Exploring the Update Universe

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Contents

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Chapter 1

Introduction

1.1 General issues

Information and Information change are very important issues for us, as we are now in an Information Age: information is produced easily and can be shared by the people around the world with very convenient ways. Different groups of people are working on information technology in various ways: Computer scientists try to build tools (e.g. hardware, software) to facilitate the processing of information; Mathematicians and Engineers try to figure out what is the limit of information a channel can transfer and then try to get that limit. It is hard for me to enumerate all the concerns from different people about information, so in the following I will only talk about the issues on which logicians are working, and the issues with which this thesis is going to deal.

The study of knowledge (or more generally information) from logic perspective began with Hintikka's *Knowledge and Belief* [11]. In epistemic logic, one can represent knowledge or belief using Kripke structures(models). In the simplest case, a Kripke structure encodes the information that a single agent has. By using multi-labeled Kripke structure, *higher-order information*, knowledge about other agents' knowledge, is modeled, which is crucial to multi-agent systems. Epistemic logic does not address the problem of information change (or update). There is another branch of logic, namely dynamic logic, which was invented to model the change of the execution of computer programs. The combination of above two, dynamic epistemic logic, addresses both the representation of information and the change of information.

This thesis is going to present some new results on Epistemic Dynamic

Logic. First we present the work this thesis is based on.

1.2 The work this thesis is based on

Our starting point is dynamic epistemic logic in the style of BMS[4], which has been developed by Baltag, Moss and Solecki since 1998.

1.2.1 Motivation

Epistemic logic usually deals with agents' uncertainty given their current information. It not only deals with agents' knowledge (or information) about the facts of the world, but also deals with *higher-order information* which is the information about the information that agents have.

One branch of Epistemic logic, namely Dynamic Epistemic logic, adds something new: modeling the information change. In [4], Baltag, Moss and Solecki introduced action structures to model various epistemic actions (or programs). In semantics, the uncertainty of each agent concerning the current state of the system is represented using the usual epistemic model and the uncertainty of each agent concerning the current action of the system is represented using an action model: a Kripke model with preconditions for all actions. The information change in such a system is modeled by update product: combining a state model and an action model to produce a new state model. So the new uncertainty of the agents is a combination of their uncertainty of the world and their uncertainty of the action. Since this brach of logics deals with information update, we also call them update logics.

1.2.2 Basic concepts

Here, we present the basic concepts which are important for understanding this thesis.

Definition 1.1 (Epistemic state model for a given language $\mathcal{L}$). Given a finite set of agents Ag, and a set of atomic propositions P in language $\mathcal{L}$, an epistemic state model is a tuple $\mathbf{M} = (W, \{\xrightarrow{i} \mid i \in Ag\}, \mathrm{Val})$ where:

- W is a set of worlds;
- $\xrightarrow{i} \subseteq W \times W$ is the accessibility relation for agent $i \in Ag$;
- $\mathrm{Val} : W \to \mathcal{P}(P)$ is a function from W to the collection of atomic propositions in $\mathcal{L}$.

To model particular situations, we use pointed model $(\mathbf{M}, S)$, where $\mathbf{M}$ is a state model and $S \subseteq W_{\mathbf{M}}$ which means the current world is among S.

For convenience, we use $(W, \xrightarrow{i}, \mathrm{Val})$ to denote a state model; if S is a singleton $\{w\}$, then $(\mathbf{M}, \{w\})$ is denoted as $(\mathbf{M}, w)$.

Definition 1.2 (Epistemic action model for a given language $\mathcal{L}$). Given a finite set of agents Ag, an action model is a tuple $\mathbf{A} = (W, \{\xrightarrow{i} \mid i \in Ag\}, \mathrm{Pre})$ where:

- W is a set of actions;
- $\xrightarrow{i} \subseteq W \times W$ is the accessibility relation for agent $i \in Ag$;
- $Pre : W \to \varphi(\mathcal{L})$ is a function from W to the collection of all formulas in language $\mathcal{L}$.

To model particular actions, we use pointed model $(\mathbf{A}, S)$, where $\mathbf{A}$ is an action model and $S \subseteq W_{\mathbf{A}}$, which means the current action is among S.

For convenience, we also use $(W, \xrightarrow{i}, \mathrm{Pre})$ to denote an action model. If S is a singleton $\{a\}$, then the pointed model is denoted as $(\mathbf{A}, a)$.

The epistemic language $\mathcal{LANG}$ is defined as follows:

Definition 1.3 (The epistemic language $\mathcal{LANG}$). Assume p ranges over the set of atomic propositions P, i ranges over the set of agents Ag and B ranges over the subsets of Ag. The formulas of $\mathcal{LANG}$ are given by:

$$\varphi \;::=\; \top \mid p \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \Box_i\varphi \mid C_B\varphi \mid \; [(\mathbf{A}, S)]\varphi$$

where $(\mathbf{A}, S)$ is a multi-pointed finite $\mathcal{LANG}$ (action) model.

We employ the usual abbreviations. In particular, $\bot$ is shorthand for $\neg\top$, $\varphi_1 \vee \varphi_2$ for $\neg(\neg\varphi_1 \wedge \neg\varphi_2)$, $\varphi_1 \to \varphi_2$ for $\neg(\varphi_1 \wedge \neg\varphi_2)$, $\Diamond_i\varphi$ for $\neg\Box_i\neg\varphi$, $\langle(\mathbf{A}, S)\rangle\varphi$ for $\neg[(\mathbf{A}, S)]\neg\varphi$.

Definition 1.4 (Update, Truth). Let $\mathbf{M} = (W, \xrightarrow{i}, \mathrm{Val})$ be a state model and $\mathbf{A} = (W, \xrightarrow{i}, \mathrm{Pre})$ an action model, then the update product of $\mathbf{M}$ and $\mathbf{A}$, denoted as $\mathbf{M} \otimes \mathbf{A}$, is $(W', \xrightarrow{i}', \mathrm{Val}')$ where:

- $W' = \{(w, a) | w \in W_{\mathbf{M}}, a \in W_{\mathbf{A}}, (\mathbf{M}, w) \models \mathrm{Pre}(a)\}$;
- $(w, a) \xrightarrow{i}' (w', a')$ iff $w \xrightarrow{i}_{\mathbf{M}} w'$ and $a \xrightarrow{i}_{\mathbf{A}} a'$ for $i \in Ag$;
- $\mathrm{Val}(w, a) = \mathrm{Val}'(w)$;

For multiple pointed state model $(\mathbf{M}, S)$, and multiple pointed action model $(\mathbf{A}, T)$, $(\mathbf{M}, S) \otimes (\mathbf{A}, T) = (\mathbf{M'}, S')$ where $\mathbf{M'} = \mathbf{M} \otimes \mathbf{A}$ and $S' = \{(w, a) \in S \times T \mid (\mathbf{M}, w) \models \mathrm{Pre}(a)\}$.

and where the truth definition is given by:

$$
\begin{array}{rcl}
(\mathbf{M}, w) \models \top & & \text{always} \\
(\mathbf{M}, w) \models p & :\equiv & p \in \mathrm{Val}_{\mathbf{M}}(w) \\
(\mathbf{M}, w) \models \neg\varphi & :\equiv & \text{not } (\mathbf{M}, w) \models \varphi \\
(\mathbf{M}, w) \models \varphi_1 \wedge \varphi_2 & :\equiv & (\mathbf{M}, w) \models \varphi_1 \text{ and } (\mathbf{M}, w) \models \varphi_2 \\
(\mathbf{M}, w) \models \Box_i\varphi & :\equiv & \text{for all } w' \text{ with } w \xrightarrow{i} w' , (\mathbf{M}, w') \models \varphi \\
(\mathbf{M}, w) \models E_B\varphi & :\equiv & \text{for all } w' \text{ with } w \xrightarrow{B} w' , (\mathbf{M}, w') \models \varphi \\
(\mathbf{M}, w) \models C_B\varphi & :\equiv & \text{for all } w' \text{ with } w \xrightarrow{B^*} w' , (\mathbf{M}, w') \models \varphi \\
(\mathbf{M}, w) \models [(\mathbf{A}, S)]\varphi & :\equiv & \text{for all } s \in S, \\
& & \text{if } (\mathbf{M}, w) \models \mathrm{Pre}(s) \text{ then } (\mathbf{M} \otimes \mathbf{A}, (w, s)) \models \varphi.
\end{array}
$$

In this definition $\xrightarrow{B}$ is the relation $\bigcup_{i \in B} \xrightarrow{i}$, and $\xrightarrow{B^*}$ its reflexive transitive closure.

Definition 1.5 (Composition of action models). Let $\mathbf{A} = (W, \xrightarrow{i}, \mathrm{Pre})$ and $\mathbf{B} = (W, \xrightarrow{i}, \mathrm{Pre})$ be two action models. The composition of $\mathbf{A}, \mathbf{B}$, denoted as $\mathbf{A} \odot \mathbf{B}$, is $(W', \xrightarrow{i}', \mathrm{Pre}')$ where:

- $W' = W_{\mathbf{A}} \times W_{\mathbf{B}}$
- $(a, b) \xrightarrow{i}' (a', b')$ iff $a \xrightarrow{i} a'$ and $b \xrightarrow{i} b'$
- $\mathrm{Pre}'(a, b) = \langle(\mathbf{A}, a)\rangle \mathrm{Pre}_{\mathbf{B}}(b)$

For multiple pointed action models $(\mathbf{A}, S)$ and $(\mathbf{B}, T)$, the composition $(\mathbf{A}, S) \odot (\mathbf{B}, T) = (\mathbf{C}, U)$, where $\mathbf{C} = \mathbf{A} \odot \mathbf{B}$ and $U = S \times T$.

Definition 1.6 (Bisimulation on state models given language $\mathcal{L}$). Let $\mathbf{M}, \mathbf{N}$ be $\mathcal{L}$ state models. Let P be the collection of atomic propositions. Then the relation $Z \subseteq W_{\mathbf{M}} \times W_{\mathbf{N}}$ is an $\mathcal{L}$ bisimulation if whenever wZv the following hold:

ValEQ $p \in \mathrm{Val}(w)$ iff $p \in \mathrm{Val}(v)$ for any $p \in P$;

Zig for any $i \in Ag$, any state w' with $w \xrightarrow{i} w'$ there is a state v' with $v \xrightarrow{i} v'$ and $w'Zv'$.

Zag same requirement vice versa.

If there is a bisimulation between $\mathbf{M}$ and $\mathbf{N}$, then we denote it as $\mathbf{M} \underline{\leftrightarrow} \mathbf{N}$.

For pointed models $(\mathbf{M}, S)$ and $(\mathbf{N}, T)$, $(\mathbf{M}, S) \underline{\leftrightarrow} (\mathbf{N}, T)$ means there is a bisimulation between $\mathbf{M}$ and $\mathbf{N}$, say Z, such that for any $s \in S$, there is $t \in T$ with $(s,t) \in Z$, and vice versa. $S \underline{\leftrightarrow} T$ is the abbreviation of $(\mathbf{M}, S) \underline{\leftrightarrow} (\mathbf{N}, T)$ when $\mathbf{M}$ and $\mathbf{N}$ are clear. A total bisimulation between $\mathbf{M}$ and $\mathbf{N}$ means $(\mathbf{M}, W_{\mathbf{M}}) \underline{\leftrightarrow} (\mathbf{N}, W_{\mathbf{N}})$.

Definition 1.7 (Bisimulation on action models given language $\mathcal{L}$). Let $\mathbf{A}, \mathbf{B}$ be $\mathcal{L}$ action models. Let $\equiv_{\mathcal{L}}$ be the appropriate equivalence notion for $\mathcal{L}$. Then relation $Z \subseteq W_{\mathbf{A}} \times W_{\mathbf{B}}$ is a $\mathcal{L}$ bisimulation if whenever aZb the following hold:

PreEQ $\mathrm{Pre}_{\mathbf{A}}(a) \equiv_{\mathcal{L}} \mathrm{Pre}_{\mathbf{B}}(b)$,

Zig for any $i \in Ag$, any state a' with $a \xrightarrow{i} a'$, there is a state b' with $b \xrightarrow{i} b'$ and $a'Zb'$.

Zag same requirement vice versa.

If there is a *total* bisimulation between $\mathbf{A}$ and $\mathbf{B}$, we denote it as $\mathbf{A} \underline{\leftrightarrow} \mathbf{B}$. Please note that we use the same symbol for the bisimulation of state models, and we think it is easy for the readers to figure out which bisimulation we are talking when they meet $\underline{\leftrightarrow}$ in this thesis.

For pointed action models $(\mathbf{A}, S)$ and $(\mathbf{B}, T)$, $(\mathbf{A}, S) \underline{\leftrightarrow} (\mathbf{B}, T)$ means there is a bisimulation between $\mathbf{A}$ and $\mathbf{B}$, say Z, such that for any $a \in S$, there is $b \in T$ with $(a,b) \in Z$, and vice versa. $S \underline{\leftrightarrow} T$ is the abbreviation of $(\mathbf{A}, S) \underline{\leftrightarrow} (\mathbf{B}, T)$ when $\mathbf{A}$ and $\mathbf{B}$ are clear. A total bisimulation can also be denoted as $(\mathbf{A}, W_{\mathbf{A}}) \underline{\leftrightarrow} (\mathbf{B}, W_{\mathbf{B}})$.

1.2.3 Some related results in BMS

Here we mention some basic results in [2],[4], which are related to this thesis.

For chapter 3:

Theorem 1 (Preservation of epistemic bisimulation; Baltag, Moss and Solecki). *Given a state model $\mathbf{M}$ and an action model $\mathbf{A}$, the following holds:*

$$\textit{if } \mathbf{M} \underline{\leftrightarrow} \mathbf{N} \textit{ then } \mathbf{M} \otimes \mathbf{A} \underline{\leftrightarrow} \mathbf{N} \otimes \mathbf{A}$$

For chapter 4:

Theorem 2. *The update induced by a composition of action models is isomorphic to the composition of the induced updates.*

1.2.4 Limitations of BMS

As any framework, the BMS framework also has certain limitations. Here we try to give the most apparent ones. However, this does not suggest that the BMS framework is very limited; in fact it is quite general, since you can have arbitrary Kripke structures with arbitrary preconditions, and you can even modify it so the fact-changing actions can be incorporated.

Logical omniscience : Notice that the agents modeled in BMS have logical omniscience property, i.e. the agent knowing all logical tautologies and all the consequences of its knowledge. This property is inherited in a lot of logic systems. However, logical omniscience is problematic when attempting to build realistic models of agent behaviour, since closure under logical consequence implies that inference or reasoning takes no time. If processes within the agent such as belief revision, planning and problem solving are modelled as derivations in a logical language, such derivations require no investment of computational resources by the agent.

Agency The framework is about the events of information updating, and it does not address the issues of agency. The agents' knowledge(or information) is modeled in a single model, and there is no place for agents' capacity, preference of the actions.

1.3 Overview of this thesis

Besides the introduction, there are four chapters in this thesis. Here, we give a brief overview.

In Chapter 2, we will study different ways message passing which can be modeled in BMS framework. First, we give some examples to show that BMS framework can model very subtle information update. Second, we focus on the case of secure private message passing, which is to pass a message to a subgroup of agents without the rest knowing it. We give an axiomatization for this case, and proof the completeness using relativized common knowledge, which is proposed in [12]. Third, we give a brief discussion of message passing by Blind Carbon Copy (BCC).

In Chapter 3, the work is jointly done with professor Jan van Eijck. We will study the relation of action models which have same update effects. A key notion of equivalence for modal and epistemic logic is bisimulation. However, to capture the update effects of action models in dynamic epistemic logic, this notion turns out to be too strong. We propose a notion of equivalence, called action emulation, which is more more appropriate for action models than bisimulation. It is proved that every bisimulation is an action emulation, but not vice versa, and that in the context of action models with propositional preconditions, action emulation provides a full characterization of update effects. Moreover, we find the necessary and sufficient conditions for having the same update effects, in the cases of action models with propositional preconditions and action models with modal preconditions.

In Chapter 4, we will study the problem of update evolution. Here update evolution means the change of the state model by iterated updating. We first study some special cases and then show some sufficient condition for stabilization.

In Chapter 5, we conclude this thesis by giving possible directions for further research.

Chapter 2

Logics of Message Passing

2.1 Modeling Message Passing by BMS

In multi-agent systems, messages play the role of changing agents' information such as knowledge, belief, etc. We abstract an agent as an information processing entity which may only has partial information of its own situation. Passing messages to the agents can reveal them the truth of reality or make them more confused of the true situation; passing messages amongst agents allows them to share their partial information and to get some jobs done. In this Chapter, we try to investigate the cases of message passing: first, we give some examples of message passing which can be modeled by BMS approach; second, we present a logic system to model particular ways of message passing; last, we do a brief analysis of an interesting case and leave it for further work.

Here we will use single pointed action models, since the message actions we study are all deterministic actions.

2.2 Examples

We start from a very simple case: Public Announcement.

- Scenario: All agents receive the same message φ and are aware of the fact that they all received this message. The corresponding pointed action model [1] is an S5 model (Assume the set of all agents is Ag):

Updating by publicly announcing φ is just eliminating the possible worlds which do not satisfy φ. We can iterate the update process, and get a sequence of state models: $\mathbf{M}$, $\mathbf{M} \otimes \mathbf{A}$,$(\mathbf{M} \otimes \mathbf{A}) \otimes \mathbf{A}$, ... In particular, the public announcement update can be represented by a sequence of non-increasing state models.

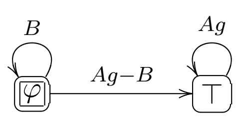
The format of φ may determine how many repetitions of an update needs to have before t he update sequence gets stabilized. If φ is a propositional formula, one update suffices and this meets the intuition that there is no point to publicly announcing a fact more than once. There are actions that are not so transparent: sending a private message to a subgroup (so the rest do not get this message); sending a message to a subgroup in such a way that rest of the agents notice what is the message about but do not know the exact content, etc. So the question is: can update logics model these actions?

The answer is confirmative. We will first briefly introduce several scenarios of passing messages and the corresponding action models in BMS style. Some of them will be discussed in detail in different sections of this thesis.

1. Message to the whole group, namely a Public Announcement: we have dealt with this above.

2. Message to a subgroup: here we distinguish 4 interesting cases:

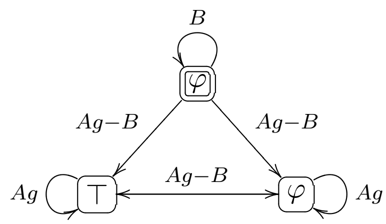
    - **secure private message to a subgroup**: A subgroup of agents, say B, receives a message φ, and the rest receive nothing. The action model is a KD45 model:

    Intuitively, the agents in group B receives φ (the left action), and the rest $Ag - B$ believes nothing happened (the right action).

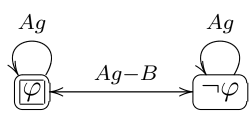
[1]From now on, the pointed world in the action model is indicated by the double border.

- **insecure private message to a subgroup**:A subgroup of agents, say B, receives a message φ, and it's possible that the message is leaked to the rest. The action model is a KD45 model:

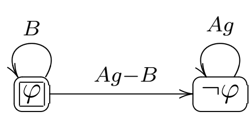


  The difference between this scenario and the above one is agents in $Ag - B$ now consider two actions possible, instead of believing only one action (the action does nothing) happened.

- **half-public message to a subgroup**: A subgroup of agents, say B, asked a question about whether φ is the case, and received a message that φ was indeed the case. The rest of the agents heard this question also notice the delivery of the answer but don't know the exact content. The action model is an S5 model:



- **misleading**:A subgroup of agents, say B, receives a message φ, and misleads the rest agents that they receive $\neg\varphi$. The action model is a KD45 model:



3. Message to more groups:

   - **BCC**(Blind Carbon Copy): If you already send some emails, you may notice that there is a column called BCC, which is supposed to hide some receivers from others. For instance, your institution may send an email to all of the members without disclose the addresses of them. In general, we have three groups: To, CC and BCC. Since there is no epistemic difference between To and CC group, we denote them with CC. For the members in CC, they commonly know that the current message is received by them, but they are not sure whether the people outside CC also get a blind carbon copy. For the members in BCC, they know that the members in CC get the current message, but they don't know who else gets this message. For the people outside the CC and BCC group, they may simply believe that nothing happens. So this case introduced more subtleties than the cases above. Please refer to the section 2.4 for further analysis.

Remark. The first thing that needs to be noticed is that here we model the message by formulas, and the different agents' uncertainty of the message that other agents get is modeled by the accessible relations of the action model. Moreover, the BMS action models above assume that messages are sent by some entity outside the agent system, not by the agents themselves. The BMS models the events of information updating and it does not address the problem of agency. It will be interesting to investigate the situation in which the sender of the messages can be modeled. **BCC** will be a nice case to study, since only the sender knows the whole situation about the message he sends and the receivers (both in CC and BCC list) are aware of this.

2.3 Case study: secure private message to a subgroup

2.3.1 Motivation

Secure private message passing introduces more twists than public announcements, because the agents who do not get the message may believe that nothing happened. It is interesting to study the logics which incorporates this kind of actions. Since common knowledge is an essential concept in the multi-agent systems that deal with knowledge (or information) and its change, we would also like to add common knowledge into the above logics.

A completeness proof for the logic of public announcement without common knowledge is easy, due to the axioms (called reduction axioms) such as $[\varphi]\Box_i\psi \leftrightarrow (\varphi \to \Box_i[\varphi]\psi)$. But the completeness proofs for dynamic epistemic logics *with* common knowledge are in general hard. The reduction axioms are missing since the logic with epistemic actions is more expressive than the logic without them, according to [3, 4].

In [12], Kooi and van Benthem proposed a method called relativization to make reduction axioms work both in the logic of public announcement with common knowledge, and in the logic of general epistemic actions. For public announcement logic, reduction axioms for formulas of the form $[\varphi]C_B\psi$ is impossible, according to [3]. So a relativized common knowledge operator $C_B(\varphi,\psi)$ is introduced to make the reduction work. The key clause in the semantic of the logic of relativized common knowledge(RCL) is :

$$(\mathbf{M},w) \models C_B(\varphi,\psi) \text{ iff } (\mathbf{M},v) \models \psi \text{ for all } v \text{ such that } (w,v) \in (\xrightarrow{B} \cap [\![\varphi]\!]^2)^*$$

where $[\![\varphi]\!] = \{w \mid (\mathbf{M},w) \models \varphi\}$, $\xrightarrow{B} = \bigcup_{i\in B} \xrightarrow{i}$, and $(\xrightarrow{B} \cap[\![\varphi]\!]^2)^*$ is the reflexive transitive closure of $\xrightarrow{B} \cap[\![\varphi]\!]^2$. The normal common knowledge $C_B\varphi$ can be expressed as $C_B(\top,\varphi)$.

The completeness proof of RCL incorporating public announcement works by the reduction axioms like $[\varphi]C_B(\psi,\chi) \leftrightarrow C_B(\varphi \wedge [\varphi]\psi, [\varphi]\chi)$.

As for the logic of general epistemic actions, automata are introduced to represent the common knowledge operator in a rather complicated way.

In the next sections, we present a logic system that models the case of secure private messages to a subgroup[2] and uses the relativization method to get the completeness proof.

[2]For simplicity we call it logic of CC. CC comes from the email button CC, which is the address list of the explicit receivers of the email. Button TO is the same as CC in the epistemic sense.

2.3.2 Logic of Relativized Common Knowledge

Now we introduce the Logic of Relativized Common Knowledge(RCL) proposed in [12].

Definition 2.1 (The language of RCL: $\mathcal{L}_{\mathbf{RCL}}$).

$$\varphi ::= \top \mid p \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \Box_i\varphi \mid C_B(\varphi,\psi)$$

We have given the key clause for the semantics of RCL above, and the rest of the semantics is standard. Now we give the proof system:

Definition 2.2 (Proof system for RCL). The proof system for RCL contains the following axioms and rules:

Taut	all instantiations of propositional tautologies	
Dist	$\Box_i(\varphi\to\psi)\to(\Box_i\varphi\to\Box_i\psi)$	(distribution)
Dist	$C_B(\varphi,\psi\to\chi)\to(C_B(\varphi,\psi)\to C_B(\varphi,\chi))$	(distribution)
Mix	$C_B(\varphi,\psi)\leftrightarrow(\varphi\to(\psi\wedge E_B(\varphi\to C_B(\varphi,\psi))))$	(mix)
Ind	$((\varphi\to\psi)\wedge C_B(\varphi,\psi\to E_B(\varphi\to\psi)))\to C_B(\varphi,\psi)$	(induction)
MP	$\dfrac{\varphi,\varphi\to\psi}{\psi}$	(modus ponens)
Nec	$\dfrac{\varphi}{\Box_i\varphi}$	(necessitation)
Nec	$\dfrac{\varphi}{C_B(\psi,\varphi)}$	(necessitation)

A proof consists of a sequence of formulas such that each formula is either an instance of an axiom, or it can be obtained from formulas that appear earlier in the sequence by applying a rule. If there is a proof of φ, we write $\vdash\varphi$.

The soundness of this proof system is easy to show by induction on the length of proofs. The completeness proof is similar to the method used for Propositional Dynamic Logic(PDL).

Theorem 3 (Completeness for RCL; Kooi and van Benthem).

If $\models\varphi$, *then* $\vdash\varphi$.

Here we do not provide the details for this proof. Please refer to [12].

2.3.3 Logic of CC(LCC)

Language and Semantics

Definition 2.3 (The language of LCC: $\mathcal{L}_{\mathbf{CC}}$). Here we use the relativized common knowledge operator instead of the normal one:

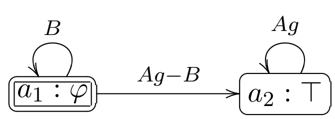
$$\varphi \quad ::= \quad \top \mid p \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \Box_i\varphi \mid C_B(\varphi_1, \varphi_2) \mid [CC_B\varphi_1]\varphi_2.$$

We employ the usual abbreviations. In particular, $\varphi_1 \vee \varphi_2$ is shorthand for $\neg(\neg\varphi_1 \wedge \neg\varphi_2)$, $\varphi_1 \to \varphi_2$ for $\neg(\varphi_1 \wedge \neg\varphi_2)$, $\Diamond_i\varphi$ for $\neg\Box_i\neg\varphi$, $\langle CC_B\varphi_1\rangle\varphi_2$ for $\neg[CC_B\varphi_1]\neg\varphi_2$, $\widehat{C}_B(\varphi_1, \varphi_2)$ for $\neg C_B(\varphi_1, \neg\varphi_2)$.

Definition 2.4. (**Semantics of LCC**): Ag is the set of all agents in consideration.$(\mathbf{M}, w)$ where $\mathbf{M} = (W, \xrightarrow{i}, \mathrm{Val})$ is a state model. Let $i \in Ag$, $\emptyset \neq B \subseteq Ag$, and $\varphi, \psi \in \mathcal{L}_{CC}$. For the atomic propositions and boolean cases, we just define them as usual.

$$\begin{array}{lll}
(\mathbf{M}, w) \models \Box_i\varphi & \text{iff} & (\mathbf{M}, v) \models \varphi \text{ for all } v \text{ such that } (w, v) \in \xrightarrow{i} \\
(\mathbf{M}, w) \models C_B(\varphi, \psi) & \text{iff} & (\mathbf{M}, v) \models \psi \text{ for all } v \text{ such that } (w, v) \in (\xrightarrow{B} \cap [\![\varphi]\!]^2)^* \\
(\mathbf{M}, w) \models [CC_B\varphi]\psi & \text{iff} & (\mathbf{M}, w) \models \varphi \text{ implies } (\mathbf{M}, w) \otimes (\mathbf{A}_B^{\varphi}, a_1) \models \psi
\end{array}$$

where
$[\![\varphi]\!] := \{w \mid (\mathbf{M}, w) \models \varphi\}$ and $\otimes$ is the product update operator. $\mathbf{A}_B^{\varphi}$ is the action model corresponding to sending a secure private message φ to a non-empty subgroup B, as we showed below:



One may notice that this logic can express everything the logic of public announcements can express, since we can take B to be the whole set of agents. Then the action model is[3]:

Moreover, the normal common knowledge $C_B\varphi$ is equivalent to $C_B(\top, \varphi)$.

[3]The right action has no update effect in the sense that it is not connected the designated action(s).

Now we explain how to find the reduction axioms for LCC. If we use the usual common knowledge operator, the semantics is as follows:

$$(\mathbf{M}, w) \models C_B\varphi \text{ iff } (\mathbf{M}, v) \models \varphi \text{ for all } v \text{ such that } (w, v) \in \xrightarrow{B}{}^{*}$$

where $\xrightarrow{B} = \bigcup_{i \in B} \xrightarrow{i}$, and $\xrightarrow{B}{}^{*}$ is its reflexive transitive closure.

Let's consider the following scenario: after subgroup B received a private message φ, subgroup D achieves common knowledge ψ. This is expressed by $[CC_B\varphi]C_D\psi$ if we use the normal common knowledge operator. Our object is to find α with:

$$(\mathbf{M}, w) \models \alpha \text{ iff } (\mathbf{M}, w) \models [CC_B\varphi]C_D\psi$$

and since

$$(\mathbf{M}, w) \models [CC_B\varphi]C_D\psi \text{ iff } (\mathbf{M}, w) \models \varphi \text{ implies } (\mathbf{M}, w) \otimes (\mathbf{A}^{\varphi}_{B}, a_1) \models C_D\psi$$

therefore

$$(\mathbf{M}, w) \models \alpha \text{ iff } (\mathbf{M}, w) \models \varphi \text{ implies } (\mathbf{M}, w) \otimes (\mathbf{A}^{\varphi}_{B}, a_1) \models C_D\psi$$

Now we try to find α by the above condition. According to the relation between B and D, we distinguish three cases:

(a) $D \cap B = \emptyset$
$[CC_B\varphi]C_D\psi$ is equivalent to $(\varphi \to C_D(\top, \psi))$
The intuition here is that the common knowledge[4] among the subgroup D, which is disjoint with B, does not change after the update, since the private message only updates the information state of the agents who get the message.

(b) $D \cap -B = \emptyset$ (i.e. $D \subseteq B$)
$[CC_B\varphi]C_D\psi$ is equivalent to $C_D(\varphi, [CC_B\varphi]\psi)$
Given $D \subseteq B$, the intuition here is that if ψ is common knowledge among a subgroup of B, say D, after a private message φ was sent to B, then it is common knowledge among the group D, in the worlds φ holds, that: after the private message φ was sent to B, ψ holds.

(c) $D \cap B \neq \emptyset$ & $D \cap -B \neq \emptyset$
$[CC_B\varphi]C_D\psi$ is equivalent to $C_{B \cap D}(\varphi, [CC_B\varphi]\psi) \wedge (\varphi \to C_D(\top, \psi))$

[4]One may notice that common belief might be more suitable than common knowledge, but we will not discuss here.

This combines two considerations above. The intuition comes from the following observation: suppose we have a state model $\mathbf{M}$, then after the update we have a state model $\mathbf{M} \otimes \mathbf{A}^{\varphi}_{B}$ which *looks like*:

$$\mathbf{M}^{B}_{a_1} \xrightarrow{Ag-B} \mathbf{M}_{a_2}$$

Where $\mathbf{M}^{B}_{a_1}$, generated from action a_1, is a sub-model of $\mathbf{M}$ with only φ-worlds and the B-relations; the $\mathbf{M}_{a_2}$ the same as $\mathbf{M}$; only $Ag - B$ relations are kept from $\mathbf{M}^{B}_{a_1}$ to $\mathbf{M}_{a_2}$.

So $C_D\psi$ should hold in the designated world in this new model: either the paths stay within $\mathbf{M}_{a_1}$ or they go to $\mathbf{M}_{a_2}$. For the paths staying within $\mathbf{M}_{a_1}$, they are actually $D \cap B$-path, so we use $[CC_B\varphi]C_{B\cap D}\psi$ to capture this, then we apply case (b) to $[CC_B\varphi]C_{B\cap D}\psi$, and get $C_{B\cap D}(\varphi, [CC_B\varphi]\psi)$, because $D \cap B$ is a subset of B. For the paths going to $\mathbf{M}_{\top}$, the intuition is captured by $\varphi \to C_D\psi$, and the equivalent expression with relativized common knowledge, $\varphi \to C_D(\top, \psi)$.

So far, we gave the intuition of how to reduce a dynamic operator with normal common knowledge to one with relativized common knowledge. Now, we have the idea on the reduction axioms with relativized common knowledge only:

C-Red-1 $D \cap B = \emptyset$
$[CC_B\varphi]C_D(\psi, \chi) \leftrightarrow (\varphi \to C_D(\psi, \chi))$

C-Red-2 $D \cap -B = \emptyset$ (i.e. $D \subseteq B$)
$[CC_B\varphi]C_D(\psi, \chi) \leftrightarrow C_D(\varphi \wedge [CC_B\varphi]\psi, [CC_B\varphi]\chi)$

C-Red-3 $D \cap B \neq \emptyset \& D \cap -B \neq \emptyset$
$[CC_B\varphi]C_D(\psi, \chi) \leftrightarrow C_{B\cap D}(\varphi \wedge \psi, [CC_B\varphi]\chi) \wedge (\varphi \to C_D(\psi, \chi))$

Now we can translate the formulas from $\mathcal{L}_{CC}$ to $\mathcal{L}_{RCL}$.

Definition 2.5 (Translation). The translation function maps a formula from the language of $\mathcal{L}_{CC}$ to a formula in the language of RCL:

$t(p)$	$=$	p	
$t(\neg\varphi)$	$=$	$\neg t(\varphi)$	
$t(\varphi \wedge \psi)$	$=$	$t(\varphi) \wedge t(\psi)$	
$t(\Box_i\varphi)$	$=$	$\Box_i t(\varphi)$	
$t(C_B(\varphi, \psi))$	$=$	$C_B(t(\varphi), t(\psi))$	
$t([CC_B\varphi]p)$	$=$	$t(\varphi) \to p$	
$t([CC_B\varphi]\neg\psi)$	$=$	$t(\varphi) \to \neg t([CC_B\varphi]\psi)$	
$t([CC_B\varphi]\psi_1 \wedge \psi_2)$	$=$	$t(\varphi) \to t([CC_B\varphi]\psi_1) \wedge t([CC_B\varphi]\psi_2)$	
$t([CC_B\varphi]\Box_i\psi)$	$=$	$t(\varphi) \to \Box_i t(\psi)$	for $i \notin B$
$t([CC_B\varphi]\Box_i\psi)$	$=$	$t(\varphi) \to \Box_i t([CC_B\varphi]\psi)$	for $i \in B$
$t([CC_B\varphi]C_D(\psi, \chi))$	$=$	$t(\varphi \to C_D(\psi, \chi))$	for $D \cap B = \emptyset$
$t([CC_B\varphi]C_D(\psi, \chi))$	$=$	$t(C_D(\varphi \wedge [CC_B\varphi]\psi, [CC_B\varphi]\chi))$	for $D \cap -B = \emptyset$
$t([CC_B\varphi]C_D(\psi, \chi))$	$=$	$t(C_{B\cap D}(\varphi \wedge \psi, [CC_B\varphi]\chi) \wedge (\varphi \to C_D(\psi, \chi)))$	for $D \cap B \neq \emptyset$ $\& D \cap -B \neq \emptyset$

Lemma 4 (Translation Correctness). *For all dynamic-epistemic formulas $\alpha \in \mathcal{L}_{CC}$ and all pointed state models $(\mathbf{M}, w)$,*

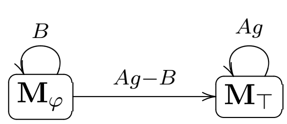
$$(\mathbf{M}, w) \models \alpha \textit{ iff } (\mathbf{M}, w) \models t(\alpha)$$

Proof. Given an arbitrary formula $\alpha \in \mathcal{L}_{CC}$ and state model $(\mathbf{M}, w)$, we prove the result by the induction on α. Here we only show the last case with common knowledge operator; the rest are easier to verify:

Let $\alpha = [CC_B\varphi]C_D(\psi, \chi)$, and assume $D \cap B \neq \emptyset \& D \cap -B \neq \emptyset$. We distinguish two cases:

- $(\mathbf{M}, w) \not\models \varphi$: it follows that $(\mathbf{M}, w) \models [CC_B\varphi]C_D(\psi, \chi)$ and $(\mathbf{M}, w) \models ([CC_B\varphi]C_{B\cap D}(\psi, \chi)) \wedge (\varphi \to C_D(\psi, \chi))$ since for two conjuncts either the update fails or the antecedent is not true.

- $(\mathbf{M}, w) \models \varphi$: we show $(\mathbf{M}, w) \models \alpha$ iff $(\mathbf{M}, w) \models C_{B\cap D}(\varphi \wedge \psi, [CC_B\varphi]\chi) \wedge (\varphi \to C_D(\psi, \chi))$. The following figure is the sketch of the new model [$(\mathbf{M}, w)$ updated by $(\mathbf{A}_B^\varphi, a_1)$]:

  It helps to understand the following proof:

  From left to right, we assume that $(\mathbf{M}, w) \models [CC_B\varphi]C_D(\psi, \chi)$. It follows that $(\mathbf{M}, w) \otimes (\mathbf{A}_B^\varphi, a_1) \models C_D(\psi, \chi)$. In the new model, we have two kinds of D-path: one is $D \cap B$-path and one is $D - B$-path. For the $B \cap D$ path, it ends within $\mathbf{M}_\varphi$, and hence $(\mathbf{M}, w) \models C_{B\cap D}(\varphi \wedge \psi, [CC_B\varphi]\chi)$. For the $D - B$-path, it ends in $\mathbf{M}_\top$, so $(\mathbf{M}, w) \models (\varphi \to C_D(\psi, \chi))$.

From right to left, it equals to show $(\mathbf{M}, w) \models \neg([CC_B\varphi]C_D(\psi, \chi))$ implies $(\mathbf{M}, w) \models \neg(C_{B\cap D}(\varphi \wedge \psi, [CC_B\varphi]\chi) \wedge (\varphi \to C_D(\psi, \chi)))$. We write it into diamond version: $(\mathbf{M}, w) \models \langle CC_B\varphi\rangle \widehat{C}_D(\psi, \chi)$ implies $(\mathbf{M}, w) \models \widehat{C}_{B\cap D}(\varphi \wedge \psi, \langle CC_B\varphi\rangle\chi) \vee (\varphi \wedge \widehat{C}_D(\psi, \chi))$.

Suppose $(\mathbf{M}, w) \models \langle CC_B\varphi\rangle \widehat{C}_D(\psi, \chi)$, it follows that $(\mathbf{M}, w) \otimes (\mathbf{A}^\varphi_B, a_1) \models \widehat{C}_D(\psi, \chi)$, which means there is a D-path (in ψ-world) from (w, a_1) such that at the end of path χ holds. So either it stays in the $\mathbf{M}_\varphi$ of the new model, or it goes to the $\mathbf{M}_\top$. For the first case $(\mathbf{M}, w) \models \widehat{C}_{B\cap D}(\varphi \wedge \psi, \langle CC_B\varphi\rangle\chi)$, and for the second case $(\mathbf{M}, w) \models (\varphi \wedge \widehat{C}_D(\psi, \chi))$.

□

Proof System

Definition 2.6 (Proof system for LCC). The proof system contains the following axioms and rules plus the ones for RCL:

At	$[CC_B\varphi]p \leftrightarrow (\varphi \to p)$	
PF	$[CC_B\varphi]\neg\psi \leftrightarrow (\varphi \to \neg[CC_B\varphi]\psi)$	
Dist	$[CC_B\varphi](\psi_1 \wedge \psi_2) \leftrightarrow ([CC_B\varphi]\psi_1 \wedge [CC_B\varphi]\psi_2)$	
$KA-1$	$[CC_B\varphi]\Box_i\psi \leftrightarrow (\varphi \to \Box_i\psi)$	for $i \notin B$
$KA-2$	$[CC_B\varphi]\Box_i\psi \leftrightarrow (\varphi \to \Box_i[CC_B\varphi]\psi)$	for $i \in B$
C-Red-1	$[CC_B\varphi]C_D(\psi, \chi) \leftrightarrow (\varphi \to C_D(\psi, \chi))$	for $D \cap B = \emptyset$
C-Red-2	$[CC_B\varphi]C_D(\psi, \chi) \leftrightarrow C_D(\varphi \wedge [CC_B\varphi]\psi, [CC_B\varphi]\chi)$	for $D \cap -B = \emptyset$
C-Red-3	$[CC_B\varphi]C_D(\psi, \chi) \leftrightarrow C_{B\cap D}(\varphi \wedge \psi, [CC_B\varphi]\chi) \wedge (\varphi \to C_D(\psi, \chi))$	for $D \cap B \neq \emptyset$ $\&D \cap -B \neq \emptyset$

Theorem 5. *(Completeness of LCC) If* $\models \varphi$ *then* $\vdash \varphi$.

Proof. This is immediate since RCL is complete and every formula in $\mathcal{L}_{LCC}$ is provably equivalent to one in $\mathcal{L}_{RCL}$. □

In [8], van Eijck proposed a way to reduce DEL to PDL by program transformation. It is interesting that for the logic of CC case, the transformation looks as follows:

$$[CC^B_\varphi][D^*]\psi$$

to

$$[(?\varphi;(B\cap D))^*][CC^B_\varphi, s_0]\psi \wedge [(?\varphi;(B\cap D))^*;(D-B);D^*][CC^B_\varphi, s_1]\psi$$

Here $D*$ corresponds to common knowledge operator. CC^B_φ is the same as CC^φ_B, and s_0 is the world with precondition φ and s_1 is the world with precondition $\top$.

This translation is quite related to my translation above; the key one is $[CC_B\varphi]C_D(\psi,\chi) \leftrightarrow C_{B\cap D}(\varphi\wedge\psi,[CC_B\varphi]\chi)\wedge(\varphi\rightarrow C_D(\psi,\chi))$ for $D\cap B\neq\emptyset \& D\cap -B\neq\emptyset$. This part $[(?\varphi;(B\cap D))^*][CC^B_\varphi, s_0]\psi$ corresponds to $C_{B\cap D}(\varphi\wedge\psi,[CC_B\varphi]\chi)$ and $[(?\varphi;(B\cap D))^*;(D-B);D^*][CC^B_\varphi, s_1]\psi$ corresponds to $\varphi\rightarrow C_D(\psi,\chi))$.

2.4 Case study: Blind Carbon Copy (BCC)

In this section, we discuss situations in which a message can be sent by blind carbon copy. The idea is inspired by the following feature of email.[5] When sending an email, we can specify the following things:

Item	Usage
From:	Sender
To:	Receiver(s)
CC:	Carbon Copy Receiver(s)
BCC:	Blind Carbon Copy Receiver(s)

Usually "From:" is filled automatically by the email program to indicate the sender. But actually you can change it in some situations; that's why some spam emails come from the addresses that not exist[6]. That is to say the credibility of the email sender can be doubted. The difference between *To:* and *CC:* is not so much: usually the receiver(s) in *To:* might be more relevant than that in *CC:*, and in the epistemic scene they are the same since they commonly know that such a message is received by each other. The meaning of "BCC:" is easy to understand if we look at what every receiver sees in email header:

Item	Usage
From:	Sender
Date:	The time when email is sent
To:	Receiver(s)
CC:	Carbon Copy Receiver(s)

Those in *BCC* list will not be indicated in the email header. This increases the uncertainties among the agents who received this email because those in *To* and *CC* list will not know who also received a BCC. Even those who received a BCC is also uncertain of who else received a BCC. In fact only the sender knows the true situation, and every receiver aware of this fact but it's not common knowledge among the receivers.

To simplify the analysis, we make some assumptions: (1) emails(or messages) are never lost; (2) sender is not an agent in the model (we can add individual sender later); (3) the *To* list and *CC* list are emerged into one; (4) those in the *BCC* list do not know whether anyone else is in the list; (5)

[5]It is an example in one of Johan's lectures.

[6]Sometimes you may get an email which inform you as the winner of a BIG lottery. But after you reply that email, you may get an error message which says "no such user exists." :-)

nobody is both in the CC and the BCC list (because if an agent is in both lists, the email looks exactly the same as if he were only in the CC).

Here is a general case: suppose we have a group of agents, say Ag. Initially, all the agents are ignorant of some fact and their ignorance is common knowledge among them. A message is sent to a subgroup of agents, say CCs, to reveal that fact, and at the same time a blind carbon copy is sent to another subgroup, say $BCCs$, and there may be some other agents, say $RSTs$, who receive nothing.[7] Now It is common knowledge among those agents in CCs that everyone in CCs receive this message and some in $Ag\backslash CCs$ may have received a blind carbon copy. For anyone in $BCCs$, he receives a blind carbon copy and knows what is known by those in CCs about this message, but is not certain whether anyone else also got such a blind carbon copy. For those in $RSTs$, they just believe nothing happened.

There are different uncertainties among the agents: those in CCs are uncertain of whether a blind carbon copy was sent and which agents got a copy; those in $BCCs$ are uncertain about who else also got the blind carbon copy, etc; those in $RSTs$ have no uncertainty about what happens, but their existence increases the uncertainties among those in CCs and $BCCs$.

The action model expends exponentially since the number of subgroups of n agents is 2^n. For the extreme case that everyone gets a blind carbon copy, each agent must consider 2^{n-1} possible situations, and the action model should have 2^n possible actions. This makes the high complexity of action models. We will leave this for further research.

[7] $Ag{=}CCs{\cup}BCCs{\cup}RSTs$, and CCs,$BCCs$,$RSTs$ are pairwise disjoint.

Chapter 3

Action Emulation

3.1 Introduction

Actions with epistemic effects, such as informing someone that something is the case, are quite similar to situations with epistemic aspects, such as models of the states of knowledge of groups of agents. Knowledge of agents is encoded in epistemic models, with transition relations $\xrightarrow{i}$ modelling the epistemic state of each agent i, and valuations over a set of proposition letters modelling factual states of affairs.

[4] proposes to model epistemic actions as epistemic models, with valuations replaced by preconditions. (See also: [1, 2, 3, 5, 6, 7, 9, 10, 13].)

This chapter addresses the question of the appropriate notion of equivalence for action models. It may seem that generalizing bisimulations to action models in the obvious way to 'precondition preserving bisimulation', as is proposed in [2], is the way to go.

In BMS, it has been proved that the action update operation $\otimes$ preserves ordinary bisimulation on epistemic models, as showed in theorem 1.

Of course, we can also look at the action models modulo $\mathcal{LANG}$ bisimulation:

Theorem 6 (Preservation of action bisimulation). *The action update operation preserves action bisimulation:*

$$\textit{if } \mathbf{A} \underline{\leftrightarrow} \mathbf{B} \textit{ then } \mathbf{M} \otimes \mathbf{A} \underline{\leftrightarrow} \mathbf{M} \otimes \mathbf{B}.$$

Proof. We have to show that for every $(w, s_i) \in \mathbf{M} \otimes \mathbf{A}$ there is a (v, t_j) among the actual worlds of $\mathbf{M} \otimes \mathbf{B}$ with $(w, s_i) \underline{\leftrightarrow} (v, t_j)$, and vice versa. This follows immediately from the existence of the bisimulation $\underline{\leftrightarrow}$ between

$\mathbf{A}$ and $\mathbf{B}$, for the relation on $\mathbf{M} \otimes \mathbf{A} \times \mathbf{M} \otimes \mathbf{B}$ defined by means of

$$(w,s)C(v,t) \text{ iff } w = v \text{ and } s \underline{\leftrightarrow} t$$

is a bisimulation. □

3.2 Same Update Effect

Thinking of the finite action models $(\mathbf{A}, S)$ as 'action programs', the basic semantic notion of equivalence between such programs is that of having the same update effect:

Definition 3.1 (Same update effect). Action models $\mathbf{A}$ and $\mathbf{B}$ have the same update effect, if given any state models $\mathbf{M}$:

$$\mathbf{M} \otimes \mathbf{A} \underline{\leftrightarrow} \mathbf{M} \otimes \mathbf{B}.$$

We denote this as $\mathbf{A} \equiv_{\mathrm{ACT}} \mathbf{B}$. Please note that the bisimulation $\underline{\leftrightarrow}$ above is a total bisimulation.

For multiple pointed action models $(\mathbf{A}, S)$ and $(\mathbf{B}, T)$, $(\mathbf{A}, S) \equiv_{\mathrm{ACT}} (\mathbf{B}, T)$ means for any multiple pointed state model $(\mathbf{M}, X)$,

$$(\mathbf{M}, X) \otimes (\mathbf{A}, S) \underline{\leftrightarrow} (\mathbf{M}, X) \otimes (\mathbf{B}, T)$$

In the following proofs, we will mostly deal with the multiple pointed cases. The reason to employ multiple pointed models for updating is that it allows us to handle choice. Suppose we want to model the action of testing whether φ followed by a public announcement of the result. This involves *choice*: if the outcome of the test is affirmative, then do this, else do that. Choice is modelled in a straightforward way in multiple pointed action models. Once we allow multiple pointed action models, it is reasonable to also take our epistemic models to be multiple pointed, with the multiple points constraining the whereabouts of the actual world.

From the update bisimulation preservation theorem it follows that:

Theorem 7. *Given pointed action models* $(\mathbf{A}, S)$ *and* $(\mathbf{B}, T)$*,*

$$(\mathbf{A}, S) \underline{\leftrightarrow} (\mathbf{B}, T) \textit{ implies } (\mathbf{A}, S) \equiv_{ACT} (\mathbf{B}, T)$$

Can we turn this around? No, we cannot. Here is a simple counterexample. Let

$$(\mathbf{A}, S) = ((\{a_0\}, \emptyset, \{a_0 \mapsto \bot\}), \{a_0\}),$$

and let

$$(\mathbf{0}, T) = ((\emptyset, \emptyset, \emptyset), \emptyset).$$

Then $(\mathbf{A}, S) \equiv_{\text{ACT}} (\mathbf{0}, T)$, but $(\mathbf{A}, S)$ and $(\mathbf{0}, T)$ are not bisimilar. Removing the inconsistent states (the states with a precondition equivalent to $\bot$) from an action model does not affect its update potential, so we might as well assume that action models contain only consistent states. This would reduce $\mathbf{A}$ to $\mathbf{0}$. However, Figure 3.1 provides another counterexample: non-bisimilar action models with consistent states and with the same update potentials.

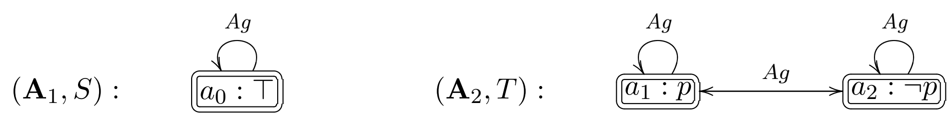


Figure 3.1: Non-bisimilar actions with the same update effects

Clearly, action a_0 in Figure 3.1 is not bisimilar to a_1, since these actions have different preconditions. Also a_0 is not bisimilar to a_2, for the same reason. Still the two action models have the same update effects:

Theorem 8. *Given pointed action models* $(\mathbf{A}_1, S), (\mathbf{A}_2, T)$ *as in Figure 3.1, for any pointed state model* $(\mathbf{M}, X)$,

$$(\mathbf{M}, X) \otimes (\mathbf{A}_1, S) \underline{\leftrightarrow} (\mathbf{M}, X) \otimes (\mathbf{A}_2, T)$$

Proof. Define a binary relation between $\mathbf{M} \otimes \mathbf{A}_1$ and $\mathbf{M} \otimes \mathbf{A}_2$: $Z := \{\langle (w, a_0), (w, a_1) \rangle \mid (\mathbf{M}, w) \models \text{Pre}(a_1)\} \cup \{\langle (w, a_0), (w, a_2) \rangle \mid (\mathbf{M}, w) \models \text{Pre}(a_2)\}$. Easy to check Z is the desired bisimulation. $\square$

For another example, consider Figure 3.2.

Each of the pointed action models in Figure 3.2 has the effect of selecting the accessibility paths with $p \vee q$ holding at every node along the paths.

Examples like these suggest that the notion of $\mathcal{LANG}$ bisimilarity is too strong to capture the 'essence' of our update actions.

In the following sections, we will first define a structural relation on action models, called action emulation, and show that this notion exactly captures the update effects of action models.

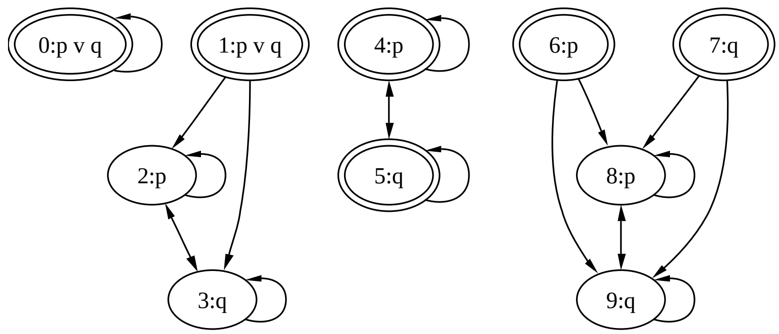


Figure 3.2: More non-bisimilar actions with the same update effects

3.3 Action Emulation

We now proceed to give a structural condition for equivalence of action models. The relation of action emulation between action models, to be defined below, can be viewed as a suitably weakened bisimulation, adapted to the case where valuations are replaced by preconditions.

Instead of insisting that the preconditions are logical equivalent, we just require that the preconditions are compatible.

Instead of insisting on a precise match in the zig and zag clauses, we merely require that an appropriate choice from a list of possible matches can be made. The idea behind this is that to match a pair (w, s) in $\mathbf{M} \otimes \mathbf{A}$, we need a pair (w, t) in $\mathbf{M} \otimes \mathbf{B}$. For (w, t) to exist, the precondition of t should be satisfied by w. Requiring that s and t have the same precondition would be too strong. Instead we require that there is a choice between finitely many t_i the preconditions of which are jointly implied by that of s.

These considerations are reflected in the following definition.

Definition 3.2 (Action Emulation). Given action models $\mathbf{A}$ and $\mathbf{B}$, a relation $E \subseteq W_{\mathbf{A}} \times W_{\mathbf{B}}$ is an action emulation if whenever sEt the following hold:

Preconditions $\mathrm{Pre}(s) \wedge \mathrm{Pre}(t)$ is consistent.

Zig If $s \xrightarrow{i} s'$ then there are $t_1, \ldots, t_n$ with

$$t \xrightarrow{i} t_1, \ldots, t \xrightarrow{i} t_n, s'Et_1, \ldots, s'Et_n \text{ and } \mathrm{Pre}(s') \models \mathrm{Pre}(t_1) \vee \cdots \vee \mathrm{Pre}(t_n).$$

Zag If $t \xrightarrow{i} t'$ then there are $s_1, \ldots, s_n$ with

$$s \xrightarrow{i} s_1, \ldots, s \xrightarrow{i} s_n, s_1Et', \ldots, s_nEt' \text{ and } \mathrm{Pre}(t') \models \mathrm{Pre}(s_1) \vee \cdots \vee \mathrm{Pre}(s_n)$$

We denote this as $\mathbf{A} \leftrightarrows \mathbf{B}$.

For multiple pointed action models $(\mathbf{A}, S)$ and $(\mathbf{B}, T)$, $(\mathbf{A}, S) \leftrightarrows (\mathbf{B}, T)$ means there is an action emulation $E \subseteq W_{\mathbf{A}} \times W_{\mathbf{B}}$ satisfying the following extra requirement: for every $s \in S(\subseteq W_{\mathbf{A}})$ there are $t_1, \ldots, t_n \in T(\subseteq W_{\mathbf{B}})$ such that $sEt_1, \ldots, sEt_n$ and $\mathrm{Pre}(s) \models \mathrm{Pre}(t_1) \vee \cdots \vee \mathrm{Pre}(t_n)$, and for every $t \in T$ there are $s_1, \ldots, s_n \in S$ with $s_1Et, \ldots, s_nEt$ and $\mathrm{Pre}(t) \models \mathrm{Pre}(s_1) \vee \cdots \vee \mathrm{Pre}(s_n)$.

A total action emulation means the emulation connects $(\mathbf{A}, W_{\mathbf{A}})$ and $(\mathbf{B}, W_{\mathbf{A}})$. For convenience, we write $\mathbf{A} \leftrightarrows \mathbf{B}$ if $(\mathbf{A}, W_{\mathbf{A}}) \leftrightarrows (\mathbf{B}, W_{\mathbf{A}})$.

Observe that the examples of actions with the same update effects all satisfy this structural requirement. Also it is easy to see that action emulation is a weakening of bisimulation, in the following sense:

Theorem 9. *Given pointed action model $(\mathbf{A}, S)$ and $(\mathbf{B}, T)$, if $(\mathbf{A}, S) \underline{\leftrightarrow} (\mathbf{B}, T)$ then $(\mathbf{A}, S) \leftrightarrows (\mathbf{B}, T)$.*

Proof. The bisimulation Z witnessing $(\mathbf{A}, S) \underline{\leftrightarrow} (\mathbf{B}, T)$, is also an action emulation witnessing $(\mathbf{A}, S) \leftrightarrows (\mathbf{B}, T)$, since the three conditions of action emulation follows from three conditions of action bisimulation respectively. □

We show that there is always a maximal action emulation. First we prove a lemma:

Lemma 10. *Suppose R, U both emulate action models $(\mathbf{A}, S)$ and $(\mathbf{B}, T)$, then $R \cup U$ is an action emulation connecting $(\mathbf{A}, S)$ and $(\mathbf{B}, T)$ too.*

Proof. For any $(s, t) \in R \cup U$, it must be the case that either $(s, t) \in R$ or $(s, t) \in U$. Without loss of generality, suppose $(s, t) \in U$, then the three conditions (Invariance, Zig, Zag) and the extra requirement follows trivially. □

Then a maximal action emulation is immediate:

Theorem 11. *There is always a maximal action emulation.*

Proof. Given action model $(\mathbf{A}, S)$ and $(\mathbf{B}, T)$, and the collection of all action emulations between them, say $\mathbf{EM}$. $\bigcup \mathbf{EM}$ is a maximal action emulation between $\mathbf{A}$ and $\mathbf{B}$, due to the fact that the union of two action emulations is still an action emulation, as showed in lemma 10. $\square$

The proof that the existence of an action emulation between $(\mathbf{A}, S)$ and $(\mathbf{B}, T)$ guarantees that they have same update effect is also straightforward:

Theorem 12. *Given pointed action models* $(\mathbf{A}, S)$ *and* $(\mathbf{B}, T)$*,*

$$\text{If } (\mathbf{A}, S) \leftrightarrows (\mathbf{B}, T) \text{ then } (\mathbf{A}, S) \equiv_{ACT} (\mathbf{B}, T).$$

Proof. Let $(\mathbf{M}, X)$ be an arbitrary pointed epistemic model. Assume $(\mathbf{A}, S) \leftrightarrows (\mathbf{B}, T)$ and let E be an action emulation witnessing this.

Define $R \subseteq \mathbf{M} \otimes \mathbf{A} \times \mathbf{M} \otimes \mathbf{B}$ by means of: $(w, s)R(v, t) :\equiv w = v \wedge sEt$. We show that R is a bisimulation: suppose $(w, s)R(v, t)$,

ValEQ From $(w, s)R(v, t)$ we get that $w = v$ and hence $\mathrm{Val}(w, s) = \mathrm{Val}(v, t)$.

Zig Let $(w, s) \xrightarrow{i} (w', s')$. Then $w \xrightarrow{i} w'$, $s \xrightarrow{i} s'$, and $(\mathbf{M}, w') \models \mathrm{Pre}(s')$. From $(w, s)R(v, t)$ we have that sEt. By sEt, there are $t_1, \ldots, t_n$ with $t \xrightarrow{i} t_1, \ldots, t \xrightarrow{i} t_n$, $s'Et_1, \ldots, s'Et_n$, and $\mathrm{Pre}(s') \models \mathrm{Pre}(t_1) \vee \cdots \vee \mathrm{Pre}(t_n)$. Since $(\mathbf{M}, w') \models \mathrm{Pre}(s')$, it follows from $\mathrm{Pre}(s') \models \mathrm{Pre}(t_1) \vee \cdots \vee \mathrm{Pre}(t_n)$ that there is some t_i with $(\mathbf{M}, w') \models \mathrm{Pre}(t_i)$. Thus $(w', s')R(w', t_i)$.

Zag Same reasoning vice versa.

Now show R connects $(\mathbf{M}, X) \otimes (\mathbf{A}, S)$ and $(\mathbf{M}, X) \otimes (\mathbf{B}, T)$. Given $(w, s) \in \mathbf{M} \otimes \mathbf{A}$ with $w \in X$ and $s \in S$, we have $(\mathbf{M}, w) \models \mathrm{Pre}(s)$. Since E connects $(\mathbf{A}, S)$ and $(\mathbf{B}, T)$, there must be $t_1, \ldots, t_n$, such that $sEt_1, \ldots, sEt_n$ and $\mathrm{Pre}(s) \models \mathrm{Pre}(t_1) \vee \ldots \vee \mathrm{Pre}(t_n)$; hence $(\mathbf{M}, w) \models \mathrm{Pre}(t_1) \vee \ldots \vee \mathrm{Pre}(t_n)$. So there must be t_i such that $(\mathbf{M}, w) \models \mathrm{Pre}(t_i)$, therefore $(w, s)R(w, t_i)$. And the other direction is similar. $\square$

The theorem shows that action emulation is a sufficient condition for having the same update effect. To see whether it is also necessary, we will make a case separation as follows.

Call an action model *propositional* if all preconditions that occur in it are purely propositional formulas. Call an action model *modal* if all preconditions that occur in it are multi-modal formulas. In the next two sections we will look at the update effects of propositional and modal action models, and show that in propositional case having the same update effect implies the existence of an action emulation, and in modal case, having the same update effect is characterized by the bisimulation of expansion defined in this thesis.

3.4 Update Effects of Propositional Actions

In this section we will show that in the case of actions with propositional preconditions, having the same update effect can be characterized in terms of the update effects in some special cases.

Let Q be a finite set of proposition letters, then a valuation over Q is a subset of Q. For $\mathbf{v} \subseteq Q$, let $\Phi(\mathbf{v}) :\equiv \bigwedge_{p \in \mathbf{v}} p \wedge \bigwedge_{p \notin \mathbf{v}} \neg p$. Then a valuation $\mathbf{v}$ models a propositional formula ψ (written as $\mathbf{v} \models \psi$) if $\Phi(\mathbf{v}) \models \psi$.

Since the preconditions of actions are propositional, we can have a set of valuations such that precondition can be modeled by this set. Given an action model $\mathbf{A}$ and let Q be the set of proposition letters occurring in the preconditions of $\mathbf{A}$, for $a \in \mathbf{A}$, $\mathrm{XP}(\mathbf{A}, a) :\equiv \{(a, \mathbf{v}) \mid \mathbf{v} \in \mathcal{P}(Q), \mathbf{v} \models \mathrm{Pre}(a)\}$, which is called the eXpansion of a Proposional action a.

Now we define the expansion of an action model with propositional preconditions by replacing all the actions with new actions in their expansions, in such a way that the expansion preserves the update effect.

Definition 3.3 (Expansion of propositional action models). Let $\mathbf{A} = (W, \xrightarrow{i}, \mathrm{Pre})$ be a finite action model with propositional precondition, Q is the set of all proposition letters occurring in $\mathbf{A}$, the expansion of $\mathbf{A}$, denoted as $\mathbf{A}^\circ$, is $(W', \xrightarrow{i}', \mathrm{Pre}')$, where:

$$\begin{array}{rcl} W' & :\equiv & \displaystyle\bigcup_{a \in W_{\mathbf{A}}} \mathrm{XP}(\mathbf{A}, a) \\ \mathrm{Pre}'(a, \mathbf{v}) & :\equiv & \displaystyle\bigwedge_{p \in \mathbf{v}} p \wedge \bigwedge_{p \notin \mathbf{v}} \neg p \\ (a, \mathbf{v}) \xrightarrow{i}' (a', \mathbf{v}') & \text{iff} & a \xrightarrow{i} a', \mathbf{v} \models \mathrm{Pre}(a), \mathbf{v}' \models \mathrm{Pre}(a') \end{array}$$

For the case of a multiple pointed action model $(\mathbf{A}, S)$, the expansion is $(\mathbf{A}^\circ, S^\circ)$ where $\mathbf{A}^\circ$ is defined above, and $S^\circ :\equiv \bigcup_{a \in S} \mathrm{XP}(\mathbf{A}, a)$

The following theorem shows that $(\mathbf{A}, S)$ and $(\mathbf{A}^\circ, S^\circ)$ have same update effect:

Theorem 13. *Given a finite action model* $(\mathbf{A}, S)$*, it has same update effect as its expansion, i.e.* $(\mathbf{A}, S) \equiv_{ACT} (\mathbf{A}^\circ, S^\circ)$*.*

Proof. Given any state model $(\mathbf{M}, X)$, we show that $(\mathbf{M}, X) \otimes (\mathbf{A}, S) \underline{\leftrightarrow} (\mathbf{M}, X) \otimes (\mathbf{A}^\circ, S^\circ)$. Define $R := \{\langle (w, s), (w, (s, \mathrm{Val}(w)))\rangle \mid w \in W_{\mathbf{M}}, s \in W_{\mathbf{A}}, (\mathbf{M}, w) \models \mathrm{Pre}(s)\}$.

Suppose $(w, s)R(w, (s, \mathrm{Val}(w)))$:

ValEQ Easy to see $\mathrm{Val}(w, s) = \mathrm{Val}(w) = \mathrm{Val}(w, (s, \mathrm{Val}(w)))$.

Zig Let $(w, s) \xrightarrow{i} (w', s')$. Then $w \xrightarrow{i} w'$, $s \xrightarrow{i} s'$, and $(\mathbf{M}, w') \models \mathrm{Pre}(s')$. Therefore, by the definition of expansion, $(s', \mathrm{Val}(w') \in \mathbf{A}^\circ$, and hence we have $(w', (s', \mathrm{Val}(w'))) \in \mathbf{M} \otimes \mathbf{A}^\circ$ complete this condition.

Zag Same reasoning vice versa.

For any $(w, s) \in \mathbf{M} \otimes \mathbf{A}$ with $w \in X, s \in S$, using the same reasoning in Zig, we find a corresponding world $(w, (s, \mathrm{Val}(w))) \in \mathbf{M} \otimes \mathbf{A}^\circ$ with $w \in X, (s, \mathrm{Val}(w)) \in S^\circ$, and vice versa. □

Thinking it from another way, we can get the expansion of $(\mathbf{A}, S)$ by updating it with a specific state model as follows: The epistemic state model VAL_Q is the model (W, R, Val) where $W = \mathcal{P}(Q)$, R is the universal relation on W for every agent $i \in Ag$, and Val is the identity function. Thus, worlds are valuations, and the valuation at each world is that world itself. For convenience, we use VAL_Q^* to denote a special pointed model $(\mathrm{VAL}_Q, W_{\mathrm{VAL}_Q})$.

Theorem 14. *Given an action model* $\mathbf{A}$*, let* Q *be the set of all proposition letters occurring in* $\mathbf{A}$*, then there is a structure preserving bijection between* $VAL_Q \otimes \mathbf{A}$ *and* $\mathbf{A}^\circ$*.*

Proof. Define a relation $Z \subseteq \mathrm{VAL}_Q \otimes \mathbf{A} \times \mathbf{A}^\circ$:

$Z :\equiv \{\langle (w, a), (a, \mathrm{Val}(w))\rangle \mid a \in W_{\mathbf{A}}, (\mathrm{VAL}_Q, w) \models \mathrm{Pre}(a)\}$

We have:

Bijection For any $(w, a) \in \mathrm{VAL}_Q \otimes \mathbf{A}$, we have $(\mathrm{VAL}_Q, w) \models \mathrm{Pre}(a)$, therefore $(a, \mathrm{Val}(w)) \in \mathbf{A}^\circ$, and vice versa. So Z is a bijection.

Relation preserving For any $(w, a) \xrightarrow{i} (w', a')$, we have $a \xrightarrow{i} a'$, then correspondingly we have $(a, \mathrm{Val}(w)) \xrightarrow{i} (a', \mathrm{Val}(w'))$. And vice versa.

□

The above theorem shows that state model $\text{VAL}_Q \otimes \mathbf{A}$ and action $\mathbf{A}^\circ$ are almost the same. The only difference is that the former one has valuation for each world and the latter one has precondition for each world, but here valuation and precondition are virtually the same, by $\Phi(\text{Val}(w, a)) \equiv \text{Pre}(a, \text{Val}(w))$.

Clearly we have:

Theorem 15. *Given propositional action models* $(\mathbf{A}, S)$ *and* $(\mathbf{B}, T)$*,*

$$(\mathbf{A}, S) \equiv_{ACT} (\mathbf{B}, T) \textit{ implies } (\mathbf{A}^\circ, S^\circ) \underline{\leftrightarrow} (\mathbf{B}^\circ, T^\circ)$$

Proof. What holds for an arbitrary epistemic model $(\mathbf{M}, X)$ certainly holds for VAL_Q^*, so $\text{VAL}_Q^* \otimes (\mathbf{A}, S) \underline{\leftrightarrow} \text{VAL}_Q^* \otimes (\mathbf{B}, T)$. By theorem 14, there is a structure preserving bijection f from $\text{VAL}_Q \otimes \mathbf{A}$ to $\mathbf{A}^\circ$, and g from $\text{VAL}_Q \otimes \mathbf{B}$ to $\mathbf{B}^\circ$. The bisimulation between $\mathbf{A}^\circ$ and $\mathbf{B}^\circ$ is established by $Z :\equiv \{(s, t) \mid s \in W_{\mathbf{A}^\circ}, t \in W_{\mathbf{B}^\circ}, f^{-1}(a) \underline{\leftrightarrow} g^{-1}(b)\}$, since the precondition equivalence and zig-zag conditions correspond to the invariance and zig-zag conditions in $\text{VAL}_Q \otimes \mathbf{A} \underline{\leftrightarrow} \text{VAL}_Q \otimes \mathbf{B}$. Also there is one-one correspondence between the pointed worlds in $\text{VAL}_Q^* \otimes (\mathbf{A}, S)$ and $(\mathbf{A}^\circ, S^\circ)$. Therefore $(\mathbf{A}^\circ, S^\circ) \underline{\leftrightarrow} (\mathbf{B}^\circ, T^\circ)$. $\square$

Next, we prove the implication from bisimulation of expanded models to having the same update effect:

Theorem 16. *Given propositional action models* $(\mathbf{A}, S), (\mathbf{B}, T)$*,*

$$(\mathbf{A}^\circ, S^\circ) \underline{\leftrightarrow} (\mathbf{B}^\circ, T^\circ) \textit{ implies } (\mathbf{A}, S) \equiv_{ACT} (\mathbf{B}, T)$$

Proof. Suppose $(\mathbf{A}^\circ, S^\circ) \underline{\leftrightarrow} (\mathbf{B}^\circ, T^\circ)$, according to theorem 14 and using similar argument in theorem 15, $\text{VAL}_Q^* \otimes (\mathbf{A}, S) \underline{\leftrightarrow} \text{VAL}_Q^* \otimes (\mathbf{B}, T)$, with Q be the set of all proposition letters occurring in $\mathbf{A}, \mathbf{B}$.

Let $(\mathbf{M}, X)$ be an arbitrary pointed epistemic model. We have to show that $(\mathbf{M}, X) \otimes (\mathbf{A}, S) \underline{\leftrightarrow} (\mathbf{M}, X) \otimes (\mathbf{B}, T)$.

Define a relation $C \subseteq W_{\mathbf{M}\otimes\mathbf{A}} \times W_{\mathbf{M}\otimes\mathbf{B}}$ by means of

$$(w, s)C(v, t) \text{ iff } w = v \;\&\; (\text{Val}(w), s) \underline{\leftrightarrow}' (\text{Val}(v), t),$$

where $\underline{\leftrightarrow}'$ is a bisimulation linking $\text{VAL}_Q^* \otimes (\mathbf{A}, S)$ to $\text{VAL}_Q^* \otimes (\mathbf{B}, T)$.

We show that C is a bisimulation. Assume $(w, s)C(v, t)$. Then $w = v$ and $(\text{Val}(w), s) \underline{\leftrightarrow}' (\text{Val}(w), t)$. We check the three bisimulation conditions:

Invariance Immediate from the fact that the valuation of (w, s) equals the valuation of w, and then equals the valuation of (w, t).

Zig Let $(w,s) \xrightarrow{i} (w',s')$. Then $w \xrightarrow{i} w'$ and $s \xrightarrow{i} s'$. It holds in $\text{VAL}_Q \otimes \mathbf{A}$ that $(\text{Val}(w),s) \xrightarrow{i} (\text{Val}(w'),s')$. By the zig condition for $(\text{Val}(w),s) \underline{\leftrightarrow}' (\text{Val}(w),t)$, it follows that there is a t' with $(\text{Val}(w),t) \xrightarrow{i} (\text{Val}(w'),t')$ and $(\text{Val}(w'),s') \underline{\leftrightarrow}' (\text{Val}(w'),t')$. So $(w',s')C(w',t')$, as desired.

Zag Similar vice versa.

For any $(w,s) \in \mathbf{M} \otimes \mathbf{A}$ with $w \in X, s \in S$, there is must be $(\text{Val}(w),s) \in \text{VAL}_Q \otimes \mathbf{A}$ due to the fact that $(\mathbf{M},w) \models \text{Pre}(s)$. According to the bisimulation between $\text{VAL}^*_Q \otimes (\mathbf{A},S)$ and $\text{VAL}^*_Q \otimes (\mathbf{B},T)$, there exists $t \in T$ such that $(\text{Val}(w),s) \underline{\leftrightarrow}' (\text{Val}(w),t)$, hence there is $(w,t) \in \mathbf{M} \otimes \mathbf{B}$ such that $(w,s)C(w,t)$. The other direction is similar. □

Combining these, we get:

Theorem 17. *Given propositional action models* $(\mathbf{A},S)$ *and* $(\mathbf{B},T)$*,*

$$(\mathbf{A}^\circ,S^\circ) \underline{\leftrightarrow} (\mathbf{B}^\circ,T^\circ) \text{ \textit{iff} } (\mathbf{A},S) \equiv_{ACT} (\mathbf{B},T)$$

Proof. Immediate from theorems 15 and 16. □

3.5 Update Effects of Modal Actions

We now turn to the case where the preconditions are multi-modal formulas, i.e., where they belong to the language defined by:

$$\varphi \quad ::= \quad \top \mid p \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \Box_i\varphi$$

We extend the definition of action model expansions, as follows. Let Π be the set of preconditions occurring in action models $\mathbf{A},\mathbf{B}$. Let Q be the set of all proposition letters occurring in Π. Let MCONS_Π be the set of all maximal consistent[1] subsets taken from $\neg\text{Sub}\Pi$, where Sub denotes taking subformulas and $\neg$ denotes closure under single negations. Let EXP_Π be the triple $(W, \xrightarrow{i}, \text{Val})$ where $W = \text{MCONS}_\Pi$, Val is the function that assigns to every maximal consistent subset $\Gamma \in \text{MCONS}_\Pi$ with the Q-valuation $\Gamma \cap Q$, and relation $\xrightarrow{i}$ is given by:

$$\Gamma \xrightarrow{i} \Gamma' \text{ iff } \forall\varphi \in \Gamma', \Box_i\neg\varphi \notin \Gamma.$$

Thus, the accessibility relations now take the modal constraints imposed by the preconditions into account. For convenience, we use EXP^*_Π to denote $(\text{EXP}_\Pi, W_{\text{EXP}_\Pi})$.

The following lemma will play a role in our proof:

[1] We use a normal multi-modal axiom system here.

Lemma 18 (Truth Lemma). *Given action model* $\mathbf{A}$ *and* EXP_Π *induced from* $\mathbf{A}$*, and* $a \in W_\mathbf{A}$*, and for any* $\varphi \in \neg Sub\Pi$*,*

$$(EXP_\Pi, \Gamma) \models \varphi \textit{ iff } \varphi \in \Gamma$$

Proof. The proof follows from a standard induction on the structure of formulas, so we do not provide it here. $\square$

Similar to the propositional case, we first give the expansion of a modal action, then define the expansion of an action model. Given a modal action model $\mathbf{A}$ and let Π be the set of preconditions in $\mathbf{A}$. Let Q be the set of proposition letters occurring in Π, for $a \in \mathbf{A}$, $\mathrm{XM}(\mathbf{A}, a) := \{(a, \Gamma) \mid \mathrm{Pre}(a) \in \Gamma \ \& \ \Gamma \in \mathrm{MCONS}_\Pi\}$, which is called the eXpansion of a Modal action a.

Now we define the expansion of a modal action model by replacing all the actions with new actions in their expansions. Then we show the expansion preserves same update effect.

Definition 3.4 (Expansion of modal action models). Let $\mathbf{A} = (W, \xrightarrow{i}, \mathrm{Pre})$ be a finite action model with modal preconditions, and Π be the set of preconditions in $\mathbf{A}$, which can be modal formulas. Let Q be the set of proposition letters occurring in Π, the expansion of $\mathbf{A}$ (denoted as $\mathbf{A}^\odot$) is $(W', \xrightarrow{i}', \mathrm{Pre}')$, where:

$$\begin{aligned}
W' &:= \bigcup_{a \in W_\mathbf{A}} \mathrm{XM}(\mathbf{A}, a) \\
\mathrm{Pre}'(a, \Gamma) &:= \bigwedge_{p \in \Gamma \cap Q} p \wedge \bigwedge_{p \notin \Gamma \cap Q} \neg p \\
(a, \Gamma) \xrightarrow{i}' (a', \Gamma') &\quad \text{iff} \quad a \xrightarrow{i} a' \text{ and } \Gamma \xrightarrow{i} \Gamma'
\end{aligned}$$

For the case of a multiple pointed action model $(\mathbf{A}, S)$, the expansion is $(\mathbf{A}^\odot, S^\odot)$ where $\mathbf{A}^\odot$ is the same as above, and $S^\odot := \bigcup_{a \in S} \mathrm{XM}(\mathbf{A}, a)$

Similar to theorem 14, we show that the expansion of $\mathbf{A}$ is very similar to EXP_Π updating with $\mathbf{A}$:

Theorem 19. *Given a modal action model* $\mathbf{A}$*, let* Π *be the set of the preconditions in* $\mathbf{A}$*, then there is a structure preserving bijection between* $EXP_\Pi \otimes \mathbf{A}$ *and* $\mathbf{A}^\odot$*.*

Proof. Define a relation $Z \subseteq \text{EXP}_\Pi \otimes \mathbf{A} \times \mathbf{A}^\odot$:

$Z := \{\langle(\Gamma, a), (a, \Gamma')\rangle \mid a \in W_\mathbf{A}, \Gamma = \Gamma' \in \text{MCONS}_\Pi\}$

We have:

Bijection By truth lemma, we have $(\Gamma, a) \in \text{EXP}_\Pi \otimes \mathbf{A}$ iff $(\text{EXP}_\Pi, \Gamma) \models \text{Pre}(a)$ iff $\text{Pre}(a) \in \Gamma$ iff $(a, \Gamma) \in \mathbf{A}^\odot$. For any $(\Gamma, a) \in \text{EXP}_\Pi \otimes \mathbf{A}$, we have one unique correspondence $(a, \Gamma) \in \mathbf{A}^\odot$. So Z is a bijection.

Relation preserving For any $(\Gamma, a) \xrightarrow{i} (\Gamma', a')$, we have $\Gamma \xrightarrow{i} \Gamma'$ and $a \xrightarrow{i} a'$, which exactly make $(a, \Gamma) \xrightarrow{i} (a', \Gamma')$; and vice versa.

$\square$

Clearly we have:

Theorem 20. *Given modal action models $(\mathbf{A}, S)$ and $(\mathbf{B}, T)$,*

$$(\mathbf{A}, S) \equiv_{ACT} (\mathbf{B}, T) \textit{ implies } (\mathbf{A}^\odot, S^\odot) \underline{\leftrightarrow} (\mathbf{B}^\odot, T^\odot)$$

Proof. What holds for an arbitrary epistemic model $(\mathbf{M}, X)$ certainly holds for EXP^*_Π, so $\text{EXP}^*_\Pi \otimes (\mathbf{A}, S) \underline{\leftrightarrow} \text{EXP}^*_\Pi \otimes (\mathbf{B}, T)$. By the theorem 19 and a similar argument as in theorem 15, it follows that $(\mathbf{A}^\odot, S^\odot) \underline{\leftrightarrow} (\mathbf{B}^\odot, T^\odot)$. $\square$

Next, we prove that for modal action models, bisimilarity of expanded models implies having the same update effect.

Theorem 21. *Given modal action models $(\mathbf{A}, S)$ and $(\mathbf{B}, T)$,*

$$(\mathbf{A}^\odot, S^\odot) \underline{\leftrightarrow} (\mathbf{B}^\odot, T^\odot) \textit{ implies } (\mathbf{A}, S) \equiv_{ACT} (\mathbf{B}, T)$$

Proof. Assume $(\mathbf{A}^\odot, S^\odot) \underline{\leftrightarrow} (\mathbf{B}^\odot, T^\odot)$. By theorem 19, we have $\text{EXP}^*_\Pi \otimes (\mathbf{A}, S) \underline{\leftrightarrow} \text{EXP}^*_\Pi \otimes (\mathbf{B}, T)$, with Π the set of preconditions occurring in $\mathbf{A}, \mathbf{B}$.

Let $(\mathbf{M}, X)$ be an arbitrary epistemic model. We have to show that $(\mathbf{M}, X) \otimes (\mathbf{A}, S) \underline{\leftrightarrow} (\mathbf{M}, X) \otimes (\mathbf{B}, T)$.

Let $\Pi^\mathbf{M}_w$ be the set $\{\varphi \in \neg\text{Sub}\Pi \mid (\mathbf{M}, w) \models \varphi\}$. Note that $\Pi^\mathbf{M}_w \in \text{MCONS}_\Pi$.

Define a relation $C \subseteq W_{\mathbf{M}\otimes\mathbf{A}} \times W_{\mathbf{M}\otimes\mathbf{B}}$ by means of

$$(w, s)C(v, t) \quad \text{iff} \quad w = v \text{ and } (\Pi^\mathbf{M}_w, s) \underline{\leftrightarrow}' (\Pi^\mathbf{M}_w, t).$$

where $\underline{\leftrightarrow}'$ is the bisimulation linking $(\mathbf{A}^\odot, S^\odot)$ to $(\mathbf{B}^\odot, T^\odot)$.

We show that C is a bisimulation. Assume $(w, s)C(v, t)$. Then $w = v$ and $(\Pi^\mathbf{M}_w, s) \underline{\leftrightarrow}' (\Pi^\mathbf{M}_w, t)$. We check the three bisimulation conditions:

Invariance Immediate from the fact that the valuation of (w, s) equals the valuation of w equals the valuation of (w, t).

Zig Let $(w, s) \xrightarrow{i} (w', s')$. Then $w \xrightarrow{i} w'$, $s \xrightarrow{i} s'$, and $(\mathbf{M}, w') \models \text{Pre}(s')$. It follows from $(\mathbf{M}, w') \models \text{Pre}(s')$ that $\text{Pre}(s') \in \Pi^{\mathbf{M}}_{w'}$.

Let $\varphi \in \Pi^{\mathbf{M}}_{w'}$. Assume, for a contradiction, that $\Box_i \neg\varphi \in \Pi^{\mathbf{M}}_{w}$. Then, because $\Pi^{\mathbf{M}}_{w}$ is maximally consistent, $\Diamond_i\varphi \notin \Pi^{\mathbf{M}}_{w}$, and contradiction with the fact that $(\mathbf{M}, w) \models \Diamond_i\varphi$. It follows that $\Box_i\neg\varphi \notin \Pi^{\mathbf{M}}_{w}$. Thus, $\Pi^{\mathbf{M}}_{w} \xrightarrow{i} \Pi^{\mathbf{M}}_{w'}$.

From $\text{Pre}(s') \in \Pi^{\mathbf{M}}_{w'}$ we get that $(\Pi^{\mathbf{M}}_{w'}, s')$ is among the states of $\mathbf{A}^{\odot}$, and from $s \xrightarrow{i} s'$ and $\Pi^{\mathbf{M}}_{w} \xrightarrow{i} \Pi^{\mathbf{M}}_{w'}$ it follows that $(\Pi^{\mathbf{M}}_{w}, s) \xrightarrow{i} (\Pi^{\mathbf{M}}_{w'}, s')$.

Since $(\Pi^{\mathbf{M}}_{w}, s) \,\underline{\leftrightarrow}'\, (\Pi^{\mathbf{M}}_{w}, t)$, it follows from $(\Pi^{\mathbf{M}}_{w}, s) \xrightarrow{i} (\Pi^{\mathbf{M}}_{w'}, s')$ that there is a t' with $(\Pi^{\mathbf{M}}_{w}, t) \xrightarrow{i} (\Pi^{\mathbf{M}}_{w'}, t')$ and $(\Pi^{\mathbf{M}}_{w'}, s') \,\underline{\leftrightarrow}'\, (\Pi^{\mathbf{M}}_{w'}, t')$. Therefore $(w', s')C(w', t')$, as desired.

Zag Similar.

Easy to check that each pointed world in $(\mathbf{M}, X) \otimes (\mathbf{A}, S)$ connects one in $(\mathbf{M}, X) \otimes (\mathbf{B}, T)$ and vice versa. $\Box$

Combining the above, we have:

Theorem 22. *Suppose* $(\mathbf{A}, S)$ *and* $(\mathbf{B}, T)$ *have modal preconditions, then*

$$(\mathbf{A}^{\odot}, S^{\odot}) \,\underline{\leftrightarrow}\, (\mathbf{B}^{\odot}, T^{\odot}) \text{ iff } (\mathbf{A}, S) \equiv_{ACT} (\mathbf{B}, T)$$

Proof. Immediate from theorems 20 and 21. $\Box$

One may note that the modal precondition case implies the propositional case. The model VAL_Q is obtained from EXP_Π when no modal operators occurred in the preconditions. The main reason we do this case separation is due to the different results for modal contraction in the following section.

3.6 Model contraction preserving update effects

Now we look at the converse of expansion, i.e. contraction. We contract the action models in such a way that same update effects are preserved. Here is a simple example in Figure 3.3.

We have shown that $(\mathbf{A}_1, a_0)$ and $(\mathbf{A}_2, \{a_1, a_2\})$ have same update effect. And it is easy to see that they are not bisimilar. However, we can view $\mathbf{A}_1$

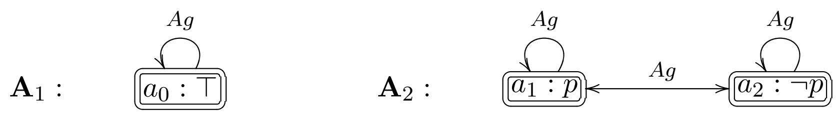


Figure 3.3: Contraction of action model

as a contracted action model from $\mathbf{A}_2$: namely the actions a_1 and a_2 in $\mathbf{A}_2$ can be glued together. The key observation here is that they have the same predecessors and successors. We make this more formal:

Definition 3.5 (Contraction of action models). Let $\mathbf{A} = (W, \xrightarrow{i}, \mathrm{Pre})$ be a finite action model, the contracted model of $\mathbf{A}$ with respect to $a \in W$, denoted as $\mathrm{CTR}(\mathbf{A}, a)$, is generated by the following procedure:
Let

$$\mathrm{T}(a) :\equiv \{b \in W \mid \forall c \in W, i \in Ag(c \xrightarrow{i} a \text{ iff } c \xrightarrow{i} b)\&(a \xrightarrow{i} c \text{ iff } b \xrightarrow{i} c)\}$$

then

We get $\mathrm{CTR}(\mathbf{A}, a)$ from $\mathbf{A}$ by deleting the actions in $\mathrm{T}(a)\backslash\{a\}$ and related links, then set $\mathrm{Pre}(a) :\equiv \bigvee\{\mathrm{Pre}(b) \mid b \in \mathrm{T}(a)\}$.

If an action model can not be contracted to a smaller model, then we call it a fully contracted model.

Now we show that $\mathbf{A}$ and $\mathrm{CTR}(\mathbf{A}, a)$ have same update effect. For the simplicity of presentation, we only treat models without pointed worlds, and therefore use total bisimulation in the following proofs. It is easy to adapt the results to pointed bisimulation.

Theorem 23. *Given action model* $\mathbf{A}$*, and* $a \in W_{\mathbf{A}}$*:*

$$\mathbf{A} \equiv_{ACT} CTR(\mathbf{A}, a)^2$$

Proof. Given any state model $\mathbf{M}$, define a binary relation between $\mathbf{M} \otimes \mathbf{A}$ and $\mathbf{M} \otimes \mathrm{CTR}(\mathbf{A}, a)$ as follows:

$$Z := \{\langle (w, a'), (w, a)\rangle \mid a' \in \mathrm{T}(a)\} \cup \{\langle (w, b), (w, b)\rangle \mid b \in W_{\mathbf{A}}\backslash \mathrm{T}(a)\}$$

Then we show Z is a total bisimulation. Suppose $\langle (w, a_1), (w, a_2)\rangle \in Z$:

ValEQ : Immediately follows from the fact that the valuations of (w, a_1) equals to the valuation of w, which equals to the valuation of (w, a_2);

[2] For the reason give above, we use total bisimulaton here.

Zig : suppose $(w, a_1) \rightarrow_i (w', a_1')$. We distinguish 4 cases:

- $a_1, a_1' \in T(a)$: So a_1, a_1' and a have the same predecessors and successors; combined with the fact that $a_1 \xrightarrow{i} a_1'$, we know that there are reflexive and transitive i-links among a_1, a_1' and a.

  By the definition of Z, $a_2 = a$. Therefore a_1 and a_2 have the same predecessors and successors. Then (w', a) will complete this condition, since $\langle (w', a_1'), (w', a) \rangle \in Z$.
- $a_1 \in T(a), a_1' \notin T(a)$: So a_1 and a have the same predecessors and successors. By the definition of Z, $a_2 = a$. Therefore a_1 and a_2 have the same predecessors and successors. Then (w', a_1') will complete this condition, since $\langle (w', a_1'), (w', a_1') \rangle \in Z$.
- $a_1 \notin T(a), a_1' \in T(a)$: By Z, $a_1 = a_2$. Then we take (w', a) to complete this condition, since a_1' and a have the same predecessors, and $\langle (w', a_1'), (w', a) \rangle \in Z$.
- $a_1 \notin T(a), a_1' \notin T(a)$: By Z, $a_1 = a_2$. Then we take (w', a_1') to complete this condition, which is easy to see.

Zag : similar argument.

Easy to check that Z is a total relation. For each $(w, s) \in \mathbf{M} \otimes \mathbf{A}$, s belongs to either $\mathrm{T}(a)$ or $W_{\mathbf{A}} \backslash \mathrm{T}(a)$, then there is (w, a) or (w, s) in $\mathbf{M} \otimes \mathrm{CTR}(\mathbf{A}, a)$ correspondingly. The other direction is similar. □

There could be different ways to contract an action model to a minimal one, due to the different order of worlds selected to do contraction. However, we can show that any fully contracted action model of $\mathbf{A}$ has the same update effect as $\mathbf{A}$:

Theorem 24. *Let* $\mathbf{A}$ *be a finite action model, and* $\mathbf{A}'$ *is a fully contracted action model started from* $\mathbf{A}$*, then*

$$\mathbf{A} \equiv_{ACT} \mathbf{A}'$$

Proof. Given any state model $\mathbf{M}$.

Since $\mathbf{A}$ is finite, the contraction procedure of $\mathbf{A}$ can only repeat finitely many times before it is contracted to $\mathbf{A}'$. Suppose the sequence of contracting $\mathbf{A}$ is as follows: $\mathbf{A}_0 (= \mathbf{A}), \mathbf{A}_1, ..., \mathbf{A}_n (= \mathbf{A}')$. From theorem 23, it follows that $\mathbf{M} \otimes \mathbf{A}_{i-1} \underline{\leftrightarrow} \mathbf{M} \otimes \mathbf{A}_i$ for $1 \leq i \leq n$. So by transitivity of bisimulation, we have $\mathbf{M} \otimes \mathbf{A} \underline{\leftrightarrow} \mathbf{M} \otimes \mathbf{A}'$. Therefore $\mathbf{A} \equiv_{\mathrm{ACT}} \mathbf{A}'$ as desired. □

We also can define a fully contracted action model directly:

Definition 3.6 (Full contraction). Let $\mathbf{A} = (W, \xrightarrow{i}, \text{Pre})$ be a finite action model, $\mathbf{A}_\circ = (W', \xrightarrow{i}{}', \text{Pre}')$ is a fully contracted model of $\mathbf{A}$, where:

- $W' = \{\text{T}(a) \mid a \in W\}$
- $\text{Pre}'(\text{T}(a)) = \bigvee_{b \in \text{T}(a)} \text{Pre}(b)$
- $\text{T}(a) \xrightarrow{i}{}' \text{T}(b)$ iff $a \xrightarrow{i} b$

Now we show different ways of contraction virtually leads to same result:

Theorem 25. *Given action model $\mathbf{A}$, suppose $\mathbf{A}'$ is a fully contracted model of $\mathbf{A}$, and $\mathbf{A}_\circ$ is the particular one defined above, then $\mathbf{A}'$ is isomorphic to $\mathbf{A}_\circ$.*

Proof. Suppose $\mathbf{A}'$ is obtained from a series one-step contraction $\mathbf{A}_0, \mathbf{A}_1, \cdots, \mathbf{A}_n$, such that $\mathbf{A}_0 = \mathbf{A}$ and $\mathbf{A}_n = \mathbf{A}'$. The universe of $\mathbf{A}$ can be divided into equivalent classes, in which worlds have same predecessors and successors. One step contraction just takes a representative world from each equivalent class and then replace that class with it, and the links to that equivalent class is preserved by this representative one. Then the isomorphism follows immediately. □

It is easy to see that the particular fully contracted model defined above has the same update effect as the other ones:

Theorem 26. *Given an action model $\mathbf{A}$, let $\mathbf{A}'$ be an arbitrary fully contracted model of $\mathbf{A}$, then $\mathbf{A}_\circ \equiv_{ACT} \mathbf{A}'$.*

Proof. This easily follows from theorem 24. □

We have shown that if two action models have the same update effect, then the expansions of two action models are bisimilar, as in theorem 15 for propositional case and in theorem 20 for modal case. Our contraction applies to both cases, then a question is: does a similar theorem hold for the contractions of action models? The answer is no, due to the counterexample in Figure 3.4.

In Figure 3.4, we have two action models $(\mathbf{A}, S)$, $(\mathbf{B}, T)$ with $S = \{0\}$ and $T = \{1, 2\}$. It is not hard to see that they are not bisimilar, and we can show that they have same update effects by theorem 12 and an easy-to-check fact $(\mathbf{A}, S) \leftrightarrows (\mathbf{B}, T)$. Please notice that their fully contracted models are identical with themselves, since no contraction is possible for them. So this

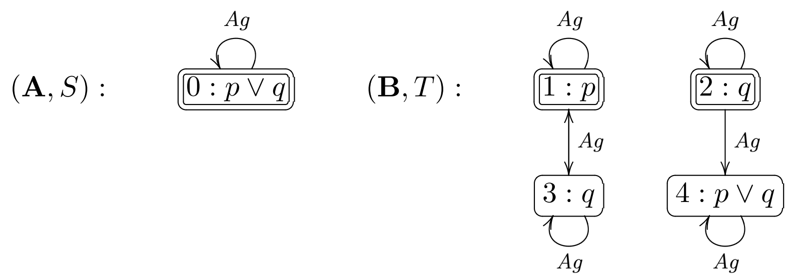


Figure 3.4: Action models with the same update effects, but their contractions are non-bisimilar.

example shows that for two action models with same update effects, their contraction are not necessary bisimilar.

But the converse does hold:

Theorem 27. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be finite action models, if* $\mathbf{A}_\circ \underline{\leftrightarrow} \mathbf{B}_\circ$ *then* $\mathbf{A} \equiv_{ACT} \mathbf{B}$*.*

Proof. Suppose $\mathbf{A}_\circ \underline{\leftrightarrow} \mathbf{B}_\circ$. It follows that $\mathbf{A}_\circ \equiv_{\mathrm{ACT}} \mathbf{B}_\circ$. And by theorem 24, we have $\mathbf{A}_\circ \equiv_{\mathrm{ACT}} \mathbf{A}$ and $\mathbf{B}_\circ \equiv_{\mathrm{ACT}} \mathbf{B}$. So $\mathbf{A} \equiv_{\mathrm{ACT}} \mathbf{B}$ as desired. $\square$

Now we try to connect our action emulation with the expansion and contraction above. We have shown that the action emulation implies same update effect, and now we show the converse also holds for propositional case.

First we prove several lemmas as follows:

Notice that emulation is not transitive, since $p \wedge \top$ and $\neg p \wedge \top$ are consistent, but $\neg p \wedge p$ is not. But for pointed bisimulation, it is transitive.

Lemma 28 (Emulation transitivity). *Given action models,* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{C}$*, suppose there are total emulation that* $\mathbf{A} \leftrightarrows \mathbf{B}$*,* $\mathbf{B} \leftrightarrows \mathbf{C}$*, then* $\mathbf{A} \leftrightarrows \mathbf{C}$*.*

Proof. Suppose E_1 is the total emulation between $\mathbf{A}$ and $\mathbf{B}$, and E_2 is the total emulation between $\mathbf{B}$ and $\mathbf{C}$. Define a binary relation $R \subseteq W_{\mathbf{A}} \times W_{\mathbf{B}}$ as follows,

$$aRc := aE_1 \circ E_2 c \;\&\; \mathrm{Pre}(a) \wedge \mathrm{Pre}(c) \text{ is consistent}$$

It is easy to verify that R is an action emulation. We show R is total: given $a \in W_{\mathbf{A}}$, since $\mathbf{A} \leftrightarrows \mathbf{B}$, we have $b_1, ..., b_n$, such that $aE_1 b_i$ and $\mathrm{Pre}(a) \models \mathrm{Pre}(b_1) \vee ... \vee \mathrm{Pre}(b_n)$. Similarly, for each b_i, there is $c_1, ..., c_j$, such that

$b_iE_2c_k$ and $\text{Pre}(b_i) \models \text{Pre}(c_1) \vee ... \vee \text{Pre}(c_j)$. From the transitivity of $\models$, it follows that there is $c_1, ..., c_m$, such that such that aE_1c_i and $\text{Pre}(a) \models \text{Pre}(c_1) \vee ... \vee \text{Pre}(c_m)$. For the other direction, the argument is similar. $\square$

Lemma 29. *Given an action model* $\mathbf{A}$*, and* $a \in W_{\mathbf{A}}$*,*

$$\mathbf{A} \leftrightarrows CTR(\mathbf{A}, a)$$

Proof. Define a relation E between $\mathbf{A}$ and $\text{CTR}(\mathbf{A}, a)$ as follows:

$$E := \{(x, x) \mid x \notin \text{T}(a)\} \cup \{(x, a) \mid x \in \text{T}(a)\}$$

It is easy to check that E is an emulation. $\square$

For the propositional case, it follows from lemma 29 that if two action models' expansions emulate each other, they should emulate each other too:

Theorem 30. *Given propositional action models* $\mathbf{A}$*,* $\mathbf{B}$*,* $\mathbf{A}^\circ \leftrightarrows \mathbf{B}^\circ$ *implies* $\mathbf{A} \leftrightarrows \mathbf{B}$*.*

Proof. As we showed in lemma 29, one step contraction preserves emulation. Since $\mathbf{A}$ is propositional, the expansion guarantees that from $\mathbf{A}^\circ$, there is a way to contract back to $\mathbf{A}$, we have $\mathbf{A}^\circ \leftrightarrows \mathbf{A}$. Similarly we have $\mathbf{B}^\circ \leftrightarrows \mathbf{B}$, therefore $\mathbf{A} \leftrightarrows \mathbf{B}$. $\square$

We do not have a similar theorem for modal case is because we may not have a way to contract the expansion for modal action models to the original one. The main reason is that $(a, \Gamma) \xrightarrow{i} (a', \Gamma')$ requires both $a \xrightarrow{i} a'$ and $\Gamma \xrightarrow{i} \Gamma'$, and $\Gamma \xrightarrow{i} \Gamma'$ depends on $\text{Pre}(a)$ and $\text{Pre}(a')$, so the expansion may lose some links which could not be found back by contraction.

Now we show same update effect implies emulation in propositional case:

Theorem 31. *Given propositional action models* $(\mathbf{A}, S)$ *and* $(\mathbf{B}, T)$*,*

$$(\mathbf{A}, S) \equiv_{ACT} (\mathbf{B}, T) \textit{ implies } (\mathbf{A}, S) \leftrightarrows (\mathbf{B}, T)$$

Proof. Suppose $(\mathbf{A}, S) \equiv_{\text{ACT}} (\mathbf{B}, T)$, then by theorem 15, we have $(\mathbf{A}^\circ, S^\circ) \underline{\leftrightarrow} (\mathbf{B}^\circ, T^\circ)$. Since emulation can be seen as a weakened bisimulation, it follows that $(\mathbf{A}^\circ, S^\circ) \leftrightarrows (\mathbf{B}^\circ, T^\circ)$. And from an easy adaption of theorem 30, we have $(\mathbf{A}, S) \leftrightarrows (\mathbf{B}, T)$, as desired. $\square$

Combine theorem 12 and 31, we have:

Theorem 32. *Given propositional action models* $(\mathbf{A}, S)$ *and* $(\mathbf{B}, T)$,

$$(\mathbf{A}, S) \equiv_{ACT} (\mathbf{B}, T) \text{ iff } (\mathbf{A}, S) \leftrightarrows (\mathbf{B}, T)$$

However in modal case, two action models having same update effects do not necessary emulate. Here is a counter example in Figure 3.5. Updating $(\mathbf{A}_1, S)$ and $(\mathbf{A}_2, T)$ with an arbitrary state model $(\mathbf{M}, X)$, the actions a_0 and a_1 will be executable in exactly same set of worlds in $\mathbf{M}$; moreover, there will be no i-links in $\mathbf{M} \otimes \mathbf{A}_2$, since the only possible case is $(m, a_1) \xrightarrow{i} (m', a_2)$ but $(\mathbf{M}, m) \models \text{Pre}(a_1)$ and $(\mathbf{M}, m') \models \text{Pre}(a_2)$ can not hold at the same time. Therefore $(\mathbf{A}_1, S) \equiv_{\text{ACT}} (\mathbf{A}_2, T)$, but not $(\mathbf{A}_1, S) \leftrightarrows (\mathbf{A}_2, T)$.

$$(\mathbf{A}_1, S) : \quad \boxed{\boxed{a_0 : \Box_i \neg\varphi}} \qquad\qquad (\mathbf{A}_2, T) : \quad \boxed{\boxed{a_1 : \Box_i \neg\varphi}} \xrightarrow{i} \boxed{a_2 : \varphi}$$

Figure 3.5: Modal actions with same update effects

In the above sections we have shown that, in the context of either propositional or modal action models, having same update effects is equivalent to bisimilarity of expanded action models, but not necessarily to bisimilarity of contracted action models. For the propositional case, our action emulation exactly characterize the same update effects. We also studied operations (expansion and contraction) on action models, by which same update effects preserved. This suggests a way of modal minimization preserving same update effects by first using the expansion, and then iterating the contraction under bisimulation and the one defined above. For instance the action model $\mathbf{B}$ in Figure 3.4 can be finally reduced to the much more simpler model $\mathbf{A}$.

Chapter 4

Evolution of Update Universe

4.1 Setting of the problems

In this chapter, we study the problem of update evolution. Here update evolution means the change of state model repeatedly updated by an action model.

Given a state model $\mathbf{M}$ and action model $\mathbf{A}$, if we iterate the update, we get a sequence of state models:

$$\mathbf{M} \otimes \mathbf{A}, (\mathbf{M} \otimes \mathbf{A}) \otimes \mathbf{A}, ..., (\cdots((\mathbf{M} \otimes \underbrace{\mathbf{A}) \otimes \mathbf{A}) \cdots \otimes \mathbf{A}}_{n}), ...$$

Then we may ask: What does the sequence looks like? When does it get stabilization in finitely many steps? Is it possible that the sequence never reaches stabilization? Here stabilization means after finite many updates, the result model is bisimilar to a state model in an earlier position of the sequence. We are also interested in the size of update universe. The question is can we determine the growth of $\mathbf{M} \otimes \mathbf{A}^k$ as a function of $|\mathbf{M}|$, $|\mathbf{A}|$.

For the target of our study, there is a choice of whether the state/action models come with pointed worlds. For generality and simplicity, we will only study the models without pointed worlds.

From [3], it has been proved that the composition of action models can achieve the following isomorphism:

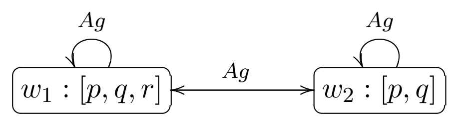
$$(\mathbf{M} \otimes \mathbf{A}) \otimes \mathbf{B} \cong \mathbf{M} \otimes (\mathbf{A} \odot \mathbf{B})$$

So it is an easy corollary that: $(\cdots((\mathbf{M} \otimes \underbrace{\mathbf{A}) \otimes \mathbf{A}) \cdots \otimes \mathbf{A}}_{n}) \cong \mathbf{M} \otimes \mathbf{A}^n$.

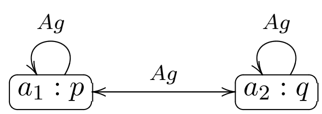
By the definition of update product, we have $|\mathbf{M} \otimes \mathbf{A}| \leq |\mathbf{M}| \times |\mathbf{A}|$. So $|\mathbf{M}| \times |\mathbf{A}|^k$ is an upper bound for the size of $\mathbf{M} \otimes \mathbf{A}^k$. This is only a first observation, and we want to find a tighter upper bound.

For the model itself, we may be interested in the preservation of some properties: reflexivity, transitivity, symmetry, seriality, the Euclidean property, etc. For the actions, we can have purely propositional preconditions, which make the problem much easier; and we can have modal preconditions, which make the problem hard since whether such actions can execute depends on the structure of the state model. In terms of epistemic precondition, we may distinguish the cases between the one with dynamic modal operator and the one without it.

Here we may want to restrict $\mathbf{M}$ and $\mathbf{A}$ both to be bisimilar minimal. So an interesting question is: will $\mathbf{M} \otimes \mathbf{A}$ also be bisimilar minimal? The answer is 'not necessary'. We can look at the following example:
State model:



Action model:

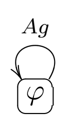


It's easy to see that in the result model, $(w_1, a_1) \underline{\leftrightarrow} (w_1, a_2)$ and $(w_2, a_1) \underline{\leftrightarrow} (w_2, a_2)$.

4.2 Special cases

Before digging into general investigation, let's look at some simple cases.

Example 4.1 (Public announcement). The action model of public announcement is very simple. (Ag is the set of all agents and φ is the sentence being announced.)



In this case, the size of the $|\mathbf{M} \otimes \mathbf{A}|$ does not increase since $|\mathbf{A}|$=1. Although after each update, we have a new state model with new worlds, we can think of updating as eliminating since the new model is isomorphic to a sub-model of the one before update. After each update, some worlds might be eliminated because they do not satisfy the precondition: φ. Once there are no worlds to be eliminated, the update reaches stabilization. It is easy to see that there are at most $|\mathbf{M}|$ updates before it gets stabilized, i.e. for any $k \geq |\mathbf{M}|$, $\mathbf{M} \otimes \mathbf{A}^k \underline{\leftrightarrow} \mathbf{M} \otimes \mathbf{A}^{k+1}$. We finally get a state model which is isomorphic to a sub-model of $\mathbf{M}$.

We find that, for some situations, the updates needed for stabilization are bounded by the length of φ. For instance, let $\varphi := p_1 \wedge (\Box_1 p_1 \rightarrow p_2) \wedge (\Box_1 p_2 \rightarrow p_3)$, the update will be stabilized in at most 3 updates, no matter how complex $\mathbf{M}$ is: each update makes p_1, p_2 and p_3 be common knowledge respectively, then φ itself becomes common knowledge and hence stabilization is reached. Now the question is: will there always be a uniform finite limit after which update has no further effect? In other words: for all φ does there always exist k such that for all $\mathbf{M}$,$n \geq k$: $\mathbf{M} \otimes [\varphi!]^n \underline{\leftrightarrow} \mathbf{M} \otimes [\varphi!]^{n+1}$?

Here is another example from Muddy Children Puzzle[1]: suppose we have 3 children(all muddy), then each update is equivalent to announce: $\neg K_1 p \wedge \neg K_2 q \wedge \neg K_3 r$ (No kids know whether they are muddy or not). The original model, where the children have been told "at least one of you has mud on your forehead", looks like a cube without one vertex. It has 7 states at first, and then each announcement will eliminate 3 states. So after two announcements, we have only one state in which all children know they are muddy and $\neg K_1 p \wedge \neg K_2 q \wedge \neg K_3 r$ becomes false. So in this case, the bound of repetitions is still determined by the the length of the announced formula. But the difference with the first example is that the times of update depends on this particular model.

The answer to the question we mentioned above is no, due to the following counterexample: Given a state model as follows:

$$\bullet \text{ —1— } \circ \text{ —2— } \bullet \text{ —1— } \circ \cdots\cdots\cdots \circ \text{ —1— } \bullet \text{ —2— } \circ \text{ —1— } \bullet$$

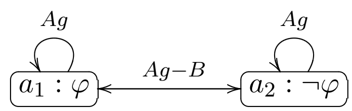
Figure 4.1: In $\bullet$-worlds, p is false; in $\circ$-worlds, p is true.

The above model has $2n$ worlds and S5 property. Suppose the publicly announced formula is $\varphi := \Box_1 \Diamond_2 p$. Easy to say that $\Diamond_2 p$ is false only in the $\bullet$-world at each end, and hence φ is only false in the two worlds at each end. Then each announcement eliminates two worlds at each end. Since

[1]If you are not familiar with this famous example, please refer to [13] or search it by Google.com

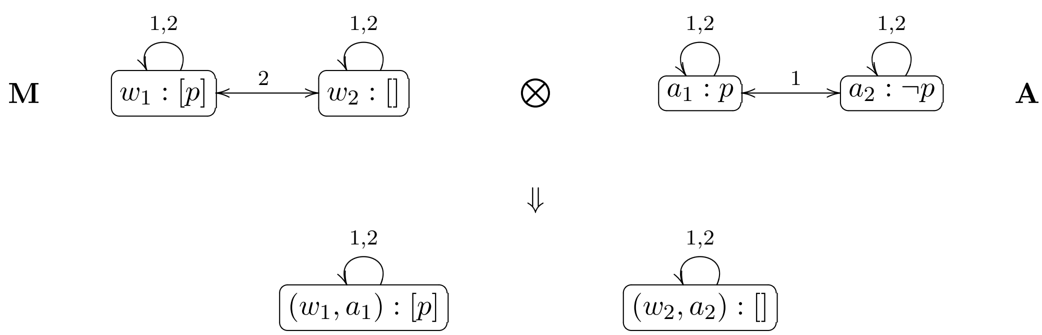
the model can be arbitrarily large, the number of repetitions before it gets stabilized can not be bounded by the length of formula.

Example 4.2 (Message to subgroup with common knowledge of suspicion). A group of agents $B(\subseteq Ag)$ asked a question about whether φ is the case, and received a truthful answer. The rest of the agents heard this question and also noticed the delivery of the answer but didn't know the content. The action model is an S5 model:



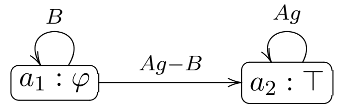
It's easy to see that $|\mathbf{M}\otimes\mathbf{A}|=|\mathbf{M}|$ since for each world $w \in \mathbf{M}$, the action a_1 can be executed iff the action a_2 can't be executed($\mathbf{M}, w \vDash \varphi$ iff $\mathbf{M}, w \nvDash \neg\varphi$). One can think this update as eliminating the accessibility relations for agents in B. Using the finiteness of the model $\mathbf{M}$, we conclude that the update will stabilize in finitely many steps. Moreover the stabilization reaches when the links between φ-worlds(the set of worlds satisfying φ) and $\neg\varphi$-worlds are all labeled within $Ag - B$.

Clearly, if the precondition φ is propositional, the stabilization only need one update. For instance: $\mathbf{M}$ is a state model with p true in w_1 and false in w_2, and $\mathbf{A}$ is an action model; the update is as follows:

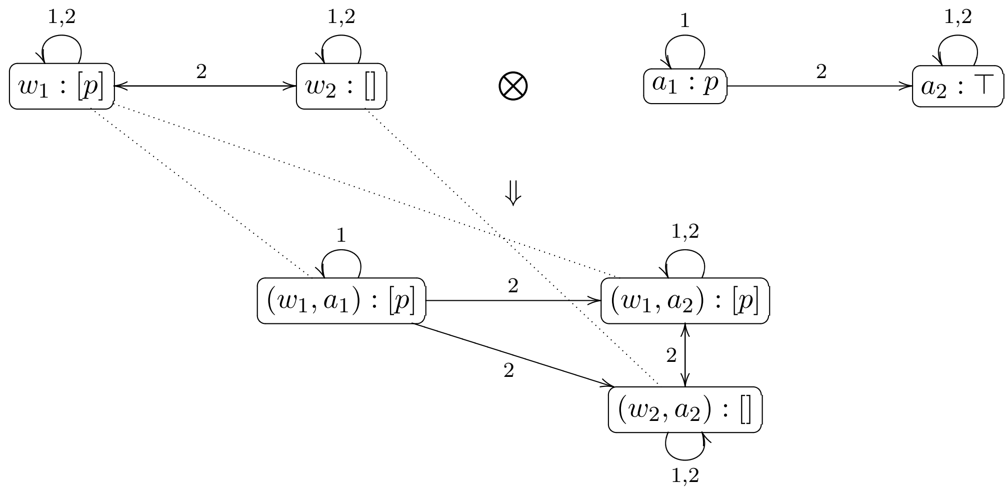


After the update, the accessible relations between w_1 and w_2 for agent 2 were deleted, so it becomes common knowledge whether p or $\neg p$ is true.

Example 4.3 (Private message to a subgroup). A subgroup of agents $B(\subseteq Ag)$ may receive a message φ, and the rest agents receive nothing. The action model is as follows:



Clearly $|\mathbf{M} \otimes \mathbf{A}| \geq |\mathbf{M}|$, since action a_2 is executable in any world. And if we assume the private message φ be truthful, i.e. action a_1 is executable in at least one world, then $|\mathbf{M} \otimes \mathbf{A}| > |\mathbf{M}|$. But this does not necessary mean that the bisimulation-minimal model of $\mathbf{M} \otimes \mathbf{A}$ is still larger than that of $\mathbf{M}$. Here is an example:



Easy to verify that the relation indicated by dotted lines is a bisimulation, so $\mathbf{M} \leftrightarrow \mathbf{M} \otimes \mathbf{A}$. This means that the bisimulation-minimal model of $\mathbf{M} \otimes \mathbf{A}$ has the same size as that of $\mathbf{M}$. Moreover, we have $\mathbf{M} \otimes \mathbf{A} \leftrightarrow \mathbf{M} \otimes \mathbf{A}^k$ for all $k \geq 1$, which means repeated update does not have new effects. Then is this necessary so? We show that in the case of preconditions being propositional, it is indeed so.

Theorem 33. *Given any state model* $\mathbf{M}$ *and an action model* $\mathbf{A}$ *of this example, for all* $k \geq 1$*,*

$$\mathbf{M} \otimes \mathbf{A} \leftrightarrow \mathbf{M} \otimes \mathbf{A}^k$$

Proof. It suffices to show that $\mathbf{M} \otimes \mathbf{A} \leftrightarrow \mathbf{M} \otimes \mathbf{A}^2$. Define $\mathbf{Z} \subseteq \mathbf{M} \otimes \mathbf{A} \times \mathbf{M} \otimes \mathbf{A}^2$ (note: $W_{\mathbf{A}} = \{a_1, a_2\}$):

$$\begin{aligned} \mathbf{Z} \ := \ & \{\langle (w, a_1), (w, a_1, a') \rangle | \mathbf{M}, w \models \varphi, a' = a_1 \text{ or } a_2\} \cup \\ & \{\langle (w, a_1), (w, a_2, a_1) \rangle | \mathbf{M}, w \models \varphi\} \cup \\ & \{\langle (w, a_2), (w, a_2, a_2) \rangle | w \in \mathbf{M}\} \end{aligned}$$

Now show $\mathbf{Z}$ is a total bisimulation. By definition, $\mathbf{Z}$ is total.
Suppose $(w, a_1)\mathbf{Z}(w, a_1, a_2)$,

- Atomic case follows easily.
- Zig: If $(w, a_1) \xrightarrow{i} (w', a')$ then (w', a', a_2) completes this condition, since $(w, a_1, a_2) \xrightarrow{i} (w', a', a_2)$ and $(w', a_1)\mathbf{Z}(w', a', a_2)$.
- Zag: If $(w, a_1, a_2) \xrightarrow{i} (w', a', a'')$ then we know that a'' must be a_2 since there is no way from a_2 to a_1. Now (w', a') completes this condition, since $(w, a_1) \xrightarrow{i} (w', a')$ and $(w', a')\mathbf{Z}(w', a', a_2)$.

For the case of $(w, a_1)\mathbf{Z}(w, a_2, a_1)$,$(w, a_1)\mathbf{Z}(w, a_1, a_1)$, and $(w, a_2)\mathbf{Z}(w, a_2, a_2)$, the argument is similar. □

There is an easier way to understand the above proof. Let's look at the elements in $\mathbf{A}^2$ and their preconditions:

$$\begin{array}{lcl}
\mathrm{Pre}(a_1, a_2) & = & \mathrm{Pre}(a_1) \wedge \langle \mathbf{A}, a_1\rangle \mathrm{Pre}(a_2) = \mathrm{Pre}(a_1); \\
\mathrm{Pre}(a_2, a_1) & = & \mathrm{Pre}(a_2) \wedge \langle \mathbf{A}, a_2\rangle \mathrm{Pre}(a_1) = \mathrm{Pre}(a_1); \\
\mathrm{Pre}(a_1, a_1) & = & \mathrm{Pre}(a_1) \wedge \langle \mathbf{A}, a_1\rangle \mathrm{Pre}(a_1); \\
\mathrm{Pre}(a_2, a_2) & = & \mathrm{Pre}(a_2) \wedge \langle \mathbf{A}, a_2\rangle \mathrm{Pre}(a_2) = \top;
\end{array}$$

And $\mathbf{A}^2$ has the following structure:

$$\begin{tikzcd}
(a_1, a_2) \arrow[loop above, "B"] \arrow[dr, "Ag-B"] & \\
(a_2, a_1) \arrow[loop above, "B"] \arrow[r, "Ag-B"] & (a_2, a_2) \arrow[loop above, "Ag"] \\
(a_1, a_1) \arrow[loop above, "B"] \arrow[ur, "Ag-B"] &
\end{tikzcd}$$

If the preconditions are propositional, we have $\mathrm{Pre}(a_1, a_2) = \mathrm{Pre}(a_2, a_1) = \mathrm{Pre}(a_1, a_1) = \mathrm{Pre}(a_1)$, which means $\mathbf{A} \underline{\leftrightarrow} \mathbf{A}^2$. Therefore $\mathbf{M} \otimes \mathbf{A} \underline{\leftrightarrow} \mathbf{M} \otimes \mathbf{A}^2$.

4.3 Propositional precondition

In this section we study the action models only with propositional conditions.

We give one sufficient condition such that the update will always reach stabilization.

Theorem 34. *Given arbitrary state model* $\mathbf{M}$*, and finite propositional action model* $\mathbf{A}$ *with only one relation and S5 property, there exists* n *such that:*

$$\mathbf{M} \otimes \mathbf{A}^n \underline{\leftrightarrow} \mathbf{M} \otimes \mathbf{A}^{n+1}$$

Proof. Let $|\mathbf{A}| = n$.

Define a binary relation between $\mathbf{M} \otimes \mathbf{A}^n$ and $\mathbf{M} \otimes \mathbf{A}^{n+1}$:

$$\begin{aligned} Z \ := \{ \ & \langle (w, a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_{n+1}), (w, a_1, \cdots, a_{n+1}) \rangle | \\ & w \in W_{\mathbf{M}}, \forall j \in [1, n+1](a_j \in W_{\mathbf{A}}), (\mathbf{M}, w) \models \mathrm{Pre}(a_1, \cdots, a_{n+1}), \\ & \exists j \neq i (a_i \rightarrow a_j) \} \end{aligned}$$

Now we show Z is a total bisimulation.

Suppose $(w, a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_{n+1}) Z (w, a_1, \cdots, a_{n+1})$, then by Z there exists $j \neq i$ such that $a_i \rightarrow a_j$:

PreEQ $\mathrm{Val}(w, a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_{n+1}) = \mathrm{Val}(w) = \mathrm{Val}(w, a_1, \cdots, a_{n+1})$.

Zig Suppose $(w, a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_{n+1}) \rightarrow (w', b_1, \cdots, b_{i-1}, b_{i+1}, \cdots, b_{n+1})$, then we have $a_j \rightarrow b_j$. Combined with $a_i \rightarrow a_j$ and S5 property, we have $a_i \rightarrow b_j$, hence $(w', b_1, \cdots, b_{i-1}, b_j, b_{i+1}, \cdots, b_{n+1})$ completes this condition.

Zag Suppose $(w, a_1, \cdots, a_{n+1}) \rightarrow (w', b_1, \cdots, b_{n+1})$, then we have $a_i \rightarrow b_i$ and $a_j \rightarrow b_j$. And by $a_i \rightarrow a_j$, we have $b_i \rightarrow b_j$. So $(w', b_1, \cdots, b_{i-1}, b_{i+1}, \cdots, b_{n+1})$ completes this condition.

Now show Z is a total relation. For arbitrary $n+1$ actions, there must be 2 actions are the same. Therefore each $(w, a_1, \cdots, a_n)$ can find $(w, a_1, \cdots, a_{n+1})$ such that there exists $i < n+1$, $a_i = a_{n+1}$, and hence $a_i \rightarrow a_{n+1}$. And vice versa. □

Due to time limit, this chapter is rather incomplete. In this direction, extensive work has been done by Tomasz Sadzik in [14].

Chapter 5

Conclusions and Further work

In the above chapters we first showed how BMS framework models the cases of message passing. And then we studied the sufficient and necessary conditions for two action models having same update effects, and proposed a structural relation between action models, namely action emulation, to capture the same update effects for propositional case. Finally we discussed the problem of update evolution and showed a sufficient condition for the stabilization of update evolution.

We also leave some open questions for further research.

In Chapter 2, we make a general discussion of BCC case, and showed that the action models of BCC will go exponentially since the number of subgroups of n agents is 2^n. It is still not clear to us how to compute the action model for a given specification. Also the logic of BCC may be an interesting topic to explore.

In Chapter 3, we end with two open questions:

Question 1. *What is the (modal) language characterization of action emulation (compare the characterization theorems for bisimulation)?*

Question 2. *What is the complexity of determining whether two action models emulate? Is this more complex than bisimulation, or is it also polynomial, like the decision problem for bisimilarity? In particular, can something like a partition refinement algorithm for bisimulation be made to work for this?*

In Chapter 4, even for the propositional case, the necessary condition for the stabilization is not clear to us. And there is also a big unknown territory for modal case. We leave all these for further research.

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