A Construction Method for Modal Logics of Space

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This thesis is dedicated to my parents Terry and Therese and sister Teena. Without whom.

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1 Introduction

Given the important role of spatial intuitions in cognition, and the apparent unreliability of these intuitions, there is something natural about looking at spatial structures from an axiomatic standpoint. Indeed, it is not surprising that the first and best known application of the axiomatic method was to provide a development of geometry.

In the past century, with the development of formal logic and subsequent discovery that elementary number theory is not axiomatizable, it became both possible and independently interesting to examine spatial structures within given logical formalizations. While the central framework for examining spatial structures axiomatically has been first order logic, from time to time other logics with spatial interpretations have been considered as well.

Recently, one popular area of investigation has been looking at modal logics with spatial interpretations. This subject can be traced back to McKinsey and Tarksi's [14] work on boolean algebras with closure operators in the 1940s. However, within the past ten years a more general program of providing a modal analysis of space has emerged.

By and large, the techniques used in investigating modal logics of space have been model theoretic in nature, involving the transfer of geometric or topological structure from the desired mathematical object to some Kripke frame. While this works well in cases where the relevant modal logic has nice Kripke frame characterizations, in other cases this way of proceeding can become quite difficult.

In this thesis we will examine a more syntactic approach to establishing completeness results in modal logics of space. The technique we will use has the virtue of constructing the desired mathematical structures directly, rather than working indirectly through Kripke frames. This allows for a good deal of control over what models of the relevant logic look like.

In the first chapter, we will use our construction method to give new proofs of the completeness of **S4** with respect to $\langle \mathbb{Q}, \tau \rangle$ and $\mathbf{S4} \oplus \mathbf{S4}$ with respect to $\langle \mathbb{Q} \times \mathbb{Q}, \tau_1, \tau_2 \rangle$. We will also provide a much simpler axiomatization of $\langle \mathbb{Q}, \tau \rangle$ in the combined language $\Box + F + P$ and an axiomatization of $\langle \mathbb{Q}, < \rangle$ in the Since/Until language. In the second chapter, we will discuss the advantages and disadvantages of the construction method in comparison to the standard model theoretic approach.

2 Modal Logics of Space

2.1 Syntax and Semantics

Although they would not have been thought of as such at the time, some of the first semantic completeness proofs for modal logic date back to McKinsey and Tarski's [14] work the on foundations of topology in the 1940s . These proofs pre-date the now standard Kripke semantics for modal logic, and in fact are spatial completeness proofs for the modal logic **S4** with a topological semantics.

Let $\mathcal{L}$ be the basic modal language, consisting of propositional variables $p, q, r..$, boolean connectives $\wedge, \vee, \neg \rightarrow$, and unary operators $\Box$ and $\Diamond$. The formulas of $\mathcal{L}$ define the least set Γ containing all propositional variables which is closed under boolean connectives and the operators $\Box$ and $\Diamond$.

Definition 2.1 A modal logic **L** is the least set of formulas of $\mathcal{L}$ containing all propositional tautologies and the axioms of **L** which is closed under substitution and proof rules :

$$\frac{\varphi, \varphi \rightarrow \psi}{\psi} \qquad \frac{\varphi}{\Box\varphi}$$

$\lhd$

Definition 2.2 A formula φ is said to be *provable* in **L** (written $\vdash_{\mathbf{L}} \varphi$) if $\varphi \in \mathbf{L}$ and a set Γ of formulas is said to be **L**-*consistent* iff there is no finite set $\{\varphi_1, \varphi_2 ... \varphi_n\} \subseteq \Gamma$ such that $\mathbf{L} \vdash \neg(\varphi_\mathbf{1} \wedge ... \wedge \varphi_\mathbf{n})$. $\lhd$

The central modal logic for our purposes is **S4**, which contains the axioms:

$\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$
$\Box(\psi \rightarrow \varphi) \rightarrow (\Box\psi \rightarrow \Box\varphi)$
$\Box\varphi \rightarrow \varphi$
$\Box\varphi \rightarrow \Box\Box\varphi$

Recall that a *topological space* is a pair $\langle W, \tau \rangle$, where W is a nonempty set and τ is a collection of subsets of W satisfying the following properties:

- $\emptyset$, $W \in \tau$;
- if $U, V \in \tau$, then $U \cap V \in \tau$;
- if $\{U_i\}_{i \in I} \in \tau$, then $\bigcup_{i \in I} U_i \in \tau$;

Definition 2.3 A *topological model* $\langle W, \tau, \nu \rangle$ is a triple, where $\langle W, \tau \rangle$ is a topological space and ν is a valuation assigning subsets of W to propositional letters. $\lhd$

Further recall that for any $U \subseteq W$,

- U is *open* if $U \in \tau$
- U is *closed* if $U \notin \tau$
- The *interior* of U ($Int(U)$) is the union of all open sets contained in U
- The *closure* of U ($Clo(U)$) is the intersection of all closed sets containing U
- $Int(U) = W - Clo(W - U)$

In the original topological semantics for $\mathcal{L}$ proposed by McKinsey and Tarski, the truth of formulas was evaluated at the level of topological models. Thus given a valuation ν assigning propositional letters to subsets of W, ν was extended to all formulas of $\mathcal{L}$ as follows:

- $\nu(p) \subseteq W$
- $\nu(\neg\varphi) = W - \nu(\varphi)$
- $\nu(\varphi \vee \psi) = \nu(\varphi) \cup \nu(\psi)$
- $\nu(\varphi \wedge \psi) = \nu(\varphi) \cap \nu(\psi)$
- $\nu(\varphi \rightarrow \psi) = (W - \nu(\varphi)) \cup \nu(\psi)$
- $\nu(\Box\varphi)) = Int(\nu(\varphi))$
- $\nu(\Diamond\varphi) = Clo(\nu(\varphi))$

A formula φ was said to be true on a topological model $\langle W, \tau, \nu \rangle$ if $\nu(\varphi) = W$.

Besides working with the notion of truth on a topological model, in this paper we will also make use of the notion of a formula φ being true at points in a topological model. This fine grained topological semantics mirrors the now prevalent Kripke semantics for modal logic, and will facilitate in transferring techniques and results later on.

Given a topological model $M = \langle W, \tau, \nu \rangle$, we define a formula φ to be *true* at a point w by induction on the length of φ. From now on we will economize on boolean connectives, considering only $\neg$ and $\vee$, the others being definable from these two alone.

- $w \models p$ iff $w \in \nu(p)$
- $w \models \psi \vee \chi$ iff $w \models \psi$ or $w \models \chi$
- $w \models \neg\psi$ iff not $w \models \psi$
- $w \models \Box\psi$ iff $\exists U \in \tau$ such that $w \in U$ and $\forall w' \in U, w' \models \psi$
- $w \models \Diamond\psi$ iff $\forall U \in \tau, w \in U$ implies that $\exists w' \in U$ such that $w' \models \psi$

Definition 2.4 We say φ is *true* on a topological model $\langle W, \tau, \nu \rangle$ (written $\langle W, \tau, \nu \rangle \models \varphi$), if $\langle W, \tau, \nu \rangle, w \models \varphi$ for every $w \in W$, or, equivalently, if $\nu(\varphi) = W$. If φ is true on $\langle W, \tau \rangle$ for any valuation ν, we say φ is *valid* on the topological space $\langle W, \tau \rangle$ (written $\langle W, \tau \rangle \models \varphi$). $\lhd$

2.2 Practical Workings

From the perspective of modal logic, the topological semantics proposed above looks quite different from the standard Kripke-style labeled transitions systems. And from the perspective of topology, it is not clear what can be expressed in the modal language $\mathcal{L}$. To sharpen intuitions on both ends, let us briefly pause to get a sense of the practical workings of the language .

At first glance, it may appear that little of mathematical interest can be expressed in $\mathcal{L}$. For example, there is no obvious way to express familiar topological properties such as connectedness, being T_0, etc.. And indeed the results that follow lend support to this belief. They show, for example, that no formula of $\mathcal{L}$ can distinguish topological structures as different as $\langle \mathbb{Q}, \tau \rangle$ and $\langle \mathbb{R}, \tau \rangle$. (So it is true that connectedness can not be expressed in $\mathcal{L}$).

On the other hand, even though many topological properties are undefinable, what formulas can be true at a point in a topological model quickly becomes quite complicated. Just how complicated becomes clear below when we try to construct a model $\langle \mathbb{R}, \tau, \nu \rangle$ which has a real number modeling an arbitrary consistent **S4** formula. But for the moment let us consider a few examples of simple formulas which can be made true at points on the real line.

Definition 2.5 A logic **L** is said to be *complete* with respect to a topological space $\langle W, \tau \rangle$ if $\vdash_{\mathbf{L}} \varphi \Leftrightarrow \langle W, \tau \rangle \models \varphi$. $\lhd$

Much of what follows will be dedicated to *proving* topological completeness results for **S4** and its extensions, but for now let us make use of a classical result.

Theorem 2.6 **S4** *is complete with respect to* $\langle \mathbb{R}, \tau \rangle$.

Viewed contrapositively, the right to left direction of Definition 2.5 says that if ψ is an **S4**-consistent formula, then it can be made true at a point on the real line.

First consider an easy example, the **S4**-consistent formula $\Diamond p \wedge \neg p$. By completeness, there exists a valuation ν such that $\langle \mathbb{R}, \tau, \nu \rangle, x \models \Diamond p \wedge \neg p$ for some $x \in \mathbb{R}$. And indeed it is not hard to think of such a ν and x. For example, if $\nu(p) = \{1/x \mid x > 0\}$, then $\langle \mathbb{R}, \tau, \nu \rangle, 0 \models \Diamond p \wedge \neg p$.

A slightly more difficult example is the formula $\Diamond\Diamond p \wedge \Diamond\neg p$. Intuitively, this says that there is a sequence of points approaching x and at every point y in this sequence there is a sequence approaching y which models p. Actually, it is easier to give a valuation to describe the situation. If

$$\nu(p) = \{\tfrac{1}{(x+\frac{1}{y})} \mid x \in \mathbb{N}/\{0\}, y \in \mathbb{N}\}$$

then $\Diamond\Diamond p \wedge \Diamond\neg p$ is true at 0.

A more interesting example is the **S4**-consistent formula $\Box(\Diamond p \wedge \Diamond\neg p)$. This formula says that every point in a given open neighborhood has both p and $\neg p$ sequences approaching it. Although at first glance it might seem difficult to describe a valuation which makes this formula true at a point in $\langle \mathbb{R}, \tau \rangle$, there is actually a quite natural one. The valuation $\nu(p) = \mathbb{Q}$ makes $\Box(\Diamond p \wedge \Diamond\neg p)$ true at every point in $\langle \mathbb{R}, \tau, \rangle$.

All of the examples considered above involve only single **S4**-consistent formulas containing one propositional letter. However, by completeness it is guaranteed that every finite consistent set of **S4** formulas containing finitely many propositional letters can be made true together at a single point on $\langle \mathbb{R}, \tau \rangle$. Now that the inner workings of the language have been explored a bit, it is time to see what properties can be established about the language.

2.3 The Historical Results

In 'The Algebra of Topology' McKinsey and Tarski [14] examine the relations between closure algebras and topological spaces. A closure algebra is a five-tuple $\langle K, \cup, \cap, -, C \rangle$ where

- K is a boolean algebra with respect to $\cup$, $\cap$, $-$
- If $x \in K$, then Cx is in K
- If $x \in K$, then $x \subseteq Cx$
- If $x \in K$, then $CCx = Cx$
- If x and y are in K, then $C(x \cup y) = Cx \cup Cy$
- $C\emptyset = \emptyset$

Clearly every topological space $\langle X, \tau \rangle$ can be viewed as a closure algebra $\mathcal{C}(X) = \langle \mathcal{P}(X), \cup, \cap, -, C \rangle$, where C is the closure operator on $\mathcal{P}(X)$. Conversely, every closure algebra of the form $\mathcal{C}(X) = \langle \mathcal{P}(X), \cup, \cap, -, C \rangle$ can be viewed as a topological space $\langle X, \tau \rangle$ such that $\langle X, \tau \rangle \models$

(i) $p \to \Diamond p$

(ii) $\Diamond\Diamond p \leftrightarrow \Diamond p$

(iii) $\Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q$

(iv) $\Diamond \perp \leftrightarrow \perp$

Furthermore, it can be proven that (i)- (iv) are equivalent to the axioms of the modal logic **S4**. So it follows that if $\vdash_{\mathbf{S4}} \varphi$, then $\langle X, \tau \rangle \models \varphi$. McKinsey and Tarski's pioneering results in 'The Algebra of Topology' show that for a wide range of topological spaces, the converse is also true.

Theorem 2.7 *(McKinsey and Tarski) For every finite topological space $\langle X, \tau \rangle$ and every normal, dense-in-itself topological space $\langle Y, \tau \rangle$ with a countable base, there exists an embedding $\mathcal{C}(X) \rightarrowtail \mathcal{C}(Y)$.*

From a logical perspective, the above theorem states that if there is a finite topological space $\langle X, \tau \rangle$ such that $\langle X, \tau \rangle \not\models \varphi$, then for any normal, dense-in-itself topological space $\langle Y, \tau \rangle$ with a countable base, $\langle Y, \tau \rangle \not\models \varphi$. It is known (and proven below in Corollary 3.6) that if $\nvdash_{\mathbf{S4}} \varphi$, then there is some finite topological space $\langle X, \tau \rangle$ such that $\langle X, \tau \rangle \not\models \varphi$. So from a logical perspective, Theorem 2.7 states that **S4** is complete with respect to every normal, dense-in-itself topological space with a countable base. This is a very general result. It shows that, for example, **S4** is the complete with respect to the rationals, the reals, the Cantor Space, and the Baire Space.

2.4 Modern Extensions

From the perspective of modal logics of space today, McKinsey and Tarski's original results are somewhat daunting in means and scope. In means, their arguments are by and large mathematical rather than metamathematical, and make use of facts not necessarily familiar to the modal logician. In scope, their results are sufficiently general to settle almost all questions about the basic modal language $\mathcal{L}$ with a topological semantics.

Thus recent investigations into modal logics of space have taken two directions. One is to make the original results of McKinsey and Tarski more accessible by using the tools built up in modal logic over the past 70 years. This project seems to have started with Mints [15], where a new completeness proof of **S4** with respect to the Cantor Space is given. Mints' paper has in turn inspired several new proofs of completeness of **S4** with respect to the real line, including Mints, Zhang [17], Aiello, van Benthem, Bezhanishvili [3] and Bezhanishvili, Gehrke [6].

The other direction is to move to richer modal languages. In the past ten years, many spatial extensions of the basic modal language $\mathcal{L}$ have been put forward. The first and perhaps most natural extension of $\mathcal{L}$ to be considered was the language $\mathcal{L}^{\Box FP}$, originally found in Shehtman [20]. $\mathcal{L}^{\Box FP}$ contains the $\Box$ operator, together with two additional unary operators F and P. In the intended semantics for $\mathcal{L}^{\Box FP}$, $\Box$ is interpreted as interior and F and P are given their standard temporal interpretation as 'future' and 'past'.

More formally, given a partial order $\langle W, < \rangle$, the standard order topology τ on $\langle W, < \rangle$ and a valuation ν, the truth of a $\mathcal{L}^{\Box FP}$ formula at a point $w \in W$ is defined as follows:

- $w \models F\psi$ iff $\exists w' > w$ such that $w' \models \psi$
- $w \models G\psi$ iff $\forall w' > w$, $w' \models \psi$
- $w \models P\psi$ iff $\exists w' < w$ such that $w' \models \psi$

- $w \models H\psi$ iff $\forall w' < w,\ w' \models \psi$
- $w \models \Box\psi$ iff $\exists U \in \tau$ such that $w \in U$ and $\forall w' \in U, w' \models \psi$
- $w \models \Diamond\psi$ iff $\forall U \in \tau, w \in U$ implies that $\exists w' \in U$ such that $w' \models \psi$

$\mathcal{L}^{\Box FP}$ has the appealing feature of combining two basic properties of mathematical objects: topological and metric structure. Plus it is topologically more expressive than $\mathcal{L}$. Since $\mathcal{L}^{\Box FP}$ will occur throughout this paper, let us pause for a moment to consider an example.

As mentioned before, it follows from the completeness of **S4** with respect to both $\langle \mathbb{R}, \tau \rangle$ and $\langle \mathbb{Q}, \tau \rangle$ that connectedness is not expressible in $\mathcal{L}$. For assume it is, and that φ expresses it. Then $\langle \mathbb{R}, \tau \rangle \models \varphi$ and $\langle \mathbb{Q}, \tau \rangle \not\models \varphi$. But

$$\langle \mathbb{R}, \tau \rangle \models \varphi \Leftrightarrow \vdash_{\mathbf{S4}} \varphi \Leftrightarrow \langle \mathbb{Q}, \tau \rangle \models \varphi.$$

However, connectedness is expressible in $\mathcal{L}^{\Box FP}$.

Lemma 2.8 $\langle W, \tau \rangle \models \Box\varphi \wedge F\neg\varphi \rightarrow F(\Diamond\varphi \wedge \Diamond\neg\varphi) \Leftrightarrow \langle W, \tau \rangle$ *is connected.*

[$\Rightarrow$] Assume $\langle W, \tau \rangle$ is not connected. Then W can be written as the union of two non empty disjoint open sets U and V. Without loss of generality, assume that there is some $x \in U$ and $y \in V$ such that $x < y$. Now consider the valuation $\nu(p) = U$.

It is clear that $\langle W, \tau, \nu \rangle, x \models \Box p \wedge F\neg p$. We want to show $\langle W, \tau, \nu \rangle, x \not\models F(\Diamond p \wedge \Diamond\neg p)$, so assume z is an arbitrary point such that $x < z$. Either $z \in U$ or $z \in V$. If $z \in U$, then $\langle W, \tau, \nu \rangle, z \models \Box p$ and if $z \in V$, then $\langle W, \tau, \nu \rangle, z \models \Box\neg p$. It follows that $\langle W, \tau, \nu \rangle, x \models G(\Box p \vee \Box\neg p)$.

[$\Leftarrow$] Assume $\langle W, \tau \rangle \not\models \Box\varphi \wedge F\neg\varphi \rightarrow F(\Diamond\varphi \wedge \Diamond\neg\varphi)$. Then there exists a valuation ν and point $x \in W$ such that $\langle W, \tau, \nu \rangle, x \models \Box\varphi \wedge F\neg\varphi$ and $\langle W, \tau, \nu \rangle, x \models G(\Box\varphi \vee \Box\neg\varphi)$. Let

$$X = \{y \mid y < x\}$$

$$Y = \{z \mid z < x\}$$

$$Z = \nu(\Box\varphi)$$

$$V = \nu(\Box\neg\varphi)$$

Then X and Y are open sets and Z and V are the unions of open sets. It follows that $U = X \cup Z$ and $T = V \cap Y$ are two disjoint open sets whose union is W. So $\langle W, \tau \rangle$ is not connected. QED

While $\mathcal{L}^{\Box FP}$ increases topological expressive power by adding metric expressivity, other strictly topological extensions of $\mathcal{L}$ have been considered as well. One recent nice example is the dynamical topological language $\mathcal{L}^{\Box\circ}$, containing the interior operator $\Box$ plus an additional operator $\circ$ for one continuous transformation on a topological space. This language is interpreted into a dynamic topological system $\langle W, \tau, T\rangle$ where $\langle W, \tau\rangle$ is a topological space and T is a continuous mapping on $\langle W, \tau\rangle$. Given a valuation ν on $\langle W, \tau, T\rangle$, a $\mathcal{L}^{\Box\circ}$ formula φ is defined to be true at a point $w \in W$ as follows:

- $w \models \Box\psi$ iff $\exists U \in \tau$ such that $w \in U$ and $\forall w' \in U, w' \models \psi$
- $w \models \Diamond\psi$ iff $\forall U \in \tau, w \in U$ implies that $\exists w' \in U$ such that $w' \models \psi$
- $w \models \circ\psi$ iff $T(w) \models \psi$

In $\mathcal{L}^{\Box\circ}$ it is possible to express not only topological properties, but also properties of functions on topological spaces. For example, $\circ\Box\varphi \to \Box \circ \varphi$ expresses the continuity of T.

$\mathcal{L}^{\Box\circ}$ was originally proposed by Artemov, Davoren, and Nerode [4] and examined by Davoren in her P.h.D thesis [9], where she showed that the logic **S4C** =

$$\mathbf{S4} + \circ\Box\varphi \to \Box \circ \varphi + \text{the interaction principles } \circ(\varphi \wedge \psi) \leftrightarrow \circ\varphi \wedge \circ\psi \text{ and } \circ\neg\varphi \leftrightarrow \neg \circ \varphi$$

is complete with respect to the class of all topological spaces. Of course given the explicitly mathematical interpretation of the language, this is not a very satisfying result. Recently, Mints, Zhang [18] established that **S4C** is also complete with respect to the Cantor Space. Otherwise, almost all questions about this language remain open.

There has also been some investigation into the combined language $\mathcal{L}^{\Box\circ\ FP}$ Mints, Kremer [16], Kremer, Mints [13] . Given the appropriate interpretation of F and P (namely that they apply only to $\mathbb{N}$), this language is strong enough to express fundamental facts of analysis, such as properties crucial to proving the Mean Value Theorem. These are only a few of a large number of possible topological extensions of $\mathcal{L}$. For a discussion of some of the other possibilities, see Aiello [1] or Aiello, van Benthem [2].

As a note before moving on, one surprising feature of moving to richer modal languages is how difficult the problems become, even when seemingly increasing the expressive power only slightly. For example, $\mathcal{L}^{\Box FP}$ does not seem like a particularly strong extension of $\mathcal{L}$. And it follows from Rabin's Theorem that $\langle \mathbb{R}, \tau\rangle$ is axiomatizable in $\mathcal{L}^{\Box FP}$. Yet no one has been able to find a complete axiomatization. Even the $\mathcal{L}^{\Box FP}$ axiomatization of $\langle \mathbb{Q}, \tau\rangle$ in Shehtman [20] is quite difficult, and something of an encyclopedia of modal logic. As mentioned above, matters are worse still in $\mathcal{L}^{\Box\circ}$, where except for the completeness of **S4C** with respect to the Cantor Space almost all questions remain open. And not even that much is known about $\mathcal{L}^{\Box\circ FP}$. It seems that by adding only a little bit more expressive power, the proofs swiftly move beyond strictly logical insight.

2.5 Modal Techniques

By and large, the techniques used in investigating modal logics of space have been model theoretic in nature. To establish completeness of a logic **L** with respect to a topological space

$\langle X, \tau \rangle$, the most common strategy has been to find a convenient class of Kripke frames the logic is complete with respect to and establish a modal equivalence between an arbitrary member of this class and the desired topological space.

For example, Aiello, van Benthem, Bezhanishvili [3] uses the fact that **S4** is complete with respect to the class of all finite transitive and reflexive trees. In outline, the proof then proceeds as follows:

- If $\nvdash_{\mathbf{S4}} \varphi$, by completeness φ can be refuted on some finite transitive and reflexive tree $\langle W, R \rangle$.
- A labeling is devised between branches of the Cantor space and points in $\langle W, R \rangle$ such that every node of $\langle W, R \rangle$ is modally equivalent to some branch of the Cantor Space.
- So the point on $\langle W, R \rangle$ which refutes φ is modally equivalent to some branch of the Cantor Space.
- So φ in turn can be refuted on the Cantor Space.

In the last chapter we will discuss this example in greater depth.

The line of reasoning sketched above has been used in Shehtman [20] , Mints [15] Aiello, van Benthem, Bezhanishvili [3] and van Benthem, Bezhanishvili, ten Cate, Sarenac [5], among other places, to give streamlined topological completeness proofs. However, the approach is not without difficulty. For one thing, when working in richer modal languages, where more topological properties can be expressed, the Kripke frame characterizations of the logic can become difficult to establish.

On the other hand, even in cases like **S4** where the logic has many simple Kripke-frame characterizations, it still may be difficult to transfer topological structure to these Kripke frames. For example, no one has ever found a simple way to transfer the topological structure of $\langle \mathbb{R}, \tau \rangle$ to an **S4** Kripke frame. Indeed, the proofs of this fact in Aiello, van Benthem, Bezhanishvili [3] and Mints [15] work even more indirectly, first establishing a modal equivalence of the real line to some subset of the Cantor Space, then establishing a modal equivalence between this subset of the Cantor Space and an arbitrary **S4** Kripke frame. The procedure of sending points on the real line to branches of the Cantor Space to points in a Kripke model, all the while preserving modal equivalence, ends up requiring some rather fancy combinatorics.

The above difficulties raise the question of whether working indirectly through Kripke frames is always the way to go. Instead of first trying to find a Kripke frame characterization of the logic and then transferring topological structure to these Kripke frames, perhaps it is better to try and think of modal techniques which work directly on the desired topological space to establish completeness.

2.6 The Construction Method

Curiously, there is a modal technique which works directly on mathematical structures rather than indirectly through Kripke frames, but it has never been applied to topological completeness proofs. This technique was developed by de Jongh and Veltman [8] and Burgess [7], among others, in the 1980s to overcome precisely the difficulties described above, but in this case for axiomatizing structures such as $\langle \mathbb{N}, < \rangle$, $\langle \mathbb{Q}, < \rangle$ and $\langle \mathbb{R}, < \rangle$ in the basic temporal language $\mathcal{L}^{FP}$.

As in the topological case, the original completeness proofs for $\langle \mathbb{N}, < \rangle$, $\langle \mathbb{Q}, < \rangle$ and $\langle \mathbb{R}, < \rangle$ in the temporal language (Segerberg [19]) provided a characterization of the relevant logic in terms of Kripke frames, then established a modal equivalence between the desired mathematical structure and its class of Kripke frames. And again as above, the expressive power of the temporal language resulted in Kripke frame characterizations which were not very nice and, as a result, somewhat technical proofs.

Faced with these difficulties, de Jongh and Veltman (among others) developed a more direct syntactic approach towards establishing completeness. The basic idea is to construct the desired mathematical structure in stages, at each stage associating every point added with a set of formulas which will be true at that point when the construction is finished. Then if the construction procedure is carried out correctly, there will be a harmony between the set of formulas associated with a point and the formulas which are actually true at that point. Provided that the sets of formulas were selected judiciously, the structure will then serve as a model of the desired state of affairs. Let us illustrate this procedure with our first completeness proof.

2.7 $\mathbf{Q_{FP}}$ on $\langle \mathbb{Q}, < \rangle$

Consider the logic $\mathbf{Q_{FP}}$ containing the axioms:

$$
\begin{array}{l}
G(\psi \to \varphi) \to (G\psi \to G\varphi) \\
H(\psi \to \varphi) \to (H\psi \to H\varphi) \\
\varphi \to GP\varphi \\
\varphi \to HF\varphi \\
GG\varphi \to G\varphi \\
G\varphi \to GG\varphi \\
F\varphi \to G(\varphi \vee P\varphi \vee F\varphi) \\
P\varphi \to H(\varphi \vee P\varphi \vee F\varphi) \\
F\top \\
P\top
\end{array}
$$

and proof rules:

$$
\frac{\varphi, \varphi \to \psi}{\psi} \qquad \frac{\varphi}{G\varphi} \qquad \frac{\varphi}{H\varphi}
$$

Theorem 2.9 $\mathbf{Q_{FP}}$ *is complete with respect to* $\langle \mathbb{Q}, < \rangle$.

Proof. To show $\vdash_{\mathbf{Q_{FP}}} \varphi \Rightarrow \langle \mathbb{Q}, < \rangle \models \varphi$ it is necessary to show that all of the axioms of $\mathbf{Q_{FP}}$ are valid on $\langle \mathbb{Q}, < \rangle$, and that the proof rules preserve validity on $\langle \mathbb{Q}, < \rangle$. This involves some routine fact checking. From now on this direction of completeness will be omitted. To show the converse direction, assume $\nvdash_{\mathbf{Q_{FP}}} \varphi$. We will construct $\langle \mathbb{Q}, < \rangle$ (modulo isomorphism) in such a way that $\langle \mathbb{Q}, < \rangle \not\models \varphi$.

Definition 2.10 A consistent set of formulas Γ is called *maximally consistent* if there is no consistent set of formulas properly containing Γ. $\lhd$

Fact 2.11 *Every consistent set of formulas* Γ *can be extended to a maximally consistent set of formulas* Γ', $\Gamma \subseteq \Gamma'$. *Furthermore,* Γ' *is maximally consistent if and only if for any formula* φ, *either* $\varphi \in \Gamma'$ *or* $\neg\varphi \in \Gamma'$ *but not both.*

Definition 2.12 Let Γ_q and $\Gamma_{q'}$ be $\mathbf{Q_{FP}}$ maximally consistent sets. We say $\Gamma_q \prec \Gamma_{q'}$ if $G\varphi \in \Gamma_q \Rightarrow \varphi \in \Gamma_{q'}$ $\lhd$

Fact 2.13 $\Gamma_q \prec \Gamma_{q'} \prec \Gamma_{q''} \Rightarrow \Gamma_q \prec \Gamma_{q''}$.

Fact 2.14 $\Gamma_q \prec \Gamma_{q'}$ *and* $\Gamma_q \prec \Gamma_{q''} \Rightarrow \Gamma_{q'} \prec \Gamma_{q''}$ *or* $\Gamma_{q'} = \Gamma_{q''}$ *or* $\Gamma_{q''} = \Gamma_{q'}$.

Fact 2.15 *The following are equivalent:*
1) $\Gamma_q \prec \Gamma_{q'}$
2) $\varphi \in \Gamma_{q'} \Rightarrow F\varphi \in \Gamma_q$
3) $H\varphi \in \Gamma_{q'} \Rightarrow \varphi \in \Gamma_q$
4) $\varphi \in \Gamma_{q'} \Rightarrow P\varphi \in \Gamma_q$

The construction of $\langle \mathbb{Q}, < \rangle$ will proceed in stages. Each stage n will consist of:

1. a finite set $Q_n = \{q_0, q_1, \ldots, q_k\}$
2. an assignment of an $\mathbf{Q_{FP}}$ maximally consistent set Γ_{q_i} to each $q_i \in Q_n$
3. a linear ordering $<$ on Q_n such that $q_j < q_h \Rightarrow \Gamma_{q_j} \prec \Gamma_{q_h}$

Let $\varphi_0\varphi_1\varphi_2$... be an enumeration of all F and P formulas in which each formula is repeated infinitely many times. The construction will proceed as follows:

- **Stage 0**: Let $Q_0 = \{q^{\#}\}$ and associate $q^{\#}$ with $\Gamma_{q^{\#}}$, a $\mathbf{Q_{FP}}$ maximally consistent extension of $\{\neg\varphi\}$.
- **Stage $\mathbf{3n+1}$**: For all $q, q'' \in Q_{3n}$ such that q'' is the immediate successor of q in Q_{3n}, add a new point q' in between q and q'' and associate q' with $\Gamma_{q'}$, a $\mathbf{Q_{FP}}$ maximally consistent extension of $\Gamma = \{\psi \mid G\psi \in \Gamma_q\} \cup \{F\varphi \mid \varphi \in \Gamma_{q''}\}$. Let Q_{3n+1} be Q_{3n} plus the points added by this procedure and let $<_{3n+1}$ be $<_{3n}$ extended in the obvious way to include these points.

- **Stage 3n + 2**: Let $F\varphi$ be the next F formula in the enumeration and let q be the greatest element in the Q_{3n+1} ordering such that $F\varphi \in \Gamma_q$ and for all $q^* \in Q_{3n+1}$ such that $q < q^*$, $\neg\varphi \in \Gamma_{q^*}$. If such a q exists, add a new point q' immediately after q and associate q' with $\Gamma_{q'}$, a $\mathbf{Q_{FP}}$ maximally consistent extension of $\Gamma' = \{\varphi\} \cup \{\psi \mid G\psi \in \Gamma_q\}$. Let Q_{3n+1} be Q_{3n} plus any points added by this procedure and let $<_{3n+1}$ be $<_{3n}$ extended in the obvious way.

- **Stage 3n + 3**. Symmetric to 3n+2 taking the first P formula in the enumeration that hasn't been used.

In order to ensure that our eventual model $\langle Q, <, \nu\rangle$ comes out right, it must be checked that Conditions $1) - 3)$ above are met at every stage in the construction. As an illuminative example, we will check that Condition 3) is met at Stage 3n+2. Fact 2.13 states that $\prec$ is transitive, so we only need to check that $\Gamma_q \prec \Gamma_{q'} \prec \Gamma_{q''}$, where q'' is the immediate successor of q in the $<_{3n+1}$ ordering and q' is a new point added between q and q'' at Stage $3n+2$.

It follows from the definition of $\Gamma_{q'}$ that $\Gamma_q \prec \Gamma_{q'}$. However, if q has an immediate successor q'' in Q_{3n+1}, the definition of $\Gamma_{q'}$ does not immediately guarantee that $\Gamma_{q'} \prec \Gamma_{q''}$. On the other hand, it does follow from Fact 2.14 that $\Gamma_{q'} \prec \Gamma_{q''}$ or $\Gamma_{q''} = \Gamma_{q'}$ or $\Gamma_{q''} \prec \Gamma_{q'}$. If $\Gamma_{q''} = \Gamma_{q'}$ holds, then $\varphi \in \Gamma_{q''}$ and this contradicts the assumption that $\neg\varphi \in \Gamma_{q^*}$ for all $q <_{3n+1} q^*$. Furthermore, if $\Gamma_{q''} \prec \Gamma_{q'}$, then $F\varphi \in \Gamma_{q''}$ and this contradicts the assumption that q is the greatest element in the $<_{3n+1}$-ordering such that $F\varphi \in \Gamma_q$. So it must be the case that $\Gamma_{q'} \prec \Gamma_{q''}$, as required.

To finish off the construction, let $\langle Q, <\rangle = \bigcup\{\langle Q_n, <_n\rangle \mid n \in \omega\}$. We first want to show our constructed frame $\langle Q, <\rangle$ can be used to refute φ. As suggested before, the fundamental idea behind the construction method is that it is possible to build mathematical structures such that the formulas associated with a point in the construction are the set of formulas that will be true at that point when the construction is finished. However, up to now no connection has been made between the syntactic objects associated with each point and the semantic notion of truth at a point. It is the valuation ν which enables us to do this. To get the ball rolling, we will choose ν such that the truth of a propositional letter at a point q corresponds to its membership in Γ_q. In other words, let $\nu(p) = \{q \mid p \in \Gamma_q\}$. Now let us check that we have performed the construction in a such a way that this harmony extends to all formulas φ.

Lemma 2.16 $\langle Q, <, \nu\rangle, q \models \varphi \Leftrightarrow \varphi \in \Gamma_q$.

Proof. By induction on the complexity of φ.

- $\varphi = p$.

  By the definition of ν, $q \models p \Leftrightarrow p \in \Gamma_q$.

- $\varphi = \psi \wedge \chi$.

$q \models \psi \wedge \chi \Leftrightarrow q \models \psi$ and $q \models \chi$
$q \models \psi$ and $q \models \chi \Leftrightarrow$ (by induction hypothesis) $\psi \in \Gamma_q$ and $\chi \in \Gamma_q$
$\psi \in \Gamma_q$ and $\chi \in \Gamma_q \Leftrightarrow$ (by maximal consistency) $\psi \wedge \chi \in \Gamma_q$.

- $\varphi = \neg\psi$

  $q \models \neg\psi \Leftrightarrow q \not\models \psi$
  $q \not\models \psi \Leftrightarrow$ (by induction hypothesis) $\psi \notin \Gamma_q$
  $\psi \notin \Gamma_q \Leftrightarrow$ (by maximal consistency) $\neg\psi \in \Gamma_q$.

- $\varphi = F\psi$

  [$\Leftarrow$]. Assume $F\psi \in \Gamma_q$. Then by construction there is a stage j such that $q \in Q_j$ and $F\psi$ is being treated. At stage $j+1$, there is a $q' \in Q_{j+1}$ such that $q < q'$ and $\psi \in \Gamma_{q'}$. It then follows from the inductive hypothesis that $q' \models \psi$, so $\langle Q, <, \nu\rangle, q \models F\psi$.

  [$\Rightarrow$]. Assume $\langle Q, <, \nu\rangle, q \models F\psi$. Then there is some $q < q'$ such that $\langle Q, <, \nu\rangle, q' \models \psi$. So it follows from the inductive hypothesis that $\psi \in \Gamma_q$. Since $q < q'$, we know by condition 3) on stages that $\Gamma_q \prec \Gamma_{q'}$. So $F\psi \in \Gamma_q$.

- $\varphi = G\psi$.

  [$\Leftarrow$] In this case it is easier to argue contrapostively. So assume $G\psi \notin \Gamma_q$. Then by maximal consistency $F\neg\psi \in \Gamma_q$ and it follows from "[$\Leftarrow$]" above that
  $\langle Q, <, \nu\rangle, q \models F\neg\psi$. So $\langle Q, <, \nu\rangle, q \not\models G\psi$.

  [$\Rightarrow$] Once again it is easier to argue contrapositively. So assume $\langle Q, <, \nu\rangle, q \not\models G\psi$. Then $\langle Q, <, \nu\rangle, q \models F\neg\psi$ and it follows from "[$\Rightarrow$]" above that $F\neg\psi \in \Gamma_q$. So by maximal consistency, $G\psi \notin \Gamma_q$.

  We will leave the H and P cases to the reader, as they are quite similar to G and F.
  QED

Now with the appropriate hindsight, it is easy to see that $\langle Q, <, \nu\rangle$ refutes φ. At Stage 0 we made sure to put $\neg\varphi$ in $\Gamma_{q^\#}$. So by the above Lemma it follows that $\langle Q, <, \nu\rangle, q^\# \models \neg\varphi$.

To finish the proof, it only remains to be shown that our constructed model is isomorphic to $\langle \mathbb{Q}, <\rangle$. To do this we will make use of Cantor's Theorem.

Theorem 2.17 *(Cantor) Every countable dense linear ordering without end points is isomorphic to* $\mathbb{Q}$.

Lemma 2.18 $\langle Q, <\rangle$ *is a countable, dense, linear order without end points.*

Proof. Linearity follows immediately from condition 2) on stages, and countability from the fact that only finitely many points are added at each stage. To see $\langle Q, <\rangle$ is dense assume that $q, q'' \in Q$ such that $q < q''$. Then there is some stage Q_k such that $q, q'' \in Q_k$ and by stage Q_{k+3} a new point q' is added such that $q < q' < q''$. For right unboundedness, assume q is the greatest element in the $<_j$ ordering for some j. Then there is a stage $\ell > k$ such that the formula $F\top$ is being treated. $F\top$ is an axiom of $\mathbf{Q_{FP}}$ and so is in every $\mathbf{Q_{FP}}$ maximally consistent set. If q does not have a successor at stage ℓ there is no $q^* \in Q_\ell$, $q < q^*$ such that $\top \in \Gamma_{q^*}$. A new point will then be added after q at this stage. Left unboundedness is similar. QED

It follows from Cantor's Theorem and the above Lemma that there is an isomorphism f from $\langle Q, < \rangle$ to $\langle \mathbb{Q}, < \rangle$. Assign $\langle \mathbb{Q}, < \rangle$ the valuation $\nu'(p) = f(\nu(p))$. It is easy to check that

$$\langle Q, <, \nu \rangle, q \models \psi \Leftrightarrow \langle \mathbb{Q}, <, \nu' \rangle, f(q) \models \psi$$

So $\langle \mathbb{Q}, <, \nu' \rangle, f(q^{\#}) \models \neg\varphi$ and $\mathbf{Q_{FP}}$ is complete with respect to $\langle \mathbb{Q}, < \rangle$. QED

In de Jongh, Veltman [8], the construction method presented above is used to give simplified completeness proofs for the structures $\langle \mathbb{N}, < \rangle$, $\langle \mathbb{Q}, < \rangle$ and $\langle \mathbb{R}, < \rangle$ in $\mathcal{L}^{FP}$. The simplicity of $\langle \mathbb{R}, < \rangle$ is particularly notable (and tempting) for as in the case of $\langle \mathbb{R}, \tau \rangle$, the completeness proofs of $\langle \mathbb{R}, < \rangle$ working indirectly through Kripke frames are quite technical.

In the next chapter we will examine how the construction method can be applied to topological completeness proofs. We will first use the method to axiomatize $\langle \mathbb{Q}, \tau \rangle$ in the the basic modal language $\mathcal{L}$ and the more difficult combined language $\mathcal{L}^{\Box FP}$. Then we will stop to consider possible extensions to $\langle \mathbb{R}, \tau \rangle$. Finally, we will examine the slightly more expressive temporal Since/ Until language on $\langle Q, < \rangle$ and return to the basic modal language $\mathcal{L}$ with two modal operators $\Box_1$ and $\Box_2$ on the topological product space $\langle \mathbb{Q} \times \mathbb{Q}, \tau_1, \tau_2 \rangle$.

3 Topological Completeness Proofs

3.1 Henkin Models

Before moving on to the topological completeness proofs, let us pause for a moment to correctly frame the construction method described above and prove a theorem made use of in Section 2.3.

The construction method is best understood as a means of adding structure to Henkin models.

Definition 3.1 Consider the basic modal language $\mathcal{L}$. A *Henkin model* is a triple $M = \langle W^{\mathbf{L}}, R^{\mathbf{L}}, \nu^{\mathbf{L}} \rangle$ where

- $\mathbf{L}$ is a logic
- $W^{\mathbf{L}}$ is the set of all $\mathbf{L}$ maximally consistent sets
- $R^{\mathbf{L}} \subset W^{\mathbf{L}} \times W^{\mathbf{L}}$ is a binary relation such that $R^{\mathbf{L}}\Gamma_x\Gamma_y$ iff $\Box\varphi \in \Gamma_x \Rightarrow \varphi \in \Gamma_y$
- $\nu^{\mathbf{L}}(p) = \{\Gamma_x \mid p \in \Gamma_x\}$

$\lhd$

Both Henkin models and the construction method contain the same ingredients: maximally consistent sets, a relation determined by the syntactic structure these sets, and a valuation which connects syntax to semantics. However, while Henkin models are rather unwieldy entities, constructed models are as well behaved as we can imagine them to be.

Yet despite their coarseness, Henkin models can be used to establish logically interesting topological completeness proofs.

Theorem 3.2 **S4** *is complete with respect to the class of all topological spaces.*

Proof. The proof will make use of the Henkin model for **S4**. Let us first define a topology on $\langle W^{\mathbf{S4}}, R^{\mathbf{S4}}, \nu^{\mathbf{S4}} \rangle$ as follows: let $\Delta_w = \{\Gamma_z \mid R^{\mathbf{S4}}\Gamma_w\Gamma_z\}$ and $\mathcal{B} = \{\Delta_w \mid w \in W^{\mathbf{S4}}\}$ serve as a basis for $\tau^{\mathbf{S4}}$.

Lemma 3.3 $\langle W^{\mathbf{S4}}, \tau^{\mathbf{S4}} \rangle$ *is a topological space.*

Proof. It suffices to check the following two properties:

- For any $\Delta_u, \Delta_v \in \mathcal{B}$ and any $\Gamma_x \in \Delta_u \cap \Delta_v$, there is some $\Delta_y \in \mathcal{B}$ such that $\Gamma_x \in \Delta_y \subseteq \Delta_u \cap \Delta_v$
- For any $\Gamma_x \in W^{\mathbf{S4}}$, there is some $\Delta_y \in \mathcal{B}$ such that $\Gamma_x \in \Delta_y$

Fact 3.4 $\langle W^{\mathbf{S4}}, R^{\mathbf{S4}} \rangle$ *is transitive and reflexive.*

Since $\langle W^{\mathbf{S4}}, R^{\mathbf{S4}} \rangle$ reflexive the second property is immediate; for any $\Gamma_x \in W^{\mathbf{S4}}$, $\Gamma_x \in \Delta_x$. The first property is also not hard to establish. If $\Gamma_x \in \Delta_u \cap \Delta_v$ then $\Delta_x = \{\Gamma_z \mid R^{\mathbf{S4}}\Gamma_x\Gamma_z\} \subseteq \Delta_u \cap \Delta_v$. QED

Lemma 3.5 $\langle W^{\mathbf{S4}}, \tau^{\mathbf{S4}}, \nu^{\mathbf{S4}} \rangle, \Gamma_x \models \varphi \Leftrightarrow \varphi \in \Gamma_x$.

Proof. By induction on the complexity of φ. The propositional and boolean cases are the same as before, and $\Diamond$ is dual to $\Box$, so we will only treat the case $\varphi = \Box\psi$.

[$\Leftarrow$] Assume $\Box\psi \in \Gamma_x$. We need to show that there is some $U \in \tau^{\mathbf{S4}}$ such that $\Gamma_x \in U$, and for every $\Gamma_y \in U$, $\Gamma_y \models \psi$. Let $U = \Delta_x = \{\Gamma_z \mid R^{\mathbf{S4}}\Gamma_x\Gamma_z\}$. Then by the definition of $R^{\mathbf{S4}}$, $\Box\psi \in \Gamma_x \Rightarrow \psi \in \Gamma_y$ for all $\Gamma_y \in U$. Furthermore, since $\langle W^{\mathbf{S4}}, R^{\mathbf{S4}} \rangle$ is reflexive, $\Gamma_x \in U$. It then follows from the inductive hypothesis that $y \models \psi$ for all $\Gamma_y \in U$. So $\Gamma_x \models \Box\psi$.

[$\Rightarrow$]. We will argue contrapositively. Assume $\Box\psi \notin \Gamma_x$. Then by maximal consistency $\Diamond\neg\psi \in \Gamma_x$. We want to show that for every $U \in \tau^{\mathbf{S4}}$, if $\Gamma_x \in U$ then there is some $\Gamma_y \in U$ such that $\Gamma_y \models \neg\psi$. Since every open set is closed under $R^{\mathbf{S4}}$ successors, it suffices to find a point Γ_z such that $R^{\mathbf{S4}}\Gamma_x\Gamma_z$ and $\Gamma_z \models \neg\psi$. It is clear what any such Γ_z must look like: $\Gamma = \{\varphi \mid \Box\varphi \in \Gamma_x\} \cup \{\neg\psi\} \subseteq \Gamma_z$. So let Γ_z be any maximally consistent extension of Γ (for the

moment we will refrain from checking that Γ is consistent). Then it follows that for all U such that $\Gamma_x \in U$, $\Gamma_z \in U$ and by the inductive hypothesis $\Gamma_z \models \neg\psi$. So $\langle W^{\mathbf{S4}}, \tau^{\mathbf{S4}}, \nu^{\mathbf{S4}} \rangle, \Gamma_x \not\models \Box\psi$.
QED

It is now easy to establish the desired completeness result. Assume $\nvdash_{\mathbf{S4}} \varphi$. We need to show there exists a topological model $\langle W, \tau, \nu \rangle$ such that $\langle W, \tau, \nu \rangle \not\models \varphi$. Since $\nvdash_{\mathbf{S4}} \varphi$, it follows by definition that $\neg\varphi$ is consistent and so is a member of some **S4** maximally consistent set Γ_y. Then by the above Lemma $\langle W^{\mathbf{S4}}, \tau^{\mathbf{S4}}, \nu^{\mathbf{S4}} \rangle, \Gamma_y \models \neg\varphi$. QED

Corollary 3.6 **S4** *is complete with respect to the class of all finite topological spaces.*

Proof. Every step in the above proof goes through if we restrict the language $\mathcal{L}$ to subformulas of $\neg\varphi$. Then there are only finitely many maximally consistent sets, each of which is finite. So the topological model refuting φ will be finite. QED

While the above theorem and corollary are logically quite useful, it is not clear how they can be used to establish completeness results for mathematically interesting structures. Corollary 3.6 does show that Henkin models can be modified in certain ways to give more structured completeness results. And indeed there exist far more subtle means for massaging Henkin models into a desired shape than just restricting the cardinality of the language. However, for our purposes such subtleties are not an issue. For one of the fundamental properties of the all the mathematical structures we will be considering, irreflexivity, is not modally definable. This means drastic measures must be taken to guarantee that a Henkin-like model is, say, linearly ordered by a relation $<$. The construction method can be seen as one such measure. It is the most fine grained means of throwing out points and removing unwanted relations from a Henkin model. Now let us see how the construction method can be used to establish mathematically interesting topological completeness results.

3.2 S4 on $\mathbb{Q}$

The simplest topological space of mathematical interest is $\langle \mathbb{Q}, \tau \rangle$. That **S4** is complete with respect to $\langle \mathbb{Q}, \tau \rangle$ follows from 'The Algebra of Topology'. Keeping with the program of simplifying McKinsey and Tarski's results, a new modal proof has been given in van Benthem, Bezhanishvili, ten Cate, Sarenac [5]. Our contribution to this project will be to give another proof using the construction method. However, using the construction method it is possible to strengthen McKinsey and Tarski's original results.

Definition 3.7 A logic **L** is said to be *strongly complete* with respect to a model $M = \langle W, R, \nu \rangle$ if for any set of formulas Δ and formula φ:

$$M \models \Delta \Rightarrow M \models \varphi \text{ (} \Delta \models_M \varphi) \text{ iff } \varphi \text{ is provable in } \mathbf{L} \text{ from } \Delta \text{ (} \Delta \vdash_{\mathbf{L}} \varphi).$$

$\lhd$

Strong completeness implies completeness (let $\Delta = \emptyset$), but the converse is not true.

Theorem 3.8 **S4** *is strongly complete with respect to* $\langle \mathbb{Q}, \tau \rangle$*.*

Proof. Assume $\Delta \nvdash_{\mathbf{S4}} \chi$. As in temporal case, we will give a construction (modulo isomorphism) of the structure $\langle \mathbb{Q}, < \rangle$. We will then view $\langle \mathbb{Q}, < \rangle$ as the topological space $\langle \mathbb{Q}, \tau \rangle$ and show that the construction has been carried out such that $\Delta \not\models_{\langle \mathbb{Q}, \tau \rangle} \varphi$.

In this case, the construction will be divided into *stages* and *steps*. A *stage* n consists of:

1. a finite set $Q_n = \{q_0, q_1, \ldots, q_k\}$
2. a linear ordering $<_n$ on Q_n
3. an assignment of an **S4** maximally consistent set Γ_{q_i} to each $q_i \in Q_n$

Each stage $n > 0$ is divided into $k + 2$ steps. A *step* j, $0 \leq j \leq k + 1$, consists of:

1. a finite set $Q_n^j = \{q_0, q_1, \ldots, q_l\}$
2. a linear ordering $<_n^j$ on Q_n^j
3. an assignment of an **S4** maximally consistent set Γ_{q_i} to each $q_i \in Q_n^j$

Note that unlike the temporal construction of $\langle \mathbb{Q}, < \rangle$, the relations $<_n$ and $<_n^j$ are independent of the syntactic structure of the maximally consistent sets. For someone accustomed to Henkin-style completeness proofs, this probably seems a bit off. But really it is no mystery. When the truth of $\mathcal{L}$ formulas is evaluated in $\langle \mathbb{Q}, \tau \rangle$, the ordering of $\langle \mathbb{Q}, < \rangle$ will play no role. During the construction we need to keep in mind the eventual topological structure and not the ordering.

With this word of advice in mind, let us begin the construction. Let $\varphi_0\varphi_1\varphi_2...$ be an enumeration of all $\Box$ and $\Diamond$ formulas in which each formula is repeated infinitely many times.

- **Stage 0**: Let $Q_0 = \{q^\#\}$ and associate $q^\#$ with $\Gamma_{q^\#}$, an **S4** maximally consistent extension of $\Delta \cup \{\neg\chi\}$.
- **Stage $\mathbf{2n+1}$**: Let $\Box\varphi$ be the next $\Box$ formula in the enumeration and $\{q_1....q_k\}$ be an enumeration of the elements in Q_{2n} ordered by $<_{2n}$.
  - *Step* 0: $Q_{2n} = Q_{2n}^0$ and $<_{2n} = <_{2n}^0$.
  - *Step* $j+1$: If $\Box\varphi \in \Gamma_{q_j}$ let $Q_{2n}^{j+1} = Q_{2n}^j \cup \{q^*, q'\}$ and let $<_{2n}^{j+1}$ be $<_{2n}^j$ extended to include new points q^* and q' immediately before and after q_j. Associate q^* and q' with Γ_{q_j}. If $\Box\varphi \notin \Gamma_{q_j}$, let $Q_{2n}^{j+1} = Q_{2n}^j$ and $<_{2n}^{j+1} = <_{2n}^j$.

  Let $Q_{2n+1} = Q_{2n}^{k+1}$ and $<_{2n+1} = <_{2n}^{k+1}$.
- **Stage $\mathbf{2n+2}$**: Let $\Diamond\varphi$ be the next $\Diamond$ formula in the enumeration and $\{q_1...q_k\}$ be an enumeration of the elements in Q_{2n+1} ordered by $<_{2n+1}$.

- *Step* $0 : Q_{2n+1} = Q^0_{2n+1}$ and $<_{2n+1} = <^0_{2n+1}$.
- *Step* $j+1$: If $\Diamond\varphi \in \Gamma_{q_j}$, let $Q^{j+1}_{2n+1} = Q^j_{2n+1} \cup \{q'\}$ and $<^{j+1}_{2n+1}$ be $<^j_{2n}$ extended to include q' immediately after q_j. Associate q' with $\Gamma_{q'}$, an **S4** maximally consistent extension of $\Gamma = \{\varphi\} \cup \{\Box\psi \mid \Box\psi \in \Gamma_{q_j}\}$. If $\Diamond\varphi \notin \Gamma_{q_j}$, add no points at step Q^{j+1}_{2n+1}.

Let $Q_{2n+2} = Q^{k+1}_{2n+1}$ and $<_{2n+2} = <^{k+1}_{2n+1}$.

To finish off the construction, let $\langle Q, < \rangle = \bigcup\{\langle Q_n, <_n\rangle \mid n \in \omega\}$. Clearly each stage and each step meets the conditions imposed above. The only thing to check is that the $\Gamma_{q'}$'s which are claimed to exist in Stage 2n+2 actually do. By Fact 2.11, it suffices to check that Γ is consistent.

Lemma 3.9 Γ *is consistent.*

Proof. Assume not. Then there are $\Box\psi_1, ..., \Box\psi_n \in \Gamma_{q_j}$ such that

$\vdash_{\mathbf{S4}} \Box\psi_1 \wedge ... \wedge \Box\psi_n \wedge \varphi \to \bot$
$\vdash_{\mathbf{S4}} \Box\psi \to \neg\varphi$ where $\Box\psi = \Box(\psi_1 \wedge ... \wedge \psi_n)$
$\vdash_{\mathbf{S4}} \Box(\Box\psi \to \neg\varphi)$
$\vdash_{\mathbf{S4}} \Box\Box\psi \to \Box\neg\varphi$
$\vdash_{\mathbf{S4}} \Box\psi \to \Box\Box\psi$
$\vdash_{\mathbf{S4}} \Box\psi \to \Box\neg\varphi$

Then since $\Box\psi \in \Gamma_{q_j}$, it follows that $\Box\neg\varphi \in \Gamma_{q_j}$. This contradicts the fact that $\Diamond\varphi \in \Gamma_{q_j}$ and Γ_{q_j} is consistent. QED

Before switching perspectives and viewing our constructed frame $\langle Q, < \rangle$ as a topological space, let us first check it has the desired structure.

Lemma 3.10 $\langle Q, < \rangle$ *is a countable dense unbounded linear order.*

Proof. It follows from condition 2) on stages that $\langle Q, < \rangle$ is linear. And again, since only finitely many points are added at every stage, $\langle Q, < \rangle$ is countable. To see $\langle Q, < \rangle$ is dense, consider two arbitrary points $q, q'' \in Q$ such that $q < q''$. Then there is some stage k such that $q, q'' \in Q_k$ and the formula $\Box\top$ is being treated. $\Box\top$ is a member of every **S4** maximally consistent set, so in particular $\Box\top \in \Gamma_q$. Thus by construction, at Stage $k+1$ there will be a new point q' such that $q < q' < q''$. The same argument shows $\langle Q, < \rangle$ is unbounded. QED

Now let us switch perspectives and view our ordered set $\langle Q, < \rangle$ as a topological space with the standard topology. Assign $\langle Q, \tau \rangle$ the expected Henkin valuation $\nu(p) = \{q \mid p \in \Gamma_q\}$.

Lemma 3.11 $\langle Q, \tau, \nu \rangle, q \models \varphi \Leftrightarrow \varphi \in \Gamma_q$.

Proof. By induction on the complexity of φ. We will only treat the case $\varphi = \Box\psi$.

[$\Rightarrow$] Assume $\Box\psi \notin \Gamma_q$. Then since Γ_q is an **S4** maximally consistent set, $\Diamond\neg\psi \in \Gamma_q$. We want to show that for every $U \in \tau$ such that $q \in U$ there is some $y \in U$ such that $y \models \neg\psi$. Let U' be an arbitrary open containing q and let q'' be a point in U' such that $q < q''$. That such a q'' exists is guaranteed by the density of $\langle Q, < \rangle$. Then there is some stage j in the construction such that $q, q'' \in Q_{j-1}$ and $\Diamond\neg\psi$ is being treated. When stage j is completed, there is a point $q' \in Q_j$ such that $q < q' < q''$ and $\neg\psi \in \Gamma_{q'}$. It then follows from the inductive hypothesis that $q' \models \neg\psi$. So $\langle Q, \tau, \nu \rangle, q \not\models \Box\psi$.

[$\Leftarrow$] Assume $\Box\psi \in \Gamma_q$. We want to show that there is some $U \in \tau$ such that $q \in U$ and for all $y \in U$, $y \models \psi$. To find such a U, consider the first stage j such that $q \in Q_{j-1}$ and $\Box\psi$ is being treated. At stage j, there is some step i where points q^{**} and q'' are added immediately before and after q and $\Gamma_{q^{**}} = \Gamma_q = \Gamma_{q''}$. We claim that for all Q_p^m, $m > i$, $p \geq j-1$, if $q' \in Q_p^m$ and $q^{**} < q' < q''$, then $\Box\psi \in \Gamma_{q'}$. We will argue by induction on the steps of the construction.

Assume at step Q_s^{r+1}, $r+1 > i$, $s \geq j-1$, a point q' is added between q^{**} and q''. By inspecting the construction, it is clear that q' is added immediately before or after the point $q_r \in Q_s$. So $q^{**} \leq q_r \leq q''$ and it follows from the inductive hypothesis and the fact that $\Box\psi$ is in $\Gamma_{q^{**}}$ and $\Gamma_{q''}$ that $\Box\psi \in \Gamma_{q_r}$. If s is odd then $\Gamma_{q'} = \Gamma_{q_r}$ and if s is even then $\{\Box\psi \mid \Box\psi \in \Gamma_{q_r}\} \subseteq \Gamma_{q'}$. So $\Box\psi \in \Gamma_{q'}$.

Finally, $\Box\psi \to \psi$ is an axiom of **S4**, so for all $q' \in Q$ such that $q^{**} < q' < q''$,$\psi \in \Gamma_{q'}$. By the inductive hypothesis on the complexity formulas, for all $q' \in Q$ such that $q^{**} < q' < q''$, $q' \models \psi$. So if we let U be the open $\{y \mid q^{**} < y < q''\}$ we immediately get that $\langle Q, \tau, \nu \rangle, q \models \Box\psi$. QED

To finish the proof, let f be the isomorphism between $\langle Q, < \rangle$ and $\langle \mathbb{Q}, < \rangle$. Assign $\langle \mathbb{Q}, \tau \rangle$ the valuation $\nu'(p) = f(\nu(p))$. It is easy to check that $\langle Q, \tau, \nu \rangle, q \models \varphi \Leftrightarrow \langle \mathbb{Q}, \tau, \nu' \rangle, f(q) \models \varphi$. Since we made sure to add $\Delta \cup \{\neg\chi\}$ to $\Gamma_{q\#}$ at the first stage of the construction, it follows from the above Lemma that $\langle Q, \tau, \nu \rangle, q^{\#} \models \Delta \cup \{\neg\chi\}$. So $\langle \mathbb{Q}, \tau, \nu' \rangle, f(q^{\#}) \models \Delta \cup \{\neg\chi\}$ and **S4** is strongly complete with respect to $\langle \mathbb{Q}, \tau \rangle$. QED

3.3 Remarks on the Construction Method

The essential feature of the above construction is that once a $\Box\varphi$ formula is treated at a step, a $\Box\varphi$ stretch is created which is preserved for the rest of the construction. Since this is the most stringent requirement on the construction, as might be guessed, there is considerable freedom in how formulas are satisfied at a point in $\langle \mathbb{Q}, \tau \rangle$.

For example, in the construction given above if $\Diamond\varphi \in \Gamma_q$, then maximally consistent sets containing φ are added arbitrarily close to q *on the right*. So if $\langle \mathbb{Q}, \tau, \nu' \rangle, q \models \Diamond\varphi$, there is guaranteed to be a sequence approaching from the right such that every point models φ. However, we could have just as well selected for a sequence from the left, or sequences from both the right and the left. Actually, this last alternative has an interesting property: for all formulas φ, $\nu'(\Box\varphi) = X$ has no least upper bound. For assume X does have a least upper bound q^*. Then $\Box\varphi \notin \Gamma_{q^*}$, so $\Diamond\neg\varphi \in \Gamma_{q^*}$. However, by construction there is a sequence of points modeling $\neg\varphi$ approaching q^* from left, so q^* is not the least upper bound of $\nu'(\Box\varphi)$ after all.

This flexibility in how formulas are to be satisfied on the desired topological space is a special feature of the construction method. When proving topological completeness results indirectly through Kripke models, the starting point is an arbitrary **S4** model M refuting a formula φ and the end result is that every point in the desired topological space is modally equivalent to some point in M. Thus we know that the desired topological space refutes φ , but we don't have any idea *how* it does this.

This flexibility might lead one to be believe that the construction method has an advantage over the model theoretic approaches in some cases. For example, by the above result $\langle\mathbb{Q}, \tau\rangle$ is a countermodel to an arbitrary non-theorem φ of **S4**. Without knowing anything about how φ is refuted on $\langle\mathbb{Q}, \tau, \nu\rangle$, it seems rather hopeless to extend $\langle\mathbb{Q}, \tau, \nu\rangle$ to a countermodel $\langle\mathbb{R}, \tau, \nu'\rangle$ refuting φ. But perhaps by constructing $\langle\mathbb{Q}, \tau\rangle$ so that φ is refuted in a particular way, it would be possible to extend $\langle\mathbb{Q}, \tau, \nu\rangle$ to a $\langle\mathbb{R}, \tau, \nu'\rangle$ countermodel . Indeed, extending a $\mathbf{Q_{FP}}$ countermodel $\langle\mathbb{Q}, <, \nu\rangle$ to an $\mathbf{R_{FP}}$ countermodel $\langle\mathbb{R}, < \nu'\rangle$ in the basic temporal language works quite naturally. We will consider such extensions shortly.

3.4 $\mathbf{Q_{\Box FP}}$ on $\mathbb{Q}$

However, before looking at possible temporal and topological extensions of $\mathbb{Q}$, let us first examine this structure in the combined temporal and topological language $\mathcal{L}^{\Box FP}$. As mentioned in the previous chapter, $\mathcal{L}^{\Box FP}$ was one of the first spatial extensions of the basic modal language to be considered. This language can be traced back to Shehtman [20], where an axiomatization of $\langle\mathbb{Q}, \tau\rangle$ was given. Since that time, little progress has been made. It turns out even the next question, axiomatizing $\langle\mathbb{R}, \tau\rangle$ in $\mathcal{L}^{\Box FP}$, is quite difficult.

The completeness proof for $\langle\mathbb{Q}, \tau\rangle$ given in Shehtman [20] works indirectly through Kripke frames, as sketched in Section 2.5. In the last chapter, we will examine this approach in some detail , but for now it suffices to say that in the case of $\mathcal{L}^{\Box FP}$ this strategy is quite demanding. This is one instance where using the construction method improves matters significantly.

Let $\mathbf{Q_{\Box FP}}$ be the logic $\mathbf{S4} + \mathbf{Q_{FP}} +$:

$$
\begin{aligned}
&F\varphi \to \Box F\varphi \\
&P\varphi \to \Box P\varphi \\
&\Box\varphi \to F\varphi \\
&\Box\varphi \to P\varphi \\
&\Diamond\varphi \to \varphi \vee F\varphi \vee P\varphi
\end{aligned}
$$

$\Box\varphi \wedge G\varphi \rightarrow \Box G\varphi$
$\Box\varphi \wedge H\varphi \rightarrow \Box H\varphi$

Theorem 3.12 $\mathbf{Q_{\Box FP}}$ *is strongly complete with respect to* $\langle \mathbb{Q}, \tau \rangle$.

Proof. Assume $\Delta \nvdash_{\mathbf{Q_{\Box FP}}} \chi$. As before, we will construct a frame $\langle Q, < \rangle \cong \langle \mathbb{Q}, < \rangle$ in steps and stages such that $\langle Q, \tau \rangle$ refutes $\Delta \cup \{\varphi\}$. However, since the ordering of $\langle Q, < \rangle$ will play a role in evaluating the truth of $\mathcal{L}^{\Box FP}$ formulas in $\langle \mathbb{Q}, \tau \rangle$, this construction will require some relation between the maximally consistent sets and $<$. We will build this interaction into the definitions of a stage and a step.

A stage n consists of:

1. a finite set $Q_n = \{q_0, q_1, ..., q_k\}$
2. a linear ordering $<_n$ on Q_n such that $q_j <_n q_l \Rightarrow \Gamma_{q_j} \prec \Gamma_{q_l}$
3. an assignment of a $\mathbf{Q_{\Box FP}}$ maximally consistent set Γ_{q_i} to each $q_i \in Q_n$

Each stage n is divided into $\leq k+2$ steps. A step j, $0 \leq j \leq k+1$, consists of:

1. a finite set $Q_n^j = \{q_0, q_1, ..., q_l\}$
2. a linear ordering $<_n^j$ on Q_n^j such that $q_h <_n^j q_i \Rightarrow \Gamma_{q_h} \prec \Gamma_{q_i}$
3. an assignment of a $\mathbf{Q_{\Box FP}}$ maximally consistent set Γ_{q_i} to each $q_i \in Q_n^j$

Once again, the salient feature of the construction will be that once a $\Box\psi$ formula is treated at a step that a $\Box\psi$ stretch is created which can be preserved for the rest of the construction. However, in the present case preserving $\Box\psi$ stretches requires some attention. For example, assume the point q is in what we intend to be a $\Box\psi$ stretch and that $H\varphi$ is in Γ_q. Furthermore, assume at some stage we add a point q'' immediately after q and $P(\neg\varphi \wedge \Diamond\neg\psi)$ is added into $\Gamma_{q''}$. Nothing so far prevents this situation from occurring. We know by the ordering relation $\prec$ that $H\varphi$ must be in all Γ_{q^*} such that $q^* < q$. So in order to ensure the truth of $P(\neg\varphi \wedge \Diamond\neg\psi)$ at q'' when the construction is finished, a point q' must be added in between q and q'' such that $\Diamond\neg\psi \in \Gamma_{q'}$. This ruins our prospective $\Box\psi$ stretch. Fortunately, guarding against this sort of occurrence alone is sufficient to preserve $\Box\psi$ stretches.

Definition 3.13 We say $\Gamma_q \prec_{\Box\psi} \Gamma_{q'}$ if

1) $\Gamma_q \prec \Gamma_{q'}$
2) $\Box\psi \in \Gamma_q, \Gamma_{q'}$
3) $H\varphi \in \Gamma_q \Rightarrow H(\varphi \vee \Box\psi) \in \Gamma_{q'}$
4) $G\varphi \in \Gamma_{q'} \Rightarrow G(\varphi \vee \Box\psi) \in \Gamma_q$
$\lhd$

Proposition 3.14 $\Gamma_{q^*} \prec_{\Box\psi} \Gamma_q \prec_{\Box\psi} \Gamma_{q'} \Rightarrow \Gamma_{q^*} \prec_{\Box\psi} \Gamma_{q'}$.

Proof. Assume $\Gamma_{q^*} \prec_{\Box\psi} \Gamma_q \prec_{\Box\psi} \Gamma_{q'}$. It is easy to see conditions 1) and 2) of $\Gamma_{q^*} \prec_{\Box\psi} \Gamma_{q'}$ are met. So assume $H\varphi \in \Gamma_{q^*}$. By assumption $\Gamma_{q^*} \prec_{\Box\psi} \Gamma_q$, so $H(\varphi \vee \Box\psi) \in \Gamma_q$. Furthermore, $\Gamma_q \prec_{\Box\psi} \Gamma_{q'}$, so $H((\varphi \vee \Box\psi) \vee \Box\psi) \in \Gamma_{q'}$. It then follows from basic propositional logic and maximal consistency that $H(\varphi \vee \Box\psi) \in \Gamma_{q'}$. Condition 4) is similar. QED

Fact 3.15 $\Gamma_q \prec_{\Box\psi} \Gamma_{q'}$ *and* $\Gamma_q \prec_{\Box\varphi} \Gamma_{q'} \Rightarrow \Gamma_q \prec_{\Box(\psi\wedge\varphi)} \Gamma_{q'}$.

Armed with our new condition $\prec_{\Box\psi}$ to preserve $\Box\psi$ stretches, we are ready to begin the construction. Let $\varphi_0\varphi_1\varphi_2...$ be an enumeration of all $F, P, \Box$ and $\Diamond$ formulas in which each formula is repeated infinitely many times.

- **Stage 0**: Let $Q_0 = \{q^\#\}$ with which we associate $\Gamma_{q^\#}$, a $\mathbf{Q_{\Box FP}}$ maximally consistent extension of $\Delta \cup \{\neg\chi\}$.

- **Stage 5n + 1**: Let $\Box\varphi$ be the next $\Box$ formula in the enumeration and let $\{q_1, ..., q_k\}$ be an enumeration of the elements in Q_{5n} ordered by $<_{5n}$.

  - *Step* 0 : $Q_{5n} = Q^0_{5n}$ and $<_{5n}=<^0_{5n}$
  - *Step* $j+1$: If $\Box\varphi \in \Gamma_{q_j}$, let q^{**} be the immediate predecessor of q_j in the $<^j_{5n}$ ordering, provided one exists. Add a new point q^* immediately before q_j and associate q^* with Γ_{q^*}, a $\mathbf{Q_{\Box FP}}$ maximally consistent extension of:

$$\Gamma^* = \{\Box\varphi\} \cup \{\Box\psi' \mid \Gamma_{q^{**}} \prec_{\Box\psi'} \Gamma_{q_j}\} \cup \{\alpha' \mid H\alpha' \in \Gamma_{q_j}\} \cup \{P\beta' \mid \beta' \in \Gamma_{q^{**}}\} \cup \\ \{P\gamma' \mid P(\gamma' \wedge \neg\Box\varphi) \in \Gamma_{q_j}\} \cup \{G\zeta' \mid G(\zeta' \vee \Box\varphi) \in \Gamma_{q_j}\}$$

  Similarly, if $\Box\varphi \in \Gamma_{q_j}$ let q'' be the immediate successor of q_j in the $<^j_{5n}$ ordering, provided one exists. Add a new point q' immediately after q_j and associate q' with $\Gamma_{q'}$, a $\mathbf{Q_{\Box FP}}$ maximally consistent extension of:

$$\Gamma' = \{\Box\varphi\} \cup \{\Box\psi \mid \Gamma_{q_j} \prec_{\Box\psi} \Gamma_{q''}\} \cup \{\alpha \mid G\alpha \in \Gamma_{q_j}\} \cup \{F\beta \mid \beta \in \Gamma_{q''}\} \cup \\ \{H(\gamma \vee \Box\varphi) \mid H\gamma \in \Gamma_{q_j}\} \cup \{F\zeta \mid F(\zeta \wedge \neg\Box\varphi) \in \Gamma_{q_j}\}$$

  Let Q^{j+1}_{5n} be Q^j_{5n} plus any points added by the above procedure, and $<^{j+1}_{5n}$ be $<^j_{5n}$ extended in the obvious way to include these points.

Let $Q_{5n+1} = Q^{k+1}_{5n}$ and $<_{5n+1}=<^{k+1}_{5n}$.

The definitions of Γ^* and Γ' ensure that for all $q \in Q_{5n}$, if $\Box\varphi \in \Gamma_q$ then at Stage $5n+1$ q will have an immediate predecessor q^* and immediate successor q' such that $\Gamma_{q*} \prec_{\Box\varphi} \Gamma_q \prec_{\Box\varphi} \Gamma_{q'}$. It only remains to be checked that Γ' is consistent.

Lemma 3.16 Γ' *is consistent.*

Proof. Assume not. Then by Proposition 3.15 there is a formula $\Box\psi$ and formulas $G\alpha, H\gamma_1, .., H\gamma_m$, $F(\zeta_1 \wedge \neg\Box\varphi), .., F(\zeta_p \wedge \neg\Box\varphi) \in \Gamma_{q_j}$ and $\beta_1, .., \beta_k \in \Gamma_{q''}$ such that:

$\vdash_{\mathbf{Q_{\Box FP}}} \Box\varphi \wedge \Box\psi \wedge \alpha \wedge F\beta_1 \wedge ... \wedge F\beta_k \wedge H(\gamma_1 \vee \Box\varphi) \wedge ... \wedge H(\gamma_m \vee \Box\varphi) \wedge F\zeta_1 \wedge ... \wedge F\zeta_p \rightarrow \perp$
$\vdash_{\mathbf{Q_{\Box FP}}} \alpha \rightarrow \neg(\Box\varphi \wedge \Box\psi \wedge F\beta_1 \wedge ... \wedge F\beta_k \wedge H(\gamma_1 \vee \Box\varphi) \wedge ... \wedge H(\gamma_m \vee \Box\varphi) \wedge F\zeta_1 \wedge ... \wedge F\zeta_p)$
$(*) \vdash_{\mathbf{Q_{\Box FP}}} G\alpha \rightarrow G\neg(\Box\varphi \wedge \Box\psi \wedge F\beta_1 \wedge ... \wedge F\beta_k \wedge H(\gamma_1 \vee \Box\varphi) \wedge ... \wedge H(\gamma_m \vee \Box\varphi) \wedge F\zeta_1 \wedge ... \wedge F\zeta_p)$

We know that:

- $\Box\varphi \rightarrow \Box\Box\varphi$ is an axiom and $\Box\varphi, \Box\psi \in \Gamma_{q_j}$, so $\Box\Box\varphi \in \Gamma_{q_j}$ and $\Box\Box\psi \in \Gamma_{q_j}$
- $\beta_1, ..., \beta_k \in \Gamma_{q''}$ and $\Gamma \prec \Gamma_{q''}$, so $F\beta_1, ..., \ F\beta_k \in \Gamma_{q''}$
- $F\beta \rightarrow \Box F\beta$ is an axiom, so $\Box F\beta_1, ..., \Box F\beta_k \in \Gamma_{q_j}$
- $\Box\varphi \in \Gamma_{q_j}$, so $\Box(\gamma_1 \vee \Box\varphi), ..., \Box(\gamma_m \vee \Box\varphi) \in \Gamma_{q_j}$
- $H\gamma_1 ... H\gamma_m \in \Gamma_{q_j}$, so $H(\gamma_1 \vee \Box\varphi), ..., H(\gamma_m \vee \Box\varphi) \in \Gamma_{q_j}$
- $\Box\psi \wedge H\psi \rightarrow \Box H\psi$ is an axiom, so $\Box H(\gamma_1 \vee \Box\varphi), ..., \Box H(\gamma_m \vee \Box\varphi) \in \Gamma_{q_j}$
- $F\zeta_1, ..., \ F\zeta_p \in \Gamma_{q_j}$ and $F\zeta \rightarrow \Box F\zeta$ is an axiom, so $\Box F\zeta_1, ..., \Box F\zeta_p \in \Gamma_{q_j}$

Putting all this together it follows that:

$$\Box(\Box\varphi \wedge \Box\psi \wedge F\beta_1 \wedge ... \wedge F\beta_k \wedge H(\gamma_1 \vee \Box\varphi) \wedge ... \wedge H(\gamma_m \vee \Box\varphi) \wedge F\zeta_1 \wedge ... \wedge F\zeta_p) \in \Gamma_{q_j}$$

and since $\Box\varphi \rightarrow F\varphi$ is an axiom:

$$F(\Box\varphi \wedge \Box\psi \wedge F\beta_1 \wedge ... \wedge F\beta_k \wedge H(\gamma_1 \vee \Box\varphi) \wedge .... \wedge H(\gamma_m \vee \Box\varphi) \wedge F\zeta_1 \wedge ... F\zeta_p) \in \Gamma_{q_j}$$

However, $G\alpha \in \Gamma_{q_j}$, so by $(*)$:

$$G\neg(\Box\varphi \wedge \Box\psi \wedge F\beta_1 \wedge ... \wedge F\beta_k \wedge H(\gamma_1 \vee \Box\varphi) \wedge ... \wedge H(\gamma_m \vee \Box\varphi) \wedge F\zeta_1 \wedge ... \wedge F\zeta_p) \in \Gamma_{q_j}$$

This contradicts the consistency of Γ_{q_j}. QED

Showing Γ^* is consistent is similar.

- **Stage $\mathbf{5n+2}$**: Let $F\varphi$ be the next F formula in the enumeration.
  - *Step* 0 : If $F\varphi \in \Gamma_{q_j}$ and $\neg\varphi \wedge G\neg\varphi \in \Gamma_{q_{j+1}}$ (or if $F\varphi \in \Gamma_{q_j}$ and $j = k$) add a new point q' immediately after q_j and associate $\Gamma_{q'}$ with a $\mathbf{Q_{\Box FP}}$ maximally consistent extension of:

$$\Lambda = \{\varphi\} \cup \{\Box\psi \mid \Gamma_{q_j} \prec_{\Box\psi} \Gamma_{q_{j+1}}\} \cup \{\alpha \mid G\alpha \in \Gamma_{q_j}\}$$

Let Q^0_{5n+1} be Q_{5n+1} plus any points added by this procedure and $<^0_{5n+1}$ be $<_{5n+1}$ extended in the obvious way to include any new points.

Let $Q_{5n+2} = Q^0_{5n+1}$ and $<_{5n+2}=<^0_{5n+1}$.

Lemma 3.17 Λ *is consistent.*

Proof. Assume not. Then there are formulas $\Box\psi$ and α in Γ_{q_j} such that:

$\vdash_{\mathbf{Q_{\Box}FP}} \varphi \wedge \Box\psi \wedge \alpha \to \perp$
$\vdash_{\mathbf{Q_{\Box}FP}} \alpha \to \neg(\Box\psi \wedge \varphi)$
$\vdash_{\mathbf{Q_{\Box}FP}} G\alpha \to G\neg(\Box\psi \wedge \varphi)$

Since $G\alpha \in \Gamma_{q_j}$, it follows that $G(\neg\Box\psi \vee \neg\varphi) \in \Gamma_{q_j}$. Furthermore, since $G\neg\varphi \in \Gamma_{q_{j+1}}$ and $\Gamma_{q_j} \prec_{\Box\psi} \Gamma_{q_{j+1}}$, it follows that $G(\Box\psi \vee \neg\varphi) \in \Gamma_{q_j}$. So:

$$G(\Box\psi \to \neg\varphi) \wedge G(\neg\Box\psi \to \neg\varphi) \in \Gamma_{q_j}$$

However, by basic propositional logic it then follows that $G\neg\varphi \in \Gamma_{q_j}$. This contradicts the fact that $F\varphi \in \Gamma_{q_j}$. QED

The same argument given in the temporal construction of $\langle Q, < \rangle$ suffices to show that $\Gamma_{q_j} \prec \Gamma_{q'} \prec \Gamma_{q_{j+1}}$.

- **Stage $\mathbf{5n+3}$**: Symmetric to $5n+2$ for the next P formula in the enumeration.

- **Stage $\mathbf{5n+4}$**: Let $\Diamond\varphi$ be the next $\Diamond$ formula in the enumeration and let $\{q_0, ..., q_k\}$ be an enumeration of the elements in Q_{5n+3} ordered by $<_{5n+3}$.

  - *Step* 0 : Let $Q_{5n+3} = Q^0_{5n+3}$ and $<_{5n+3}=<^0_{5n+3}$.

  - *Step* $j+1$: If $\Diamond\varphi \in \Gamma_{q_j}$ let q'' be the immediate successor and q^{**} be the immediate predecessor of q_j in the $<^j_{5n+3}$ ordering, provided these exist. Let

$$\Lambda' = \{\varphi\} \cup \{\Box\psi \mid \Gamma_{q_j} \prec_{\Box\psi} \Gamma_{q''}\} \cup \{\alpha \mid G\alpha \in \Gamma_{q_j}\} \cup \{F\beta \mid \beta \in \Gamma_{q''}\}$$

and

$$\Lambda^* = \{\varphi\} \cup \{\Box\psi' \mid \Gamma_{q_j} \prec_{\Box\psi'} \Gamma_{q^{**}}\} \cup \{\alpha' \mid H\alpha' \in \Gamma_{q_j}\} \cup \{P\beta' \mid \beta' \in \Gamma_{q^{**}}\}$$

  If $\varphi \in \Gamma_{q_j}$ add no points. If $\varphi \notin \Gamma_{q_j}$ and Λ' is consistent, then add a new point q' immediately after q_j and associate q' with a maximally consistent extension of Λ'. Otherwise, add a new point q^* immediately before q_j and associate q^* with a maximally consistent extension of Λ^*.

Let $Q_{5n+4} = Q^{k+1}_{5n+3}$ and $<_{5n+4}= Q^{k+1}_{5n+3}$.

Proposition 3.18 *Either $\varphi \in \Gamma_{q_j}$, or Λ' is consistent, or Λ^* is consistent.*

Proof. Assume not. Then there are formulas $\Box\psi, \Box\psi', G\alpha, H\alpha' \in \Gamma_{q_j}$, $\beta_1, ...\beta_n \in \Gamma_{q''}$ and $\beta'_1, ..., \beta'_k \in \Gamma_{q^{**}}$ such that

- $\vdash_{\mathbf{Q_{\Box FP}}} \varphi \wedge \Box\psi \wedge \alpha \wedge\ F\beta_1 \wedge ... \wedge F\beta_n \rightarrow \perp$
- $\vdash_{\mathbf{Q_{\Box FP}}} \varphi \wedge \Box\psi' \wedge \alpha' \wedge P\beta'_1 \wedge ... \wedge P\beta'_k \rightarrow \perp$

As in the previous stages, it follows that:

- $\vdash_{\mathbf{Q_{\Box FP}}} G\alpha \rightarrow G\neg(\varphi \wedge \Box\psi \wedge\ F\beta_1 \wedge ... \wedge F\beta_n)$
- $\vdash_{\mathbf{Q_{\Box FP}}} H\alpha' \rightarrow H\neg(\varphi \wedge \Box\psi' \wedge P\beta'_1 \wedge ... \wedge P\beta'_k)$

and thus (**):

- $\varphi \notin \Gamma_{q_j}$ and
- $G\neg(\varphi \wedge \Box\psi \wedge\ F\beta_1 \wedge ... \wedge F\beta_n) \in \Gamma_{q_j}$ and
- $H\neg(\varphi \wedge \Box\psi' \wedge P\beta'_1 \wedge ... \wedge P\beta'_k) \in \Gamma_{q_j}$

Since $\Gamma_{q^{**}} \prec \Gamma_{q_j} \prec \Gamma_{q''}$, it follows that $F\beta_1, ..., F\beta_n, P\beta'_1, ..., P\beta'_k \in \Gamma_{q_j}$. Furthermore, since $\Box\psi, \Box\psi' \in \Gamma_{q_j}$ and $P\varphi \rightarrow \Box P\varphi$, $F\varphi \rightarrow \Box F\varphi$, $\Box\varphi \rightarrow \Box\Box\varphi$ are axioms, it follows that:

$$\Box(\Box\psi \wedge \Box\psi' \wedge F\beta_1 \wedge ... \wedge F\beta_n \wedge P\beta'_1 \wedge ... \wedge P\beta'_k) \in \Gamma_{q_j}$$

It is easy to show that $\vdash_{\mathbf{S4}} \Box\gamma \wedge \Diamond\varphi \rightarrow \Diamond(\gamma \wedge \varphi)$, and by substituting

$$\Box\psi \wedge \Box\psi' \wedge\ F\beta_1 \wedge ... \wedge F\beta_n \wedge P\beta'_1 \wedge ... \wedge P\beta'_k$$

for γ we get:

$$\vdash_{\mathbf{Q_{\Box FP}}} \Box(\Box\psi \wedge \Box\psi' \wedge\ F\beta_1 \wedge ... \wedge F\beta_n \wedge P\beta'_1 \wedge ... \wedge P\beta'_k) \wedge \Diamond\varphi \rightarrow \\ \Diamond(\varphi \wedge \Box\psi \wedge \Box\psi' \wedge\ F\beta_1 \wedge ... \wedge F\beta_n \wedge P\beta'_1 \wedge ... \wedge P\beta'_k).$$

So

$$\Diamond(\varphi \wedge \Box\psi \wedge \Box\psi' \wedge\ F\beta_1 \wedge ... \wedge F\beta_n \wedge P\beta'_1 \wedge ... \wedge P\beta'_k) \in \Gamma_{q_j}$$

$\Diamond\gamma \rightarrow \gamma \vee F\gamma \vee P\gamma$ is an axiom, so:

- $(\varphi \wedge \Box\psi \wedge \Box\psi' \wedge F\beta_1 \wedge ... \wedge F\beta_n \wedge\ P\beta'_1 \wedge ... \wedge P\beta'_k) \in \Gamma_{q_j}$ or
- $F(\varphi \wedge \Box\psi \wedge \Box\psi' \wedge F\beta_1 \wedge ... \wedge F\beta_n \wedge\ P\beta'_1 \wedge ... \wedge P\beta'_k) \in \Gamma_{q_j}$ or

- $P(\varphi \wedge \Box\psi \wedge \Box\psi' \wedge F\beta_1 \wedge ... \wedge F\beta_n \wedge \; P\beta'_1 \wedge ... \wedge P\beta'_k) \in \Gamma_{q_j}$

This contradicts $(**)$ above. QED

To finish off the construction, let $\langle Q, < \rangle = \bigcup\{\langle Q_n, <_n \rangle \mid n \in \omega\}$. It is not difficult to check that $\langle Q, < \rangle$ is a countable dense unbounded linear order (the formula $\Box\top$ ensures both unboundedness and density), so we will move directly on to showing that $\langle Q, \tau \rangle$ satisfies $\Delta \cup \{\neg\varphi\}$. Again let $\langle Q, \tau \rangle$ have the Henkin valuation $\nu(p) = \{q \mid p \in \Gamma_q\}$.

Proposition 3.19 $\langle Q, \tau, \nu \rangle, q \models \varphi \Leftrightarrow \varphi \in \Gamma_q$.

Proof. By induction on the complexity of φ. We will treat the cases $\varphi = \Box\psi$ and $\varphi = F\psi$.

$\varphi = \Box\psi$

[$\Rightarrow$] Assume $\Diamond\neg\psi \in \Gamma_q$. Let U' be an arbitrary open containing q and let q^* and q' be points in U' such that $q^* < q < q'$. Then there is a stage j in the construction such that $q, q^*, q' \in Q_{j-1}$ and $\Diamond\neg\psi$ is being treated. When stage j is completed, there is a point $x \in Q_j$ such that $q^* < x < q'$ and $\neg\psi \in \Gamma_x$. It then follows from the inductive hypothesis that $x \models \neg\psi$. So $\langle Q, \tau, \nu \rangle, q \models \Diamond\neg\psi$.

[$\Leftarrow$] Assume $\Box\psi \in \Gamma_q$. We know at some stage j and step i such there are points q^{**} and q'' immediately before and after q in the $<^i_j$ ordering such that $\Gamma_{q^{**}} \prec_{\Box\psi} \Gamma_q \prec_{\Box\psi} \Gamma_{q''}$. We claim that for all $r, s, t \in Q^m_p$, $m > i$, $p \geq j$, if $q^{**} \leq r < s < t \leq q''$, then $\Gamma_r \prec_{\Box\psi} \Gamma_s \prec_{\Box\psi} \Gamma_t$. We will argue by induction on the steps of the construction.

Assume at step Q^{m+1}_p, $m \geq i$, $p \geq j$ a point q' is added between q^{**} and q''. By inspecting the construction, it is clear that q' is added immediately before or after the point $q_m \in Q_p$. Without loss of generality, assume it is placed immediately after q_m. By Proposition 3.14 $\prec_{\Box\psi}$ is transitive, so it suffices to show that $\Gamma_{q_m} \prec_{\Box\psi} \Gamma_{q'} \prec_{\Box\psi} \Gamma_{q_v}$, where q_v is the immediate successor of q_m in the $<^m_p$ ordering. We will only check that $\Gamma_{q_m} \prec_{\Box\psi} \Gamma_{q'}$, showing $\Gamma_{q'} \prec_{\Box\psi} \Gamma_{q_v}$ is similar.

By condition 2) on stages, we know that $\Gamma_{q_m} \prec \Gamma_{q'} \prec \Gamma_{q_v}$. Furthermore, by the inductive hypothesis we know that $\Gamma_{q_m} \prec_{\Box\psi} \Gamma_{q_v}$. By inspecting each stage of the construction, we see that $\{\Box\varphi \mid \Gamma_{q_m} \prec_{\Box\varphi} \Gamma_{q_v}\} \subseteq \Gamma_{q'}$, so $\Box\psi \in \Gamma_{q'}$. Now assume $H\varphi \in \Gamma_{q_m}$. As $\Gamma_{q_m} \prec_{\Box\psi} \Gamma_{q_v}$, it follows that $H(\varphi \vee \Box\psi) \in \Gamma_{q_v}$. It is easy to check that this entails $HH(\varphi \vee \Box\psi)$ is also in Γ_{q_v}. So by the fact $\Gamma_{q'} \prec \Gamma_{q_v}$, $H(\varphi \vee \Box\psi) \in \Gamma_{q'}$. Showing $G\varphi \in \Gamma_{q'} \Rightarrow G(\varphi \vee \Box\psi) \in \Gamma_{q_m}$ is similar.

So for all r, s, t such that $q^{**} \leq r < s < t \leq q''$, $\Gamma_r \prec_{\Box\psi} \Gamma_s \prec_{\Box\psi} \Gamma_t$. It follows immediately that for all q' such that $q^{**} < q' < q''$, $\Box\psi \in \Gamma_{q'}$. And since $\Box\psi \rightarrow \psi$ is an axiom, it follows that ψ is also in all $\Gamma_{q'}$. Then by the inductive hypothesis on the complexity of formulas, $q' \models \psi$ for all q', $q^{**} < q' < q''$. By letting U be the open $U = \{x \mid q^{**} < x < q''\}$, we get that $\langle Q, \tau, \nu \rangle, q \models \Box\psi$.

$\varphi = F\psi$

[$\Rightarrow$] Assume $F\psi \in \Gamma_q$. Then there is some stage j such that $q \in Q_{j-1}$ and $F\psi$ is being treated at j . After this stage, there is a point $q' > q$ such that $\psi \in \Gamma_{q'}$. Then by the inductive hypothesis $q' \models \psi$, so $\langle Q, \tau, \nu \rangle, q \models F\psi$.

[$\Leftarrow$] Assume $\langle Q, \tau, \nu \rangle, q \models F\psi$. Then there is some $q' > q$ such that $q' \models \psi$ and by the inductive hypothesis $\psi \in \Gamma_{q'}$. Since q and q' are related by $\prec$ it follows that $F\psi \in \Gamma_q$. QED

It follows as before that $\mathbf{Q_{\Box FP}}$ is strongly complete with respect to $\langle \mathbb{Q}, \tau \rangle$. QED

3.5 Extensions To $\mathbb{R}$: The Temporal Case

So far we have seen that $\mathbb{Q}$ is complete with respect to the logics $\mathbf{Q_{FP}}$, **S4** and $\mathbf{Q_{\Box FP}}$. While this is interesting, $\mathbb{Q}$ is not a particularly rich topological structure. From a mathematical perspective, $\mathbb{R}$ is more interesting. And indeed extensions to $\mathbb{R}$ have been considered for the logics $\mathbf{Q_{FP}}$, **S4**, and $\mathbf{Q_{\Box FP}}$. Working indirectly through Kripke frames, the corresponding cases are found to be technical, difficult, and open, respectively.

As mentioned before, one impressive feature of the temporal construction method is the ease in which completeness on $\langle \mathbb{Q}, < \rangle$ is extended to completeness on $\langle \mathbb{R}, < \rangle$. Thus in considering topological extensions of $\langle \mathbb{Q}, \tau \rangle$ to $\langle \mathbb{R}, \tau \rangle$ for $\mathcal{L}$ and $\mathcal{L}^{\Box FP}$ using the construction method, it is useful to have the original $\mathcal{L}^{FP}$ proof in mind. The proof makes use of the following well known facts.

Fact 3.20 *$\langle \mathbb{R}, < \rangle$ is the unique complete linear ordering that has a countable dense subset isomorphic to $\langle \mathbb{Q}, < \rangle$.*

Fact 3.21 *If $\langle W, < \rangle$ is a dense unbounded linearly ordered set, then there exists a continuous unbounded linearly ordered set $\langle W' <' \rangle$ such that:*

- *$W \subset W'$ and $<$ and $<'$ agree on W*
- *W is dense in W'*

It follows that our constructed dense, unbounded, linear order $\langle Q, < \rangle$ constructed in Section 2.6 can be extended to a frame $\langle R, < \rangle \cong \langle \mathbb{R}, < \rangle$. Using our intuitions from first order logic as a guide, after extending $\langle Q, < \rangle$ to $\langle R, < \rangle$ we should associate every new point with a $\mathbf{Q_{FP}}$ maximally consistent set. Then we should extend our Henkin valuation to include these new points, and check that our new model $\langle R, < \rangle$ still satisfies $\Delta \cup \{\neg\chi\}$. However, in this case our first order intuitions fail us. This is one of the rare practical occurrences where the basic modal language is, in some respects, more expressive than its first order counterpart.

Theorem 3.22 *Every dense unbounded linear order is elementary equivalent.*

Let **DULO** be the first order theory containing axioms for density, unboundedness and linearity. Then by the above classical theorem:

$$\langle \mathbb{R}, < \rangle \models \varphi \Leftrightarrow \vdash_{\mathbf{DULO}} \varphi \Leftrightarrow \langle \mathbb{Q}, < \rangle \models \varphi$$

It follows immediately that continuity is not first order definable. However, continuity is definable in $\mathcal{L}^{FP}$. Let $A\psi$ denote $\psi \wedge G\psi \wedge H\psi$.

Proposition 3.23 $\langle W, < \rangle \models A(G\varphi \to PG\varphi) \to (G\varphi \to H\varphi) \Leftrightarrow \langle W, < \rangle$ *is continuous.*

Proof. [$\Rightarrow$] Assume $\langle W, < \rangle$ is not continuous. Then there is some bounded set $X \subseteq W$ such that $\sup X \notin W$. Let $\langle W, < \rangle$ have the valuation $\nu(p) = W/X$. Then for all y such that $y < \sup X$, $y \models F\neg p$ and for all y such that $y > \sup X$, $y \models Gp \wedge PGp$. Let z be a point greater than $\sup X$. Then $z \models A(Gp \to PGp)$ and $z \not\models Gp \to Hp$.

[$\Leftarrow$] Assume $\langle W, < \rangle$ is continuous and $\langle W, < \nu \rangle, w \models A(G\varphi \to PG\varphi)$. Furthermore, assume towards a contradiction that $w \models G\varphi \wedge P\neg\varphi$. Then the set $X = \{x \mid x < w \text{ and } x \models \neg\varphi\}$ is nonempty. Since W is continuous, $\sup X = w' \in W$. So $w' \models G\varphi \wedge \neg PG\varphi$. But since $w' \leq w$, this means $w \models P\neg(G\varphi \to PG\varphi) \vee \neg(G\varphi \to PG\varphi)$. So $w \not\models A(G\varphi \to PG\varphi)$, contrary to assumption. QED

It should be clear that with continuity we are already getting more than expected out of $\mathcal{L}^{FP}$ and that no additional properties of $\langle \mathbb{R}, < \rangle$ can be expressed. However, we still want to prove this fact. Let $\mathbf{R_{FP}}$ be the logic $\mathbf{Q_{FP}} + A(G\varphi \to PG\varphi) \to (G\varphi \to H\varphi)$.

Theorem 3.24 $\mathbf{R_{FP}}$ *is strongly complete with respect to* $\langle \mathbb{R}, < \rangle$.

Proof. Assume $\Delta \nvdash_{\mathbf{R_{FP}}} \chi$. First carry out the temporal construction procedure for $\langle Q, < \rangle$ as in Section 2.6, except this time replacing $\mathbf{Q_{FP}}$ maximally consistent sets with $\mathbf{R_{FP}}$ ones. The end result of the construction will be a countable, dense, unbounded linear order $\langle Q, < \rangle$ which satisfies $\Delta \cup \{\neg\chi\}$. Now extend $\langle Q, < \rangle$ to a structure $\langle R, < \rangle \cong \langle \mathbb{R}, < \rangle$. We know many new points will be added by this procedure, and we want to associate each of them with a $\mathbf{R_{FP}}$ maximally consistent set such that $\langle R, < \rangle$ still satisfies $\Delta \cup \{\neg\chi\}$.

An obvious candidate for the irrational newcomer $x \in R \backslash Q$ is Γ_x, an $\mathbf{R_{FP}}$ maximally consistent extension of:

$$\Gamma = \{\alpha \mid \exists q \in Q \text{ such that } q < x \text{ and } G\alpha \in \Gamma_q\} \cup \\ \{\beta \mid \exists q' \in Q \text{ such that } x < q' \text{ and } H\beta \in \Gamma_{q'}\}$$

In this case the obvious candidate turns out to be the correct one.

Lemma 3.25 Γ *is consistent.*

Proof. Assume not. Then there are $q_1, ...q_m < x$ with $G\alpha_i \in \Gamma_{q_i}$ for $1 \leq i \leq m$ and $q'_1, ..., q'_n > x$ with $H\beta_j \in \Gamma_{q'_j}$ for $1 \leq j \leq n$ such that

$$\vdash_{\mathbf{RFP}} \neg(\alpha_1 \wedge ...\alpha_m \wedge \beta_1 \wedge ... \wedge \beta_n)$$

However, since $\langle Q, < \rangle$ is dense, there must be some q^* such that $q_1, ...q_m < q^* < q'_1, ..., q'_n$ and since the $<$ ordering on $\langle Q, < \rangle$ respects $\prec$, $\alpha_1 \wedge ...\alpha_m \wedge \beta_1 \wedge ... \wedge \beta_n \in \Gamma_{q^*}$. QED

So we only need to check that extending the Henkin valuation to include the new points preserves the desired harmony between the syntactic and semantic.

Lemma 3.26 *For all $r \in R \backslash Q$, if $F\psi \in \Gamma_r$ then there is some $q \in Q$, $r < q$, such that $\psi \in \Gamma_q$.*

Proof. Assume $F\psi \in \Gamma_r$, $r \in R\backslash Q$, and for all $q'' \in Q$ such that $r < q''$, $\neg\psi \in \Gamma_{q''}$. Then we know from the properties of our constructed model $\langle Q, <, \nu \rangle$ that for all $q'' \in Q$ such that $r < q''$, $G\neg\psi \wedge PG\neg\psi \in \Gamma_{q''}$. Furthermore, we also know that for all $q^* \in Q$ such that $q^* < r$, $F\psi \in \Gamma_{q^*}$, for otherwise $G\neg\psi \in \Gamma_r$ by the definition of Γ_r. So for every $q \in Q$, either $PG\neg\psi \in \Gamma_q$ or $F\psi \in \Gamma_q$. It follows that for all $q \in Q$, $A(G\neg\psi \rightarrow PG\neg\psi) \in \Gamma_q$. Now let q' be a point in Q such that $q < q'$. From $A(G\neg\psi \rightarrow PG\neg\psi) \wedge G\neg\psi \in \Gamma_{q'}$ and the continuity axiom it follows that $H\neg\psi \in \Gamma_{q'}$. But this contradicts the fact that $F\psi \in \Gamma_{q^*}$ for all $q^* < r$ and $\langle Q, <, \nu \rangle$ obeys the $\prec$ ordering. QED

Lemma 3.27 $\langle R, <, \nu \rangle \models \varphi \Leftrightarrow \varphi \in \Gamma_r$.

Proof. By induction on the complexity of φ. We will only treat the case $\varphi = F\psi$.

[$\Leftarrow$] Assume $F\psi \in \Gamma_r$. Then by the above Lemma and the $\prec$ ordering on $\langle Q, < \rangle$, there is some $q \in Q$ such that $r < q$ and $\psi \in \Gamma_q$. By induction hypothesis, $q \models \psi$, so $\langle R, <, \nu \rangle, r \models F\psi$.

[$\Rightarrow$] Assume $F\psi \notin \Gamma_r$. It is not hard to check that $\neg\psi \in \Gamma_q$ for all $q \in Q$ such that $r < q$. So assume towards a contradiction that there is some $r' \in R\backslash Q$ such that $r < r'$ and $\psi \in \Gamma_{r'}$. Since $\langle Q, < \rangle$ is a dense subset of $\langle R, < \rangle$, there exists a point $q' \in Q$ such that $r < q' < r'$. However, we know $G\neg\psi \in \Gamma_{q'}$ and thus by the definition of $\Gamma_{r'}$ that $\neg\psi \in \Gamma_{r'}$. This contradicts our assumption. So for all $r'' \in R$ such that $r < r''$, $\neg\psi \in \Gamma_{r''}$. By the inductive hypothesis all $r'' \models \neg\psi$ and $\langle R, < \nu \rangle, r \not\models F\psi$. QED

Since $\Delta \cup \{\neg\chi\} \subseteq \Gamma_{q\#}$, it readily follows that $\mathbf{R_{FP}}$ is strongly complete with respect to $\langle \mathbb{R}, < \rangle$.

3.6 Further Extensions to $\mathbb{R}$

So we have seen that our constructed $\mathcal{L}^{FP}$ model $\langle Q, <, \nu\rangle$ can be extended to a new model $\langle \mathbb{R}, <, \nu\rangle$ while preserving the truth of $\mathcal{L}^{FP}$ formulas on $\langle Q, <, \nu\rangle$. Now let us consider this procedure for $\langle Q, \tau, \nu\rangle$ and $\langle \mathbb{R}, \tau, \nu\rangle$ in the languages $\mathcal{L}$ and $\mathcal{L}^{\Box FP}$.

Let us first consider extending $\langle Q, \tau, \nu\rangle$ to $\langle \mathbb{R}, \tau, \nu\rangle$ in $\mathcal{L}$. Unlike the temporal case, we know that **S4** is the logic of both $\langle \mathbb{Q}, \tau\rangle$ and $\langle \mathbb{R}, \tau\rangle$, so we only need to fill in $\langle Q, \tau\rangle$ with **S4** maximally sets. Otherwise, let us proceed as above.

Assume $\Delta \nvdash_{\mathbf{S4}} \chi$ and let $\langle Q, \tau, \nu\rangle$ be our constructed topological model which satisfies $\Delta \cup \{\neg\chi\}$. Extend $\langle Q, <\rangle$ to a structure $\langle R, <\rangle \cong \langle \mathbb{R}, <\rangle$. Now associate each $x \in R\backslash Q$ with Γ_x, an **S4** maximally consistent extension of:

$$\begin{aligned}\Gamma = \{\Box\psi \mid \exists q_1, q_2 \in Q \text{ such that } q_1 < x < q_2 \text{ and } \forall q \in Q \text{ such that } q_1 < q < q_2,\ \psi \in \Gamma_q\} \cup \\ \{\Diamond\varphi \mid \forall q_3, q_4 \in Q \text{ such that } q_3 < x < q_4,\ \exists q' \in Q \text{ such that } q_3 < q' < q_4 \text{ and } \varphi \in \Gamma_{q'}\}\end{aligned}$$

Lemma 3.28 *$\langle R, \tau, \nu\rangle, r \models \varphi \Leftrightarrow \varphi \in \Gamma_r$.*

Proof. By induction on the complexity of φ. We will only treat the case $\varphi = \Box\psi$.

[$\Leftarrow$] Assume $\Box\psi \in \Gamma_r$. We claim that $\exists q_1, q_2 \in Q$ such that $q_1 < r < q_2$ and $\forall q \in Q$, $q_1 < q < q_2$, $\psi \in \Gamma_q$. If $r \in Q$, this holds by the construction of $\langle Q, \tau\rangle$. And if $r \in R\backslash Q$ this follows from the definition of Γ_r (since $\Diamond\neg\psi \notin \Gamma_r$).

Now assume towards a contradiction that there is some $r' \in R\backslash Q$ such that $q_1 < r' < q_2$ and $\neg\psi \in \Gamma_{r'}$. Then by maximal consistency $\Box\psi \notin \Gamma_{r'}$. It then follows from the definition of $\Gamma_{r'}$ that for all $q_3, q_4 \in Q$ such that $q_3 < r' < q_4$, there exists some $q' \in Q$, $q_3 < q' < q_4$, and $\neg\psi \in \Gamma_{q'}$. But this means there is a $q' \in Q$ such that $q_1 < q' < q_2$ and $\neg\psi \in \Gamma_{q'}$, contradicting the previous paragraph.

So for all $x \in R$ such that $q_1 < x < q_2$, $\psi \in \Gamma_x$. By letting $U = \{y \mid q_1 < x < q_2\}$ and the inductive hypothesis we get that $\langle R, \tau, \nu\rangle, r \models \Box\psi$.

[$\Rightarrow$] Assume $\Box\psi \notin \Gamma_r$. We claim that $\forall q_3, q_4 \in Q$ such that $q_3 < r < q_4$, $\exists q' \in Q$, $q_3 < q' < q_4$, and $\neg\psi \in \Gamma_{q'}$. If $r \in Q$ this holds by the construction of $\langle Q, <\rangle$ and if $r \in R\backslash Q$ then this holds by definition of Γ_r (since $\Box\psi \notin \Gamma_r$). So it follows from the inductive hypothesis that $\langle R, \tau, \nu\rangle, r \not\models \Box\psi$. QED

In order to show our constructed model $\langle Q, \tau, \nu\rangle$ can be extended to $\langle \mathbb{R}, \tau, \nu\rangle$ while preserving the truth of $\mathcal{L}$ formulas, it only remains to be shown that Γ is consistent. Unfortunately, this can not be done. For unlike the previous cases, our natural syntactic candidate to associate with a new point may be inconsistent. For example, assume $x \in R\backslash Q$ and $\langle Q, \tau\rangle$ has the valuation $\nu(p) = \{q \mid q < x\}$. Then according to the definition of Γ_x, $\Diamond p$ and $\Diamond\neg p$ are in

Γ_x. Furthermore, for every $q \in Q$, either $q \models \Box p$ or $q \models \Box\neg p$, so $\Box(\Box p \vee \Box\neg p) \in \Gamma_x$. But $\vdash_{\mathbf{S4}} \Box(\Box p \vee \Box\neg p) \to \neg(\Diamond p \wedge \Diamond\neg p)$ and this entails that $\neg(\Diamond p \wedge \Diamond\neg p) \wedge (\Diamond p \wedge \Diamond\neg p) \in \Gamma_x$

So already things are worse than the temporal case, but one might still hope that they are better than using the model theoretic approach. As mentioned before, the completeness of **S4** with respect to $\langle \mathbb{R}, \tau \rangle$ can be established model theoretically, but with tricky diversions through finite transitive and reflexive trees and subsets of the Cantor Space. As we have seen, using the construction method we have some control over how formulas are satisfied on $\langle Q, \tau \rangle$ and perhaps we can make use this control when jumping to $\langle \mathbb{R}, \tau \rangle$.

Looking at the failure of Γ, it is clear what property we want our constructed model $\langle Q, \tau, \nu \rangle$ to possess: if there are rational $\varphi_1, .., \varphi_n$ sequences approaching an irrational point x, then there is a rational sequence $\varphi_1 \wedge ... \wedge \varphi_n$ approaching x. We can then show Γ is consistent. For assume not. Then there are formulas $\Box\psi, \Diamond\varphi_1, ..., \Diamond\varphi_n$ such that

- $\exists q_1, q_2 \in Q$, $q_1 < x < q_2$ and for all $q' \in Q$, $q_1 < q' < q_2$, $\psi \in \Gamma_{q'}$
- $\forall q_3, q_4 \in Q$, $q_3 < x < q_4$, there exists some $q'_i \in Q$ such that $q_3 < q'_i < q_4$ and $\varphi_i \in \Gamma_{q'_i}$, for $1 \leq i \leq n$
- $\vdash_{\mathbf{S4}} \Box\psi \to \neg(\Diamond\varphi_1 \wedge ... \wedge \Diamond\varphi_n)$

But as there are rational sequences $\varphi_1 ... \varphi_n$ approaching x, by assumption there is a rational sequence $\Diamond(\varphi_1 \wedge ... \wedge \varphi_n)$ approaching x. This implies there is some rational point q' in between q_1 and q_2 such that: $\Box\psi, \Diamond\varphi_1, ..., \Diamond\varphi_n \in \Gamma_{q'}$.

However, after some reflection, it is clear that the desired property of $\langle Q, \tau, \nu \rangle$ is (at least) very difficult to ensure. For while we have control on how formulas are satisfied on the rationals, this tells us very little about how they will line up on the irrationals. There is no natural way to force a construction of $\langle Q, \tau, \nu \rangle$ to tell us what kinds of sequences are being created towards irrational points.

The underlying difficulty of extending $\langle Q, \tau, \nu \rangle$ to $\langle \mathbb{R}, \tau, \nu \rangle$ in $\mathcal{L}$ is that there is no way of expressing any topological relation between the points in the language. We can see this defect clearly when comparing our failed attempt in $\langle \mathbb{R}, \tau \rangle$ to the successful temporal proof for $\langle \mathbb{R}, < \rangle$. In the the temporal construction, each state of affairs expressible in $\mathbf{R_{FP}}$ (i.e. each maximal consistent set) is related to every other state of affairs in a way that respects order structure on $\langle \mathbb{Q}, < \rangle$. Furthermore, each description of an $\mathbf{R_{FP}}$ state of affairs existing in $\langle \mathbb{Q}, < \rangle$ says that if there is a bounded φ sequence, then this φ sequence has a least upper bound respecting the $\prec$ ordering. Thus when extending $\langle \mathbb{Q}, < \rangle$ to $\langle \mathbb{R}, < \rangle$, we know we can associate an irrational point x with its natural syntactic candidate Γ_x, since this state of affairs has already been described to exist in $\langle \mathbb{Q}, < \rangle$.

The temporal construction for $\langle \mathbb{R}, < \rangle$ goes so smoothly because there is a kind of expressive harmony between what can be said to occur in the language and the structure $\langle \mathbb{R}, < \rangle$.

Clearly this is not the case in $\langle \mathbb{R}, \tau \rangle$, where we are working with a fairly rich topological structure and a language that can express almost no topological properties of the structure. In this sense, one might think that the seemingly more difficult construction of $\langle \mathbb{R}, \tau \rangle$ in $\mathcal{L}^{\Box FP}$ would fare better, since additional topological properties such as connectedness are expressible in $\mathcal{L}^{\Box FP}$. However, $\mathcal{L}^{\Box FP}$ is still to weak of a language for things to go smoothly. In order to successfully extend $\langle Q, \tau \rangle$ to $\langle \mathbb{R}, \tau \rangle$ we need to be able to express in the language when a particular $\Box$ stretch ends, which is a much stronger property than connectedness.

3.7 $\mathbf{Q_{SU}}$ on $\mathbb{Q}$

Perhaps at some point in the preceding discussion, the reader noted the resemblance between $\mathcal{L}^{\Box FP}$ and the popular temporal Since and Until language $\mathcal{L}^{SU}$. Indeed, $\mathcal{L}^{SU}$ has all the expressive power of $\mathcal{L}^{\Box FP}$, plus the additional property we desired in our attempted constructions of $\langle \mathbb{R}, \tau \rangle$: the ability to express up to what point a $\Box$ stretch holds. In this section we will examine how this additional property simplifies completeness proofs using the construction method.

The Since/Until language $\mathcal{L}^{SU}$ contains two binary operators U and S, and is built up in the usual way from a set of propositional letters and these two operators. Intuitively, the formulas $U(\varphi, \psi)$ and $S(\varphi, \psi)$ are supposed to mean 'until φ is true, ψ holds' and 'since φ was true, ψ has held'. More formally, given a model $M = \langle W, <, \nu \rangle$, where $<$ is a partial order, we define when an $\mathcal{L}^{SU}$ formula γ is true at a point w as follows:

- $w \models U(\varphi, \psi)$ iff $\exists w'' > w$ such that $w'' \models \varphi$ and $\forall w'$, $w < w' < w''$, $w' \models \psi$
- $w \models S(\varphi, \psi)$ iff $\exists w'' < w$ such that $w'' \models \varphi$ and $\forall w'$, $w > w' > w''$, $w' \models \psi$

As the reader may have noticed, $\mathcal{L}^{SU}$ is expressively complete over $\mathcal{L}^{\Box FP}$. We can define $\Box$, F and P as follows:

$$F\psi := U(\psi, \top)$$

$$P\psi := S(\psi, \top)$$

$$\Box\psi := S(\top, \psi) \wedge \psi \wedge U(\top, \psi)$$

However, in $\mathcal{L}^{SU}$ it is also possible to say additional things like "until we are in a $\Box\varphi$ stretch, we are in a $\Box\psi$ stretch" ($U(\Box\varphi, \Box\psi)$).

$\mathcal{L}^{SU}$ was first introduced by Kamp in [12] and, as may be guessed from its name, the original interest was temporal rather than topological. After the first most general completeness results for partial orders were discovered, attention naturally turned to mathematical structures such as $\langle \mathbb{N}, < \rangle$, $\langle \mathbb{Q}, < \rangle$ and $\langle \mathbb{R}, < \rangle$. Kamp made progress in these problems but never published his results, as his proofs were regarded as very unwieldy. The first published results in the area are found in Burgess [7], where axiomatizations of $\langle \mathbb{N}, < \rangle$ and $\langle \mathbb{Q}, < \rangle$ are given.

However, even Burgess proofs are not very conspicuous and make crucial use high powered technical results, such as Rabin's theorem. Let us see how the construction method can be used to simplify matters.

Let $\mathbf{Q_{SU}}$ be the logic containing axioms:

1a) $G(\varphi \to \psi) \to (U(\varphi, \gamma) \to U(\psi, \gamma))$
2a) $G(\varphi \to \psi) \to (U(\gamma, \varphi) \to U(\gamma, \psi))$
3a) $\varphi \wedge U(\psi, \gamma) \to U(\psi \wedge S(\varphi, \gamma), \gamma)$
4a) $U(\varphi, \psi) \wedge \neg U(\gamma, \psi) \to U(\varphi \wedge \neg\gamma, \psi)$
5a) $U(\varphi, \psi) \wedge \neg U(\varphi, \gamma) \to U(\psi \wedge \neg\gamma, \psi)$
6a) $U(\varphi, \psi) \to U(\varphi, \psi \wedge U(\varphi, \psi))$
7a) $U(\psi \wedge U(\varphi, \psi), \psi) \to U(\varphi, \psi)$
8a)$U(\varphi, \psi) \wedge U(\gamma, \zeta) \to U(\gamma \wedge \psi \wedge U(\varphi, \psi), \psi \wedge \zeta) \vee U(\varphi \wedge \gamma, \psi \wedge \zeta) \vee U(\varphi \wedge \zeta \wedge U(\gamma, \zeta), \psi \wedge \zeta)$
9a) $U(\top, \top)$
10 $\neg U(\top, \bot)$

and 1b) $-$ 9b), where all occurrences of S are replaced by U in 1a) $-$ 9a) , and vice versa.

Theorem 3.29 $\mathbf{Q_{SU}}$ *is strongly complete with respect to* $\langle \mathbb{Q}, < \rangle$.

Proof. Assume $\Delta \nvdash_{\mathbf{Q_{SU}}} \chi$. We will construct a frame $\langle Q, < \rangle \cong \langle \mathbb{Q}, < \rangle$ in stages which satisfies $\Delta \cup \{\neg\chi\}$.

A stage n consists of:

1) a finite set $Q_n = \{q_0, q_1 ..., q_k\}$
2) a linear ordering $<_n$ on Q_n
3) an assignment of a $\mathbf{Q_{SU}}$ maximally consistent set Γ_{q_i} to each $q_i \in Q_n$

Definition 3.30 We say $\Gamma_q \prec_\varphi \Gamma_{q'}$ if $\psi \in \Gamma_{q'} \Rightarrow U(\psi, \varphi) \in \Gamma_q$ $\lhd$

Note the simplicity of the $\prec$ relation compared to $\prec_{\Box\psi}$ in $\mathcal{L}^{\Box FP}$. By adding slightly more expressive power, we no longer have to force the temporal operators G and H to interact correctly with $\Box$. The language now takes care of this for us.

Proposition 3.31 $\Gamma_q \prec_\varphi \Gamma_{q'}$ *and* $\Gamma_q \prec_\psi \Gamma_{q'}$ $\Rightarrow \Gamma_q \prec_{\varphi\wedge\psi} \Gamma_{q'}$

Proof. Assume not. Then there is a $\gamma \in \Gamma_{q'}$ such that $U(\gamma, \varphi) \in \Gamma_q$, $U(\gamma, \psi) \in \Gamma_q$ and $\neg U(\gamma, \varphi \wedge \psi) \in \Gamma_q$. By axiom 8$a$)

i) $U(\gamma, \varphi \wedge \psi) \in \Gamma_q$ or

ii) $U(\gamma \wedge \varphi \wedge U(\varphi, \psi), \varphi \wedge \psi) \in \Gamma_q$ or

iii) $U(\gamma \wedge \psi \wedge U(\gamma, \psi), \varphi \wedge \psi) \in \Gamma_q$

i) contradicts the assumption that $\neg U(\gamma, \varphi \wedge \psi) \in \Gamma_q$ straight away, so assume either ii) or iii) hold instead. It is a basic fact that $\vdash_{\mathbf{Q_{SU}}} U(\alpha \wedge \beta, \chi) \leftrightarrow U(\alpha, \chi) \wedge U(\beta, \chi)$, so it follows again that $U(\gamma, \varphi \wedge \psi) \in \Gamma_q$, contradicting the consistency of Γ_q. QED

Similar to previous constructions, when adding a new point q' in between points q and q'' such that $\Gamma_q \prec_\varphi \Gamma_{q''}$, we want to take care that $\varphi \in \Gamma_{q'}$ and $\Gamma_q \prec_\varphi \Gamma_{q'} \prec_\varphi \Gamma_{q''}$. In other words, we want to be sure that

$$\{\varphi\} \cup \{\neg\psi \mid \neg U(\psi, \varphi) \in \Gamma_q\} \cup \{U(\gamma, \varphi) \mid \gamma \in \Gamma_{q''}\} \subseteq \Gamma_{q'}$$

In order to show that this is possible it is useful to prove the following Proposition.

Proposition 3.32 *Assume* $\Gamma_q \prec_\varphi \Gamma_{q''}$, $\neg U(\psi, \varphi) \in \Gamma_q$ *and* $\gamma \in \Gamma_{q''}$. *Then* $\Gamma_q \prec_{\varphi \wedge \neg\psi \wedge U(\gamma, \varphi)} \Gamma_{q''}$.

Proof. Assume $\alpha \in \Gamma_{q''}$. We want to show $U(\alpha, \varphi \wedge \neg\psi \wedge U(\gamma, \varphi)) \in \Gamma_q$.

Since $\Gamma_q \prec_\varphi \Gamma_{q''}$, we know that $U(\alpha, \varphi) \in \Gamma_q$.

Now we will show that $U(\alpha, \neg\psi) \in \Gamma_q$. Assume towards a contradiction that $\neg U(\alpha, \neg\psi) \in \Gamma_q$. From $\alpha \in \Gamma_{q''}$ and $\Gamma_q \prec_\varphi \Gamma_{q''}$, we know that $U(\alpha, \varphi) \in \Gamma_q$. Then it follows from axiom 5a) that $U(\varphi \wedge \psi, \varphi) \in \Gamma_q$. But this implies $U(\psi, \varphi) \in \Gamma_q$, contradicting our assumption.

Finally we will show $U(\alpha, U(\gamma, \varphi)) \in \Gamma_q$. Assume towards a contradiction that $\neg U(\alpha, U(\gamma, \varphi)) \in \Gamma_q$. From $\alpha, \gamma \in \Gamma_{q''}$ and $\Gamma_q \prec_\varphi \Gamma_{q''}$, we know that $U(\alpha \wedge \gamma, \varphi) \in \Gamma_q$. Then from axiom 6$a$) it follows that $U(\alpha \wedge \gamma, \varphi \wedge U(\alpha \wedge \gamma, \varphi)) \in \Gamma_q$. But this implies that $U(\alpha, U(\gamma, \varphi)) \in \Gamma_q$, again contradicting the consistency of Γ_q.

It follows from the above and Proposition 3.31 that $U(\alpha, \varphi \wedge \neg\psi \wedge U(\gamma, \varphi)) \in \Gamma_q$. QED

We are now ready to begin the construction of $\langle Q, < \rangle$. Let $\varphi_0\varphi_1\varphi_2\ldots$ be an enumeration of all U, $\neg U$, S and $\neg S$ formulas, in which each formula is repeated infinitely many times.

- **Stage 0**: Let $Q_0 = \{q^\#\}$ and associate $q^\#$ with $\Gamma_{q^\#}$, a $\mathbf{Q_{SU}}$ maximally consistent extension of $\Delta \cup \{\neg\chi\}$.
- **Stage 4n + 1**: Let $\neg U(\varphi, \psi)$ be the next $\neg U$ formula in the enumeration. For all $q \in Q_{4n}$, if
  - i) $\neg U(\varphi, \psi) \in \Gamma_q$ and
  - ii) there is a $u' > q$ such that $\varphi \in \Gamma_{u'}$ and

iii) for all $t \in Q_{4n}$ such that $q < t < u'$, $\psi \in \Gamma_t$

then let u be the least point $q < u$ with $\varphi \in \Gamma_u$ and let r be the greatest point $q \leq r < u$ such that $\neg U(\varphi, \psi) \in \Gamma_r$. Let s be the immediate successor of r in $<_{4n}$. Add a new point q' in between r and s and associate q' with $\Gamma_{q'}$, a $\mathbf{Q_{SU}}$ maximally consistent extension of:

$$\Gamma = \{\neg\psi\} \cup \{\gamma \mid \Gamma_r \prec_\gamma \Gamma_s\} \cup$$
$$\{\neg\alpha \mid \Gamma_r \prec_\gamma \Gamma_s \text{ and } \neg U(\alpha, \gamma) \in \Gamma_r\} \cup \{U(\beta, \gamma) \mid \Gamma_r \prec_\gamma \Gamma_s \text{ and } \beta \in \Gamma_s\}$$

Proposition 3.33 Γ *is consistent.*

Proof. Assume not. Then there are formulas $\gamma_1..., \gamma_k, \neg U(\alpha_1, \gamma_1'), ..., \neg U(\alpha_l, \gamma_l'), \gamma_1'', ..., \gamma_m'' \in \Gamma_r$ and $\beta_1...\beta_m \in \Gamma_s$ such that

$\vdash_{\mathbf{Q_{SU}}} \neg(\neg\psi \wedge \gamma_1 \wedge ... \wedge \gamma_k \wedge \neg\alpha_1 \wedge ... \wedge \neg\alpha_l \wedge U(\beta_1, \gamma_1'') \wedge ... \wedge U(\beta_m, \gamma_m''))$
$\vdash_{\mathbf{Q_{SU}}} G\neg(\neg\psi \wedge \gamma_1 \wedge ... \wedge \gamma_k \wedge \neg\alpha_1 \wedge ... \wedge \neg\alpha_l \wedge U(\beta_1, \gamma_1'') \wedge ... \wedge U(\beta_m, \gamma_m''))$
$\vdash_{\mathbf{Q_{SU}}} \neg F(\neg\psi \wedge \gamma_1 \wedge ... \wedge \gamma_k \wedge \neg\alpha_1 \wedge ... \wedge \neg\alpha_l \wedge U(\beta_1, \gamma_1'') \wedge ... \wedge U(\beta_m, \gamma_m''))$

From now on let $\zeta = \gamma_1 \wedge ... \wedge \gamma_k \wedge \neg\alpha_1 \wedge ... \wedge \neg\alpha_l \wedge U(\beta_1, \gamma_1'') \wedge ... \wedge U(\beta_m, \gamma_m'')$.

We know from assumptions i)-iii) above that either:

1) $\Gamma_s = \Gamma_u$ and $\varphi \in \Gamma_s$ or

2) $\Gamma_s \neq \Gamma_u$ and $\psi \wedge U(\varphi, \psi) \in \Gamma_s$

First assume 1) holds. Then since $\varphi \in \Gamma_s$, it follows from Propositions 3.31 and 3.32 that $U(\varphi, \zeta) \in \Gamma_r$. Furthermore, since $\neg U(\varphi, \psi) \in \Gamma_r$, it follows from axiom $5a$) that $U(\neg\psi \wedge \zeta, \zeta) \in \Gamma_r$. But this implies $F(\neg\psi \wedge \zeta) \in \Gamma_r$, contradicting the consistency of Γ_r.

So assume 2) holds instead. By contraposition of axiom $7a$) we have that $\neg U(\varphi, \psi) \rightarrow \neg U(\psi \wedge U(\varphi, \psi), \psi)$. Since $\neg U(\varphi, \psi) \in \Gamma_r$, $\neg U(\psi \wedge U(\varphi, \psi), \psi) \in \Gamma_r$. Furthermore, since $\psi \wedge U(\varphi, \psi) \in \Gamma_s$, by Propositions 3.31 and 3.32 we have that $U(\psi \wedge U(\varphi, \psi), \zeta) \in \Gamma_r$. So by axiom $5a$), $U(\neg\psi \wedge \zeta, \zeta) \in \Gamma_r$. But this implies $F(\neg\psi \wedge \zeta) \in \Gamma_r$, again contradicting the consistency of Γ_r. QED

- **Stage 4n + 2**: Let $U(\varphi, \psi)$ be the next U formula in the enumeration. For all $q \in Q_{4n+1}$ if $U(\varphi, \psi) \in \Gamma_q$ and there is no $w > q$ such that:

  - $\varphi \in \Gamma_w$
  - $\forall r \forall s$, $q \leq r < s \leq w$, $\Gamma_r \prec_\psi \Gamma_s$
  - $\forall q'$, $q < q' < w$, $\psi \in \Gamma_{q'}$

then we need to add a new point somewhere after q such that all of the above properties hold at that point. Let t be the greatest element in the $<_{4n+1}$ ordering, $q \leq t$, such that

- $\forall r \; q \leq r \leq t$, $U(\varphi, \psi) \in \Gamma_r$
- $\forall r \forall s \; q \leq r < s \leq t$, $\Gamma_r \prec_\psi \Gamma_s$
- $\forall r \; q < r \leq t$, $\psi \in \Gamma_r$

Add a new point u immediately after t. Let v be the immediate successor of t in Q_{4n+1}, provided one exists. Associate u with Γ_u, a $\mathbf{Q_{SU}}$ maximally consistent extension of:

$$\Gamma' = \{\varphi\} \cup \{\neg\alpha \mid \neg U(\alpha, \psi) \in \Gamma_t\} \cup$$
$$\cup \{\neg\beta \mid \Gamma_t \prec_\gamma \Gamma_v \text{ and } \neg U(\beta, \gamma) \in \Gamma_t\} \cup \{U(\xi, \gamma) \mid \Gamma_t \prec_\gamma \Gamma_v \text{ and } \xi \in \Gamma_v\}$$

Proposition 3.34 Γ' *is consistent*

Proof. Assume not. Then there are $\neg U(\alpha_1, \psi), ..., \neg U(\alpha_j, \psi), \gamma_1, ..., \gamma_k, \neg U(\beta_1, \gamma'_1), ..., \neg U(\beta_l, \gamma'_l)$, $\gamma''_1, ... \gamma''_m \in \Gamma_t$ and $\xi_1 .... \xi_m \in \Gamma_v$ such that:

$\vdash_{\mathbf{Q_{SU}}} \neg(\varphi \wedge \neg\alpha_1 \wedge ... \wedge \neg\alpha_j \wedge \gamma_1 \wedge ... \wedge \gamma_k \wedge \neg\beta_1 \wedge ... \wedge \neg\beta_l \wedge U(\xi_1, \gamma''_1) \wedge ... \wedge U(\xi_m, \gamma''_m))$
$\vdash_{\mathbf{Q_{SU}}} G\neg(\varphi \wedge \neg\alpha_1 \wedge ... \wedge \neg\alpha_j \wedge \gamma_1 \wedge ... \wedge \gamma_k \wedge \neg\beta_1 \wedge ... \wedge \neg\beta_l \wedge U(\xi_1, \gamma''_1) \wedge ... \wedge U(\xi_m, \gamma''_m))$

Let $\zeta = \gamma_1 \wedge ... \wedge \gamma_k \wedge \neg\beta_1 \wedge ... \wedge \neg\beta_l \wedge U(\xi_1, \gamma''_1) \wedge ... \wedge U(\xi_m, \gamma''_m)$.

We know that the point t was chosen for one of the following reasons:

i) $\Gamma_t \not\prec_\psi \Gamma_v$

ii) $\Gamma_t \prec_\psi \Gamma_v$ but $\neg\varphi \wedge \neg U(\varphi, \psi) \in \Gamma_v$

iii) $\Gamma_t \prec_\psi \Gamma_v$ but $\neg\varphi \wedge \neg\psi \in \Gamma_v$

First assume i) holds. Then there is some $\chi \in \Gamma_v$ such that $\neg U(\chi, \psi) \in \Gamma_t$. By Propositions 3.31 and 3.32, it follows that $U(\chi, \zeta) \in \Gamma_t$ and thus by axiom 5a), we have that $U(\zeta \wedge \neg\psi, \zeta) \in \Gamma_t$. Furthermore, since $U(\varphi, \psi), \neg U(\alpha_1, \psi), ..., \neg U(\alpha_j, \psi) \in \Gamma_t$, by repeated use of axiom 4a) we have that $U(\varphi \wedge \alpha_1 \wedge ... \wedge \alpha_n, \psi) \in \Gamma_t$. Putting these facts together, it follows from axiom 8a) that:

a) $U(\zeta \wedge \neg\psi \wedge \psi \wedge U(\varphi \wedge \neg\alpha_1 \wedge ... \wedge \neg\alpha_j, \psi), \psi \wedge \zeta) \in \Gamma_t$ or

b) $U(\varphi \wedge \neg\alpha_1 \wedge ... \wedge \neg\alpha_j \wedge \zeta \wedge \neg\psi, \psi \wedge \zeta) \in \Gamma_t$ or

c) $U(\varphi \wedge \neg\alpha_1 \wedge ... \wedge \neg\alpha_j \wedge \zeta \wedge U(\zeta \wedge \neg\psi, \zeta), \psi \wedge \zeta) \in \Gamma_t$

Choice $a)$ straightforwardly contradicts the consistency of Γ_t. However, both $b)$ and $c)$ imply that $F(\varphi \wedge \neg\alpha_1 \wedge ... \wedge \neg\alpha_j \wedge \zeta) \in \Gamma_t$, which also contradicts the consistency of Γ_t.

So assume ii) holds. By Propositions 3.31 and 3.32, it follows that $U(\neg\varphi \wedge \neg U(\varphi, \psi), \zeta) \in \Gamma_t$. As before, from $U(\varphi, \psi), \neg U(\alpha_1, \psi), ..., \neg U(\alpha_j, \psi) \in \Gamma_t$ we have $U(\varphi \wedge \neg\alpha_1 \wedge ... \wedge \neg\alpha_j, \psi) \in \Gamma_t$. Then by axiom $8a)$

a) $U(\varphi \wedge \neg\alpha_1 \wedge ... \wedge \neg\alpha_j \wedge \zeta \wedge U(\neg\varphi \wedge \neg U(\varphi, \psi), \zeta), \psi \wedge \zeta) \in \Gamma_t$ or

b) $U(\varphi \wedge \neg\alpha_1 \wedge ... \wedge \neg\alpha_j \wedge \neg\varphi \wedge \neg U(\varphi\wedge, \psi), \psi \wedge \zeta) \in \Gamma_t$ or

c) $U(\neg\varphi \wedge \neg U(\varphi, \psi) \wedge \psi \wedge U(\varphi \wedge \neg\alpha_1 \wedge ... \wedge \neg\alpha_j, \psi), \psi \wedge \zeta) \in \Gamma_t$

Clearly $b)$ and $c)$ contradict the consistency of Γ_t. So only $a)$ is possible. However by assumption $F(\varphi \wedge \neg\alpha_1 \wedge ... \wedge \neg\alpha_j \wedge \zeta)$ is inconsistent.

Finally assume iii) holds. It follows from Propositions 3.31 and 3.32 that $U(\neg\psi \wedge \neg\varphi, \zeta) \in \Gamma_t$. As above, we have that $U(\varphi \wedge \neg\alpha_1 \wedge ... \wedge \neg\alpha_j, \psi) \in \Gamma_t$. Then it follows from axiom $8a)$ that

a) $U(\varphi \wedge \neg\alpha_1 \wedge ... \wedge \neg\alpha_j \wedge \neg\psi \wedge \neg\varphi, \psi \wedge \zeta) \in \Gamma_t$ or

b) $U(\neg\psi \wedge \neg\varphi \wedge \psi \wedge U(\varphi \wedge \neg\alpha_1 \wedge ... \wedge \neg\alpha_j, \psi), \psi \wedge \zeta) \in \Gamma_t$ or

c) $U(\varphi \wedge \neg\alpha_1 \wedge ... \wedge \neg\alpha_j \wedge \zeta \wedge U(\neg\psi \wedge \neg\varphi, \zeta), \psi \wedge \zeta) \in \Gamma_t$

$a)$ and $b)$ contradict the consistency of Γ_t directly, and $c)$ implies $F(\varphi \wedge \neg\alpha_1 \wedge ... \wedge \neg\alpha_j \wedge \zeta)$. QED

Definition 3.35 We say $\Gamma_q \succ_\varphi \Gamma_{q'}$ if $\psi \in \Gamma_q \Rightarrow S(\psi, \varphi) \in \Gamma_{q'}$. $\lhd$

- **Stages $\mathbf{4n+3}$** and **$\mathbf{4n+4}$** treat the next $\neg S$ and S formulas in the enumeration. They are mirror images of $4n+1$ and $4n+2$, using the $\succ$ relation in place of $\prec$.

Proposition 3.36 $\Gamma_q \prec_\varphi \Gamma_{q'} \Leftrightarrow \Gamma_q \succ_\varphi \Gamma_{q'}$

Proof. [$\Rightarrow$] Assume towards a contradiction that $\gamma \in \Gamma_{q'} \Rightarrow U(\gamma, \psi) \in \Gamma_q$ and there is some $\varphi \in \Gamma_q$ such that $\neg S(\varphi, \psi) \in \Gamma_{q'}$. It follows that $\varphi \wedge U(\neg S(\varphi, \psi), \psi) \in \Gamma_q$. But from axiom $3a)$ we know that $\psi \wedge U(\neg S(\varphi, \psi), \psi) \to U(\neg S(\varphi, \psi) \wedge S(\varphi, \psi), \psi)$, so $U(\neg S(\varphi, \psi) \wedge S(\varphi, \psi), \psi) \in \Gamma_q$, which contradicts the consistency of Γ_q. The converse direction is symmetric. QED

By the above Proposition, it follows that if a new point q' is added between points q and q'' at S or $\neg S$ stages that:

$$\{\varphi \mid \Gamma_q \prec_\varphi \Gamma_{q''}\} \cup \{\neg\psi \mid \Gamma_q \prec_\varphi \Gamma_{q''} \text{ and } \neg U(\psi, \varphi) \in \Gamma_q\} \cup$$
$$\{U(\gamma, \varphi) \mid \Gamma_q \prec_\varphi \Gamma_{q''} \text{ and } \varphi \in \Gamma_{q''}\} \subseteq \Gamma_{q'}$$

To finish the construction, let $\langle Q, < \rangle = \bigcup \{ \langle Q_n, <_n \rangle \mid n \in \omega \}$. Again it is not hard to check that $\langle Q, < \rangle$ is a countable dense unbounded linear order; the axioms $S(\top, \top)$ and $U(\top, \top)$ ensure unboundedness and $\neg U(\top, \bot)$ ensures density. So let us move straight way to showing that $\langle Q, < \rangle$ satisfies $\Delta \cup \{\neg\chi\}$.

Proposition 3.37 $\langle Q, <, \nu \rangle, q \models \gamma \Leftrightarrow \gamma \in \Gamma_q$.

Proof. By induction on the complexity of γ. We will only treat the case $\gamma = U(\varphi, \psi)$.

[$\Rightarrow$] We will argue contrapositively. Assume $\neg U(\varphi, \psi) \in \Gamma_q$. If $\varphi \notin \Gamma_r$ for all $r > q$, then by the inductive hypothesis $r \not\models \varphi$ for all $r > q$. In this case, clearly $\langle Q, <, \nu \rangle, q \models \neg U(\varphi, \psi)$. So assume there is an $s > q$ such that $\varphi \in \Gamma_s$. Then there is some stage k such that $q, s \in Q_{k-1}$ and $\neg U(\varphi, \psi)$ is being treated. If at stage k there is no q'', $q < q'' < s$, such that $\neg\psi \in \Gamma_{q''}$, then a new point q' is added such that $q < q' < s$ and $\neg\psi \in \Gamma_{q'}$. By inductive hypothesis $s \models \varphi$ and $q' \models \neg\psi$, so $\langle Q, <, \nu \rangle, q \models \neg U(\varphi, \psi)$.

[$\Leftarrow$] Assume $U(\varphi, \psi) \in \Gamma_q$. By construction, there is a stage j with $t \in Q_j$, $q < t$, such that

i) $\varphi \in \Gamma_t$

ii) $\forall r \forall s, q \le r < s \le t$, $\Gamma_r \prec_\psi \Gamma_s$

iii) $\forall r, q < r < t, \psi \in \Gamma_r$

Furthermore, for any point v' added in the construction immediately between arbitrary points v and v''

$$\{\psi \mid \Gamma_v \prec_\psi \Gamma_{v''}\} \cup \{\neg\varphi \mid \Gamma_q \prec_\psi \Gamma_{v''} \text{ and } \neg U(\varphi, \psi) \in \Gamma_v\} \cup \\ \{U(\gamma, \psi) \mid \Gamma_v \prec_\psi \Gamma_{v''} \text{ and } \psi \in \Gamma_{v''}\} \subseteq \Gamma_{v'}$$

So it follows immediately from ii) and iii) that for all q', $q < q' < t$, $\psi \in \Gamma_{q'}$. By the inductive hypothesis, every $q' \models \psi$ and $t \models \varphi$, so $\langle Q, <, \nu \rangle, q \models U(\varphi, \psi)$. QED

It then follows as before that $\mathbf{Q_{SU}}$ is strongly complete with respect to $\langle \mathbb{Q}, < \rangle$.

3.8 $\mathbf{S4} \oplus \mathbf{S4}$ on $\mathbb{Q} \times \mathbb{Q}$

For one final application of the construction method, let us return to our starting point: the basic modal language $\mathcal{L}$. However, this time we will consider the basic modal language with two modal operators $\Box_1$ and $\Box_2$. In the past fifteen years, such bimodal languages have been studied extensively on products of Kripke frames. For a survey of the literature, see Gabbay and Shehtman [10], or the encyclopedic Gabbay, Kurucz, Wolter and Zakharyaschev [11].

In the standard product semantics for mutimodal languages, each operator is interpreted in the usual manner on a single dimension of a Kripke product frame. Thus given a bimodal language and two Kripke frames $\mathcal{F} = \langle W, S \rangle$ and $\mathcal{G} = \langle V, T \rangle$, $\Box_1$ and $\Box_2$ are interpreted as would be expected on $\langle W \times V, R_1, R_2 \rangle$ where for all $w, w' \in W$ and $v, v' \in V$,

$$(w, v)R_1(w', v') \text{ iff } wSw' \text{ and } v = v'$$
$$(w, v)R_2(w', v') \text{ iff } w = w' \text{ and } vTv'$$

Although we will not delve into the results in this area, one relevant theorem of Gabbay and Shehtman [10] is that if two logics L_1 and L_2 are complete with respect to frame classes $\mathbb{F}'$ and $\mathbb{F}^*$ which are defined by universal Horn conditions, then

$$\mathbb{F}' \times \mathbb{F}^* = \{\langle W \times V, R_1, R_2\rangle : \langle W, S\rangle \in \mathbb{F}' \text{ and } \langle V, T\rangle \in \mathbb{F}^*\}$$

is axiomatized by $L_1 \oplus L_2$ plus the interaction principles $com = \Box_1\Box_2\varphi \leftrightarrow \Box_2\Box_1\varphi$ and $chr = \Diamond_1\Box_2\varphi \rightarrow \Box_2\Diamond_1\varphi$.

More recently, the notion of an $\mathbf{S4} \times \mathbf{S4}$ Kripke product frame has been generalized to that of a topological product space preserving one dimensional topologies. For convenience, we will call such spaces *topological product frames*. Thus given two topological spaces $\mathcal{X} = \langle X, \eta\rangle$ and $\mathcal{Y} = \langle Y, \theta\rangle$, a topological product frame is defined to be $\langle X \times Y, \tau_1, \tau_2\rangle$ where

$$\tau_1 = \{U \times \{y\} \mid U \in \eta \text{ and } y \in Y\}$$
$$\tau_2 = \{V \times \{x\} \mid V \in \theta \text{ and } x \in X\}$$

Given the usual topological semantics for $\Box$ on $\mathcal{X} = \langle X, \eta\rangle$ and $\mathcal{Y} = \langle Y, \theta\rangle$, the topological semantics for the bimodal language on $\langle X \times Y, \tau_1, \tau_2\rangle$ is given as follows:

- $(x, y) \models \Box_1\psi \Leftrightarrow \exists U \in \tau_1$ such that $(x, y) \in U$ and for all $(x', y) \in U$, $(x', y) \models \psi$
- $(x, y) \models \Box_2\psi \Leftrightarrow \exists U \in \tau_2$ such that $(x, y) \in U$ and for all $(x, y') \in U$, $(x, y') \models \psi$

Topological product frames are proposed and investigated in van Benthem, Bezhanishvili, ten Cate and Sarenac [5] where it is shown, among other things, that interaction principles such as *com* and *chr*, which are valid on $\mathbf{S4} \times \mathbf{S4}$ Kripke product frames by the above mentioned result, fail in the more general topological setting. The main result of [5] is that the $\mathbf{S4}$ axioms for $\Box_1$ and $\Box_2$ alone suffice to axiomatize the topological product frame $\langle \mathbb{Q} \times \mathbb{Q}, \tau_1, \tau_2\rangle$. Thus any purported interaction between the horizontal and vertical topologies can be refuted on simple topological spaces. As a final illustration of the construction method, we will provide another short proof of the above fact, which illustrates the lack of interaction between $\Box_1$ and $\Box_2$ on $\langle \mathbb{Q} \times \mathbb{Q}, \tau_1, \tau_2\rangle$.

Theorem 3.38 $\mathbf{S4} \oplus \mathbf{S4}$ *is strongly complete with respect to* $\langle \mathbb{Q} \times \mathbb{Q}, \tau_1, \tau_2\rangle$.

Proof. Assume $\Delta \nvdash_{\mathbf{S4}\oplus\mathbf{S4}} \chi$. We will construct two frames $\langle Q_a, <_a\rangle \cong \langle Q_b, <_b\rangle \cong \langle \mathbb{Q}, <\rangle$ in steps and stages such that $\langle Q_a \times Q_b, \tau_1, \tau_2\rangle$ satisfies $\Delta \cup \{\neg\chi\}$. For the construction, it is useful to have a linear ordering of $Q_a \times Q_b$ on hand. Let us define such a relation $<$ as follows. For all $(x, y), (x', y') \in Q_a \times Q_b$:

$$(x, y) < (x', y') \text{ iff } x <_a x', \text{ or } x = x' \text{ and } y <_b y'$$

A stage n consists of :

1. a finite set $Q_{a_n} = \{x_0, x_1..., x_k\}$ linearly ordered by $<_{a_n}$
2. a finite set $Q_{b_n} = \{y_0, y_1..., y_m\}$ linearly ordered by $<_{b_n}$
3. a finite set $Q_n \subseteq Q_{a_n} \times Q_{b_n}$ linearly ordered by $<_n$, where $<_n = < \restriction Q_n$
4. an assignment of an $\mathbf{S4} \oplus \mathbf{S4}$ maximally consistent set $\Gamma_{(x,y)}$ to all $(x, y) \in Q_n$

Each stage is divided into $\leq k+2$ steps. A step j, $0 \leq j \leq k+1$, consists of:

1. a finite set $Q^j_{a_n} = \{x_0, x_1..., x_l\}$ linearly ordered by $<_{a_n}$
2. a finite set $Q^j_{b_n} = \{y_0, y_1..., y_p\}$ linearly ordered by $<_{b_n}$
3. a finite set $Q^j_n \subseteq Q^j_{a_n} \times Q^j_{b_n}$ linearly ordered by $<^j_n$, where $<^j_n = < \restriction Q^j_n$
4. an assignment of an $\mathbf{S4} \oplus \mathbf{S4}$ maximally consistent set $\Gamma_{(x,y)}$ to all $(x, y) \in Q^j_n$

Let $\varphi_0\varphi_1\varphi_2...$ be an enumeration of all $\Box_1, \Box_2, \Diamond_1$ and $\Diamond_2$ formulas in which each formula is repeated infinitely many times. The construction will proceed as follows:

- **Stage 0**: Let $Q_{a_0} = \{x^\#\}$, $Q_{b_0} = \{y^\#\}$ and $Q_0 = \{(x^\#, y^\#)\}$. Associate $(x^\#, y^\#)$ with $\Gamma_{(x^\#,y^\#)}$, an $\mathbf{S4} \oplus \mathbf{S4}$ maximally consistent extension of $\Delta \cup \{\neg\chi\}$.

- **Stage $\mathbf{5n+1}$**: Let $\Box_1\varphi$ be the next $\Box_1$ formula in the enumeration and let $\{(x_1, y_1)....(x_k, y_k)\}$ be an enumeration of the elements in Q_{5n} ordered by $<_{5n}$.

  - *Step* 0 : Let $Q^0_{a_{5n}} = Q_{a_{5n}}$, $Q^0_{b_{5n}} = Q_{b_{5n}}$ and $Q^0_{5n} = Q_{5n}$.
  - *Step* $j+1$: If $\Box_1\varphi \in \Gamma_{(x_j,y_j)}$, let $Q^{j+1}_{a_{5n}} = Q^j_{a_{5n}} \cup \{x^*, x'\}$, $Q^{j+1}_{b_{5n}} = Q^j_{b_{5n}}$ and $Q^{j+1}_{5n} = Q^j_{5n} \cup \{(x^*, y_j), (x', y_j)\}$. Let $<^{j+1}_{a_{5n}}$ be $<^j_{a_{5n}}$ extended to include x^* and x' immediately before and after x_j, and associate the new points $(x^*, y_j), (x', y_j)$ in Q^{j+1}_{5n} with $\Gamma_{(x_j,y_j)}$.

Let $Q_{a_{5n+1}} = Q^{k+1}_{a_{5n}}$, $Q_{b_{5n+1}} = Q^{k+1}_{b_{5n}}$ and $Q_{5n+1} = Q^{k+1}_{5n}$.

- **Stage $\mathbf{5n+2}$**: Let $\Diamond_1\varphi$ be the next $\Diamond_1$ formula in the enumeration and let $\{(x_1, y_1)...(x_k, y_k)\}$ be an enumeration of the elements in Q_{5n+1} ordered by $<_{5n+1}$.

  - *Step* 0 : $Q^0_{a_{5n+1}} = Q_{a_{5n+1}}$, $Q^0_{b_{5n+1}} = Q_{b_{5n+1}}$ and $Q^0_{5n+1} = Q_{5n+1}$.
  - *Step* $j+1$: If $\Diamond\varphi \in \Gamma_{(x_j,y_j)}$, let $Q^{j+1}_{a_{5n+1}} = Q^j_{a_{5n+1}} \cup \{x'\}$, $Q^{j+1}_{b_{5n+1}} = Q^j_{b_{5n+1}}$ and $Q^{j+1}_{5n+1} = Q^j_{5n+1} \cup \{(x', y_j)\}$. Let $<^{j+1}_{a_{5n+1}}$ be $<^j_{a_{5n+1}}$ extended to include x' immediately after x. Associate the new point $(x', y_j) \in Q^{j+1}_{5n+1}$ with $\Gamma_{(x',y_j)}$, an $\mathbf{S4} \oplus \mathbf{S4}$ maximally consistent extension of $\{\varphi\} \cup \{\Box_{1\psi} \mid \Box_1\psi \in \Gamma_{(x_j,y_j)}\}$. By Lemma 3.9 , $\Gamma_{(x',y_j)}$ is consistent.

Let $Q_{a_{5n+2}} = Q^{k+1}_{a_{5n+1}}$, $Q_{b_{5n+2}} = Q^{k+1}_{b_{5n+1}}$ and $Q_{5n+2} = Q^{k+1}_{5n+1}$.

- **Stage $\mathbf{5n+3}$**: Repeat $\Box_1$ procedure for $\Box_2$ formulas, adding points to Q_b instead of Q_a.

- **Stage $\mathbf{5n+4}$**: Repeat $\Diamond_2$ procedure for $\Diamond_2$ formulas, adding points to Q_b instead of Q_a.

- **Stage $\mathbf{5n+5}$**:

  - *Step* 0: Let $Q^0_{a_{5n+5}} = Q_{a_{5n+4}}$, $Q^0_{b_{5n+5}} = Q_{b_{5n+4}}$ and $Q^0_{5n+5} = Q_{a_{5n+4}} \times Q_{b_{5n+4}}$.

Let $Q_{a_{5n+5}} = Q^0_{a_{5n+5}}$, $Q_{b_{5n+5}} = Q^0_{b_{5n+5}}$ and $Q_{5n+5} = Q^0_{5n+5}$.

We want to associate all the new points $\mathrm{Q}_{5n+5} \backslash Q_{5n+4}$ with maximally consistent sets such that all of the horizontal $\Box_1$ stretches and vertical $\Box_2$ stretches are preserved. Because of the way $\Box_1$ and $\Box_2$ stretches are created, we know that a newly added point (x, y) will not occur in both a designated $\Box_1$ stretch and a designated $\Box_2$ stretch. Thus it suffices to associate $(x, y) \in Q_{5n+5} \backslash Q_{5n+4}$ with a maximally consistent set in the most recently created $\Box_1$ or $\Box_2$ stretch containing (x, y). More formally, for all $(x, y) \in Q_{5n+5} \backslash Q_{5n+4}$, let $m \leq 5n + 4$ be the greatest $\Box_1$ or $\Box_2$ stage and i be the greatest step where points $(x^*, y), (x', y)$ or (x, y^*), (x, y') are added such that $x^* <_a x <_a x'$ or $y^* <_b y <_b y'$. If such an m and i exist, let $\Gamma_{(x,y)} = \Gamma_{(x^*,y)}$ or $\Gamma_{(x,y)} = \Gamma_{(x,y^*)}$. If not, associate $\Gamma_{(x,y)}$ with an arbitrary **S4**$\oplus$**S4** maximally consistent set.

To finish the construction, let $\langle Q_a, <_a \rangle = \bigcup \{ \langle Q_{a_n}, <_{a_n} \rangle \mid n \in \omega \}$,
$\langle Q_b, <_b \rangle = \bigcup \{ \langle Q_{b_n}, <_{b_n} \rangle \mid n \in \omega \}$ and $\langle Q \times Q, < \rangle = \bigcup \{ \langle Q_n, <_n \rangle \mid n \in \omega \}$. It is easy to check that $\langle Q_a, <_a \rangle$ and $\langle Q_b, <_b \rangle$ are countable dense unbounded linear orders. So it follows from $\langle Q_a, <_a \rangle \cong \langle Q_b, <_b \rangle \cong \langle \mathbb{Q}, < \rangle$ that $\langle Q_a \times Q_b, < \rangle \cong \langle \mathbb{Q} \times \mathbb{Q}, < \rangle$. Furthermore, it is not hard to see that $\langle Q \times Q, < \rangle = \langle Q_a \times Q_b, < \rangle$. By definition, $\langle Q \times Q, < \rangle \subseteq \langle Q_a \times Q_b, < \rangle$. So assume $(x, y) \in \langle Q_a \times Q_b, < \rangle$. Then there is some stage j such that $x \in Q_{a_j}$ and $y \in Q_{b_j}$ and by stage $j + 5$, $(x, y) \in Q_{j+5} \subseteq Q \times Q$. So $\langle Q \times Q, < \rangle \cong \langle \mathbb{Q} \times \mathbb{Q}, < \rangle$.

Proposition 3.39 $\langle Q \times Q, \tau_1, \tau_2 \rangle, (x, y) \models \varphi \Leftrightarrow \varphi \in \Gamma_{(x,y)}$.

Proof. By induction on the complexity of φ. We will only treat the case $\varphi = \Box_1 \psi$.

[$\Rightarrow$] Assume $\Box_1 \psi \notin \Gamma_{(x,y)}$ and assume $(x_1, y), (x_2, y) \in Q \times Q$ such that $x_1 <_a < x <_a x_2$. Then there is a stage j such that $(x_1, y), (x, y), (x_2, y) \in Q_{j-1}$ and $\Diamond_1 \neg \psi$ is being treated. After stage j, there is a point $(x', y) \in Q_j$ such that $x_1 <_a x' <_a x_2$ and $\neg \psi \in \Gamma_{(x',y)}$. So $\langle Q \times Q, \tau_1, \tau_2 \rangle, (x, y) \not\models \Box_1 \psi$.

[$\Leftarrow$] Assume $\Box_1 \psi \in \Gamma_{(x,y)}$. Then at some stage j and step k there are points $(x^*, y), (x', y) \in Q^k_j$ such that x^* and x' occur immediately before and after x in the $<^k_{a_j}$ ordering and $\Gamma_{(x^*,y)} = \Gamma_{(x',y)} = \Gamma_{(x,y)}$. We claim that for all $(r, y) \in Q \times Q$ such that $x^* <_a r, <_a x'$, $\Box_1 \psi \in \Gamma_{(r,y)}$. We will argue by induction on the steps of the construction.

Assume we are at at some $\Box$ or $\Diamond$ stage $l \geq j$ and step $m+1$ and a point (r, y) is added such that $x^* <_a r <_a x'$. If l is a $\Box_1$ or $\Diamond_1$ stage, then (r, y) is added immediately before or after (x_m, y), $x^* \leq_a x_m \leq_a x'$, and $\{\Box\varphi \mid \Box\varphi \in \Gamma_{(x_m,y)}\} \subseteq \Gamma_{(r,y)}$. By the inductive hypothesis and the fact that $\Box\psi \in \Gamma_{(x^*,y)}, \Gamma_{(x',y)}$, it follows that $\Box\psi \in \Gamma_{(r,y)}$. If l is a $\Box_2$ or $\Diamond_2$ stage, no points (r, y) are added such that $x^* <_a r <_a x'$.

So finally assume $l = 5n + 5$ for some n and a point (r, y) is added such that $x^* <_a r <_a x'$. At no $\Box_2$ stage less than l are points (r, y^*), (r, y') added such that $y^* <_b y <_b y'$, so (r, y) must be associated with $\Gamma_{(r,y)} = \Gamma_{(r^*,y)} = \Gamma_{(r',y)}$ such that $x^* \leq_a r^* <_a r <_a r' \leq_a x'$ and $\{\Box\varphi \mid \Box\varphi \in \Gamma_{(r,y)}\} \subseteq \Gamma_{(r^*,y)}$. By the inductive hypothesis and the fact that $\Box\psi \in \Gamma_{(x^*,y)}, \Gamma_{(x',y)}$, it follows that $\Box\psi \in \Gamma_{(r,y)}$. QED

So **S4** $\oplus$ **S4** is strongly complete with respect to $\langle \mathbb{Q} \times \mathbb{Q}, \tau_1, \tau_2 \rangle$.

4 Method Comparison

Throughout this paper we have made remarks contrasting the construction method to its more established counterpart, model theoretic approach to proving topological completeness. However, thus far we have only sketched what the model theoretic approach is and have not seen how the two approaches differ on any concrete examples. The purpose of this final chapter will be to remedy this state of affairs. For a specific example of method comparison, we will turn to the nicest application of the model theoretic approach to topological completeness: **S4** on the Cantor Space. We will then see how the construction method handles this case and discuss more generally in which cases the construction method can be seen to have an advantage over the standard model theoretic approach. Finally, we will speculate on further applications of the construction method in modal logics of space.

4.1 S4 on 2^ω, the Model Theoretic Approach

As mentioned in the introduction, one current area of investigation in modal logics of space involves finding ways to simplify results found in McKinsey and Tarski's 'The Algebra of Topology'. This project seems to have started with Mints [15], where a new completeness proof of **S4** with respect to the Cantor Space is given. Since that time, a number of new proofs of completeness with respect to $\mathbb{R}$ and $\mathbb{Q}$ have discovered, but none of them are quite as nice as the Cantor Space proof. The reason for this is clear: **S4** has a large number of simple tree frame characterizations. It is just much more natural to transfer topological structure from one tree to another than from a tree to a strict linear order. Now let us examine how this model theoretic transfer of topological structure from the Cantor Space to an **S4** tree works. As a note, the proof given below does not follow Mints [15], but rather a slightly different argument from Aiello, van Benthem, Bezhanishvili [3].

Fact 4.1 S4 *is complete with respect to the class of all rooted finite transitive and reflexive trees.*

Fact 4.2 *Every* **S4** *frame* $\langle W, R\rangle$ *can be viewed as a topological space* $\langle W, \tau'\rangle$ *where* $\Delta_x = \{y \mid Rxy\}$ *and* $\mathcal{B} = \{\Delta_y \mid y \in W\}$ *serves as a basis for* τ'.

Recall that the domain of the Cantor Space $\langle 2^\omega, \tau\rangle$ is the set of all infinite sequences of 0s and 1s. Such infinite sequences $\sigma \in 2^\omega$ are called *branches* of the Cantor Space and finite initial sequences $s \subset \sigma$ are called *nodes* of the Cantor Space. Let $B_t = \{\sigma \mid t \subset \sigma\}$. Then $\mathcal{B} = \{B_t \mid t \text{ is a node of the Cantor Space}\}$ serves as a basis for τ.

In outline, the model theoretic completeness argument proceeds as follows:

- Assume $\nvdash_{\mathbf{S4}} \varphi$
- By Fact 4.1, φ can be refuted on a rooted finite transitive and reflexive tree $\langle W, R\rangle$
- By Fact 4.2, this tree can be viewed as a topological space $\langle W, \tau'\rangle$
- A homeomorphism is devised between $\langle 2^\omega, \tau\rangle$ and $\langle W, \tau'\rangle$
- Homeomorphism is shown to be stronger than the suitable notion of modal equivalence
- So $\langle 2^\omega, \tau\rangle \not\models \varphi$

Devising the homeomorphism between $\langle 2^\omega, \tau\rangle$ and $\langle W, \tau'\rangle$ is the step which requires work, and which is the nice part of model theoretic proof. Essential use is made of the fact that $\langle W, R\rangle$ is a *finite* tree refuting φ.

Theorem 4.3 $\langle 2^\omega, \tau\rangle$ *is complete with respect to* **S4**.

Proof. Assume $\nvdash_{\mathbf{S4}} \varphi$ and $\langle W, R\rangle$ is a rooted finite transitive and reflexive tree refuting φ. Intuitively, $\langle W, R\rangle$ can be viewed as tree in which some nodes are points in W and other nodes are sets of points $C \subseteq W$, where for every $x, y \in C$, Rxy and Ryx. We call such $C \subseteq W$ *clusters*. In other words, the nodes of the tree can be viewed as equivalence classes, some singletons and others proper.

Now let us view our Kripke frame $\langle W, R\rangle$ refuting φ as the topological space $\langle W, \tau'\rangle$. The first step is to devise a labeling between *nodes* of the Cantor Space and *points* in W. The labeling proceeds recursively as follows:

1) Label the root of the Cantor Space with a root of W (since the root of W may be a cluster, there could be more than one point to choose from).

2) If a node t of the Cantor Space is labeled with a point $w \in W$ then label every node $s \supseteq t$ of the branch $t000...$ with w.

3) If t is labeled with w, label the nodes occurring immediately to the right of the branch $t000...$ as follows: let $\{w_0, ..., w_k\}$ be an enumeration of Δ_w and label the node $t1$ with w_0, $t01$ with w_1, ... ,$t(0)^k1$ with w_k, $t(0)^k01$ with w_0..., and so on.

Now we want to transform this labeling of nodes of the Cantor Space with points in W into a labeling of *branches* of the Cantor Space with points of W. We know that that if a branch $\sigma \in 2^\omega$ is of the form $\sigma = t000$ and t is labeled with w, then every node s, $t \subseteq s \subset \sigma$, is labeled w. In this case, we say w *stabilizes* σ. Otherwise, since all extensions $t \subseteq s \subset \sigma$ of a given node t labeled with w are labeled with members of Δ_w (and Δ_w is finite), we know that from some point on every node of σ is labeled with elements in some cluster $C \subseteq W$. With this in mind, let us define the following function $F : 2^\omega \to W$.

$$F(\sigma) = \begin{cases} w & \text{if } w \text{ stabilizes } \sigma \\ \rho(C) & \text{if } \sigma \text{ keeps cycling in } C \subseteq W\text{, where } \rho(C) \text{ is an arbitrary member of } C \end{cases}$$

Lemma 4.4 *For any node t labeled with w , $F[B_t] = \Delta_w$.*

Proof. As stated above, all extensions $s \supseteq t$ are labeled with points from Δ_w, so it follows immediately from the definition of F that $F[B_t] \subseteq \Delta_w$. For the converse, assume $w' \in \Delta_w$. Further assume $w' = w_j$ in the enumeration $\{w_0....w_k\}$ of Δ_w. Then the branch $t(0)^j 1000...$ is labeled w'. So $\Delta_w \subseteq F[B_t]$. QED

Lemma 4.5 *F is a homeomorphism from $\langle 2^\omega, \tau\rangle$ to $\langle W, \tau'\rangle$.*

Proof. Clearly F is onto, so let us move on to showing F preserves and reflects opens.

F preserves opens: It suffices to check that F preserves basic opens. However, by the above Lemma F sends basic opens B_t of $\langle 2^\omega, \tau\rangle$ to basic opens Δ_w of $\langle W, \tau'\rangle$.

F reflects opens: Let Δ_w be a basic open in $\langle W, \tau'\rangle$ and let

$$V = \bigcup\{B_t : t \text{ is labeled with a point } w \in \Delta_w\}$$

Clearly V is open in $\langle 2^\omega, \tau\rangle$, so it suffices to check that $F^{-1}(\Delta_w) = V$. It follows easily from Lemma 4.4 that $F(V) \subseteq \Delta_w$, so $V \subseteq F^{-1}(\Delta_w)$. Now assume $\sigma \in F^{-1}(\Delta_w)$. Then $F(\sigma) \in \Delta_w$ and we know by the labeling that from some point $t \subset \sigma$ on, all nodes $t \subseteq s \subset \sigma$ are labeled with members of Δ_w. So $\sigma \in B_t \subseteq V$ and $F^{-1}(\Delta_w) \subseteq V$. QED

Definition 4.6 Suppose two topological models $\langle X, \tau, \nu\rangle$ and $\langle X^*, \tau^*, \nu^*\rangle$ are given. A *topo − bisimulation* is a non-empty relation $T \subseteq X \times X^*$ such that if xTx^* then

- $x \in \nu(p)$ iff $x^* \in \nu'(p)$ for any propositional letter p
- if $x \in U \in \tau$ then $\exists U^* \in \tau^*$ such that $x^* \in U^*$ and $\forall y^* \in U^*$, $\exists y \in U$ such that yTy^*
- if $x^* \in U^* \in \tau'$ then $\exists U \in \tau$ such that $x \in U$ and $\forall y \in U$, $\exists y^* \in U^*$ such that yTy^*

$\lhd$

Fact 4.7 *Let f be a homeomorphism between $\langle X, \tau \rangle$ and $\langle X^*, \tau^* \rangle$. Then given a valuation ν^* on $\langle X^*, \tau^* \rangle$, the valuation $\nu(p) = f^{-1}(\nu^*(p))$ defines a total topo-bisimulation between the models $\langle X^*, \tau^*, \nu^* \rangle$ and $\langle X, \tau, \nu \rangle$.*

Fact 4.8 *If T is a topo-bisimulation between two models X, X^* such that xTx^*, then x and x^* satisfy the same modal formulas.*

To finish the proof, let ν' be the valuation and w be the point in W such that $\langle W, \tau', \nu' \rangle, w \models \neg\varphi$. Then by Lemma 4.5 and Fact 4.7, we know that assigning $\langle 2^\omega, \tau \rangle$ the valuation $\nu(p) = F^{-1}(\nu'(p))$ defines a total topo-bisimulation between $\langle W, \tau', \nu' \rangle$ and $\langle 2^\omega, \tau, \nu \rangle$. So by Fact 4.8, $\langle 2^\omega, \tau, \nu \rangle, F^{-1}(w) \models \neg\varphi$. QED

4.2 S4 on 2^ω, the Construction Method

We have now seen how the model theoretic approach can be used to show **S4** is complete with respect to the Cantor Space. Let us examine this proof from the perspective of the construction method.

Theorem 4.9 *$\langle 2^\omega, \tau \rangle$ is complete with respect to* **S4**.

Proof. Assume $\nvdash_{\mathbf{S4}} \chi$. Let $\mathcal{L}'$ be the language $\mathcal{L}$ restricted to subformulas of $\neg\chi$. Then there are only finitely many $\mathcal{L}'$ maximally consistent sets, each of which is finite. We will label nodes of the Cantor Space with $\mathcal{L}'$ maximally consistent sets as follows:

1) Label the root r of the Cantor Space with Γ_r, an $\mathcal{L}'$ maximally consistent extension of $\{\neg\chi\}$.

2) If a node t is labeled with Γ_t then label every node $s \supseteq t$ of the branch $t000...$ with Γ_t.

3) If a node t is labeled with Γ_t then label the nodes occurring immediately to the right of the branch $t000...$ as follows. Let $\Sigma = \{\Diamond\psi_0, \Diamond\psi_1 ... \Diamond\psi_k\}$ be an enumeration of the diamond formulas in Γ_t. Label the node $t1$ with a $\mathcal{L}'$ maximally consistent extension of $\{\psi_0\} \cup \{\Box\varphi \mid \Box\varphi \in \Gamma_t\}$, the node $t01$ with a maximally consistent extension of $\{\psi_1\} \cup \{\Box\varphi \mid \Box\varphi \in \Gamma_t\}$, $t(0)^k 1$ with a maximally consistent extension of $\{\psi_k\} \cup \{\Box\varphi \mid \Box\varphi \in \Gamma_t\}$, $t(0)^k 01$ with a maximally consistent extension of $\{\psi_0\} \cup \{\Box\varphi \mid \Box\varphi \in \Gamma_t\}$, ..., and so on.

Associate $\sigma \in 2^\omega$ with any $\mathcal{L}'$ maximally consistent set Γ which occurs infinitely many times on the nodes of σ (since there are only finitely many $\mathcal{L}'$ maximally consistent sets, such a Γ is guaranteed to exist). Assign $\langle 2^\omega, \tau \rangle$ the usual Henkin valuation.

Lemma 4.10 $\langle 2^\omega, \tau, \nu \rangle, \sigma \models \varphi \Leftrightarrow \varphi \in \Gamma_\sigma$.

Proof. By induction on the complexity of φ. We will only treat the case $\varphi = \Box\psi$.

[$\Leftarrow$] Assume $\Box\psi \in \Gamma_\sigma$. By examining the labeling procedure, it is clear that $s \subset t \Rightarrow \{\Box\psi \mid \Box\psi \in \Gamma_s\} \subseteq \{\Box\psi \mid \Box\psi \in \Gamma_t\}$. Since each $\mathcal{L}'$ maximally consistent set is finite, there must be some $t' \subset \sigma$ such that for all $t'' \supseteq t'$, $\{\Box\psi \mid \Box\psi \in \Gamma_{t'}\} = \{\Box\psi \mid \Box\psi \in \Gamma_{t''}\}$. It then follows from the labeling of branches that for every $\sigma' \in B_{t'}$, $\Box\psi \in \Gamma_{\sigma'}$. Furthermore, $\Box\psi \to \psi$ is an axiom (and ψ is in $\mathcal{L}'$ since it is a subformula of $\Box\psi$), so for all $\sigma' \in B_{t'}$, $\psi \in \Gamma_{\sigma'}$. Thus by the inductive hypothesis all $\sigma' \models \psi$. It follows that $\langle 2^\omega, \tau, \nu\rangle, \sigma \models \Box\psi$.

[$\Rightarrow$] Assume $\Diamond\neg\psi \in \Gamma_\sigma$ and B_t is an arbitrary open such that $t \subset \sigma$. We know there is some $t' \supset t$ labeled with Γ_σ. Assume $\Diamond\neg\psi = \Diamond\varphi_j$ in the enumeration $\{\Diamond\varphi_0, \Diamond\varphi_1, ..., \Diamond\varphi_n\}$ of $\Diamond$ formulas in Γ_σ. Let $\sigma' = t'(0)^j100....$ Then $\sigma' \in B_t$ and $\neg\psi \in \Gamma_{\sigma'}$, so by the inductive hypothesis, $\sigma' \models \neg\psi$. Thus $\langle 2^\omega, \tau, \nu\rangle, \sigma \models \Diamond\neg\psi$.

Let $\sigma_0 = 000....$ We know that σ_0 is associated with Γ_r and $\neg\chi \in \Gamma_r$, so it follows from the above Lemma that $\langle 2^\omega, \tau, \nu\rangle, \sigma_0 \models \neg\chi$. QED

4.3 Method Comparison

When using the construction method to establish that **S4** is complete with respect to the Cantor Space, no mention is made of Kripke frame characterizations of **S4**, viewing these frames as topological spaces, establishing a homeomorphism between the Cantor Space and an arbitrary member of a class of Kripke frames, etc.. The only relevant fact, that there are only finitely many maximally consistent sets, each of which is finite, is used in exactly the way you would expect given how $\Box$ and $\Diamond$ formulas are evaluated on branches of the Cantor Space. In this case, the construction method avoids a detour through Kripke semantics.

However, the mere streamlining of existing topological completeness proofs does not prove the utility of the construction method. This comes only when we moving to modal languages richer than $\mathcal{L}$, where the requisite Kripke frame characterizations are often very difficult to come by. In such cases, the construction method can be used to considerable advantage.

One instance where the construction method simplifies matters significantly is axiomatizing $\langle \mathbb{Q}, \tau\rangle$ in $\mathcal{L}^{\Box FP}$. To get a sense of the difficulty of working model theoretically, consider one of the common Kripke frame characterizations of **S4**. For example, recall that **S4** is complete with respect to the class of all finite transitive and reflexive trees. Call this frame class $\mathbb{F}$. To make the model theoretic argument go forward, it is necessary to add structure to $\mathbb{F}$ to make the temporal axioms of $\mathbf{Q_{\Box FP}}$ true. As it stands, axioms such as $F\varphi \to G(\varphi \vee P\varphi \vee F\varphi)$ (right-linearity) and $P\varphi \to H(\varphi \vee P\varphi \vee F\varphi)$ (left-linearity) need not be true on an arbitrary $\langle W, R\rangle \in \mathbb{F}$. The natural way to do this is to add an additional relation S to all $\langle W, R\rangle \in \mathbb{F}$ such that the temporal axioms (density, unboundedness, right linearity, left linearity, etc.) are true on S.

However, once this new frame class $\mathbb{F}'$ is conceived such that the axioms of **S4** and $\mathbf{Q_{FP}}$ are true on $\langle W, R, S\rangle \in \mathbb{F}'$, it still must be taken care that all the interaction principles between $\Box$ and F and P are true on $\langle W, R, S\rangle \in \mathbb{F}'$. This is guaranteed to be difficult, since

$\mathbf{Q_{\Box FP}}$ is a somewhat expressive language and a tree containing two relations R and S bears little resemblance to the intended structure $\langle \mathbb{Q}, \tau \rangle$. Once this is done, it remains to be shown that $\langle \mathbb{Q}, \tau \rangle$ is modally equivalent to the relevant Kripke countermodel $\langle W, R, S \rangle \in \mathbb{F}'$.

In this case, rather than wrestling with Kripke frames it is better to go to the intended structure $\langle \mathbb{Q}, \tau \rangle$ directly. It is much simpler to force $\Box$ and F and P to interact correctly on $\langle \mathbb{Q}, \tau \rangle$, as we did with the syntactic condition $\prec_{\Box\psi}$, than establish that $\langle \mathbb{Q}, \tau \rangle$ is modally equivalent to an arbitrary member of some class of Kripke frames.

In general, when attempting to establish modal completeness results on mathematical structures it seems preferable to avoid Kripke semantics. In a large number of cases, the relevant Kripke frame characterizations are difficult to establish and the necessary structural transfer proves awkward [19] [20]. The construction method provides a means to avoid Kripke semantics. Thus when working in modal languages more expressive than $\mathcal{L}$, it seems reasonable that the construction method would be the first choice to establish spatial completeness results. Furthermore, since most questions concerning the basic modal language $\mathcal{L}$ with a spatial interpretation have already been answered, it is reasonable to believe that the construction method will be a useful tool in the future of modal logics of space.

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