Axiomatization of **ML** and **Cheq**

MSc Thesis (*Afstudeerscriptie*)

written by

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1 Introduction

Intuitionistic logic was introduced at the beginning of the 20th century. Its original goal was to describe the laws of constructive reasoning. The main principle of this approach is that in order to establish the truth of a statement, one has to produce a "constructive proof" for it (or less formally: to show the existence of something one has to provide a way of constructing it).

Around 1930, Brouwer gave an informal definition of the intuitionistic logic. Heyting [7] made Brouwer's ideas precise by introducing a formal calculus, the intuitionistic logic **Int**. Later Tarski [18] provided a topological interpretation of intuitionistic calculus, which was developed in the forties by McKinsey and Tarski [13, 14] into a full algebraic semantics for intuitionistic logic. In 1965, Kripke [10] introduced the relational semantics for intuitionistic logic.

Gödel [6] noticed that there are infinitely many logics between the intuitionistic logic and the classical logic. Since then these logics have been broadly studied and are nowadays called intermediate logics. Jankov [8, 9] showed that in fact there are continuum many intermediate logics. In this thesis we focus our attention on two intermediate logics, namely Medvedev's logic and the logic of chequered subsets of $\mathbb{R}^\infty$.

The motivation behind Medvedev's logic is closely related to Brouwer's motivation for introducing intuitionistic logic. This logic was defined by Medvedev [15] in 1962. His idea was to consider intuitionistic formulas as finite problems. A finite problem is defined as a pair $\langle X, Y\rangle$, where $X \neq \emptyset$ is the set of possibles solutions to the problem and where $Y \subseteq X$ is the set of solutions. The operations on finite problems are defined as follows:

$$
\begin{aligned}
\langle X_1, Y_1\rangle \wedge \langle X_2, Y_2\rangle &= \langle X_1 \times X_2, Y_1 \times Y_2\rangle \\
\langle X_1, Y_1\rangle \vee \langle X_2, Y_2\rangle &= \langle X_1 \uplus X_2, Y_1 \uplus Y_2\rangle \\
\langle X_1, Y_1\rangle \to \langle X_2, Y_2\rangle &= \langle X_2^{X_1}, \{f \in X_2^{X_1} \mid f(Y_1) \subseteq Y_2\}\rangle \\
\bot &= \langle \{\emptyset\}, \emptyset\rangle.
\end{aligned}
$$

Recall that $X \uplus Y$ is the disjoint union of X and Y and that X^Y is the set of all maps from Y to X. Given a formula φ, we interpret its propositional variables as finite problems, whereas its connectives are interpreted as described above. A formula is finitely valid if for any interpretation of its propositional variables, the result is a finite problem whose set of solutions is non-empty. The intermediate logic **ML** is the set of all finitely valid formulas. Medvedev [16] proved that it can be characterized in terms of Kripke semantics. He showed that there is a class of Kripke frames such that **ML** is the set of formulas that are valid in each of the frames. We will give this definition later and in fact, this definition will be the one that will be used throughout the paper.

It is known that **ML** has the finite model property, the disjunction property, contains the so-called Kreisel-Putnam and Scott logics, and is contained in the so-called logic of weak excluded middle (see, e.g., [3]). In the late 1970's Maksimova et al. [12] showed that **ML** is not axiomatizable by any set of formulas with finitely many variables. The question whether **ML** is decidable is one of the most long-standing open problems in the field of intermediate logics; see, e.g., [3, §16] for further discussion.

Recently, van Benthem et al. [1] introduced the modal logic $\mathbf{L}_\infty$ of chequered subsets of the countable product of the real line $\mathbb{R}^\infty$. The chequered subsets of $\mathbb{R}^\infty$ are the finite unions of products $\Pi_{i\in\mathbb{N}} C_i$, where each C_i is a convex subset of $\mathbb{R}$ and all but finitely many C_i's are equal to $\mathbb{R}$ (recall that $C \subseteq \mathbb{R}$ is convex if for all $x, y \in C$, $[x, y] = \{z \mid x \leq z \leq y\}$ is a subset of C). Given a modal formula, we can interpret its propositional variables as chequered subsets of $\mathbb{R}^\infty$, the connectives $\wedge$, $\vee$ and $\neg$ as standard boolean operators and the modal operator $\Diamond$ as the closure operator (that is, it associates a set $X \subseteq \mathbb{R}^\infty$ with the smallest closed set containing X). A formula is $\mathbf{L}_\infty$-valid if for any interpretation of its propositional variables, the result is equal to $\mathbb{R}^\infty$. The modal logic $\mathbf{L}_\infty$ is the set of all modal formulas that are $\mathbf{L}_\infty$-valid.

Van Benthem et al. [1] showed that $\mathbf{L}_\infty$ has the finite model property, is not tabular and is a logic over the so-called Grzegorczyk logic. Moreover, they proved that there is a set of frames $\{\mathcal{F}_n \mid n \in \mathbf{N}\}$ characterizing $\mathbf{L}_\infty$ in the sense that a formula φ is $\mathbf{L}_\infty$-valid iff φ is valid in any of the $\mathcal{F}_n$'s.

Litak [11] introduced the "intuitionistic counterpart" of $\mathbf{L}_\infty$. This logic is defined using the standard correspondence between the modal logics over **S4** and intermediate logics (see, e.g., [3, §9]). This correspondence associates an intermediate logic with a class of modal logics and maps a modal logic **L** over **S4** to an intermediate logic, called the intermediate fragment of **L**. Litak denoted the intermediate fragment of $\mathbf{L}_\infty$ by **Cheq** and showed that it has the disjunction property and contains the Scott logic.

Medvedev's logic **ML** and **Cheq** have similar properties and they are both determined by recursive sequences of finite rooted frames. Besides, it is known that **Cheq** is contained in Medvedev's logic (see [11]). In fact, every finite rooted **Cheq**-frame is a p-morphic image of a finite rooted **ML**-frame. This raises a question how closely related the two logics are and whether the methods used to investigate **ML** could be applied to **Cheq**.

Litak [11] raised a question whether **ML** is finitely axiomatizable over **Cheq**. If this were the case, it would imply that **Cheq** is not finitely axiomatizable. We will give a negative solution to Litak's question by proving that **ML** is not finitely axiomatizable over **Cheq**. Thus, the connection between the Medvedev's logic and **Cheq** is not as strong as it first appeared.

It still remains an open problem whether **Cheq** is finitely axiomatizable. We looked at this question by using a similar approach to the one of Maksimova et al. [12]. At the moment, we can only prove that **Cheq** is not axiomatizable with four variables. Decidability of **Cheq** is also an open question.

The thesis is organized as follows. In Section 2 we recall the basics of intuitionistic logic. In Section 3 we give the proof of Maksimova et al. that Medvedev's logic is not finitely axiomatizable. In Section 4 we show that **ML** is not finitely axiomatizable over **Cheq**. The proof that **Cheq** is not axiomatizable with four variables is given in Section 5. Finally, in Section 6 we investigate the modal companions of **ML** and **Cheq** and specify a problem which implies that **Cheq** is not finitely axiomatizable.

2 Preliminaries

2.1 Syntax and semantics of intermediate logics

We denote by P the set of propositional variables. The formulas of our language $\mathcal{L}$ are given by the rule

$$\varphi ::= \bot \mid p \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \varphi \to \varphi,$$

where p ranges over elements of P. We let $\neg\varphi$ abbreviate $\varphi \to \bot$ and $\top$ abbreviate $\neg\bot$.

The intuitionistic logic **Int** is the smallest set of formulas that contains the axioms

i. $p_0 \to (p_1 \to p_0)$

ii. $(p_0 \to (p_1 \to p_2)) \to ((p_0 \to p_1) \to (p_0 \to p_2))$

iii. $p_0 \wedge p_1 \to p_0$

iv. $p_0 \wedge p_1 \to p_1$

v. $p_0 \to p_0 \vee p_1$

vi. $p_1 \to p_0 \vee p_1$

vii. $(p_0 \to p_2) \to ((p_1 \to p_2) \to ((p_0 \vee p_1) \to p_2))$

viii. $\bot \to p_0$

and closed under the inference rules

$$\mathbf{MP} : \frac{\varphi, \varphi \to \psi}{\psi} \qquad \text{and} \qquad \mathbf{Subst} : \frac{\varphi(p_0, \dots, p_n)}{\varphi(\psi_0, \dots, \psi_n)}.$$

A set of formulas closed under **MP** and **Subst** is called an *intermediate logic* if it is a subset of the classical logic and contains the intuitionistic logic. An intermediate logic **L** is said to be *finitely axiomatizable* if there is a finite set of formulas Γ such that the least intermediate logic containing Γ is **L**.

We briefly recall the basic notions of the Kripke semantics for intuitionistic logic. An *intuitionistic Kripke frame* is a pair $\mathcal{F} = \langle W, \leq \rangle$ such that W is a non-empty set and $\leq$ is a partial order, that is, a reflexive, transitive and anti-symmetric binary relation on W. A *valuation* in a frame $\mathcal{F} = \langle W, \leq \rangle$ is a map V associating with each variable $p \in P$ some subset $V(p)$ of W such that, for every $x \in V(p)$ and $y \in W$, $x \leq y$ implies $y \in V(p)$. An *intuitionistic Kripke model* is a pair $\mathcal{M} = \langle \mathcal{F}, V \rangle$, where $\mathcal{F}$ is an intuitionistic Kripke frame and V a valuation in $\mathcal{F}$.

Let $\mathcal{M} = \langle \mathcal{F}, V \rangle$ be a model and x a point in the frame $\mathcal{F} = \langle W, \leq \rangle$. We inductively define $x \Vdash \varphi$ as follows:

$\mathcal{M}, x \Vdash p$	iff	$x \in V(p)$
$\mathcal{M}, x \Vdash \varphi \wedge \psi$	iff	$\mathcal{M}, x \Vdash \varphi$ and $\mathcal{M}, x \Vdash \psi$
$\mathcal{M}, x \Vdash \varphi \vee \psi$	iff	$\mathcal{M}, x \Vdash \varphi$ or $\mathcal{M}, x \Vdash \psi$
$\mathcal{M}, x \Vdash \varphi \to \psi$	iff	for all y, ($x \leq y$ and $\mathcal{M}, y \Vdash \varphi$) implies $\mathcal{M}, y \Vdash \psi$
$\mathcal{M}, x \nVdash \bot$.		

A formula φ is *true* in $\mathcal{M}$ if $\mathcal{M}, x \Vdash \varphi$ for every $x \in \mathcal{F}$; in this case we write $\mathcal{M} \Vdash \varphi$. If φ is not true then we say that φ is *refuted* in $\mathcal{M}$. The formula φ is *valid* in a frame $\mathcal{F}$ if φ is true in all models based on $\mathcal{F}$; in this case we write $\mathcal{F} \Vdash \varphi$. Next if $\mathbf{L}$ is an intermediate logic, a frame $\mathcal{F}$ is an $\mathbf{L}$-*frame* if all formulas of $\mathbf{L}$ are valid in $\mathcal{F}$. Finally we say that φ is *valid* in a class of Kripke frames K, and write $K \Vdash \varphi$, if $\mathcal{F} \Vdash \varphi$, for every $\mathcal{F} \in K$. The logic $Log(K)$ is the set of formulas that are valid in K.

It is well-known that the set of formulas valid in all Kripke frames coincides with the logic **Int**.

2.2 Operations on Kripke frames

We recall some basic operations on Kripke frames and models.

Generated subframes A *subframe* of a frame $\mathcal{F} = \langle W, \leq \rangle$ is a frame $\mathcal{F}' = \langle W', \leq' \rangle$ such that W' is a subset of W and $x \leq' y$ iff $x \leq y$, for all $x, y \in W'$. In that case, we say that the subframe $\mathcal{F}'$ is *based* on W'.

Moreover, $\mathcal{F}'$ is a *generated subframe* of $\mathcal{F}$ if $\mathcal{F}'$ is a subframe of $\mathcal{F}$ and W' is an upset (recall that W' is an upset iff for every $x \in W'$ and every $y \in W$, $x \leq y$ implies that y belongs to W').

If $\mathcal{F}'$ is a generated subframe of $\mathcal{F}$ and if W' is the least upset of W that contains some set X, we say that $\mathcal{F}'$ is *generated* by X. If $\mathcal{F}$ is generated by some singleton $\{x\}$, then $\mathcal{F}$ is said to be a *rooted frame* and x is called the *root*. A class K of Kripke frames is *closed under rooted generated subframes* if for any $\mathcal{F} \in K$ and any rooted generated subframe $\mathcal{F}'$ of $\mathcal{F}$, we have that $\mathcal{F}'$ is isomorphic to a frame in K.

p-morphisms A map f from $\mathcal{F}$ to $\mathcal{F}'$ is a *p-morphism* if

i. for all $x, y \in \mathcal{F}$, $x \leq y$ implies $f(x) \leq f(y)$,

ii. for all $x \in \mathcal{F}$ and all $z \in \mathcal{F}'$, $f(x) \leq z$ implies that there exists a $y \in \mathcal{F}$ such that $x \leq y$ and $f(y) = z$.

In case f is onto, we say that $\mathcal{F}'$ is a *p-morphic image* of $\mathcal{F}$. Moreover, f is a *p-morphism* from a model $\langle \mathcal{F}, V \rangle$ on a model $\langle \mathcal{F}', V' \rangle$ if f is a p-morphism from $\mathcal{F}$ to $\mathcal{F}'$ such that

$$x \in V(p) \quad \text{iff} \quad f(x) \in V'(p),$$

for all $x \in \mathcal{F}$ and all variables p.

Recall that p-morphisms preserve validity. That is, if f is a p-morphism from $\mathcal{F}$ onto $\mathcal{F}'$, then

$$\mathcal{F} \Vdash \varphi \quad \text{implies} \quad \mathcal{F}' \Vdash \varphi,$$

for every formula φ. Recall also that if f is a p-morphism from a model $\mathcal{M}$ to a model $\mathcal{M}'$, then

$$\mathcal{M}, x \Vdash \varphi \quad \text{iff} \quad \mathcal{M}', f(x) \Vdash \varphi,$$

for all $x \in \mathcal{F}$ and all formulas φ.

2.3 Jankov-de Jongh theorem

Both proofs concerning axiomatizations of **ML** and **Cheq** make use of the Jankov-de Jongh formulas. We recall the main property of these formulas; see, e.g., [2] and [3, Proposition 9.41]. In fact, the Jankov-de Jongh theorem can be formulated not only for Kripke frames but also for the so-called descriptive frames.

Theorem 1 (Jankov-de Jongh theorem). *For every finite rooted frame $\mathcal{F}$, there is a formula $\chi(\mathcal{F})$ such that for every frame $\mathcal{G}$,*

$$\mathcal{G} \nVdash \chi(\mathcal{F}) \quad \textit{iff} \quad \mathcal{F} \textit{ is a p-morphic image of a generated subframe of } \mathcal{G}.$$

The formula $\chi(\mathcal{F})$ is called the Jankov-de Jongh formula of $\mathcal{F}$.

We will only make use of the following corollary of Theorem 1.

Corollary 2. *If K is a class of finite Kripke frames closed under rooted generated subframes, then for every finite rooted frame $\mathcal{F}$,*

$$\mathcal{F} \Vdash Log(K) \quad \textit{iff} \quad \mathcal{F} \textit{ is a p-morphic image of some frame in } K.$$

Proof. Let K be a class of finite Kripke frames closed under rooted generated subframes and let $\mathcal{F}$ be a finite rooted frame. For the direction from right to left, suppose that $\mathcal{F}$ is a p-morphic image of $\mathcal{G} \in K$. Since p-morphisms preserve validity, all formulas valid in $\mathcal{G}$ are valid as well in $\mathcal{F}$. In particular, any formula of $Log(K)$ is valid in $\mathcal{F}$.

Conversely, suppose that $\mathcal{F}$ is not a p-morphic image of any of the members of K. Let $\chi(\mathcal{F})$ be the Jankov-de Jongh formula of $\mathcal{F}$. We show that for any $\mathcal{G} \in K$, $\mathcal{G} \Vdash \chi(\mathcal{F})$. Fix $\mathcal{G} \in K$. By the Jankov-de Jongh theorem, $\mathcal{G} \Vdash \chi(\mathcal{F})$ iff $\mathcal{F}$ is not a p-morphic image of a generated subframe of $\mathcal{G}$. Thus, it is sufficient to show that $\mathcal{F}$ is not a p-morphic image of a generated subframe of $\mathcal{G}$. Suppose for contradiction that $\mathcal{F}$ is a p-morphic image of a generated subframe $\mathcal{G}'$ of $\mathcal{G}$. Let f be a p-morphism from $\mathcal{G}'$ onto $\mathcal{F}$. Since f is surjective, there is some $x \in \mathcal{G}'$ such that $f(x)$ is the root of $\mathcal{F}$. It is not hard to see that f is a p-morphism from the subframe $\mathcal{G}''$ of $\mathcal{G}$ generated by $\{x\}$ onto $\mathcal{F}$. Since K is closed under rooted generated subframes, $\mathcal{G}''$ belongs to K. Thus $\mathcal{F}$ is a p-morphic image of $\mathcal{G}'' \in K$, which is impossible. This completes the proof that for any $\mathcal{G} \in K$, $\mathcal{G} \Vdash \chi(\mathcal{F})$.

Therefore, $\chi(\mathcal{F})$ belongs to $Log(K)$. Note that $\mathcal{F}$ is a p-morphic image of a generated subframe of $\mathcal{F}$ (the identity map is clearly a p-morphism from $\mathcal{F}$ onto $\mathcal{F}$). It follows from the Jankov-de Jongh theorem that $\mathcal{F} \nVdash \chi(\mathcal{F})$. Putting everything together we obtain that $\chi(\mathcal{F})$ is a formula in $Log(K)$ such that $\mathcal{F} \nVdash \chi(\mathcal{F})$. Thus $\mathcal{F} \nVdash Log(K)$. $\square$

2.4 General terminology

We recall some basic notation that will be used later. Suppose $\mathcal{F} = \langle W, \leq \rangle$ is a finite frame. The *depth of* $\mathcal{F}$ is the maximal natural number n such that there is a chain of n points (recall that a set E is a chain if given two points x and y of E, either $x \leq y$ or $y \leq x$). The *depth of a point* $x \in W$ (denoted $d(x)$) is the depth of the subframe generated by x.

We say that $y \in \mathcal{F}$ is an *immediate successor* of $x \in \mathcal{F}$ if $x \neq y$, $x \leq y$ and there is no $z \notin \{x, y\}$ such that $x \leq z$ and $z \leq y$. The number of immediate successors of a point x is called the *branching degree* of x.

We will use the notation $x\uparrow$ to denote the principal upset $\{y \in \mathcal{F} \mid x \leq y\}$ and the notation $x\downarrow$ to denote the principal downset $\{y \in W \ \ |y \leq x\}$. We remark that with these notations, a frame is rooted if there is some x such that $x\uparrow$ is equal to $\mathcal{F}$. Special cases of rooted frames are trees. The frame $\mathcal{F}$ is a *tree* if $\mathcal{F}$ is rooted and for every $x \in \mathcal{F}$, the set $x\downarrow$ is a finite chain. We recall also that every rooted frame is a p-morphic image of some tree, which is finite if the rooted frame is finite (see, e.g., [3, Theorem 2.19]).

2.5 Medvedev's logic

As mentioned before there are several characterizations of Medvedev's logic. We will use the following definition.

Definition 3 (Maksimova et al. [12]). For a finite non-empty set D, let $\mathcal{P}^0(D)$ denote the Kripke frame

$$\mathcal{P}^0(D) = \langle \{X \subseteq D | X \neq \emptyset\}, \supseteq \rangle.$$

$\mathcal{P}^0(D)$ is the frame whose elements are non-empty subsets of D and whose relation is the reverse inclusion. We call $\mathcal{P}^0(D)$ a *Medvedev's frame.* The intermediate logic **ML** is the logic of all Medvedev frames, that is, the set of formulas that are valid in all Medvedev frames. As usual, a frame $\mathcal{F}$ is called an **ML**-*frame* if all the theorems of **ML** are valid in $\mathcal{F}$.

It is not hard to see that the class of Medvedev frames is closed under rooted generated subframes. Let $\mathcal{P}^0(D)$ be a Medvedev frame and let $\mathcal{F}$ be the subframe of $\mathcal{P}^0(D)$ generated by some E ($E \subseteq D$). Clearly $\mathcal{F}$ is isomorphic to the Medvedev frame $\mathcal{P}^0(E)$.

Therefore, by Corollary 2, a frame is an **ML**-frame iff it is a p-morphic image of some Medvedev frame.

2.6 The logic Cheq

We will use the following characterization of the logic **Cheq**.

Definition 4 (van Benthem et al. [1]). Let $\mathcal{F}$ denote the two-fork Kripke frame shown in Figure 1. Let $\mathcal{F}_0$ be the frame $\langle \{x\}, \{(x,x)\} \rangle$ and let $\mathcal{F}_n = \underbrace{\mathcal{F} \times \cdots \times \mathcal{F}}_{n \text{ times}}$. The logic **Cheq** is the logic of $\{\mathcal{F}_n \mid n \in \mathbb{N}\}$, that is, the set of formulas valid in any of the $\mathcal{F}_n$'s. As usual, a frame $\mathcal{F}$ is called a **Cheq**-*frame* if all the theorems of **Cheq** are valid in $\mathcal{F}$.

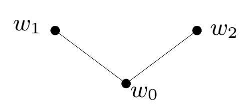

Figure 1: The frame $\mathcal{F}_1$.

We introduce some notation that we will use subsequently. Let x be an element of $\mathcal{F}_n$. We use the notation $x(j)$ to refer to the jth component of x. Next if x is an immediate successor of $(w_0, \ldots, w_0)$, it has only one component $x(i)$ that differs from w_0 and we denote it by $\delta(x)$. Finally we denote by $N_i(x)$ the number of w_i that occur in x. One can easily show by induction on $N_0(x)$ that $d(x) = N_0(x) + 1$. If $N_0(x) = 0$, then x is maximal and $d(x) = 1$. If $N_0(x) = k + 1$, then any immediate successor y of x is such that $N_0(y) = k$. So by induction, $d(y) = k + 1$. Since $d(x) = max\{d(y) \mid y \text{ immediate successor of } x\}+1$, we get that $d(x) = k+2$.

Now we show that the set of $\mathcal{F}_n$'s is closed under rooted generated subframes. Therefore, by Corollary 2, a frame is a **Cheq**-frame iff it is a p-morphic image of some $\mathcal{F}_n$.

Proposition 5. *The set of $\{\mathcal{F}_n \mid n \in \mathbb{N}\}$ of Kripke frames is closed under rooted generated subframes.*

Proof. Suppose $\mathcal{F}$ is a generated subframe of $\mathcal{F}_n$ with root x. We have to show that $\mathcal{F}$ is isomorphic to some $\mathcal{F}_l$. If x is maximal, then $\mathcal{F}$ is isomorphic to $\mathcal{F}_0$. So from now on we suppose that x is not maximal. Without loss of generality, we may assume that $x = (w_0, \ldots, w_0, x(l), \ldots, x(n-1))$, where $0 < l < n$ and $x(j)$ belongs to $\{w_1, w_2\}$ for $j \geq l$. Define a map f from $\mathcal{F}$ to $\mathcal{F}_l$ by

$$f(y) = (y(0), \ldots, y(l-1)).$$

We check that f is an isomorphism. It is sufficient to show that f is order preserving and to find an order preserving map $g : \mathcal{F}_l \to \mathcal{F}$ such that $f \circ g$ and $g \circ f$ are the identity maps. Define a map g form $\mathcal{F}_l$ to $\mathcal{F}$ by

$$g(z) = (z(0), \ldots, z(l-1), x(l), \ldots, x(n-1)).$$

Obviously f and g are order preserving. Checking that $f \circ g$ is the identity map is trivial. Thus, it remains to prove that for all y above x, we have $g(f(y)) = y$. Fix y in $\mathcal{F}$ and $i < n$. We show that $y(i) = g(f(y))(i)$. If i is less than l, it is immediate that $g(f(y))(i)$ and $y(i)$ coincide. So suppose $i \geq l$. By definition, $g(f(y))(i)$ is equal to $x(i)$. Thus we have to show that $y(i) = x(i)$. Since y is above x, $y(i)$ is above $x(i)$. As $x(i)$ is either w_1 or w_2, this can only happen in case $y(i) = x(i)$. This completes the proof that $g \circ f$ is the identity map. □

3 Medvedev's logic is not finitely axiomatizable

In this section we give an overview of the result of Maksimova, Skvorcov and Shehtman (see [12]) that Medvedev's logic is not finitely axiomatizable.

The proof is organized as follows. First, for each $k \neq 0$ and each $i \leq k$, we introduce finite rooted frames $\mathcal{G}_k$ and $\mathcal{G}_k^i$. Next we prove that the $\mathcal{G}_k$'s are not frames for the logic **ML**, whereas the $\mathcal{G}_k^i$'s make true all the theorems of **ML**. Finally we show that for any formula $\varphi(p_1, \ldots, p_k)$, there is a natural number i such that φ is valid in $\mathcal{G}_k$ if and only if φ is valid in $\mathcal{G}_k^i$.

From these results, it is not hard to derive that **ML** is not finitely axiomatizable. Suppose for contradiction that there is a finite set of formulas axiomatizing **ML**. Without loss of generality we may assume that **ML** is axiomatized by a single formula φ with k variables (for some natural number $k \neq 0$). So there exists a natural number $i \leq k$ such that φ is valid in $\mathcal{G}_k$ iff φ is valid in $\mathcal{G}_k^i$. Recall that $\mathcal{G}_k^i$ is an **ML**-frame. Thus, φ is valid in $\mathcal{G}_k^i$. Therefore, φ is valid in $\mathcal{G}_k$. But $\mathcal{G}_k$ is not an **ML**-frame. This contradiction proves that such a φ does not exist.

For each natural number $k \neq 0$ and each $i \leq k$, let $\mathcal{G}_k$ and $\mathcal{G}_k^i$ be the frames shown in Figure 2 and 3, respectively.

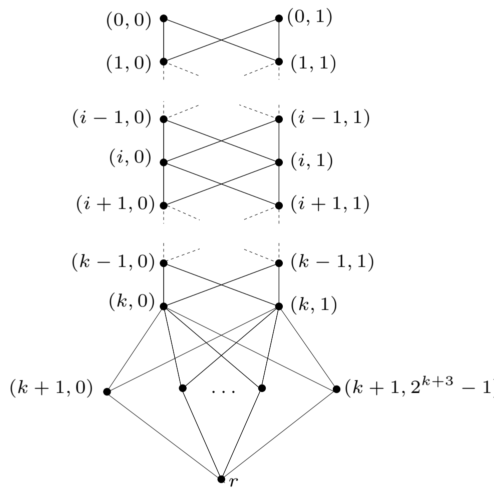

Figure 2: The frame $\mathcal{G}_k$.

Proposition 6. *For each natural number $k > 0$, the frame $\mathcal{G}_k$ is not an* **ML***-frame.*

Proof. Recall that $\{\mathcal{P}^0(D) \mid D \text{ finite non-empty set}\}$ is closed under rooted generated subframes. So by Corollary 2, $\mathcal{G}_k$ is an **ML**-frame iff $\mathcal{G}_k$ is a p-morphic image of some $\mathcal{P}^0(D)$. Suppose that $\mathcal{G}_k$ is an **ML**-frame, i.e. there is some finite non-empty set D_0 such that $\mathcal{G}_k$ is a p-morphic image of $\mathcal{P}^0(D_0)$. In order to arrive at a contradiction, we prove the following claim.

Claim 7. *Let D be a finite non-empty set and let $\mathcal{F}$ be a finite rooted frame. If $\mathcal{F}$ is a p-morphic image of some $\mathcal{P}^0(D)$, then either $\mathcal{F}$ has some point with a single immediate successor or the branching degree of any x in $\mathcal{F}$ is less than $2^{d(x)}$.*

Proof. Assume that f is a p-morphism from $\mathcal{P}^0(D)$ onto $\mathcal{F}$ and that no point of $\mathcal{F}$ has a single immediate successor. First, we prove that for any x in $\mathcal{F}$ there exists a subset E_x of D such that x is the image of E_x and the cardinality of E_x is less than $2^{d(x)}$. This is done by induction on the depth of x. The case $d(x) = 1$ is easy. Indeed, for some $E \subseteq D$, we have that

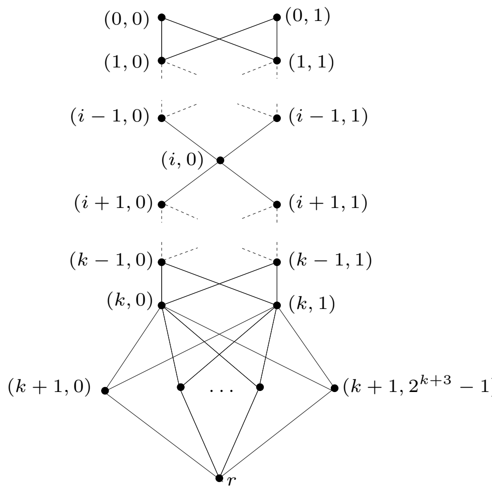

Figure 3: The frame $\mathcal{G}_k^i$.

$f(E)$ is equal to x. Let e be a point in E and define E_x as $\{e\}$. Obviously the cardinality of E_x is less than $2^{d(x)} = 2^1$. Also since E_x is a subset of E, $f(E_x)$ is above $f(E) = x$. As x is maximal, this can only happen if $f(E_x) = x$.

For the case $d(x) = n + 1$, let E be a subset of D such that $f(E) = x$. Hence f is a p-morphism from the subframe of $\mathcal{P}^0(D)$ generated by E to the subframe of $\mathcal{F}$ generated by x. That is, f is a p-morphism from $\mathcal{P}^0(E)$ to the subframe of $\mathcal{F}$ generated by x. By induction, for any proper successor y of x, there is a subset E_y of E such that y is the image of E_y and the cardinality of E_y is less than $2^{d(y)}$.

Since no point of $\mathcal{F}$ has a single immediate successor, we may assume that there are two distinct points x_1 and x_2 that are immediate successors of x. Therefore, there are subsets E_{x_1} and E_{x_2} of E such that $f(E_{x_i})$ is equal to x_i and the cardinality of E_{x_i} is less than $2^{d(x_i)} = 2^n$.

Now consider the set $E_x = E_{x_1} \cup E_{x_2}$. Observe that the cardinality of E_x is bounded by the sum of the cardinality of E_{x_1} and the cardinality of E_{x_2}. As the cardinality of E_{x_i} is less than 2^n, we obtain that the cardinality of E_x is bounded by $2^n + 2^n$. Thus the cardinality of E_x is less than $2^{n+1} = 2^{d(x)}$.

Next we prove that $f(E_x)$ is equal to x. As E_x is a subset of E, $x = f(E)$ is below $f(E_x)$. Moreover, since E_x contains E_{x_i}, we get that $f(E_x)$ is less than or equal to $f(E_{x_i}) = x_i$. Putting everything together we obtain that x is below $f(E_x)$ and that $f(E_x)$ is below x_1 and x_2. Since x_1 and x_2 are distinct immediate successors of x, this can only happen if $f(E_x) = x$.

It remains to show that the branching degree of any element x of $\mathcal{F}$ is less than $2^{d(x)}$. Fix a point x in $\mathcal{F}$. Then, there is a set $E \subseteq D$ such that $f(E) = x$ and the cardinality of E is less than $2^{d(x)}$. By taking E as small as possible, we may assume that there is no F strictly included in E such that $f(F) = x$. It is not hard to see that the branching degree of E in $\mathcal{P}^0(D)$ is equal to the cardinality of E. Thus, the branching degree of E is less than $2^{d(x)}$.

To show that the branching degree of x is less than $2^{d(x)}$, it is then sufficient to prove that the branching degree of x is less than or equal to the branching degree of E. Assume that x' is an immediate successor of x. Then there is a subset E' of E such that $f(E') = x'$. Let E'' be an immediate successor of E, which contains E'. Clearly, $f(E'')$ is above $f(E) = x$ and below $f(E') = x'$. Thus either $f(E'') = x$ or $f(E'') = x'$. Since there is no F strictly included in E such that $f(F) = x$, $f(E'')$ is equal to x'. Therefore, with any immediate successor x' of x, we can associate an immediate successor E'' of E. It immediately follows that the branching degree of x is less than or equal to the branching degree of E and this finishes the proof. $\square$

Using this claim together with the fact that $\mathcal{G}_k$ is a p-morphic image of $\mathcal{P}^0(D_0)$, we obtain that the branching degree of any point x in $\mathcal{G}_k$ is less than $2^{d(x)}$. But this contradicts the fact that the root of $\mathcal{F}$ has depth $k+3$ and branching degree 2^{k+3}. $\square$

Proposition 8. *For each natural number $k > 0$ and each $i \leq k$, the frame $\mathcal{G}_k^i$ is an* **ML***-frame.*

Proof. It is sufficient to show that $\mathcal{G}_k^i$ is a p-morphic image of some $\mathcal{P}^0(D)$ and this will be done in two steps. First, we show in Claim 9 that the subframe of $\mathcal{G}_k^i$ based on $(i,0)\!\downarrow$ is a p-morphic image of some $\mathcal{P}^0(D)$. Next we prove that by adding two maximal points to the top of a frame which is a p-morphic image of some $\mathcal{P}^0(D)$, we obtain a frame that is a p-morphic image of some $\mathcal{P}^0(D)$.

More formally, here are the two claims.

Claim 9. *If $\mathcal{F}$ is a finite rooted frame with a greatest element, then $\mathcal{F}$ is a p-morphic image of some $\mathcal{P}^0(D)$.*

Claim 10. *If a finite rooted frame $\mathcal{F} = \langle W, R\rangle$ is a p-morphic image of some $\mathcal{P}^0(D)$, then the frame $\mathcal{G} = \langle V, S\rangle$ defined by*

$$\begin{array}{rcl} V & = & W \cup \{a, b\} \\ S & = & R \cup \{(x,a),(x,b) | x \in W\} \end{array}$$

is also a p-morphic image of some $\mathcal{P}^0(D')$.

These two claims are sufficient to show that $\mathcal{G}_k^i$ is a p-morphic image of some $\mathcal{P}^0(D)$. Indeed, by Claim 9, the subframe of $\mathcal{G}_k^i$ based on $(i,0)\!\downarrow$ is a p-morphic image of some $\mathcal{P}^0(D)$ (this subframe has $(i,0)$ as maximum element). Moreover, it is easy to see that by applying repeatedly (i times)

Claim 9 to this subframe we will finally get that the frame $\mathcal{G}_k^i$ is a p-morphic image of some $\mathcal{P}^0(D')$.

So it remains to prove the claims.

Proof of Claim 9. Let $\mathcal{F}$ be a finite rooted frame with a greatest element 1. In case $\mathcal{F}$ consists of a single element, Claim 9 is immediate. So from now we will assume that $\mathcal{F}$ has at least two elements.

Recall also that any finite rooted frame is a p-morphic image of some finite tree. Thus, there is a finite tree $\mathcal{T}$ and a p-morphism from $\mathcal{T}$ onto $\mathcal{F}\backslash\{1\}$. Now consider the frame $\mathcal{T}'$ obtained by adding a greatest element 1 to the tree $\mathcal{T}$. Obviously, there is a p-morphism from $\mathcal{T}'$ onto $\mathcal{F}$. Consequently, it is enough to show that $\mathcal{T}'$ is a p-morphic image of some $\mathcal{P}^0(D)$. This will be done by induction on the depth (in $\mathcal{T}$) of the root r of $\mathcal{T}$.

The case $d(r) = 1$ is immediate. As for the case $d(r) = n + 1$, let $r_0, \ldots, r_k$ be the immediate successors of r. By induction hypothesis there are finite sets $E_0, \ldots, E_k$ so that the subframe of $\mathcal{T}'$ generated by r_i is a p-morphic image of $\mathcal{P}^0(E_i)$. Without loss of generality, we can assume that if $i \neq j$, then $E_i \cap E_j$ is empty.

For any $i \leq k$, let E'_i be a finite set such that E_i is a proper subset of E'_i. We may suppose that if $i \neq j$, then $E'_i \cap E'_j$ is empty. Let D be the union of the E'_i's.

First, we prove that for any $i \leq k$, there is a p-morphism g_i from $\mathcal{P}^0(E'_i)$ onto the subframe of $\mathcal{T}'$ based on $\{r\} \cup r_i\uparrow$. Moreover, we show that there is only one point mapped to r and that if E contains $D\backslash E'_i$ and $D\backslash E'_j$, then $g_i(E \cap E'_i)$ coincide with $g_j(E \cap E'_j)$.

By the definition of the E_i's, there is a p-morphism f_i from $\mathcal{P}^0(E_i)$ onto $r_i\uparrow$. Now define g_i by

$$g_i(E) = \begin{cases} r & \text{if } E = E'_i \\ f_i(E \cap E_i) & \text{if } E \neq E'_i \text{ and } E \cap E_i \neq \emptyset \\ 1 & \text{otherwise,} \end{cases}$$

for all $E \in \mathcal{P}^0(E'_i)$. It is routine to check that g_i is a surjective p-morphism such that only E'_i is mapped to r.

It remains to show that if E contains $D\backslash E'_i$ and $D\backslash E'_j$, then $g_i(E \cap E'_i)$ coincide with $g_j(E \cap E'_j)$. So suppose $i \neq j$ and E contains $D\backslash E'_i$ and $D\backslash E'_j$. Therefore E is a supperset of $(D\backslash E'_i) \cup (D\backslash E'_j) = D\backslash(E'_i \cap E'_j)$. From the construction of the E'_l's, it follows that $E'_i \cap E'_j = \emptyset$ and thus, E is equal to D. It is easy to see that for all l, $g_l(D \cap E'_l)$ is equal to r; that is, $g_l(E \cap E'_l)$ is equal to r. This completes the proof that $g_i(E \cap E'_i)$ and $g_j(E \cap E'_j)$ coincide.

Finally, we define f from $\mathcal{P}^0(D)$ onto $\mathcal{T}'$ and show that it is a p-morphism. For all $E \in \mathcal{P}^0(D)$, $f(E)$ is defined by

$$f(E) = \begin{cases} g_i(E \cap E'_i) & \text{if } E \text{ contains } D\backslash E'_i \\ 1 & \text{otherwise.} \end{cases}$$

Recall that if E contains $D\backslash E_i'$ and $D\backslash E_j'$, then $g_i(E \cap E_i')$ coincide with $g_j(E \cap E_j')$. Thus f is a well-defined map. Next we show that f is a surjection. Suppose that x is an element of $\mathcal{T}'$. If x is the root, then $f(D)$ is equal to x. If x belongs to $r_i\!\uparrow$, then there is a E' in $\mathcal{P}^0(E_i')$ such that $g_i(E')$ is equal to x. Consider $E = E' \cup (D\backslash E_i')$. It follows from the definition of f that $f(E)$ is equal to x.

For the forth condition of p-morphisms, suppose that $E \in \mathcal{P}^0(D)$ contains the set $E' \in \mathcal{P}^0(D)$. We will prove that $f(E)$ is related to $f(E')$. If E' contains some $D\backslash E_i'$, then E also contains $D\backslash E_i'$. Therefore, $f(E')$ is equal to $g_i(E' \cap E_i')$ and $f(E)$ is equal to $g_i(E \cap E_i')$. Using the fact that g_i is a p-morphism, we may conclude that $f(E)$ is below $f(E')$. Suppose finally that E' does not contain any of the $D\backslash E_i'$'s. It follows from the definition of f that $f(E')$ is the greatest element and obviously, $f(E)$ is below.

For the back condition, suppose that $f(E)$ sees x'. We have to show that there exists a non-empty subset E' of D such that $E \supseteq E'$ and $f(E') = x'$. If $E = D$, take E' as a subset of D which is mapped to x'. So from now on we will assume that E is a proper subset of D. If E does not contain any of the $D\backslash E_i'$'s, then $f(E)$ is the greatest element, which implies that x' is also equal to 1. So we can define E' as E.

Finally, assume that E contains $D\backslash E_i'$. Thus, $f(E)$ is equal to $g_i(E\cap E_i')$. We show that $E \cap E_i' \neq E_i'$. If not, E contains E_i'. Putting that together with the fact that E contains $D\backslash E_i'$, we obtain that $E = D$. This contradicts our assumption on E and completes the proof that $E \cap E_i' \neq E_i'$.

Recall that the only set mapped by g_i to the root is the set E_i'. It follows that $g_i(E \cap E_i')$ is not equal to r and therefore, belongs to $r_i\!\uparrow$. Now $f(E) = g_i(E \cap E_i')$ is below E'. Hence, E' also belongs to $r_i\!\uparrow$. Putting everything together, we get that $g_i(E\cap E_i')$ and E' belongs to $r_i\!\uparrow$, $g_i(E\cap E_i')$ is below E' and g_i is a p-morphism from $\mathcal{P}^0(E_i')$ onto the subframe of $\mathcal{T}'$ based on $\{r\} \cup r_i\!\uparrow$. Therefore, there is some F in $\mathcal{P}^0(E_i')$ such that $E \cap E_i'$ contains F and $g_i(F) = x'$. It is then immediate that $E' = F \cup (D\backslash E_i')$ is such that $E \supseteq E'$ and $f(E') = x'$. $\square$

Proof of Claim 10. Suppose that f is a p-morphism from $\mathcal{P}^0(D)$ onto $\mathcal{F}$ and let $\mathcal{G}$ be as in the statement of the claim. We may assume that D is a set of the form $\{1, \ldots, n\}$ for some n.

Now consider the set $D' = \{0, \ldots, n\}$ and the map g from $\mathcal{P}^0(D)$ onto $\mathcal{G}$ defined by

$$g(E) = \begin{cases} a & \text{if } E = \{0\} \\ b & \text{if } E \subseteq D \\ f(E \cap D) & \text{otherwise} \end{cases}$$

for all $E \subseteq D'$. It is routine to check that g is a p-morphism. $\square$

This finishes the proof of Proposition 8.

$\square$

Proposition 11. *Let φ be a formula with k variables. There exists a natural number $i \leq k$ such that*

$$\mathcal{G}_k \Vdash \varphi \quad \textit{iff} \quad \mathcal{G}_k^i \Vdash \varphi. \tag{1}$$

Proof. Assume that φ is a formula with k variables ($k > 0$). For each $i \leq k$, we define a map $f_i : \mathcal{G}_k \to \mathcal{G}_k^i$ by

$$f_i(x) = \begin{cases} (i,0) & \text{if } x = (i,0) \text{ or } x = (i,1) \\ x & \text{otherwise.} \end{cases}$$

It is routine to check that f_i is an onto p-morphism. Since p-morphisms preserve validity, if φ is valid in $\mathcal{G}_k$, then φ is also valid in $\mathcal{G}_k^i$. Therefore, to prove that there is some $i \leq k$ that satisfies (1), it is sufficient to show that if φ is not valid in $\mathcal{G}_k$, then there is a natural number $i \leq k$ such that φ is refuted in $\mathcal{G}_k^i$.

Suppose that $\varphi(p_1, \ldots, p_k)$ is not valid in $\mathcal{G}_k$. Hence, there is a valuation V such that φ is not true in the model $\langle \mathcal{G}_k, V\rangle$. We prove that there is some i_0 so that $(i_0, 0)$ and $(i_0, 1)$ agree on $p_1, \ldots, p_k$. Recall that two points x and y agree on p if

$$x \Vdash p \quad \text{iff} \quad y \Vdash p.$$

Assume that $(i, 0)$ and $(i, 1)$ do not agree on p ; say for instance that $(i, 0) \Vdash p$ and $(i, 1) \nVdash p$. Since $V(p)$ is an upset containing $(i, 0)$, we obtain that for all x above $(i, 0)$, $x \Vdash p$. So for all $j > i$, $(j, 0) \Vdash p$ and $(j, 1) \Vdash p$. In particular, $(j, 0)$ and $(j, 1)$ agree on p.

Next observe that $\mathcal{G}_k \backslash V(p)$ is a downset containing $(i, 1)$. So for all x below $(i, 1)$, we get that $x \nVdash p$. Therefore, for all $j < i$, $(j, 0) \nVdash p$ and $(j, 1) \nVdash p$. It follows that $(j, 0)$ and $(j, 1)$ agree on p.

Putting everything together we obtain that for all $j \neq i$, $(j, 0)$ and $(j, 1)$ agree on p. So for every propositional variable p, there is at most one i such that $(i, 0)$ and $(i, 1)$ do not agree on p. It follows that the cardinality of $\{i \mid (i, 0) \text{ and } (i, 1) \text{ do not agree on some } p\}$ is at most equal to the number of propositional variables, that is, is at most equal to k. Therefore, there is some i_0 in $\{0, \ldots, k\}$ such that $(i_0, 0)$ and $(i_0, 1)$ agree on all propositional variables.

Define a valuation V' on $\mathcal{G}_k^{i_0}$ by

$$V'(p_j) = V(p_j) \backslash \{(i_0, 1)\},$$

for all $1 \leq j \leq k$. We show that f_{i_0} is a p-morphism from $\langle \mathcal{G}_k, V\rangle$ onto $\langle \mathcal{G}_k^{i_0}, V'\rangle$. It is in fact enough to prove that for all $x \in \mathcal{G}_k$ and all $1 \leq j \leq k$, we have

$$x \in V(p_j) \quad \text{iff} \quad f(x) \in V'(p_j).$$

Fix $1 \leq j \leq k$. Remark that for all $x \in \mathcal{G}_k$, $f(x) = x$ iff $x \neq (i_0, 1)$. Thus for any $x \neq (i_0, 1)$, it is obvious that $x \in V(p_j)$ iff $f(x) \in V'(p_j)$. It remains then to show that $(i_0, 1) \in V(p_j)$ iff $f((i_0, 1)) \in V'(p_j)$. That is, $(i_0, 1) \in V(p_j)$ iff $(i_0, 0) \in V'(p_j)$. This is immediate, since $(i_0, 0)$ and $(i_0, 1)$ agree on p_j.

Therefore, φ is true in $\langle \mathcal{G}_k, V \rangle$ iff φ is true in $\langle \mathcal{G}_k^{i_0}, V' \rangle$. We can then conclude that φ is not valid in $\mathcal{G}_k^{i_0}$ and this finishes the proof. □

4 ML is not finitely axiomatizable over Cheq

In this section, which is based on [5], we prove that **ML** is not finitely axiomatizable over **Cheq**. To show that, it is in fact enough to prove that $\mathcal{G}_k$ is a **Cheq**-frame. Indeed, assume that for each $k > 1$, $\mathcal{G}_k$ is a **Cheq**-frame and suppose for contradiction that there is a finite set of formulas that axiomatizes **ML** over **Cheq**. Without loss of generality we may then assume that there is a single formula φ with k variables such that $\mathbf{ML} = \mathbf{Cheq} + \varphi$. By Proposition 11, there exists a natural number $i \leq k$ such that φ is valid in $\mathcal{G}_k$ iff φ is valid in $\mathcal{G}_k^i$. By Proposition 8, $\mathcal{G}_k^i$ is an **ML**-frame. Thus, φ is valid in $\mathcal{G}_k^i$. Therefore, φ is valid in $\mathcal{G}_k$. By our assumption, $\mathcal{G}_k$ is a **Cheq**-frame. Thus, $\mathcal{G}_k$ is a **ML**-frame, which contradicts Proposition 6. This finishes the proof that under the assumption that the $\mathcal{G}_k$'s are **Cheq**-frames, **ML** is not finitely axiomatizable over **Cheq**.

To show that the $\mathcal{G}_k$'s are **Cheq**-frame, we proceed by induction on k. The basic case is covered by Proposition 12 and the induction step is covered by Proposition 13. For every $k > 0$ and every $l > 0$, let $\mathcal{G}_{k,l}$ denote the frame shown in Figure 4 (note that $\mathcal{G}_k = \mathcal{G}_{k,2^{k+3}-1}$).

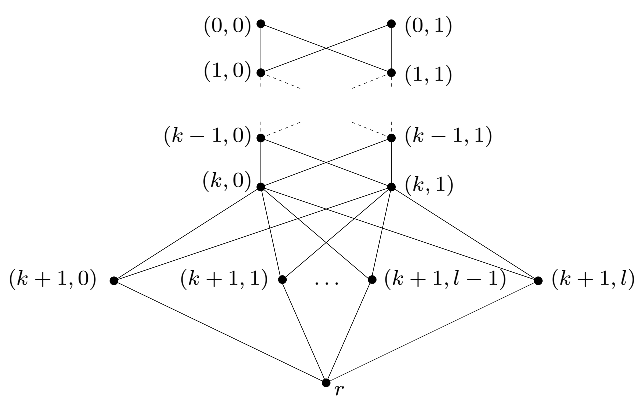

Figure 4: The frame $\mathcal{G}_{k,l}$.

Proposition 12. *For every $l > 0$, the frame $\mathcal{G}_{2,l}$ is a p-morphic image of some $\mathcal{F}_n$. Moreover, there is a p-morphism f from $\mathcal{F}_n$ onto $\mathcal{G}_{2,l}$ such that*

$f^{-1}\{(3,i) \mid i \leq l\}$ is the set of all immediate successors of $(w_0, \dots, w_0)$.

Proof. Fix $l > 0$ and an arbitrary n so that $2n \geq l + 1$ and $n > 3$. We show that there is a p-morphism f from $\mathcal{F}_n$ onto $\mathcal{G}_{2,l}$ such that $f^{-1}\{(3,i) \mid i \leq l\}$ is the set of all immediate successors of $(w_0, \dots, w_0)$. Since $2n \geq l + 1$, there is a map g from the set of immediate successors of $(w_0, \dots, w_0)$ onto $\{(3,i) \mid i \leq l\}$.

Define f by

$$f(x) = \begin{cases} r & \text{if } x = (w_0, \dots, w_0) \\ g(x) & \text{if } x \text{ is an immediate successor of } (w_0, \dots, w_0) \\ (2,0) & \text{if } N_0(x) = n-2,\ x(i) = w_1,\ x(j) = w_2 \text{ and } i+j \text{ is even} \\ (2,1) & \text{if } N_0(x) = n-2,\ x(i) = w_1,\ x(j) = w_2 \text{ and } i+j \text{ is odd} \\ (1,0) & \text{if } x \text{ is not maximal, } N_1(x) > 1 \text{ and } N_2(x) \leq 1 \\ (1,1) & \text{if } x \text{ is not maximal, } N_2(x) > 1 \text{ and } N_1(x) \leq 1 \\ (0,0) & \text{if } x \text{ is maximal and either } N_1(x) = 1 \text{ or } N_2(x) = 1 \\ (0,1) & \text{otherwise.} \end{cases}$$

Observe that if $N_0(x) = 2$ then $f(x)$ belongs to $\{(2,0), (2,1), (1,0)(1,1)\}$. Indeed, if there are components $x(i)$ and $x(j)$ such that $\{x(i), x(j)\} = \{w_1, w_2\}$, then $f(x)$ is either $(2,0)$ or $(2,1)$. Otherwise it is not hard to see that $f(x)$ is either $(1,0)$ or $(1,1)$.

Obviously, f is a well-defined onto map such that $f^{-1}\{(3,i) \mid i \leq l\}$ is the set of all immediate successors of $(w_0, \dots, w_0)$. We show that f is a p-morphism; that is, if $f(x) \leq u$, then there is a y such that $x \leq y$ and $f(y) = u$ and if $x \leq y$, then $f(x) \leq f(y)$. First, we verify the former condition.

For $x \in \mathcal{F}_n$ and $u \in \mathcal{G}_{2,l}$, let $f(x) \leq u$. Then we need to find a $y \in \mathcal{F}_n$ such that $x \leq y$ and $f(y) = u$. If $f(x) = u$, then take y as x. So from now on we assume that $f(x) < u$. Since $\mathcal{G}_{2,l}$ is finite, there are $k \in \mathbb{N}$ and $u_0, \dots, u_k \in \mathcal{G}_{2,l}$ such that $f(x) \leq u_0 \leq \cdots \leq u_k = u$, u_0 is an immediate successor of $f(x)$ and each u_{i+1} is an immediate successor of u_i. We show the existence of y by induction on k. If $k = 0$, u is an immediate successor of $f(x)$ and there are nine cases possible.

1. $x = (w_0, \dots, w_0)$. Take any y such that $f(y) = u$.

2. x is an immediate successor of $(w_0, \dots, w_0)$ and $u = (2,0)$. Without loss of generality we may assume that $x(i_0) = w_1$. Since $n > 3$, there is an index $i_1 \neq i_0$ such that $i_0 + i_1$ is even. Then take y such that $y(i_1) = w_2$ and $y(i) = x(i)$ for all $i \neq i_1$.

3. x is an immediate successor of $(w_0, \dots, w_0)$ and $u = (2,1)$. Then the argument is similar to case (2).

4. $N_0(x) = n-2$, $x(i) = w_1$, $x(j) = w_2$ and $u = (1,0)$. Since $n > 3$, there is an index i_0 such that $x(i_0) = w_0$. Then take y such that $y(i_0) = w_1$ and $y(i) = x(i)$ for all $i \neq i_0$.

5. $N_0(x) = n - 2$, $x(i) = w_1$, $x(j) = w_2$ and $u = (1,1)$. Then the argument is similar to case (4).

6. $N_1(x) > 1$, $N_2(x) \leq 1$ and $u = (0,0)$. If $N_2(x) = 1$, there exists an index i_0 such that $x(i_0) = w_2$. Then take y such that $y(i_0) = w_2$ and $y(i) = w_1$, for all $i \neq i_0$. If $N_2(x) = 0$, fix an index i_0 such that $x(i_0) = w_0$ and take y such that $y(i_0) = w_2$ and $y(i) = w_1$ for all $i \neq i_0$.

7. $N_2(x) > 1$, $N_1(x) \leq 1$ and $u = (0,0)$. Then the argument is similar to case (6).

8. $N_1(x) > 1$, $N_2(x) \leq 1$ and $u = (0,1)$. If $N_2(x) = 0$, then define y as $(w_1, \ldots, w_1)$. If $N_2(x) = 1$, then there exists an index i_0 such that $x(i_0) = w_0$. Take y such that $y(i_0) = w_2$ and $y(i) = x(i)$ for all $i \neq i_0$.

9. $N_2(x) > 1$, $N_1(x) \leq 1$ and $u = (0,1)$. Then the argument is similar to case (8).

Next suppose that $k = k' + 1$. By the induction hypothesis, there is a y' such that $x \leq y'$ and $f(y') = u_{k'}$. Recall that $u = u_{k'+1}$ is an immediate successor of $u_{k'} = f(y')$. In the same way as we showed above, we can prove that there is a y such that $y' \leq y$ and $f(y) = u$. Therefore, we obtain that $x \leq y' \leq y$ and $f(y) = u$.

Finally we verify that if $x \leq y$, then $f(x) \leq f(y)$. Suppose $x, y \in \mathcal{F}_n$ are two distinct points such that $x \leq y$. We show that $f(x) \leq f(y)$. There are six cases possible.

1. $x = (w_0, \ldots, w_0)$. Then $f(x) = r$ and $r \leq f(y)$.

2. x is an immediate successor of $(w_0, \ldots, w_0)$. By the definition of f, $f(x)$ is equal to some $(3, i)$. Since y is not an immediate successor of $(w_0, \ldots, w_0)$, $f(y)$ is also not an immediate successor of r. Hence, $f(x) \leq f(y)$.

3. $N_0(x) = n - 2$, $x(i) = w_1$ and $x(j) = w_2$. By the definition of f $f(x)$ is either $(2,0)$ or $(2,1)$. Since $x \leq y$, we can deduce that either $N_1(y) > 1$ or $N_2(y) > 1$. In both cases this implies that $f(y)$ belongs to $\{(1,0), (1,1), (0,0), (0,1)\}$. So $f(x) \leq f(y)$.

4. x is not maximal, $N_1(x) > 1$ and $N_2(x) \leq 1$. From the definition of f it follows that $f(x) = (1,0)$. Moreover, since $x \leq y$, we also have that $N_1(y) > 1$. So $f(y)$ belongs to $\{(1,0), (0,0), (0,1)\}$. In any case, $f(x) \leq f(y)$.

5. x is not maximal, $N_2(x) > 1$ and $N_1(x) \leq 1$. Then the argument is similar to case (4).

6. $N_1(x) > 1$ and $N_2(x) > 1$. By the definition of f, we have that $f(x) = (0, 1)$. Moreover, $x \leq y$ implies $N_1(y) > 1$ and $N_2(y) > 1$. So $f(y)$ is also equal to $(0, 1)$.

□

Proposition 13. *For every $k > 1$, for every $l > 0$, the frame $\mathcal{G}_{k,l}$ is a p-morphic image of some $\mathcal{F}_n$, where $n > 2$. Moreover, there is a p-morphism f from $\mathcal{F}_n$ onto $\mathcal{G}_{k,l}$ such that $f^{-1}\{(k+1, i) \mid i \leq l\}$ is the set of all immediate successors of $(w_0, \ldots, w_0)$.*

Proof. The proof is by induction on k. If $k = 2$, apply Proposition 12. Suppose $k = k'+1$ and there is a p-morphism f from $\mathcal{F}_n$ onto $\mathcal{G}_{k',l}$ such that $f^{-1}\{(k'+1, i) \mid i \leq l\}$ is the set of all immediate successors of $(w_0, \ldots, w_0)$.

If $x \in \mathcal{F}_{n+1}$, $x^- = (x(0), \ldots, x(n-1))$ and $j \in \{1, 2\}$, define $g(x)$ by

$$g(x) = \begin{cases} (k'+2, 0) & \text{if } x = (w_0, \ldots, w_0, w_j) \\ (k'+2, i) & \text{if } x \neq (w_0, \ldots, w_0, w_j),\ N_0(x) = n \text{ and} \\ & f(x^-) = (k'+1, i) \\ (k'+1, 0) & \text{if } N_0(x) = n-1,\ N_0(x^-) = n-1 \text{ and } \delta(x^-) = x(n) \\ (k'+1, 1) & \text{if } N_0(x) = n-1,\ N_0(x^-) = n-1 \text{ and } \delta(x^-) \neq x(n) \\ f(x^-) & \text{if } N_0(x^-) < n-1. \end{cases}$$

Intuitively, the frame $\mathcal{G}_{k'+1,l}$ is obtained from the frame $\mathcal{G}_{k',l}$ by adding two points between the points of depth $k'+1$ and the points of depth $k'+2$. In general, if $x = (x^-, w)$ belongs to $\mathcal{F}_{n+1}$, we map x on the same point on which x^- was mapped before. The only exceptions are when $w \neq w_0$ and x^- is either $(w_0, \ldots, w_0)$ or an immediate successor of $(w_0, \ldots, w_0)$. In the case x^- is equal to $(w_0, \ldots, w_0)$ and w is either w_1 or w_2, we map x to an immediate successor of r, namely $(k'+2, 0)$. In the case x^- is an immediate successor of $(w_0, \ldots, w_0)$ and w is either w_1 or w_2, we map x to one of the two added points.

Obviously $g : \mathcal{F}_{n+1} \to \mathcal{G}_{k'+1,l}$ is a well-defined onto map such that $g^{-1}\{(k+1, i) \mid i \leq l\}$ is the set of all immediate successors of $(w_0, \ldots, w_0)$. We check that g is a p-morphism. For $x \in \mathcal{F}_{n+1}$ and $u \in \mathcal{G}_{k,l}$, let $g(x) \leq u$. Then we need to find a $y \in \mathcal{F}_{n+1}$ such that $x \leq y$ and $g(y) = u$. As in the previous proof we may assume that u is an immediate successor of $g(x)$. There are five cases possible.

1. $g(x) = r$ and $u = (k'+2, i)$. By the definition of g, we have that $x = (w_0, \ldots, w_0)$. Moreover, by the induction hypothesis, there is a s such that $N_0(s) = 1$ and $f(s) = (k'+1, i)$. Then put $y = (s, w_0)$.

2. $g(x) = (k'+2, i)$ and $u = (k'+1, 0)$. If $g(x) = (k'+2, i)$, then either $x = (w_0, \ldots, w_0, w_j)$ (where $1 \leq j \leq 2$) or $x = (x^-, w_0)$ and $N_0(x) = n$. If $x = (x^-, w_0)$ and $N_0(x) = n$, put $y = (x^-, \delta(x^-))$. If $x = (w_0, \ldots, w_0, w_j)$ and j belongs to $\{1, 2\}$, then define y as $(w_j, w_0, \ldots, w_0, w_j)$.

3. $g(x) = (k'+2, i)$ and $u = (k'+1, 1)$. Then the argument is similar to case (2).

4. $g(x) = (k'+1, i)$ and $u = (k', i')$. Then by definition of g, we get that $x = (x^-, w_j)$, where $N_0(x^-) = n-1$ and j belongs to $\{1, 2\}$. Recall that by assumption on f, $f(x^-)$ is equal to some $(k'+1, i'')$. Since f is a p-morphism and $f(x^-)$ is below u, there is some s such that $x^- \leq s$ and $f(s) = u$. We show that $N_0(s)$ is less than $n-1$. Since $f(x^-) = (k+1, i'')$ and $f(s) = (k', i')$ are distinct, we have $x^- \neq s$. Putting that together with the fact that x^- sees s, we obtain that $x^- < s$. Thus $N_0(s)$ is less than $N_0(x^-)$. As $N_0(x^-) = n-1$, this implies that $N_0(s)$ is less than $n-1$. Define y by (s, w_j). Clearly, $x = (x^-, w_j)$ is below $y = (s, w_j)$. Since $N_0(s)$ is less than $n-1$, we have that $g(y) = f(s)$, that is, $g(y) = u$.

5. $g(x) = (i_1, i_2)$ and $u = (i_1 - 1, i_2')$, where $1 \leq i_1 \leq k'$. By the definition of g, we have that $x = (x^-, w_j)$, where $N_0(x^-) < n-1$ and $f(x^-) = g(x)$. Since f is a p-morphism and $f(x^-)$ is below u, there is some s such that $x^- \leq s$ and $f(s) = u$. We show that $N_0(s)$ is less than $n-1$. Since x^- is related to s, $N_0(s)$ is less than or equal to $N_0(x)$. As $N_0(x)$ is less than $n-1$, so does $N_0(s)$. We put $y = (s, w_j)$. It is not hard to check that x is below y and $g(y) = f(s)$, that is, $g(y) = u$.

Next suppose that $x, y \in \mathcal{F}_{n+1}$ are two distinct points such that $x \leq y$. We show that $g(x) \leq g(y)$. Let x^-, y^-, j and j' be such that $x = (x^-, w_j)$ and $y = (y^-, w_{j'})$. There are four cases possible.

1. $x = (w_0, \ldots, w_0)$. Then $g(x) = r$ and $g(x)$ is below $g(y)$.

2. x is an immediate successor of $(w_0, \ldots, w_0)$, that is, $N_0(x) = n$. As observed before, $g^{-1}\{(k'+2, i) \mid i \leq l\}$ is the set of all immediate successors of $(w_0, \ldots, w_0)$. Thus, $g(x)$ is equal to $(k'+2, i_0)$ for some i_0. Since y is not the root, $g(y)$ is not equal to r. Moreover, as y is not an immediate successor of $(w_0, \ldots, w_0)$, $g(y)$ does not belong to $\{(k'+2, i) \mid i \leq l\}$. Putting everything together, we obtain that $g(y) = (i_1, i_2)$, where $i_1 \leq k'+1$ and $i_2 \leq 1$. Hence, $g(x) = (k'+2, i_0)$ is below $g(y) = (i_1, i_2)$.

3. $N_0(x^-) = n-1$ and $x = (x^-, w_j)$, where $1 \leq j \leq 2$. By the definition of g, $g(x)$ is either $(k'+1, 0)$ or $(k'+1, 1)$. We show that that $N_0(y^-)$ is less than $n-1$. Since y is above x, $y(n)$ is above $x(n)$. As $x(n)$ is either w_1 or w_2, this can only happen in case $y(n) = x(n)$. Since $x \neq y$ and $x(n) = y(n)$, x^- and y^- are distinct. Putting that together with the fact that x is below y, we obtain that $x^- < y^-$. It follows that $N_0(y^-)$ is less than $N_0(x^-)$. As $N_0(x^-) = n-1$, this implies that $N_0(s)$ is less than $n-1$. Thus $g(y)$ is equal to $f(y^-)$. As y^- is neither the root nor an immediate successor of $(w_0, \ldots, w_0)$, we can deduce that $f(y^-)$ is equal to some (i_1, i_2), where $i_1 \leq k'$ (recall that $f^{-1}\{(k'+1, i) \mid i \leq l\}$ is the set of all immediate successors of $(w_0, \ldots, w_0)$). It follows that $g(x) = (k'+1, i)$ is below $g(y) = (i_1, i_2)$.

4. $N_0(x^-) < n-1$. By the definition of g, $g(x)$ is equal to $f(x^-)$. Also since x^- is below y^-, $N_0(y^-)$ is less than or equal to $N_0(x^-)$. As $N_0(x^-)$ is less than $n-1$, so does $N_0(y^-)$. Thus, $g(y) = f(y^-)$. Using the fact that f is a p-morphism, we obtain that $g(x) = f(x^-)$ is below $g(y) = f(y^-)$.

□

Corollary 14. *For each $k > 1$, the frame $\mathcal{G}_k$ is a p-morphic image of some $\mathcal{F}_n$. Thus, for each $k > 1$, $\mathcal{G}_k$ is a* **Cheq***-frame.*

Proof. The result follows from Proposition 13. □

5 Cheq is not axiomatizable with four variables

It is still an open problem whether **Cheq** is finitely axiomatizable. At the moment, by adapting the method used by Maksimova et al., we can only prove that **Cheq** is not axiomatizable with four variables.

The idea is the following. We define frames $\mathcal{H}_k$ and $\mathcal{H}_k^i$ (for each $k > 0$ and each $i \leq k$). Next we prove that none of the $\mathcal{H}_k$'s is a **Cheq**-frame, whereas $\mathcal{H}_4^i$ is a **Cheq**-frame (for any $i \leq 4$). Finally we show that for any set of formulas $\Gamma(p_1, \ldots, p_k)$, there is some i such that Γ is valid in $\mathcal{H}_k$ if and only if Γ is valid in $\mathcal{H}_k^i$.

From these results we can deduce that **Cheq** is not axiomatizable with four variables. Suppose for contradiction that there exists a set of formulas $\Gamma(p_1 \ldots, p_4)$ axiomatizing **Cheq**. Thus, there exists an $i \leq 4$ such that Γ is valid in $\mathcal{H}_4$ iff Γ is valid in $\mathcal{H}_4^i$. Since $\mathcal{G}_4^i$ is a **Cheq**-frame, Γ is valid in $\mathcal{H}_4^i$. Therefore, Γ is valid in $\mathcal{H}_4$. This contradicts the fact that $\mathcal{H}_4$ is not a **Cheq**-frame.

For each natural number $k \neq 0$ and each $i \leq k$, let $\mathcal{H}_k$ and $\mathcal{H}_k^i$ be the frames shown in Figure 5 and 6, respectively.

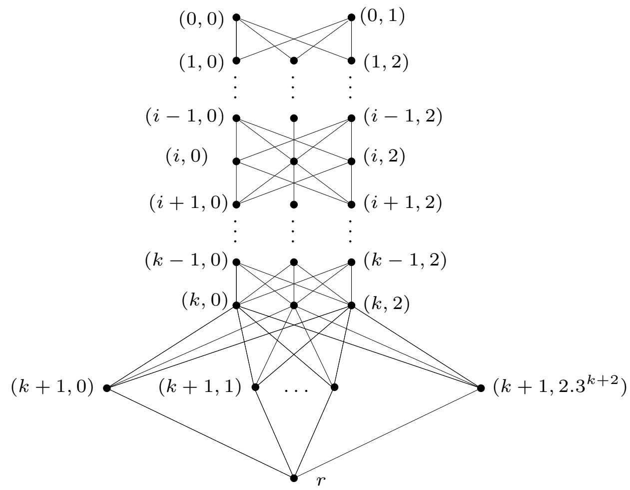

Figure 5: The frame $\mathcal{H}_k$.

Proposition 15. *For every $k \geq 1$, the frame $\mathcal{H}_k$ is not a* **Cheq***-frame.*

Proof. In the preliminary section, we observed that $\{\mathcal{F}_n \mid n > 0\}$ is closed under rooted generated subframes. So by Corollary 2, $\mathcal{H}_k$ is a **Cheq**-frame iff $\mathcal{H}_k$ is a p-morphic image of some $\mathcal{F}_n$. Suppose that $\mathcal{H}_k$ is a **Cheq**-frame, that is, is a p-morphic image of some $\mathcal{F}_n$. In order to arrive at a contradiction, we prove the following claim.

Claim 16. *Let $\mathcal{F}$ be a finite frame and let f be a p-morphism from $\mathcal{F}_n$ onto $\mathcal{F}$. Suppose that every point in $\mathcal{F}$ of depth two has branching degree two and that every point in $\mathcal{F}$ of depth greater than two has branching degree greater than or equal to three. Then for every u in $\mathcal{F}$, the branching degree of u is less than or equal to $2(3^{d(u)-1})$.*

Proof. Let $\mathcal{F}$ be as in the statement of the claim. First we show that for every u in $\mathcal{F}$, there is some x in $\mathcal{F}_n$ such that $f(x) = u$ and $d(x)$ is less than or equal to $3^{d(u)-1} + 1$. Recall that $d(x) = N_0(x) + 1$.

This is done by induction on the depth of u. Suppose first that u is a maximal point. Since f is surjective, there is some x' in $\mathcal{F}_n$ such that $f(x') = u$. Define x as a maximal point in $\mathcal{F}_n$ such that $x' \leq x$. Then $f(x)$ is above $f(x') = u$. As u is maximal, this implies that $f(x) = u$. Moreover, since x is maximal, $d(x) = 1$ and in particular, $d(x)$ is less than or equal to $3^{d(u)-1} + 1 = 3^0 + 1$.

Suppose next that u is a point of depth two. Let $x' \in \mathcal{F}_n$ be such that $f(x') = u$ and let u_1 and u_2 be the two immediate successors of u. We prove that there are $x_1, x_2 \in \mathcal{F}_n$ such that $x' \leq x_i$, $f(x_i) = u_i$ and x_1, x_2 are two

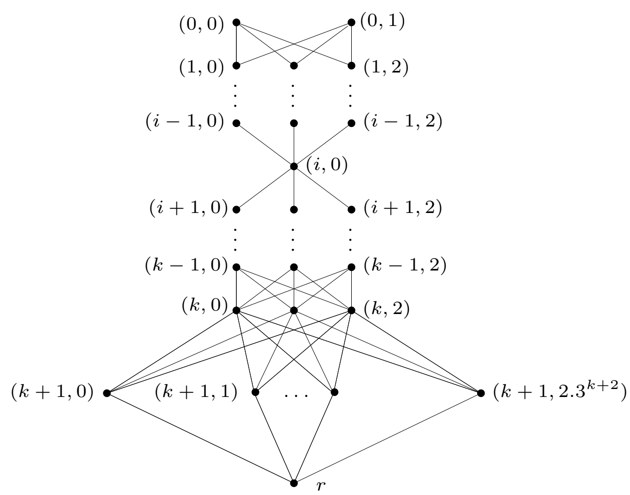

Figure 6: The frame $\mathcal{H}_k^i$.

maximal points that do not agree on exactly one component (recall that x and y do not agree on the ith component if $x(i) \neq y(i)$).

Let x_1', x_2' be such that $x' \leq x_i'$ and $f(x_i') = u_i$. Define x_i'' as a maximal point in $\mathcal{F}_n$ such that $x_i' \leq x_i''$. Then $f(x_i'')$ is above $f(x_i') = u_i$. Since u_i is maximal, this can only happen if $f(x_i'') = u_i$. Thus, x_1'' and x_2'' are two maximal points in $\mathcal{F}_n$ such that $x' \leq x_i''$ and $f(x_i'') = u_i$.

For all $j \leq n$, define y_j by $(x_1''(0), \dots, x_1''(j-1), x_2''(j), \dots, x_2''(n-1))$. As x_1'', x_2'' are maximal points above x', so are the y_j's. Hence, $f(y_j)$ is a maximal point above $f(x') = u$; that is, $f(y_j)$ is either u_1 or u_2. Next observe that $y_0 = x_2''$ and $y_n = x_1''$. Thus, $f(y_0) = u_2$ and $f(y_n) = u_1$. Putting everything together, we obtain that $\{f(y_j) \mid j \leq n\} = \{u_1, u_2\}$. Therefore there exists some $j_0 < n$ such that $\{f(y_{j_0}), f(y_{j_0+1})\} = \{u_1, u_2\}$. Remark finally that for all $j < n$, y_j and y_{j+1} do not agree on at most one component. So y_{j_0} and y_{j_0+1} are two maximal points above x', they do not agree on exactly one component and $\{f(y_{j_0}), f(y_{j_0+1})\} = \{u_1, u_2\}$. This completes the proof that there are $x_1, x_2 \in \mathcal{F}_n$ such that $x' \leq x_i$, $f(x_i) = u_i$ and x_1, x_2 are two maximal points that do not agree on exactly one component.

Without loss of generality we may assume that $x_1 = (w_1, x_1(1), \dots, x_1(n-1))$ and $x_2 = (w_2, x_1(1), \dots, x_1(n-1))$. Define x by $(w_0, x_1(1), \dots, x_1(n-1))$. Since x_1 is maximal, $N_0(x)$ is equal to 1. Next we prove that $x' \leq x$. Clearly for all $j > 0$, $x'(j) \leq x_1(j) = x(j)$. To prove that x' is below x, it is then sufficient to show that $x'(0) \leq x(0)$; that is, $x'(0) \leq w_0$. Since $x' \leq x_1$, we get that $x'(0) \leq x_1(0) = w_1$. Using the fact that $x' \leq x_2$, we can also show that $x'(0) \leq w_2$. But this can only happen if $x'(0) = w_0$.

So x' is below x. Also it is immediate that $x \leq x_1$ and $x \leq x_2$. Hence, from the fact that f is a p-morphism, we can deduce that $u = f(x') \leq f(x)$,

$f(x) \leq f(x_1) = u_1$ and $f(x) \leq f(x_2) = u_2$. As u_1 and u_2 are two distinct immediate successors of u, this implies that $f(x)$ is equal to u. Thus, we found a point x in $\mathcal{F}_n$ such that $x' \leq x$, $f(x) = u$ and $N_0(x) = 1$. Notice that $d(x)$ $(= N_0(x) + 1)$ is equal to 2 and in particular, $d(x)$ is less than or equal to $3^{d(u)-1} + 1 = 3^1 + 1$.

For the induction step, suppose that u is a point of depth $k + 1$ (where $k > 1$). Let u_1, u_2 and u_3 be three distinct successors of u and let $x \in \mathcal{F}_n$ be such that $f(x) = u$. By taking a maximal such x, we can assume that there is no y such that $x < y$ and $f(y) = u$. Obviously f is a p-morphism from the subframe of $\mathcal{F}_n$ generated by x onto the subframe of $\mathcal{F}$ generated by u. Recall that $\{\mathcal{F}_n \mid n \in \mathbb{N}\}$ is closed under rooted generated subframes. In particular, the subframe of $\mathcal{F}_n$ generated by x is isomorphic to some $\mathcal{F}_l$.

So $\langle x\uparrow, \leq\rangle$ is isomorphic to some $\mathcal{F}_l$ and there is a p-morphism from $\langle x\uparrow, \leq\rangle$ onto the subframe of $\mathcal{F}$ generated by u. By induction hypothesis there are x_1, x_2 and x_3 in $x\uparrow$ such that $f(x_i) = u_i$ and $d(x_i)$ is less than or equal to $3^{d(u_i)-1} + 1 = 3^{k-1} + 1$.

It remains to show that $d(x)$ is less than or equal to $3^k + 1$, which is equivalent to prove that $N_0(x)$ is less than or equal to 3^k. Suppose for contradiction that $l = N_0(x)$ is greater than $3k$. To make our notation easier, we will assume that x coincides with the n-tuple $(w_0, \dots, w_0, x(l), \dots, x(n-1))$ (where $x(i)$ belongs to $\{w_1, w_2\}$ for $i \geq l$).

Recall that $N_0(x_i) = d(x_i) + 1$ is less than or equal to 3^{k-1}. Hence $N_0(x_1) + N_0(x_2) + N_0(x_3)$ is less than or equal to 3^k and, in particular, is smaller than l. Next remark that the set $I = \{j \mid \exists\, 1 \leq i \leq 3, x_i(j) = w_0\}$ has at most $N_0(x_1) + N_0(x_2) + N_0(x_3)$ elements. Thus, its cardinality is less than l and there exists some j in $\{0, \dots, l-1\}$ such that j does not belong to I. In other words, there is an index $j < l$ such that $x_i(j) \neq w_0$, for every $i \in \{1, 2, 3\}$. Since $\{x_1(j), x_2(j), x_3(j)\} = \{w_1, w_2\}$, there exist distinct $i_1, i_2 \in \{1, 2, 3\}$ such that $x_{i_1}(j) = x_{i_2}(j)$. Without loss of generality, we may even assume that $j = 0$, $i_1 = 1$, $i_2 = 2$ and $x_1(0) = w_1 = x_2(0)$.

Consider now the n-tuple $y = (w_1, w_0, \dots, w_0, x(l), \dots, x(n-1))$. We show that $x \leq y$, $y \leq x_1$ and $y \leq x_2$. Obviously, x is below y. Next for any $j > 0$, we have that $y(j) = x(j) \leq x_1(j)$. Remark also that $y(0) = x_1(0)$. Putting everything together, we obtain that $y \leq x_1$. In a similar way, one can show that y is below x_2.

As f is p-morphism, it follows that $u = f(x) \leq f(y)$, $f(y) \leq f(x_1) = u_1$ and $f(y) \leq f(x_2) = u_2$. Since u_1 and u_2 are immediate successors of u, this can only happen in case $f(y)$ is equal to u. Putting everything together, we obtain that $x < y$ and $f(y) = u$. But we defined x so that there is no $y > x$ that satisfies $f(y) = u$. We arrived at the desired contradiction.

It remains to deduce that the branching degree of any $u \in \mathcal{F}$ is less than or equal to $2(3^{d(u)-1})$. Fix u in $\mathcal{F}$. We previously showed that there is a point x in $\mathcal{F}_n$ such that $f(x) = u$ and $N_0(x) \leq 3^{d(u)-1}$. By taking a maximal x such that $f(x) = u$, we can also assume that there is no $y > x$

such that $f(y) = u$. It is not difficult to see that the branching degree of x is equal to $2(N_0(x))$. Thus, the branching degree of x is less than or equal to $2(3^{d(u)-1})$.

To prove that the branching degree of u is below $2(3^{d(u)-1})$, it is then enough to show that the branching degree of u is less than or equal to the branching degree of x. Suppose that u' is an immediate successor of u. Then there is some x' in $\mathcal{F}_n$ such that $x \leq x'$ and $f(x') = u'$. Let x'' be an immediate successor of x such that $x'' \leq x'$. Clearly, $f(x'')$ is above $f(x) = u$ and below $f(x') = u'$. Thus either $f(x'') = u$ or $f(x'') = u'$. Since there is no $y > x$ such that $f(y) = u$, $f(x'')$ is equal to u'. Therefore, with any immediate successor u' of u, we can associate an immediate successor x'' of x. It immediately follows that the branching degree of u is less than or equal to the branching degree of x and this finishes the proof. □

It follows from Claim 16 that the branching degree of $x \in \mathcal{H}_k$ is less than or equal to $2(3^{d(x)-1})$. This contradicts the fact that r has depth $k+3$ and branching degree $2(3^{k+2}) + 1$. □

Proposition 17. *For any $i \leq 4$, $\mathcal{H}_4^i$ is a* **Cheq**-*frame.*

Proof. As in the case of Medvedev's logic, this is done in three steps. First we show in Claim 18 that a frame with a unique maximal point is a p-morphic image of some $\mathcal{F}_n$. Next we prove that the subframe of $\mathcal{H}_4^i$ based on $(i, 0)\uparrow$ is a p-morphic image of some $\mathcal{F}_n$. Finally we put these two results together to show that $\mathcal{H}_4^i$ is a p-morphic image of some $\mathcal{F}_n$.

Claim 18. *If $\mathcal{F}$ has a unique maximal point, then $\mathcal{F}$ is a p-morphic image of some $\mathcal{F}_n$.*

Proof. Suppose $\mathcal{F}$ has a greatest element. By Claim 9, $\mathcal{F}$ is a p-morphic image of a Medvedev frame. Recall that any Medvedev frame is a p-morphic image of some $\mathcal{F}_n$ (see for instance [11]). Hence, $\mathcal{F}$ is a p-morphic image of some $\mathcal{F}_n$. □

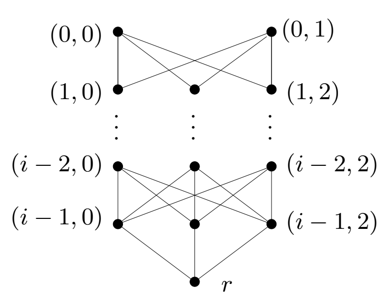

Figure 7: The frame $\mathcal{H}_i'$.

For each natural number $i > 0$, let $\mathcal{H}'_i$ be the frame shown in Figure 7. We show that $\mathcal{H}'_4$ is a **Cheq**-frame. In the proof, we will use the following definition. If X and Y are subsets of a frame $\mathcal{F}$, then $\{X_0, \ldots, X_{n-1}\}$ is a *full n-partition of X with respect to Y* if $\{X_0, \ldots, X_{n-1}\}$ is a partition of X and for all $0 \leq i < n$ and all $y \in Y$, there is some x_i in X_i such that $y \leq x_i$.

Claim 19. *$\mathcal{H}'_4$ is a p-morphic image of $\mathcal{F}_4$.*

Proof. For any $i \leq 4$, let D_i be the set of points of depth $i+1$ in $\mathcal{F}_4$. First we show that there is a full 2-partition of D_0 with respect to D_1 and that for any $i \in \{1, 2, 3\}$, there is a full 3-partition of D_i with respect to D_{i+1}.

We begin by proving that there is a full 2-partition of D_0 with respect to D_1. Define $D_{0,0}$ and $D_{0,1}$ by

$$\begin{aligned} D_{0,0} &= \{x \in D_0 \mid N_1(x) \text{ is even}\} \\ D_{0,1} &= \{x \in D_0 \mid N_1(x) \text{ is odd}\}. \end{aligned}$$

Obviously $\{D_{0,0}, D_{0,1}\}$ is a partition of D_0. So it remains to show that for any $x \in D_1$ and any $i \in \{0, 1\}$, there is some x_i in $D_{0,i}$ such that $x \leq x_i$. Let x be a point in D_1, that is, $N_0(x) = 1$. Without loss of generality we may assume that $x = (w_0, x(1), \ldots, x(3))$. If $N_1(x)$ is odd, define x_1 by $(w_1, x(1), \ldots, x(3))$ and x_2 by $(w_2, x(1), \ldots, x(3))$. If $N_1(x)$ is even, put $x_1 = (w_2, x(1), \ldots, x(3))$ and $x_2 = (w_1, x(1), \ldots, x(3))$. It is easy to check that $x_i \in D_{0,i}$ and $x \leq x_i$.

Now we prove that there is a full 3-partition of D_1 with respect to D_2. Note that we did not use any particular method to find this partition. We basically looked at random partitions and checked whether they were full. Define $D_{1,0}$, $D_{1,1}$ and $D_{1,2}$ by

$$\begin{aligned} D_{1,0} = \{ & (w_0, w_1, w_1, w_1), (w_1, w_2, w_1, w_0), (w_2, w_0, w_1, w_2), (w_0, w_2, w_2, w_2), \\ & (w_2, w_2, w_0, w_2), (w_2, w_1, w_0, w_1), (w_2, w_0, w_2, w_1), (w_1, w_1, w_2, w_0), \\ & (w_1, w_2, w_0, w_1), (w_1, w_0, w_1, w_2), (w_0, w_1, w_2, w_2)\}, \\ D_{1,1} = \{ & (w_1, w_0, w_1, w_1), (w_2, w_1, w_1, w_0), (w_0, w_2, w_1, w_2), (w_1, w_2, w_2, w_0), \\ & (w_2, w_2, w_0, w_1), (w_2, w_0, w_2, w_2), (w_0, w_1, w_2, w_1), (w_1, w_1, w_0, w_2) \\ & (w_0, w_2, w_1, w_1), (w_1, w_0, w_2, w_1)\}, \\ D_{1,2} = \{ & (w_2, w_0, w_1, w_1), (w_1, w_1, w_1, w_0), (w_2, w_2, w_1, w_0), (w_1, w_2, w_0, w_2), \\ & (w_2, w_2, w_2, w_0), (w_2, w_1, w_2, w_0), (w_1, w_1, w_0, w_1), (w_1, w_0, w_2, w_2), \\ & (w_2, w_1, w_0, w_2), (w_0, w_2, w_2, w_1), (w_0, w_1, w_1, w_2)\}. \end{aligned}$$

Although tedious, it is easy to check that these sets form a full partition of D_1 with respect to D_2.

Next we prove that there is a full 3-partition of D_2 with respect to D_3. Define $D_{2,0}$, $D_{2,1}$ and $D_{2,2}$ by

$$\begin{aligned}
D_{2,0} &= \{(w_1,w_1,w_0,w_0),(w_2,w_0,w_1,w_0),(w_0,w_2,w_2,w_0),(w_2,w_0,w_2,w_0),\\
&\quad (w_1,w_0,w_0,w_1),(w_1,w_0,w_0,w_2),(w_0,w_2,w_0,w_1),(w_0,w_2,w_0,w_2)\},\\
D_{2,1} &= \{(w_1,w_0,w_1,w_0),(w_2,w_2,w_0,w_0),(w_0,w_1,w_2,w_0),(w_0,w_2,w_1,w_0)\\
&\quad (w_2,w_0,w_0,w_1),(w_2,w_0,w_0,w_2),(w_0,w_0,w_1,w_1),(w_0,w_0,w_1,w_2)\},\\
D_{2,2} &= \{(w_1,w_2,w_0,w_0),(w_2,w_1,w_0,w_0),(w_0,w_1,w_1,w_0),(w_1,w_0,w_2,w_0)\\
&\quad (w_0,w_1,w_0,w_1),(w_0,w_1,w_0,w_2),(w_0,w_0,w_2,w_1),(w_0,w_0,w_2,w_2)\}.
\end{aligned}$$

It is routine to check that these sets form a full partition of D_2 with respect to D_3.

Finally we show that there is a full 3-partition of D_3 with respect to D_4. Remark that $D_4 = \{(w_0,w_0,w_0,w_0)\}$. Thus any partition $\{D_{3,0},D_{3,1},D_{3,2}\}$ of D_3 such that $D_{3,i} \neq \emptyset$, is a full partition of D_3 with respect to D_4.

Now we define a map f from $\mathcal{F}_4$ to $\mathcal{H}'_4$ by

$$f(x) = \begin{cases} r & \text{if } x = (w_0,\dots,w_0) \\ (i,j) & \text{if } x \in D_{i,j}. \end{cases}$$

We check that f is a well-defined surjective p-morphism. Remark that if $\{i,j\} \neq \{i',j'\}$, then $D_{i,j} \cap D_{i',j'}$ is empty. Observe also that the union of the $D_{i,j}$'s is equal to $D_0 \cup \dots \cup D_3$ $(= \mathcal{F}_4 \backslash \{(w_0,\dots,w_0)\})$. Thus f is a well-defined total map. It is also easy to see that f is onto since none of the $D_{i,j}$'s is empty.

We show that f is a p-morphism; that is, if $x \leq y$, then $f(x) \leq f(y)$ and if $f(x) \leq u$, then there is a y such that $x \leq y$ and $f(y) = u$. First, we verify the former condition.

Fix x and y such that $x \leq y$. If $x = y$, then $f(x) \leq f(y)$. So from now on we will assume that $x < y$. Thus there exist i, i' such that $i < i'$, $x \in D_i$ and $y \in D_{i'}$. Since x belongs to D_i, there is some j such that $f(x) = (i,j)$. Also, since y belongs to $D_{i'}$, there is some j' such that $f(x) = (i',j')$. Since $i < i'$, we obtain that $f(x) = (i,j)$ is below $f(y) = (i',j')$.

Now we fix x and u such that $f(x) \leq u$. We have to find a y such that $x \leq y$ and $f(y) = u$. Note that if $f(x) = r$, then $x = (w_0,\dots,w_0)$ and any y that satisfies $f(y) = u$ is such that $x \leq y$. So we may assume that $f(x) \neq r$. Since $\mathcal{H}'_4$ is finite, there are $k \in \mathbb{N}$ and $u_0,\dots,u_k \in \mathcal{H}'_4$ such that $f(x) \leq u_0 \leq \dots \leq u_k = u$, u_0 is an immediate successor of $f(x)$ and each u_{i+1} is an immediate successor of u_i. We show the existence of y by induction on k.

As for the case $k = 0$, u is an immediate successor of $f(x)$. Thus there exist i_0, j_0 and j_1 such that $f(x) = (i_0+1, j_0)$ and $u = (i_0, j_1)$. Note that this implies that x belongs to D_{i_0+1}. Recall that the $D_{i_0,j}$'s form a full partition of D_{i_0} with respect to D_{i_0+1}. Hence there is some y in D_{i_0,j_1} such that $x \leq y$. So y satisfies $x \leq y$ and $f(y) = (i_0, j_1)$, that is $f(y) = u$.

Next suppose that $k = k' + 1$. By the induction hypothesis, there is a y' such that $x \leq y'$ and $f(y') = u_{k'}$. Recall that $u = u_{k'+1}$ is an immediate successor of $u_{k'} = f(y')$. In the same way as we showed above, we can prove that there is a y such that $y' \leq y$ and $f(y) = u$. Therefore, we obtain that $x \leq y' \leq y$ and $f(y) = u$. $\square$

Before moving to the next claim, we recall the definition of a linear sum and a vertical sum (see, e.g., [4, §1]). The *linear sum* of $\mathcal{E}_1 = \langle W_1, \leq \rangle$ and $\mathcal{E}_2 = \langle W_2, \leq \rangle$ is the frame $\langle W_1 \uplus W_2, \leq \rangle$ such that $W_1 \uplus W_2$ is a disjoint union of W_1 and W_2 and for every $x, y \in W_1 \uplus W_2$, we have

$$\begin{aligned} x \leq y \qquad \text{iff} \qquad & (x \in W_2 \text{ and } y \in W_1) \\ & \text{or } (x, y \in W_1 \text{ and } x \leq y) \\ & \text{or } (x, y \in W_2 \text{ and } x \leq y). \end{aligned}$$

The *vertical sum* $\mathcal{E}$ of $\mathcal{E}_1$ and $\mathcal{E}_2$ is obtained from the linear sum of $\mathcal{E}_1$ and $\mathcal{E}_2$ by identifying the greatest element of $\mathcal{E}_2$ with the least element of $\mathcal{E}_1$ (provided they exist).

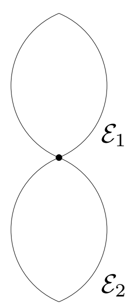

Figure 8: The vertical sum of $\mathcal{E}_1$ and $\mathcal{E}_2$.

Claim 20. *Let $\mathcal{E}_1$ be a frame with a least element and $\mathcal{E}_2$ a frame with a greatest element. Suppose that $\mathcal{E}_1$ is a p-morphic image of $\mathcal{F}_{n_1}$ and that $\mathcal{E}_2$ is a p-morphic image of $\mathcal{F}_{n_2}$. Then the vertical sum $\mathcal{E}$ of $\mathcal{E}_1$ and $\mathcal{E}_2$ is a p-morphic image of some $\mathcal{F}_n$.*

Proof. Let f_1 be a p-morphism from $\mathcal{F}_{n_1}$ onto $\mathcal{E}_1$ and let f_2 be a p-morphism from $\mathcal{F}_{n_2}$ onto $\mathcal{E}_2$. Now let n be $n_1 + n_2$ and define a map f from $\mathcal{F}_n$ onto the vertical sum of $\mathcal{F}_1$ and $\mathcal{F}_2$. If $x_1 \in \mathcal{F}_{n_1}$, $x_2 \in \mathcal{F}_{n_2}$ and $x = (x_2, x_1)$, then $f(x)$ is defined by

$$f(x) = \begin{cases} f_2(x_2) & \text{if } x_1 = (w_0, \ldots, w_0) \\ f_1(x_1) & \text{otherwise.} \end{cases}$$

We check that f is a surjective p-morphism. First we show that f is onto. Let $u \in \mathcal{E}$. If $u \in \mathcal{E}_2$, there is some x_2 in $\mathcal{F}_{n_2}$ such that $f_2(x_2) = u$. Define x by $(x_2, w_0, \dots, w_0)$. Clearly $f(x) = u$. If u does not belong to $\mathcal{E}_2$, there is some x_1 in $\mathcal{F}_1$ such that $f_1(x_1) = u$. We prove that $x_1 \neq (w_0, \dots, w_0)$. If not, $u = f_1(w_0, \dots, w_0)$ is the least point of $\mathcal{E}_1$. In particular, u belongs to $\mathcal{E}_2$, which contradicts our assumption on u. Define x by $(w_0, \dots, w_0, x_1)$. As $x_1 \neq (w_0, \dots, w_0)$, we obtain that $f(x) = u$.

We show that f is a p-morphism; that is, if $f(x) \leq u$, then there is a y such that $x \leq y$ and $f(y) = u$ and if $x \leq y$, then $f(x) \leq f(y)$. First, we verify the former condition.

For $x \in \mathcal{F}_n$ and $u \in \mathcal{E}$, let $f(x) \leq u$. Then we need to find a $y \in \mathcal{F}_n$ such that $x \leq y$ and $f(y) = u$. Suppose $x = (x_2, x_1)$, where $x_1 \in \mathcal{F}_{n_1}$ and $x_2 \in \mathcal{F}_{n_2}$. Assume first that $x_1 \neq (w_0, \dots, w_0)$. Then by definition of f, $f(x) = f_1(x_1)$. As $f(x)$ belongs to $\mathcal{E}_1$, so does u. Since f_1 is a p-morphism, there is some y_1 in $\mathcal{F}_{n_1}$ such that $f_1(y_1) = u$ and $x_1 \leq y_1$. Define y as (x_2, y_1). It is not hard to check that $x \leq y$ and $f(y) = f_1(y_1) = u$.

Next suppose that $x_1 = (w_0, \dots, w_0)$. Then $f(x) = f_2(x_2)$. If u does not belong to $\mathcal{E}_2$, take $y_1 \in \mathcal{F}_{n_1}$ such that $f_1(y_1) = u$. We show that $y_1 \neq (w_0, \dots, w_0)$. If not, $u = f_1(y_1)$ is the least point of $\mathcal{E}_1$. Hence u belongs to $\mathcal{E}_2$, which is impossible. Define y as (x_2, y_1). It is not hard to see that $x \leq y$ and $f(y) = u$. It remains to consider the case in which u belongs to $\mathcal{E}_2$. Since f_2 is a p-morphism and $f_2(x_2) \leq u$, there is some y_2 in $\mathcal{F}_{n_2}$ such that $f_2(y_2) = u$ and $x_2 \leq y_2$. Define y as $(y_2, w_0, \dots, w_0)$. Obviously $x \leq y$ and $f(y) = f_2(y_2) = u$.

Finally we verify that if $x \leq y$, then $f(x) \leq f(y)$. Suppose $x, y \in \mathcal{F}_n$ are two points such that $x \leq y$. We show that $f(x) \leq f(y)$. Suppose $x = (x_2, x_1)$ and $y = (y_2, y_1)$, where $x_1, y_1 \in \mathcal{F}_{n_1}$ and $x_2, y_2 \in \mathcal{F}_{n_2}$. Assume first that x_1 and y_1 are equal to $(w_0, \dots, w_0)$. By the definition of f, we have that $f(x) = f_2(x_2)$ and $f(y) = f_2(y_2)$. Since f_2 is a p-morphism and $x_2 \leq y_2$, we obtain that $f(x) = f_2(x_2)$ is below $f(y) = f_2(y_2)$.

Next suppose that $x_1 = (w_0, \dots, w_0)$ and $y_1 \neq (w_0, \dots, w_0)$. Thus $f(x) = f_2(x_2)$ and $f(y) = f_1(y_1)$. As $f(x)$ belongs to $\mathcal{E}_2$ and $f(y)$ belongs to $\mathcal{E}_1$, we get that $f(x) \leq f(y)$. Finally consider the case where $x_1 \neq (w_0, \dots, w_0)$ and $x_2 \neq (w_0, \dots, w_0)$. Then by the definition of f, $f(x) = f_1(x_1)$ and $f(y) = f_1(y_1)$. Since f_1 is a p-morphism and $x_1 \leq y_1$, we obtain that $f(x) = f_1(x_1)$ is below $f(y) = f_1(y_1)$. $\square$

Fix $i \leq 4$. We show that $\mathcal{H}_4^i$ is a **Cheq**-frame. By Claim 18, the subframe $\mathcal{H}'$ of $\mathcal{H}_k^i$ based on $(i,0)\downarrow$ is a p-morphic image of some $\mathcal{F}_n$.

Let $\mathcal{H}''$ be the subframe of $\mathcal{H}_4^i$ based on $(i,0)\uparrow$. It is not hard to see that $\mathcal{H}''$ is a generated subframe of $\mathcal{H}_4'$. By Claim 19, $\mathcal{H}_4'$ is a p-morphic image of $\mathcal{F}_4$. Thus $\mathcal{H}''$ is a p-morphic image of a generated subframe of $\mathcal{F}_4$. Recall that $\{\mathcal{F}_n \mid n > 0\}$ is closed under rooted generated subframes. Thus any generated subframe of $\mathcal{F}_4$ is isomorphic to some $\mathcal{F}_n$. Putting everything together we obtain that $\mathcal{H}''$ is a p-morphic image of some $\mathcal{F}_n$.

Observe finally that $\mathcal{H}_4^i$ is the vertical sum of $\mathcal{H}''$ and $\mathcal{H}'$. Thus by Claim 20, $\mathcal{H}_4^i$ is a p-morphic image of some $\mathcal{F}_n$. It follows immediately that $\mathcal{H}_4^i$ is a **Cheq**-frame. $\square$

Proposition 21. *Let Γ be a set of formulas with k variables. Then there exists $i \leq k$ such that*

$$\mathcal{H}_k \Vdash \Gamma \quad \textit{iff} \quad \mathcal{H}_k^i \Vdash \Gamma.$$

Proof. The proof is similar to the one of Proposition 11. $\square$

6 Further questions

6.1 Axiomatization of Cheq

To prove that **Cheq** is not finitely axiomatizable we tried to generalize the method used in Section 5. This eventually led us to a combinatorial problem which is completely independent of intermediate logics. A positive solution to this problem would imply that **Cheq** is not finitely axiomatizable.

Recall that $\{X_0, \ldots, X_{n-1}\}$ is a full n-partition of X with respect to Y if $\{X_0, \ldots, X_{n-1}\}$ is a partition of X and for all $0 \leq i < n$ and all $y \in Y$, there is some x_i in X_i such that $y \leq x_i$. Also we denote by $D(i,j)$ the set of elements x in $\mathcal{F}_i$ such that $N_0(x) = j$ (we recall that $N_0(x)$ is the number of w_0 that occur in x).

Proposition 22. *If for every $i > 1$, there exists a 3-full partition of $D(i,1)$ with respect to $D(i,2)$, then* **Cheq** *is not finitely axiomatizable.*

Proof. The proof is organized as follows. First, we show that under the assumption that $\mathcal{H}_k^i$ is a **Cheq**-frame, **Cheq** if not finitely axiomatizable. The second step is to prove that if the $\mathcal{H}_i'$'s are **Cheq**-frames, so are the $\mathcal{H}_k^i$'s. Next we show that if for all $0 < j < i$, there is a 3-full partition of $D(i,j)$ with respect to $D(i,j+1)$, then the $\mathcal{H}_i'$'s are **Cheq**-frames. We finish by proving that if for all $i > 1$, there is a 3-full partition of $D(i,1)$ with respect to $D(i,2)$, then for all $i > 0$ and all $i > j > 0$, there is a 3-full partition of $D(i,j)$ with respect to $D(i,j+1)$. It is not hard to see that putting everything together, we obtain the desired result.

So the first step is to show that if the $\mathcal{H}_k^i$'s are **Cheq**-frames, then **Cheq** is not finitely axiomatizable. For, assume that the $\mathcal{H}_k^i$'s are **Cheq**-frames. Suppose for contradiction that **Cheq** is axiomatized by a formula φ with k variables. By Proposition 21, there is some $i \leq k$ such that φ is valid in $\mathcal{H}_k$ iff φ is valid in $\mathcal{H}_k^i$. By our assumption, $\mathcal{G}_k^i$ is an **ML**-frame. Thus, φ is valid in $\mathcal{H}_k^i$. Therefore, φ is valid in $\mathcal{H}_k$. But $\mathcal{H}_k$ is not a **Cheq**-frame by Proposition 15. This contradiction proves that such a φ does not exist.

Next we prove that if the $\mathcal{H}_i'$'s are **Cheq**-frames, so are the $\mathcal{H}_k^i$'s. Assume that the $\mathcal{H}_i'$'s are **Cheq**-frames and fix $k \neq 0$ and $i \leq k$. It suffices to show that $\mathcal{H}_k^i$ is a p-morphic image of some $\mathcal{F}_n$. Observe that $\mathcal{H}_k^i$ is the vertical sum of $\mathcal{H}_i'$ and the subframe of $\mathcal{H}_k^i$ based on $(i,0)\!\downarrow$. Thus by Claim 20, we only have to prove that $\mathcal{H}_i'$ and the subframe of $\mathcal{H}_k^i$ based on $(i,0)\!\downarrow$ are p-morphic images of some $\mathcal{F}_n$'s. By assumption, $\mathcal{H}_i'$ is a **Cheq**-frame. Hence, by Corollary 2, $\mathcal{H}_i'$ is a p-morphic image of some $\mathcal{F}_n$. Finally by Claim 18, the subframe of $\mathcal{H}_k^i$ based on $(i,0)\!\downarrow$ is a p-morphic image of some $\mathcal{F}_n$. This completes the proof that $\mathcal{H}_k^i$ is a **Cheq**-frame.

Now we show that if for all $i > 0$ and all $i > j > 0$, there is a 3-full partition of $D(i,j)$ with respect to $D(i,j+1)$, then the $\mathcal{H}_i'$'s are **Cheq**-frames. The idea is to generalize the method used to show that $\mathcal{H}_4'$ is a **Cheq**-frame. Suppose that for all $i > 0$ and all $i > j > 0$, there is a 3-full partition of $D(i,j)$ with respect to $D(i,j+1)$. Fix $i > 0$. We have to prove that $\mathcal{H}_i'$ is a **Cheq**-frame. For any $i > j > 0$, there is a full partition $\{D_0(i,j), D_1(i,j), D_2(i,j)\}$ of $D(i,j)$ with respect to $D(i,j+1)$. Next define $D_0(i,0)$ and $D_1(i,0)$ by

$$\begin{aligned} D_0(i,0) &= \{x \in D(i,0) \mid N_1(x) \text{ is even}\} \\ D_1(i,0) &= \{x \in D(i,0) \mid N_1(x) \text{ is odd}\}. \end{aligned}$$

As in Claim 19, we can show $\{D_0(i,0), D_0(i,1)\}$ is a full partition of $D(i,0)$ with respect to $D(i,1)$.

Now we define a map f from $\mathcal{F}_i$ to $\mathcal{H}_i'$ by

$$f(x) = \begin{cases} r & \text{if } x = (w_0, \dots, w_0) \\ (j,k) & \text{if } x \in D_k(i,j). \end{cases}$$

As in Claim 19, we can prove that f is a p-morphism from $\mathcal{F}_i$ onto $\mathcal{H}_i'$. Hence $\mathcal{H}_i'$ is a **Cheq**-frame.

To finish the proof, it remains to show that if for all $i > 1$, there is a 3-full partition of $D(i,1)$ with respect to $D(i,2)$, then for all $i > 0$ and all $i > j > 0$, there is a 3-full partition of $D(i,j)$ with respect to $D(i,j+1)$. Suppose that for all $i > 1$, there is a 3-full partition of $D(i,1)$ with respect to $D(i,2)$. We prove by induction on i that for all $i > j > 0$, there is a 3-full partition of $D(i,j)$ with respect to $D(i,j+1)$.

If $i = 2$, then any partition $\{D_0(2,1), D_1(2,1), D_2(2,1)\}$ of $D(2,1)$ so that $D_k(2,1) \neq \emptyset$, is a 3-full partition of $D(2,1)$ with respect to $D(2,2) = \{(w_0, w_0)\}$.

As for the case $i = i' + 1$ (where $i' \geq 2$), we show by induction on j that for all $i > j > 0$, there is a 3-full partition of $D(i,j)$ with respect to $D(i,j+1)$. If $j = 1$, then it immediately follows from our assumption that there is a 3-full partition of $D(i,1)$ with respect to $D(i,2)$.

Suppose $j = j' + 1$ (where $j' \geq 1$). By the induction hypothesis, there is a full partition $\{D_0(i',j'), D_1(i',j'), D_2(i',j')\}$ of $D(i',j')$ with respect to $D(i',j)$ and there is a full partition $\{D_0(i',j), D_1(i',j), D_2(i',j)\}$ of $D(i',j)$

with respect to $D(i', j+1)$. For every k in $\{0,1,2\}$, we define $D_k(i,j)$ by

$$D_k(i,j) = \{(x, w_0) \mid x \in D_k(i', j')\} \cup \{(x, w_1), (x, w_2) \mid x \in D_k(i', j)\}.$$

We prove that $\{D_0(i,j), D_1(i,j), D_2(i,j)\}$ is a 3-full partition of $D(i,j)$ with respect to $D(i, j+1)$. First we check that $\{D_0(i,j), D_1(i,j), D_2(i,j)\}$ is a partition of $D(i,j)$. Let x be an element of $D(i,j)$. We have to show that x belongs to $D_k(i,j)$, for some $k \in \{0,1,2\}$. If $x = (x^-, w_0)$, then x^- belongs to $D(i', j')$. Recall that $\{D_0(i', j'), D_1(i', j'), D_2(i', j')\}$ is a partition of $D(i', j')$. Thus there is a $k \leq 3$ such that x^- belongs to $D_k(i', j')$. It immediately follows that $x = (x^-, w_0)$ belongs to $D_k(i,j)$. Assume next that $x = (x^-, w_l)$ (where $l \in \{1,2\}$). Hence x^- belongs to $D(i', j)$. Since $\{D_0(i', j), D_1(i', j), D_2(i', j)\}$ is a partition of $D(i', j)$, there is a $k \leq 3$ such that x^- belongs to $D_k(i', j)$. By the definition of $D_k(i,j)$, $x = (x^-, w_l)$ belongs to $D_k(i,j)$. This completes the proof that $\{D_0(i,j), D_1(i,j), D_2(i,j)\}$ is a partition of $D(i,j)$.

It remains to show that for all $x \in D(i, j+1)$ and all $k \in \{0,1,2\}$, there is some x_k in $D_k(i,j)$ such that $x \leq x_k$. Fix $x \in D(i, j+1)$ and $k \in \{0,1,2\}$. Suppose first that $x = (x^-, w_0)$. Thus x^- belongs to $D(i', j)$. Since $\{D_0(i', j'), D_1(i', j'), D_2(i', j')\}$ is a full partition of $D(i', j')$ with respect to $D(i', j)$, there is a point x_k^- in $D_k(i', j')$ such that $x^- \leq x_k^-$. Define x_k as the i-tuple (x_k^-, w_0). It is easy to check that $x \leq x_k$ and $x_k \in D_k(i,j)$.

Next assume that $x = (x^-, w_1)$. Hence x^- belongs to $D(i', j+1)$. Since $\{D_0(i', j), D_1(i', j), D_2(i', j)\}$ is a full partition of $D(i', j)$ with respect to $D(i', j+1)$, there is a point x_k^- in $D_k(i', j)$ such that $x^- \leq x_k^-$. Define x_k^- as the i-tuple (x_k^-, w_1). It is not hard to see that $x \leq x_k$ and $x_k \in D_k(i,j)$. The case $x = (x^-, w_2)$ is similarly handled.

$\square$

In proof of Claim 19, we showed that there is a 3-full partition of $D(4,1)$ with respect to $D(4,2)$. It is unknown whether there is a 3-full partition of $D(5,1)$ with respect to $D(5,2)$.

We will not prove it but by using the results of these two last sections, one can show that if there exists a 3-full partition of $D(i,1)$ with respect to $D(i,2)$, then **Cheq** is not axiomatizable with i variables.

6.2 The modal case

With each intermediate logic we can associate its modal companions – modal logics obtained via the Gödel translation (see, e.g., [3, §9]). This translation maps every intuitionistic formula φ to a modal formula $T(\varphi)$ such that for any Kripke frame $\mathcal{F}$, $\mathcal{F} \Vdash \varphi$ iff $\mathcal{F} \Vdash T(\varphi)$. So, given an intermediate logic $\mathbf{L}$, we obtain the class of its modal companions; these are normal modal logics $\mathbf{L}'$ containing **S4** and such that for any intuitionistic formula φ, $\varphi \in \mathbf{L}$ iff $T(\varphi) \in \mathbf{L}'$. Note that the least modal companion of an intermediate logic

L is the least modal logic containing **S4** and $\{T(\varphi) \mid \varphi \in \mathbf{L}\}$. Moreover, if **L** is characterized by a class K of finite frames, then the greatest modal companion of **L** is the logic consisting of the modal formulas valid in K.

Shehtman [17] showed that none of the modal companions of **ML** is finitely axiomatizable. We sketch the proof, which is similar to the one for the intuitionistic case. First it is proved that for any modal companion **L** of **ML**, a Kripke frame is an **L**-frame iff it is an **ML**-frame. The second step it to show that $\mathcal{G}_{2^k-3,4^k-1}$ is not an **ML**-frame. In fact it can be proved in the same way as we established that $\mathcal{G}_k$ is not an **ML**-frame. Next Shehtman shows that $\mathcal{G}_{2^k-3,2^k-1}$ is an **ML**-frame. The proof is rather long and involves some graph theory. The last step is done by proving that for any modal formula φ with k variables and for any $l \geq 2^k - 1$, we have $\mathcal{G}_{m,l} \Vdash \varphi$ iff $\mathcal{G}_{m,2^k-1} \Vdash \varphi$. The proof is similar to the one establishing that for any formula φ with k variables, there is some $i \leq k$ such that $\mathcal{G}_k \Vdash \varphi$ iff $\mathcal{G}_k^i \Vdash \varphi$.

From these results, one can show that no companion of **ML** is finitely axiomatizable. Fix a modal companion **L** of **ML** and suppose for contradiction that there is a finite axiomatization of **L**. We may assume that **L** is axiomatized by a single formula φ with k variables (where $k \geq 2$). Recall that a frame is an **L**-frame iff it is an **ML**-frame. As $\mathcal{G}_{2^k-3,4^k-1}$ is not an **ML**-frame, $\mathcal{G}_{2^k-3,4^k-1}$ is not an **L**-frame. Hence φ is not valid in $\mathcal{G}_{2^k-3,4^k-1}$. Recall also that for any $l \geq 2^k - 1$, we have $\mathcal{G}_{m,l} \Vdash \varphi$ iff $\mathcal{G}_{m,2^k-1} \Vdash \varphi$. Since $4^k - 1$ is greater than $2^k - 1$, we can deduce that φ is not valid in $\mathcal{G}_{2^k-3,2^k-1}$. Therefore $\mathcal{G}_{2^k-3,2^k-1}$ is not an **L**-frame. But this contradicts the fact that $\mathcal{G}_{2^k-3,2^k-1}$ is an **ML**-frame and that any **ML**-frame is an **L**-frame. This completes the proof that **L** is not finitely axiomatizable.

Note that all the statements formulated in the second paragraph of this section remain true for **Cheq**, except the one establishing that $\mathcal{G}_{2^k-3,2^k-1}$ is an **ML**-frame. In fact, we can only show that $\mathcal{H}_{2^2-3,2^2-1}$ is a **Cheq**-frame. We can then deduce that none of the modal companions of **Cheq** is axiomatizable with two variables. In particular, $\mathbf{L}_\infty$ is not axiomatizable with two variables. We will skip the details.

Finally, we remark that if for every $i > 1$, there exists a 3-full partition of $D(i, 1)$ with respect to $D(i, 2)$, then no modal companion of **Cheq** is finitely axiomatizable.

7 Conclusion

We proved that **ML** is not finitely axiomatizable over **Cheq**, which shows that these two logics are not as closely related as previously thought. We also proved that **Cheq** is not axiomatizable with four variables and found a combinatorial problem a positive solution to which would imply that **Cheq** is not axiomatizable by any set of formulas with finitely many variables.

It still remains an open question whether **Cheq** is finitely axiomatizable and/or decidable. Of course, the decidability of **ML** is still an interesting (but difficult) open problem.

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