Models of the Polymodal Provability Logic

MSc Thesis (*Afstudeerscriptie*)

written by

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1 Introduction

Arguably, the most successful applications of modal logic to other areas of mathematics have been the so called "provability" interpretations, and various topological interpretations.[1] This thesis concerns both of these interpretations, albeit each only indirectly. The Polymodal Provability Logic known as **GLP** has held significant interest in the study of strong provability predicates in arithmetical theories, as well as in the study of ordinal notation systems for these theories. In particular, the closed fragment of **GLP**, which we shall denote $\mathbf{GLP}_0$, simply **GLP** restricted to the language without variables, has found applications in mainstream proof theory (See Section 1.2). The majority of this thesis is therefore dedicated to exploring relational and topological models of this logic. At the same time, however, an underlying impetus for this study is the fact that the full logic **GLP** with variables is frame-incomplete (See Section 1.1.2). While a thorough treatment of the full logic is beyond the scope of the thesis, it is hoped that a better understanding of the closed fragment, and in particular the interaction between relational and topological interpretations of the closed fragment, will shed some light on the full fragment. Last but not least, some of the structures we will come across are sufficiently intriguing and natural, we believe, so as to merit interest in and of themselves.

1.1 GL and GLP

1.1.1 Classical Provability Logic

The idea of a *logic of provability* goes back to a short paper by Kurt Gödel ([Gödel, 1933]). Let $B(x, y)$ be (any reasonable variation of) Gödel's Δ_0 predicate formalizing, "y is the code of a proof in Peano Arithmetic of the sentence with Gödel number x," with $Bew(x) = \exists y B(x, y)$, and let $A^{\#}$ be the numeral of the Gödel code of A. Gödel first showed that the following schema is valid:

$$Bew((A \to B)^{\#}) \Rightarrow (Bew(A^{\#}) \Rightarrow Bew(B^{\#}))$$

In fact, not only is the schema true, it can be proven in Peano Arithmetic (henceforth PA) itself, whereby the meta-conditional "$\Rightarrow$" becomes "$\to$" in the language of arithmetic. The following rule also holds for all A (as an instance of so called Σ_1-completeness):

If A is provable, then $Bew(A^{\#})$ is provable.

This fact is also formalizable in PA (by so called *provable* Σ_1-completeness), so we have one more schema:

$$Bew(A^{\#}) \to Bew(Bew(A^{\#})^{\#})$$

Ignoring the distinction between a formula and its Gödel number, and replacing *Bew* with $\Box$, we can write each of these schemas and rules in what looks like the standard modal language, giving each a name suggestive thereof.

[1] At any rate, so much is claimed in, e.g. [Artemov, 2006].

K: $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$

4: $\Box\phi \to \Box\Box\phi$

And the rule,

Nec: If ϕ, then $\Box\phi$.

If we add to this the rule *modus ponens*, we obtain the modal system **K4**. Later, building on work by Hilbert and Bernays, Martin Löb identified ([Löb, 1955]) one last principle that he showed is sufficient, given the other axioms, for a schematic proof of Gödel's Second Incompleteness Theorem. The extra principle is known as Löb's Axiom (stated here in the anachronistic modal language):

L: $\Box(\Box\phi \to \phi) \to \Box\phi$

The resulting extension of **K4** is called **GL** (named after Gödel and Löb). Essentially what Gödel, Löb, and Hilbert and Bernays showed is that if ϕ is a theorem of **GL**, then for all functions f that send propositional variables to arbitrary arithmetical formulas, $\Box\psi$ to $Bew(\psi^{\#})$, and that commute with boolean operations, $f(\phi)$ is a theorem of PA. And this is indeed enough to prove the Second Incompleteness Theorem using purely modal reasoning: If PA could prove its consistency statement, $\neg\Box\bot$, then $\Box\bot \to \bot$ would follow. By rule *Nec*, we would have $\Box(\Box\bot \to \bot)$, and finally by Axiom L and one application of *modus ponens*, $\Box\bot$, that is, PA would be inconsistent.

The question eventually arose whether **GL** encompasses all schemata provable of the provability predicate. Provoked by a notice to the American Mathematical Society written by George Boolos, Robert Solovay proved **GL** is arithmetically complete.[2]

Theorem 1.1.1 ([Solovay, 1976]). **GL** $\vdash \phi$ *iff* $\mathsf{PA} \vdash f(\phi)$, *for all* f.

The logic **GL** is sound and complete with respect to the class of finite, rooted, partially ordered Kripke frames (henceforth **GL**-frames), a result first published by Krister Segerberg ([Segerberg, 1971]). The basic idea of Solovay's proof is to embed an arbitrary **GL**-model into Peano Arithmetic, so that any refuting Kripke model becomes an arithmetical counterexample.

This striking correspondence between modal logic and arithmetic opened up the possibility of proving interesting arithmetical results using purely modal methods, one of the most famous examples being the de Jongh-Sambin Fixed Point Theorem (in fact proven even before the publication of Solovay's paper), which shows when and why certain formulas have explicit and unique solutions, thus generalizing the original fixed point theorem. For, in a very definite sense, **GL** tells us everything there is to know about what a reasonable arithmetical theory can say about the behavior of its own provability predicate.

[2]Solovay also proved that the non-normal logic **GLS**, obtained by extending **GL** with the schema $\Box\phi \to \phi$ and rejecting the necessitation principle, encompasses all *true* schemata involving the provability predicate. For a historical overview of the early development of provability logic, see [Boolos and Sambin, 1991].

1.1.2 From Provability to ω-Provability to Reflection Principles

Gödel's original proofs of the incompleteness theorems did not apply to all consistent arithmetical theories, but merely to theories that were ω-consistent. Recall that T is ω-inconsistent if there is an arithmetical formula $A(x)$ such that, $\mathsf{T} \vdash \exists x A(x)$, but for each natural number n, $\mathsf{T} \vdash \neg A(\mathbf{n})$.[3] T is ω-consistent if it is not ω-inconsistent. ω-consistency is stronger than consistency: If T is ω-consistent, then there is some formula that T does not prove (either $\neg A(\mathbf{n})$ for some n, or $\exists x A(x)$, for every predicate $A(x)$), so it is certainly consistent; however, there are examples of consistent theories that are not ω-consistent (e.g. add the sentence $Bew(\mathbf{0{=}1})$ to PA). And it was not until later that John Rosser extended the range of incompleteness to consistent theories *tout court*.

We say A is *ω-provable* in a theory T, if the theory $\mathsf{T} + \neg A$ is ω-inconsistent.[4] As ω-consistency implies consistency, so also ω-provability implies provability: If A is not provable, then $\mathsf{T} + \neg A$ is consistent, i.e. not inconsistent, hence not ω-inconsistent, and so A is not ω-provable. George Boolos was the first to address the logic of ω-provability for PA ([Boolos, 1980]). He found that it is also **GL**, and with some minor adjustments the Solovay-style arithmetical completeness proof can be used.

The next natural question was to determine the joint logic of provability and ω-provability so as to capture the exact relationship between them. Several principles we can see right away must be valid. Let us write [0] for normal provability and [1] for the natural arithmetical formalization of ω-provability as a Σ_3-predicate. We know both [0] and [1] satisfy all of the axioms of **GL**. Moreover, by our earlier observation, we obviously have,

$$[0]\phi \to [1]\phi$$

It is also not difficult to see that this principle should also be part of the logic:

$$\neg[0]\phi \to [1]\neg[0]\phi$$

Reasoning in PA, suppose A is not provable. Then no number n is the code of a proof of A, that is, $\neg B(\mathbf{n}, A^{\#})$ is true for all n, which means that $\mathsf{PA} \vdash \neg B(n, A^{\#})$ for all n. But then it is easy to see that $\mathsf{PA} + \exists x B(x, A^{\#})$ will be ω-inconsistent, that is to say, $\neg\exists x B(x, A^{\#})$ is ω-provable. And $\exists x B(x, A^{\#})$ is exactly the definition of $Bew(A^{\#})$, so $\neg Bew(A^{\#})$ is ω-provable.

Giorgi Japaridze was the first to prove that these principles are indeed sufficient ([Japaridze, 1985]). The proof is a non-trivial variation on Solovay's method for proving arithmetical completeness of **GL**.

In a certain sense, the set of ω-provable sentences of PA gets us one step closer to the set of *true arithmetical sentences* than the set of normally provable sentences. This can be better seen by considering an alternative, but equivalent,

[3] For convenience, we use bold notation for the arithmetical representation of numerals and logical symbols.

[4] The presentation in this section is very much based on that in [Boolos, 1993]. This is also an excellent source for the proof of Japaridze's Theorem 1.1.3 in the bimodal case.

formulation of ω-provability. Let us say A is *provable by one application of the ω-rule* if there is some formula $B(x)$ such that $\mathsf{PA} \vdash B(\mathbf{n})$ for all n, and $\mathsf{PA} \vdash \forall x B(x) \to A$. It is easy to see that these two concepts coincide. That provability by one application of the ω-rule implies ω-provability is obvious. For the other direction, if A is ω-provable, there is some $B(x)$ such that (using the Deduction Rule), for all n, $\mathsf{PA} \vdash \neg A \to \neg B(\mathbf{n})$, but $\mathsf{PA} \vdash \neg A \to \exists x B(x)$. It follows that, $\mathsf{PA} \vdash \forall x(\neg A \to \neg B(x)) \leftrightarrow \bot$, and in particular $\mathsf{PA} \vdash \forall x(\neg A \to \neg B(x)) \to A$. So A is provable by one application of the ω-rule.

Taking provability by one application of the ω-rule as an alternative definition of ω-provability, this notion naturally gives rise to a whole succession of stronger and stronger provability predicates. We could then let [2] correspond to the formalization of *provable by two applications of the ω-rule*, and so on for all natural numbers. In this way, adding more and more sentences to this list of "provable" formulas, we gradually approach the standard model of all true arithmetical sentences, since infinitely many applications of the rule gives all such sentences. Obviously the logic of "n-provability" for each n will mirror that of normal provability, just as 1-provability does. And the relationship between n- and $n+1$-provability is analogously captured by that between 0- and 1-provability. Thus, we are ready to give the formal definition of **GLP** (**P** for "polymodal"):

Definition 1.1.2. The logic **GLP** is defined by the following axioms, for $n < \omega$:

(i) All tautologies

(ii) $[n](\phi \to \psi) \to ([n]\phi \to [n]\psi)$

(iii) $[n]([n]\phi \to \phi) \to [n]\phi$

(iv) $\langle n \rangle\phi \to [n+1]\langle n \rangle\phi$

(v) $[n]\phi \to [n+1]\phi$

GLP is closed under modus ponens, and $[n]$-necessitation for all $n < \omega$.

As in the case of **GL**, we put a natural restriction on the functions f from modal formulas to arithmetical formulas, in particular so that, e.g. $[n]\phi$ is always mapped to the formalization of "ϕ is provable by n applications of the ω-rule." Then, Japaridze's Theorem can be stated:

Theorem 1.1.3 ([Japaridze, 1985]). *$\mathbf{GLP} \vdash \phi$ iff $\mathsf{PA} \vdash f(\phi)$, for all f.*

The interpretation of **GLP** we have been discussing so far is mostly of historical interest, e.g. given Gödel's original proof of the incompleteness theorem. This is also the original interpretation that Japaridze considered. However, from a proof theoretic point of view, the formalization of ω-provability is somewhat of an incidental object. As Craig Smoryński has argued ([Smoryński, 1975]), the notable historical uses of ω-consistency, e.g. in the first incompleteness theorem, have been dispensable; and moreover the notion of ω-consistency is naturally encompassed by the more general and canonical notion of *reflection principles*, sentences of the form,

$$Bew(A^{\#}) \to A$$

When such principles are restricted to subclasses of the arithmetical hierarchy, a more natural definition of n-provability arises.

Where $\mathsf{Th}_{\Pi_n}(\mathbb{N})$ is the set of *true* Π_n *arithmetical sentences*, let us say a theory T is n-consistent if $\mathsf{T} + \mathsf{Th}_{\Pi_n}(\mathbb{N})$ is consistent, otherwise it is n-inconsistent. It is known that the statement of a theory's n-consistency is equivalent (provably in a weak arithmetic) to Σ_n-reflection over that theory. And a natural notion of n-provability of a sentence can be defined analogously to the case of ω-provability:

A sentence A is n-provable in T if the theory $\mathsf{T} + \neg A$ is n-inconsistent.

So n-provability becomes a natural Σ_{n+1} predicate meaning, "provable from T along with all true Π_n sentences," and the connection to reflection principles thereby becomes more evident.[5]

Theorem 1.1.3 was extended by Konstantin Ignatiev ([Ignatiev, 1993]) to a wider class of possible interpretations of $[n]\phi$. He isolated minimal requirements for an interpretation to give rise to **GLP**, and n-provability turns out to be the weakest possible. All of these considerations together suggest that n-provability takes precedence as the standard arithmetical interpretation of **GLP**. The application discussed in the next section strengthens this suggestion yet further.

1.2 GLP and Ordinal Analysis

One of the traditional aims of proof theory has been to assign appropriate ordinals to arithmetical theories. By appropriate, it is meant, on the one hand, that the assignment should provide a comparative measure of "strength" of different theories, most notably consistency strength. On the other, it should be possible to glean computational information about such theories from their respective ordinals, e.g. a characterization of the class of functions the theory is able to prove total. Ordinal analysis began with Gerhard Gentzen, who proved that Peano Arithmetic is consistent by using induction up to the ordinal ϵ_0, that is, the least ordinal fixed point of the equation $\omega^\alpha = \beta$. In the same paper, Gentzen showed that PA is able to verify transfinite induction for any arithmetical predicate up to any ordinal less than ϵ_0. Thus, a reasonable idea for the definition of the *proof-theoretic ordinal* of a theory T could be something like, "the least ordinal needed to prove the consistency of T," or else, "the supremum of the ordinals (order types) that T is able to prove are well-founded."

The problem with these definitions is that pathologies creep in where details, such as how ordinals are to be represented in the theory, are left vague and open-ended. For instance, Georg Kreisel has shown that it is possible to define a (primitive recursive) ordering of type ω so that even very weak theories can prove the consistency of quite strong theories, like PA, using induction on this

[5] See [Beklemishev, 2005] for more details.

ordering. As a converse to this, Lev Beklemishev has shown that for any ordinal $\lambda < \omega_1^{CK}$ there is a (primitive recursive) well-order of order type λ such that no sufficiently modest theory can prove the consistency of PA, even with transfinite induction up to λ.[6] One lesson that could be drawn from such examples is that the notion of a proof-theoretic ordinal should abstract away as much as possible from the particular syntactic details of the theory in question, and that the assignment of an ordinal should be somehow *canonical*. The problem then becomes, what sort of information *should be* relevant to determining a theory's ordinal? And what exactly does, or should, *canonical* mean here?

In [Beklemishev, 2004a], Beklemishev proposes that the relevant information is captured, roughly, by the logic **GLP**. More particularly, he considers the Lindenbaum-Tarski Algebra of **GLP** (See Definition 3.1.10), in which terms corresponds to polymodal formulas and the identities are exactly the theorems of **GLP**. On the 0-generated free subalgebra, corresponding exactly to the set of closed formulas of **GLP**, a primitive recursive relation $<_0$ is defined:

$$\phi <_0 \psi \;\Leftrightarrow\; \mathbf{GLP} \vdash \psi \to \langle 0 \rangle \phi$$

As we shall see later in this thesis (Section 2.3), the order type of $<_0$ is exactly ϵ_0 when T is a sound arithmetical theory. Given the arithmetical interpretation of **GLP** as n-provability, a direct connection can be made between n-provability and quantifier complexity. Finally, by defining a bijection between elements of this algebra and ordinals less than ϵ_0 (see Definition 2.4.4), an ordinal notation system is established that essentially depends only on the natural quantifier complexity levels of the arithmetical language. This treatment allows for a very simple, and "high-level" proof of Gentzen's result on the consistency of PA, as well as a more refined ordinal analysis of theories weaker than PA.

As this seems to be a promising and attractive approach to one of the central problems in proof theory and mathematical logic generally, the closed fragment of **GLP**, and the associated modal algebra, holds special interest.

1.3 Outline of the Thesis

The thesis is divided into three main sections. In the first section, we investigate relational models of **GLP**. After presenting a simplified treatment of Beklemishev's blow-up model construction, we exploit completeness for such models to obtain a new, purely semantic proof that $\mathbf{GLP}_0$ is complete with respect to Ignatiev's universal frame $\mathcal{U}$. Following this, we investigate *formula definable subsets* of $\mathcal{U}$ in anticipation of our work in the next two sections.

In the second section, we explore the connection between the frame $\mathcal{U}$ and the canonical frame of $\mathbf{GLP}_0$. Using the theory of descriptive frames, we extend $\mathcal{U}$ to a frame $\mathcal{V}^c$ that is isomorphic to the canonical frame, thus obtaining a detailed definition of this object in terms of a coordinate structure we develop in Section 2.

[6]Recall that ω_1^{CK} is the least ordinal that cannot be represented as the order type of a recursive well-ordering. For details on these two examples, see [Pohlers, 1999].

Finally, in the last section, we explore topological models of **GLP**. The central result of this section is an analog of the Abashidze-Blass Theorem for **GL**, to the effect that $\mathbf{GLP}_0$ enjoys topological completeness with respect to a simply defined polytopology on the ordinal $\epsilon_0 + 1$. This space can be seen as a condensed, and much simplified, version of the canonical frame of $\mathbf{GLP}_0$. We then consider the possibility of an extension of this theorem to full **GLP**. However, such an extension would require large cardinal assumptions beyond ZFC, so we leave this further question for future work.

2 Relational Models

2.1 Frame Incompleteness

GLP is the normal polymodal logic defined as follows:

Definition 2.1.1 (The Logic GLP). For each $n < \omega$, extend **K** by the axioms:

(iii) $[n]([n]\phi \to \phi) \to [n]\phi$

(iv) $\langle n \rangle \phi \to [n+1]\langle n \rangle \phi$

(v) $[n]\phi \to [n+1]\phi$

As usual, we would like to know what class of frames, if any, this logic defines. Unfortunately, in this case, the answer is that there is no single frame on which **GLP** is sound. Recall the relational semantics of **GL**:[7]

Fact 2.1.2 ([Segerberg, 1971]). *$\mathbb{F} \vDash [n]([n]\phi \to \phi) \to [n]\phi$, if and only if R_n is transitive and converse-well-founded.*

Axiom (iv) also defines a property on frames:[8]

Fact 2.1.3. *$\mathbb{F} \vDash \langle n \rangle \phi \to [n+1]\langle n \rangle \phi$, if and only if xR_ny, $xR_{n+1}z$ imply zR_ny.*

Axiom (v) does as well:

Fact 2.1.4. *$\mathbb{F} \vDash [n]\phi \to [n+1]\phi$, if and only if $xR_{n+1}y$ implies xR_ny.*

Remark 2.1.5. Let R_n be the set of pairs (x, y) such that xR_ny, and let R_n^{-1} be its inverse. Then $R_n(X) := \{y : \exists x \in X, xR_ny\}$, and $R_n^{-1}(X) := \{x : \exists y \in X : xR_ny\}$. We call a set X an R_n-*upset* if it is upwards closed with respect to R_n, i.e. $x \in X$ and xR_ny implies $y \in X$.

Axioms (iv) and (v) have interpretations in this terminology: Axiom (iv) means every set $R_n^{-1}(X)$ is an R_{n+1}-upset. And Axiom (v) means $R_{n+1} \subseteq R_n$.[9]

[7] $\mathbb{F}$ denotes an arbitrary frame in the polymodal language, and "$\vDash$" denotes modal validity.
[8] Facts 2.1.3 and 2.1.4 are very easily proven. See, e.g. [Boolos, 1993].
[9] This formulation is due to Guram Bezhanishvili.

No non-trivial frame satisfies all of these requirements. Suppose for a contradiction that **GLP** is sound with respect to a frame $\mathbb{F}$ with R_1 non-empty. Then there are x, y such that xR_1y, and by Fact 2.1.4, xR_0y. By Fact 2.1.3, yR_0y, which contradicts Fact 2.1.2. Consequently, on any frame for which **GLP** is sound, R_n must empty, so $[n]\bot$ becomes valid for all $n > 0$. We state this result as a limitative theorem:

Theorem 2.1.6. **GLP** *is incomplete with respect to its class of frames. In particular,* **GLP** *is not sound on any frame for which $R_n \neq \emptyset$ for $n > 0$.*

2.2 Models of GLP

Faced with the impossibility of obtaining a class of frames for **GLP**, one might nonetheless hope a reasonable class of models for **GLP** is obtainable. Beklemishev has isolated such a class for which **GLP** is sound and complete. We shall outline the basic idea of this approach, simplifying where possible, and use it to study models of the closed fragment. Since many of the results in this first part are derived from [Beklemishev, 2007a], we often refer to this paper when proofs would take us too far afield. We are most concerned with presenting a readable treatment that gives the basic idea of the construction, and of what models of **GLP** look like.

Following [Beklemishev, 2007a], our strategy will be as follows. We first identify a logic **J** that approximates **GLP**, but in which the "monotonicity axiom," Axiom (v), is weakened. **J** enjoys completeness with respect to a simple class of frames, which satisfy all of the axioms of **GLP**, with the exception of Axiom (v). Models of **GLP** will then be obtained by defining a certain operation, called the *blow-up operation*, over models based on **J**-frames. The resulting models, called *blow-up models*, will simultaneously be models of **J** and satisfy Axiom (v); that is, they will be models of **GLP**.

Before going further, we define a model property that serves as a sufficient condition for the monotonicity axiom to hold. Recall the definition of m-bisimilarity:

Definition 2.2.1 (n-bisimilarity). We define $x \sim_m x'$ by recursion on m:

- $x \sim_0 x'$ if and only if x and x' force all the same variables
- $x \sim_{m+1} x'$ if and only if each of the following holds:
  - $x \sim_m x'$
  - $\forall k \geq 0 \, \forall y(xR_ky \Rightarrow \exists y'(x'R_ky' \; \& \; y \sim_m y'))$
  - $\forall k \geq 0 \, \forall y'(x'R_ky' \Rightarrow \exists y(xR_ky \; \& \; y' \sim_m y))$

Let us write $dp(\phi)$ for the *depth* of ϕ, that is, the maximal number of nested modalities in ϕ. The following fact is well known.[10]

[10]See, e.g. [Beklemishev, 2001].

Fact 2.2.2. *If $x \sim_n x'$, then if $dp(\phi) \leq n$, $x \vDash \phi \iff x' \vDash \phi$.*

Definition 2.2.3. A model $\mathcal{A}$ satisfies the *m-similarity property* for R_n if,

$$\forall x, y \in \mathcal{A}(xR_{n+1}y \Rightarrow \exists y'(xR_n y' \ \& \ y \sim_m y'))$$

Lemma 2.2.4. *If $\mathcal{A}$ satisfies the m-similarity property for R_n, then*

$$\mathcal{A} \vDash [n]\phi \to [n+1]\phi$$

for every ϕ with $dp(\phi) \leq m$.

Proof. If $\mathcal{A}, x \nvDash [i+1]\phi$, then there is some $y \in \mathcal{A}$ such that $xR_{i+1}y$ and $\mathcal{A}, y \nvDash \phi$. By the n-similarity property, there is some y' such that $xR_i y'$ and $y \sim_n y'$. Hence $\mathcal{A}, x \nvDash [i]\phi$. ⊣

Given the completeness result that we will obtain in Theorem 2.2.36, we can give relatively abstract conditions for a class of models for which **GLP** is sound and complete:

Proposition 2.2.5. **GLP** $\vdash \phi$*, if and only if ϕ is valid in all converse-well-founded, transitive models with the m-similarity property for all m and all R_n, and in which each $R_n^{-1}(A)$ is an R_{n+1}-upset.*

This section is dedicated to giving a sense of how such models are obtained.

2.2.1 The Logic J

Definition 2.2.6. **J** is the sublogic of **GLP** defined by the weaking the monotonicity axiom (v) of Definition 2.1.1 to axioms (vi) and (vii) below:

(vi) $[n]\phi \to [n+1][n]\phi$

(vii) $[n]\phi \to [n][n+1]\phi$

As usual, we close under modus ponens and $[n]$-necessitation.

One can easily show that, indeed, (vi) and (vii) are already theorems of **GLP**. As we said earlier, **J** enjoys a relatively simple frame semantics:

Definition 2.2.7. A modal frame is called a **J**-frame if, for all n, R_n is a converse-well-founded, transitive relation, and, moreover, it satisfies the properties (I) and (J) for all such n and m:

(I) $\forall x, y(xR_n y \Rightarrow \forall z(xR_m z \Leftrightarrow yR_m z))$, if $m < n$.

(J) $\forall x, y(xR_m y \ \& \ yR_n z \Rightarrow xR_m z)$, if $m \leq n$.

Condition (I) simply says that each $R_n^{-1}(A)$ is always an R_{n+1}-upset, thus ensuring the validity of Axiom (iv). See Figures 1 and 2.

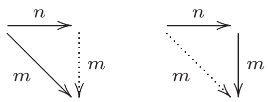

Figure 1: Frame Condition (I)

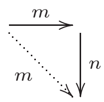

Figure 2: Frame Condition (J)

Theorem 2.2.8 ([Beklemishev, 2007a]). **J** *is sound and complete with respect to (finite)* **J***-frames.*

This theorem is proven using a variation of the standard filtration method used to prove completeness of **GL** ([Segerberg, 1971]). The class of **J**-frames can be improved to another, more restricted class.

Definition 2.2.9. A frame is called a *stratified frame* if it is a **J**-frame and moreover satisfies the property (S):

(S) $\forall x, y, z(zR_m x \;\&\; yR_n x \Rightarrow zR_m y)$, if $m < n$

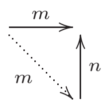

Figure 3: Frame Condition (S)

Stratified frames have a certain structure that makes them especially pleasant to work with. For any stratified frame, the relation R_0 defines a strict, well-founded partial ordering on what we shall call 1-*sheets*; 1-sheets are simply submodels all of whose members are R_i-related to one another, for $i > 0$.

Definition 2.2.10. A subframe of a stratified frame is an *n-sheet* if it is closed under the operation R_n^*, where xR_n^*y if for some $m \geq n$, $xR_m y$ or $yR_m x$.

In turn, each 1-sheet can be seen as a partial ordering of 2-sheets, and so on for all n. This also means, in general, we can talk about $n+1$-sheets being R_n-related to one another: Letting ordinals serve as variables for sheets, if α and β are $n+1$-sheets, $x \in \alpha$, $y \in \beta$, and $xR_n y$, then, by properties (J) and (S), all elements of α are R_n-related to all elements of β.

Within the class of stratified frames, we can single out an even more restrictive class:

Definition 2.2.11. A stratified frame is *hereditarily rooted* if for all n-sheets, there is an $n+1$-sheet α, such that $\alpha R_n \beta$ for all other $n+1$-sheets β.

Theorem 2.2.12 ([Beklemishev, 2007a]). **J** *is sound and complete with respect to (finite) stratified, hereditarily rooted frames.*

Our interest in the closed fragment leads us to the following subclass:

Definition 2.2.13. Let us say a stratified frame is *hereditarily linear*, or *h.l.* for short, if it satisfies property (L):

(L) $\forall x, y, z\ (xR_n y \ \&\ xR_n z \ \Rightarrow\ \exists m \geq n(yR_m z \text{ or } zR_m y \text{ or } z = y))$

In other words, h.l. stratified frames can be visualized as those stratified frames in which R_n defines a transitive, linear ordering of R_{n+1}-sheets for all n. This class holds particular interest because it is exactly the class of frames for the closed fragment of **J**. Just as the closed fragment of **GL** is complete with respect to converse-well-founded, transitive linear frames, the closed fragment of **J** is complete with respect to h.l. stratified frames. In particular, we have:

Lemma 2.2.14. *The root point of any rooted, stratified frame is point-wise bisimilar to the root point of some h.l. stratified frame.*

Proof. The R_n*-depth* of a point in a stratified frame is the length of the greatest R_n-chain beginning at that point. Given a rooted stratified frame $\mathcal{A}$, define an equivalence relation on $\mathcal{A}$ so that xEy if and only if x and y have the same R_n-depth for all n. Let $\mathcal{A}_E$ be the frame consisting of these equivalence classes, and let $[x]R_n[y]$ in $\mathcal{A}_E$ if and only if there is some $x \in [x]$ and some $y \in [y]$ such that $xR_n y$ in $\mathcal{A}$ (or equivalently, if and only if for all $x \in [x]$ there is some $y \in [y]$ such that $xR_n y$ in $\mathcal{A}$). Where r is the root point of $\mathcal{A}$, we must check that r and $[r]$ are bisimilar, and that $\mathcal{A}_E$ is a h.l. stratified frame.

That r and $[r]$ are point-wise (frame) bisimilar follows immediately from the definition of $\mathcal{A}$.[11] Noting that all paths are finite, if $rR_i y..R_j x$ is any path in $\mathcal{A}$ it can be imitated by the path $[r]R_i[y]...R_j[x]$ in $\mathcal{A}_E$. For the other direction, if $[r]R_i[y]...R_j[x]$, then taking any representative w of $[r]$ will give us some z in $[y]$ such that $wR_i z$, and so on up to $[x]$.

$\mathcal{A}_E$ is a stratified frame because $\mathcal{A}$ is. E.g., to see condition (S), suppose $[z]R_n[x]$ and $[y]R_{n+1}[x]$. Then there are $z \in [z]$, $x \in [x]$, and $y \in [y]$ such that $zR_n x$ and $yR_{n+1} x$ in $\mathcal{A}$, so since (S) holds in $\mathcal{A}$, $zR_n y$, and so $[x]R_n[y]$ in $\mathcal{A}_E$.

It remains to show hereditary linearity. Suppose $[x]R_n[y]$ and $[x]R_n[z]$. We must show either $[z]R_k[y]$ or $[z]R_k[y]$ for some $k \geq n$, or $[z] = [y]$. Since $[y]$ and $[z]$ are in the same n-sheet, they have the same R_i-depth for all $i < n$. If $[z] \neq [y]$, suppose without loss that k is the least such that $[z]$ has greater R_k-depth than $[y]$. Then $[z]R_k[w]$, where $[w]$ has the same R_k-depth as $[y]$. $[w]$ also has the same R_i-depth for all $i < n$, again because $[w]$ and $[y]$ are in the same n-sheet. We must show they have the same R_i-depth for $n \leq i < k$ as well: This follows because, for any such i, $[z]$ and $[y]$ have the same R_i-depth,

[11] Yet, not all points in $\mathcal{A}$ will be bisimilar to the corresponding equivalence class in $\mathcal{A}_E$.

and if $[z]R_i[u]$, then by Condition (I), $[w]R_i[u]$ as well. Consequently, we can continue in this way: If there is a j such that $[w]$ and $[y]$ have different R_j-depth for $j > k$, then repeat the same process. Since $\mathcal{A}_E$ is well-founded, this will eventually come to an end. We will obtain some point $[v]$ such that (using (J)), either $[z]R_i[v]$ for $i > n$ or $[z] = [v]$, and either $[y]R_j[v]$ for $j > n$ or $[y] = [v]$. If either equality holds, we are done. We cannot have $i = j$, because then they would be in the same i-sheet, which contradicts the fact that, e.g. $[z]$ has greater R_i-depth than $[y]$. So, if $i > j$, then by (S), $[y]R_j[z]$. And if $i < j$, then again by (S), $[z]R_i[y]$. $\dashv$

Corollary 2.2.15. *If ϕ is a closed formula and $\mathbf{J} \nvdash \phi$, then there is a h.l. stratified frame $\mathcal{A}$ such that $\mathcal{A} \nvDash \phi$.*

Proof. If $\mathbf{J} \nvdash \phi$, then ϕ is falsified at the root x of some hereditarily rooted stratified frame $\mathcal{A}$. By Lemma 2.2.14, x is bisimilar to some point x' in a h.l. stratified frame $\mathcal{A}'$. As a closed formula, ϕ is obviously falsified at x' in $\mathcal{A}'$. $\dashv$

2.2.2 Beklemishev's Blow-up Models

We present a different treatment of blow-up models from [Beklemishev, 2007a]. First of all, whereas Beklemishev defines blow-ups to be finite objects and obtains models of **GLP** as inverse limits of these objects, our construction is slightly more general. Our blow-up operation applies directly to infinite objects, and we obtain infinite objects already in the first stage of the construction. This has advantages and disadvantages. On the one hand, it is significantly simpler and less involved than the treatment in [Beklemishev, 2007a]. On the other hand, we are unable to reason about our construction in a finitistic manner. Another difference is that we are mainly interested in blow-ups of h.l. stratified frames. Therefore, while we believe our treatment applies equally well to the case of arbitrary hereditarily rooted stratified models, we focus on this case.

Blow-up models will be introduced in two stages. First we define what is called *sheet-wise blow-up*, which takes an $n+1$-sheet and turns it into a much larger n-sheet satisfying the m-similarity property for R_n. Then, in order to ensure that the m-similarity property holds for all R_n simultaneously, we will introduce *global blow-up*.

For the next two definitions, suppose $(I, \mathcal{R})$ is a converse-well-founded, transitive, linearly ordered set, and we have a 0-sheet α_i for each $i \in I$.

Definition 2.2.16. Let $\sum_{i<\omega} \alpha_i$, the *disjoint sum*, denote the disjoint union of the α_i's, R_0-ordered so that xR_0y if and only if $x, y \in \alpha_i$ and xR_0y, or $x \in \alpha_i$ and $y \in \alpha_j$ and $i\mathcal{R}j$. For the binary case, we write $\alpha_i + \alpha_j$.

We call an injective p-morphism an *end-embedding*.[12] We define the direct limit of end-embeddings as follows:

[12]See [Chagrov and Zakharyaschev, 1997] or [Blackburn et al., 2001] for the definition of p-morphism, in the latter referred to as *bounded morphism*.

Definition 2.2.17 (Direct Limit). Suppose whenever $i\mathcal{R}j$, there is an associated end-embedding $\upsilon_{ij} : \alpha_i \longrightarrow \alpha_j$, such that:

- υ_{ii} is the identity map on α_i.
- $\upsilon_{jk} \circ \upsilon_{ij} = \upsilon_{ik}$, if $i\mathcal{R}j\mathcal{R}k$.

Then we define $\lim_{i\in I} \alpha_i$ as the disjoint union of the α_i's, modulo the equivalence relation $\sim_I$ defined as the transitive, symmetric closure of:

$$x \sim_I y \iff \exists i, j \in I(i\mathcal{R}j \text{ or } i = j, x \in \alpha_i, y \in \alpha_j, \text{ and } y = \upsilon_{ij}(x))$$

The relations are defined for equivalence classes:

$$[x]R_n^I[y] \iff \exists i, \exists x', y' \in \alpha_i(x \sim_I, y \sim_I y' \text{ and } x'R_n y')$$

Finally, validity is defined:

$$\lim_{i\in I} \alpha_i, [x] \vDash \phi \iff \exists i, \exists x' \in \alpha_i(x \sim_I x' \text{ and } \alpha_i, x' \vDash \phi)$$

Essentially, the direct limit identifies exactly those points that are mapped one to the other by the end-embeddings υ_{ij}. The key fact about such limits is the following:

Lemma 2.2.18. *If each of the α_i's is a hereditarily linear stratified model, then $\lim_{i\in I} \alpha_i$ is a hereditarily linear stratified model.*

Proof. This is a straightforward consequence of the Definition 2.2.17. $\dashv$

Remark 2.2.19. We shall find it convenient to abuse notation somewhat and write $\bigcup_{i\in I} \alpha_i$, instead of $\lim_{i\in I} \alpha_i$. End-embeddings are, after all, embeddings, so it is often more intuitive to see each α_i as a submodel of the direct limit.

Without further ado, we introduce the first blow-up notion, *sheet-wise blow-up*. Where α is a 1-sheet, and ρ its root 2-sheet (we write α_ρ to make this more apparent), the resulting 0-sheet, $\alpha_\rho^{(\omega)}$, will give us the m-similarity property for R_0 without leaving the class of hereditarily linear stratified models.[13]

Definition 2.2.20 (Sheet-Wise Blow-Up). We define the *sheet-wise blow-up* of α_ρ, denoted $\alpha_\rho^{(\omega)}$, by induction on R_1-successors of ρ. If α_ρ consists only of the 2-sheet ρ, then $\alpha_\rho^{(\omega)}$ is defined to be the same 1-sheet, now considered as a 0-sheet with R_0 empty. Otherwise, suppose $\alpha_\sigma^{(\omega)}$ is defined for any σ such that $\rho R_1 \sigma$. For each such σ, let $\mathcal{B}_\sigma := \sum_{i<\omega} \alpha_\sigma^{(\omega)}$, that is, ω-many copies of $\alpha_\sigma^{(\omega)}$ conversely-well-founded and linearly ordered by R_0. We inductively assume whenever $\rho R_1 \sigma R_1 \sigma'$, there are natural end-embeddings $\upsilon_{\sigma'\sigma} : \mathcal{B}_{\sigma'} \longrightarrow \mathcal{B}_\sigma$. Appealing to Definition 2.2.17 and Lemma 2.2.18, $\alpha_\rho^{(\omega)}$ is obtained by putting $\bigcup_{\rho R_1 \sigma} \mathcal{B}_\sigma$, the

[13]It will often be convenient to give definitions and prove results for 0-sheets, tacitly using the obvious fact that these notions can be "lifted" to n-sheets.

direct limit, R_0-above a copy of α_ρ. Thus, the inductive end-embedding $\upsilon_{\sigma\rho}$: $\mathcal{B}_\sigma \longrightarrow \mathcal{B}_\rho$ becomes the composition of the following obvious end-embeddings:

$$\mathcal{B}_\sigma \longrightarrow \bigcup_{\sigma R_1 \sigma'} \mathcal{B}_{\sigma'} \longrightarrow \alpha_\rho^{(\omega)} \longrightarrow \sum_{i<\omega} \alpha_\rho^{(\omega)} = \mathcal{B}_\rho$$

Corollary 2.2.21. *If α_ρ is a hereditarily linear, stratified model, so is $\alpha_\rho^{(\omega)}$.*

Effectively, the blow-up operation turns a 1-sheet ($n+1$-sheet) into a much larger 0-sheet (n-sheet). To work with such models, it will help to label certain parts of the structure. In that direction we shall let $[\alpha_\sigma^i]^{(\omega)}$ stand for the i^{th} copy of $\alpha_\sigma^{(\omega)}$ within $\sum_{i<\omega} \alpha_\sigma^{(\omega)}$. So we will have, e.g.,

$$\alpha_\rho R_0 ... R_0 [\alpha_\sigma^{i+1}]^{(\omega)} R_0 [\alpha_\sigma^i]^{(\omega)} R_0 ... R_0 [\alpha_\sigma^0]^{(\omega)}$$

(and the transitive closure thereof) for any σ for which $\rho R_1 \sigma$. Furthermore, each $[\alpha_\sigma^i]^{(\omega)}$ has a copy of α_σ as its root, and this will be denoted α_σ^i.

We say a point $x \in \alpha_\rho^{(\omega)}$ has *level* i, written $l(x) = i$, if $x \in [\alpha_\sigma^i]^{(\omega)}$ for $\rho R_1 \sigma$. If x is in the root, then $l(x) = \infty$. We also define natural projection functions for this structure:

Definition 2.2.22. We recursively define a *projection* function $\pi_\rho : \alpha_\rho^{(\omega)} \to \alpha_\rho$. First, π_ρ is identity on α_ρ. Assume $\pi_\sigma^i : [\alpha_\sigma^i]^{(\omega)} \to \alpha_\sigma^i$ has been defined for each σ for which $\rho R_1 \sigma$ and for all $i < \omega$. If $x \in [\alpha_\sigma^i]^{(\omega)}$, then $\pi_\rho(x) := y$ where $y \in \alpha_\sigma$ is the point corresponding to $\pi_\sigma^i(x)$ in the isomorphic structure α_σ^i.

Fact 2.2.23. *π_ρ restricted to any 1-sheet in $\alpha_\rho^{(\omega)}$ is an end-embedding.*

Proof. See [Beklemishev, 2007a]. $\dashv$

With these definitions and facts in hand, we can show that blowing up sheet-wise gets us part of the way to the global k-similarity property.

Lemma 2.2.24. *Suppose $a \in \alpha_\rho$. Each point $x \in \alpha_\rho^{(\omega)}$ such that $x \in \pi_\rho^{-1}(a)$ and $l(x) \geq k$ is k-bisimilar to a (and thus also to every other such point).*

Proof. We show this by induction on k. The case of $k = 0$ follows trivially since the valuation on all points in $\pi_\rho^{-1}(a)$ is the same. So consider the case of $k+1$. We show each of the conditions from Definition 2.2.1 in turn. For the first condition, we have $x \sim_k a$ by the induction hypothesis.

Next, suppose $xR_m y$. If $m = 0$, then certainly $aR_m y$ by the construction. If $m > 0$, then $aR_m b$ for $b = \pi_\rho(y)$, and by induction hypothesis $b \sim_k y$.

For the last condition, suppose $aR_m b$. If $m > 0$, then b is in the root as well, so we can take a $y \in \pi_\rho^{-1}(b)$ in the same 2-sheet as x, and $xR_0 y$ and $y \sim_k b$ by induction hypothesis and by Fact 2.2.23. On the other hand, if $m = 0$, then b must be in $[\alpha_\sigma^i]^{(\omega)}$ for some σ and some i. If $i \leq k$, then automatically $xR_0 b$ since $l(x) \geq k+1$; but if $i > k$, then we can still find the corresponding b' in $[\alpha_\sigma^k]^{(\omega)}$, and clearly $xR_0 b'$, and $b \sim_k b'$ by induction hypothesis. $\dashv$

We also have the following elementary fact about n-bisimilations:

Fact 2.2.25. *Suppose $\mathcal{B} \subseteq \mathcal{A}$ is a submodel such that for all $k \geq 0$, we have:*

$$\forall x, y \in \mathcal{B}\, \forall z \in \mathcal{A} \setminus \mathcal{B}\, (xR_k z \Rightarrow yR_k z)$$

Then $x \sim_n y$ in $\mathcal{B}$ if and only if $x \sim_n y$ in $\mathcal{A}$.

Lemma 2.2.26. *The model $\alpha_\rho^{(\omega)}$ satisfies the k-similarity property for R_0.*

Proof. This is proven by induction on the construction of $\alpha_\rho^{(\omega)}$. Suppose $x \in \alpha_\rho^{(\omega)}$ and xR_1y. If $x \in \alpha_\rho$, then so is y and we can find an appropriate $y' \in \pi_\rho^{-1}(y)$ such that $l(y') \geq k$. Then, by Lemma 2.2.24, $y' \sim_k y$.

If $x \in [\alpha_\sigma^i]^{(\omega)}$, by the inductive hypothesis, we can find a y' such that xR_0y' and $y \sim_k y'$ in $[\alpha_\sigma^i]^{(\omega)}$. Then $[\alpha_\sigma^i]^{(\omega)}$, as a submodel of $\alpha_\rho^{(\omega)}$, satisfies the conditions of Fact 2.2.25. So we have $y \sim_k y'$ in $\alpha_\rho^{(\omega)}$ as well. ⊣

Lemma 2.2.27. *If $m > 0$ and α_ρ satisfies the k-similarity property for R_m, then so does $\alpha_\rho^{(\omega)}$.*

Proof. By Lemma 2.2.23, every 1-sheet of $\alpha_\rho^{(\omega)}$ can be end-embedded into α_ρ, and so each such 1-sheet satisfies the k-similarity property for R_m. Since this is generally true for 1-sheets in $\alpha_\rho^{(\omega)}$, it also holds for $\alpha_\rho^{(\omega)}$ itself. ⊣

What we have argued so far is that, if we sheet-wise blow-up a 1-sheet α, we get a new 0-sheet $\alpha^{(\omega)}$, that satisfies the k-similarity property for R_0. As we noted earlier, this can automatically be lifted to the case of $n+1$-sheets, for any n. However, to obtain models of **GLP** or **GLP**$_0$, we need the k-similarity property for all R_n simultaneously. For that, we introduce the *global blow-up* operation. If $\mathcal{A}$ is a 0-sheet (i.e. a stratified model), we write $\alpha \in \mathcal{A}$ to mean α is a 1-sheet in $\mathcal{A}$, and so on for 1-sheets, 2-sheets, etc.

Definition 2.2.28 (Global Blow-Up). For any hereditarily linear, stratified model $\mathcal{A}$, we define,

$$\mathfrak{B}_\omega(\mathcal{A}) := \sum_{\alpha \in \mathcal{A}} \mathfrak{B}_\omega(\alpha)^{(\omega)}$$

That is, to obtain the blow-up of a model $\mathcal{A}$, we take the ordered sum of infinitely many copies of the blow-up of each 1-sheet in $\mathcal{A}$, and order the resulting sum by mimicking the R_0-order of the 1-sheets in $\mathcal{A}$. In turn, the blow-up of each 1-sheet is defined analogously in terms of blow-ups of its 2-sheets, and so on. As there is a hidden recursion in the definition, the operation $\mathfrak{B}_\omega$ is ambiguous in a certain sense: we must know whether the model we we are blowing up is being considered as a 0-sheet, a 1-sheet, etc. The operation is defined in such a way as to apply to n-sheets for any n. It is this operation that will give us the k-similarity property for all R_m.

Corollary 2.2.29. *If $\mathcal{A}$ is hereditarily linear and stratified, so is $\mathfrak{B}_\omega(\mathcal{A})$.*

Proof. This follows by Corollary 2.2.21 and Definition 2.2.28. ⊣

Lemma 2.2.30. *If $\mathcal{A}$ is a finite, hereditarily linear, stratified model, then $\mathfrak{B}_\omega(\mathcal{A})$ satisfies the k-similarity property for all R_m.*

Proof. Assume this holds for all 1-sheets $\alpha \in \mathcal{A}$. So, for each α, $\mathfrak{B}_\omega(\alpha)$ satisfies the k-similarity property for all R_n. We have two cases:

If $n = 0$, then indeed $\mathfrak{B}_\omega(\alpha)^{(\omega)}$ satisfies the k-similarity property by Lemma 2.2.27. By Fact 2.2.25 so does $\sum_{\alpha\in\mathcal{A}} \mathfrak{B}_\omega(\alpha)^{(\omega)}$.

If $n > 0$, then by Lemma 2.2.27, each $\mathfrak{B}_\omega(\alpha)^{(\omega)}$ also satisfies the k-similarity property. Once again, by Fact 2.2.25 this carries over to $\sum_{\alpha\in\mathcal{A}} \mathfrak{B}_\omega(\alpha)^{(\omega)}$. ⊣

When we took sheet-wise blowups of $n+1$-sheets, we maintained models of **J**, in fact, even stronger, we remained in the class of hereditarily linear stratified models (Corollary 2.2.21). Lemma 2.2.30, coupled with Lemma 2.2.4, ensures that blow-up models also satisfy each of the monotonicity schemas.

Corollary 2.2.31. *For any hereditarily linear, stratified model $\mathcal{A}$, $\mathfrak{B}_\omega(\mathcal{A})$ is a model of* **GLP**.

Definition 2.2.32. Let $M(\phi) := \bigwedge_{i<s}([m_i]\phi_i \to [m_i+1]\phi_i)$, where $[m_i]\phi_i$ for $i < s$ are all subformulas of ϕ of the form $[k]\psi$. And if $n = \max_{i<s} m_i$, let $M^+(\phi) := M(\phi) \wedge \bigwedge_{i\leq n}[i]M(\phi)$.

We clearly have:[14]

Fact 2.2.33. *If* $\mathbf{J} \vdash M^+(\phi) \to \phi$, *then* $\mathbf{GLP} \vdash \phi$.

Given Corollary 2.2.15 and Fact 2.2.33, completeness of $\mathbf{GLP}_0$ is relatively straightforward.

Theorem 2.2.34. *If* $\mathbf{GLP}_0 \nvdash \phi$, *there is an h.l. stratified model $\mathcal{A}$, such that $\mathfrak{B}_\omega(\mathcal{A}) \nvDash \phi$.*

Proof. If ϕ is a closed formula and $\mathbf{GLP}_0 \nvdash \phi$, then $\mathbf{J} \nvdash M^+(\phi) \to \phi$. By Corollary 2.2.15, there is a h.l. stratified model $\mathcal{A}$ such that $\mathcal{A} \nvDash M^+(\phi) \to \phi$. We must see that $\mathfrak{B}_\omega(\mathcal{A}) \nvDash \phi$.

Define the natural projection function $\pi^* : \mathfrak{B}_\omega(\mathcal{A}) \to \mathcal{A}$, so that π^* is the identity mapping when $\mathcal{A}$ is trivial. Otherwise, recall by Definition 2.2.28,

$$\mathfrak{B}_\omega(\mathcal{A}) = \sum_{\alpha\in\mathcal{A}} \mathfrak{B}_\omega(\alpha)^{(\omega)}$$

By inductive hypothesis, we are given a function π^*_α for $\mathfrak{B}_\omega(\alpha)$. And let $\pi_\alpha : \mathfrak{B}_\omega(\alpha)^{(\omega)} \to \mathfrak{B}_\omega(\alpha)$ be the projection function from Definition 2.2.22. Then $\pi^*(x) := \pi^*_\alpha(\pi_\alpha(x))$, whenever $x \in \mathfrak{B}_\omega(\alpha)^{(\omega)}$. We state the following lemma:

[14] As a result of Theorem 2.2.36, this is actually a biconditional. This fact allows for a proof of the Craig Interpolation Property and the Fixed Point Property for **GLP** by reasoning formalizable in PA ([Beklemishev, 2007b]). Ignatiev established in [Ignatiev, 1992] and [Ignatiev, 1993] the Craig Interpolation Property for **GLP**, but the proof used arithmetical soundness of **GLP** in the form of reflection principles, and so was not formalizable in Peano Arithmetic. Whether this could be done in PA alone was noted as an open question.

Lemma 2.2.35. *If ψ is a subformula of ϕ, then for all $x \in \mathfrak{B}_\omega(\mathcal{A})$,*

$$\mathfrak{B}_\omega(\mathcal{A}), x \vDash \psi \iff \mathcal{A}, \pi^*(x) \vDash \psi$$

Proof. The proof of this lemma is identical to that for Lemma 9.3 in the original [Beklemishev, 2007a], so we shall not repeat it here. The rough idea is simply that the composition of the natural projection functions from the "outermost" sheets all the way to the root n-sheet isomorphic to $\mathcal{A}$ is exactly the function π^*. Since each of these functions preserves and reflects modal validity (Lemma 2.2.24), so does π^*. ⊣

Lemma 2.2.35 thus completes the proof, as $\mathfrak{B}_\omega(\mathcal{A}) \nvDash M^+(\phi) \to \phi$, but since $\mathfrak{B}_\omega(\mathcal{A}) \vDash M^+(\phi)$ by Corollary 2.2.31, it follows that $\mathfrak{B}_\omega(\mathcal{A}) \nvDash \phi$. ⊣

As we noted earlier, the treatment here is significantly simpler than that in [Beklemishev, 2007a], both because our operation is well defined over infinite models, and because we concentrate on the hereditarily linear case. While we believe these results also apply equally well to arbitrary hereditarily rooted stratified models, nevertheless such a completeness result for full **GLP** has been established, and we state this result here.

Theorem 2.2.36 ([Beklemishev, 2007a]). **GLP** $\vdash \phi$*, if and only if, for all hereditarily rooted stratified models $\mathcal{A}$, $\mathfrak{B}_\omega(\mathcal{A}) \vDash \phi$.*

2.3 Ignatiev's Frame $\mathcal{U}$

As a historical point, the frame we shall introduce in this section, due to Ignatiev, was actually the prototype on which the models discussed in the last section were based. Indeed, we shall prove that, ignoring valuations, Ignatiev's frame $\mathcal{U}$ is a special case of the blow-ups of h.l. stratified frames (Corollary 2.3.11 below). However, $\mathcal{U}$ is of special interest because it is universal for the closed fragment of **GLP**; that is, every non-theorem is falsifiable on $\mathcal{U}$. By showing that the blow-up of any h.l. stratified frame is embeddable in $\mathcal{U}$, completeness of $\mathbf{GLP}_0$ with respect to the former, proven in the previous section, will translate to completeness with respect to the latter.

Remark 2.3.1. If $\mathcal{A}$ is a relational structure in the polymodal language, $\mathcal{A}^+$ is the same structure obtained by letting $xR_{n+1}y$ in $\mathcal{A}^+$ if and only if xR_ny in $\mathcal{A}$, so that R_n is empty in $\mathcal{A}^+$, for the least non-empty R_n in $\mathcal{A}$.

Recall that every ordinal can be put into normal form:

Theorem 2.3.2 (Cantor Normal Form). *Every ordinal > 0 can be written in Cantor Normal Form: in the form $\omega^{\lambda_k} + ... + \omega^{\lambda_0}$, where $\lambda_i \geq \lambda_j$ for $i > j$.*

The following function e, defined on the Cantor Normal Form of an ordinal will be used heavily in our study. It is important to notice that, for any ordinal $\alpha < \epsilon_0$, after some finite number n of iterations of e, $e^n(\alpha) = 0$.

Definition 2.3.3. If $\alpha \leq \epsilon_0$ has Cantor Normal Form $\omega^{\lambda_k} + ... + \omega^{\lambda_0}$, then we define $e(\alpha) := \lambda_0$. In particular, $e(\epsilon_0) = \epsilon_0$. We furthermore stipulate $e(0) = 0$.

We shall give two different definitions of $\mathcal{U}$. The first is a variation of Ignatiev's original definition ([Ignatiev, 1992]), due to [Beklemishev et al., 2005].

Definition 2.3.4. Define $\mathcal{U} = (U, R_n : n < \omega)$ in ϵ_0-many stages so that $\mathcal{U} = \bigcup_{\alpha<\epsilon_0} \mathcal{U}_\alpha$. $\mathcal{U}_0$ is the structure consisting of a single point with empty relations. Assuming $\mathcal{U}_\beta$ has been defined for $\beta < \alpha$, let $\mathcal{W}_\alpha$ be an isomorphic copy of $\mathcal{U}^+_{e(\alpha)}$, which is already defined, since $e(\alpha) < \alpha$ whenever $\alpha < \epsilon_0$. Then $\mathcal{U}_\alpha$ is obtained by taking the disjoint union of $\bigcup_{\beta<\alpha} \mathcal{U}_\beta$ and $\mathcal{W}_\alpha$, and by extending R_0 so that each element of $\mathcal{W}_\alpha$ is R_0-related to each element in $\bigcup_{\beta<\alpha} \mathcal{U}_\beta$.

Recalling Remark 2.2.19, the definition of U_α can simply be stated:

$$\mathcal{U}_\alpha := \bigcup_{\beta<\alpha} \mathcal{U}_\beta + \mathcal{U}^+_{e(\alpha)}$$

Since $\mathcal{W}_\alpha$ is isomorphic to $\mathcal{U}^+_{e(\alpha)}$, we let $p : \mathcal{U} \to \mathcal{U}$ be a function mapping any point of $\mathcal{W}_\alpha$ to the corresponding point in $\mathcal{U}_{e(\alpha)}$ (the same frame with the original relations).

For ease of reference,[15] we associate with every element $x \in \mathcal{U}$ a sequence of ordinals $(\alpha_0, \alpha_1, \alpha_2, ...)$, each $\alpha_i < \epsilon_0$. In general, α_0 will signify at which stage x was constructed, and the rest of the coordinates will locate where x is within $\mathcal{W}_{\alpha_0}$. More precisely, suppose α_0 is the first ordinal such that $x \in \mathcal{U}_{\alpha_0}$. Since α_0 is least, $x \notin \bigcup_{\beta<\alpha_0} \mathcal{U}_\beta$, and so $x \in \mathcal{W}_{\alpha_0}$. Then consider $p(x)$, and let α_1 be the first ordinal such that $p(x) \in \mathcal{U}_{\alpha_1}$. Continuing in this way we obtain our sequence of ordinals α_0, α_1, α_2, ... Recall that, for example, since $p(x) \in \mathcal{U}_{e(\alpha)}$, we must have $\alpha_1 \leq e(\alpha)$. In general, then, our sequence satisfies the requirement that $\forall i$, $\alpha_{i+1} \leq e(\alpha_i)$. Conversely, if we consider any such sequence satisfying this requirement, we can easily find an appropriate point x.

Let Ω be the set of ω-sequences $\vec{\alpha} := (\alpha_0, \alpha_1, \alpha_2, ...)$, where each $\alpha_i < \epsilon_0$. The correspondence described in the previous paragraph leads to our second definition of $\mathcal{U}$.[16]

Definition 2.3.5 (Ignatiev's Frame $\mathcal{U}$). $\mathcal{U} := (U, R_n : n < \omega)$ is defined:

$$U := \{\vec{\alpha} \in \Omega : \forall n < \omega, \alpha_{n+1} \leq e(\alpha_n)\}$$

$$\vec{\alpha} R_n \vec{\beta} \Leftrightarrow (\forall m < n, \alpha_m = \beta_m \ \& \ \alpha_n > \beta_m)$$

Thus, each point in U can be seen as a strictly decreasing finite sequence of ordinals. We label the n-th "coordinate" of a point $\vec{\alpha}$ by α_n. Then, for each $\vec{\alpha} \in U$, there is some m such that $\alpha_n = 0$ for all $n \geq m$.

[15] And following [Beklemishev et al., 2005].

[16] Definition 2.3.5 is based on the treatment in [Beklemishev et al., 2005], which is itself a variation on that in [Ignatiev, 1992].

We define validity of formulas in the usual way:

$$\mathcal{U}, \vec{\alpha} \vDash \langle n \rangle \phi \iff \exists \vec{\beta} \in U \ (\vec{\alpha} R_n \vec{\beta} \ \& \ \mathcal{U}, \vec{\beta} \vDash \phi)$$

For a visualization of the very top part of $\mathcal{U}$, namely $\mathcal{U}_{\omega^{\omega+1}}$, see Figure 4.[17] Single arrows represent R_0, double arrows represent R_1, and triple arrows represent R_2. While each R_n is transitive, we leave out many of the arrows that should be there as a consequence.

The 1-sheets on this structure are easily identifiable. They are simply those clusters of points that are R_1-related to one another. E.g. $(\omega, 1)$ and $(\omega, 0)$ form a 1-sheet. $\mathcal{U}$ is also clearly a h.l. stratified frame. Indeed, $\mathcal{U}$ is a model of $\mathbf{GLP}_0$, and $\mathbf{GLP}_0$ is complete with respect to $\mathcal{U}$, as Ignatiev was the first to show. As our proof of this fact makes use of blow-up models, it differs significantly from those in [Ignatiev, 1993] and [Beklemishev et al., 2005], which are both based on syntactic arguments.

Proposition 2.3.6. $\mathbf{GLP}_0 \vdash \phi$ *if and only if, for all* $\vec{\alpha} \in \mathcal{U}$, $\mathcal{U}, \vec{\alpha} \vDash \phi$.

We first prove soundness, then completeness.

Lemma 2.3.7. $\mathbf{GLP}_0$ *is sound w.r.t.* $\mathcal{U}$.

Proof. We reason by induction on proofs in $\mathbf{GLP}_0$, treating only axioms (iii), (iv), and (v). To see Axiom (iv), it is sufficient to note that each relation R_{n+1} satisfies the condition:

$$\forall \vec{\alpha}, \vec{\beta} \in U (\vec{\alpha} R_{n+1} \vec{\beta} \Rightarrow \forall \vec{\gamma} (\vec{\alpha} R_n \vec{\gamma} \Leftrightarrow \vec{\beta} R_n \vec{\gamma}))$$

This is obvious by the definition of R_{n+1}.

An easy proof that Löb's Axiom is valid relies on the fact that each relation R_n is transitive and converse-well-founded. It is converse-well-founded simply because all ordinals are well-founded under the $<$-relation. So given a point $\vec{\alpha} \in \mathcal{U}$, α_n can decrease only finitely often.[18]

Finally, while it is possible to prove the validity of Axiom (v), the "Monotonicity Axiom", in a number of ways, we refer the reader to Corollary 3.2.14 below, where we prove that the validity of closed formulas does not change when we augment relations so that $\vec{\alpha} R_{n+1} \vec{\beta}$ implies $\vec{\alpha} R_n \vec{\beta}$ for all n. This is clearly sufficient for Axiom (v) to hold. $\dashv$

Our method of showing completeness is to reduce the completeness for $\mathcal{U}$ to that for blow-ups of h.l. stratified frames. So we must first prove a number of auxiliary lemmas.

Lemma 2.3.8. *For all* $\alpha, \beta < \epsilon_0$, $\mathcal{U}_\alpha + \mathcal{U}_\beta \cong \mathcal{U}_{\alpha+1+\beta}$.

[17] This diagram is due to Joost Joosten, having originally appeared in [Joosten, 2004].

[18] This fact, however, is not formalizable in PA. In [Beklemishev et al., 2005], a proof of the soundness of Axiom (iii) is given using reasoning formalizable in a weak subsystem of PA.

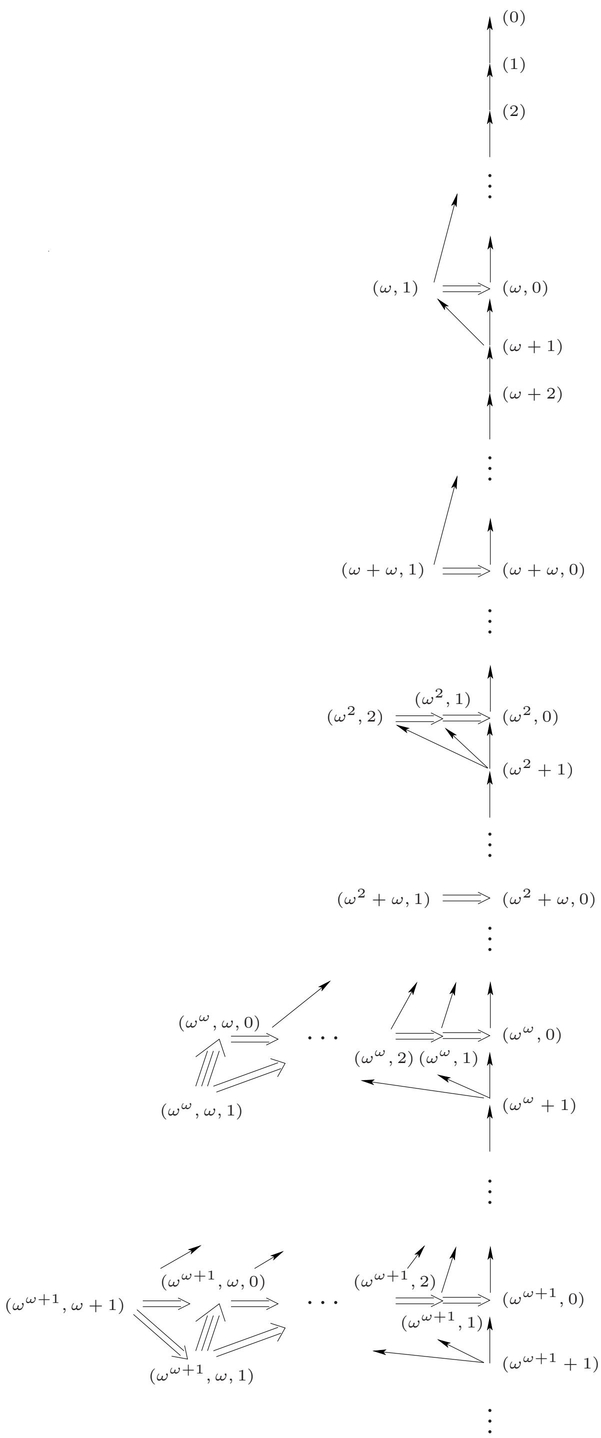

Figure 4: Ignatiev's Frame $\mathcal{U}$

Proof. Transfinite induction on β. For the basic case, $\mathcal{U}_\alpha + \mathcal{U}_0$ is isomorphic to $\mathcal{U}_{\alpha+1}$ by definition.

Suppose $\beta = \gamma + 1$. The induction hypothesis ($\times 2$) gives the result:

$$\begin{aligned}
\mathcal{U}_\alpha + \mathcal{U}_{\gamma+1} &\cong \mathcal{U}_\alpha + \bigcup_{\delta<\gamma+1} \mathcal{U}_\delta + \mathcal{U}_{e(\gamma+1)} \\
&\cong \mathcal{U}_\alpha + \mathcal{U}_\gamma + \mathcal{U}_0 \\
&\cong \mathcal{U}_{\alpha+1+\gamma} + \mathcal{U}_0 \\
&\cong \mathcal{U}_{\alpha+1+\gamma+1}
\end{aligned}$$

When $\beta = \lambda$ is a limit, we have the following:

$$\begin{aligned}
\mathcal{U}_\alpha + \mathcal{U}_\lambda &\cong \mathcal{U}_\alpha + \bigcup_{\gamma<\lambda} \mathcal{U}_\gamma + \mathcal{U}^+_{e(\lambda)} \\
&\cong \bigcup_{\gamma<\lambda} (\mathcal{U}_\alpha + \mathcal{U}_\gamma) + \mathcal{U}^+_{e(\lambda)} \\
&\cong \bigcup_{\gamma<\lambda} \mathcal{U}_{\alpha+1+\gamma} + \mathcal{U}^+_{e(\lambda)} \\
&\cong \bigcup_{\gamma<\alpha+1+\lambda} \mathcal{U}_\gamma + \mathcal{U}^+_{e(\lambda)} \\
&\cong \bigcup_{\gamma<\alpha+1+\lambda} \mathcal{U}_\gamma + \mathcal{U}^+_{e(\alpha+1+\lambda)} \\
&\cong \mathcal{U}_{\alpha+1+\lambda}
\end{aligned}$$

The penultimate step follows because the e function only depends on the last summand of the Cantor Normal Form. $\dashv$

Lemma 2.3.9. *For all α, $(\mathcal{U}^+_\alpha)^{(\omega)} \cong \mathcal{U}_{\omega^\alpha}$.*

Proof. This is shown by transfinite induction on α. First of all, notice we have the following by the definition of sheet-wise blow-up:

$$(\mathcal{U}^+_\alpha)^{(\omega)} := \bigcup_{\widehat{\alpha} R_0 \vec{\beta}} \sum_{i<\omega} (\mathcal{U}^+_{\vec{\beta}})^{(\omega)} + \mathcal{U}^+_\alpha$$

Otherwise put,

$$(\mathcal{U}^+_\alpha)^{(\omega)} = \bigcup_{\beta<\alpha} \sum_{i<\omega} (\mathcal{U}^+_\beta)^{(\omega)} + \mathcal{U}^+_\alpha$$

By the inductive hypothesis,

$$\bigcup_{\beta<\alpha} \sum_{i<\omega} (\mathcal{U}^+_\beta)^{(\omega)} \cong \bigcup_{\beta<\alpha} \sum_{i<\omega} \mathcal{U}_{\omega^\beta}$$

We claim that,

$$\bigcup_{\beta<\alpha} \sum_{i<\omega} \mathcal{U}_{\omega^\beta} = \bigcup_{\beta<\omega^\alpha} \mathcal{U}_\beta$$

There are two cases to check: either α is a limit or a successor ordinal.

Suppose α is a successor ordinal. In particular, suppose $\alpha = \gamma + 1$:

$$\begin{aligned} \bigcup_{\beta<\alpha} \sum_{i<\omega} \mathcal{U}_{\omega^\beta} &= \sum_{i<\omega} \mathcal{U}_{\omega^\gamma} \\ &= \bigcup_{\beta<\omega^\gamma \cdot\, \omega} \mathcal{U}_\beta \\ &= \bigcup_{\beta<\omega^\alpha} \mathcal{U}_\beta \end{aligned}$$

On the other hand, if α is a limit ordinal:

$$\begin{aligned} \bigcup_{\beta<\alpha} \sum_{i<\omega} \mathcal{U}_{\omega^\beta} &= \bigcup_{\beta<\alpha} \mathcal{U}_{\omega^{\beta+1}} \\ &= \bigcup_{\beta<\alpha} \mathcal{U}_{\omega^\beta} \\ &= \bigcup_{\beta<\omega^\alpha} \mathcal{U}_\beta \end{aligned}$$

Finally, putting all of this together,

$$(\mathcal{U}_\alpha^+)^{(\omega)} \quad \cong \quad \bigcup_{\beta<\omega^\alpha} \mathcal{U}_\beta + \mathcal{U}_\alpha^+$$

But this is exactly the definition of $\mathcal{U}_{\omega^\alpha}$. ⊣

Lemma 2.3.10. *Suppose $\mathcal{A}$ is a h.l. stratified model and α is an n-sheet with R_m empty for all $m > n$. Then there is some β such that $\mathfrak{B}_\omega(\alpha) = \mathcal{U}_\beta$.*

Proof. Any such n-sheet can be written as $\mathcal{U}_k^{+\cdots++}$, where k is the length of the single n-chain in α and there are n '+' symbols. This is simply an n-sheet isomorphic to the very top part of the Ignatiev frame. For simplicity denote it $\mathcal{U}_k^{+n}$. By the definition of global blowup, we have:

$$\begin{aligned} \mathfrak{B}_\omega(\alpha) &\cong \mathfrak{B}_\omega(\mathcal{U}_k^{+n}) \\ &\cong (\mathfrak{B}_\omega(\mathcal{U}_k^{+n-1})^+)^{(\omega)} \\ &\vdots \quad (n-1 \text{ applications of Definition 2.2.28}) \\ &\cong ((\ldots(\mathfrak{B}_\omega(\mathcal{U}_k)^+)^{(\omega)}\ldots)^+)^{(\omega)} \\ &\vdots \quad (n-1 \text{ applications of Lemma 2.3.9}) \\ &\cong \mathcal{U}_\beta \end{aligned}$$

This β is therefore determined as in Lemma 2.3.9. ⊣

Seeing as $\mathfrak{B}_\omega(\mathcal{U}_1) \cong \mathcal{U}_\omega$ is obvious, the argument in Lemma 2.3.10 gives rise to the following corollary (noted in [Beklemishev, 2007a]), which shows that blowups of simple two point frames approximate the entire structure $\mathcal{U}$:

Corollary 2.3.11. *For all* n, $\mathfrak{B}_\omega(\mathcal{U}_1^{+n}) \cong \mathcal{U}_{\omega_n}$.

This next theorem shows that $\mathcal{U}$ is in a strong sense universal for the class of finite h.l. stratified frames.

Theorem 2.3.12. *For any finite h.l. stratified frame* $\mathcal{A}$, *there is a* β *such that*

$$\mathfrak{B}_\omega(\mathcal{A}) \cong \mathcal{U}_\beta.$$

Proof. We show this by induction on rank.[19] The case of rank 0 is exactly Lemma 2.3.10. So, consider any h.l. stratified frame $\mathcal{A}$ with positive rank. Every such frame can be written as the 0-linear ordering of 1-sheets:

$$\alpha_0 R_0 \alpha_1 R_0 ... R_0 \alpha_n$$

The blowup, $\mathfrak{B}_\omega(\mathcal{A})$, can be written similarly as the 0-linearly ordered sum of blow-ups of 1-sheets:

$$\mathfrak{B}_\omega(\alpha_0) + \mathfrak{B}_\omega(\alpha_1) + ... + \mathfrak{B}_\omega(\alpha_n)$$

By induction hypothesis, each such $\mathfrak{B}_\omega(\alpha_i)$ is isomorphic to some $\mathcal{U}_{\beta_i}$. So:

$$\begin{aligned} \mathfrak{B}_\omega(\mathcal{A}) &\cong \mathcal{U}_{\beta_0} + \mathcal{U}_{\beta_1} + ... + \mathcal{U}_{\beta_n} \\ &\cong \mathcal{U}_{\beta_0+1+\beta_1+1+...+\beta_n} \end{aligned}$$

The last step is by Lemma 2.3.8. ⊣

Theorem 2.3.13. $\mathbf{GLP}_0$ *is complete with respect to* $\mathcal{U}$.

Proof. If $\mathbf{GLP}_0 \nvdash \phi$, then $\mathbf{J} \nvdash M^+(\phi) \to \phi$, and by Corollary 2.2.15 there is a h.l. stratified frame $\mathcal{A}$ such that $\mathcal{A} \nvDash M^+(\phi) \to \phi$. Correspondingly, $\mathfrak{B}_\omega(\mathcal{A}) \nvDash \phi$. Theorem 2.3.12 ensures that $\mathfrak{B}_\omega(\mathcal{A})$ is isomorphic to some generated subframe of $\mathcal{U}$. Therefore, $\mathcal{U} \nvDash \phi$. ⊣

In fact Theorem 2.3.13 can be improved to a special subset of points in $\mathcal{U}$. We only mention the result here, and defer the proof to the next section, after further study of $\mathcal{U}$.

Definition 2.3.14 (Main Axis). $\vec{\alpha} \in \mathcal{U}$ is a *root point* if and only if $\forall i$, $\alpha_{i+1} = e(\alpha_i)$. The *main axis* of $\mathcal{U}$, denoted M, is the set of all root points.

We shall write $\widehat{\alpha}$ for the root point "generated by" a given ordinal α:

$$\widehat{\alpha} := (\alpha, e(\alpha), e(e(\alpha)), ...)$$

Proposition 2.3.15 ([Ignatiev, 1993]). *If* $\mathbf{GLP}_0 \nvdash \phi$, *then there is a root point* $\widehat{\alpha}$ *on the main axis* M, *such that* $\mathcal{U}, \widehat{\alpha} \nvDash \phi$.

[19] We have not needed to use the notion of *rank* ([Beklemishev, 2007a]) up to this point, but it is a sufficiently natural notion. Essentially, it is just the maximal depth of nestings of sheets. The formal definition is as follows: For an m-sheet $\mathcal{A}$, if all R_n are empty, then $rk_m(\mathcal{A}) = 0$; otherwise $rk_m(\mathcal{A}) := \max_{\alpha \in \mathcal{A}} rk_{m+1}(\alpha) + 1$.

2.4 Words and Ordinals

Our study of $\mathcal{U}$ will center around a class of formulas called *words*, introduced in [Beklemishev, 2004a].[20] An important fact, analogous to the Normal Form Theorem for **GL**,[21] is that all closed formulas are $\mathbf{GLP}_0$-equivalent to a boolean combination of words. Syntactic proofs of this fact were given by Ignatiev and Beklemishev ([Beklemishev, 2004a],[Beklemishev, 2004b],[Ignatiev, 1992]). As a result of Theorem 2.3.13, knowing that $\mathbf{GLP}_0$ is sound and complete for $\mathcal{U}$, we automatically gain a normal form result for $\mathcal{U}$:

Theorem 2.4.1 (Normal Form Theorem). *Every closed formula is equivalent in $\mathcal{U}$ to a boolean combination of words.*

We shall use words to study the subsets of $\mathcal{U}$ that correspond to the validity set of a closed formula. We call such sets *closed formula definable subsets*, or simply *definable subsets*, of $\mathcal{U}$. The notation S_ϕ denotes the set of points where ϕ is true. As a result of Theorem 2.4.1, words will provide a very useful tool in studying such subsets.

By a *word* we mean a modal formula of the form $\langle n_m\rangle ... \langle n_o\rangle\top$. Instead of writing out words in this way we shall abbreviate, letting any sequence of numerals denote a word. In particular, the empty sequence Λ corresponds to $\top$. This will allow us to perform various operations on sequences such as concatenation: e.g. if 12 denotes $\langle 1\rangle\langle 2\rangle\top$ and 01 denotes $\langle 0\rangle\langle 1\rangle\top$, then 1201 denotes $\langle 1\rangle\langle 2\rangle\langle 0\rangle\langle 1\rangle\top$. We shall let $n, m, ...$ serve as variables for individual diamonds, and $A, B...$ serve as variables for sequences of diamonds. We will also write, e.g. $\mathcal{U}, \vec{\beta} \vDash A$, even though, strictly speaking, this is an abuse of the notation. Finally, let S denote the set of all words, and S_n denote the set of words with only modalities $\langle m\rangle$ for $m \geq n$.

In order to investigate validity of words, and closed formulas in general, in $\mathcal{U}$, we introduce the following ordering on U.

Definition 2.4.2. The relation $\preceq$ defines a rooted partial order on points in U so that, for any two points $\vec{\alpha}$ and $\vec{\beta}$,

$$\vec{\beta} \preceq \vec{\alpha} \Leftrightarrow (\forall i, \beta_i \leq \alpha_i)$$

Lemma 2.4.3. *For any word A, if $\mathcal{U}, \vec{\alpha} \vDash A$, then for all $\vec{\beta} \in \mathcal{U}$ such that $\vec{\beta} \succeq \vec{\alpha}$, we have $\mathcal{U}, \vec{\beta} \vDash A$.*

Proof. By induction on the length of words. If $A = \Lambda$, this is obvious. Suppose this holds for formulas of length m, and $\vec{\alpha} \vDash nA$, for A of length $\leq m$. Suppose in particular that $\vec{\alpha} R_n \vec{\delta}$ for some $\vec{\delta} \vDash A$. Now take any $\vec{\beta} \in \mathcal{U}$ such that $\vec{\beta} \succeq \vec{\alpha}$. We see $\vec{\beta} R_n \vec{\gamma}$, where $\vec{\gamma}$ is such that $\forall i < n\ \gamma_i = \beta_i$, and $\forall i \geq n\ \gamma_i = \delta_i$. Then we have that $\vec{\gamma} \succeq \vec{\delta}$, so by the induction hypothesis $\vec{\gamma} \vDash A$, and hence $\vec{\beta} \vDash nA$. $\dashv$

[20] Although, similar formulas were introduced in [Ignatiev, 1992].

[21] See [Boolos, 1993].

Next, we define a surjective function o from words to ordinals, originally used by Beklemishev to establish a one-to-one ordinal notation system for ϵ_0 (See Section 1.2).[22] The function o is defined by recursion on the width of a word, denoted $w(A)$, where $w(A)$ is the number of different numerals occurring in A, with a subsidiary recursion on $\min(A)$, where $\min(A) = \min(n_m, ..., n_1)$ if $A = n_m...n_1$. We write A^- for the result of reducing each element of A by one.

Definition 2.4.4 ([Beklemishev, 2004a],[Beklemishev, 2005]). The function $o : S \to \epsilon_0$ is defined:

- If $A = 0^k$, then $o(A) = k$
- Otherwise, if $A = A_1 0...0 A_k$, with each $A_i \in S_1$ and not all A_i empty, then $o(A) = \omega^{o(A_k^-)} + ... + \omega^{o(A_1^-)}$.

To see that this is a well defined function, note that $w(A_i^-) < w(A)$ whenever $k > 1$. And when $k = 1$, $\min(A_i^-) = \min(A_i) - 1 < \min(A)$.

We shall prove that o gives us a means of specifying at what points in $\mathcal{U}$ a given closed formula is validated. First, an auxiliary lemma:[23]

Lemma 2.4.5. *If $A \in S_1$, then for any word B, we have $\mathcal{U}, \vec{\beta} \vDash A0B$ if and only if $\mathcal{U}, \vec{\beta} \vDash A$ and $\mathcal{U}, \vec{\beta} \vDash 0B$.*

Proof. For the left-to-right direction, if $\vec{\beta} \vDash A0B$, certainly $\vec{\beta} \vDash A$. Moreover, there is some sequence

$$\vec{\beta} R_i \vec{\gamma} ... R_j \vec{\delta} R_0 \vec{\alpha}$$

such that each $i, j... > 0$ and $\vec{\alpha} \vDash B$. Since therefore $\alpha_0 < \beta_0$, we have $\vec{\beta} R_0 \vec{\alpha}$, and thus $\vec{\beta} \vDash 0B$.

For the other direction, if $\vec{\beta} \vDash A$, we have a sequence witnessing this fact:

$$\vec{\beta} R_i \vec{\gamma} ... R_j \vec{\delta}$$

As $A \in S_1$, each $i, j, .. > 0$. Therefore, $\delta_0 = \beta_0$. Since $\vec{\beta} \vDash 0B$, it is also clear that $\vec{\delta} \vDash 0B$. This means $\vec{\beta} \vDash A0B$. $\dashv$

Definition 2.4.6. Let $\iota(A) := \widehat{o(A)}$, so that $\iota(A)$ is in M.

Proposition 2.4.7. *For all words $A \in S$ and points $\vec{\beta} \in \mathcal{U}$,*

$$\mathcal{U}, \vec{\beta} \vDash A \Longleftrightarrow \vec{\beta} \succeq \iota(A)$$

Proof. Induction on the length of A. When A is the empty sequence, this is clear, as $\iota(\Lambda) = (0)$.

Supposing A is not empty, we now induct on $\min(A)$.

[22] Ignatiev has also defined such a surjection ([Ignatiev, 1993]), however that in [Beklemishev, 2004a] is simpler.

[23] C.f. [Beklemishev, 2004a], Lemma 5.9(iv) for a syntactic proof of this fact.

If $\min(A) = 0$, then A can be written in the form $A_0 0 A'$, with $A_0 \in S_1$. Let

$$\iota(A') = \widehat{\alpha}$$

Then by induction hypothesis, $\vec{\beta} \vDash A'$ if and only if $\vec{\beta} \succeq \widehat{\alpha}$. As $A_0 \in S_1$, we have $o(A_0) = \omega^\gamma$, where $\gamma = o(A_0^-)$. Thus,

$$\iota(A_0) = (\omega^\gamma, \gamma, e(\gamma), e(e(\gamma)), ...)$$

And by the induction hypothesis, $\vec{\beta} \vDash A_0$ if and only if $\vec{\beta} \succeq \widehat{\omega^\gamma}$. Using Lemma 2.4.5, we have

$$\vec{\beta} \vDash A_0 0 A' \iff (\vec{\beta} \vDash A_0 \;\; \& \;\; \vec{\beta} \vDash 0A')$$

We know exactly when $\vec{\beta} \vDash A_0$; To see when $\vec{\beta} \vDash 0A'$, we have:

$$\begin{aligned} \vec{\beta} \vDash 0A' &\iff \exists \vec{\beta'} (\vec{\beta} R_0 \vec{\beta'} \;\; \& \;\; \vec{\beta'} \vDash A') \\ &\iff \exists \vec{\beta'} (\vec{\beta} R_0 \vec{\beta'} \;\; \& \;\; \vec{\beta'} \succeq \widehat{\alpha}) \\ &\iff \beta_0 > \alpha_0 \end{aligned}$$

We can thus conclude, $\vec{\beta} \vDash A_0 0 A'$ if and only if $\vec{\beta} \succeq \widehat{\omega^\gamma}$ and $\beta_0 > \alpha_0$.

On the other hand, $o(A) = o(A') + \omega^{o(A_0^-)} = \alpha_0 + \omega^\gamma$. So,

$$\iota(A) = (\alpha_0 + \omega^\gamma, \gamma, e(\gamma), e(e(\gamma)), ...)$$

Hence, $\vec{\beta} \succeq \iota(A)$ if and only if $\vec{\beta} \succeq \widehat{\omega^\gamma}$ and $\beta_0 \geq \alpha_0 + \omega^\gamma$. It remains to show $\beta_0 > \alpha_0$ is equivalent to $\beta_0 \geq \alpha_0 + \omega^\gamma$.

The right-to-left direction is obvious. For the other direction, if $\beta_0 > \alpha_0$, then $\beta_0 = \alpha_0 + \delta$ for $\delta > 0$. Since $e(\beta_0) = e(\delta)$, it follows

$$\delta \geq \omega^{e(\delta)} = \omega^{e(\beta_0)} \geq \omega^{\beta_1} \geq \omega^\gamma$$

And so $\beta_0 \geq \alpha_0 + \omega^\gamma$, thus concluding the basic case.

Finally suppose $\min(A) > 0$. Since 0 does not occur in A, $\vec{\beta} \vDash A$ if and only if $\vec{\beta'} \vDash A^-$, where $\vec{\beta'} := (\beta_1, \beta_2, ...)$ (each coordinate shifted once to the left). By the induction hypothesis (on $\min(A)$), this is equivalent to $\vec{\beta'} \succeq \iota(A^-)$. Let $\iota(A^-) = \widehat{\alpha}$. Then $\iota(A) = \widehat{\omega^\alpha}$. But clearly $\vec{\beta} \succeq \widehat{\omega^\alpha}$ if and only if $\vec{\beta'} \succeq \widehat{\alpha}$. Putting all of this together:

$$\vec{\beta} \vDash A \iff \vec{\beta'} \vDash A^- \iff \vec{\beta'} \succeq \widehat{\alpha} \iff \vec{\beta} \succeq \widehat{\omega^\alpha} \iff \vec{\beta} \succeq \iota(A)$$

This concludes the inductive step. $\dashv$

Proposition 2.4.7 is of central importance. One immediate consequence is that $\mathcal{U}$ is optimal in the sense that it does not have any superfluous points:[24]

[24] In the proof of Lemma 2.4.8, as elsewhere, we write α_n^β to mean an n-stack of α's with a β as the top exponent, rather than, e.g. $(\alpha_n)^\beta$, which would denote an n-stack of α's raised to the power of β.

Lemma 2.4.8. *For any distinct points $\vec{\alpha}$ and $\vec{\beta}$ in $\mathcal{U}$, there is a word A such that $\mathcal{U}, \vec{\alpha} \vDash A$ but $\mathcal{U}, \vec{\beta} \nvDash A$.*

Proof. Without loss of generality, suppose $\forall i < n, \alpha_i = \beta_i$, and $\alpha_n > \beta_n$. Let

$$\widehat{\gamma} := (\omega_n^{\beta_n}, ..., \omega^{\beta_n}, \beta_n, e(\beta_n), e(e(\beta_n)), ...)$$

and suppose $o(A) = \gamma = \omega_n^{\beta_n}$. Then clearly $\mathcal{U}, \vec{\alpha} \vDash nA$ (since $\vec{\alpha} R_n \vec{\delta}$ for $\vec{\delta}$ exactly like $\vec{\alpha}$ up to the n-th coordinate and like $\vec{\gamma}$ from the n-th on). But we cannot have $\mathcal{U}, \vec{\beta} \vDash nA$, because if $\vec{\beta} R_n \vec{\delta}$, for some appropriate $\vec{\delta}$, then $\delta_n < \beta_n = \gamma_n$. $\dashv$

We are also now in a position to prove Ignatiev's strengthening of completeness, to the effect that it is possible to falsify any non-theorem of $\mathbf{GLP}_0$ on the main axis.

Theorem 2.4.9. *If $\mathbf{GLP}_0 \nvdash \phi$, there is $\widehat{\alpha} \in M$ such that $\mathcal{U}, \widehat{\alpha} \nvDash \phi$.*

Proof. If $\mathbf{GLP}_0 \nvdash \phi$, then by Theorem 2.3.13, $\mathcal{U} \nvDash \phi$. By Corollary 2.4.1, we can assume ϕ is in conjunctive normal form, where each "atom" is a word or the negation thereof. It follows that $\mathcal{U} \nvDash A \to (A_1 \vee ... \vee A_j)$ for some such conjunct of ϕ. I.e, there is some point $\vec{\alpha}$ such that $\vec{\alpha} \vDash A$, but $\vec{\alpha} \nvDash (A_1 \vee ... \vee A_j)$. By Proposition 2.4.7, $\vec{\alpha} \succeq \iota(A)$, and $\vec{\alpha} \not\succeq \iota(A_i)$, for each $i \leq j$. Putting these together gives $\iota(A) \not\succeq \iota(A_i)$, for each $i \leq j$, so $\iota(A) \nvDash (A_1 \vee ... \vee A_j)$. In other words, $\iota(A) \nvDash A \to (A_1 \vee ... \vee A_j)$, and hence $\iota(A) \nvDash \phi$, as desired. $\dashv$

3 The Canonical Frame of $\mathbf{GLP}_0$

Recall the definition of the *canonical frame*:

Definition 3.0.10. The *canonical frame* for $\mathbf{GLP}_0$ is defined:

$$\mathbb{C}^0 := (W^0, R_n^0 : n < \omega)$$

- W^0 is the set of maximal-$\mathbf{GLP}_0$-consistent sets.
- $xR_n^0 y$ if and only if for all closed formulas ϕ, $\phi \in y$ implies $\langle n \rangle \phi \in x$.

A very natural question, given completeness and soundness of $\mathbf{GLP}_0$ with respect to $\mathcal{U}$, is what the relationship is between $\mathcal{U}$ and $\mathbb{C}^0$. Certainly every finite $\mathbf{GLP}_0$-consistent set is a subset of the formulas true at some point in $\mathcal{U}$. But we might also wonder whether every *maximal* $\mathbf{GLP}_0$-consistent set is satisfiable at a single point. The answer turns out to be no, as we shall see.

After discussing the relationship between descriptive frames and canonical frames, we will be able to show that $\mathcal{U}$ is not the canonical frame of $\mathbf{GLP}_0$. The rest of the section will be dedicated to turning $\mathcal{U}$ into a descriptive frame $\mathcal{V}^c$ whose logic is still $\mathbf{GLP}_0$. We will show that $\mathcal{V}^c$ is isomorphic to $\mathbb{C}^0$, and will thereby have obtained a detailed, coordinate-wise definition of the latter.

3.1 Descriptive General Frames

3.1.1 Basic Facts

Definition 3.1.1 (General Frames). A *general frame*[25] in the basic modal language is a pair $(\mathbb{F}, A)$ such that $\mathbb{F} = (F, R)$ and $A \subseteq \mathcal{P}(F)$, which is closed under finite union, finite intersection, complement, and R^{-1}.

In other words, a general frame is simply a frame along with a distinguished set of subsets, which also defines an algebra over the frame. For the extension to the polymodal case, we must simply add closure under R_n^{-1} for each n.

Remark 3.1.2. Though we will not be concerned with valuations in this section, as we are only dealing with the closed fragment, it is worth mentioning that a so called *admissible* valuation on a general frame, giving rise to a *model based on a general frame*, is one whose values are restricted to sets in A. For any logic $\mathcal{L}$, if we consider the canonical frame $\mathbb{C}^{\mathcal{L}}$ and let C be the set of sets of the form $\{\Gamma : \phi \in \Gamma, \Gamma \text{ is maximal-}\mathcal{L}\text{-consistent}\}$, with ϕ ranging over all formulas, it is easily seen that $(\mathbb{C}^{\mathcal{L}}, C)$ is a general frame. From this it follows that any logic is sound and complete with respect to some class of general frames.[26] This of course means that we do have a very abstract completeness result for full **GLP**. However, a challenge would be to describe what such a class looks like in some detail, analogous to our treatment of $\mathbb{C}^0$ in the coming pages.

The algebraic flavor of general frames is by no means merely superficial. Given a general frame $\mathbb{G} = (\mathbb{F}, A)$, we can form a *modal algebra* $\mathbb{G}^*$, taking A as the underlying set and R^{-1} as an operator. A formula ϕ will be valid in $\mathbb{G}$ just in case the identity $\phi = \top$ is a valid identity of $\mathbb{G}^*$, so $\mathbb{G}$ and $\mathbb{G}^*$ are in a sense modally equivalent.

In the other direction, it is also always possible to form a general frame from a modal algebra. From a modal algebra $\mathfrak{A}$ we obtain the so called *general ultrafilter frame* $\mathfrak{A}_*$, the underlying set being the set of ultrafilters of $\mathfrak{A}$, with relations and the set A defined appropriately. As before, for a formula ϕ, $\phi = \top$ is an identity of $\mathfrak{A}$ if and only if ϕ is valid in $\mathfrak{A}_*$.

This explanation is of course far from comprehensive, but the details of this correspondence are not crucial. An important fact about these operations is that the following isomorphism always holds, where $\mathfrak{A}$ is any modal algebra:

$$(\mathfrak{A}_*)^* \cong \mathfrak{A}$$

That is, if we form the general ultrafilter frame and then take the associated algebra of this general frame, we will always end up with something isomorphic to $\mathfrak{A}$. The converse, however, does not hold for arbitrary general frames:

$$(\mathbb{G}^*)_* \cong \mathbb{G}$$

[25] The definitions and background results in this section are drawn from a combination of [Blackburn et al., 2001] and [Chagrov and Zakharyaschev, 1997].

[26] See [Blackburn et al., 2001] for details.

General frames that satisfy this condition are called *descriptive general frames.*

Descriptive frames can be distinguished by three properties:[27]

Definition 3.1.3. A general frame $\mathbb{G} = (\mathbb{F}, A)$ is *differentiated* if for all distinct points x and y there is some $X \in A$ such that $x \in X$, but $y \notin X$.

Definition 3.1.4. A general frame is *tight* if for all points x, y, xRy if and only if: $y \in X$ implies $x \in R^{-1}(X)$ for all $X \in A$.

Definition 3.1.5. A general frame $(\mathbb{F}, A)$ is *compact* if for every subset A_0 of A with the finite intersection property, $\bigcap A_0$ is non-empty.

Theorem 3.1.6 (Descriptive Frames). *A general frame is descriptive if and only if it is differentiated, tight, and compact.*[28]

3.1.2 Descriptive Frames and Canonical Frames

Consider the canonical frame $\mathbb{C}^{\mathcal{L}}$ for an arbitrary logic $\mathcal{L}$ in a variable-free language. The general frame defined on $\mathbb{C}^{\mathcal{L}}$ in Remark 3.1.2 by letting C consist of sets of the form,

$$\{\Gamma : \phi \in \Gamma, \Gamma \text{ is maximal-}\mathcal{L}\text{-consistent}\}$$

with ϕ ranging over all formulas of the language, can also be defined by letting $C := \{S_\phi : \phi \text{ is a closed formula}\}$.[29] These two definitions are clearly equivalent, since the above set is simply the set of points in the frame, themselves maximal consistent sets, where ϕ is valid.

When the algebra A of a general frame $(\mathbb{F}, A)$ is simply the set of closed formula definable subsets of $\mathbb{F}$, the definitions of *differentiated* and *tight* become more perspicuous:[30]

Definition 3.1.7. A general frame $(\mathbb{F}, A)$ is *differentiated* if for all distinct points x and y there is some (closed) formula ϕ such that $\mathbb{F}, x \vDash \phi$, but $\mathbb{F}, y \nvDash \phi$.

Definition 3.1.8. A general frame $(\mathbb{F}, A)$ is *tight* if for all points x, y in $\mathbb{F}$, xRy if and only if: $\mathbb{F}, y \vDash \phi$ implies $\mathbb{F}, x \vDash \Diamond\phi$ for all ϕ.

The following fact then becomes clear:

Fact 3.1.9. *For any logic $\mathcal{L}$, the general frame $(\mathbb{C}^{\mathcal{L}}, C)$, where $\mathbb{C}^{\mathcal{L}}$ is the canonical frame and C is the set of formula definable subsets of $\mathbb{C}^{\mathcal{L}}$, is descriptive.*

[27] Here we follow [Chagrov and Zakharyaschev, 1997] in defining descriptive frames by their duality with modal algebras, and taking these to be properties of such general frames, rather than the other way around as in [Blackburn et al., 2001]. However, as we are not proving the equivalence, it does not really matter.

[28] For proofs, see [Blackburn et al., 2001] or [Chagrov and Zakharyaschev, 1997].

[29] Recall S_ϕ is the set of points where ϕ is valid.

[30] As usual, the adjustment to the polymodal case is obvious.

Proof Sketch. Obviously $(\mathbb{C}^{\mathcal{L}}, C)$ is differentiated, since every two maximal-$\mathcal{L}$-consistent sets differ by at least one formula. It is clearly tight because the definition of tightness coincides exactly with the definition of a relation in $\mathbb{C}^{\mathcal{L}}$. To see that it is compact, consider the following intuition: If $C_0 \subseteq C$ has the finite intersection property, this just means for some (possibly infinite) collection of formulas, every finite set of these formulas is consistent. By a familiar argument, this means the whole collection is consistent since any contradiction would follow from some finite subset. Consequently there is a maximal-$\mathcal{L}$-consistent set containing all of the formulas. $\dashv$

Recall the definition of the Lindenbaum-Tarski Algebra:

Definition 3.1.10. The *Lindenbaum-Tarski Algebra* $\mathfrak{T}_{\mathcal{L}}$ of a logic $\mathcal{L}$ is defined as the set of formulas of $\mathcal{L}$, modulo $\mathcal{L}$-equivalence, along with boolean operations and an operation $\Diamond$ for each modality, bringing an element ϕ to the element $\Diamond\phi$.

The *0-generated Lindenbaum-Tarski Algebra* $\mathfrak{T}^0_{\mathcal{L}}$ is the free subalgebra of $\mathfrak{T}_{\mathcal{L}}$ obtained from the constant $\top$ by closing under all operations. Otherwise put, $\mathfrak{T}^0_{\mathcal{L}}$ is the Lindenbaum-Tarski Algebra of the closed fragment of $\mathcal{L}$.

Lemma 3.1.11. *Given a logic $\mathcal{L}$, suppose for all closed formulas ϕ, $\mathcal{L} \vdash \phi$ if and only $\mathbb{F} \vDash \phi$. Then $(\mathbb{F}, A)^* \cong \mathfrak{T}^0_{\mathcal{L}}$, where $A = \{S_\phi : \phi \text{ is a closed formula}\}$.*

Proof. As we said above, $(\mathbb{F}, A)^*$ is the algebra on A with the operator Q. The desired isomorphism should then be clear. Let i be the function that sends every closed formula to the set of points on $\mathbb{F}$ where the formula is valid: $i : \phi \mapsto S_\phi$. Then i is a bijection, because

$$\begin{aligned} \mathcal{L} \vdash \phi \leftrightarrow \psi &\iff \mathbb{F} \vDash \phi \leftrightarrow \psi \\ &\iff S_\phi = S_\psi \\ &\iff i(\phi) = i(\psi). \end{aligned}$$

Obviously i respects boolean operations. Finally, since A is the set of formula definable subsets, $R^{-1}(S_\phi) = \{x : \mathbb{F}, x \vDash \Diamond\phi\}$. That is, $i(R^{-1}(S_\phi)) = \Diamond(i(S_\phi))$. $\dashv$

Proposition 3.1.12. *Given a logic $\mathcal{L}$ and frame $\mathbb{F}$, suppose $\mathcal{L} \vdash \phi \iff \mathbb{F} \vDash \phi$ for all closed formulas ϕ. Then $\mathbb{F} \cong \mathbb{C}^{\mathcal{L}}$ if and only if $(\mathbb{F}, A)$ is descriptive.*

Proof. The left-to-right direction is obvious given Fact 3.1.9. For the other direction, suppose $(\mathbb{F}, A)$ is descriptive. Then $((\mathbb{F}, A)^*)_* \cong (\mathbb{F}, A)$. By Fact 3.1.9, $(\mathbb{C}^{\mathcal{L}}, C)$ is also descriptive, so $((\mathbb{C}^{\mathcal{L}}, A)^*)_* \cong (\mathbb{C}^{\mathcal{L}}, C)$. By Lemma 3.1.11,

$$(\mathbb{F}, A) \cong ((\mathbb{F}, A)^*)_* \cong (\mathfrak{T}^0_{\mathcal{L}})_* \cong ((\mathbb{C}^{\mathcal{L}}, C)^*)_* \cong (\mathbb{C}^{\mathcal{L}}, C)$$

From this it obviously follows that $\mathbb{F} \cong \mathbb{C}^{\mathcal{L}}$. $\dashv$

3.2 The Frame $\mathcal{V}^c$

Proposition 3.1.12 provides us with a convenient test to see whether $\mathcal{U}$, or any other purported structure, is isomorphic to $\mathbb{C}^0$. While we know $\mathcal{U}$ and $\mathbb{C}^0$ are modally equivalent, we must check whether $(\mathcal{U}, A)$ is descriptive.

Lemma 2.4.8 tells us that $(\mathcal{U}, A)$ is differentiated, in fact stronger, that every two points differ by a word.

Proposition 3.2.1. *$(\mathcal{U}, A)$ is differentiated.*

However, $(\mathcal{U}, A)$ is neither tight nor compact. To see that it is not tight, consider the very top part of $\mathcal{U}$, the substructure $\mathcal{U}_\omega$. Given any formula ϕ, it is easily seen that if $\mathcal{U}, (\omega, 0) \vDash \phi$, then $\mathcal{U}, (\omega, 1) \vDash \langle 0 \rangle \phi$. But as things stand, we do not have $(\omega, 1) R_0 (\omega, 0)$. Similarly, compactness fails, since the following set, while finitely satisfiable, is not satisfiable at any single point:

$$\{\langle n \rangle \top : n < \omega\}$$

Therefore, $\mathcal{U}$ is not descriptive, so it cannot be isomorphic to $\mathbb{C}^0$.

Intuitively, this tells us that $\mathcal{U}$ does not have "enough" points for all maximal consistent sets to be satisfied, and that the relations are a proper subset of what we have in the canonical frame. Put otherwise, not all ultrafilters over the algebra on A are represented in $\mathcal{U}$, and the relations R_n are not closed in the topology associated with (U, A) (see below). As a consequence, the dual of $(\mathcal{U}, A)^*$, which by Lemma 3.1.11 is isomorphic to the dual of $\mathfrak{T}^0_{\mathcal{L}}$, is not (isomorphic to) $(\mathcal{U}, A)$. In pictorial form, we would like to know what it takes to get directly from $(\mathcal{U}, A)$ to $(\mathbb{C}^0, C)$. Knowing that A and C give rise to

$$\begin{tikzcd}
(\mathbb{C}^0, C) \arrow[r, shift left, "*"] & \mathfrak{T}^0_{\mathcal{L}} \arrow[l, shift left, "*"] \\
(\mathcal{U}, A) \arrow[u, squiggly, "?"] \arrow[ur, "*"']
\end{tikzcd}$$

isomorphic algebras, this will only require modification of the frame $\mathcal{U}$ itself, and it will correspond exactly to turning $(\mathcal{U}, A)$ into a descriptive frame.

3.2.1 Compactness

The most natural way to approach this problem is to address it at the topological level. In that direction, recall the standard interval topology on an ordinal λ. This topology is generated by the subbasis of open *rays* of the form, for $\beta < \lambda$:

$$\{\alpha : \beta < \alpha\}$$

$$\{\alpha : \beta > \alpha\}$$

Alternatively, the same topology can be defined by a basis of such open rays in addition to all open intervals:

$$\{\alpha : \beta < \alpha < \gamma\}$$

It is well known that such a topology is compact if and only if λ is a successor ordinal ([Willard, 1970]). Now consider the following topological space:

Definition 3.2.2. Let $(\mathcal{E}, \tau^{\mathcal{E}})$ be such that $\mathcal{E} := \prod_{i<\omega}(\epsilon_0)$, and $\tau^{\mathcal{E}}$ is the product topology on $\mathcal{E}$. That is, where $\pi_i : \mathcal{E} \to (\epsilon_0)_i$ is the standard *projection function*, open sets of $\tau^{\mathcal{E}}$ correspond to *cylinder sets*, or finite intersections of sets of the form $\pi_i^{-1}(I)$, where I is an open interval or ray of ϵ_0.

At the same time, we can define a space (U, τ^U) by declaring sets of the form $S_\phi \in A$ to determine a clopen subbase. Then we have the following theorem:

Theorem 3.2.3. *(U, τ^U) is a subspace of $(\mathcal{E}, \tau^{\mathcal{E}})$.*

Proof. Let $D_\beta := \{\vec{\alpha} : \vec{\alpha} \succeq \widehat{\beta}\}$ and $\overline{D}_\beta := \{\vec{\alpha} : \vec{\alpha} \not\succeq \widehat{\beta}\}$. We first observe that by Lemma 2.4.7, an equivalent subbasis for τ^U would consist of all sets D_β and $\overline{D}_\beta$, for $\beta < \epsilon_0$. Therefore, it suffices to show that every such set D_β, or $\overline{D}_\beta$, is $\tau^{\mathcal{E}}$-open, and conversely that every $\tau^{\mathcal{E}}$-subbasic ray $\{\vec{\alpha} : \beta < \alpha_i\}$, or $\{\vec{\alpha} : \beta > \alpha_i\}$, is τ^U-open.

First suppose we have some set $D_\beta = \{\vec{\alpha} : \vec{\alpha} \succeq \widehat{\beta}\}$. Notice the last non-zero coordinate β_i of $\widehat{\beta}$ cannot be a limit ordinal. Thus $\beta_i = \gamma + 1$ for some γ. Suppose β has the following Cantor Normal Form:

$$\kappa_i + \omega^{\cdot^{\cdot^{\cdot^{\kappa_1 + \omega^{\gamma+1}}}}}$$

Let κ be the ordinal,

$$\kappa_i + \omega^{\cdot^{\cdot^{\cdot^{\kappa_1 + \omega^{\gamma}}}}}$$

Then we take the set $\{\vec{\alpha} : \alpha_0 > \kappa\} \cap \{\vec{\alpha} : \alpha_i > \gamma\}$, clearly equivalent to D_β.

Next suppose we have some set $\overline{D}_\beta = \{\vec{\alpha} : \vec{\alpha} \not\succeq \widehat{\beta}\}$. Let β_i be the last non-zero coordinate of $\widehat{\beta}$. Then our equivalent set is obtained as in the last case: $\{\vec{\alpha} : \alpha_0 < \beta_0\} \cup \{\vec{\alpha} : \alpha_i < \beta_i\}$.

For the other direction, take some set $\{\vec{\alpha} : \alpha_i > \beta\}$. Then we consider the following root point:

$$\widehat{\gamma} := (\omega_i^{\beta+1}, ..., \omega^{\beta+1}, \beta + 1, 0, ...)$$

That is, $\widehat{\gamma}$ is the $\preceq$-least root point with $\beta + 1$ as the i-th coordinate. Our equivalent set is thus D_γ.

If we have a set $\{\vec{\alpha} : \alpha_i < \beta\}$ for some ordinal β, take the root point $\widehat{\delta}$ with β as the i-th coordinate:

$$\widehat{\delta} := (\omega_i^\beta, ..., \omega^\beta, \beta, e(\beta), ...)$$

Then, $\{\vec{\alpha} : \alpha_i < \beta\} = \overline{D}_\delta$. $\dashv$

Theorem 3.2.3 suggests a probable cause of incompactness, as well as a natural solution. Instead of ϵ_0, consider $\epsilon_0 + 1$, and let $(\mathcal{E}', \tau^{\mathcal{E}'})$ be the corresponding product, which is compact by Tychonoff's Theorem. We will obtain a closed subspace of $(\mathcal{E}', \tau^{\mathcal{E}'})$ if we augment $\mathcal{U}$ to the following frame $\mathcal{V}$:[31]

[31] Something similar to this was done in [Ignatiev, 1992] to a different end.

Definition 3.2.4. Let $\mathcal{V} = (V, R_n : n < \omega)$ be defined exactly like $\mathcal{U}$, except that in V the α_i's range over all ordinals less than *or equal to* ϵ_0. That is,

$$V := \{\vec{\alpha} : \forall i, \alpha_i \leq \epsilon_0, \alpha_{i+1} \leq e(\alpha_i)\}$$

The relations R_n are defined exactly the same as in $\mathcal{U}$.

All of our work on $\mathcal{U}$ carries over without change, as we have not added any root points:[32]

Proposition 3.2.5. *For all words $A \in S$ and points $\vec{\beta} \in V$,*

$$\mathcal{V}, \vec{\beta} \vDash A \Leftrightarrow \vec{\beta} \succeq \iota(A)$$

Proposition 3.2.6. *For any distinct points $\vec{\alpha}$, $\vec{\beta}$ in V, there is a word A such that $\mathcal{V}, \vec{\alpha} \vDash A$ but $\mathcal{V}, \vec{\beta} \nvDash A$.*

Proposition 3.2.7. *For all closed formulas ϕ, $\mathbf{GLP}_0 \vDash \phi$, if and only if $\mathcal{V}, \widehat{\alpha} \vDash \phi$, for all root points $\widehat{\alpha} \in M$.*

If we let (V, τ^V) be the space with τ^V the subspace topology inherited from $\tau^{\mathcal{E}'}$, then as a closed subspace of a compact space, (V, τ^V) becomes compact. Moreover, as Theorem 3.2.3 carries over without change, we can think of A as exactly the clopen sets of τ^V, and we have:[33]

Corollary 3.2.8. *$(\mathcal{V}, A)$ is a differentiated, compact general frame.*

The frame $\mathcal{V}$ mimics the structure of $\mathcal{U}$ in a certain sense. We obviously have an isomorphic copy of $\mathcal{U}$ as a subframe of $\mathcal{V}$. Let $V_0 := \{\vec{\alpha} \in V : \alpha_0 = \epsilon_0 \;\&\; \alpha_1 < \epsilon_0\}$ and $\mathcal{V}_0$ be the frame with relations restricted to points in V_0. Then it is clear that $\mathcal{V}_0$ is isomorphic to $\mathcal{U}^+$, and all elements of V_0 are R_0-related to all points in (the isomorphic copy of) U. We can then define V_1 to be $\{\vec{\alpha} \in V : \alpha_0 = \alpha_1 = \epsilon_0 \;\&\; \alpha_2 < \epsilon_0\}$, and we notice that $\mathcal{V}_1$ is isomorphic to $\mathcal{U}^{++}$, with each element of V_1 R_1-related to all points in V_0. We can continue in this way obtaining a series of natural subframes of $\mathcal{V}$, with each point in V_{n+1} R_{n+1}-related to each point in V_n, and so on. Finally, when we consider the set $V_\omega := \{\vec{\alpha} \in V : \forall i, \alpha_i = \epsilon_0\}$, there is but one point: $\widehat{\epsilon_0}$. A picture of the situation can be seen in Figure 5.

Of course, $\mathcal{V}$ inherits from $\mathcal{U}$ the problem that it is not tight. We still do not have, for example, $(\omega, 1) R_0 (\omega, 0)$. Or, to take a curious example, it is easy to see that whenever $(\omega, 0) \vDash \phi$, we have $(\omega, 0) \vDash \langle 0 \rangle \phi$. This means, after we tighten $\mathcal{V}$, there will actually be reflexive points.

[32] One could consider the point $\widehat{\epsilon_0}$ as a root point since it formally satisfies the definition. However, there is obviously no word A such that $\iota(A) = \widehat{\epsilon_0}$. Thus, all of our work on validity in $\mathcal{U}$ really is left unchanged.

[33] Strictly speaking, the set of subsets S_ϕ in $\mathcal{U}$ is not the same as that in $\mathcal{V}$. However, as they give rise to isomorphic modal algebras, we use the same letter A to denote this algebra.

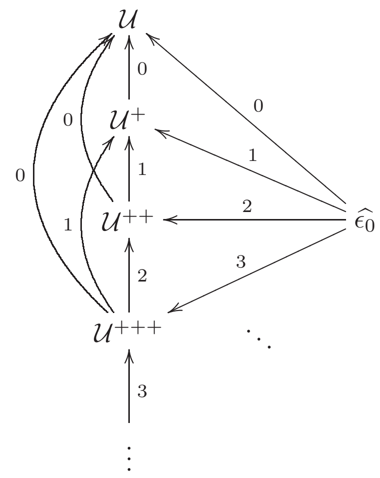

Figure 5: The Frame $\mathcal{V}$

3.2.2 Tightness

In general we have the following sufficient condition for tightness, stated for an arbitrary general frame $(\mathbb{F}, A)$:

Proposition 3.2.9 ([Blackburn et al., 2001]). *If* $(\mathbb{F}, A)$ *is differentiated and compact, it is descriptive if and only if* R_x *is closed in* τ^F *for all* x.

Proof. For the left-to-right direction, we refer the reader to the text cited. We prove the other direction, but only for the case where A is the set of formula-definable subsets.[34]

Suppose for all formulas ϕ, if $y \vDash \phi$ then $x \vDash \Diamond\phi$. We must show xRy. Consider an enumeration of formulas valid at y: $\phi_0, \phi_1, \phi_2, ...$ By hypothesis, for each such formula ϕ_k there is a point y_k such that xRy_k and $y_k \vDash \phi_k$. Since R_x is closed in τ^F, and $(\mathbb{F}, A)$ is compact, it follows that (R_x, B) is compact, where B is the restriction of A to R_x. For each k, let $B_{\phi_k} := R_x \cap S_{\phi_k}$, and define $B' := \{B_{\phi_k} : k > 0\}$. As B' is closed under finite intersection, and (R_x, B) is compact, it follows $\bigcap B'$ is not empty, say $z \in \bigcap B'$. This means, for each ϕ_k, $z \vDash \phi_k$. But as $(\mathbb{F}, A)$ is differentiated, this means $z = y$, and so xRy. $\dashv$

Once again, the generalization to multiple modalities is unproblematic. We are now ready to introduce the final amendment to our universal frame. The following we claim is isomorphic to the canonical frame:

Definition 3.2.10. Let $\mathcal{V}^c$ be the frame $(V, R^c_n : n < \omega)$, where,

$$\vec{\alpha} R^c_n \vec{\beta} \iff \vec{\alpha} R_n \vec{\beta}, \text{ or } \exists k \geq n, \forall m \leq k, \alpha_m = \beta_m \ \& \ e(\alpha_k) > \beta_{k+1}$$

Thus, $\mathcal{V}^c$ is exactly like $\mathcal{V}$, with the exception of new pairs added to each R_n.

[34] This proposition holds for the more general case that A is any algebra over F and τ^F is the corresponding topology. However we are only concerned with this particular case.

Lemma 3.2.11. *$R^c_{n,\vec{\alpha}}$ is closed in τ^V.*

Proof. We prove this indirectly by showing that $V \setminus R^c_{n,\vec{\alpha}}$ is open in τ^V. We consider two cases: when α_n is a successor ordinal, and when α_n is a limit ordinal.

Suppose $\alpha_n = \gamma + 1$. Then $e(\alpha_n) = 0$, and there is consequently no $\vec{\beta}$ such that $e(\alpha_k) > \beta_{k+1}$ for any $k \geq n$. We let $U := \{\vec{\beta} : \beta_n > \gamma\}$. Let

$$Y := \bigcup_{m<n} \{\vec{\beta} : \beta_m < \alpha_m\} \cup \bigcup_{m<n} \{\vec{\beta} : \beta_m > \alpha_m\} = \bigcup_{m<n} \{\vec{\beta} : \beta_m \neq \alpha_m\}$$

Then we clearly have

$$V \setminus R^c_{n,\vec{\alpha}} = Y \cup U$$

If α_n is a limit ordinal, then, for any $\vec{\beta}$, if $\beta_m = \alpha_m$ for all $m \leq n$, then $\vec{\beta} \in R^c_{n,\alpha}$ if and only if there is some $k > n$ such that $e(\alpha_k) > \beta_{k+1}$, and $\beta_m = \alpha_m$ for all $m \leq k$. Let p be greatest such that $e(\alpha_p) \neq 0$, so that we have a bound on how high we check whether $e(\alpha_k) \leq \beta_{k+1}$. Our desired set is thus,

$$Z := \bigcap_{n \leq k \leq p} (\{\vec{\beta} : e(\alpha_k) < \beta_{k+1} + 1\} \cup \bigcup_{m \leq k} \{\vec{\beta} : \beta_m \neq \alpha_m\})$$

Note that this intersection is finite, so that Z is still open. We have, then,

$$V \setminus R^c_{n,\vec{\alpha}} = Z$$

In both cases, as $V \setminus R^c_{n,\vec{\alpha}}$ is open, $R^c_{n,\vec{\alpha}}$ is closed. ⊣

Corollary 3.2.12. *$(\mathcal{V}^c, A)$ is a descriptive general frame.*

Lemma 3.2.13. *For all words A and points $\vec{\alpha} \in V$*

$$\mathcal{V}, \vec{\alpha} \vDash A \iff \mathcal{V}^c, \vec{\alpha} \vDash A$$

Proof. By induction on the length of A. The basic case is obvious, as is the left-to-right direction of the inductive case, since R^c_n extends R_n.

Suppose $\mathcal{V}^c, \vec{\alpha} \vDash nA$. Moreover, suppose the reason is that $\vec{\alpha} R^c_n \vec{\beta}$, where $e(\alpha_k) > \beta_{k+1}$ for some $k \geq n$, and $\beta_m = \alpha_m$ for all $m \leq k$, and $\mathcal{V}^c, \vec{\beta} \vDash A$. Otherwise, we would have $\vec{\alpha} R_n \vec{\beta}$, which would give us the result immediately by the inductive hypothesis. Consider $\iota(A) = \widehat{\gamma}$. Since $\widehat{\gamma} \preceq \vec{\beta}$, we have $\gamma_{k+1} \leq \beta_{k+1}$, which implies $\gamma_k < \beta_k$, since $\beta_{k+1} < e(\beta_k)$ and $\widehat{\gamma}$ is a root point. As a matter of fact, $\gamma_m < \beta_m = \alpha_m$ for all $m \leq k$. From this we conclude $\vec{\alpha} R_n \vec{\delta}$ where,

$$\vec{\delta} := (\alpha_0, \alpha_1, ...\alpha_{n-1}, \gamma_n, \gamma_{n+1}, ...)$$

As therefore $\mathcal{V}, \vec{\delta} \vDash A$, we have $\mathcal{V}, \vec{\alpha} \vDash nA$. ⊣

Corollary 3.2.14. *For all closed formulas ϕ and points $\vec{\alpha} \in V$*

$$\mathbf{GLP} \vdash \phi \iff \mathcal{V}, \vec{\alpha} \vDash \phi \iff \mathcal{V}^c, \vec{\alpha} \vDash \phi$$

Proof. By Corollary 2.4.1 and Lemma 3.2.13. ⊣

Theorem 3.2.15. $\mathcal{V}^c \cong \mathbb{C}^0$.

4 Topological Models

In the previous section, we considered a single topology τ^V on the frame $\mathcal{V}^c$, which allowed us to prove that $\mathcal{V}^c$ gives rise to a descriptive frame and is isomorphic to the canonical frame of $\mathbf{GLP}_0$. In this final section, τ^V will be divided into infinitely many topologies, which will allow us to interpret the modalities of our language directly in topological terms. Topological semantics of a given modal logic is often simpler than relational semantics, and the topological model of $\mathbf{GLP}_0$ we will obtain in this section will be no exception.

Ordinarily, when modal logics are interpreted topologically, the diamond operator is translated to topological closure: $\Diamond$ becomes c in the language of topology. However, this only works if the logic in question contains the reflexivity axiom T, since every set is a subset of its closure. For logics that do not contain T, of which **GL** and **GLP** are of course examples, $\Diamond$ can instead be translated to the topological derivative operator, an idea originating with McKinsey and Tarski and explored thoroughly by the "Tbilisi Group."[35] Recall the definition of derivative:

Definition 4.0.16 (Derived Set). The *derivative* of a set A, written $d(A)$, is the set of points x, such that, for all open neighborhoods I_x of x, $I_x \setminus \{x\}$ has non-empty intersection with A. These points x are called *limit points* of A.

The closure operator is easily definable in terms of the derived set operator:

$$cA = A \cup dA$$

Definition 4.0.17 (Topological Validity). Given a topological space $\mathcal{X}$ and a point $x \in \mathcal{X}$, let us say, $\mathcal{X}, x, f \Vdash \phi$ if $f(\phi)$ is true at x and f is a function from modal formulas to $\mathcal{P}(\mathcal{X})$ such that:

$$f(p) \subseteq \mathcal{P}(\mathcal{X})$$

$$f(\top) = \mathcal{X}$$

$$f(\neg\phi) = \mathcal{X} \setminus f(\phi)$$

$$f(\phi \vee \psi) = f(\phi) \cup f(\psi)$$

$$f(\Diamond\phi) = d(f(\phi))$$

Correspondingly, $\mathcal{X}, x \Vdash \phi$ if $\mathcal{X}, x, f \Vdash \phi$ for all such f. Finally, $\mathcal{X} \Vdash \phi$ if $\mathcal{X}, x, f \Vdash \phi$ for all f and $x \in \mathcal{X}$.

Remark 4.0.18. When working in the closed fragment, f does not vary. So these definitions can be somewhat simplified. E.g. $\mathcal{X} \Vdash \phi$ iff $f(\phi) = \mathcal{X}$.

Topological semantics of **GL** in terms of derivative was first investigated by Leo Esakia, who proved that the logic corresponds to a natural class of spaces.

[35] An interesting overview of the development of this work can be found in [Esakia, 2003].

Definition 4.0.19. A space is *scattered* if every subspace has an isolated point.

Theorem 4.0.20 ([Esakia, 1981]). **GL** $\vdash \phi$*, if and only if* ϕ *is valid in all scattered spaces.*

This result was later improved, independently by Andreas Blass and Merab Abashidze. As in the previous section, we consider ω^ω as an ordinal space with the interval topology. Every subspace of ω^ω has some $<$-least point, say β, so the ray $\{\alpha : \alpha < \beta + 1\}$ isolates β. In other words, ω^ω is clearly scattered, so **GL** is sound. In fact, it is also complete.[36]

Theorem 4.0.21 ([Abashidze, 1985],[Blass, 1990]). **GL** $\vdash \phi$*, iff* $\omega^\omega \Vdash \phi$.

One can imagine obtaining topological models of **GLP** by generalizing this theorem. Since the logic of each $[n]$ considered separately is exactly **GL**, one could start with τ_0 as the interval topology on some sufficiently large ordinal, and we will automatically have soundness and completeness for this particular fragment of the logic. Then, for τ_n, $n > 0$, we would find stronger and stronger topologies that enforce completeness for the fragment containing only $[n]$, and that "fit together" exactly right with all the topologies τ_m, $m < n$, so that the two "bridge axioms" (iv) and (v) are also valid. In the next section, we will show that something very similar to this idea is possible in the case of the closed fragment. In fact, our earlier work on $\mathcal{U}$ and $\mathcal{V}^c$ will bring us a long way toward such a result, and (perhaps unsurprisingly) we will only need to go up to the ordinal ϵ_0. In the case of the full fragment the situation is quite different. By results of Beklemishev and Blass, if we begin with τ_0 as the interval topology, obtaining a stronger, non-trivial topology τ_1 with the appropriate properties will already require going up to the first uncountable ordinal. The question of completeness even for the restricted fragment of **GLP** containing only the modality [1] hinges on facts about large cardinals independent of ZFC.

4.1 Ordinal Completeness of $\mathbf{GLP}_0$

4.1.1 The Space Θ

We saw in Section 2.4 that every non-theorem of $\mathbf{GLP}_0$ is falsifiable somewhere on the main axis M of either $\mathcal{V}^c$ or $\mathcal{U}$, which suggests that in a sense both frames contain "too many" points. This is of course necessary in relational models. But by moving to topological models, it can be avoided, and the underlying space will become much simpler. Instead of a subspace of the product topology on $\epsilon_0 + 1$, we will be able to take $\epsilon_0 + 1$ itself as the underlying set. In doing so, we will be rid of all points outside of the main axis.

The most obvious way of defining a polytopology on $\mathcal{V}^c$ is to let τ_n consist of all R_n-upsets of $\mathcal{V}^c$. Then we will have,

$$\begin{aligned} \vec{\alpha} R_n \vec{\beta} &\iff \text{for all } \tau_n\text{-open } R_n\text{-upsets } A \text{ of } \vec{\alpha},\ \vec{\beta} \in A \\ &\iff \vec{\alpha} \in d_n(\{\vec{\beta}\}) \end{aligned}$$

[36] For a very readable proof of this theorem, see [Bezhanishvili and Morandi, 2008].

So modal validity and topological validity will coincide, trivially. This is nothing more than an alternative way of talking about $\mathcal{V}^c$ as a relational model. To obtain a more interesting polytopology, recall the single topology τ^V defined on the underlying set V of $\mathcal{V}^c$. As in all spaces, closed sets correspond to those of the form $X \cup d(X)$. This suggests a definition of a more fine-grained topology, based on τ^V, that will allow us to differentiate among derivative operators. We adopt the convention that $[n]^+\phi := \phi \wedge [n]\phi$.

Definition 4.1.1. Let $\Lambda = (V, \nu_n : n < \omega)$ be defined so that ν_n is generated by the open basis of sets $S_{[n]^+\phi}$, with ϕ ranging over closed formulas.

The following proposition reinforces the suggestion that this is a reasonable topology to consider.

Proposition 4.1.2. *For all points $\widehat{\alpha} \in M$, $\Lambda, \widehat{\alpha} \Vdash \phi$ if and only if $\mathcal{V}^c, \widehat{\alpha} \vDash \phi$.*

Proof. We check the case of $\langle n \rangle \phi$.

Suppose $\mathcal{V}^c, \widehat{\alpha} \nvDash \langle n \rangle \phi$. Then for all $\vec{\beta}$ such that $\widehat{\alpha} R^c_n \vec{\beta}$, $\mathcal{V}^c, \vec{\beta} \nvDash \phi$. To show $\Lambda, \widehat{\alpha} \nVdash \langle n \rangle \phi$, we must find some ν_n-open neighborhood of $\widehat{\alpha}$ such that all other elements of this neighborhood falsify ϕ. By the definition of ν_n, it suffices to identify an appropriate formula. Because $\widehat{\alpha}$ is a root point, there is some word A such that $\iota(A) = \widehat{\alpha}$. Let

$$\psi := (A \vee \neg\phi) \wedge [0]\neg A$$

Clearly, $\mathcal{V}^c, \widehat{\alpha} \vDash [n]^+\psi$, so this formula defines an open neighborhood of $\widehat{\alpha}$. We must now show, if $\vec{\beta} \vDash [n]^+\psi$, then $\vec{\beta} \nvDash \phi$. Assume for a contradiction $\vec{\beta} \neq \widehat{\alpha}$ and $\vec{\beta} \vDash A \wedge [0]\neg A$ (since otherwise $\vec{\beta} \vDash \neg\phi$ and we are done). This means $\vec{\beta} \succeq \widehat{\alpha}$, but also $\beta_0 = \alpha_0$, because otherwise we would have $\vec{\beta} R^c_0 \widehat{\alpha}$, which would imply $\vec{\beta} \vDash \langle 0 \rangle A$. As we are assuming $\vec{\beta} \neq \widehat{\alpha}$, this implies $\vec{\beta}$ cannot be a root point. That is, there is some $k \geq 0$ such that $e(\beta_k) > \beta_{k+1}$. By the definition of R^c_n, this means in fact $\vec{\beta} R^c_0 \vec{\beta}$, which gives the desired contradiction. We conclude that if $\vec{\beta} \vDash [n]^+\psi$ and $\vec{\beta} \neq \widehat{\alpha}$, then $\vec{\beta} \nvDash \phi$. That is to say, $\Lambda, \widehat{\alpha} \nVdash \langle n \rangle \phi$.

If $\Lambda, \widehat{\alpha} \nVdash \langle n \rangle \phi$, then there is some formula $[n]^+\psi$ such that $\vec{\beta} \neq \widehat{\alpha}$ and $\vec{\beta} \vDash [n]^+\psi$ imply $\vec{\beta} \nvDash \phi$. Given any such formula, it is clear that if $\widehat{\alpha} \vDash [n]^+\psi$, then for all $\vec{\beta}$ such that $\widehat{\alpha} R^c_n \vec{\beta}$, $\vec{\beta} \vDash [n]^+\psi$. Therefore, we automatically have $\widehat{\alpha} R^c_n \vec{\beta} \Rightarrow \vec{\beta} \nvDash \phi$, i.e. $\mathcal{V}^c, \widehat{\alpha} \nvDash \langle n \rangle \phi$. $\dashv$

Corollary 4.1.3. GLP$_0$ *is complete with respect to* Λ.

As the topologies of Λ are defined *via* validity on $\mathcal{V}^c$, and therefore do not depend on other points in Λ, Corollary 4.1.3 tells us we may just as well consider the following subspace:

Definition 4.1.4. Let $\mathcal{M} := (M', \nu_n : n < \omega)$ be the subspace of Λ, where $M' \subset V$ is the *main axis* M of $\mathcal{V}$, in addition to the single point $\widehat{\epsilon_0}$.

By Proposition 4.1.2, $\mathbf{GLP}_0$ is complete with respect to $\mathcal{M}$, and in fact it is also sound: If $\mathbf{GLP}_0 \vdash \phi$, then $\mathcal{V}^c, \widehat{\alpha} \vDash \phi$ for all root points $\widehat{\alpha}$, so $\Lambda, \widehat{\alpha} \Vdash \phi$ by Proposition 4.1.2, and hence $\mathcal{M}, \widehat{\alpha} \Vdash \phi$. Since M' only consists of root points and $\widehat{\epsilon_0}$, this means $\mathcal{M} \Vdash \phi$. We state this as a theorem:

Theorem 4.1.5. $\mathbf{GLP}_0$ *is sound and complete with respect to* $\mathcal{M}$.

Another consequence of dealing only with root points is that we no longer need to speak about points as vectors; we can refer to root points simply by their first coordinate, since the rest of the coordinates are thereby determined. This means that the underlying set M' can simply be seen as the ordinal $\epsilon_0 + 1$.

What we would like now is a more conspicuous definition of the topologies ν_n, in particular a definition that does not depend on the frame $\mathcal{V}^C$ and that relates in a more natural way to the ordinal $\epsilon_0 + 1$. In that direction we introduce:[37]

Definition 4.1.6 (The Space Θ). Let Θ denote the space $(\epsilon_0 + 1, \theta_n : n < \omega)$, where θ_n is defined by the subbasis of rays, for $m \leq n$, $k < n$, and $\beta \leq \epsilon_0$:[38]

$$\{\alpha : e^m(\alpha) < \beta\}$$

$$\{\alpha : e^k(\alpha) > \beta\}$$

Remark 4.1.7. Recall that for any root point $\widehat{\alpha}$, $\alpha_n = e^n(\alpha)$. Definition 4.1.6 therefore has an equivalent formulation in terms of points $\widehat{\alpha}$ and their corresponding coordinates. That would be: $\Theta := (M', \theta_n : n < \omega)$, where a subbasis for θ_n is the set of all *n-cylinder sets* of (M', τ^V),[39] in addition to all rays of the following form (restricted to root points),

$$\{\widehat{\alpha} : \alpha_n < \beta\}$$

This means the topology τ^V on the canonical frame (restricted to root points) is the union of all such topologies:

$$\tau^V = \bigcup_{n<\omega} \theta_n.$$

In the proof of the following theorem, let us assume we are working with this definition, so as to keep the notation consistent among Θ, $\mathcal{M}$, and $\mathcal{V}^c$.

Lemma 4.1.8. Θ *and* $\mathcal{M}$ *are homeomorphic.*

Proof. Take some ν_n-basic set $S_{[n]^+\phi}$. By Theorem 3.2.3, we know $S_{[n]^+\phi}$ is equivalent to some cylinder set of V, i.e. some finite intersection of rays of the

[37] Leo Esakia has observed that the bimodal fragment of **GLP**, that is, the fragment containing only [0] and [1], is sound with respect to a similar space to this one, taking τ_0 to be the topology of "downsets" and τ_1 to be the interval topology, on ω^ω.

[38] We write $e^m(\alpha)$ to mean $e(...(e(\alpha))...)$, m times e. So $e^0(\alpha) = \alpha$.

[39] In an n-cylinder set we only allow finite intersections of sets $\pi_m^{-1}(I)$ for $m < n$, instead of for arbitrary m as in Definition 3.2.2 of τ^V.

form $\{\vec{\alpha} : \beta < \alpha_m\}$ or $\{\vec{\alpha} : \beta > \alpha_m\}$. To prove this cylinder is equivalent to a θ_n-open set, it suffices to show that any ray $\{\vec{\alpha} : \beta < \alpha_m\}$ with $m \geq n$ is equivalent (as a subset of $S_{[n]^+\phi}$) to one with $m < n$, and that any ray $\{\vec{\alpha} : \alpha_m < \beta\}$ with $m > n$ is equivalent (as a subset of $S_{[n]^+\phi}$) to one with $m \leq n$. For the case of $\{\vec{\alpha} : \beta < \alpha_m\}$, this is obvious, because this set is equivalent *tout court* to $\{\vec{\alpha} : \omega_m^\beta < \alpha_0\}$. In the other case, if $m > n$, we claim,

$$S_{[n]^+\phi} \subseteq \{\vec{\alpha} : \alpha_m < \beta\} \iff S_{[n]^+\phi} \subseteq \{\vec{\alpha} : \alpha_n < \omega_{m-n}^\beta\}.$$

The right-to-left direction is obvious. Suppose for a contradiction $\widehat{\alpha} \vDash [n]^+\phi$ and $\alpha_m < \beta$, but $\alpha_n > \omega_{m-n}^\beta$. We observe $\widehat{\alpha} R_n^c \vec{\gamma}$, where

$$\vec{\gamma} := (\alpha_0, ..., \alpha_{n-1}, \omega_{m-n}^\beta, ..., \beta, ...).$$

So $\vec{\gamma} \vDash [n]^+\phi$, which is a contradiction since $\gamma_m = \beta$.

For the other direction, let $I = \{\vec{\alpha} : \beta < \alpha_m\}$ be any θ_n-open ray, so that $m < n$. Define,

$$\widehat{\delta} := (\omega_m^\beta, ..., \omega^\beta, \beta, e(\beta), ...).$$

Stipulate $\iota(A) = \widehat{\delta}$ and $\phi := mA$. We show $\widehat{\alpha} \vDash [n]^+\phi$ if and only if $\widehat{\alpha} \in I$.

First note $\alpha_m > \beta$ if and only if $\widehat{\alpha} \vDash mA$: If $\alpha_m > \beta$, then $\widehat{\alpha} R_m^c \vec{\kappa}$, where $\vec{\kappa} := (\alpha_0, ..., \alpha_{m-1}, \beta, e(\beta), ...)$, so $\vec{\kappa} \vDash A$ and $\widehat{\alpha} \vDash mA$. On the other hand, if $\widehat{\alpha} \vDash mA$, then $\widehat{\alpha} R_m^c \vec{\kappa}$ for some $\vec{\kappa}$ identical to $\widehat{\alpha}$ up to the m-th coordinate, and either $\alpha_m > \kappa_m \geq \beta$, or there is a $k \geq m$ such that $\kappa_{k+1} < e(\kappa_k)$. If the latter, then $\alpha_m \geq \kappa_m > \beta$.

Thus, $\widehat{\alpha} \in I$ if and only if $\widehat{\alpha} \vDash \phi$. This in turn is true if and only if $\widehat{\alpha} \vDash [n]^+\phi$. For, if $\widehat{\alpha} \vDash mA$ and $\widehat{\alpha} R_n^c \vec{\kappa}$, since $\alpha_k = \kappa_k$ for all $k < n$, also $\vec{\kappa} \vDash mA$.

Let $J = \{\vec{\alpha} : \alpha_m < \beta\}$ be any θ_n-open ray, so that $m \leq n$. Define,

$$\widehat{\eta} := (\omega_m^\gamma, ..., \omega^\gamma, \gamma, e(\gamma), ...).$$

Let $\iota(B) = \widehat{\eta}$ and $\phi := \neg mB \wedge \neg B$. We show $\widehat{\alpha} \vDash [n]^+\phi$ if and only if $\widehat{\alpha} \in J$.

Observe $\alpha_m < \gamma$ if and only if $\widehat{\alpha} \vDash \phi$: Supposing $\alpha_m < \gamma$, obviously $\widehat{\alpha} \nvDash B$, and if $\widehat{\alpha} R_m^c \vec{\kappa}$, then $\kappa_m \leq \alpha_m < \gamma$, so $\vec{\kappa} \nvDash B$. Conversely, if $\alpha_m \geq \gamma$, then $\alpha_k \geq \eta_k$ for all $k < m$, since $\widehat{\eta}$ is the $\preceq$-minimal root point with $\eta_m = \gamma$. If $\alpha_m = \gamma$, then $\widehat{\alpha} \vDash B$; if $\alpha_m > \gamma$, then $\widehat{\alpha} R_m^c \vec{\kappa}$, where $\vec{\kappa} := (\alpha_0, ..., \alpha_{m-1}, \gamma, e(\gamma), ...)$. Since $\vec{\kappa} \vDash B$, $\widehat{\alpha} \vDash mB$. Either way, $\widehat{\alpha} \vDash mB \vee B$, so $\widehat{\alpha} \nvDash \phi$.

Finally we show $\widehat{\alpha} \vDash \phi \;\Rightarrow \widehat{\alpha} \vDash [n]^+\phi$: Since $\widehat{\alpha} R_n^c \vec{\kappa}$ implies $\alpha_k = \kappa_k$ for all $k < n$ and $\alpha_n \geq \kappa_n$, it follows $\widehat{\alpha} \nvDash mB \Rightarrow \vec{\kappa} \nvDash mB$. For $\neg B$ we use the previous fact: If $\widehat{\alpha} \vDash \phi$, then $\alpha_m < \gamma$, so if $\widehat{\alpha} R_n^c \vec{\kappa}$, then $\kappa_m \leq \alpha_m < \gamma$, and $\vec{\kappa} \nvDash B$.

We conclude that every θ_n-basic-open interval is equivalent to some set $S_{[n]^+\phi}$, basic in ν_n. $\dashv$

Thus we come to our central theorem of this section.

Theorem 4.1.9 (Topological Completeness). $\mathbf{GLP}_0 \vdash \phi \iff \Theta \Vdash \phi$.

The space Θ can be seen simply as a condensed version of the canonical frame of $\mathbf{GLP}_0$. It consists of root points and nothing else, yet topological validity is defined so that an ordinal α of Θ and the corresponding root point $\widehat{\alpha}$ of $\mathcal{V}^c$ are modally equivalent. To see what Θ looks like in more detail, we end our discussion with a short investigation into the θ_n-derived sets of $\epsilon_0 + 1$.

4.1.2 Limit Points in Θ

We prove that the θ_n-non-isolated points of $\epsilon_0 + 1$ generalize the notion of limit. Of course, the only point that is θ_0-isolated from $\epsilon_0 + 1$ is 0. The set of θ_1-non-isolated points is exactly the set of ordinary limit points, as θ_1 coincides with the interval topology on $\epsilon_0 + 1$. As we shall see, θ_2-non-isolated are those ordinals that are limits of limits of limits..., infinitely many times. And so on for all greater n. First we prove a number of auxiliary lemmas.

Definition 4.1.10 (Fundamental Sequences). Suppose $\alpha = \kappa + \omega^\lambda$ is any limit ordinal greater than 0, less than ϵ_0. We define $\alpha[n]$ by recursion on $e(\alpha) = \lambda$, noting that for any α, there is some m such that $e^m(\alpha)$ is a successor ordinal:

- If $\lambda = \mu + 1$, $\alpha[n] = \kappa + \omega^\mu \cdot (n + 1)$.
- If λ is a limit ordinal, $\alpha[n] = \kappa + \omega^{\lambda[n]}$.

Let FS_α denote the set $\{\alpha[n] : n < \omega\}$. Clearly $\lim_{n<\omega} \alpha[n] = \alpha$.

Lemma 4.1.11. *For $n > 0$, if $e^n(\alpha) > 0 = e^{n+1}(\alpha)$, then $\alpha \in d_n(FS_\alpha)$.*

Proof Sketch. Suppose $e^n(\alpha) > 0 = e^{n+1}(\alpha)$. Say, $e^n(\alpha) = \lambda = \mu + 1$, and:

$$\alpha = \kappa_n + \omega^{\cdots^{\kappa_1 + \omega^\lambda}}$$

Take any θ_n-open I. Clearly I contains such an interval, for some m:

$$(\kappa_n + \omega^{\cdots^{\kappa_1 + \omega^\mu \cdot m}}, \alpha]$$

That is, I contains infinitely many ordinals in any fundamental sequence of α, which is to say $\alpha \in d_n(FS_\alpha)$. $\dashv$

In the previous lemma, we can omit the requirement that $e^{n+1}(\alpha) = 0$, since $d_{n+1}(A) \subseteq d_n(A)$ for all $A \subseteq \epsilon_0$.

Lemma 4.1.12. *For $n > 0$, if $e^n(\alpha) > 0$, then $\alpha \in d_n(FS_\alpha)$.*

Definition 4.1.13. We define a sequence of classes, $\mathfrak{L}_\alpha$, indexed by the ordinals:

- Every ordinal is $\mathfrak{L}_0$.
- An ordinal α is $\mathfrak{L}_{\beta+1}$ if there is a strictly increasing sequence of ordinals $\gamma_0, \gamma_1, \gamma_2, ...$, such that $\lim_{n<\omega} \gamma_n = \alpha$ and, for all n, $\gamma_n \in \mathfrak{L}_\beta$.
- An ordinal α is $\mathfrak{L}_\lambda$, for λ a limit ordinal, if α is $\mathfrak{L}_\kappa$ for all $\kappa < \lambda$.

Thus $\mathfrak{L}_1$ is the set of ordinary limits, $\mathfrak{L}_2$ is the set of "limits of limits," $\mathfrak{L}_3$ the set of "limits of limits of limits," and so on.

Lemma 4.1.14. *For $n \geq 0$, $\mathfrak{L}_{\omega_n} = \{\alpha : e^{n+1}(\alpha) > 0\}$.*

Proof Sketch. The case of $n = 0$ is straightforward, so consider $n + 1$.

If $\alpha \in \mathfrak{L}_{\omega_{n+1}}$, then there is a sequence of ordinals $\lambda_0, \lambda_1, \lambda_2, ...$, such that $\lim_{i<\omega} \lambda_i = \alpha$ and each $\lambda_i \in \mathfrak{L}_{\omega_n}$. By inductive hypothesis, $e^{n+1}(\lambda_i) > 0$. Since the λ_i's are strictly increasing and $\lim_{i<\omega} \lambda_i = \alpha$, this forces $e^{n+1}(\alpha)$ to be a limit ordinal, which is to say $e^{n+2}(\alpha) > 0$.

Conversely, if $e^{n+2}(\alpha) > 0$, then $e^{n+1}(\alpha)$ is some limit ordinal λ. Given any κ such that $\omega_n < \kappa < \omega_{n+1}$, we can always find a fundamental sequence $n_0 < n_1 < n_2 ...$, such that each $\alpha[n_i] \in \mathfrak{L}_\kappa$. Then $\lim_{i<\omega} \alpha[n_i] = \alpha$, so $\alpha \in \mathfrak{L}_{\omega_{n+1}}$. $\dashv$

Proposition 4.1.15. *For $n \geq 0$, $d_{n+1}(\epsilon_0 + 1) = \mathfrak{L}_{\omega_n}$.*

Proof. If $\alpha \in \mathfrak{L}_{\omega_n}$, then by Lemma 4.1.14, $e^{n+1}(\alpha) > 0$, so using Lemma 4.1.12, $\alpha \in d_{n+1}(FS_\alpha) \subseteq d_{n+1}(\epsilon_0 + 1)$.

If $\alpha \notin \mathfrak{L}_{\omega_n}$, then $e^{n+1}(\alpha) = 0$, which means $e^n(\alpha)$ is either a successor ordinal or equal to 0. Suppose α is of the form,

$$\kappa_n + \omega^{\cdot^{\cdot^{\cdot^{\kappa_1 + \omega^{\beta+1}}}}}.$$

Then take the following θ_{n+1}-open set:

$$\begin{array}{rcl} \{\delta : \beta < e^n(\delta) < \beta + 2\} & \cap & \{\delta : \kappa_1 < e^{n-1}(\delta) < \kappa_1 + \omega^{\beta+1} + 1\} \\ & \cap & ... \\ & \cap & \{\delta : \kappa_n < \delta < \alpha + 1\} \end{array}$$

Certainly α is the only element of this set. If $e^n(\alpha) = 0$, then some κ_m, $m \leq n$ is a successor ordinal and we can take the same isolating set. $\dashv$

4.2 GLP-Spaces

In this last part, we consider topological models of full **GLP**. We first discuss what properties such spaces would have to satisfy, and then we make some speculative remarks concerning the prospective of an analog of the Abashidze-Blass Theorem for the full fragment of **GLP**.

Each of the axioms of **GLP** corresponds to a reasonably simple topological condition. Theorem 4.0.20 tells us exactly which spaces satisfy Löb's Axiom, namely the scattered spaces. The following theorems give analogous conditions for axioms (iv) and (v):

Lemma 4.2.1. *For all $A \subseteq \mathcal{X}$, $d_{n+1}(A) \subseteq d_n(A)$, if and only if $\tau_n \subseteq \tau_{n+1}$.*

Proof. Suppose $\tau_n \subseteq \tau_{n+1}$ and take any point $x \in d_{n+1}(A)$ and τ_n-open neighborhood I_x. Since I_x is also a τ_{n+1} neighborhood, we know $I_x \setminus \{x\} \cap A \neq \emptyset$. Thus $x \in d_n(A)$. Conversely, suppose B is τ_n-open but not τ_{n+1}-open. Since

B is not τ_{n+1}-open there must be some $x \in B$ such that, for any τ_{n+1}-open I_x containing x, $I_x \cap \mathcal{X} \setminus B \neq \emptyset$. Let $A := X \setminus B$. Then what we have seen is that $B \cap d_{n+1}(A) \neq \emptyset$. Yet $B \cap d_n(A) = \emptyset$. Thus $d_{n+1}(A) \nsubseteq d_n(A)$. ⊣

Theorem 4.2.2. *$\mathcal{X} \Vdash [n]\phi \to [n+1]\phi$, if and only if $\tau_n \subseteq \tau_{n+1}$.*

Remark 4.2.3. In the following, we adopt the convention that

$$t_n(A) := \mathcal{X} \setminus (d_n(\mathcal{X} \setminus A))$$

In other words, t_n is the interpretation of $[n]$: $f([n]\phi) = t_n(f(\phi))$.

Lemma 4.2.4. *For all $A \subseteq \mathcal{X}$, $d_n(A) \subseteq t_{n+1}(d_1(A))$, if and only if $d_n(A)$ is τ_{n+1}-open.*

Proof. For the right-to-left direction we must show, for any $x \in d_n(A)$, there is some τ_{n+1}-neighborhood I_x containing x such that $I_x \subseteq d_n(A)$. However, assuming $d_n(A)$ itself is τ_{n+1}-open, this is immediate. For the converse direction, suppose for any $x \in d_n(A)$ there is an open neighborhood I_x such that $I_x \subseteq d_n(A)$. Then let,

$$I := \bigcup_{x \in d_n(A)} I_x$$

As I is the union of τ_{n+1}-open sets, it too is open. And clearly $I = d_n(A)$. ⊣

Theorem 4.2.5. *$\mathcal{X} \Vdash \langle n\rangle\phi \to [n+1]\langle n\rangle\phi$, if and only if $d_n(A)$ is τ_{n+1}-open for all $A \subseteq \mathcal{X}$.*

These results give rise to a natural definition of **GLP**-space.

Definition 4.2.6. A **GLP**-*space* is a polytopological space $(\mathcal{X}, \tau_n : n < \omega)$, in which τ_n is scattered, $\tau_n \subseteq \tau_{n+1}$, and $d_n(A)$ is τ_{n+1}-open, for all $A \subseteq \mathcal{X}$ and n.

It is still an open question whether full **GLP** is complete with respect to its topological semantics. However, we do have limitative results that show a wide class of commonly considered spaces cannot be non-trivial **GLP**-spaces. Because of the duality between relational structures and Alexandrov topological spaces, a corollary of Theorem 2.1.6 is the following:

Corollary 4.2.7. *Every* **GLP**-*space in which τ_0 and τ_n, $n > 0$, are both Alexandrov must be τ_n-discrete.*

Given a topology τ_0 on some space, there is always a weakest topology τ_1 that is generated by τ_0 and sets of the form $d_0(A)$. To find non-trivial **GLP**-spaces we must ensure that our τ_0 does not force such a τ_1 to be discrete. A corollary of the following theorem is, in case τ_0 is the interval topology on an ordinal, this ordinal must be at least uncountable for τ_1 to be non-discrete.[40]

[40] Observations in Theorems 4.2.8 and 4.2.14 are due to Lev Beklemishev.

Theorem 4.2.8. *If $(\mathcal{X}, \tau_n : n < \omega)$ is a* **GLP***-space and τ_0 is Hausdorff and first-countable, then τ_n is discrete for all $n > 0$.*[41]

Proof. By Theorem 4.2.2, it suffices to show that τ_1 is discrete.

To prove this we show for every $x \in d_0(\mathcal{X})$, there is some $A \subseteq \mathcal{X}$ such that $d_0(A) = \{x\}$. By Theorem 4.2.5, this will imply that every such singleton is τ_1-open. However, since $d_1(\mathcal{X}) \subseteq d_0(\mathcal{X})$, $d_1(\mathcal{X}) = \emptyset$, i.e. τ_1 is discrete.

Consider any such $x \in d_0(\mathcal{X})$, and let $I_0, I_1, I_2, ...,$ be a basis of open neighborhoods of x with $I_n \subset I_m$ whenever $n > m$. For each such I_n, we know there is some $i_n \in I_n$ such that $i_n \neq x$, since $x \in d_0(\mathcal{X})$. So let $A := \{i_n : n \geq 0\}$.

Clearly, $x \in d_0(A)$, since for any neighborhood J of x there is an n such that $x \in I_n \subseteq J$, and so $i_n \in J \cap A$.

It remains to show that $y \neq x$ implies $y \notin d_0(A)$. If $y \neq x$, then since τ_0 is Hausdorff there are neighborhoods $I \ni x$ and $J \ni y$ such that $I \cap J = \emptyset$. Take some m such that $I_m \subseteq I$. Then for all $n \geq m$, $i_n \in I$, which means $i_n \notin J$. As therefore J contains at most m elements of A, $J \cap A$ is finite. Again using the fact that τ_0 is Hausdorff, it is possible to select a smaller neighborhood $J' \subseteq J$ of y such that $J' \cap A = \emptyset$. Hence $y \notin d_0(A)$. $\dashv$

On the other hand, if one is willing to give up one of these properties, it becomes possible to define non-trivial **GLP**-spaces. For example, the bitopology defined on ω^ω with τ_0 the Alexandrov topology of downsets and τ_1 the interval topology is a **GLP**-space, in which τ_0 is not first-countable.[42] However, there is obviously no hope for completeness as the 0-linearity axiom becomes valid:

$$[0]([0]\phi \to \psi) \vee [0]([0]^+\psi \to \phi).$$

A different approach to circumvent Theorem 4.2.8 would to be to move to uncountable ordinals. In the following, we shall assume τ_0 is always the interval topology on any given ordinal. The following definitions and facts are standard:

Definition 4.2.9. A set A is *cofinal in* λ if $\forall \gamma < \lambda, \exists \xi \in A$, such that $\gamma < \xi < \lambda$.

The *cofinality* of λ, $\mathrm{cf}(\lambda)$, is the smallest cardinality of a cofinal subset of λ.

Definition 4.2.10. A set A is *unbounded in* λ if $\forall \gamma < \lambda, \exists \xi \in A$, such that $\gamma \leq \xi < \lambda$. Otherwise A is *bounded in* λ.

Definition 4.2.11. A set A is *closed unbounded*, or simply *club*, *in* λ if it is τ_0-closed and unbounded in λ.

We shall use the notation $\aleph_1$ to denote the first uncountable ordinal, and assume the successor $\aleph_1 + 1$ can be written $\aleph_1 \cup \{\aleph_1\}$.

Fact 4.2.12 ([Jech, 1978]). *A countable intersection of clubs is club in $\aleph_1$.*

We are now ready to define a **GLP**-space in which τ_1 is non-trivial.

[41] A space is *first-countable* if for every x there is a sequence of neighborhoods I_0, I_1, I_2,..., such that, for any neighborhood J of x, $I_k \subseteq J$ for some k.

[42] This is the space mentioned above in Footnote 37, due to Leo Esakia.

Definition 4.2.13. Let $(\aleph_1 + 1, \tau_n : n < \omega)$ be defined so that τ_0 is the interval topology on $\aleph_1 + 1$, and $A \subseteq \aleph_1 + 1$ is τ_1-open if whenever $\aleph_1 \in A$ there is a club set $C \subseteq A \cap \aleph_1$. All other τ_n are discrete.

By Fact 4.2.12, τ_1 gives rise to a well defined topology. And τ_1 is not discrete, since $d_1(\aleph_1 + 1) = \{\aleph_1\}$.

Theorem 4.2.14. *$(\aleph_1 + 1, \tau_n : n < \omega)$ is a* **GLP***-space.*

Proof. Obviously $\tau_0 \subseteq \tau_1$, since every set $A \subseteq \aleph_1 + 1$ is τ_1-open unless $\aleph_1 \in A$. In that case, if $(\alpha, \aleph_1)$ is a τ_0-open interval, then it contains a club $[\alpha + 1, \aleph_1)$ and is therefore τ_1-open. As τ_0 is scattered, it thus follows that τ_1 is as well.

Finally, to show $d_0(A)$ is τ_1-open for all A, first note that $d_0(A)$ is always τ_0-closed.[43] If $A \cap \aleph_1$ is bounded, then $\aleph_1 \notin d_0(A)$, so $d_0(A)$ is τ_1-open. If $A \cap \aleph_1$ is unbounded, we claim $d_0(A) \cap \aleph_1$ is as well.

Indeed, for any $\alpha < \aleph_1$, we can find a sequence $\alpha < \alpha_0 < \alpha_1 < \alpha_2 < \ldots$ in A. Let $\beta := \lim_{n<\omega} \alpha_n$, so that $\beta \in d_0(A)$. Clearly $\beta < \aleph_1$, since our sequence is countable. Consequently, $d_0(A) \cap \aleph_1$ itself is club in $\aleph_1$, so $d_0(A) \in \tau_1$. $\dashv$

This leaves us with two obvious questions. The first is how to obtain yet stronger topologies in order to interpret $\langle 2 \rangle$, $\langle 3 \rangle$, and so on. The second, more immediate question is how high we would need to go in order to obtain more τ_1-non-isolated points. Our answer to the second question should cast doubt on the likelihood of a satisfactory answer to the first question.

A topology identical to our τ_1 has been considered by Blass in the context of topological interpretations of **GL** ([Blass, 1990]). For an arbitrary ordinal κ, the interpretation of $\langle 1 \rangle$ is simply, for $A \subseteq \kappa$:

$$d_1(A) = \{\lambda < \kappa : \mathrm{cf}(\lambda) > \aleph_0, \text{ and for all club sets } C \text{ of } \lambda, \ C \cap A \neq \emptyset\}.$$

In a sense, club sets take the place of open intervals. To introduce further terminology, a set A is called *stationary* in λ if A intersects all club sets of λ. So derived set can instead be written,

$$d_1(A) = \{\lambda < \kappa : \mathrm{cf}(\lambda) > \aleph_0, \text{ and } A \text{ is stationary in } \lambda\}.$$

Blass proves that **GL** is sound with respect to any ordinal under such an interpretation and considers the question whether it is also complete for a suitably large ordinal. Surprisingly, he shows the answer to this question is independent of ZFC. In fact, completeness of **GL** is shown to be equiconsistent with a large cardinal hypothesis, the existence of a so called Mahlo cardinal.[44] In particular, he isolates two well known infinitary combinatorial principles (Jensen's $\Box$ Principle and the Stationary Reflection Principle), both independent from ZFC, one

[43] As Guram Bezhanishvili has pointed out to us, necessary and sufficient conditions for $d(A)$ to be closed are that τ is a so called T_D-space, which means every singleton is the intersection of a closed and an open set. Our τ_0 clearly satisfies this requirement.

[44] For the definition, and discussion, of Mahlo cardinals, see [Jech, 1978].

of which implies the completeness and the other the incompleteness of **GL**.[45] Thus, from Blass' result we know that it is consistent with ZFC that **GLP** be incomplete as well. However, we cannot conclude from his work any positive result on topological completeness of **GLP**. Whether it is consistent with ZFC that **GLP** be complete is left for future work.

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[45] It is worth mentioning that **GL** is complete with respect to $\aleph_\omega$ if V=L is assumed, as this implies Jensen's $\Box$ Principle.

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