complete. Then since $F \subseteq F$, $\bigcap F \in F$. So $a \in F$ for all $a \supseteq \bigcap F$. And if $a \in \mathcal{P}(W)$ is such that $a \not\supseteq \bigcap F$ then $a \notin F$. Thus $F = \uparrow(\bigcap F)$ and so F is principal.

Conversely, suppose F is principal, so $F = \uparrow a$ for some $a \in \mathcal{P}(W)$. For $B \subseteq F$ we have $a \subseteq b$ for all $b \in B$. Thus $a \supseteq \bigcap B$ and thus $\bigcap B \in F$. $\square$

Now it is easy to see that there are incomplete modal filters (and hence that there are incomplete congruences): if $[R]a = W$ for all $a \in \mathcal{P}(W)$ then any nonprincipal filter on $\mathbb{W}^{\#}$ is an incomplete modal filter. However:

Proposition 2.22 $\mathbb{W}^{\#}$ has a least nontrivial modal filter iff $\mathbb{W}^{\#}$ has a least nontrivial complete modal filter.

Proof. First observe that if F is a modal filter, then $\uparrow(\bigcap F)$ is the least complete modal filter containing F. Certainly $\uparrow(\bigcap F)$ is a filter. By proposition 2.20, to show that $\uparrow(\bigcap F)$ is a modal filter, it is enough to show that $\bigcap F \subseteq [R](\bigcap F)$. Consider $w \in \bigcap F$, so that $w \in a$ for all $a \in F$. Since F is a modal filter, $\{[R]a : a \in F\} \subseteq F$. Thus $w \in [R]a$ for all $a \in F$. That is, $w \in \bigcap\{[R]a : a \in F\} \subseteq [R](\bigcap F)$. So $\uparrow(\bigcap F)$ is a modal filter, and is complete by proposition 2.21. And if G is a complete modal filter with $F \subseteq G$ then $\bigcap F \in G$, and thus $\uparrow(\bigcap F) \subseteq G$.

Now the left to right direction of the proposition is immediate. For suppose F is the least nontrivial modal filter on $\mathbb{W}^{\#}$. Then $\uparrow(\bigcap F)$ is a complete modal filter as above, and is nontrivial as it contains the nontrivial F. Consider a nontrivial complete modal filter G with $G \subseteq \uparrow(\bigcap F)$. Then $F \subseteq G$, since F is the smallest nontrivial modal filter. And hence $G = \uparrow(\bigcap F)$ since $\uparrow(\bigcap F)$ is the smallest complete modal filter containing F. Thus the least nontrivial complete modal filter must be $\uparrow(\bigcap F)$.

For the converse, suppose $\mathbb{W}^{\#}$ has a least nontrivial complete modal filter F. Then by proposition 2.21, $F = \uparrow a$ for some $a \in \mathcal{P}(W)$. Since F is nontrivial, $\uparrow a$ is not the trivial filter $\{W\}$. Thus $a \neq W$ and hence there is some $w \in W \setminus a$. Now since the intersection of a collection of modal filters is again a filter, if the intersection of all nontrivial modal filters is nonempty then that intersection must be the least nontrivial modal filter. Thus to show that there is a least nontrivial modal filter, it suffices to ensure that $W \setminus \{w\}$ is in every nontrivial modal filter. So consider a nontrivial modal filter G. Since G is nontrivial, there is some

$b \in G$ such that $b \neq W$. Then $\bigcap G \subseteq b$ and so $b \in \uparrow(\bigcap G)$. Thus $\uparrow(\bigcap G)$ is nontrivial. By proposition 2.20 $\uparrow(\bigcap G)$ is complete, and by the reasoning of the previous paragraph $\uparrow(\bigcap G)$ is modal. Thus, since F is the least nontrivial complete modal filter, $F = \uparrow a \subseteq \uparrow(\bigcap G)$. Hence $\bigcap G \subseteq a$, and so $w \notin \bigcap G$. That is, $w \notin c$ for some $c \in G$. From this it follows that $c \subseteq W \setminus \{w\}$, and thus $W \setminus \{w\} \in G$. □

Proposition 2.23 $\mathbb{W}^{\#}$ has a least nontrivial congruence iff $\mathbb{W}^{\#}$ has a least nontrivial complete congruence.

Proof. This is an immediate consequence of proposition 2.19 and proposition 2.22. □

Thus we have established proposition 2.16.

Chapter 3

Representing Distributive Modal Algebras

In this chapter the category $\mathcal{PR}$ of extended Priestley spaces with continuous order-preserving bounded morphisms is shown to be dually equivalent to the category $\mathcal{DMA}$ of *DMA*'s with *DMA*-homomorphisms. This duality is an extension of Priestley duality, as presented in the appendix. The chapter follows the same plan as that of chapter 1: the "object part" of the duality is established in the first section, and the "morphism part" in the second.

3.1 Extended Priestley Spaces

In this section we see that *DMA*'s can be represented as extended Priestley spaces, to be introduced below. The dual $\mathbb{X}^*$ of an extended Priestley space $\mathbb{X}$ is defined and shown to be a *DMA*, and the dual $\mathbb{A}_*$ of a *DMA* is defined and shown to be an extended Priestley space. We see then that $\mathbb{X} \cong (\mathbb{X}^*)_*$ for an extended Priestley space $\mathbb{X}$, and that $\mathbb{A} \cong (\mathbb{A}_*)^*$ for a *DMA* $\mathbb{A}$. That is, the "object part" of the duality of this section is established.

An extended Priestley space consists of a Priestley space together with some additional relations. See section A.1 of the appendix for a definition of Priestley space, as well as definitions of the upper and lower topologies for a Priestley space.

Definition 3.1 An *extended Priestley space* is a tuple $\mathbb{X} = (X, \leqslant, \tau, R_\Diamond, R_\Box, R_\rhd, R_\lhd)$ such that $(X, \leqslant, \tau)$ is a Priestley space, $R_\Diamond, R_\Box, R_\rhd, R_\lhd \subseteq X \times X$ satisfy the relation conditions of definition 1.1 and

(CL) If $a \in \mathrm{cl}\mathcal{D}(\mathbb{X})$ then so are $\langle R_\Diamond\rangle a, [R_\Box]a, [R_\rhd\rangle a$ and $\langle R_\lhd]a$,

(TOP) For all $x \in X$, $R_\Diamond[x]$ and $R_\rhd[x]$ are closed in the upper topology for $(X, \leqslant, \tau)$, and $R_\Box[x]$ and $R_\lhd[x]$ are closed in the lower topology for $(X, \leqslant, \tau)$.

Recall that the definitions of $\langle R\rangle, [R], [R\rangle$ and $\langle R]$ for a binary relation R are given by definition 1.8. And for $R \subseteq X \times X$ with $x \in X$, $R[x]$ denotes the set of x's R-successors. That is, $R[x] = \{y \in X : xRy\}$. $\mathrm{cl}\mathcal{D}(\mathbb{X})$ is the set of clopen down-sets of $(X, \leqslant, \tau)$.

The dual $\mathbb{X}^*$ of an extended Priestley space $\mathbb{X} = (X, \leqslant, \tau, R_\Diamond, R_\Box, R_\rhd, R_\lhd)$ is defined by

$$\mathbb{X}^* = (\mathrm{cl}\mathcal{D}(\mathbb{X}), \cup, \cap, \emptyset, X, \langle R_\Diamond\rangle, [R_\Box], [R_\rhd\rangle, \langle R_\lhd]).$$

We must check that $\mathbb{X}^*$ is indeed a distributive modal algebra.

Proposition 3.2 If $\mathbb{X}$ is an extended Priestley space, $\mathbb{X}^*$ is a *DMA*.

Proof. Let $\mathbb{X} = (X, \leqslant, \tau, R_\Diamond, R_\Box, R_\rhd, R_\lhd)$ so that $\mathbb{X}^* = (\mathrm{cl}\mathcal{D}(\mathbb{X}), \cup, \cap, \emptyset, X, \langle R_\Diamond\rangle, [R_\Box], [R_\rhd\rangle, \langle R_\lhd])$. By proposition A.3, $(\mathrm{cl}\mathcal{D}(\mathbb{X}), \cup, \cap, \emptyset, X)$ is a bounded distributive lattice. It remains to check that $\langle R_\Diamond\rangle, [R_\Box], [R_\rhd\rangle, \langle R_\lhd]$ are modal operators satisfying the conditions of definition 1.2. $\langle R_\Diamond\rangle$ is an operator on $\mathrm{cl}\mathcal{D}(\mathbb{X})$ by (CL) above, and the conditions we must check are:

(i) $\langle R_\Diamond\rangle\emptyset = \emptyset$, and

(ii) $\langle R_\Diamond\rangle(a \cup b) = \langle R_\Diamond\rangle a \cup \langle R_\Diamond\rangle b$.

For (i) we have $\langle R_\Diamond\rangle\emptyset = \{x : \exists y(xR_\Diamond y \text{ and } y \in \emptyset)\} = \emptyset$. And for (ii) observe that

$$\begin{aligned}\langle R_\Diamond\rangle(a \cup b) &= \{x : \exists y(xR_\Diamond y \text{ and } y \in a \cup b)\}\\ &= \{x : \exists y(xR_\Diamond y \text{ and } y \in a)\} \cup \{x : \exists y(xR_\Diamond y \text{ and } y \in b)\}\\ &= \langle R_\Diamond\rangle a \cup \langle R_\Diamond\rangle b.\end{aligned}$$

Checking that the relevant conditions are satisfied for the other modal operators is similarly straightforward. □

The dual $\mathbb{A}_*$ of a *DMA* $\mathbb{A} = (A, \vee, \wedge, 0, 1, \Diamond, \Box, \rhd, \lhd)$ is defined by

$$\mathbb{A}_* = (\mathcal{I}_\mathcal{P}(\mathbb{A}), \subseteq, \tau, Q_\Diamond, Q_\Box, Q_\rhd, Q_\lhd),$$

where $(\mathcal{I}_{\mathcal{P}}(\mathbb{A}), \subseteq, \tau)$ is the Priestley dual of $(A, \vee, \wedge, 0, 1)$ as in section A.1. and $Q_\diamond, Q_\Box, Q_\rhd, Q_\lhd \subseteq \mathcal{I}_{\mathcal{P}}(\mathbb{A}) \times \mathcal{I}_{\mathcal{P}}(\mathbb{A})$ are defined by

$$\begin{aligned} IQ_\diamond J \text{ iff } \quad & \diamond^{-1} I && \subseteq J, \\ IQ_\Box J \text{ iff } \quad & \Box J && \subseteq I, \\ IQ_\rhd J \text{ iff } \quad & \rhd(A \setminus J) && \subseteq I, \\ IQ_\lhd J \text{ iff } \quad & \lhd J && \subseteq A \setminus I. \end{aligned}$$

Here $\vartriangle I$ is the set $\{\vartriangle a : a \in I\}$ for $I \subseteq A$ and modal operator $\vartriangle$. Similarly, $\vartriangle^{-1} I$ denotes the set $\{a \in A : \vartriangle a \in I\}$.

Remark Notice that since I is a prime ideal iff $A \setminus I$ is a prime filter, there is a closely related alternative notion of the dual of a *DMA*, stated in terms of filters rather than ideals. This dual is

$$\mathbb{A}_{\tilde{*}} = (\mathcal{F}_P(\mathbb{A}), \subseteq, \tau, \tilde{Q}_\diamond, \tilde{Q}_\Box, \tilde{Q}_\rhd, \tilde{Q}_\lhd),$$

where τ is as above, $\mathcal{F}_P(\mathbb{A})$ is the set of prime filters of $\mathbb{A}$ and for each modal operator $\vartriangle$ $\tilde{Q}_\vartriangle \subseteq \mathcal{F}_P(\mathbb{A}) \times \mathcal{F}_P(\mathbb{A})$ is given by

$$F\tilde{Q}_\vartriangle G \text{ iff } A \setminus F \ Q_\vartriangle \ A \setminus G.$$

Thus, for example, $F\tilde{Q}_\vartriangle G$ iff $\diamond a \in F$ for all $a \in G$. Similarly, $F\tilde{Q}_\Box G$ iff $a \in G$ for all $\Box a \in F$. This approach might look more familiar to readers used to the ultrafilter frame representation of boolean algebras with operators, as in [BdRV01] pp.287-291.

We must ensure that $\mathbb{A}_*$ is an extended Priestley space. Propositions 3.3 and 3.4 below are used to prove proposition 3.5, which is in turn used to show that $\mathbb{A}_*$ satisfies (TOP). All three of these next propositions will also come in useful later.

Proposition 3.3 Suppose I is an ideal and F is a filter of a *DMA* $\mathbb{A}$. If $F \cap I = \emptyset$ then there is a $J \in \mathcal{I}_{\mathcal{P}}(\mathbb{A})$ such that $I \subseteq J$ and $F \cap J = \emptyset$.

Remark. This last proposition, known as the prime ideal theorem, is introduced as an additional assumption rather than proved. It follows from the axiom of choice, and in fact is equivalent to a finitary version according to which every family of non-empty finite sets has a choice function. See [DP02] pp.187-190 for a discussion of this. $\Box$

Proposition 3.4 Let $\mathbb{A} = (A, \vee, \wedge, 0, 1, \Diamond, \Box, \rhd, \lhd)$ be a *DMA* with $I \in \mathcal{I}_{\mathcal{P}}(\mathbb{A})$. Then

(*i*) $\Diamond^{-1}I$ is an ideal of $\mathbb{A}$,

(*ii*) $A \backslash \Box I$ is a filter of $\mathbb{A}$,

(*iii*) $A \backslash \rhd I$ is an ideal of $\mathbb{A}$, and

(*iv*) $\lhd^{-1}I$ is a filter of $\mathbb{A}$.

Proof. For (i), suppose $a \in \Diamond^{-1}I$ and $b \leqslant a$. Then $\Diamond a \in I$ and by proposition 1.7 $\Diamond b \leqslant \Diamond a$. So $\Diamond b \in I$ and thus $b \in \Diamond^{-1}I$. Next consider $a, b \in \Diamond^{-1}I$. Then $\Diamond a, \Diamond b \in I$ and so $\Diamond a \vee \Diamond b = \Diamond(a \vee b) \in I$. Thus $a \vee b \in \Diamond^{-1}I$. Finally, since $0 = \Diamond 0 \in I$, also $0 \in \Diamond^{-1}I$.

For (ii), suppose $a \in F := \{a \in A : \Box a \notin I\}$ and $a \leqslant b$. Then $\Box a \notin I$ and by proposition 1.7 $\Box a \leqslant \Box b$. So $\Box b \notin I$, and thus $b \in F$. Next consider $a, b \in F$. Then $\Box a, \Box b \notin I$ and so, since I is prime, $\Box a \wedge \Box b = \Box(a \wedge b) \notin I$. Thus $a \wedge b \in F$. Finally, since I is proper, $1 = \Box 1 \notin I$ and hence $1 \in F$.

For (iii), suppose $a \in J := \{a \in A : \rhd a \notin I\}$ and $b \leqslant a$. Then $\rhd a \notin I$ and by proposition 1.7 $\rhd a \leqslant \rhd b$. So $\rhd b \notin I$ and thus $b \in J$. Next consider $a, b \in J$. then $\rhd a, \rhd b \notin I$ and so, since I is prime, $\rhd a \wedge \rhd b = \rhd(a \vee b) \notin I$. Thus $a \vee b \in J$. Finally, since I is proper $1 = \rhd 0 \notin I$ and thus $0 \in J$.

And for (iv), suppose $a \in \lhd^{-1}I$ and $a \leqslant b$. Then $\lhd a \in I$ and by proposition 1.7 $\lhd b \leqslant \lhd a$. So $\lhd b \in I$ and thus $b \in \lhd^{-1}I$. Next consider $a, b \in \lhd^{-1}I$. Then $\lhd a, \lhd b \in I$ and so $\lhd a \vee \lhd b = \lhd(a \wedge b) \in I$. Thus $a \wedge b \in \lhd^{-1}I$. Finally, since $0 = \lhd 1 \in I$, $1 \in \lhd^{-1}I$. $\Box$

Recall that $\hat{a}$ denotes $\{I \in \mathcal{I}_{\mathcal{P}}(\mathbb{A}) : a \notin I\}$ for $a \in A$.

Proposition 3.5 Let $\mathbb{A} = (A, \vee, \wedge, 0, 1, \Diamond, \Box, \rhd, \lhd)$ be a *DMA*. Then

(*i*) $\langle Q_{\Diamond} \rangle \hat{a} = \widehat{\Diamond a}$,

(*ii*) $[Q_{\Box}]\hat{a} = \widehat{\Box a}$,

(*iii*) $[Q_{\rhd}\rangle \hat{a} = \widehat{\rhd a}$, and

(*iv*) $\langle Q_{\lhd}]\hat{a} = \widehat{\lhd a}$.

Proof. For the first claim we have

$$\begin{aligned}\widehat{\Diamond a} &= \{I \in \mathcal{I}_{\mathcal{P}}(\mathbb{A}) : \Diamond a \notin I\}, \text{ and}\\ \langle Q_\Diamond\rangle \hat{a} &= \{I \in \mathcal{I}_{\mathcal{P}}(\mathbb{A}) : \exists J \in \hat{a} \text{ such that } IQ_\Diamond J\}\\ &= \{I \in \mathcal{I}_{\mathcal{P}}(\mathbb{A}) : \exists J \in \mathcal{I}_{\mathcal{P}}(\mathbb{A}) \text{ such that } a \notin J \text{ and } \Diamond^{-1} I \subseteq J\}.\end{aligned}$$

Consider $I \in \mathcal{I}_{\mathcal{P}}(\mathbb{A})$. First suppose $I \in \langle Q_\Diamond\rangle \hat{a}$. Then there is a $J \in \mathcal{I}_{\mathcal{P}}(\mathbb{A})$ such that $a \notin J$ and $\Diamond^{-1} I \subseteq J$. Assume $\Diamond a \in I$. Then since $\Diamond^{-1} I \subseteq J$ we have $a \in J$, a contradiction. So $\Diamond a \notin I$ and thus $I \in \widehat{\Diamond a}$. Conversely suppose $I \in \widehat{\Diamond a}$, so that $\Diamond a \notin I$. $\Diamond^{-1} I$ is an ideal by proposition 3.4 (i), and $a \notin \Diamond^{-1} I$. Thus we have $\uparrow a \cap \Diamond^{-1} I = \emptyset$, with $\uparrow a$ a filter and $\Diamond^{-1} I$ an ideal. So by proposition 3.3, there is a prime ideal J such that $\Diamond^{-1} I \subseteq J$ and $\uparrow a \cap J = \emptyset$. Thus there is a $J \in \mathcal{I}_{\mathcal{P}}(\mathbb{A})$ such that $a \notin J$ and $\Diamond^{-1} I \subseteq J$, and so $I \in \langle Q_\Diamond\rangle \hat{a}$. This establishes ($i$).

For the second claim observe that

$$\begin{aligned}\widehat{\Box a} &= \{I \in \mathcal{I}_{\mathcal{P}}(\mathbb{A}) : \Box a \notin I\} \text{ and}\\ [Q_\Box]\hat{a} &= \{I \in \mathcal{I}_{\mathcal{P}}(\mathbb{A}) : \forall J \in \mathcal{I}_{\mathcal{P}}(\mathbb{A})(IQ_\Box J \to J \in \hat{a})\}\\ &= \{I \in \mathcal{I}_{\mathcal{P}}(\mathbb{A}) : \forall J \in \mathcal{I}_{\mathcal{P}}(\mathbb{A})(\Box J \subseteq I \to a \notin J)\}.\end{aligned}$$

Consider $I \in \mathcal{I}_{\mathcal{P}}(\mathbb{A})$. First suppose $I \in \widehat{\Box a}$ so that $\Box a \notin I$. Suppose $\Box J \subseteq I$ for $J \in \mathcal{I}_{\mathcal{P}}(\mathbb{A})$. Then $a \notin J$, for otherwise $\Box a \in I$. So $I \in [Q_\Box]\hat{a}$. Conversely suppose $I \notin \widehat{\Box a}$, so that $\Box a \in I$. $F := \{b \in A : \Box b \notin I\}$ is a filter by proposition 3.4 (ii), and $a \notin F$. So we have $F \cap \downarrow a = \emptyset$, with F a filter and $\downarrow a$ an ideal. So by proposition 3.3, there is a prime ideal J such that $\downarrow a \subseteq J$ and $F \cap J = \emptyset$. Notice that for this J we have $\Box J \subseteq I$. For consider $b \in J$. Since $F \cap J = \emptyset$, $b \notin F$ and thus $\Box b \in I$. But since $\downarrow a \subseteq J$, $a \in J$. Thus $J \in \mathcal{I}_{\mathcal{P}}(\mathbb{A})$ is such that $\Box J \subseteq I$ but $a \in J$. So $I \notin [Q_\Box]\hat{a}$.

Next consider claim (iii) observing that

$$\begin{aligned}\widehat{\rhd a} &= \{I \in \mathcal{I}_{\mathcal{P}}(\mathbb{A}) : \rhd a \notin I\} \text{ and}\\ [Q_\rhd\rangle \hat{a} &= \{I \in \mathcal{I}_{\mathcal{P}}(\mathbb{A}) : \forall J \in \mathcal{I}_{\mathcal{P}}(\mathbb{A})(IQ_\rhd J \to J \notin \hat{a})\}\\ &= \{I \in \mathcal{I}_{\mathcal{P}}(\mathbb{A}) : \forall J \in \mathcal{I}_{\mathcal{P}}(\mathbb{A})(\rhd(\backslash J) \subseteq I \to a \in J)\}.\end{aligned}$$

Consider $I \in \mathcal{I}_{\mathcal{P}}(\mathbb{A})$. First suppose $I \in \widehat{\rhd a}$, so that $\rhd a \notin I$. Suppose $\rhd(\backslash J) \subseteq I$ for $J \in \mathcal{I}_{\mathcal{P}}(\mathbb{A})$. If $a \notin J$ then $\rhd a \in I$. Thus $a \in J$, and so $I \in [Q_\rhd\rangle \hat{a}$. Conversely suppose $I \notin \widehat{\rhd a}$, so that $\rhd a \in I$. $K := \{b \in A : \rhd b \notin I\}$ is an ideal by proposition 3.4 (iii), and $a \notin K$.

Thus we have $\uparrow a \cap K = \emptyset$ with $\uparrow a$ a filter and K an ideal. So by proposition 3.3, there is a $J \in \mathcal{I}_{\mathcal{P}}(\mathbb{A})$ such that $K \subseteq J$ and $\uparrow a \cap J = \emptyset$. Notice that for this J we have $\rhd(\backslash J) \subseteq I$. For suppose $\rhd b \notin I$. Then $b \in K \subseteq J$. And since $\uparrow a \cap J = \emptyset$, also $a \notin J$. So $J \in \mathcal{I}_{\mathcal{P}}(\mathbb{A})$ is such that $\rhd(\backslash J) \subseteq I$ and yet $a \notin J$. Thus $I \notin [Q_{\rhd}\rangle \hat{a}$.

For the fourth and final claim we have

$$\begin{aligned} \widehat{\lhd a} &= \{I \in \mathcal{I}_{\mathcal{P}}(\mathbb{A}) : \lhd a \notin I\}, \text{ and} \\ \langle Q_{\lhd}]\hat{a} &= \{I \in \mathcal{I}_{\mathcal{P}}(\mathbb{A}) : \exists J \notin \hat{a} \text{ such that } IQ_{\lhd}J\} \\ &= \{I \in \mathcal{I}_{\mathcal{P}}(\mathbb{A}) : \exists J \in \mathcal{I}_{\mathcal{P}}(\mathbb{A}) \text{ such that } a \in J \text{ and } \lhd J \subseteq \backslash I\}. \end{aligned}$$

Consider $I \in \mathcal{I}_{\mathcal{P}}(\mathbb{A})$. First suppose $I \notin \widehat{\lhd a}$, so that $\lhd a \in I$. Suppose $a \in J$ for some $J \in \mathcal{I}_{\mathcal{P}}(\mathbb{A})$. Then $\lhd J \nsubseteq \backslash I$, since $\lhd a \notin \backslash I$. So $I \notin \langle Q_{\lhd}]\hat{a}$. Conversely suppose $I \in \widehat{\lhd a}$, so that $\lhd a \notin I$. $\lhd^{-1} I$ is a filter by proposition 3.4, and $a \notin \lhd^{-1} I$. Thus we have $\lhd^{-1} I \cap \downarrow a = \emptyset$ with $\lhd^{-1} I$ a filter and $\downarrow a$ an ideal. So by proposition 3.3 there is a $J \in \mathcal{I}_{\mathcal{P}}(\mathbb{A})$ such that $\downarrow a \subseteq J$ and $\lhd^{-1} I \cap J = \emptyset$. Notice that $\lhd J \subseteq \backslash I$. For consider $b \in J$. Then since $\lhd^{-1} I \cap J = \emptyset$, $b \notin \lhd^{-1} I$; so $\lhd b \in \backslash I$. And since $\downarrow a \subseteq J$, also $a \in J$. So $J \in \mathcal{I}_{\mathcal{P}}(\mathbb{A})$ is such that $a \in J$ and $\lhd J \subseteq \backslash I$. Thus $I \in \langle Q_{\lhd}]\hat{a}$. $\square$

With these preliminary results in place, we are ready to prove that the dual of a *DMA* is indeed an extended Priestley space:

Proposition 3.6 Where $\mathbb{A}$ is a *DMA*, $\mathbb{A}_*$ is an extended Priestley space.

Proof. Let $\mathbb{A} = (A, \vee, \wedge, 0, 1, \Diamond, \Box, \rhd, \lhd)$ so that $\mathbb{A}_* = (\mathcal{I}_{\mathcal{P}}(\mathbb{A}), \subseteq, \tau, Q_{\Diamond}, Q_{\Box}, Q_{\rhd}, Q_{\lhd})$. By proposition A.5, $(\mathcal{I}_{\mathcal{P}}(\mathbb{A}), \subseteq, \tau)$ is a Priestley space. It remains to check that the additional relations satisfy the relation conditions, (CL) and (TOP). The details for (CL) and the relation conditions will be given only for the case of $Q_{\Diamond}$.

The relation conditions for $Q_{\Diamond}$ require that $(\subseteq \circ Q_{\Diamond} \circ \subseteq) \subseteq Q_{\Diamond}$. So consider $H, I, J, K \in \mathcal{I}_{\mathcal{P}}(\mathbb{A})$ for which $H \subseteq IQ_{\Diamond}J \subseteq K$; we must ensure that $HQ_{\Diamond}K$. Consider $\Diamond a \in H$. Since $H \subseteq I$, also $\Diamond a \in I$. Then since $IQ_{\Diamond}J$, $\Diamond^{-1} I \subseteq J$. So $a \in J$. And so since $J \subseteq K$, also $a \in K$. Thus $\Diamond^{-1} H \subseteq K$, and so $HQ_{\Diamond}K$.

To see that (CL) is satisfied, consider $U \in \mathrm{cl}\mathcal{D}(\mathcal{I}_{\mathcal{P}}(\mathbb{A}))$. We must ensure that also $\langle Q_{\Diamond}\rangle U$, $[Q_{\Box}]U$, $[Q_{\rhd}\rangle U$, $\langle Q_{\lhd}]U \in \mathsf{cl}\mathcal{D}(\mathcal{I}_{\mathcal{P}}(\mathbb{A}))$. Since $U \in \mathsf{cl}\mathcal{D}(\mathcal{I}_{\mathcal{P}}(\mathbb{A}))$, by proposition A.1, $U = \hat{a}$ for some $a \in A$. So by proposition 3.5(i) we have $\langle Q_\diamond \rangle U = \widehat{\diamond a}$. And thus, again by proposition A.1, $\langle Q_\diamond \rangle U \in \mathrm{cl}\mathcal{D}(\mathcal{I}_\mathcal{P}(\mathbb{A}))$. The cases for $Q_\square, Q_\triangleright$ and $Q_\triangleleft$ are by analogous reasoning making use of proposition 3.5(ii), (iii) and (iv), respectively.

Turning attention to (TOP), consider $I \in \mathcal{I}_\mathcal{P}(\mathbb{A})$. We have

$$\begin{aligned} Q_\diamond[I] &= \{J \in \mathcal{I}_\mathcal{P}(\mathbb{A}) : IQ_\diamond J\} \\ &= \{J \in \mathcal{I}_\mathcal{P}(\mathbb{A}) : \diamond^{-1} I \subseteq J\} \\ &= \mathcal{I}_\mathcal{P}(\mathbb{A}) \setminus \{J \in \mathcal{I}_\mathcal{P}(\mathbb{A}) : \exists a \in A(\diamond a \in I \text{ and } a \notin J)\} \\ &= \mathcal{I}_\mathcal{P}(\mathbb{A}) \setminus \bigcup_{\diamond a \in I} \{J \in \mathcal{I}_\mathcal{P}(\mathbb{A}) : a \notin J\}, \end{aligned}$$

and thus by proposition A.10(i), $Q_\diamond[I]$ is closed in the upper topology for $(\mathcal{I}_\mathcal{P}(\mathbb{A}), \subseteq, \tau)$.

Next for $Q_\square$ we have

$$\begin{aligned} Q_\square[I] &= \{J \in \mathcal{I}_\mathcal{P}(\mathbb{A}) : IQ_\square J\} \\ &= \{J \in \mathcal{I}_\mathcal{P}(\mathbb{A}) : \square J \subseteq I\} \\ &= \mathcal{I}_\mathcal{P}(\mathbb{A}) \setminus \{J \in \mathcal{I}_\mathcal{P}(\mathbb{A}) : \exists a \in A(a \in J \text{ and } \square a \notin I)\} \\ &= \mathcal{I}_\mathcal{P}(\mathbb{A}) \setminus \bigcup_{\square a \notin I} \{J \in \mathcal{I}_\mathcal{P}(\mathbb{A}) : a \in J\}, \end{aligned}$$

which is closed in the lower topology for $(\mathcal{I}_\mathcal{P}(\mathbb{A}), \subseteq, \tau)$ by proposition A.10(ii).

Next for $Q_\triangleright$ observe that

$$\begin{aligned} Q_\triangleright[I] &= \{J \in \mathcal{I}_\mathcal{P}(\mathbb{A}) : IQ_\triangleright J\} \\ &= \{J \in \mathcal{I}_\mathcal{P}(\mathbb{A}) : \triangleright(\backslash J) \subseteq I\} \\ &= \mathcal{I}_\mathcal{P}(\mathbb{A}) \setminus \{J \in \mathcal{I}_\mathcal{P}(\mathbb{A}) : \exists a \in A(a \notin J \text{ and } \triangleright a \in I)\} \\ &= \mathcal{I}_\mathcal{P}(\mathbb{A}) \setminus \bigcup_{\triangleright a \in I} \{J \in \mathcal{I}_\mathcal{P}(\mathbb{A}) : a \notin J\}, \end{aligned}$$

which is closed in the upper topology for $(\mathcal{I}_\mathcal{P}(\mathbb{A}), \subseteq, \tau)$ by proposition A.10(i).
And finally for $Q_\triangleleft$ we have

$$\begin{aligned} Q_\triangleleft[I] &= \{J \in \mathcal{I}_\mathcal{P}(\mathbb{A}) : IQ_\triangleleft J\} \\ &= \{J \in \mathcal{I}_\mathcal{P}(\mathbb{A}) : \triangleleft J \subseteq \backslash I\} \\ &= \mathcal{I}_\mathcal{P}(\mathbb{A}) \setminus \{J \in \mathcal{I}_\mathcal{P}(\mathbb{A}) : \exists a \in A(a \in J \text{ and } \triangleleft a \notin I)\} \\ &= \mathcal{I}_\mathcal{P}(\mathbb{A}) \setminus \bigcup_{\triangleleft a \notin I} \{J \in \mathcal{I}_\mathcal{P}(\mathbb{A}) : a \in J\}, \end{aligned}$$

which is closed in the lower topology for $(\mathcal{I}_{\mathcal{P}}(\mathbb{A}), \subseteq, \tau)$ by proposition A.10(i). $\square$

Next we will see that $\mathbb{X}$ is isomorphic to $(\mathbb{X}^*)_*$.

Proposition 3.7 Where $\mathbb{X}$ is an extended Priestley space, $\mathbb{X} \cong (\mathbb{X}^*)_*$.

Proof. Let $\mathbb{X} = (X, \leqslant, \tau, R_\diamond, R_\Box, R_\rhd, R_\lhd)$ so that $(\mathbb{X}^*)_* = (\mathcal{I}_{\mathcal{P}}(\mathbb{X}^*), \subseteq, \tau, Q_{\langle R_\diamond \rangle}, Q_{[R_\Box]}, Q_{[R_\rhd\rangle}, Q_{\langle R_\lhd]})$. It will be shown that $f : x \mapsto \{a \in \mathrm{cl}\mathcal{D}(\mathbb{X}) : x \notin a\}$ is an isomorphism from $\mathbb{X}$ onto $(\mathbb{X}^*)_*$.

By proposition A.8 f is an isomorphism between $(\mathbb{X}, \leqslant, \tau)$ and $(\mathcal{I}_{\mathcal{P}}(\mathbb{X}^*), \subseteq, \tau)$. So it remains to establish that for $x, y \in X$,

(i) $xR_\diamond y$ iff $f(x)Q_{\langle R_\diamond \rangle}f(y)$,

(ii) $xR_\Box y$ iff $f(x)Q_{[R_\Box]}f(y)$,

(iii) $xR_\rhd y$ iff $f(x)Q_{[R_\rhd\rangle}f(y)$,

(iv) $xR_\lhd y$ iff $f(x)Q_{\langle R_\lhd]}f(y)$.

Consider $x, y \in X$. For (i) we have

$$\begin{aligned} f(x)Q_{\langle R_\diamond \rangle}f(y) &\text{ iff } \langle R_\diamond \rangle^{-1}\{a \in \mathrm{cl}\mathcal{D}(\mathbb{X}) : x \notin a\} \subseteq \{a \in \mathrm{cl}\mathcal{D}(\mathbb{X}) : y \notin a\}, \\ &\text{ iff for all } a \in \mathrm{cl}\mathcal{D}(\mathbb{X})(x \notin \langle R_\diamond \rangle a \text{ only if } y \notin a), \\ &\text{ iff for all } a \in \mathrm{cl}\mathcal{D}(\mathbb{X})(y \in a \to R_\diamond[x] \cap a \neq \emptyset). \end{aligned}$$

Suppose $xR_\diamond y$, and consider $a \in \mathrm{cl}\mathcal{D}(\mathbb{X})$ with $y \in a$. Then trivially $y \in R_\diamond[x] \cap a$, so $f(x)Q_{\langle R_\diamond \rangle}f(y)$. Conversely, suppose $xR_\diamond y$ does not hold so that $y \notin R_\diamond[x]$. Since (TOP) ensures that $R_\diamond[x]$ is closed in the upper topology with basis $\mathrm{cl}\mathcal{D}(\mathbb{X})$, we have $R_\diamond[x] = X \setminus \bigcup U$ for some $U \subseteq \mathrm{cl}\mathcal{D}(\mathbb{X})$. Since $y \notin R_\diamond[x]$, $y \in \bigcup U$. Thus $y \in a$ for some $a \in \mathrm{cl}\mathcal{D}(\mathbb{X})$ with $a \cap R_\diamond[x] = \emptyset$. But then $f(x)Q_{\langle R_\diamond \rangle}f(y)$ does not hold.

For the second claim we have

$$\begin{aligned} f(x)Q_{[R_\Box]}f(y) &\text{ iff } [R_\Box]\{a \in \mathrm{cl}\mathcal{D}(\mathbb{X}) : y \notin a\} \subseteq \{a \in \mathrm{cl}\mathcal{D}(\mathbb{X}) : x \notin a\}, \\ &\text{ iff for all } a \in \mathrm{cl}\mathcal{D}(\mathbb{X}), (y \notin a \to x \notin [R_\Box]a), \\ &\text{ iff for all } a \in \mathrm{cl}\mathcal{D}(\mathbb{X}), (x \in [R_\Box]a \to y \in a), \\ &\text{ iff for all } a \in \mathrm{cl}\mathcal{D}(\mathbb{X}), (R_\Box[x] \subseteq a \to y \in a). \end{aligned}$$

Suppose $xR_\Box y$ and consider $a \in \mathrm{cl}\mathcal{D}(\mathbb{X})$ such that $R_\Box[x] \subseteq a$. Then since $y \in R_\Box[x], y \in a$. Thus $f(x)Q_{[R_\Box]}f(y)$. Conversely, suppose $xR_\Box y$ does not hold, so that $y \notin R_\Box[x]$. Since (TOP) ensures that $R_\Box[x]$ is closed in the lower topology with basis $\{X \setminus a : a \in \mathrm{cl}\mathcal{D}(\mathbb{X})\}$, we have $R_\Box[x] = X \setminus \bigcup\{X \setminus a : a \in U\}$ for some $U \subseteq \mathrm{cl}\mathcal{D}(\mathbb{X})$. Thus, since $\bigcup\{X \setminus a : a \in U\}$ is an open up-set, $R_\Box[x]$ is a closed down-set. So, since also $y \notin R_\Box[x]$, by proposition A.1 there is an $a \in \mathrm{cl}\mathcal{D}(\mathbb{X})$ such that $R_\Box[x] \subseteq a$ but $y \notin a$. Thus $f(x)Q_{[R_\Box]}f(y)$ does not hold.

Next, for claim (iii), observe that

$$
\begin{aligned}
f(x)Q_{[R_\triangleright\rangle}f(y) \text{ iff } & [R_\triangleright\rangle\{a \in \mathrm{cl}\mathcal{D}(\mathbb{X}) : y \in a\} \subseteq \{a \in \mathrm{cl}\mathcal{D}(\mathbb{X}) : x \notin a\}, \\
& \text{iff for all } a \in \mathrm{cl}\mathcal{D}(\mathbb{X}), (y \in a \to x \notin [R_\triangleright\rangle a), \\
& \text{iff for all } a \in \mathrm{cl}\mathcal{D}(\mathbb{X}), (y \in a \to R_\triangleright[x] \cap a \neq \emptyset).
\end{aligned}
$$

Suppose $xR_\triangleright y$ and consider $a \in \mathrm{cl}\mathcal{D}(\mathbb{X})$ such that $y \in a$. Then $y \in R_\triangleright[x] \cap a$. So $R_\triangleright[x] \cap a \neq \emptyset$ and thus $f(x)Q_{[R_\diamond\rangle}f(y)$. Conversely, suppose $xR_\triangleright y$ does not hold, so that $y \notin R_\triangleright[x]$. Since (TOP) ensures that $R_\triangleright[x]$ is closed in the upper topology with basis $\mathrm{cl}\mathcal{D}(\mathbb{X})$, we have $R_\triangleright[x] = X \setminus \bigcup U$ for some $U \subseteq \mathrm{cl}\mathcal{D}(\mathbb{X})$. Since $y \notin R_\triangleright[x]$, $y \in \bigcup U$. Thus $y \in a$ for some $a \in \mathrm{cl}\mathcal{D}(\mathbb{X})$ with $R_\triangleright[x] \cap a = \emptyset$. But then $f(x)Q_{[R_\triangleright\rangle}f(y)$ does not hold.

And for the final claim,

$$
\begin{aligned}
f(x)Q_{\langle R_\triangleleft]}f(y) \text{ iff } & \langle R_\triangleleft]\{a \in \mathrm{cl}\mathcal{D}(\mathbb{X}) : y \notin a\} \subseteq \{a \in \mathrm{cl}\mathcal{D}(\mathbb{X}) : x \in a\}, \\
& \text{iff for all } a \in \mathrm{cl}\mathcal{D}(\mathbb{X})(y \notin a \to x \in \langle R_\triangleleft]a), \\
& \text{iff for all } a \in \mathrm{cl}\mathcal{D}(\mathbb{X})(R_\triangleleft[x] \subseteq a \to y \in a).
\end{aligned}
$$

Suppose $xR_\triangleleft y$ and consider $a \in \mathrm{cl}\mathcal{D}(\mathbb{X})$ such that $R_\triangleleft[x] \subseteq a$. Then since $y \in R_\triangleleft[x]$, also $y \in a$. Thus $f(x)Q_{\langle R_\diamond]}f(y)$. Conversely suppose $xR_\triangleleft y$ does not hold so that $y \notin R_\triangleleft[x]$. Since (TOP) ensures that $R_\triangleleft[x]$ is closed in the lower topology with basis $\{X \setminus a : a \in \mathrm{cl}\mathcal{D}(\mathbb{X})\}$, we have $R_\triangleleft[x] = X \setminus \bigcup\{X \setminus a : a \in U\}$ for some $U \subseteq \mathrm{cl}\mathcal{D}(\mathbb{X})$. Thus, since $\bigcup\{X \setminus a : a \in U\}$ is an open up-set, $R_\triangleleft[x]$ is a closed down-set. So, since also $y \notin R_\triangleleft[x]$, by proposition A.1 there is an $a \in \mathrm{cl}\mathcal{D}(\mathbb{X})$ such that $R_\triangleleft[x] \subseteq a$ but $y \notin a$. Thus $f(x)Q_{\langle R_\triangleleft]}f(y)$ does not hold. $\Box$

All of the work for the final proposition of this section, which is all that remains to establish the "object part" of the duality, has already been done.

Proposition 3.8 Where $\mathbb{A}$ is a distributive modal algebra, $\mathbb{A} \cong (\mathbb{A}_*)^*$.

Proof. Let $\mathbb{A} = (A, \vee, \wedge, 0, 1, \Diamond, \Box, \rhd, \lhd)$ so that $(\mathbb{A}_*)^* = (\mathrm{cl}\mathcal{D}(\mathcal{I}_{\mathcal{P}}(\mathbb{A})), \cup, \cap, \emptyset, \mathcal{I}_{\mathcal{P}}(\mathbb{A}), \langle Q_\Diamond \rangle, [Q_\Box],$ $[Q_\rhd\rangle, \langle Q_\lhd])$. Where $\hat{a} = \{I \in \mathcal{I}_{\mathcal{P}}(\mathbb{A}) : a \notin I\}$ as before, it will be shown that $g : a \mapsto \hat{a}$ is an isomorphism from $\mathbb{A}$ onto $(\mathbb{A}_*)^*$.

It follows from the Priestley duality results that g is an isomorphism between $(A, \vee, \wedge, 0, 1)$ and $(\mathrm{cl}\mathcal{D}(\mathcal{I}_{\mathcal{P}}(\mathbb{A})), \cup, \cap, \emptyset, \mathcal{I}_{\mathcal{P}}(\mathbb{A}))$. It remains to establish that for $a \in A$,

(*i*) $\langle Q_\Diamond \rangle g(a) = g(\Diamond a)$,

(*ii*) $[Q_\Box] g(a) = g(\Box a)$,

(*iii*) $[Q_\rhd\rangle g(a) = g(\rhd a)$, and

(*iv*) $\langle Q_\lhd] g(a) = g(\lhd a)$.

But this has been established already in proposition 3.5. □

3.2 Morphisms

The morphisms for extended Priestley spaces, continuous order-preserving bounded morphisms, are introduced next. Recall that the morphisms for *DMA*'s, *DMA*-homomorphisms, have already been defined in chapter 1. This section contains the "morphisms part" of the duality between the category of extended Priestley spaces with continuous order-preserving bounded morphisms and the category of *DMA*'s with *DMA*-homomorphisms.

For the remainder of this section, let $\mathbb{X} = (X, \leqslant, \tau, R_\Diamond, R_\Box, R_\rhd, R_\lhd)$ and $\mathbb{X}' = (X', \leqslant', \tau',$ $R'_\Diamond, R'_\Box, R'_\rhd, R'_\lhd)$ be extended Priestley spaces; let $\mathbb{A} = (A, \vee, \wedge, 0, 1, \Diamond, \Box, \rhd, \lhd)$ and $\mathbb{A}' =$ $(A', \vee', \wedge', 0', 1', \Diamond', \Box', \rhd', \lhd')$ be *DMA*'s.

Definition 3.9 A *continuous order-preserving bounded morphism* between $\mathbb{X}$ and $\mathbb{X}'$ is a function $\chi : X \to X'$ satisfying the following conditions:

(Mi) $x \leqslant y$ implies $\chi(x) \leqslant' \chi(y)$ for $x, y \in X$,

(Mii) χ is continuous,

$(Miii)$ $xR_\vartriangle y$ implies $\chi(x)R'_\vartriangle \chi(y)$ for $x, y \in X$ and $\vartriangle \in \{\Diamond, \Box, \rhd, \lhd\}$,

(Miv)(a) If $\chi(x)R'_\Diamond y$ for some $x \in X$ and $y \in X'$ then $\exists z \in X$ with $xR_\Diamond z$ and $\chi(z) \leqslant' y$,

(Miv)(b) If $\chi(x)R'_\Box y$ for some $x \in X$ and $y \in X'$ then $\exists z \in X$ with $xR_\Box z$ and $y \leqslant' \chi(z)$,

(Miv)(c) If $\chi(x)R'_\rhd y$ for some $x \in X$ and $y \in X'$ then $\exists z \in X$ with $xR_\rhd z$ and $\chi(z) \leqslant' y$,

(Miv)(d) If $\chi(x)R'_\lhd y$ for some $x \in X$ and $y \in X'$ then $\exists z \in X$ with $xR_\lhd z$ and $y \leqslant' \chi(z)$.

Notice that this definition combines aspects we have seen before: conditions (Mi), $(Miii)$ and (Miv) are as in the definition of order-preserving bounded morphism for frames, and conditions (Mi) and $(M2)$ are as required of the morphisms for Priestley spaces (see section A.4).

For a function $\chi : X \to X'$ between extended Priestley spaces $\mathbb{X}$ and $\mathbb{X}'$, define the dual $\chi^* : \mathrm{cl}\mathcal{D}(\mathbb{X}') \to \mathcal{P}(X)$ by

$$\chi^* : a \mapsto \{x \in X : \chi(x) \in a\}.$$

And the dual $\eta_* : \mathcal{I}_\mathcal{P}(\mathbb{A}') \to \mathcal{P}(A)$ of a function $\eta : A \to A'$ is defined by

$$\eta_* : I \mapsto \{a \in A : \eta(a) \in I\}.$$

Next we check that the dual of a continuous order-preserving bounded morphism is a *DMA*-homomorphism, and that the dual of a *DMA*-homomorphism is a continuous order-preserving bounded morphism.

Proposition 3.10 If χ is a continuous order-preserving bounded morphism between $\mathbb{X}$ and $\mathbb{X}'$ then χ^* is a *DMA*-homomorphism from $\mathbb{X}'^*$ to $\mathbb{X}^*$.

Proof. By proposition A.11, χ^* is a homomorphism between the lattice reducts of $\mathbb{X}'^*$ and $\mathbb{X}^*$. So it remains to ensure that for $a \in \mathrm{cl}\mathcal{D}(\mathbb{X}')$,

(i) $\chi^*(\langle R'_\Diamond\rangle a) = \langle R_\Diamond\rangle(\chi^*(a))$,

(ii) $\chi^*([R'_\Box]a) = [R_\Box](\chi^*(a))$,

(iii) $\chi^*([R'_\rhd\rangle a) = [R_\rhd\rangle(\chi^*(a))$, and (*iv*) $\chi^*(\langle R'_\lhd]a) = \langle R_\lhd](\chi^*(a))$.

Consider $a \in \mathrm{cl}\mathcal{D}(\mathbb{X}')$. For the first claim we have

$$\begin{aligned}\chi^*(\langle R'_\Diamond\rangle a) &= \chi^*(\{x' \in X' : \exists y' \in a \text{ such that } x'R'_\Diamond y'\}) \\ &= \{x \in X : \exists y' \in a \text{ such that } \chi(x)R'_\Diamond y'\}, \text{ and} \\ \langle R_\Diamond\rangle(\chi^*(a)) &= \{x \in X : \exists z(xR_\Diamond z \text{ and } z \in \chi^*(a))\} \\ &= \{x \in X : \exists z(\chi(z) \in a \text{ and } xR_\Diamond z).\end{aligned}$$

First suppose $x \in \chi^*(\langle R'_\Diamond\rangle a)$ so that $\chi(x)R'_\Diamond y'$ for some $y' \in a$. By condition $(Miv)(a)$ of the definition of continuous order-preserving bounded morphism, $xR_\Diamond z$ and $\chi(z) \leqslant' y'$ for some $z \in X$. Since a is a down-set with $y' \in a$, also $\chi(z) \in a$. So $\chi(z) \in a$ and $xR_\Diamond z$. Thus $x \in \langle R_\Diamond\rangle(\chi^*(a))$. Conversely, suppose $x \in \langle R_\Diamond\rangle(\chi^*(a))$ so that $xR_\Diamond z$ for some $z \in X$ with $\chi(z) \in a$. Since $xR_\Diamond z$, clause $(Miii)$ ensures that $\chi(x)R'_\Diamond\chi(z)$. Thus $x \in \chi^*(\langle R'_\Diamond\rangle a)$.

Next consider the second claim, observing that

$$\begin{aligned}\chi^*([R'_\Box]a) &= \chi^*(\{x' \in X' : \forall y'(x'R'_\Box y' \to y' \in a)\}) \\ &= \{x \in X : \forall y'(\chi(x)R'_\Box y' \to y' \in a)\}, \text{ and} \\ [R_\Box](\chi^*(a)) &= \{x \in X : \forall z(xR_\Box z \to z \in \chi^*(a))\} \\ &= \{x \in X : \forall z(xR_\Box z \to \chi(z) \in a)\}.\end{aligned}$$

First suppose $x \notin \chi^*([R'_\Box]a)$ so that $\chi(x)R'_\Box y'$ for some $y' \in X'$ with $y' \notin a$. Then by condition $(Miv)(b)$, $xR_\Box z$ and $y' \leqslant' \chi(z)$ for some $z \in X$. Since a is a down-set with $y' \notin a$, also $\chi(z) \notin a$. Thus $x \notin [R_\Box](\chi^*(a))$. And conversely, suppose $x \notin [R_\Box](\chi^*(a))$ so that $xR_\Box z$ for some $z \in X$ with $\chi(z) \notin a$. But then by condition $(Miii)$ $\chi(x)R'_\Box\chi(z)$. So $x \notin \chi^*([R'_\Box]a)$.

Now for claim (*iii*) we have

$$\begin{aligned}\chi^*([R'_\Box\rangle a) &= \chi^*(\{x' \in X' : \forall y'(x'R'_\rhd y' \to y' \notin a)\}) \\ &= \{x \in X : \forall y'(\chi(x)R'_\rhd y' \to y' \notin a)\}, \text{ and} \\ [R_\rhd\rangle(\chi^*(a)) &= \{x \in X : \forall z(xR_\rhd z \to z \notin \chi^*(a))\} \\ &= \{x \in X : \forall z(xR_\rhd z \to \chi(z) \notin a)\}.\end{aligned}$$

First suppose $x \notin \chi^*([R_\rhd\rangle a)$ so that $\chi(x)R'_\rhd y'$ for some $y' \in X'$ with $y' \in a$. Then by condition $(Miv)(c)$, $xR_\rhd z$ for some $z \in X$ with $\chi(z) \leqslant' y'$. Since a is a down-set with $y' \in a$, also $\chi(z) \in a$. So $xR_\rhd z$ but $\chi(z) \in a$. So $x \notin [R_\rhd\rangle(\chi^*(a))$. And conversely, suppose

$x \notin [R_{\triangleright}\rangle(\chi^*(a))$ so that $xR_{\triangleright}z$ for some $z \in X$ with $\chi(z) \in a$. Then by condition $(Miii)$ we have $\chi(x)R'_{\triangleright}\chi(z)$. So $x \notin \chi^*([R'_{\triangleright}\rangle a)$.

Turning attention to the fourth claim, observe that

$$\begin{aligned}\chi^*(\langle R'_{\triangleleft}]a) &= \chi^*(\{x' \in X' : \exists y'(x'R'_{\triangleleft}y' \text{ and } y' \notin a)\}) \\ &= \{x \in X : \exists y'(\chi(x)R'_{\triangleleft}y' \text{ and } y' \notin a)\}, \text{ and} \\ \langle R_{\triangleleft}](\chi^*(a)) &= \{x \in X : \exists z(xR_{\triangleleft}z \text{ and } z \notin \chi^*(a))\} \\ &= \{x \in X : \exists z(\chi(z) \notin a \text{ and } xR_{\triangleleft}z).\end{aligned}$$

First suppose $x \in \chi^*(\langle R_{\triangleleft}]a)$ so that $\chi(x)R'_{\triangleleft}y'$ for some $y' \in X'$ with $y' \notin a$. Then by condition $(Miv)(d)$, $xR_{\triangleleft}z$ for some $z \in X$ with $y' \leqslant \chi(z)$. Since a is a down-set with $y' \notin a$, also $\chi(z) \notin a$. So $xR_{\triangleleft}z$ and $\chi(z) \notin a$; thus $x \notin \langle R_{\triangleleft}](\chi^*(a))$. And conversely, suppose $x \in \langle R_{\triangleleft}](\chi^*(a))$ so that $xR_{\triangleleft}z$ for some $z \in X$ with $\chi(z) \notin a$. Since $xR_{\triangleleft}z$, condition $(Miii)$ guarantees that $\chi(x)R'_{\triangleleft}\chi(z)$. Thus $x \in \chi^*(\langle R'_{\triangleleft}]a)$. □

Checking that η_* satisfies (Miv) is the hard part of showing that η_* is a continuous order-preserving bounded morphism. This involves establishing the existence of certain prime ideals. Thus the prime ideal theorem (proposition 3.3 above), together with proposition 3.4 and the following result, is useful.

Proposition 3.11 Where $\eta : \mathbb{A} \to \mathbb{A}'$ is a *DMA*-homomorphism and $J \in \mathcal{I}_{\mathcal{P}}(\mathbb{A})$,

(*i*) $F := \{b \in A' : b \geqslant \eta(a) \text{ for some } a \notin J\}$ is a filter, and

(*ii*) $H := \{\, b \in A' : b \leqslant \eta(a) \text{ for some } a \in J\}$ is an ideal.

Proof. Certainly F is an up-set, and it is nonempty since $1 \notin J$. Consider $b_1, b_2 \in F$. Then $b_1 \geqslant \eta(a_1)$ and $b_2 \geqslant \eta(a_2)$ for some $a_1, a_2 \notin J$. So $b_1 \wedge b_2 \geqslant \eta(a_1) \wedge \eta(a_2) = \eta(a_1 \wedge a_2)$. Since $a_1, a_2 \notin J$ and J is prime, also $a_1 \wedge a_2 \notin J$. Thus $b_1 \wedge b_2 \geqslant \eta(a_1 \wedge a_2)$ while $a_1 \wedge a_2 \notin J$. So $b_1 \wedge b_2 \in F$.

For the second claim, notice that H is a down-set which is nonempty as $0 \in J$. Consider $b_1, b_2 \in H$. Then $b_1 \leqslant \eta(a_1)$ and $b_2 \leqslant \eta(a_2)$ for some $a_1, a_2 \in J$. So $b_1 \vee b_2 \leqslant \eta(a_1) \vee \eta(a_2) = \eta(a_1 \vee a_2)$. And since $a_1, a_2 \in J$ also $a_1 \vee a_2 \in J$. Thus $b_1 \vee b_2 \in H$. □

Proposition 3.12 If η is a *DMA*-homomorphism from $\mathbb{A}$ to $\mathbb{A}'$ then η_* is a continuous order-preserving bounded morphism between $\mathbb{A}'_*$ and $\mathbb{A}_*$.

Proof. By proposition A.12 η_* is a Priestley morphism between the underlying Priestley spaces of $\mathbb{A}'_*$ and $\mathbb{A}_*$. Thus it remains to ensure that η_* satisfies conditions $(Miii)$ and (Miv) in the definition of order-preserving bounded morphism.

For $\diamond$, $(Miii)$ is the condition that for $I, J \in \mathcal{I}_{\mathcal{P}}(\mathbb{A}')$,

$$IQ'_\diamond J \text{ implies } \eta_*(I)Q_\diamond\eta_*(J).$$

Suppose $IQ'_\diamond J$ for $I, J \in \mathcal{I}_{\mathcal{P}}(\mathbb{A})$. Then $\diamond'^{-1}I \subseteq J$; we must check that $\diamond^{-1}\eta_*(I) \subseteq \eta_*(J)$. Consider $x \in \diamond^{-1}\eta_*(I)$, so that $\diamond x \in \eta_*(I)$. Then $\eta(\diamond x) = \diamond'(\eta(x)) \in I$. So since $\diamond'^{-1}I \subseteq J$, we have $\eta(x) \in J$. But then $x \in \eta_*(J)$, as required. The cases for $\Box, \rhd$ and $\lhd$ are similarly straightforward.

Finally, we must check that η_* satisfies conditions (a), (b), (c) and (d) of (Miv). For (a) suppose $\eta_*(I)Q_\diamond J$ for $I \in \mathcal{I}_{\mathcal{P}}(\mathbb{A}')$ and $J \in \mathcal{I}_{\mathcal{P}}(\mathbb{A})$. Then $a \in J$ for all $a \in A$ with $\eta(\diamond a) \in I$. Now $\diamond'^{-1}I$ is an ideal by proposition 3.4(i) and $F := \{b \in A' : b \geqslant \eta(a) \text{ for some } a \notin J\}$ is a filter by proposition 3.11(i). Notice that these two sets are disjoint. For consider $b \in F$. Then $b \geqslant \eta(a)$ for some $a \notin J$. Since $a \notin J, \eta(\diamond a) = \diamond'\eta(a) \notin I$. And since $b \geqslant \eta(a)$, by proposition 1.7 (i) also $\diamond' b \geqslant \diamond'\eta(a)$. So $\diamond' b \notin I$, and thus $b \notin \diamond'^{-1}I$. So by proposition 3.3 there is a $K \in \mathcal{I}_{\mathcal{P}}(\mathbb{A}')$ with $\diamond'^{-1}I \subseteq K$ and $K \cap F = \emptyset$. Since $\diamond'^{-1}I \subseteq K$, we have $IQ'_\diamond K$. Consider $a \notin J$. Then $\eta(a) \in F$, and so $\eta(a) \notin K$. Thus $\eta_*(K) \subseteq J$. So $K \in \mathcal{I}_{\mathcal{P}}(\mathbb{A}')$ is as required in (a).

For (b) suppose $\eta_*(I)Q_\Box J$ for $I \in \mathcal{I}_{\mathcal{P}}(\mathbb{A}')$ and $J \in \mathcal{I}_{\mathcal{P}}(\mathbb{A})$. Then $\eta(\Box a) \in I$ for all $a \in J$. Now $\{a \in A' : \Box' a \notin I\}$ is a filter by proposition 3.4(ii) and $H := \{b \in A' : b \leqslant \eta(a) \text{ and } a \in J\}$ is an ideal by proposition 3.11(ii). Notice that these two sets are disjoint. For consider $b \in H$. Then $b \leqslant \eta(a)$ for some $a \in J$. Since $a \in J$, $\eta(\Box a) = \Box'\eta(a) \in I$. And since $b \leqslant \eta(a)$, by proposition 1.7 (ii) also $\Box' b \leqslant \Box'\eta(a)$. So $\Box' b \in I$. Thus $b \notin \{a \in A' : \Box' a \notin I\}$. So by proposition 3.3, there is a $K \in \mathcal{I}_{\mathcal{P}}(\mathbb{A}')$ with $H \subseteq K$ and $K \cap \{a \in A' : \Box' a \notin I\} = \emptyset$. Since $H \subseteq K$, we have $J \subseteq \eta_*(K)$. And to ensure that $IQ'_\Box K$, consider $a \in K$. Since $K \cap \{a \in A' : \Box' a \notin I\} = \emptyset$ we have $\Box' a \in I$, and so $\Box' K \subseteq I$. So there is a $K \in \mathcal{I}_{\mathcal{P}}(\mathbb{A}')$ such that $IQ_\Box K$ and $J \subseteq \eta_*(K)$, as required to establish (b).

For (c) suppose $\eta_*(I)Q_\rhd J$ for $I \in \mathcal{I}_{\mathcal{P}}(\mathbb{A}')$ and $J \in \mathcal{I}_{\mathcal{P}}(\mathbb{A})$. Then $\eta(\rhd a) \in I$ for $a \in A \setminus J$. Now $\{a \in A' : \rhd' a \notin I\}$ is an ideal by proposition 3.4(iii) and $F := \{b \in A' : b \geqslant \eta(a) \text{ and } a \notin J\}$

is a filter by proposition 3.11(i). Notice that these two sets are disjoint. For consider $b \in F$. Then $b \geqslant \eta(a)$ for some $a \notin J$. Since $a \notin J$, $\eta(\rhd a) = \rhd'\eta(a) \in I$. And since $b \geqslant \eta(a)$ by proposition 1.7 (iii) $\rhd' b \leqslant \rhd'\eta(a)$. So also $\rhd' b \in I$. But then $b \notin \{a \in A' : \rhd' a \notin I\}$. So by proposition 3.3 there is a $K \in \mathcal{I}_{\mathcal{P}}(\mathbb{A}')$ with $\{a \in A' : \rhd' a \notin I\} \subseteq K$ and $K \cap F = \emptyset$. Since $K \cap F = \emptyset$ we have $\eta_*(K) \subseteq J$. And since $\{a \in A' : \rhd' a \notin I\} \subseteq K$, also $\rhd'(\backslash K) \subseteq I$ and so $IQ_{\rhd'}K$. So $K \in \mathcal{I}_{\mathcal{P}}(\mathbb{A})$ is as required in (c).

For (d) suppose $\eta_*(I)Q_{\lhd}J$ for $I \in \mathcal{I}_{\mathcal{P}}(\mathbb{A}')$ and $J \in \mathcal{I}_{\mathcal{P}}(\mathbb{A})$. Then $\eta(\lhd a) \notin I$ for $a \in J$. Now ${\lhd'}^{-1}I$ is a filter by proposition 3.4(iv) and $H := \{b \in A' : b \leqslant \eta(a) \text{ and } a \in J\}$ is an ideal by proposition 3.11(ii). Notice that these two sets are disjoint. For consider $b \in H$. Then $b \leqslant \eta(a)$ for some $a \in J$. Then $\eta(\lhd a) = \lhd'\eta(a) \notin I$. Since $b \leqslant \eta(a)$ by proposition 1.7 (iv) $\lhd' b \geqslant \lhd'\eta(a)$. But then $\lhd' b \notin I$ and so $b \notin {\lhd'}^{-1}I$. So by proposition 3.3 there is a $K \in \mathcal{I}_{\mathcal{P}}(\mathbb{A}')$ with $H \subseteq K$ and $K \cap {\lhd'}^{-1}I = \emptyset$. Since $K \cap {\lhd'}^{-1}I = \emptyset$, we have $\lhd' K \subseteq \backslash I$ and so $IQ'_{\lhd}K$. And since $H \subseteq K$, also $J \subseteq \eta_*(K)$. So K is as required in (d). $\square$

The following two propositions follow immediately from propositions A.13 and A.14, respectively.

Proposition 3.13 Where $f_{\mathbb{X}}$ and $f'_{\mathbb{X}}$ are bijections as in proposition 3.7 from $\mathbb{X}$ to $(\mathbb{X}^*)_*$ and $\mathbb{X}'$ to $(\mathbb{X}'^*)_*$ respectively, and χ is a continuous order-preserving bounded morphism between $\mathbb{X}$ and $\mathbb{X}'$,

$$f_{\mathbb{X}'}(\chi(x)) = (\chi^*)_*(f_{\mathbb{X}}(x)), \text{ for all } x \in X.$$

Proposition 3.14 Where $g_{\mathbb{A}}$ and $g'_{\mathbb{A}}$ are bijections as in proposition 3.8 from $\mathbb{A}$ to $(\mathbb{A}_*)^*$ and $\mathbb{A}'$ to $(\mathbb{A}'_*)^*$ respectively, and η is a *DMA*-homomorphism from $\mathbb{A}$ to $\mathbb{A}'$,

$$g_{\mathbb{A}'}(\eta(a)) = (\eta_*)^*(g_{\mathbb{A}}(a)), \text{ for all } a \in A.$$

This concludes the proof of the "morphism part" of the duality; together with the "object part" of the previous section, this gives us:

Theorem 3.15 The category $\mathcal{PR}$ of extended Priestley spaces with continuous order-preserving bounded morphisms, and the category $\mathcal{DMA}$ of *DMA*'s with *DMA*-homomorphisms are dually equivalent.

Chapter 4

Subdirect Irreducibility of Distributive Modal Algebras

A characterization of subdirectly irreducible *DMA*'s is provided (in § 2). Next some steps are made (in § 3) towards a more transparent characterization, and such a characterization is established for a class of *DMA*'s satisfying a certain condition. In the final section of the chapter, an example of a class of algebras that does satisfy that condition, Ockham algebras, is discussed. First, though, the notion of topo-heredity is introduced.

4.1 Topo-Heredity

Here the notions of topo-heredity and topo-root are introduced. For the remainder of this section, let $\mathbb{X} = (X, \leqslant, \tau, R_\Diamond, R_\Box, R_\lhd, R_\rhd)$ be an extended Priestley space.

Recall that the order-heredity of a subset of the underlying set of a frame played a key role in the subdirect irreducibility characterization of chapter 2. We also make use of the notion of order-heredity for a subset of the underlying set of an extended Priestley space:

Definition 4.1 A subset $c \subseteq X$ is an *order-hereditary* subset of $\mathbb{X}$ if it is such a set with respect to the underlying frame of $\mathbb{X}$.

A subset of X is a *topo-hereditary* subset of $\mathbb{X}$ if a is a closed order-hereditary subset of $\mathbb{X}$. An proper topo-hereditary subset of $\mathbb{X}$ is a topo-hereditary subset of $\mathbb{X}$ which is also a proper subset of X. Now we are ready to define the appropriate notion of root:

Definition 4.2 A *topo-root* of $\mathbb{X}$ is an element of X not contained in any proper topo-hereditary subset of $\mathbb{X}$.

The set of topo-hereditary subsets of $\mathbb{X}$ is denoted $t\mathcal{H}$. The following states a connection between the notions of order-heredity and of continuous order-preserving bounded morphisms.

Proposition 4.3 If $\chi : X' \to X$ is a continuous order-preserving bounded morphism then $\chi[X']$ is topo-hereditary.

Proof. Let $c := \chi[X']$. Then since χ is continuous and (X, τ) is compact and Hausdorff, standard topological considerations guarantee that $\chi[X']$ is closed. The proof that c satisfies (i) in the definition of order-hereditary is as in the proof of proposition 2.3. $\square$

This last proposition should be compared to proposition 2.3. Notice that an analogue of proposition 2.3 would also state a converse to proposition 4.3, according to which any topo-hereditary subset of $\mathbb{X}$ is the image of some continuous order-preserving bounded morphism. This does indeed obtain, but is not proved here as it will not be needed later; more will be said about this in the next section.

4.2 Subdirect Irreducibility Results

Consider a Priestley space $\mathbb{X} = (X, \leqslant, \tau)$. Notice that since $a \cup b$ and $a \cap b$ are both closed for closed subsets a, b of $\mathbb{X}$, the set of closed subsets of $\mathbb{X}$ forms a lattice under $\supseteq$. It is known (see [DP02] p.266) that this lattice is isomorphic to the congruence lattice of the bounded distributive lattice $\mathbb{X}_*$:

Proposition 4.4 Let $\mathbb{X} = (X, \leqslant, \tau)$ be a Priestley space. Then $\varepsilon : c \mapsto \theta_c$ is an isomorphism from the lattice of closed subsets of $\mathbb{X}$ under $\supseteq$ to the congruence lattice of $\mathbb{X}_*$, where θ_c is defined by

$$(a, b) \in \theta_c \text{ iff } a \cap c = b \cap c.$$

Readers interested in the converse of proposition 4.3 can find it proved in the course of proving (as in [DP02], p.266) the just stated proposition.

The congruences of a *DMA* correspond to those closed subsets of its dual that are order-hereditary. For the remainder of this section, let $\mathbb{X} = (X, \leqslant, \tau, R_\Diamond, R_\Box, R_\lhd, R_\rhd)$ be an extended Priestley space.

Proposition 4.5 $(t\mathcal{H}(\mathbb{X}), \supseteq)$ is isomorphic to the congruence lattice $(Con(\mathbb{X}^*), \subseteq)$ of $\mathbb{X}^*$.

Proof. Let ε be as defined in proposition 4.4, for $a \in t\mathcal{H}(\mathbb{X})$. By proposition 4.4, ε satisfies $\varepsilon(a) \leqslant \varepsilon(b)$ iff $b \subseteq a$ for $a, b \in t\mathcal{H}(\mathbb{X})$. This proves the injectivity of ε. Hence it is left to show the ε is surjective. For this purpose, take an arbitrary $\theta \in Con(\mathbb{X}^*)$. Let $\eta : \mathbb{X}^* \to \mathbb{X}'^*$ be a *DMA*-homomorphism with $\ker \eta = \theta$. Then the duality results ensure the existence of a continuous order-preserving bounded morphism $\chi : \mathbb{X}' \to \mathbb{X}$ such that χ^* is η. Let $c := \chi(X')$, the image of χ in $\mathbb{X}$. Then by proposition 4.3, $c \in t\mathcal{H}(\mathbb{X})$.
Finally, observe that $\theta = \theta_c$. For we have

$$\begin{aligned}
(a, b) \in \theta \text{ iff } & (a, b) \in \ker \chi^*, \\
\text{iff } & \chi^*(a) = \chi^*(b), \\
\text{iff } & \{x \in X' : \chi(x) \in a\} = \{x \in X' : \chi(x) \in b\}, \\
\text{iff } & a \cap \chi(X') = b \cap \chi(X'), \\
\text{iff } & (a, b) \in \theta_{\chi(X')} = \theta_c.
\end{aligned}$$

$\square$

Notice that it follows from this last proposition that $t\mathcal{H}(\mathbb{X})$ forms a lattice under $\supseteq$; this lattice has top element $\emptyset$ and bottom element X. Thus we have so far seen how the lattice of topo-hereditary subsets of an extended Priestley space corresponds to the congruence lattice of its dual. From this correspondence it follows that a *DMA* has a smallest nontrivial complete congruence iff its dual has a greatest proper topo-hereditary subset. Thus we can arrive at a dual characterization of subdirect irreducibility for *DMA*'s by investigating the conditions under which an extended Priestley space has a greatest proper topo-hereditary subset.

Proposition 4.6 $\mathbb{X}$ has a greatest proper topo-hereditary subset iff the set of topo-roots of $\mathbb{X}$ is open and non-empty.

Proof. First suppose $c \subseteq X$ is the greatest proper topo-hereditary subset of $\mathbb{X}$. Let $a := X \setminus c$. Certainly a is open and nonempty, as c is closed and proper; it will be shown that a is the set of topo-roots of $\mathbb{X}$. Consider $x \in a$. Let b be any topo-hereditary subset of $\mathbb{X}$ with $x \in b$. Then $b \nsubseteq c$; so as c is the greatest proper topo-hereditary subset of $\mathbb{X}$, b is not a proper subset of $\mathbb{X}$. Thus x is a topo-root. Hence $a \subseteq X$ is an open and non-empty set of topo-roots.

Suppose for the converse that the set of topo-roots of $\mathbb{X}$ is open and non-empty, so that the set c of non topo-roots is a proper closed subset of $\mathbb{X}$. To see that c satisfies clause (i) of the definition of order-hereditary, consider $x, y \in X$ such that $x \in c$ and $xR_\diamond y$. We must ensure that there is some $z \in c$ for which $xR_\diamond z$ and $z \leqslant y$. Since $x \in c$, x isn't a topo-root. Thus there is some proper topo-hereditary subset b containing x. Then using $x \in b$ and $xR_\diamond y$, the order-heredity of b guarantees the existence of some $z \in b$ such that $xR_\diamond z$ and $z \leqslant y$. Since b is a proper topo-hereditary subset, z is not a topo-root and hence $z \in c$. Establishing that c satisfies the remaining conditions for being order-hereditary proceeds by analogous reasoning. Thus c is a proper topo-hereditary subset of $\mathbb{X}$. Now if $a \nsubseteq c$ for $a \subseteq X$ then a must contain some topo-root; so if a is topo-hereditary then $a = X$. This establishes that c is the greatest proper topo-hereditary subset of $\mathbb{X}$. □

This affords the following characterization:

Theorem 4.7 A nontrivial *DMA* $\mathbb{A}$ is subdirectly irreducible iff the set of topo-roots of its dual $\mathbb{A}_*$ is open and non-empty.

Proof. Since $\mathbb{A}$ is nontrivial, $\mathbb{A}$ is subdirectly irreducible iff $\mathbb{A}$ has a least nontrivial congruence, iff (by proposition 4.5) $\mathbb{A}_*$ has a greatest proper topo-hereditary subset, iff (by proposition 4.6) the set of topo-roots of $\mathbb{A}_*$ is open and non-empty. □

4.3 M-Heredity

This section makes steps towards a more transparent characterization of subdirectly irreducible *DMA*'s. It turns out, we will see, that a set is order-hereditary iff it is hereditary with respect to a certain restriction $M \subseteq R_\diamond \cup R_\Box \cup R_\triangleright \cup R_\triangleleft$, to be defined in terms of the maximal and minimal elements of the set of successors of an element of $\mathbb{X}$.

For a partially ordered set $(X, \leqslant)$ with $a \subseteq X$, $\mathsf{min}a$ is the set of all minimal element of a. That is, $\mathsf{min}a = \{x \in a : \forall y \in a(y \leqslant x \rightarrow y = x)\}$. Similarly, $\mathsf{max}a = \{x \in a : \forall y \in a(x \leqslant y \rightarrow y = x)\}$. If R is a binary relation on X and $x \in X$ then $R[x] := \{y \in X : xRy\}$ is the set of R-successors of x.

Definition 4.8 Let $M \subseteq X \times X$ be defined by $M := M_\diamond \cup M_\Box \cup M_\triangleright \cup M_\triangleleft$, where $M_\diamond$, $M_\Box$,

$M_\triangleright$, $M_\triangleleft \subseteq X \times X$ are given by

$$\begin{aligned} xM_\Diamond y &\quad \text{iff } y \in \mathsf{min} R_\Diamond[x], \\ xM_\Box y &\quad \text{iff } y \in \mathsf{max} R_\Box[x], \\ xM_\triangleright y &\quad \text{iff } y \in \mathsf{min} R_\triangleright[x], \text{ and} \\ xM_\triangleleft y &\quad \text{iff } y \in \mathsf{max} R_\triangleleft[x]. \end{aligned}$$

A subset $a \subseteq X$ is *M-hereditary* if $y \in a$ for all $x, y \in X$ with $x \in a$ and xMy.

Notice that the following is an immediate consequence of the above definition:

Proposition 4.9 A subset $a \subseteq X$ is M-hereditary iff for all $x \in X$,

(i) if $x \in a$ then $\mathsf{min} R_\Diamond[x] \subseteq a$,

(ii) if $x \in a$ then $\mathsf{max} R_\Box[x] \subseteq a$,

(iii) if $x \in a$ then $\mathsf{min} R_\triangleright[x] \subseteq a$, and

(iv) if $x \in a$ then $\mathsf{max} R_\triangleleft[x] \subseteq a$.

The following result is what we need to show that being M-hereditary coincides with being order-hereditary.

Proposition 4.10 Consider $x \in X$. Then:

(i) For all $y \in R_\Diamond[x]$, there is a $z \in \mathsf{min} R_\Diamond[x]$ such that $z \leqslant y$,

(ii) For all $y \in R_\Box[x]$, there is a $z \in \mathsf{max} R_\Box[x]$ such that $y \leqslant z$,

(iii) For all $y \in R_\triangleright[x]$, there is a $z \in \mathsf{min} R_\triangleright[x]$ such that $y \leqslant z$,

(iv) For all $y \in R_\triangleleft[x]$, there is a $z \in \mathsf{max} R_\triangleleft[x]$ such that $y \leqslant z$.

Proof. A proof will be supplied for only (i); the remaining cases are similar. Let $\mathbb{A}$ be a *DMA* such that $\mathbb{A}_* \cong \mathbb{X}$. Consider $I \in \mathcal{I}_\mathcal{P}(\mathbb{A})$. We will see that every descending chain in $Q_\Diamond[I]$ has a lower bound; (i) will then follow by Zorn's lemma and duality results. So let $\{J_i : i \in \alpha\} \subseteq Q_\Diamond[I]$ be such that $J_i \subseteq J_j$ for $j \leqslant i$. Let $K := \bigcap\{J_i : i \in \alpha\}$. Since each J_i is an ideal, K is also an ideal. Moreover, since each J_i is prime, $0 \in J_i$ and thus K is nonempty. To see that K is prime, suppose $x, y \notin K$. Then for some $i, j \in \alpha$ we have $x \notin J_i$ and $y \notin J_j$. Without loss of generality, we may assume that $j \leqslant i$ so that $J_i \subseteq J_j$. Thus $x, y \notin J_i$. So since J_i is prime, $x \wedge y \notin J_i$. But then also $x \wedge y \notin K$. So $K \in \mathcal{I}_{\mathcal{P}}(\mathbb{A})$. To see that $K \in Q_\diamond[I]$, notice that the definition of $Q_\diamond$, as given in chapter 3, ensures that

$$Q_\diamond[I] = \{J \in \mathcal{I}_{\mathcal{P}}(\mathbb{A}) : \diamond x \in I \text{ implies } x \in J, \text{ for all } x \in A\}.$$

Consider $\diamond a \in I$; it follows from $J_i \in Q_\diamond[I]$ that a belongs to each J_i, so that $a \in K = \bigcap\{J_i : i \in \alpha\}$. This shows that $IQ_\diamond K$. So every descending chain in $Q_\diamond[I]$ has a lower bound. Thus, since $\mathbb{X} \cong \mathbb{A}_*$ for some $\mathbb{A}$, every descending chain in $R_\diamond[x]$ has a lower bound.

Now consider $y \in R_\diamond[x]$ and let $\mathcal{C}$ by the class of all descending chains in $R_\diamond[x]$ with maximum element y. Since each element of $\mathcal{C}$ has a lower bound, by Zorn's lemma $\mathcal{C}$ has a minimum $z \in \mathsf{min}R_\diamond[x]$. So there is a $z \leqslant y$ with $z \in \mathsf{min}R_\diamond[x]$. $\square$

Remark This last result is due to the fact that $R_\triangle[x]$ is closed (as is guaranteed by the condition (TOP) in chapter 3) for $x \in X$ and $\triangle \in \{\diamond, \square, \triangleright, \triangleleft\}$. To see this, consider replacing $Q_\diamond[I]$ in the above proof with an arbitrary closed set $u \subseteq \mathcal{I}_{\mathcal{P}}(\mathbb{A})$. Then where $\{J_i : i \in \alpha\} \subseteq u$ is such that $J_i \subseteq J_j$ for $j \leqslant i$, the reasoning of the above proof establishes that $K := \bigcap\{J_i : i \in \alpha\}$ is a prime ideal. As in the above proof, it remains to ensure that $K \in u$. For this, proposition A.7 is useful, according to which

$$u = \bigcap_{(x,y)\in\beta} \{J \in \mathcal{I}_{\mathcal{P}}(\mathbb{A}) : x \in J \rightarrow y \in J\}$$

for some $\beta \subseteq A \times A$. Consider $(x, y) \in \beta$; it follows from $J_i \in u$ that for each J_i we have $x \in J_i$ implies $y \in J_i$. Thus if $x \in \bigcap\{J_i : i \in \alpha\} = K$ also $y \in K$, and hence $K \in u$ as required.

Proposition 4.11 For $a \subseteq X$, a is order-hereditary iff a is M-hereditary.

Proof. First suppose a is order-hereditary. To see that a satisfies (i) in proposition 4.9, consider $x \in a$ and $y \in \mathsf{min}R_\diamond[x]$; we must ensure that $y \in a$. Since $xR_\diamond y$, by condition (i) of the definition of order-hereditary, there is some $z \leqslant y$ such that $z \in R_\diamond[x]$ and $z \in a$. But since $y \in \mathsf{min}R_\diamond[x]$, $z = y$. Thus $y \in a$ as required. The remaining conditions (ii), (iii) and (iv) follow similarly.

Now suppose a is M-hereditary. To see that a satisfies (i) of the definition of order-hereditary, consider $x \in a$ and $y \in R_\diamond[x]$. By proposition 4.10 there is a $z \in \mathsf{min}R_\diamond[x]$ such that $z \leqslant y$.

Then by condition (i) in proposition 4.9, $z \in a$. Thus $xR_\diamond z$ and $x \leqslant z$. The remaining conditions (ii), (iii) and (iv) follow similarly. $\square$

Thus we have:

Proposition 4.12 $\mathbb{X}^*$ is subdirectly irreducible iff $\mathbb{X}$ has a greatest proper closed M-hereditary subset.

Proof. Since $X \neq \emptyset$, $\mathbb{X}^*$ cannot be trivial and thus $\mathbb{X}^*$ is subdirectly irreducible iff $\mathbb{X}^*$ has a least nontrivial congruence. And $\mathbb{X}^*$ has a least nontrivial congruence iff (by proposition 4.5) $\mathbb{X}$ has a greatest proper closed order-hereditary subset, iff (by proposition 4.11) $\mathbb{X}$ has a greatest proper closed M-hereditary subset. $\square$

Next the notion of M-toporoot, depending on the topology on $\mathbb{X}$ as well as M, is introduced. Some notation is useful for this. For a binary relation R on a set X with $x \in X$, $R^\omega[x]$ denotes the set of elements of X that can be reached from x in any finite number of R-steps. That is, $R^\omega[x] = \bigcup_{n\in\mathbb{N}} R^n[x]$, where

$$\begin{aligned} R^0[x] =& \{x\}, \\ R^{n+1}[x] =& \{y \in X : xRy \text{ and } x \in R^n[x]\}, \text{ for } n \neq 0. \end{aligned}$$

If $a \subseteq X$ then $\overline{a}$ denotes the topological closure of a.

Definition 4.13 For $x \in X$, x is an *M-toporoot* of $\mathbb{X}$ if $\overline{M^\omega[x]} = X$.

Proposition 4.14 A *DMA* $\mathbb{A}$ is subdirectly irreducible if the set of M-toporoots of $\mathbb{A}_*$ is open and nonempty.

Proof. Suppose that the set a of M-toporoots of $\mathbb{X} := \mathbb{A}_*$ is open and nonempty. Let $c := X \setminus a$. It will be shown that c is the greatest proper closed M-hereditary subset; the result then follows by proposition 4.12. Certainly c is closed and proper, as a is open and nonempty. To see that c is M-hereditary, suppose otherwise, assuming $x \in c$ but $y \notin c$ for $y \in \mathsf{min}R_\diamond[x]$ so that c violates condition (i) of proposition 4.9. Then $y \in a$ and thus y is an M-toporoot. Thus $\overline{M^\omega[y]} = X$. Now since $y \in \mathsf{min}R_\diamond$, we have xMy. Thus $M^\omega[y] \subseteq M^\omega[x]$, and hence $\overline{M^\omega[y]} \subseteq \overline{M^\omega[x]} = X$. Thus x is an M-toporoot and so $x \in a$, a contradiction. Supposing that v violates any of conditions (ii), (iii) and (iv) similarly leads to a contradiction. This shows that c is a proper closed M-hereditary subset.

Now suppose that $b \nsubseteq c$ for some closed M-hereditary subset b of $\mathbb{X}$. To ensure that c is the greatest proper closed M-hereditary subset of b, we must check that b is not proper. Consider $x \in b \setminus c$. Since $x \notin c$, x is an M-toporoot. Thus $\overline{M^\omega[x]} = X$. Now since b is M-hereditary, $M^\omega[x] \subseteq b$. So

$$X = \overline{M^\omega[x]} \subseteq \overline{b} = b,$$

where this last equality follows because b is closed. Thus $b = X$, as required. $\square$

The following states a partial converse to proposition 4.14.

Proposition 4.15 Suppose that for all x in $\mathbb{A}_* := \mathbb{X}$, $\overline{M^\omega[x]}$ is M-hereditary. Then the set of M-toporoots of $\mathbb{A}_*$ is open and nonempty if $\mathbb{A}$ is subdirectly irreducible.

Proof. Let c be the greatest proper closed M-hereditary subset of $\mathbb{X}$. Let $a := X \setminus c$. Certainly a is open and nonempty, as c is closed and proper; it will be shown that a is the set of M-toporoots of $\mathbb{X}$.
Consider $x \in a$. Now $\overline{M^\omega[x]}$ is closed, and by assumption is also M-hereditary. But $x \in \overline{M^\omega[x]} \setminus c$, and thus $\overline{M^\omega[x]} \nsubseteq c$. So since c is the greatest proper closed M-hereditary subset, $\overline{M^\omega[x]} = X$. Thus x is an M-toporoot.

Now consider $x \notin a$; we must ensure that x is not an M-toporoot. Since c is M-hereditary and $x \in c$, $M^\omega[x] \subseteq c$. So $\overline{M^\omega[x]} \subseteq \overline{c} = c$, where this last equality follows from c's being closed. Thus, since c is proper, $\overline{M^\omega[x]} \neq X$. And thus x is not an M-toporoot of $\mathbb{X}$. $\square$

4.4 Example: Ockham Algebras

Here a characterization of subdirect irreducibility is obtained for a class of algebras satisfying the hypothesis of proposition 4.15.

An Ockham algebra $\mathbb{A} = (A, \vee, \wedge, 0, 1, \sim)$ is an algebra with $(A, \vee, \wedge, 0, 1)$ a bounded distributive lattice with a weak negation $\sim$ reversing top and bottom elements and satisfying de Morgan's laws. That is, $\sim$ is a unary operator on A such that

(1) $\sim(a \vee b) = \sim a \wedge \sim b$, $\sim 0 = 1$,

(2) $\sim(a \wedge b) = \sim a \vee \sim b$, $\sim 1 = 0$.

Notice that the conditions (1) are those that must be satisfied by the $\rhd$ operator in a *DMA*, and the conditions (2) are those required of $\lhd$. Thus Ockham algebras can be seen as *DMA*'s $(A, \vee, \wedge, 0, 1, \rhd, \lhd)$ satisfying $\rhd a = \lhd a$ for all $a \in A$.

The dual of such an algebra is $\mathbb{A}_* = (\mathcal{I_P}(\mathbb{A}), \subseteq, \tau, R_\rhd, R_\lhd)$, where $R_\rhd$ and $R_\lhd$ are binary relations on $\mathcal{I_P}(\mathbb{A})$ satisfying

$$IR_\rhd J \text{ iff } \quad a \notin J \to {\sim}a \in I \text{ for all } a \in A,$$

$$IR_\lhd J \text{ iff } \quad a \in J \to {\sim}a \notin I \text{ for all } a \in A.$$

Here ${\sim}a := \rhd a = \lhd a$ for $a \in A$.

Consider $R_\sim \subseteq \mathcal{I_P}(\mathbb{A}) \times \mathcal{I_P}(\mathbb{A})$ defined by $R_\sim := R_\rhd \cap R_\lhd$. $\mathbb{A}$'s being subdirectly irreducible will be characterized it terms of this relation $R_\sim$ on the dual $\mathbb{A}_*$. The characterization will be established via propositions 4.14 and 4.15 above. We will see that the relation M figuring in these propositions coincides in this special case with $R_\sim$, and that $\overline{M^\omega[I]}$ is M-hereditary for $I \in \mathcal{I_P}(\mathbb{A})$. It will then follow that $\mathbb{A}$ is subdirectly irreducible iff the set of $R_\sim$-toporoots of $\mathbb{A}_*$ is open and nonempty. $R_\sim$-toporoots coincide with M-toporoots, so that

Definition 4.16 $I \in \mathcal{I_P}(\mathbb{A})$ is an $R_\sim$-toporoot of $\mathbb{A}_*$ if $\mathcal{I_P}(\mathbb{A}) = \overline{R_\sim^\omega[I]}$.

Proposition 4.17 For every $I \in \mathcal{I_P}(\mathbb{A})$, $R_\sim(I) := \{a \in A : {\sim}a \notin I\}$ is the unique $R_\sim$-successor of I. Moreover, for all $a \in A$,

$$a \in R_\sim(I) \text{ iff } {\sim}a \notin I.$$

Proof. It is immediate from the conditions on $R_\rhd$ and $R_\lhd$, and the definition $R_\sim = R_\rhd \cap R_\lhd$ that $IR_\sim J$ iff for all $a \in A$,

$$a \in J \text{ iff } {\sim}a \notin I.$$

Thus $J = \{a \in A : {\sim}a \notin I\}$ if J is an $R_\sim$-successor of I. It remains to ensure that $\{a \in A : {\sim}a \notin I\} \in \mathcal{I_P}(\mathbb{A})$. By proposition 3.4($iv$), $\{a \in A : {\sim}a \notin I\}$ is an ideal. To check that it is prime, consider $a, b \notin \{a \in A : {\sim}a \notin I\}$. Then ${\sim}a, {\sim}b \in I$, and so ${\sim}a \vee {\sim}b = {\sim}(a \wedge b) \in I$. But then $a \wedge b \notin \{a \in A : {\sim}a \notin I\}$. $\square$

The prime ideal $\underbrace{R_\sim(\cdots R_\sim}_{n}(I)\cdots)$, reachable from I in n $R_\sim$-steps, is denoted $R_\sim^n(I)$.

Next we see how to express $R_{\triangleright}$ and $R_{\triangleleft}$ in terms of $R_{\sim}$; this is useful in establishing that $R_{\sim}$ coincides with M.

Proposition 4.18

(i) $R_{\triangleright} = R_{\sim} \circ \subseteq$

(ii) $R_{\triangleleft} = R_{\sim} \circ \supseteq$

Proof. Consider $I, J \in \mathcal{I}_{\mathcal{P}}(\mathbb{A})$. For (i) observe that (by proposition 4.17) we have $I(R_{\sim} \circ \subseteq)J$ iff $R_{\sim}(I) \subseteq J$, iff $\{a \in A : \sim a \notin I\} \subseteq J$, iff for all $a \in A$, $a \notin J$ implies $\sim a \in I$, iff $IR_{\triangleright}J$.

And for (ii) observe that (again by proposition 4.17) we have $I(R_{\sim} \circ \supseteq)J$ iff $R_{\sim}(I) \supseteq J$, iff $\{a \in A : \sim a \notin I\} \supseteq J$, iff for all $a \in A, a \in J$ implies $\sim a \notin I$, iff $IR_{\triangleleft}J$. $\square$

For these algebras, with only the two relations $R_{\triangleright}$ and $R_{\triangleleft}$, the relation M defined in the previous section is $M_{\triangleright} \cup M_{\triangleleft}$. Recall that

$IM_{\triangleright}J$ iff $J \in \mathsf{min}R_{\triangleright}[I]$, and

$IM_{\triangleleft}J$ iff $J \in \mathsf{max}R_{\triangleleft}[I]$,

for $I, J \in \mathcal{I}_{\mathcal{P}}(\mathbb{A})$. We will now see that M and $R_{\sim}$ coincide, and that $\overline{R^{\omega}_{\sim}[I]}$ is $R_{\sim}$-hereditary. Recall that a set u is $R_{\sim}$-hereditary iff for all $I, J \in \mathcal{I}_{\mathcal{P}}(\mathbb{A})$ we have $I \in u$ and $IR_{\sim}J$ implies $J \in u$.

Proposition 4.19 For $I, J \in \mathcal{I}_{\mathcal{P}}(\mathbb{A})$, IMJ iff $IR_{\sim}J$.

Proof. This follows from proposition 4.18, for we have

$$\begin{aligned}
IMJ \text{ iff } & IM_{\triangleright}J \text{ or } IM_{\triangleleft}J,\\
\text{iff } & J \in \mathsf{min}R_{\triangleright}[I] \text{ or } J \in \mathsf{max}R_{\triangleleft}[I],\\
\text{iff } & J \in \mathsf{min}((R_{\sim} \circ \subseteq)[I]) \text{ or } J \in \mathsf{max}((R_{\sim} \circ \supseteq)[I]),\\
\text{iff } & J \in R_{\sim}[I] \text{ or } J \in R_{\sim}[I],\\
\text{iff } & IR_{\sim}J.
\end{aligned}$$

$\square$

Proposition 4.20 For $I \in \mathcal{I}_{\mathcal{P}}(\mathbb{A})$, $\overline{R^{\omega}_{\sim}[I]}$ is $R_{\sim}$-hereditary.

Proof. Consider $I \in \mathcal{I}_{\mathcal{P}}(\mathbb{A})$. Suppose $J \in \overline{R_\sim^\omega[I]}$ and $JR_\sim K$. To see that $K \in \overline{R_\sim^\omega[I]}$, suppose otherwise, considering some closed set u for which $\overline{R_\sim^\omega[I]} \subseteq u$ but $K \notin u$. To obtain a contradiction, it suffices to find a closed set v such that $\overline{R_\sim^\omega[I]} \subseteq v$ but $J \notin v$, for then $J \notin \overline{R_\sim^\omega[I]}$.

Since u is closed, proposition A.7 ensures that $u = \bigcap_{(a,b)\in\alpha}\{I_0 \in \mathcal{I}_{\mathcal{P}}(\mathbb{A}) : a \in I_0 \to b \in I_0\}$ for some $\alpha \subseteq A \times A$. Thus, as $K \notin u$, we have $a \in K$ and $b \notin K$ for some $(a,b) \in \alpha$. For this a, b, also

$$a \in R_\sim^n(I) \to b \in R_\sim^n(I) \text{ for all } n,$$

as $R_\sim^n(I) \in R_\sim^\omega[I] \subseteq u$ for all n. Let $v = \{I_0 \in \mathcal{I}_{\mathcal{P}}(\mathbb{A}) : \sim b \in I_0 \to \sim a \in I_0\}$. Since v is closed by proposition A.7, it remains to show that $R_\sim^\omega[I] \subseteq v$ but $J \notin v$. Now since $a \in K, b \notin K$ and $JR_\sim K$, proposition 4.17 ensures that $\sim a \notin J$ and $\sim b \in J$; thus $J \notin v$. To see that $\overline{R_\sim^\omega[I]} \subseteq v$, suppose otherwise, assuming $R_\sim^n(I) \notin v$ for some n. Then $\sim b \in R_\sim^n(I)$ and $\sim a \notin R_\sim^n(I)$. But then by proposition 4.17 we have $a \in R_\sim^{n+1}(I)$ and $b \notin R_\sim^{n+1}(I)$, contradicting $a \in R_\sim^{n+1}(I) \to b \in R_\sim^{n+1}(I)$. □

Proposition 4.21 An Ockham algebra $\mathbb{A} = (A, \vee, \wedge, 0, 1, \sim)$ is subdirectly irreducible iff the set of $R_\sim$-toporoots of its dual $\mathbb{X} = (\mathcal{I}_{\mathcal{P}}(\mathbb{A}), \subseteq, \tau, R_\sim)$ is open and nonempty.

Proof. Observe that an equivalence relation θ on A is a congruence of $(A, \vee, \wedge, 0, 1, \sim)$ iff it is a congruence of the corresponding *DMA* $\mathbb{A}' = (A, \vee, \wedge, 0, 1, \rhd, \lhd)$ satisfying $\rhd a = \lhd a$ for all $a \in A$. Thus $\mathbb{A}$ is subdirectly irreducible iff $\mathbb{A}'$ is subdirectly irreducible. And by propositions 4.19 and 4.20, $\overline{M^\omega[I]}$ is M-hereditary. So $\mathbb{A}$ is subdirectly irreducible iff $\mathbb{A}'$ is subdirectly irreducible, iff (by propositions 4.14 and 4.15) the set of M-toporoots of $\mathbb{A}'_*$ is open and nonempty, iff (by proposition 4.19) the set of $R_\sim$-toporoots of $\mathbb{X}$ is open and nonempty.
□

Appendix A

Priestley Duality

All results here without proof are proved in chapter 11 of [DP02].

A.1 Priestley Spaces

An *ordered topological space* is a triple $\mathbb{X} = (X, \leqslant, \tau)$ where $(X, \leqslant)$ is a partially ordered set and (X, τ) is a topological space. Terminology for partially ordered sets and topological spaces is applied also to ordered topological spaces. For example, where $\mathbb{X} = (X, \leqslant, \tau)$ is an ordered topological space, $\mathbb{X}$ is closed iff the associated topological space (X, τ) is closed. And a closed down-set of $\mathbb{X}$ is a subset of X that is both a closed set of (X, τ) and a down-set of $(X, \leqslant)$.

The set of clopen down-sets of an ordered topological space $\mathbb{X}$ is denoted $\mathsf{cl}\mathcal{D}(\mathbb{X})$.

An ordered topological space $\mathbb{X} = (X, \leqslant, \tau)$ is a *Priestley space* if it is compact and such that for any $x, y \in X$ with $x \not\geqslant y$ there is a clopen down-set a such that $x \in a$ and $y \notin a$.

The following states a useful property of Priestley spaces:

Proposition A.1 Let $\mathbb{X} = (X, \leqslant, \tau)$ be a Priestley space. Let $c \subseteq X$ be a closed down-set of $\mathbb{X}$ and let $x \notin c$. Then there is a clopen down-set a of $\mathbb{X}$ for which $c \subseteq a$ and $x \notin a$.

A Priestley space $\mathbb{X} = (X, \leqslant, \tau)$ gives rise to two "weaker" topologies $T_1, T_2 \subseteq T$ on X. Let $T_1 = \{\bigcup U : U \subseteq \mathsf{cl}\mathcal{D}(\mathbb{X})\}$ and let $T_2 = \{\bigcup\{(X \setminus a) : a \in U\} : U \subseteq \mathsf{cl}\mathcal{D}(\mathbb{X})\}$.

Proposition A.2 T_1 and T_2 are topologies on X

Proof. Since $\emptyset, X \in \mathsf{cl}\mathcal{D}(\mathbb{X})$, also $\emptyset = \bigcup\{\emptyset\}$, $X = \bigcup\{X\} \in T_1$. And certainly T_1 is closed under arbitrary union. To see that T_1 is closed under finite intersection, consider $U_1, \ldots, U_n \subseteq \mathsf{cl}\mathcal{D}(\mathbb{X})$. Observe that since $a_1 \cap \cdots \cap a_n \in \mathsf{cl}\mathcal{D}(\mathbb{X})$ for $a_1, \ldots, a_n \in \mathsf{cl}\mathcal{D}(\mathbb{X})$, we have

$$\bigcup U_1 \cap \cdots \cap \bigcup U_n = \bigcup\{a_1 \cap \cdots \cap a_n : a_1 \in U_1, \ldots, a_n \in U_n\} \in T_1.$$

Thus T_1 is a topology on X; the reasoning to establish that T_2 is a topology on X is similar.
□

T_1 is the *upper topology (for* $\mathbb{X}$*)*, and has basis $\mathsf{cl}\mathcal{D}(\mathbb{X})$. T_2 is the *lower topology (for* $\mathbb{X}$*)*, and has basis $\{X \setminus a : a \in \mathsf{cl}\mathcal{D}(\mathbb{X})\}$.

A.2 Representing Bounded Distributive Lattices

The dual of a Priestley space is a bounded distributive lattice:

Proposition A.3 Where $\mathbb{X} = (X, \leqslant, \tau)$ is a Priestley space, $\mathbb{X}^\star := (\mathsf{cl}\mathcal{D}(\mathbb{X}), \cup, \cap, \emptyset, X)$ is a bounded distributive lattice.

Next the notion of the Priestley dual of a bounded distributive lattice is formulated, with some useful notation introduced along the way:

Definition A.4 Let $\mathbb{A} = (A, \vee, \wedge, 0, 1)$ be a bounded distributive lattice. For $a \in A$,

$$\hat{a} := \{I \in \mathcal{I}_\mathcal{P}(\mathbb{A}) : a \notin I\}.$$

The *Priestley dual* of $\mathbb{A}$ is $\mathbb{A}_\star := (\mathcal{I}_\mathcal{P}(\mathbb{A}), \subseteq, \tau)$, where τ is the topology on $\mathcal{I}_\mathcal{P}(\mathbb{A})$ with basis $\{\hat{a} \cap (\mathcal{I}_\mathcal{P}(\mathbb{A}) \setminus \hat{b}) : a, b \in A\}$.

Here $\mathcal{I}_\mathcal{P}(\mathbb{A}$ denotes the set of prime ideals of $\mathbb{A}$.

Proposition A.5 The Priestley dual $\mathbb{A}_\star$ of a bounded distributive lattice $\mathbb{A}$ is a Priestley space.

The next proposition concerns the set $\mathsf{cl}\mathcal{D}(\mathbb{A}_\star)$ of clopen down-sets of the dual $\mathbb{A}_\star$ of a bounded distributive lattice $\mathbb{A}$.

Proposition A.6 Let $\mathbb{A}$ be a bounded distributive lattice with carrier A. Then $\{\hat{a} : a \in A\} = \mathsf{cl}\mathcal{D}(\mathbb{A}_\star)$.

Proposition A.7 Let $\mathbb{A} = (A, \vee, \wedge, 0, 1)$ be a bounded distributive lattice. Then $u \subseteq \mathcal{I}_{\mathcal{P}}(\mathbb{A})$ is a closed subset of $\mathbb{A}_\star$ iff $u = \bigcap_{(a,b)\in\alpha}\{I \in \mathcal{I}_{\mathcal{P}}(\mathbb{A}) : b \in I \to a \in I\}$ for some $\alpha \subseteq A \times A$.

Proof. Since the topology on $\mathbb{A}_\star$ has basis $\{\hat{a} \cap (\mathcal{I}_{\mathcal{P}}(\mathbb{A}) \setminus \hat{b}) : a, b \in A\}$, the open sets of $\mathbb{A}_\star$ are those sets $\bigcup_{(a,b)\in\alpha}\{I \in \mathcal{I}_{\mathcal{P}}(\mathbb{A}) : a \notin I \text{ and } b \notin I\}$ for $\alpha \subseteq A \times A$. The complements of such sets are the sets $\bigcap_{(a,b)\in\alpha}\{I \in \mathcal{I}_{\mathcal{P}}(\mathbb{A}) : b \in I \to a \in I\}$ for some $\alpha \subseteq A \times A$. □

The following pair of propositions states that the bidual of a Priestley space $\mathbb{X}$ is isomorphic to $\mathbb{X}$, and similarly that the bidual of a bounded distributive lattice $\mathbb{A}$ is isomorphic to $\mathbb{A}$.

Proposition A.8 Where $\mathbb{X}$ is a Priestley space, $\mathbb{X} \cong (\mathbb{X}^\star)_\star$ via $f : x \mapsto \{a \in \mathsf{cl}\mathcal{D}(\mathbb{X}) : x \notin a\}$.

Proposition A.9 Where $\mathbb{A}$ is a bounded distributive lattice, $\mathbb{A} \cong (\mathbb{A}_\star)^\star$ via $g : a \mapsto \hat{a}$.

A.3 Upper and Lower Topologies

The next proposition states an immediate consequence of the definition (in section A.1 above) of upper and lower topologies; it is recorded here for use in chapter 3.

Proposition A.10 Let $\mathbb{A} = (A, \vee, \wedge, 0, 1)$ be a bounded distributive lattice. Consider $Y \subseteq A$. Then

(*i*) $\mathcal{I}_{\mathcal{P}}(\mathbb{A}) \setminus \bigcup\{\hat{a} : a \in Y\}$ is closed in the upper topology for $\mathbb{A}_\star$.

(*ii*) $\mathcal{I}_{\mathcal{P}}(\mathbb{A}) \setminus \bigcup\{\mathcal{I}_{\mathcal{P}}(\mathbb{A}) \setminus \hat{a} : a \in Y\}$ is closed in the lower topology for $\mathbb{A}_\star$.

Proof. For $a \in A$, proposition A.6 ensures that $\hat{a} \in \mathsf{cl}\mathcal{D}(\mathbb{X})$. So $\bigcup\{\hat{a} : a \in Y\}$ is open in the topology with basis $\mathsf{cl}\mathcal{D}(\mathbb{X})$, and thus $\mathcal{I}_{\mathcal{P}}(\mathbb{A}) \setminus \bigcup\{\hat{a} : a \in Y\}$ is closed in the upper topology for $\mathbb{X}$. This establishes (*i*); the reasoning for (*ii*) is similar. □

A.4 Morphisms

The representation above is extended to a full duality between the category of bounded distributive lattices with homomorphisms and the category of Priestley spaces with Priestley morphisms.

Let $\mathbb{X} = (X, \leqslant, \tau)$ and $\mathbb{X}' = (X', \leqslant', \tau')$ be Priestley spaces. Let $\mathbb{A} = (A, \vee, \wedge, 0, 1)$ and $\mathbb{A}' = (A', \vee', \wedge, 0', 1')$ be bounded distributive lattices.

A *Priestley morphism* between $\mathbb{X}$ and $\mathbb{X}^*$ is a continuous map $\chi : X \to X'$ from $\mathbb{X}$ to $\mathbb{X}'$ such that $x \leqslant y$ implies $\chi(x) \leqslant \chi(y)$ for all $x, y \in X$. A *homomorphism* from $\mathbb{A}$ to $\mathbb{A}'$ is a map $\eta : A \to A'$ satisfying $\eta(0) = 0'$, $\eta(1) = 1'$, and such that for $a, b \in A$ we have $\eta(a \vee b) = \eta(a) \vee \eta(b)$ and $\eta(a \wedge b) = \eta(a) \wedge \eta(b)$.

The dual $\chi^* : \mathsf{cl}\mathcal{D}(\mathbb{X}') \to \mathcal{P}(X)$ of a Priestley morphism χ between $\mathbb{X}$ and $\mathbb{X}'$ is defined by $\chi^* : a \mapsto \{x \in X : \chi(x) \in a\}$. The dual $\eta_* : \mathcal{I}_{\mathcal{P}}(A)$ of a homomorphism from $\mathbb{A}$ to $\mathbb{A}'$ is defined by $\eta_* : I \mapsto \{a \in a : \eta(a) \in I\}$.

Proposition A.11 The dual χ^* of a Priestley morphism χ between $\mathbb{X}$ and $\mathbb{X}'$ is a homomorphism from $\mathbb{X}'^*$ to $\mathbb{X}^*$.

Proposition A.12 The dual η_* of a homomorphism η from $\mathbb{A}$ to $\mathbb{A}'$ is a Priestley morphism between $\mathbb{A}'_*$ and $\mathbb{A}_*$.

Proposition A.13 Where $f_{\mathbb{X}}$ and $f'_{\mathbb{X}}$ are bijections as in proposition A.8 from $\mathbb{X}$ to $(\mathbb{X}^*)_*$ and $\mathbb{X}'$ to $(\mathbb{X}'^*)_*$ respectively, and χ is a Priestley morphism between $\mathbb{X}$ and $\mathbb{X}'$,

$$f_{\mathbb{X}'}(\chi(x)) = (\chi^*)_*(f_{\mathbb{X}}(x)), \text{ for all } x \in X.$$

Proposition A.14 Where $g_{\mathbb{A}}$ and $g'_{\mathbb{A}}$ are bijections as in proposition A.9 from $\mathbb{A}$ to $(\mathbb{A}_*)^*$ and $\mathbb{A}'$ to $(\mathbb{A}'_*)^*$ respectively, and η is a homomorphism from $\mathbb{A}$ to $\mathbb{A}'$,

$$g_{\mathbb{A}'}(\eta(a)) = (\eta_*)^*(g_{\mathbb{A}}(a)), \text{ for all } a \in A.$$

Thus we arrive at:

Theorem A.15 The category of Priestley spaces with Priestley morphisms is dually equivalent to the category of bounded distributive lattices with homomorphisms.

Appendix B

Categories

Here some notions pertaining to category theory are discussed. The first section contains an explanation of what is involved in specifying a category, and the second section explains what must be proved in order to establish that two categories are dually equivalent. The third section contains a more careful exposition of the working definition of subdirect irreducibility, as mentioned in the introduction; the notion of a category is used for this.

B.1 Categories

A *Category* $\mathcal{C} = (\mathcal{O}_{\mathcal{C}}, \mathcal{M}_{\mathcal{C}}, \circ_{\mathcal{C}})$ consist of a collection $\mathcal{O}_{\mathcal{C}}$ of *objects* together with a collection $\mathcal{M}_{\mathcal{C}}$ of *morphisms* and *compositions* $\circ_{\mathcal{C}}$. The set $\mathcal{M}_{\mathcal{C}}$ is the union of a collection of families $\{Mor_{\mathcal{C}}(\mathbb{A}, \mathbb{B})\}$ of mutually disjoint sets, where there is one such family for each pair $\mathbb{A}, \mathbb{B} \in \mathcal{O}_{\mathcal{C}}$. The set $\circ_{\mathcal{C}}$ consists of the union of a collection of families of maps

$$\{Mor_{\mathcal{C}}(\mathbb{A}, \mathbb{B}) \times Mor_{\mathcal{C}}(\mathbb{B}, \mathbb{C}) \to Mor_{\mathcal{C}}(\mathbb{A}, \mathbb{C})\},$$

where there is one such family for each $\mathbb{A}, \mathbb{B}, \mathbb{C} \in \mathcal{O}_{\mathcal{C}}$. In order to be a category, such a triple $(\mathcal{O}_{\mathcal{C}}, \mathcal{M}_{\mathcal{C}}, \circ_{\mathcal{C}})$ must satisfy certain conditions (see [Par70] p.2) concerning the associativity of the compositions, and the existence of an identity morphism.

In specifying a category here, only the objects and morphisms will be given; the compositions will be clear from context and will never be explicitly discussed. Similarly, the conditions mentioned above will not be discussed. For example, the collection of partially ordered sets with order-preserving maps forms a category. Here the objects are the partially ordered sets, the morphisms are the order-preserving maps, and the compositions take $(f : \mathbb{A} \to \mathbb{B}, g : \mathbb{B} \to C)$ to the usual composition $g \circ f : \mathbb{A} \to \mathbb{C}$, where $\mathbb{A}, \mathbb{B}, \mathbb{C}$ are partially ordered sets and f, g

are order-preserving maps. The above mentioned conditions here are satisfied by $f : x \mapsto x$ being an order-preserving map, and by the identity $[h \circ (g \circ f)](x) = [(h \circ g) \circ f](x)$ obtaining for all order-preserving maps $f : \mathbb{A} \to \mathbb{B}, g : \mathbb{B} \to \mathbb{C}, h : \mathbb{C} \to \mathbb{C}$, partially ordered sets $\mathbb{A}, \mathbb{B}, \mathbb{C}$, and $x \in \mathbb{A}$. We will be concerned with examples of categories such as this, where the objects consist of an underlying set with additional structure, and the morphisms are maps between these sets preserving aspects of the structure.

B.2 Duality

A *contravariant functor* $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ consists of an object map $\mathcal{F}_{\mathcal{O}} : \mathcal{O}_{\mathcal{C}} \to \mathcal{O}_{\mathcal{D}}$ and morphism maps

$$\mathcal{F}_{\mathbb{A},\mathbb{B}} : Mor_{\mathcal{C}}(\mathbb{A}, \mathbb{B}) \to Mor_{\mathcal{D}}(\mathcal{F}_{\mathcal{O}}(\mathbb{B}), \mathcal{F}_{\mathcal{O}}(\mathbb{A})),$$

for $\mathbb{A}, \mathbb{B}, \in \mathcal{O}_{\mathcal{C}}$. Certain conditions, concerning the existence of identity morphism maps and the composition of morphism maps, must be satisfied (see [Par70] p.7). Again, such conditions will not be mentioned in introducing a functor. In defining a functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ we must check that it does indeed map members of $\mathcal{C}$ to members of $\mathcal{D}$. That is:

$\mathcal{O}1$ $\mathcal{F}_{\mathcal{O}}(\mathbb{C}) \in \mathcal{O}_{\mathcal{D}}$ for $\mathbb{C} \in \mathcal{O}_{\mathcal{C}}$, and

$\mathcal{M}1$ $\mathcal{F}_{\mathbb{A},\mathbb{B}}(f) \in \mathcal{M}_{\mathcal{D}}$, for $\mathbb{A}, \mathbb{B} \in \mathcal{O}_{\mathcal{C}}$ and $f \in Mor_{\mathcal{C}}(\mathbb{A}, \mathbb{B})$.

Henceforth subscripts will be omitted in denoting such maps as $\mathcal{F}_{\mathbb{A},\mathbb{B}}$ and $\mathcal{F}_{\mathcal{O}}$.

Two categories $\mathcal{C}, \mathcal{D}$ are *dually equivalent* if there are contravariant functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ such that their compositions $\mathcal{F}\mathcal{G}$ and $\mathcal{G}\mathcal{F}$ are isomorphic to identities. That is, for the categories we will be concerned with, it is sufficient that there are isomorphisms $\varepsilon_{\mathcal{C}}, \varepsilon_{\mathcal{D}}$ for $D \in \mathcal{D}, C \in \mathcal{C}$ such that:

$\mathcal{O}2$ $\varepsilon_{\mathcal{C}} : \mathbb{C} \to \mathcal{G}(\mathcal{F}(\mathbb{C}))$ for $\mathbb{C} \in \mathcal{C}$,

$\mathcal{O}3$ $\varepsilon_{\mathcal{D}} : \mathbb{D} \to \mathcal{F}(\mathcal{G}(\mathbb{D}))$ for $\mathbb{D} \in \mathcal{D}$,

$\mathcal{M}2$ $\varepsilon_{\mathcal{C}'}(g(c)) = \mathcal{G}(\mathcal{F}(g(\varepsilon_C(c)))$ for c in C, $g : \mathbb{C} \to \mathbb{C}'$,

$\mathcal{M}3$ $\varepsilon_{\mathcal{D}'}(g(d)) = \mathcal{G}(\mathcal{F}(g(\varepsilon_D(d)))$ for d in D, $g : \mathbb{D} \to \mathbb{D}'$.

Establishing that two categories are dually equivalent, then, involves defining maps $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \to C$ and ensuring that the labeled conditions above are satisfied. In proving that those conditions labeled with an $\mathcal{O}$, we establish the "object part" of the duality; the "morphism part" is established by proving that those conditions labeled with a $\mathcal{M}$ are satisfied.

B.3 Categories and Subdirect Irreducibility

Consider a category $\mathcal{C}$ of algebras $\mathcal{A}$ with homomorphisms $\mathcal{M} : \mathcal{A} \to \mathcal{A}$. Suppose $\mathbb{A}$, $\mathbb{A}' \in \mathcal{A}$ and $\mathbb{A}$ has underlying set A. A *congruence* of $\mathbb{A}$ is the kernel $\theta_h := \{(a, b) \in A \times A : h(a) = h(b)\}$ of a homomorphism $h : \mathbb{A} \to \mathbb{A}'$.

The set of congruences $Con(\mathbb{A})$ of $\mathbb{A}$ forms a lattice under $\subseteq$. This *congruence lattice* of $\mathbb{A}$ has top element $A \times A$, and bottom element $\{(a, a) : a \in A\}$. This bottom element is known as the *trivial* congruence.

The following states a "working definition" of subdirect irreducibility:

Proposition B.1 An algebra $\mathbb{A}$ is subdirectly irreducible iff $\mathbb{A}$ is trivial or $\mathbb{A}$ has a smallest nontrivial congruence.

This statement is sometimes taken as a definition of subdirect irreducibility, and sometimes (as in [BS81] section II.8) established as an immediate consequence of a differently formulated definition.

Suppose $\mathbb{A}$ belongs also to another category $\mathcal{C}'$ of algebras $\mathcal{A}'$ with morphisms $\mathcal{M}' : \mathcal{A}' \to \mathcal{A}'$. This category also gives rise to a congruence lattice for $\mathbb{A}$, this time considering the kernels of elements of $\mathcal{M}'$ rather than or $\mathcal{M}$. This congruence lattice need not be the same as the congruence lattice considered previously. In particular, $\mathbb{A}$ may have a smallest nontrivial kernel of an element of $\mathcal{M}$, but not have a smallest nontrivial kernel of an element of $\mathcal{M}'$. Thus where the relevant category $\mathcal{C}$ to which $\mathbb{A}$ belongs is not clear from context, we write of "subdirect irreducibility (with respect to $\mathcal{C}$)". That is, where a $\mathcal{C}$-congruence is the kernel of a morphism of $\mathcal{C}$,

Proposition B.2 An algebra $\mathbb{A}$ is subdirectly irreducible (with respect to $\mathcal{C}$) iff $\mathbb{A}$ is trivial or $\mathbb{A}$ has a smallest nontrivial $\mathcal{C}$-congruence.

Bibliography

[BD74] R. Balbes and P. Dwinger. *Distributive Lattices.* University of Missouri Press, Columbia, Mo., 1974.

[BdRV01] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic.* Cambridge University Press, Cambridge, UK, 2001.

[BS81] S. Burris and H. Sankappanavar. *A Course in Universal Algebra.* Graduate Texts in Mathematics. Springer-Verlag, New York, 1981.

[Cig91] R. Cignoli. Quantifiers on distributive lattices. *Discrete Mathematics*, 96:183–197, 1991.

[CJ99] S. Celani and R. Jansana. Priestley duality, a Sahlqvist theorem and a Goldblatt-Thomason theorem for positive modal logic. *Logic Journal of the IGPL*, 7(6):683–715, 1999.

[DP02] B. Davey and H. Priestley. *Introduction to Lattices and Order.* Cambridge University Press, New York, 2002.

[Dun95] M. Dunn. Positive modal logic. *Studia Logica*, 55(2):301–317, 1995.

[GNV02] M. Gehrke, H. Nagahashi, and Y. Venema. A Sahlqvist theorem for distributive modal logic. ILLC report PP-2002-09, 2002.

[Gol89] R. Goldblatt. Varieties of complex algebras. *Annals of Pure and Applied Logic*, 44(3):173–242, 1989.

[Hal62] P. Halmos. *Algebraic Logic.* Chelsea Publishing Co., New York, 1962.

[Kra99] M. Kracht. *Tools and Techniques in Modal Logic.* Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1999.

[Par70] B. Pareigis. *Categories and Functors.* Pure and Applied Mathematics. Academic Press, New York, 1970.

[Sam99] G. Sambin. Subdirectly irreducible modal algebras and initial frames. *Studia Logica*, 62(2):269–282, 1999.

[SS00] V. Sofronie-Stokkermans. Duality and canonical extensions of bounded distributive lattices with operators, and applications to the semantics of non-classical logics I. *Studia Logica*, 64:93–132, 2000.

[SV88] G. Sambin and V. Vaccaro. Topology and duality in modal logic. *Annals of Pure and Applied Logic*, 37(3):249–296, 1988.

[Tho75] S. Thomason. Categories of frames for modal logic. *Journal of Symbolic Logic*, 40(3):439–442, 1975.

[Ven02] Y. Venema. A dual characterization of subdirectly irreducible BAOs. ILLC report PP-2002-10, 2002.