Game values and equilibria for undetermined sentences of Dependence Logic

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written by

Pietro Galliani
(born December 22, 1983 in Bologna, Italy)

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Members of the Thesis Committee:
Prof Dr Jouko Väänänen
Prof Dr Piet Rodenburg
Prof Dr Ulle Endriss
Prof Dr Peter van Emde Boas

INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

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Abstract

Logics of imperfect information, such as IF-Logic or Dependence Logic, admit a game-theoretic semantics: every formula ϕ corresponds to a game $H(\phi)$ between a Verifier and a Falsifier, and the formula is true [false] if and only if the Verifier [Falsifier] has a winning strategy.
Since the rule of the excluded middle does not hold in these logics, it is possible for a game $H(\phi)$ to be undetermined; this thesis attempts to examine the values of such games, that is, the maximum expected payoffs that the Verifier is able to guarantee.
For finite models, the resulting "Probabilistic Dependence Logic" can be shown, by means of the Minimax Theorem, to admit a compositional semantics similar to Hodges' one for Slash Logic.

1 Introduction

Logics of imperfect information, such as IF-logic, DF-logic or Dependence Logic, are extensions of First-Order Logic which allow more general patterns of dependence and independence between quantifiers.

In particular, Dependence Logic highlights the fundamental concept of *functional dependence*, introducing *dependence atomic formulas* $=(t_1, \dots t_n)$, meaning "The value of the term t_n is determined by the values of the terms $t_1 \dots t_{n-1}$".

However, in all of these logics the principle of the excluded middle fails: indeed, it is easy to find formulas ϕ such that neither player has an (uniform) winning strategy in the associated game.

Following a suggestion by Professor Väänänen and Professor Buhrman, this thesis attempts, by means of the Minimax Theorem, to associate to undetermined formula ϕ the values of the corresponding semantic games, that is, the maximum expected payoffs that one of the players is able to guarantee.
A similar proposal has been previously advanced by Miklos Ajtai, as Blass and Gurevich state at the very end of [4], but as far as I have been able to find out it has never been carried on.

The first part of this thesis recalls some basics about these logics; then, we define the values $V(\phi)$ and use the Minimax Theorem to prove a few results about them - in particular, we will be able to find a correspondence between dependence atomic formulas and one of the measures of approximate functional dependency used in Database Theory.
Finally, we will consider some possible extensions of the resulting "Probabilistic Dependence Logic".

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2 Game-theoretical semantics and Independence-Friendly logic

2.1 Preliminaries

In this work I will make frequent use of the notions of *first-order model* and *variable assignment*; moreover, if ϕ is a first-order formula, that is, an expression of the form

$$\phi := R(t_1 \dots t_k) \mid \phi \wedge \phi \mid \phi \vee \phi \mid \neg \phi \mid \exists x_j \phi \mid \forall x_j \phi$$

where the terms t are of the form

$$t := x_j \mid f(t_1 \dots t_k)$$

$\mathcal{M}$ is a model, and s is a variable assignment with domain $dom(s)$ containing the set

$$FV(\phi) = \{x : x \text{ is a free variable in } \phi\}$$

then the statement

$$\mathcal{M}, s \models_{FO} \phi$$

will mean that, according to the traditional Tarskian semantics for first-order logic, ϕ holds in the model $\mathcal{M}$ with respect to the assignment s.

I do not believe it necessary to recall here the formal definition of this semantics. However, it is worth observing that it has the very noteworthy property of *compositionality* [25] [23] [24]: if we define the *meaning* of a formula ϕ in the model $\mathcal{M}$ as the set of all assignments s such that $\mathcal{M}, s \models \phi$, we have that *the meaning of a non-atomic formula is determined by its structure and the meanings of its components.*
This is a principle which has led to considerable debate in the logical and linguistic communities: opinions range from believing it an essential requirement of any learnable truth definition[7] to regarding it "a lost cause in the study of semantics of natural languages"[12][13] [1].

When there is no ambiguity about the model $\mathcal{M}$, I will use $s \models_{FO} \phi$ as a shorthand for $\mathcal{M}, s \models_{FO} \phi$.

Another fact that I will use is that any first-order formula ϕ is logically equivalent to a formula ϕ' in *Negation Normal Form*, that is, such that no negation operators $\neg$ occur unless immediately before atomic formulas.

[1] Of course, it is formally possible to make any truth definition "compositional" by associating to every subexpression a sufficiently convoluted meaning-carrying structure.
However, this is of little relevance: the problem here is if a given logic admits a reasonably *informative* and *simple* compositional semantics, not whether it is possible to characterize the truth value of a formula in terms of ad-hoc properties of arbitrarily defined components.

The existence of such a form for all ϕ is a direct consequence of the well-known equivalencies

$$\neg(\phi \vee \psi) \equiv \neg\phi \wedge \neg\psi;$$
$$\neg(\phi \wedge \psi) \equiv \neg\phi \vee \neg\psi;$$
$$\neg\forall x\phi \equiv \exists x\neg\phi;$$
$$\neg\exists x\phi \equiv \forall x\neg\phi.$$

In most of this work, I will assume that all formulas ϕ, either first-order or belonging to some logic of imperfect information, are in negation normal form; in this way, I will be able to avoid - until the last chapter - some difficulties that the negation operator offers for these extensions of first-order logic.

A more thorough introduction to first-order logic and Tarskian semantics can be found in any book on general mathematical logic, for example in[3].

In some places, I will write $\mathcal{D}(A)$ as the set of all probability distributions over the set A, that is, all the functions

$$f : A \to [0,1]$$

such that

$$\sum_{a\in A} f(a) = 1$$

Also, I will sometimes need to speak about different *instances* of a subformula ψ in a formula ϕ.
The intuition behind this concept is clear: we can identify an instance of ψ in ϕ as a tuple (ψ, n), where $n \in \mathcal{N}$ indicates the position of ψ in the formula ϕ. For example, in the formula

$$\phi = \forall x P(x) \vee \forall x P(x)$$

the subformula $\forall x P(x)$ occurs in two different instances, named respectively as $(\forall x P(x), 1)$ and $(\forall x P(x), 4)$, and the subformula $P(x)$ occurs in the two instances $(P(x), 2)$ and $(P(x), 5)$.

There is another, equivalent method for representing instances, which has the advantage of not requiring us to keep explicitly track of tuples (subformula, position).

Let us associate to every occurrence of an atomic formula of our sentence an unique index: for example, we can write ϕ as

$$\forall x P(x)^{(1)} \vee \forall x P(x)^{(2)}$$

Then, two subexpressions ψ_1 and ψ_2 of a formula ϕ correspond to the same *subformula instance* if and only if ψ_1 and ψ_2 coincide, while they correspond to the same *subformula* if and only if we can transform ψ_1 in ψ_2 by changing the values of the indexes.

This is clearly equivalent to the previous formulation of the concepts of subformula and subformula instances: for example, the expressions $\forall x P(x)^{(1)}$ and $\forall x P(x)^{(2)}$ correspond to the same subformula, but to different subformula instances.

In the following work, I will adopt this second convention, as it will somewhat simplify the definitions of the semantic games $H(\phi)$.

2.2 Game-theoretic semantics for first-order logic

Instead of analyzing the truth of a sentence ϕ in terms of the sets of assignments which satisfy its components, as in Tarskian semantics, it has been long known that it is possible to associate ϕ to a *game* $H(\phi)$ between a *doubter* (called also *falsifier*, or $\forall$*belard*), who attempts to disprove the sentence, and a *verifier* (called also $\exists$*loise*), who instead wishes the opposite.

These two players, which with a great leap of fancy I will call Player **I** and Player **II**, fight ferociously for their cause; however, contrarily to what perhaps was the case in the trials by combat of yore, what matters here is not the outcome of a specific battle, but rather the existence of an unbeatable strategy which allows Abelard to effortlessly thwart any plot of Eloise, or vice versa.

Without further ado, let us lay out the rules of the duel:

Definition 1 (Game for first-order logic)
Let ϕ be a first-order formula in NNF, let $\mathcal{M}$ be our model and let the assignment s_0 be such that $dom(s_0) = FV(\phi)$.
Then, a position of the game $H^{\mathcal{M}}_{s_0}(\phi)$ is a tuple (ψ, s), where ψ is a subformula of ψ and $dom(s) = FV(\psi)$.
The game starts at (ϕ, s_0), and the possible successors of a given position are determined as follows:

1. *If the position is (ψ, s) and ψ is a literal (that is, an atomic formula or its negation) then Player* **II** *wins if and only if ψ holds in our model under the variable assignment s; otherwise, Player* **I** *wins;*

2. *If the position is $(\psi \vee \theta, s)$, Player* **II** *decides if the next position is (ψ, s) or (θ, s);*

3. *If the position is $(\psi \wedge \theta, s)$, Player* **I** *decides if the next position is (ψ, s) or (θ, s);*

4. *If the position is $(\exists x\psi, s)$, Player* **II** *picks an element c of our domain, and the next position is $(\psi, s[c/x])$;*

5. *If the position is $(\forall x\psi, s)$, Player* **I** *picks an element c of our domain, and the next position is $(\psi, s[c/x])$.*

When there is no risk of ambiguity about the model $\mathcal{M}$ or the initial assignment s_0, I will abbreviate $H_{s_0}^{\mathcal{M}}(\psi)$ as $H(\psi)$.

Definition 2 (Play, partial play, complete play, winning play)
A play *of $H(\phi)$ is a sequence of positions $(p_1, p_2, \ldots p_n)$ such that*

1. *p_1 is the starting position (ϕ, s_0);*
2. *For $i > 1$, p_i can be reached from p_{i-1} by applying the game rules above.*

Such a play is said to be complete *if the last position p_n is a terminal one - that is, p_n corresponds to an atomic formula or its negation; otherwise, it is said to be* partial.
Moreover, a complete play is said to be winning *for Player $\alpha \in \{\mathbf{I}, \mathbf{II}\}$ if and only if the last position p_n is a winning one for player α; and since we are more interested in truth conditions than in falsity conditions, a* winning *play will usually be intended as a winning strategy for Player* **II**.

Definition 3 (Strategy, winning strategy)
A strategy σ for Player **I** *is a collection of functions σ_i from partial plays $(p_1 \ldots p_i)$, where according to our rules it is Player* **I** *who must make a move in p_i, to positions p_{i+1} which can be reached from p_i.*

A strategy τ for Player **II** *is defined analogously.*

In a play $(p_1, \ldots p_n)$, Player **I** [**II**] *is said to* follow *the strategy σ $[\tau]$ if for all partial plays $(p_1, \ldots p_i)$, $i < n$, if Player* **I** [**II**] *needs to move in p_i then*

$$p_{i+1} = \sigma_i(p_1 \ldots p_i) \ [\tau_i(p_1 \ldots p_i)]$$

The only complete play of $H(\phi)$ in which Player **I** *follows σ and Player* **II** *follows τ is called $(\sigma; \tau)$; moreover, it is easy to see that every possible play is of the form $(\sigma; \tau)$ for some σ and τ.*

A strategy σ $[\tau]$ is said to be winning *for Player* **I** [**II**] *if and only if every complete play in which σ $[\tau]$ is used is winning for Player* **I** [**II**].

The game $H(\phi)$ is *zero-sum*, as either player wins if and only if the other one loses, and it is of *perfect information*, as the strategies can depend on the whole sequence of previous positions.
Moreover, one can verify by induction that no play of $H(\phi)$ may exceed in length the depth of ϕ $d(\phi)$ defined by

- If ϕ is a literal, $d(\phi) = 1$;
- If ϕ is $\psi \vee \theta$ or $\psi \wedge \theta$,

$$d(\phi) = \max\{d(\psi), d(\theta)\} + 1;$$

- If ϕ is $\forall x\phi$ or $\exists x\phi$,
$$d(\phi) = d(\psi) + 1.$$

Thus, we say that the game $H(\phi)$ is *finite*.
This allows us to apply *Zermelo's Theorem*([29]; [8] generalizes this result):

Theorem 1 (Zermelo's Theorem)
Every two-player, finite, zero-sum game of perfect information G is determined, that is,

$$\textit{Player } \mathbf{I} \textit{ has a w.s. in } G \Leftrightarrow \textit{Player } \mathbf{II} \textit{ has no w.s. in } G.$$

Proof:
The formal proof of this result will be omitted, but the intuition behind it is the following: if Player **II** has no winning strategy, then no matter how she plays Player **I** will be able to counteract in such a way that Player **II** does not have a winning strategy in the new position. But since the game is finite, the play will eventually end; and as the game is zero-sum, if it does not end in a victory for **II** it will certainly end in a victory for **I**.
The same reasoning holds if we substitute **I** for **II** and vice versa, and this implies the result.

□

The fact that the game $H(\phi)$ is determined is somewhat surprising: in general, there is no reason why one of the players of a game should have a winning strategy, and in effect assumptions of determinacy for some games has been shown to be equivalent to highly nontrivial assumptions in axiomatic set theory[19].
Moreover, we will see in the next section that a relatively minor modification of $H(\phi)$ is sufficient to lose this property.

As the next theorem shows, the determinacy of $H(\phi)$ follows from the fact that first-order logic satisfies the principle of the excluded middle:

Theorem 2 (Game-theoretical and Tarskian semantics are equivalent)
If ϕ is a F.O. formula in NNF,

$$\mathcal{M}, s \models_{FO} \phi \Leftrightarrow \textit{Player } \mathbf{II} \textit{ has a w.s. in } H_s^{\mathcal{M}}(\phi)$$

Proof:
The proof is by structural induction on ϕ:

- ϕ is a literal:
  In this case, the result follows directly from the definition of $H(\phi)$.

- $\phi = \psi \vee \theta$:
  Suppose that $\mathcal{M}, s \models_{FO} \psi \vee \theta$: then either $\mathcal{M}, s \models_{FO} \psi$ or $\mathcal{M}, s \models_{FO} \theta$.

Thus, by induction hypothesis, Player **II** has a winning strategy in either $H_s^{\mathcal{M}}(\psi)$ or in $H_s^{\mathcal{M}}(\theta)$; but then, Player **II** has also a w.s. for $H_s^{\mathcal{M}}(\theta)$, which consists in selecting the conjunct for which she has a winning strategy and then playing accordingly.

Conversely, suppose that Player **II** has a winning strategy for $H_s^{\mathcal{M}}(\psi \vee \theta)$, and suppose that the first move of this strategy selects ψ.
Then Player **II** has a winning strategy in $H_s^{\mathcal{M}}(\psi)$; but then, by induction hypothesis, $\mathcal{M}, s \models_{FO} \psi$ and therefore $\mathcal{M}, s \models_{FO} \psi \vee \theta$.
If Player **II** selects the second conjunct instead, the proof is analogous.

- $\phi = \psi \wedge \theta$:
  Suppose that $\mathcal{M}, s \models_{FO} \psi \wedge \theta$: then $\mathcal{M}, s \models_{FO} \psi$ and $\mathcal{M}, s \models_{FO} \theta$.
  But then Player **II** has a w.s. in both $H_s^{\mathcal{M}}(\psi)$ and $H_s^{\mathcal{M}}(\theta)$, and thus she has a w.s. in $H_s^{\mathcal{M}}(\psi \wedge \theta)$ too: no matter which conjunct $\chi \in \{\psi, \theta\}$ Player **I** selects, Player **II** can play according to her winning strategy for $H_s^{\mathcal{M}}(\chi)$.

  Conversely, if Player **II** has a winning strategy for $H_s^{\mathcal{M}}(\psi \wedge \theta)$ she also has winning strategies for $H_s^{\mathcal{M}}(\psi)$ and $H_s^{\mathcal{M}}(\theta)$; but then, by induction hypothesis $\mathcal{M}, s \models_{FO} \psi$ and $\mathcal{M}, s \models_{FO} \theta$, and in conclusion $\mathcal{M}, s \models_{FO} \psi \wedge \theta$.

- $\phi = \exists x \psi$:
  Suppose that $\mathcal{M}, s \models_{FO} \exists x \psi$: then there exists an element $c \in M$ such that $\mathcal{M}, s[c/x] \models_{FO} \psi$.
  By induction hypothesis, Player **II** has a winning strategy in $H_{s[c/x]}^{\mathcal{M}}(\psi)$; but then she also has a w.s. τ in $H_s^{\mathcal{M}}(\exists x \psi)$, which consists in selecting c for x as the first move, and then playing as in τ.

  Conversely, if Player **II** has a winning strategy τ for $H_s^{\mathcal{M}}(\exists x \psi)$, let $c \in M$ be the element that is selected for x in the first move, according to τ.
  Then Player **II** has a winning strategy for $H_{s[c/x]}^{\mathcal{M}}(\psi)$; but by induction hypothesis, this implies that $\mathcal{M}, s[c/x] \models_{FO} \psi$, and thus $\mathcal{M}, s \models_{FO} \exists x \psi$.

- $\phi = \forall x \psi$:
  Suppose that $\mathcal{M}, s \models_{FO} \forall x \psi$; then, for all $m \in M$ we have that $\mathcal{M}, s[m/x] \models_{FO} \psi$.
  Then, by induction hypothesis, for every $m \in M$ Player **II** has a winning strategy τ_m for $H_{s[m/x]}^{\mathcal{M}}(\psi)$, and therefore she also has a w.s. for $H_s^{\mathcal{M}}(\forall x \psi)$ - if Player **I** selects m for x, Player **II** only needs to play according to τ_m.

  Conversely, if Player **II** has a w.s. in $H_s^{\mathcal{M}}(\forall x \psi)$, she also has w.s. for $H_{s[m/x]}^{\mathcal{M}}(\psi)$, for every m; but then $\mathcal{M}, s[m/x] \models_{FO} \psi$ for every m, and thus $\mathcal{M}, s \models_{FO} \forall x \psi$, as required.

□

2.3 Games of Imperfect Information, IF-Logic, and DF-Logic

One of the main reasons for the expressive power of first-order logic, and one of the main differences between it and many of the various forms of syllogistic logic, is that it allows to express patterns of *dependence* and *independence* between quantified variables.
From a technical point of view, this is a result of the possibility of nesting quantifiers: as a (very popular) example, let us compare the interplay of the variables in the definition of *continuous function*

$$\forall x \forall \epsilon \exists \delta \forall x' (|x - x'| < \delta \Rightarrow |f(x) - f(x')| < \epsilon)$$

and that of *uniformly continuous function*

$$\forall \epsilon \exists \delta \forall x \forall x' (|x - x'| < \delta \Rightarrow |f(x) - f(x')| < \epsilon)$$

In the former, ϵ may depend on both x and δ, whereas in the latter it depends only on δ: as a consequence, there exist functions - such as x^{-1} in the interval $(0, 1]$ - which are continuous but not uniformly so.

It is easy to see that, in a first-order logic formula ϕ, a quantified variable Qy depends on a quantified variable $Q'x$ if and only if Qy occurs in the *syntactic scope* of Qx, that is, the part contained between the parentheses of

$$\phi = \ldots Q'x(\ldots)\ldots$$

This poses some constraints on the patterns of dependence and independence which can occur in a first-order formula: in particular, the set of all quantifier Qx in the scope of which a given subformula instance ψ occurs is always linearly ordered.
In [11], Henkin introduced the *branching quantifier*

$$\begin{pmatrix} \forall x_1 & \exists y_1 \\ \forall x_2 & \exists y_2 \end{pmatrix} \psi(x_1, y_1, x_2, y_2)$$

whose interpretation is that y_1 depends only on x_1 and y_2 depends only on x_2, or, by writing the formula in the second-order Skolem normal form,

$$\exists f \exists g\ \forall x_1 \forall x_2 \psi(x_1, f(x_1), x_2, g(x_2))$$

where f and g range over all functions $M \to M$, where $M = dom(\mathcal{M})$.
One of the questions about this branching quantifier and others of its kind is

whether they increase the expressive power of first-order logic or not: more precisely, is there any class $\mathcal{K}$ of models such that

$$\mathcal{K} = \{\mathcal{M} : \mathcal{M} \models \phi\}$$

for some formula ϕ with branching quantifiers, but such that this is not the case for any first-order formula?
This can be answered by considering the formula

$$\exists z \begin{pmatrix} \forall x_1 & \exists y_1 \\ \forall x_2 & \exists y_2 \end{pmatrix} (x_1 = x_2 \leftrightarrow y_1 = y_2 \wedge y_1 \neq z)$$

which holds in a model $\mathcal{M}$ if and only if its domain is infinite.
Indeed, its Skolem normal form is

$$\exists f \exists g\ \exists z \forall x_1 \forall x_2 (x_1 = x_2 \leftrightarrow f(x_1) = g(x_2) \wedge f(x_1) \neq z)$$

This formula states that there exists a function f from the domain M to itself such that

- f is $1-1$;
- The range of f is a proper subset of M.

and this is the case if and only if f is infinite, as required.
Since, by the compactness theorem, no first-order formula may characterize the set of all infinite models, this settles the matter.

Now, branching quantifiers pose a few difficulties.
First of all, while their syntax is certainly evocative it is also somewhat cumbersome: a single branching quantifier may contain many first-order quantifiers, whose interplay can be fairly complex in the general case.
Because of this, even relatively simple notions such as variable substitution need to be handled with much care, and in any case it is quite apparent that branching quantifiers do not seem to be primitive notions in the same sense of connectives or first-order logic.
A solution for this issue was found by Hintikka an Sandu in [14]: their main observation was that branching quantifiers can be explained away by distinguishing the notions of *binding scope* - that is, which occurrences of the variable x are bound by the quantifier Qx - and *priority scope* - that is, which other quantifiers $Q'y$ depend on Qx.
Formally, this distinction is achieved by introducing *Independence Friendly Logic* and its *slashed quantifiers*

$$(Q'y/\{Q_1x_1 \ldots Q_kx_k\})$$

which specify that $Q'y$ is not in the priority scope of a quantifier Q_ix_i, even though it is in its syntactic scope.

For example, it can be readily seen that[2]

$$\begin{pmatrix} \forall x_1 & \exists y_1 \\ \forall x_2 & \exists y_2 \end{pmatrix} \psi(x_1, y_1, x_2, y_2) \equiv \forall x_1 \exists y_1 \forall x_2 (\exists y_2 / \{\forall x_1, \exists y_1\}) \psi(x_1, y_1, x_2, y_2)$$

In [16], Hodges also suggests to simplify the slash operator by putting in it only the hidden variables, and not the whole quantified expression, e.g., the previous formula would write as

$$\forall x_1 \exists y_1 \forall x_2 (\exists y_2 / \{x_1, x_2\})$$

this will reduce the book-keeping[3]; thus, in what follows I will use this form instead.

Finally, a well-known variation of slashed quantification consists of *back-slashed quantification*

$$Q' y \backslash \{x_1 \dots x_k\}$$

whose interpretation is that the choice of y depends *only* on the values of $x_1 \dots x_k$.
Clearly, the expressive logic of first-order logic with slashed quantifiers is equivalent to that of first-order logic with backslashed quantifiers; however, I believe backslashed quantification to be more intuitive than slashed quantification, as *functional dependence* is a more natural concept than the form of functional independence used in slashed quantifiers.[4]

A further problem with the slashed quantifier is that truth may depend on values of variables which do not occur in the formula, unlike with backslashed quantification[28].

[2] Actually, Hintikka's original formulation of IF-logic [see [12], page 63, point (vi)] assumes that an existential quantifier may depend only on *universal quantifiers*, and vice versa. As a consequence, there is no need of shielding existential quantifiers from other existential quantifiers.
So, the Henkin quantifier would correspond to $\forall x_1 \exists y_1 \forall x_2 (\exists y_2 / \forall x_1) \psi(\dots)$ instead, as there is no more possibility of *signalling* - that is, the choice of y_2 may not anymore make use of information about x_1 inferred by observing the value of y_1.
However, as Hodges observes in [16], this more sophisticated form of the slash operator does not offer much advantage, and in this work I will assume that, unless specified otherwise, a quantifier Qx has priority over all quantifiers $Q'y$ in its syntactic scope.

[3] Otherwise, the interpretation of the subformula $(\exists y / \forall x) R(x, y)$ in

$$\forall x (\exists y / \forall x) R(x, y)$$

would be different from the interpretation of the same subformula in

$$\exists x (\exists y / \forall x) R(x, y)$$

This would pose a few difficulties for the compositional semantics of the next section, as [5] shows.

[4] Moreover, in the next chapter it will be simpler to define the dependence atomic formulas of Dependence Logic through backslashed quantification than through slashed quantification.

The logic with backslashed quantifiers is called *dependence-friendly logic* (DF-Logic) in [27] and *backslash logic* in [28].

This said, we still have to formally define what the interpretation of these "incomplete information quantifiers" *is.*
Of course, it is easy to specify the interpretation of those quantifiers in second order logic, through their Skolem functions, but this seems overkill - as Hintikka observed ([12]) , there is nothing explicitly second-order at play here, as all we did was allowing more general patterns of dependence and independence between *first-order* quantifiers.
However, it is not immediately obvious how to extend first-order Tarskian semantics to this case: given an assignment s, how could we determine whether a variable y is dependent only on $\{x_1 \dots x_k\}$ or not? There is no sane reason why the choice of a given $m \in dom(\mathcal{M})$ should be more (or less) determined from $s(x_1) \dots s(x_k)$ than any other m'.

Luckily, the extension of the *game-theoretic* semantics for first-order logic to imperfect information logic is instead relatively straightforward:

Definition 4 (The game $H_{s_0}^{\mathcal{M}}(\phi)$ for DF-Logic)
Let ϕ be a DF-logic formula in NNF, let $\mathcal{M}$ be a first-order model and let $dom(s_0) = FV(\phi)$.
A position of the game $H(\phi)$ is a tuple (ψ, s), where ψ is a subformula of ψ and s is an assignment over the free variables of ψ.
The first position of the game is (ϕ, s_0), and the rules are as follows:

1. *If the position is (ψ, s) and ψ is a literal (that is, an atomic formula or its negation) then Player* **II** *wins if and only if $s \models_{FO} \psi$; otherwise, Player* **I** *wins;*
2. *If the position is $(\psi \vee \theta, s)$, Player* **II** *decides if the next position is (ψ, s) or (θ, s);*
3. *If the position is $(\psi \wedge \theta, s)$, Player* **I** *decides if the next position is (ψ, s) or (θ, s);*
4. *If the position is $(\exists x \psi, s)$, Player* **II** *picks an element c of our domain, and the next position is $(\psi, s[c/x])$;*
5. *If the position is $(\exists x \backslash \{x_1, \dots x_k\} \psi, s)$, Player* **II** *picks an element c of our domain, and the next position is $(\psi, s[c/x])$;*
6. *If the position is $(\forall x \psi, s)$, Player* **I** *picks an element c of our domain, and the next position is $(\psi, s[c/x])$;*
7. *If the position is $(\forall x \backslash \{x_1, \dots x_k\} \psi, s)$, Player* **II** *picks an element c of our domain, and the next position is $(\psi, s[c/x])$;*

As usual, we will use $H(\phi)$ or $H_{s_0}(\phi)$ as shorthands for $H_{s_0}^{\mathcal{M}}(\phi)$, when there is no risk of ambiguity.

The rule for backslashed quantifiers is identical to that for first-order quantifiers. This is because imperfect information does not limit the possible *moves* of the players, but rather the possible *strategies*:

Definition 5 (Uniform strategy)
A strategy τ for Player **II** *is* uniform *if and only if, for every two partial plays $(p_1 \dots p_i)$ and $(p'_1 \dots p'_i)$ in which* **II** *uses τ, if*

- *It holds that*
$$p_i = (\exists x \backslash \{x_1 \dots x_k\}\psi, s)$$
and
$$p'_i = (\exists x \backslash \{x_1 \dots x_k\}\psi, s')$$
for the same instance[5] *of $\exists x \backslash \{x_1 \dots x_k\}\psi$;*

- *The assignment s and s' coincide over $\{x_1 \dots x_k\}$;*

then τ selects the same $c \in M$ for x in both p_i and $p'_{i'}$.

The definition of uniform winning strategy for Player **I** *is completely analogous.*

Then we require that both players use only uniform strategies, and we say that a formula ϕ is *true* if Player **II** has an uniform winning strategy (u.w.s.) τ in $H_s(\phi)$; analogously, ϕ will be *false* if **I** has a winning strategy.

The following simple observation is worth noticing:

Proposition 1
If σ is a strategy for Player $\alpha \in \{\mathbf{I}, \mathbf{II}\}$ which is winning against all uniform *strategies τ for the opponent α^*, then σ is also winning against* all *(possibly non-uniform) strategies of α^*.*

Proof:
Suppose that $(\sigma; \tau)$ is a losing play for Player **I**, where τ is not uniform.
Then, let τ^u be the strategy for Player **II** defined as follows:

- If in the first position p_1 Player **II** needs to make a choice,
$$\tau^u_1(p_1) = \tau(p_1);$$

[5] Here we use the concept of instance of a subformula defined in the preliminaries.
This is necessary, because Player **II** may select different strategies for different occurrences of the same subformula.
For example, in
$$(\exists y \backslash \{\})(x = y) \wedge (\exists y \backslash \{\})(x = y)$$
the two instances of $(\exists y \backslash \{\})(x = y)$ correspond to different instances, and an uniform strategy for Player **II** may well select different values for y when Player **I** chooses the left or the right conjunct.

Väänänen, in [27], solves this difficulty by defining the positions of the game $H(\phi)$ as tuples (ψ, n, s), where (ψ, n) denotes the instance of ψ starting at position n of our original formula; as we saw in the preliminaries, this is equivalent to our definition of instance.

- If $(p_1 \dots p_i)$ is an initial segment of $(\sigma; \tau)$ and Player **II** needs to make a choice in p_i,

$$\tau_i^u(p_1 \dots p_i) = \text{ the immediate successor of } (p_1 \dots p_i) \text{ in } (\sigma; \tau);$$

- Otherwise, play in such a way that the uniformity condition is respected.

This τ^u is a "blind" strategy, in the sense that its choices do not depend on Player **I**'s moves, as it just assumes that he follows σ: thus, it is uniform, and $(\sigma; \tau^u) = (\sigma; \tau)$

The case for the other player is analogous.
□

Because of this, if we want to check whether ϕ is true we can let the strategies σ of Player **I** in $H(\phi)$ range even over non-uniform strategies, and we can do the same to the strategies τ of Player **II** if we wish to verify the falsity of ϕ.

For DF-logic formulas ϕ, $H(\phi)$ is not anymore a game of perfect information: therefore, the conditions of Zermelo's theorem do not hold, and thus it could be that neither player has an uniform winning strategy.
In this case, ϕ would be *undetermined* - neither true nor false.

A very simple example of an undetermined formula is

$$\phi := \forall x (\exists y \backslash \{\})(x = y)$$

in a model $\mathcal{M}$ with at least two elements: indeed, it is easy to see that Player **II** has a winning strategy, but neither player has an uniform winning strategy.

2.4 A compositional semantics for IF Logic

Logics of imperfect information have been long thought to admit no "natural" compositional semantics - for example, in [12] Hintikka suggested that the failure of compositionality in IF-logic mirrors the failure of compositionality in human languages, and is a reason for believieving that for those language the Tarskian style of truth-definition is not adequate.

However, in [16] Hodges was able to find a semantics for a logic of this kind which, apart from being compositional, was able to shed some light about what the meaning of the patterns of variable dependence and independence in IF-logic.

I will now introduce a variation of this semantics, tailored to the logic described above, and at the same time we will verify its equivalence with the game-theoretic semantics of the previous section.

The deep idea behind Hodges' semantics is that functional dependence can be characterized in terms of *sets of assignments*:

Definition 6 (Team)
A team *X with domain $dom(X) = \{x_1 \dots x_n\}$ is simply a set of assignments s_i with $dom(s_i) = \{x_1 \dots x_n\}$.*

Then, let us consider a variant of the game $H_s^{\mathcal{M}}$:

Definition 7 ($H_X^{\mathcal{M}}(\phi)$)
Let ϕ be a formula in NNF, let $\mathcal{M}$ be a model, and let X be a team. The game $H_X^{\mathcal{M}}(\phi)$ is then played as follows:

1. *First, an assignment $s \in X$ is selected from a third player, which we will call Nature;*

2. *Then, the game $H_s^{\mathcal{M}}(\phi)$ is played.*

*Therefore, Player **II** has a winning strategy in $H_X^{\mathcal{M}}(\phi)$ if and only if she has a winning strategy in $H_s^{\mathcal{M}}(\phi)$ for all $s \in X$.*
The definition of uniform strategy *σ for $H_X^{\mathcal{M}}(\phi)$ is as in $H_s^{\mathcal{M}}(\phi)$, except that now σ must also be uniform with respect to the first move of Nature - that is, if the positions*

$$(\exists x \backslash \{x_1 \dots x_k\}\psi, s)$$

and

$$((\exists x \backslash \{x_1 \dots x_k\})\psi, s'),$$

*for the same instance of $(\exists x \backslash \{x_1 \dots x_k\})\psi$, are reached in two plays in which Player **II** uses σ and Nature selects two different initial assignment, and s, s' coincide on $x_1 \dots x_k$, then the choice of $c \in M$ for x must be the same in the two plays.*

In particular, it is clear that $H_{\{s\}}^{\mathcal{M}}(\phi) = H_s^{\mathcal{M}}(\phi)$.

Since we want to find a technique for finding true formulas, we are interested in the existence of uniform winning strategies in these games:

Definition 8 (Trumps and $\mathcal{T}$) *Given a NNF formula ϕ, a team X is a* trump *of ϕ if and only if the Verifier **II** has a u.w.s. in $H_X(\phi)$.*
Then, we define

$$\mathcal{T} = \{(\phi, X) : X \text{ is a trump of } \phi\}$$

In order to characterize $\mathcal{T}$, we first need a few preliminary definitions:

Definition 9 (Supplementation)
If X is a team and F is a function

$$F : X \to M$$

then the supplement team *$X[F/x]$ is given by*

$$X[F/x] = \{s[F(s)/x] : s \in X\}$$

where $s[F(s)/x]$ *is the assignment with domain* $dom(s) \cup \{x\}$ *such that*

$$s[F(s)/x](y) = \begin{cases} F(s) & \text{if } y = x; \\ s(y) & \text{otherwise.} \end{cases}$$

Definition 10 (Duplication)
If X *is a team, its* duplicate $X[M/x]$ *is given by*

$$X[M/x] = \{s[a/x] : s \in X, a \in M\}$$

Then we have the following results about $\mathcal{T}$:

Theorem 3 (Hodges' Compositional semantics)
Let ϕ *be a formula in negation normal form, and let* X *be a team with* $dom(X) \supseteq FV(\phi)$*.*
Then the following results hold:

1. *If* ϕ *is a literal,* $(\phi, X) \in \mathcal{T}$ *if and only if* $s \models_{FO} \phi$ *for all* $s \in X$*;*

2. $(\psi \vee \theta, X) \in \mathcal{T}$ *if and only if there exist two teams* Y *and* Z *such that*
   - $X = Y \cup Z$*;*
   - $(\psi, Y) \in \mathcal{T}$*;*
   - $(\theta, Z) \in \mathcal{T}$*.*

3. $(\psi \wedge \theta, X) \in \mathcal{T}$ *if and only if* $(\psi, X) \in \mathcal{T}$ *and* $(\theta, X) \in \mathcal{T}$*;*

4. $(\exists x \psi, X) \in \mathcal{T}$ *if and only if there exists a* $F : X \to M$ *such that* $(\psi, X[F/x]) \in \mathcal{T}$*;*

5. $(\exists x \backslash \{x_1 \ldots x_k\} \psi, X) \in \mathcal{T}$ *if and only if there exists a* $F : X \to M$ *such that* $(\psi, X[F/x]) \in \mathcal{T}$ *and moreover*

   $$s(x_i) = s'(x_i) \text{ for all } x_i \in \{x_1 \ldots x_k\} \cap dom(X) \Rightarrow F(s) = F(s')$$

   for all $s, s' \in X$*;*

6. $(\forall x \psi, X) \in \mathcal{T}$ *if and only if* $(\psi, X[M/x]) \in \mathcal{T}$*;*

7. $(\forall x \backslash \{x_1 \ldots x_k\} \psi, X) \in \mathcal{T}$ *if and only if* $(\forall x \psi, X) \in \mathcal{T}$*.*

Proof:

1. If ϕ is a literal, then Player **II** has a u.w.s. in $H_X(\phi)$ if and only if $s \models_{FO} \phi$ for all $s \in X$, as required;

2. Suppose that $(\psi \vee \theta, X) \in \mathcal{T}$: then, let τ be a u.w.s. for Player **II** in $H_X(\psi \vee \theta)$, and let

$$Y = \{s \in X : \tau_1(\psi \vee \theta, s) = (\psi, s)\}$$

and

$$Z = X \backslash Y$$

we have that $X = Y \cup Z$, and it is easy to see that the (still uniform) strategy τ' described by

$$\tau'_i(p_1 \dots p_i) = \tau_{i+1}(p_0 p_1 \dots p_i)$$

where $p_0 = (\psi \vee \theta, s)$ for the same assignment s occurring in p_1, is winning for Player **II** in both $H_Y(\psi)$ and $H_Z(\theta)$.
Thus, (ψ, Y) and $(\theta, Z) \in \mathcal{T}$, as required.

Conversely, suppose that Player **II** has a u.w.s. τ' for $H_Y(\psi)$ and another u.w.s. τ'' for $H_Z(\theta)$.
Then I state that Player **II** has also a u.w.s. τ in $H_{Y \cup Z}(\psi \vee \theta)$, defined by

$$\tau_1(\psi \vee \theta, s) = \begin{cases} (\psi, s) & \text{if } s \in Y; \\ (\theta, s) & \text{if } s \in Z. \end{cases}$$

and

$$\tau_{i+1}(p_1 p_2 \dots p_{i+1}) = \begin{cases} \tau'_i(p_2 \dots p_{i+1}) & \text{if } p_2 = (\psi, s); \\ \tau''_i(p_2 \dots p_{i+1}) & \text{if } p_2 = (\theta, s). \end{cases}$$

This strategy is clearly still uniform, and as every play of $H_{Y \cup Z}(\psi \vee \theta)$ in which Player **II** uses τ contains properly either a play of $H_Y(\psi)$ in which Player **II** uses τ' or a play of $H_Z(\theta)$ in which Player **II** uses τ'' it is also winning, as required.

3. Suppose that $(\psi \wedge \theta, X) \in \mathcal{T}$: then there exists a u.w.s. τ for Player **II** in $H_X(\psi \wedge \theta)$.
Now, the strategy τ' defined by

$$\tau'_i(p_1 \dots p_i) = \tau_{i+1}(p_0 p_1 \dots p_i)$$

where $p_0 = (\psi \wedge \theta, s)$ for the same s of p_1, is winning for Player **II** in both $H_X(\psi)$ and $H_X(\theta)$, as required.

Conversely, suppose that Player **II** has uniform winning strategies τ' and τ'' for $H_X(\psi)$ and $H_X(\theta)$: then, the strategy τ defined by

$$\tau_{i+1}(p_1 p_2 \dots p_{i+1}) = \begin{cases} \tau'_i(p_2 \dots p_{i+1}) & \text{if } p_2 = (\psi, s); \\ \tau''_i(p_2 \dots p_{i+1}) & \text{if } p_2 = (\theta, s). \end{cases}$$

is a u.w.s. for[6] $H_X(\psi \wedge \theta)$.

[6] In this case, there is no need of specifying τ_1, as Player **I** moves first.

4. Suppose that $(\exists x\psi, X) \in \mathcal{T}$, and let τ be the u.w.s. for **II** in the corresponding game.
Then, let us find the function $F : X \to M$ such that

$$F(s) = m \Leftrightarrow \tau_1(\exists x\psi, s) = (\psi, s[m/x])$$

Then, let us consider the following strategy τ': given a partial play $p_1 \dots p_i$ with $p_1 = (\psi, s)$, let s_0 be obtained from s by removing the variable x from the domain, and let

$$\tau'_i(p_1 \dots p_i) = \tau_{i+1}(p_0 p_1 \dots p_i)$$

where $p_0 = (\exists x\psi, s_0)$.
This strategy is still uniform, and it is winning for $H_{X[F/x]}(\psi)$ - as usual, this can be verified by observing that every play of $H_{X[F/x]}(\psi)$ in which τ' is used is properly contained in a play of $H_X(\exists x\psi)$ in which τ is used instead.

Conversely, let τ' be a u.w.s. for $H_{X[F/x]}(\psi)$; then, if we define the strategy τ as

$$\tau_1(\exists x\psi, s) = (\psi, s[F(s)/x]);$$
$$\tau_{i+1}(p_1 p_2 \dots p_{i+1}) = \tau'_i(p_2 \dots p_{i+1})$$

it is easy to verify that τ is a u.w.s. for $H_X(\exists x\psi)$, as required.

5. Suppose that $(\exists x\backslash\{x_1 \dots x_k\}\psi, X) \in \mathcal{T}$, and let τ be the u.w.s. for **II** in the corresponding game.
Then, let us find the function $F : X \to M$ such that

$$F(s) = m \Leftrightarrow \tau_1(\exists x\psi, s) = (\psi, s[m/x])$$

Since τ is uniform (even with respect to the moves of Nature), we immediately have that, for all $s, s' \in X$,

$$s(x_i) = s'(x_i) \text{ for } i = 1 \dots k \Rightarrow F(s) = F(s')$$

Then, the strategy τ' defined as in the previous case is our required u.w.s. for $H_{X[F/x]}(\psi)$.

Conversely, let τ' be a u.w.s. for $H_{X[F/x]}(\psi)$, where F satisfies the functional dependence condition above, and let τ be defined as

$$\tau_1(\exists x\backslash\{x_1 \dots x_k\}\psi, s) = (\psi, s[F(s)/x]);$$
$$\tau_{i+1}(p_1 p_2 \dots p_{i+1}) = \tau'_i(p_2 \dots p_{i+1})$$

Then it only remains to show that τ is uniform: indeed, let $(p_1 \dots p_i)$ and $(p'_1 \dots p'_i)$ be two partial plays, in which Player **II** follows τ, such that

$$p_i = ((\exists y \backslash V)\theta, s);$$
$$p'_i = ((\exists y \backslash V)\theta, s')$$

for some set V of variables, and for the same instance of $(\exists y \backslash V)\theta$, and moreover

$$s(x_i) = s'(x_i) \text{ for all } x_i \in V$$

If $i = 1$ then $p_1 = ((\exists x \backslash \{x_1 \dots x_k\})\psi, s)$ and $p'_1 = ((\exists x \backslash \{x_1 \dots x_k\})\psi, s')$, and thus in both cases Player **II** selects the value of x according to the distribution $F(s) = F(s')$.

If instead $i > 1$ then, since $(p_2 \dots p_i)$, $(p'_2 \dots p'_i)$ are partial plays of $H_{X[F/x]}(\psi)$ in which Player **II** follows τ' and τ' is uniform, we have that Player **II** chooses y according to the same distribution in both cases, as required.

6. Let τ be a u.w.s. for $H_X(\forall x\psi)$, and let us define the strategy τ' for $H_{X[M/x]}(\psi)$ as

$$\tau'_i(p_1 \dots p_i) = \tau_{i+1}(p_0 p_1 \dots p_i)$$

where $p_0 = (\forall x\psi, s_0)$ and, as in the existential case, s_0 is obtained by removing the variable x from the assignment of position p_1.
Then τ' is a u.w.s. for $H_{X[M/x]}$, as required.

Conversely, given a u.w.s τ' for $H_{X[M/x]}(\psi)$, we can build a u.w.s. τ for $H_X(\forall x\psi)$ simply by letting

$$\tau_{i+1}(p_1 p_2 \dots p_i) = \tau'_i(p_2 \dots p_i)$$

where, as for the conjunction case, there is no need to specify τ_1, since the Player **I** moves first.

7. Suppose that $(\forall x \backslash \{x_1 \dots x_k\}\psi, X) \in \mathcal{T}$.
By our previous proposition, we know that this is possible if and only if there is a uniform strategy τ for Player **II** which wins against all strategies σ of Player **I**, *even nonuniform ones.*
Thus, this case reduces immediately to the previous one.

□

Thus, the set of the trumps of a formula ϕ is determined by the set of the trumps of its components; and since ϕ is *true* under the assignment s if and only if $(\phi, \{s\}) \in \mathcal{T}$, we have a compositional semantics for this logic of imperfect information, in which trumps cover the same role that satisfying assignments cover in the Tarskian semantics for first-order logic.

Of particular interest is point 7. of the above proof, which shows that slashed universal quantification is indistinguishable from nonslashed universal quantification as long as we are only concerned with the notion of true formulas.

The reason for this is that, as we saw, a strategy τ is winning against all strategies σ if and only if it is winning against all *uniform* strategies σ; however, this will not be the case anymore for the analysis of behavioral strategies and undetermined formulas which constitutes the bulk of this work.

2.5 Dependence atomic formulas and Dependence Logic

The team semantics of the previous section illustrates, perhaps more clearly than its game semantics, what the main conceptual difference between first-order logic and these logics of imperfect information is.
Indeed, it is easy to localize the reason for the increased expressive power of IF logic in the truth condition for backslashed existential quantifiers, and more precisely in the requirement that

$$s(x_1) = s'(x_1), \ldots s(x_k) = s'(x_k) \Rightarrow F(s) = F(s')$$

Väänänen, in [27], examines a generalization of this expression, introducing the *dependence atomic formulas* $=(t_1, \ldots t_n)$ with their semantics

- $(=(t_1 \ldots t_{n-1}, t_n), X) \in \mathcal{T}$ if and only if for all $s, s' \in X$

$$t_1\langle s\rangle = t_1\langle s'\rangle, \ldots t_{n-1}\langle s\rangle = t_{n-1}\langle s'\rangle \Rightarrow t_n\langle s\rangle = t_n\langle s'\rangle$$

Clearly, backslashed quantifiers can be represented in terms of dependence atomic formulas, as

$$\exists x \backslash \{x_1 \ldots x_k\}\psi \equiv \exists x(= (x_1 \ldots x_k, x) \wedge \psi)$$

The proof is obtained simply by unwinding the definitions:

$(\exists x(=(x_1 \ldots x_k, x) \wedge \psi), X) \in \mathcal{T}$ iff
iff $(=(x_1 \ldots x_k, x) \wedge \psi, X[F/x]) \in \mathcal{T}$ for some $F : X \to M$ iff
iff $(=(x_1 \ldots x_k, x), X[F/x]), (\psi, X[F/x]) \in \mathcal{T}$, iff
iff $s(x_1) = s'(x_1), \ldots s(x_k) = s'(x_k) \Rightarrow s(x) = s'(x)\ \forall s, s' \in X[F/x]$ and $(\psi, X[F/x]) \in \mathcal{T}$ iff
iff $s(x_1) = s'(x_1), \ldots s(x_k) = s'(x_k) \Rightarrow F(s) = F(s')\ \forall s, s' \in X$ and $(\psi, X[F/x]) \in \mathcal{T}$ iff
iff $(\exists x \backslash \{x_1 \ldots x_k\}\psi, X) \in \mathcal{T}$

On the other hand, these dependence formulas can be represented in terms of backslashed quantifiers:

$$=\!(t_1, \dots t_n) \equiv \exists y_1 \dots \exists y_{n-1} (\exists y_n \backslash \{y_1 \dots y_{n-1}\}) \left(\bigwedge_{i=1}^{n} y_i = t_i \right)$$

and this can also be verified by unwinding the definitions.

Thus, Dependence Logic can be seen as a different but equivalent formulation of the logic of imperfect information seen above - but a variant which highlights the main non-first-order characteristics of this kind of logic: functional dependence.

2.6 Independence atomic formulas

One could also attempt to introduce "independence atomic formulas" $\doteqdot (t_1 \dots t_n)$, generalizing the independence condition

$$s(x_i) = s'(x_i) \text{ for all } x_i \in dom(s) \backslash \{x_1, \dots x_k\} \Rightarrow F(s) = F(s')$$

of the slashed quantifier $(\exists x / \{x_1 \dots x_k\})\psi$.
Now, the above condition says that F must behave in the same way for all assignments which coincinde in all variables *other than* $x_1 \dots x_k$: because of this, it would make sense to require that

- $(\doteqdot(x_1 \dots x_n), X) \in \mathcal{T}$ if and only if

  $$s(x_i) = s'(x_i) \text{ for all } x_i \in dom(X) \backslash \{x_1, \dots x_n\} \Rightarrow s(x_n) = s'(x_n)$$

  for all $s \in X$.

so that

$$(\exists x / \{x_1 \dots x_k\})\psi \equiv \exists x (\doteqdot(x_1 \dots x_k, x) \wedge \psi)$$

However, it is not clear how to interpret a formula of the kind $\doteqdot(t_1 \dots t_n)$ where the t_i are not variables.
The problem is that the expression $\doteqdot(t_1 \dots t_n)$ tells us that t_n does *not* depend on $t_1 \dots t_{n-1}$, but it does not tell us what it *does* depend on.
The more straightforward answer would be "the value of t_n must be determined by the values of *all terms other than* $t_1 \dots t_n$", but this cannot be correct: otherwise, for every model $\mathcal{M} = (M, f, \dots)$ where $f : M \to M$ is a bijection we would have that

$$(\doteqdot(t_1 \dots t_n), X) \in \mathcal{T}$$

for all teams X and for all terms $t_1 \dots t_n$.

More in general, the difficulty lies in the fact that the interpretation of the independence atom depends on the value of variables not occurring in it - for example, the formula $\backsimeq(x, y)$ holds in the team

$X =$

	x	y	z
s_1	a	a	a
s_2	b	b	b

but not in the team

$Y =$

	x	y	z
s_1	a	a	a
s_2	b	b	a

Thus, the verification of an independence relation presupposes the specification of a context, of a set of terms to be independent with respect to; in the case of variables, a natural context is the set of all variables in our domain, but for arbitrary terms I have not been able to notice any obvious candidate.

We could just let this context be specified in the independence formula, by considering formulas of the kind $\backsimeq(t_1, \ldots t_n \mid C)$ with the semantics

- If C is a finite set of terms, $t_1 \ldots t_n \in T$, and X is a team, $\backsimeq(t_1, \ldots t_n \mid C) \in \mathcal{T}$ if and only if

  $$t\langle s\rangle = t\langle s'\rangle \ \forall t \in T \backslash \{t_1 \ldots t_n\} \Rightarrow t_n\langle s\rangle = t_n\langle s'\rangle$$

  for all $s, s' \in X$.

This more complex definition, in any case, can be directly translated to the formalism of Dependence Logic: indeed, it is easy to see that if $C \backslash \{t_1 \ldots t_n\} = \{t'_1, \ldots t'_w\}$ then $\backsimeq(t_1, \ldots t_n \mid C)$ is equivalent to $=(t'_1, \ldots t'_w, t_n)$.

This result appears to lend further credibility to the idea that functional dependence is a simpler, more fundamental notion than the kind of functional independence occurring in IF-logic.

2.7 Infinitary dependence formulas

In the previous section, the condition that the context C is finite may appear somewhat arbitrary: why should we prohibit expressions of the form $\backsimeq(t_1, \ldots t_n \mid C)$, where C is an infinite set of terms?

As we saw, this expression would hold if and only if every two assignments of our team which coincide over $C \backslash \{t_1 \ldots t_n\}$ also coincide over t_n.
This justifies the following generalization of the dependence atomic formulas:

Definition 11 (Infinitary Dependence Atomic Formulas)
Let t be a term, let T be a (possibly infinite) set of terms, and let X be a team. Then $(=(T, t), X) \in \mathcal{T}$ if and only if, for all $s, s' \in X$,

$$t'\langle s\rangle = t'\langle s'\rangle \text{ for all } t' \in T \Rightarrow t\langle s\rangle = t\langle s'\rangle$$

The already-seen translation of finitary dependence atomic formulas into expressions with backslashed quantifiers cannot be applied to infinitary dependence atomic formulas, as doing so would result in a formula with infinitely many nested existential quantifiers; thus, it makes sense to ask if adding infinitary dependence formulas would increase the descriptive power of Dependence Logic.

The answer is positive: indeed, let T be the set

$$\{g(x), g(f(x)), g(f(f(x))), g(f(f(f(x)))), \ldots\}$$

and let us consider the formula

$$\phi := \exists x \exists y \forall z (g(z) = x \vee g(z) = y) \wedge \forall x (=(T, x))$$

Then ϕ holds in the model $(\{0,1\}^*, f, g)$, where $f(x_1 x_2 x_3 \ldots) = x_2 x_3 \ldots$ and $g(x_1 x_2 x_3 \ldots) = x_1 x_1 x_1 \ldots$: indeed, for every $\bar{x}$ we have that $g(\bar{x}) = 000\ldots$ or $111\ldots$, and every $\bar{x}$ is determined by the list of its components.

On the other hand, ϕ does not hold on any model of cardinality greater than $2^{\aleph_0}$, as it states that every element of our domain is determined by a sequence in $\{a, b\}^*$, for two $a, b \in M$.

This contradicts the Löwenheim-Skolem Theorem ([27], page 91):

Theorem 4 (Löwenheim-Skolem Theorem)
If ϕ is a sentence of DF-logic which holds in an infinite model or in arbitrarily large finite models, then ϕ has models of all infinite cardinalities.

Proof:

For First-Order logic, this is one of the central theorems of Model Theory and can be found in any book on the topic, for example in [3] or in [17].

Now, any formula ϕ of DF-logic is equivalent to an existential second-order formula

$$\exists S_1 \ldots \exists S_n \psi$$

where ψ is a first-order formula: in order to verify this, it suffice to consider its Skolem normal form.

But this formula holds in a model $\mathcal{M}$ if and only if $\mathcal{M}$ is the reduct of a $\mathcal{M}' = (\mathcal{M}, S_1, \ldots S_n)$ such that

$$\mathcal{M}' \models_{FO} \psi$$

Then, by the Löwenheim-Skolem Theorem for first-order logic, ψ holds in models of all infinite cardinalities, and in conclusion the same holds for ϕ.

□

Thus, infinitary dependence formulas are not expressible in DF-logic, and add to their expressive power.

In the rest of this work, I will not concern myself with this kind of formulas; however, this seems a fairly interesting extension of Dependence Logic, and it is easy to see how to adapt Hodges's semantics to this case.
Moreover, it also allows a game-theoretic semantics, similar to that in ([27], pages 80–85): in brief, it suffices to let all positions $=(T, t)$ be winning for Player **II** and modify the definition of uniform strategy so that for every two plays $(\sigma; \tau)$ and $(\sigma'; \tau)$ ending respectively in $(=(T, t), s)$ and $(=(T, t), s')$ we require that

$$t'\langle s\rangle = t'\langle s'\rangle \text{ for all } t' \in T \Rightarrow t\langle s\rangle = t\langle s'\rangle$$

However, a dependence pattern among terms as the one appearing in our example cannot be reduced to a dependence pattern among variables without having to resort to infinitely nested quantification: this strongly suggests that there is more in the dependence atom formalism than a syntactic variant of the slashed quantifier formalism, and that perhaps the former may be more suitable for the study of the logic of such patterns.

3 Behavioral strategies for undetermined formulas

3.1 Behavioral strategies and Game values

Given a game $H_s(\phi)$, one of the questions we could wish to inquire about is its *value* for a given player - that is, the greatest payoff she is able to guarantee. Let us formalize this concept:

Definition 12 (Payoffs for pure strategies)
*Given a game $H_s(\psi)$, a strategy σ for Player **I**, a strategy τ for Player **II** and*

$$\alpha \in \{\mathbf{I}, \mathbf{II}\}$$

we define the payoff[7]

$$P_\alpha(H_s(\psi); \sigma; \tau) = \begin{cases} 1 & \textit{if the play } (\sigma; \tau) \textit{ is winning for Player } \alpha; \\ 0 & \textit{otherwise.} \end{cases}$$

As $H_s(\psi)$ is a game of imperfect information, it may be useful for one of the players to randomize some of his choices:

[7]One could worry that, according to these payoffs, our games are *not* zero-sum, but rather "one-sum".
This is not an issue: as all theorems about zero-sum games that we will use also hold for this case - in order to see this, we can simply consider the payoffs $P'(H_s(\psi); \sigma; \tau) = P(H_s(\psi); \sigma; \tau) - 1/2$, and then apply the theorem to this new payoff function.

Definition 13 (Behavioral strategy, Uniform behavioral strategy)
A behavioral strategy β *for Player* $\alpha \in \{\mathbf{I}, \mathbf{II}\}$ *in the game* $H_s(\psi)$ *is a family of functions* β_i *from partial plays* $\bar{p} = (p_1 \dots p_i)$*, where Player* α *has to make a choice in* p_i*, to* distributions *of possible successors* p_{i+1}*.*

A behavioral strategy γ *for Player* $\mathbf{II}$ *is said to be* uniform *if and only if, for every two partial plays* $(p_1 \dots p_i)$ *and* $(p'_1 \dots p'_i)$ *in which Player* $\mathbf{II}$ *follows* γ*, if*

- *It holds that*
$$p_i = ((\exists x \backslash V)\psi, s)$$
and
$$p'_i = ((\exists x \backslash V)\psi, s')$$
for the same instance of the subformula $(\exists x \backslash V)\psi$*;*
- *The assignments* s *and* s' *coincide over the set of variables* V*;*

then γ *induces the same distribution over* x *in both plays, that is,*

$$\gamma(p_1 \dots p_i)(\psi, s[m/x]) = \gamma(p'_1 \dots p'_i)(\psi, s'[m/x]), \text{ for all } m \in M$$

The definition of behavioral strategy for Player $\mathbf{I}$ *is analogous.*

Clearly, a behavioral strategy β induces a probability distribution β^* over pure strategies σ; stretching the notation, we will write

$$\beta^*(\sigma) = Prob(\text{For all } \bar{p} = p_1 \dots p_i, \beta(\bar{p}) \text{ selects } \sigma(\bar{p})) = \prod_{\bar{p}} \beta(\bar{p})(\sigma(\bar{p}))$$

The *payoffs* for behavioral strategies are weighed averages of the payoffs for the corresponding pure strategies:

Definition 14 (Payoffs for behavioral strategies)
Let σ *and* τ *be pure strategies for Players* $\mathbf{I}$ *and* $\mathbf{II}$*, and let* β *and* γ *be behavioral strategies for Players* $\mathbf{I}$ *and* $\mathbf{II}$*.*
Then we define

$$P_\alpha(H_s(\phi); \beta; \tau) = \sum_{\sigma'} \beta^*(\sigma') P_\alpha(H_s(\phi); \sigma'; \tau)$$

$$P_\alpha(H_s(\phi); \sigma; \gamma) = \sum_{\tau' \in T} \gamma^*(\tau') P_\alpha(H_s(\phi); \sigma; \tau')$$

and
$$P_\alpha(H_s(\phi); \beta; \gamma) = \sum_{\sigma' \in S} \sum_{\tau' \in T} \beta^*(\sigma') \gamma^*(\tau') P_\alpha(H_s(\phi); \sigma; \tau)$$

It can be easily verified that

$$P_\alpha(H_s(\phi);\beta;\gamma) = \sum_{\sigma'\in S}\beta^*(\sigma')P_\alpha(H_s(\phi);\sigma';\gamma) = \sum_{\tau'\in T}\gamma^*(\tau')P_\alpha(H_s(\phi);\beta;\tau')$$

and that

$$P_\alpha(H_s(\phi);\beta;\gamma) = Prob(\text{Player } \alpha \text{ wins} \mid \mathbf{I} \text{ uses } \beta, \mathbf{II} \text{ uses } \gamma)$$

Then the *value* of $H_s(\phi)$ for Player α is defined as

Definition 15 (Value of a game)

$$V_I(H_s(\phi)) = \sup_\beta \inf_\gamma P_I(H_s(\phi);\beta;\gamma)$$

and

$$V_{II}(H_s(\phi)) = \sup_\gamma \inf_\beta P_{II}(H_s(\phi);\beta;\gamma)$$

In the above formulas, it is possible to restrict the inner minimization to pure strategies:

Proposition 2

$$\sup_\beta \inf_\gamma P_I(H_s(\phi);\beta;\gamma) = \sup_\beta \inf_\tau P_I(H_s(\phi);\beta;\tau)$$

and

$$\sup_\gamma \inf_\beta P_{II}(H_s(\phi);\beta;\gamma) = \sup_\gamma \inf_\sigma P_{II}(H_s(\phi);\sigma;\gamma)$$

Proof:
Fix a behavioral strategy β for Player **I**.
Since each pure strategy τ for Player **II** is equivalent to the behavioral strategy which, for every $\bar{p}$, selects $\tau(\bar{p})$ with probability one, we clearly have that

$$\inf_\gamma P_I(H_s(\phi);\beta;\gamma) \leq \inf_\tau P_I(H_s(\phi);\beta;\tau)$$

On the other hand, since $\sum_{\tau\in T}\gamma^*(\tau) = 1$ for all behavioral strategies γ,

$$\inf_\gamma P_I(H_s(\phi);\beta;\gamma) = \inf_\gamma \sum_{\tau\in T}\gamma^*(\tau)P_I(H_s(\phi);\beta;\tau) \geq$$
$$\geq \inf_\gamma \inf_\tau P_I(H_s(\phi);\beta;\tau) = \inf_\tau P_I(H_s(\phi);\beta;\tau)$$

Thus,

$$\inf_\gamma P_I(H_s(\phi);\beta;\gamma) \geq \inf_\tau P_I(H_s(\phi);\beta;\tau)$$

and, as this holds for all β,

$$\sup_\beta \inf_\gamma P_I(H_s(\phi); \beta; \gamma) = \sup_\beta \inf_\tau P_I(H_s(\phi); \beta; \tau)$$

as required.
The other equivalence is proved similarly.
□

3.2 The minimax theorem

For games $H_s(\phi)$ in finite models, the *minimax theorem for behavioral strategies* [26] implies the following result:

Theorem 5 (The Minimax Theorem for $H_s(\psi)$)
For every game $H_s(\phi)$ in a finite model $\mathcal{M}$, there exist two behavioral strategies β^e and γ^e such that

$$P_I(H_s(\phi); \beta^e; \gamma^e) \geq P_I(H_s(\phi); \beta; \gamma^e) \text{ for all behavioral strategies } \beta$$

and

$$P_{II}(H_s(\phi); \beta^e; \gamma^e) \geq P_{II}(H_s(\phi); \beta^e; \gamma) \text{ for all behavioral strategies } \gamma$$

A pair of strategies (β^e, γ^e) as above is called an equilibrium pair.

Proof:
These two conditions can be easily seen to be equivalent to the requirement that

$$P_{II}(H_s(\phi); \beta^e; \gamma) \leq P_{II}(H_s(\phi); \beta^e; \gamma^e) \leq P_{II}(H_s(\phi); \beta; \gamma^e)$$

for all behavioral strategies β and γ.

The original proof, found in ([26], pages 14–23), will not be reported here: rather, I will directly verify the result for the semantic games $H_s(\phi)$.

Given a game $H_s(\phi)$, let us define its *associated n-person non-cooperative game* $K_s(\phi)$ as follows:

- For every non-literal subformula instance ψ there exists a distinct player $[\psi]$. The players are divided into two leagues

  $I = \{[\psi] :$ In $H_s(\phi)$, Player **I** must make a choice in $(\psi, s')\}$;
  $II = \{[\psi] :$ In $H_s(\phi)$, Player **II** must make a choice in $(\psi, s')\}$.

  that is, the players corresponding to disjunctions or existential quantifications belong to II, and those corresponding to conjunctions or universal quantifications belong to I.

- As in $H_s(\phi)$, the positions are of the form (ψ, s'), where ψ is a subformula instance of ϕ and s' is an assignment.
- The game starts at the position (ϕ, s), and follows these rules:
  1. If the current position is (ψ, s') for a literal ψ and an assignment s' such that $s' \models_{FO} \psi$, all players in II win (and receive a payoff of 1), and all players in I lose (and receive a payoff of 0); if instead $s' \not\models_{FO}$, all players in I win and all players in II lose;
  2. If the current position is $(\psi \vee \theta, s')$, Player $[\psi \vee \theta]$ chooses whether the next position is (ψ, s') or (θ, s');
  3. If the current position is $(\psi \wedge \theta, s')$, Player $[\psi \wedge \theta]$ chooses whether the next position is (ψ, s') or (θ, s');
  4. If the current position is $(\exists x\psi, s')$ or $(\exists x \backslash V \psi, s')$, Player $[\exists x\psi]$ (or $[\exists x \backslash V \psi]$) chooses an element m of our domain, and the next position is $(\psi, s'[m/x])$;
  5. If the current position is $(\forall x\psi, s')$ or $(\forall x \backslash V \psi, s')$, Player $[\forall x\psi]$ (or $[\forall x \backslash V \psi]$) chooses an element m of our domain, and the next position is $(\psi, s'[m/x])$.
- A *pure strategy* τ_ψ for Player $[\psi]$ is simply a function from partial plays ending in (ψ, s') to possible successors of (ψ, s'); such a strategy is *uniform* if ψ does not begin with a backslashed quantifier, or if ψ is of the form $(\exists x \backslash V)\theta$ or $(\forall x \backslash V)\theta$ and

  $$\tau_\psi(\dots(\psi, s)) = (\theta, s[m/x]) \Rightarrow \tau_\psi(\dots(\psi, s')) = (\theta, s'[m/x])$$

  for all s, s' such that

  $$s(x_i) = s'(x_i) \text{ for all } x_i \in V.$$

  In general, we will indicate with σ_ψ the strategies of the players $[\psi] \in I$, and with τ_σ the strategies of the players $[\sigma] \in II$.

Then, we define an *uniform mixed strategy* τ^*_ψ for Player $[\psi]$ as a probability distribution over uniform pure strategies for $[\psi]$.

By these definitions, it is easy to see that every uniform behavioral strategy γ $[\beta]$ for Player **II** [**I**] in $H_s(\phi)$ corresponds to a set of uniform mixed strategies $\mathcal{G} = \{\tau^*_\theta : [\theta] \in II\}$ $[\mathcal{B} = \{\sigma^*_\psi : [\psi] \in I\}]$, and vice versa; and moreover,

$$P_{II}(H_s(\phi); \beta; \gamma) = P_{II}(K_s(\phi); \mathcal{B}; \mathcal{G})$$

where the right hand expression is defined as usual.

Thus, in order to prove the result it suffices to prove that there exist sets of uniform mixed strategies $\mathcal{B}^e$ and $\mathcal{G}^e$ for the two leagues such that

$$\max_{\mathcal{G}} P_{II}(K_s(\phi); \mathcal{B}^e; \mathcal{G}) \le P_{II}(K_s(\phi); \mathcal{B}^e; \mathcal{G}^e) \le \min_{\mathcal{B}} P_{II}(K_s(\phi); \mathcal{B}; \mathcal{G}^e)$$

Now, this result can be proved using the same techinque used for proving the usual minimax theorem for mixed strategies[8]: in brief, the main idea is to define a continuous transformation T from the set of all tuples of uniform mixed strategies $(\mathcal{B};\mathcal{G})$ to itself such that

$$T(\mathcal{B};\mathcal{G}) = (\mathcal{B};\mathcal{G}) \text{ iff } (\mathcal{B};\mathcal{G}) \text{ is an equilibrium pair.}$$

and then applying Brouwer's fixed point theorem [15] [9]

Theorem 6 (Brouwer's fixed point theorem) *Let B^n be the closed unit ball in $\mathbf{R}^n$, and let $f : B^n \to B^n$ be an arbitrary continuous function. Then f has at least one fixed point.*

For completeness, we will now formally define the transformation T, and verify its properties.

Let us enumerate the players of $K_s(\phi)$ in such a way that $\mathcal{B} = (\sigma_1^* \ldots \sigma_k^*)$ and $\mathcal{G} = (\tau_{k+1}^* \ldots \tau_t^*)$; moreover, let us define with σ_{ij} $[\tau_{ij}]$ the j-th uniform pure strategy available for the i-th player[10]
Then, for $i \in 1 \ldots k$ let c_{ij} be the amount by which the payoff of the i-th player would increase should he switch from the mixed strategy σ_i^* to the pure strategy σ_{ij} (or zero, if this value would be negative):

$$c_{ij} = \max\{P_{II}(K_s(\phi), \sigma_1^* \ldots \sigma_k^*, \tau_{k+1}^* \ldots \tau_t^*) - P_{II}(K_s(\phi), \sigma_1^* \ldots \sigma_{ij} \ldots \sigma_k^*, \tau_{k+1}^* \ldots \tau_t^*), 0\}$$

Analogously, for $i \in k+1 \ldots t$ let d_{ij} be the amount by which the payoff of the i-th player would increase should he switch from τ_i^* to τ_{ij} (or, again, zero if this value would be negative):

$$d_{ij} = \max\{P_{II}(K_s(\phi), \sigma_1^* \ldots \sigma_k^*, \tau_{k+1}^* \ldots \tau_{ij} \ldots \tau_t^*) - P_{II}(K_s(\phi), \sigma_1^* \ldots \sigma_k^*, \tau_{k+1}^* \ldots \tau_t^*), 0\}$$

Now we can define T as the transformation such that

$$T(\sigma_1^*; \ldots \sigma_k^*; \tau_{k+1}^*; \ldots \tau_t^*) = (\sigma_1^{**}; \ldots \sigma_k^{**}; \tau_{k+1}^{**}; \ldots \tau_t^{**})$$

where the σ_i^{**} are defined as

$$\sigma_i^{**}(\sigma_{ij}) = \frac{\sigma_i^*(\sigma_{ij}) + c_{ij}}{1 + \sum_j c_{ij}};$$

and, analogously, the τ_i^{**} are

$$\tau_i^{**}(\tau_{ij}) = \frac{\sigma_i^*(\tau_{ij}) + d_{ij}}{1 + \sum_j d_{ij}};$$

[8]This part of the proof follows very closely the proof of the minimax theorem found in [9] and [21].

[9]Of course, the set of all pairs of mixed strategies is not B^n, but it is easy to verify that it is homeomorphic to it.

[10]Of course, it is not at all necessary that all players have the same number of possible uniform pure strategies.

These new σ^{**} and τ^{**} are still mixed strategies: indeed,

$$\sum_j \sigma_i^{**}(\sigma_{ij}) = \frac{\sum_j(\sigma_i^*(\sigma_{ij}) + c_{ij})}{1 + \sum_j c_{ij}} = 1$$

and the same holds for the τ_i^{**}.

Now, it only remains to show that $(\sigma_1^* \dots \sigma_k^*, \tau_{k+1}^* \dots \tau_t^*)$ is a fixed point of T if and only if it is an equilibrium tuple.
Suppose that $(\sigma_1^* \dots \sigma_k^*, \tau_{k+1}^* \dots \tau_t^*)$ is an equilibrium tuple: then, by definition, no player can increase his payoff by changing his strategy, and thus every c_{ij} and d_{ij} is 0 and this is a fixed point of T.

On the other hand, suppose that $(\sigma_1^* \dots \sigma_k^*, \tau_{k+1}^* \dots \tau_t^*)$ is a fixed point of T: then, I need to prove that this tuple of strategies is in equilibrium.

Now, let us consider any player in the League II, as for example the one corresponding to the strategy τ_i^*.
Since the payoff of this tuple can be expressed as the weighed average

$$P_{II}(K_s(\phi), \sigma_1^* \dots \sigma_k^*, \tau_{k+1}^* \dots \tau_t^*) = \sum_{\tau_{ij}} \tau_i^*(\tau_{ij}) P_{II}(K_s(\phi), \sigma_1^* \dots \sigma_k^*, \tau_{k+1}^* \dots \tau_{ij} \dots \tau_t^*)$$

there exists at least one pure strategy τ_{ij} such that $\tau_i^*(\tau_{ij}) > 0$ and

$$P_{II}(K_s(\phi), \sigma_1^* \dots \sigma_k^*, \tau_{k+1}^* \dots \tau_t^*) \geq P_{II}(K_s(\phi), \sigma_1^* \dots \sigma_k^*, \tau_{k+1}^* \dots \tau_{ij} \dots \tau_t^*)$$

Because of this, $d_{ij} = 0$; and moreover, since by hypothesis $(\sigma_1^* \dots \sigma_k^*, \tau_{k+1}^* \dots \tau_t^*)$ is a fixed point of T we have that

$$\tau_i^*(\tau_{ij}) = \frac{\tau_i^*(\tau_{ij}) + 0}{1 + \sum_j d_{ij}}$$

Now, $\tau^*(\tau_{ij}) > 0$: therefore, $\sum_j d_{ij} = 0$ and, therefore, all $d_{ij} = 0$ for all j.
Thus, no player in II can increase her payoff by switching to a pure strategy, and, since the payoff for a mixed strategy is the weighed average of the payoffs for all corresponding pure strategies, no such player can increase her payoff at all.

An analogous reasoning takes care of the payoffs of the players in I: for every i, there exists at least a pure strategy σ_{ij} such that $\sigma_i^*(\sigma_{ij}) > 0$ and

$$P_{II}(K_s(\phi), \sigma_1^* \dots \sigma_k^*, \tau_{k+1}^* \dots \tau_t^*) \leq P_{II}(K_s(\phi), \sigma_1^* \dots \sigma_k^*, \tau_{k+1}^* \dots \tau_{ij} \dots \tau_t^*)$$

Thus, $c_{ij} = 0$, and since we are in a fixed point

$$\sigma_i^*(\sigma_{ij}) = \frac{\sigma_i^*(\sigma_{ij}) + 0}{1 + \sum_j c_{ij}}$$

Then, all c_{ij} are 0, and in conclusion no player in I can increase his payoff by changing strategy.
□

Now, the above theorem has the following important, well known corollary:

Corollary 1
For every game $H_s(\psi)$ in a finite model $\mathcal{M}$,

$$\forall\gamma\exists\sigma P_I(H_s(\psi);\sigma;\gamma) \geq r \Leftrightarrow \exists\beta\forall\tau P_I(H_s(\psi);\beta;\tau) \geq r$$

and

$$\forall\beta\exists\tau P_{II}(H_s(\psi);\beta;\tau) \geq r \Leftrightarrow \exists\gamma\forall\sigma P_{II}(H_s(\psi);\sigma;\gamma) \geq r$$

Proof:
Let us verify the second case; the proof of the first one will be essentially the same, *mutatis mutandis.*
First of all, since - as we already observed in the proof of Proposition 2 -

$$\forall\beta\exists\tau P_{II}(H_s(\psi);\beta;\tau) \geq r \Leftrightarrow \forall\beta\exists\gamma P_{II}(H_s(\psi);\beta;\gamma) \geq r$$

and

$$\exists\gamma\forall\sigma P_{II}(H_s(\psi);\sigma;\gamma) \geq r \Leftrightarrow \exists\gamma\forall\beta P_{II}(H_s(\psi);\beta;\gamma) \geq r$$

it suffices to prove that

$$\forall\beta\exists\gamma P_{II}(H_s(\psi);\beta;\gamma) \geq r \Leftrightarrow \exists\gamma\forall\beta P_{II}(H_s(\psi);\beta;\gamma) \geq r$$

The right-to-left direction is then trivial, as it is nothing more than an instantiation of the logical principle $\exists x\forall y R(x,y) \Rightarrow \forall y\exists x R(x,y)$.
In order to verify the left-to-right direction of the proof, let us suppose that for all β there is a γ such that

$$P_{II}(H_s(\psi);\beta;\gamma) \geq r$$

and, moreover, let (β^e, γ^e) be an equilibrium pair of the game.
Then, by hypothesis, there exists a strategy γ such that

$$P_{II}(H_s(\psi);\beta^e;\gamma) \geq r$$

and thus, by the definition of equilibrium pair,

$$P_{II}(H_s(\psi);\beta^e;\gamma^e) \geq r$$

too.
But then, again by the definition of equilibrium pair,

$$\forall\beta P_{II}(H_s(\psi);\beta;\gamma^e) \geq r$$

and this concludes the proof.
□

In the rest of this work, we will make use of the following abbreviations:

$$V_{I,s}(\phi) := V_I(H_s(\phi)) = \sup_\beta \inf_\tau P_I(H_s(\phi); \beta; \tau);$$
$$V_{II,s}(\phi) := V_{II}(H_s(\phi)) = \sup_\gamma \inf_\sigma P_{II}(H_s(\phi); \sigma; \gamma).$$

The following observation will be of some use later: if the model $\mathcal{M}$ is finite, by the theorem above

$$V_{II,s}(\phi) = \sup_\gamma \inf_\sigma P_{II}(\sigma; \gamma) = \sup_\gamma \inf_\sigma (1 - P_I(\sigma; \gamma)) =$$
$$= 1 - \inf_\gamma \sup_\sigma P_I(\sigma; \gamma) = 1 - \sup_\beta \inf_\tau P_I(\beta; \tau) = 1 - V_{I,s}(\phi)$$

As we will usually be more interested in the payoffs for Player **II** than in those for Player **I**, all the expressions will be calculated for Player **II** unless otherwise indicated - e.g., I will write $V_s(\phi)$ for $V_{II,s}(\phi)$, and so on.
Moreover, as usual, the model $\mathcal{M}$ will be indicated as a superscript as necessary - e.g., I will write $V_s^{\mathcal{M}}(\psi)$ for the value $V_s(\psi)$ calculated in the model $\mathcal{M}$.

3.3 An example

Let us consider the formula

$$\phi := \forall x (\exists y \backslash \{\})(x = y)$$

in a model $\mathcal{M}$ with n elements: I state that

$$V_{I,\emptyset}^{\mathcal{M}}(\phi) = 1 - \frac{1}{n}$$

and that

$$V_{II,\emptyset}^{\mathcal{M}}(\phi) = \frac{1}{n}$$

where $\emptyset$ is the empty assignment over no variables.

Indeed, let β be the behavioral strategy which selects the value of x according to the uniform probability distribution over $M = dom(\mathcal{M})$, and let τ be any pure strategy for Player **II**.
By definition of our game, τ selects, for y, a value $c \in M$ independent of x; therefore,

$$P_I(H_\emptyset(\phi); \beta; \tau) = Prob(x \neq c) = 1 - 1/n$$

and, since this is the case for every c,

$$V_{I,\emptyset}^{\mathcal{M}}(\phi) = \sup_\beta \inf_\tau P_I(H_\emptyset(\phi); \beta; \tau) \geq 1 - 1/n$$

On the other hand, consider an arbitrary behavioral strategy β and the corresponding probability distribution of x in M.
Since $|M| = n$, it is easy to see that there exists a $m \in M$ such that

$$Prob(x = m) \geq 1/n$$

Then consider the strategy τ for Player **II** which selects this very m for y: then,

$$P_I(H_s(\phi); \beta; \tau) = Prob(x \neq m) \leq 1 - 1/n$$

Since for every β it is possible to find such a τ, we have that

$$V^{\mathcal{M}}_{I,\emptyset}(\phi) = \sup_{\beta} \inf_{\tau} P_I(H_\emptyset(\phi); \beta; \tau) \leq 1 - 1/n$$

and finally we can conclude that

$$V_{I,\emptyset}(\phi) = 1 - 1/n$$

Then, we could immediately conclude that $V_{II,\emptyset}(\phi) = 1/n$.

As an exercise, let us briefly verify this: if γ selects, as the value of y, every element of M with uniform probability, and if the pure strategy σ picks a $m \in M$, then

$$P_{II}(H_\emptyset(\phi); \sigma; \gamma) = Prob(y = m) \geq 1/n$$

and therefore

$$V^{\mathcal{M}}_{II,\emptyset}(\phi) = \sup_{\gamma} \inf_{\sigma} P_{II}(H_\emptyset(\phi); \sigma; \gamma) \geq 1/n$$

On the other hand, given any distribution for y in M there exists an element c such that

$$Prob(y = c) \leq 1/n$$

Therefore, if σ selects this very c for x we have that

$$P_{II}(H_\emptyset(\phi); \sigma; \gamma) \leq 1/n$$

and, finally,

$$V^{\mathcal{M}}_{II,\emptyset}(\phi) \leq 1/n$$

as required.

3.4 Infinite models

The minimax theorem and its consequences do not hold for $\mathcal{M}$ infinite: for example, let us consider the formula

$$\phi := \forall x (\exists y \backslash \{\})(y > x)$$

in the model $\mathcal{N} = (\mathbb{N}, >)$.
I then state that

$$V_{I,\emptyset}(\phi) = V_{II,\emptyset}(\phi) = 0$$

This contradicts the minimax theorem, which - as we saw - would instead imply that

$$V_{I,\emptyset}(\phi) + V_{II,\emptyset}(\phi) = 1;$$

This can be verified as follows: let β be any behavioral strategy for Player **I**: clearly, β induces a probability distribution of the variable x over $\mathbb{N}$, that is

$$Prob(x = m) = \beta(\phi, \emptyset)(\exists y \backslash \{\}(y > x), \emptyset[m/x])$$

Now, let r be any real such that $0 < r \leq 1$, and let us find a $n_0 \in \mathbb{N}$ such that

$$Prob(x \geq n_0) < r$$

Such a n_0 necessarily exists: indeed, suppose instead that $Prob(x \geq n) \geq r$ for all $n \in \mathbb{N}$, that is,

$$Prob(x \in \{1 \dots n-1\}) \leq 1 - r \text{ for all } n \in \mathbb{N}$$

But then, as $(\{1 \dots n-1\})_{n \in \mathbb{N}}$ is a countable, directed family of sets, we have that

$$\begin{aligned} Prob(x \in \mathbb{N}) &= Prob(x \in \bigcup_{n \in \mathbb{N}} \{1 \dots n-1\}) = \\ &= \sup_{n \in \mathbb{N}} Prob(x \in \{1 \dots n-1\}) \leq 1 - r < 1 \end{aligned}$$

which is impossible[11] .
Now, let τ be the pure strategy for Player **II** which chooses this n_0 for y: then,

$$P_I(H_\emptyset(\phi); \beta; \tau) = Prob(x \geq n_0) \leq r$$

and, since r ranges over $(0, 1]$,

$$\inf_\tau P_I(H_\emptyset(\phi); \beta; \tau) = 0$$

This holds for all β, so in conclusion

$$V_{I,\emptyset}(\phi) = \sup_\beta \inf_\tau P_I(H_\emptyset(\phi); \beta; \tau) = 0$$

[11] The fact that, if $\{A_i\}_{i \in \mathbb{N}}$ is directed, $Prob(x \in \cup_i A_i) = \sup_i Prob(x \in A_i)$ is well known, and follows easily from the Kolmogorov axioms for probability.
In brief,

$$\begin{aligned} Prob(x \in \cup_i A_i) &= Prob(x \in A_1) + Prob(x \in A_2 \backslash A_1) + \dots = \\ &= \lim_{k \to \infty} (Prob(x \in A_1) + \dots + Prob(x \in A_k \backslash A_{k-1})) = \\ &= \lim_{k \to \infty} Prob(x \in A_k) = \sup_k Prob(x \in A_k) \end{aligned}$$

On the other hand, let us compute $V_{II,\emptyset}(\phi)$.
Let γ be any behavioral strategy for Player **II**: then, since the choice of y does not depend on the choice of x, τ^* induces a distribution

$$Prob(y = m) = \gamma(\exists y \backslash \{\}(y > x), s)(y > x, s[m/x]), \text{ for all } s \text{ with } dom(s) = \{x\}$$

Using the same argument of above, for any $r \in (0, 1]$ we can find a $n_0 \in \mathbf{N}$ such that

$$Prob(y > n_0) \leq r$$

Then, if σ is the strategy for Player **I** which chooses this very n_0 for x, we have that

$$P_{II}(H_\emptyset(\phi); \sigma; \gamma) = Prob(n_0 < y) \leq r$$

Therefore, for all γ,

$$\inf_\sigma P_{II}(H_\emptyset(\phi); \sigma; \gamma) = 0$$

and in conclusion

$$V_{II,\emptyset}(\phi) = \sup_\gamma \inf_\sigma P_{II}(H_\emptyset(\phi); \sigma; \gamma) = 0$$

3.5 The range of the value function

For every finite model $\mathcal{M}$, we now have defined a function $\phi \mapsto V(\phi)$.
Let us now try to obtain some results about the range of this mapping:

Theorem 7 *If the domain of $\mathcal{M}$ is finite and contains at least two elements,*

$$\{r \in \mathbb{R} : V(\phi) = r \textit{ for some } \phi\} = \mathbb{Q} \cap [0, 1]$$

Proof:

- $\{r \in \mathbb{R} : V(\phi) = r \text{ for some } \phi\} \supseteq \mathbb{Q} \cap [0, 1]$:
  Let $r = p/q$, where $p < q$, and let $s = \lceil \log_2(q) \rceil$.

Then, let us consider the following sentence:

$$\begin{aligned}
&\phi \equiv \exists x_0 \exists x_1 ((x_0 \neq x_1) \wedge \\
&\quad \wedge (\exists y_{1,1} \exists y_{1,2} \ldots \exists y_{1,s})(\exists y_{2,1} \exists y_{2,2} \ldots \exists y_{2,s}) \ldots (\exists y_{q,1} \exists y_{q,2} \ldots \exists y_{q,s}) \\
&\quad \left(\bigwedge_{i=1}^{q} \bigwedge_{k=1}^{s} y_{i,k} = x_0 \vee y_{i,k} = x_1 \right) \wedge \left(\bigwedge_{i=1}^{q} \bigwedge_{j=i+1}^{q} \bigvee_{k=1}^{s} y_{i,k} \neq y_{j,k} \right) \wedge \\
&\quad \forall z_1 \forall z_2 \ldots \forall z_s \left(\bigwedge_{i=1}^{q} \bigvee_{k=1}^{s} (z_k \neq y_{i,k}) \vee \right. \\
&\quad \vee (\exists w_{1,1}/\{z_1 \ldots z_s\}) \ldots (\exists w_{1,s}/\{z_1 \ldots z_s\}) \\
&\quad (\exists w_{2,1}/\{z_1 \ldots z_s\}) \ldots (\exists w_{2,s}/\{z_1 \ldots z_s\}) \\
&\quad \ldots \\
&\quad (\exists w_{p,1}/\{z_1 \ldots z_s\}) \ldots (\exists w_{p,s}/\{z_1 \ldots z_s\}) \\
&\quad \left. \left(\bigvee_{i=1}^{p} \bigvee_{j=1}^{q} \bigwedge_{k=1}^{s} w_{i,k} = z_{j,k} \right) \right) \Bigg) \Bigg)
\end{aligned}$$

where $\exists w/\{z_1 \ldots z_s\}$ is the *slashed quantifier* of IF-logic, which requires the choice of w to be *independent* from the choice of $z_1 \ldots z_s$, and can be clearly interpreted in terms of our usual backslashed quantifier.

Then I state that $V(\phi) = p/q$: indeed, the game $H(\phi)$ can be described as follows:

1. First, Player **II** selects two distinct elements $x_0, x_1 \in M$;
2. Then, Player **II** selects q distinct strings $y_1, \ldots y_q$ in $\{x_0, x_1\}^s$;
3. Then, Player **I** selects a string $z \in \{y_1, \ldots y_q\}$;
4. Finally, Player **II** selects p strings $w_1 \ldots w_p$, without knowing z, and wins if and only if $w_i = z$ for some $i = 1 \ldots p$.

Now, let γ be the following strategy for Player **II**:

1. First, select two fixed distinct elements x_0 and x_1.
2. Then, select q fixed distinct strings $y_1, \ldots, y_q \in \{x_0, x_1\}^s$;
3. Then, extract p strings $w_1, \ldots w_p$ from $\{y_1 \ldots y_q\}$, with uniform probability and without repetition - that is, w_1 can be each y_i with probability $1/q$, w_2 can be each remaining element with probability $1/(q-1)$, and so on.

Now, consider any strategy σ for Player **I**: by definition, σ selects an element $z \in \{y_1 \ldots y_q\}$, and Player **II** wins if it is one of $\{w_1 \ldots w_p\}$.

According to our behavioral strategy γ,

$$
\begin{aligned}
&P(H(\phi);\sigma;\gamma) = Prob(w_i = z \text{ for some } i) = \\
&= Prob(w_1 = z) + Prob(w_1 \neq z \ \&\ w_2 = z) + \ldots + \\
&+ Prob(w_1 \neq z \ \&\ w_2 \neq z \ \&\ \ldots \ \&\ w_{p-1} \neq z \ \&\ w_p = z) = \\
&= 1/q + (q-1)/q \cdot 1/(q-1) + \ldots + (q-1)/q \cdot (q-2)/(q-1) \cdot \ldots \\
&\ldots \cdot 1/(q-p+1) = p/q
\end{aligned}
$$

Since this holds for any σ,

$$V(\phi) = \sup_{\gamma} \inf_{\sigma} P(H(\phi);\sigma;\gamma) \geq p/q$$

On the other hand, consider any strategy γ for Player **II**: since the choice of $w_1 \ldots w_p$ is independent on the choice of z, γ induces a probability distribution of $(w_1, \ldots w_p)$ over[12] $\{y_1 \ldots y_q\}^p$.
For any such distribution, there exists a string y_j such that

$$Prob(w_i = y_j \text{ for some } i = 1 \ldots p) \leq p/q$$

Indeed, if

$$Prob(w_i = y_j \text{ for some } i = 1 \ldots p) > p/q \text{ for all } j = 1 \ldots q$$

then, since

$$Prob(w_i = y_j \text{ for some } i = 1 \ldots p) \leq \sum_{i=1}^{p} Prob(w_i = y_j)$$

we can infer that, for all $j = 1 \ldots q$,

$$\sum_{i=1}^{p} Prob(w_i = y_j) > p/q$$

Then, let us sum the above equations for all j: we obtain

$$\sum_{j=1}^{q} \sum_{i=1}^{p} Prob(w_i = y_j) > p$$

and, since

$$\sum_{j=1}^{q} \sum_{i=1}^{p} Prob(w_i = y_j) = \sum_{i=1}^{p} \sum_{j=1}^{q} Prob(w_i = y_j) = \sum_{i=1}^{p} 1 = p$$

[12] Player **II** could also select a string outside $\{y_1 \ldots y_q\}$ for a w_i, but such a strategy would always be disadvantageous, as z is a member of $\{y_1, \ldots y_q\}$.

we obtained the contradiction $p > p$.

Now, let the strategy σ for Player **I** select, for z, an y_i which is selected with probability $\leq p/q$ by Player **II** as one of the w_i: then

$$P(H(\phi); \sigma; \gamma) \leq p/q$$

and thus

$$V(\phi) = \sup_{\gamma} \inf_{\sigma} P(H(\phi); \sigma; \gamma) \leq p/q$$

In conclusion,

$$V(\phi) = p/q$$

as required.

- $\{r \in \mathbb{R} : V(\phi) = r \text{ for some } \phi\} \subseteq \mathbb{Q} \cap [0, 1]$:
Since the model is finite, in $H^{\mathcal{M}}(\phi)$ there exists a finite set $\{\sigma_1, \sigma_2, \ldots \sigma_k\}$ of pure strategies for Player **I**, and a finite set $\{\tau_1, \tau_2, \ldots \tau_t\}$ of pure strategies for Player **II**.

Now, let us consider all uniform behavioral strategies γ for Player **II**, or, more precisely, the corresponding distributions of pure strategies γ^*.
Of course, not all distributions derive from a behavioral strategy: more precisely, if γ is required to be uniform then, for all partial plays $(p_1 \ldots p_i)$ and $(p'_1 \ldots p'_i)$ with

$p_i = (\exists x \backslash V\psi, s), p'_i = (\exists x \backslash V\psi, s')$ for the same instance of $\exists x \backslash V\psi$;
$s(x_i) = s'(x_i)$ for all $x_i \in V$

we must have that, for all $m \in M$,

$$\gamma_i(p_1 \ldots p_i)(\psi, s[m/x]) = \sum\{\gamma^*(\tau_i) : \tau_i(p_1 \ldots p_i) = (\psi, s[m/x])\} =$$
$$= \sum\{\gamma^*(\tau_i) : \tau_i(p'_1 \ldots p'_i) = (\psi, s'[m/x])\} = \gamma_i(p'_1 \ldots p'_i[m/x])$$

Now, as our model is finite there exist only finitely many possible partial plays $(p_1 \ldots p_i)$ and $(p'_1 \ldots p'_i)$ as above, and therefore the requirement that a vector

$$\bar{\gamma}^* = \begin{pmatrix} \gamma^*(\tau_1) \\ \gamma^*(\tau_2) \\ \ldots \\ \gamma^*(\tau_t) \end{pmatrix}$$

corresponds to an uniform behavioral strategy γ can be expressed by a linear equation

$$A\bar{\gamma}^* = c$$

for a suitable matrix A and vector c *with rational coefficients.*

Then our value $V(\phi)$ is the result of the following linear programming problem:

$$\begin{array}{ll} \text{maximize} & v, \text{ with respect to the variables } (v, \lambda_1, \dots \lambda_t), \\ \text{and under the conditions} & \begin{cases} \sum_{i=1}^{t} \lambda_1 = 1; & \\ \sum_{i=1}^{t} \lambda_i P(H(\phi); \sigma_j; \tau_i) \geq v, & \text{for all } j = 1 \dots k; \\ A(\lambda_1, \dots \lambda_t)^T = c; & \\ \lambda_i \geq 0, & \text{for all } i = 1 \dots t. \end{cases} \end{array}$$

where the tuple $(\lambda_1, \dots \lambda_t)$ represents the probability distribution over pure strategies induced by a uniform behavioral strategy γ.

In other words, the problem of calculating $V(\phi)$ is equivalent to the problem of finding the maximum of the linear function z in the $t+1$-dimensional polytope described by the linear inequalities and equalities with rational coefficients described above.
It is then clear that the maximum is always reached at one of the vertices of the polytope[13]; but since the linear inequalities have rational coefficients, the coordinates of these vertices are also rational, and thus the value of our target function z at this point will also be rational.

Moreover, the value function always assumes values between 0 and 1, and this concludes the proof.

□

Theorem 8 *For every $r \in [0, 1]$, there exists a (possibly infinite) model $\mathcal{M}$ and a formula ϕ such that*

$$V(\phi) = r$$

Proof:
Let $r \in [0, 1]$, and let us consider the model

$$\mathcal{M} = ([0, 1], I, E)$$

where I is the two-place predicate given by[14]

$$I = \{(a, b) \in [0, 1]^2 : b = a + r \mod 1\}$$

and E is the three-place predicate given by

$$E = \{(a, b, c) \in [0, 1]^3 : a \leq b \leq c \vee c \leq a \leq b \vee b \leq c \leq a\}$$

[13] This is also the basis of the *simplex method* for solving linear optimization problems.

[14] Given two $x, y \in \mathbb{R}$, we say that $x = y \mod 1$ if there exists a $k \in \mathbb{Z}$ such that $x + k = y$. For example, it is simple to verify that the definition of I could be rewritten as $I = \{(a, b) \in [0, 1]^2 : b = a + r \text{ or } b + 1 = a + r\}$.

that is, $E(a,b,c)$ if and only if b is in the interval (arc) $[a,c]$ of the circumference of radius $1/2\pi$.
Now, let us consider the sentence

$$\phi := \forall xy(\exists z\backslash\{\})(\neg I(x,y) \vee E(x,z,y))$$

Then I state that, in $\mathcal{M}$,

$$V(\phi) = r$$

Indeed, for $k \in \mathbb{N}$, let γ^k be the following strategy for Player **II**:

- Select z in the set $\{0, 1/k, 2/k, \dots 1\}$ with uniform probability;
- If $I(x,y)$ does not hold, choose the first conjunct of $\neg I(x,y) \vee E(x,z,y)$; otherwise, choose the second one.

It is easy to see that

$$\lim_{k\to\infty} \inf_{\sigma} P(\sigma;\gamma^k) \geq r$$

Indeed, if σ selects x and y such that $\neg I(x,y)$ then Player **II** always wins the play; and otherwise, the probability that z is in the interval $[x,y]$ tends to r, for k which tends to ∞.[15]
Thus,

$$V(\phi) = \sup_{\gamma} \inf_{\beta} P(\beta;\gamma) = \sup_{\gamma} \inf_{\sigma} P(\sigma;\gamma) \geq r$$

On the other hand, let γ be any behavioral strategy for Player **II**: since z is selected independently from x, y, γ induces a probability distribution over z.
Now, I state that, for any such probability distribution, there exists an arc $[a,b]$ of length r such that

$$Prob(z \in [a,b]) \leq r$$

Then, if σ is the strategy which selects $x = a, y = b$ we have that

$$P(H_\emptyset(\phi);\sigma;\gamma) \leq r$$

Therefore,

$$V(\phi) = \sup_{\gamma} \inf_{\sigma} P(H_\mu(\phi);\sigma;\gamma) \leq r$$

and in conclusion

$$V(\phi) = r$$

as required.

[15] Indeed, this probability is

$$\frac{1}{k}\left|\{p \in \mathbb{N} : \frac{p}{k} \in [a,b]\}\right| = \frac{1}{k}\left|\{p \in \mathbb{N} : \frac{p}{k} \in [0,r]\}\right| = \frac{\lfloor kr \rfloor}{k}$$

and thus it tends to r for $k \to \infty$.

It remains to verify that, for any probability distribution over z in our circle, there exists an arc $[a, b]$ of length r such that

$$Prob(z \in [a, b]) \leq r$$

Indeed, take an arc $arc_1 = [a_0, a_1]$ of length r; if $Prob(z \in arc_1) \leq r$, we are done.
Otherwise, let $\epsilon > 0$ be defined as

$$\epsilon = Prob(z \in arc_1) - r$$

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can find $p, q \in \mathbb{N}$ such that

$$r \in \left[\frac{p - \epsilon}{q}, \frac{p}{q}\right]$$

and, therefore,

$$0 \leq p - qr \leq \epsilon$$

Now, let us take arc_1 as the starting point of a sequence of q consecutive arcs of length r

$$arc_1 = [a_0, a_1]; \ arc_2 = [a_1, a_2]; \ \ldots; \ arc_q = [a_{q-1}, a_q].$$

These q arcs cover our circumference p times, except for a gap of at most ϵ between the end of the last arc a_q and the start of the first arc a_0.

Again, if one of these arc_i is such that $Prob(z \in arc_i) \leq r$ we are done; and otherwise,

$$p \geq \sum_{i=1}^{q} Prob(z \in arc_i) > qr + \epsilon \geq p$$

where the second inequality holds because $Prob(z \in arc_1) = r + \epsilon$, and for all other arc_i $Prob(z \in arc_i) > r$.
Thus, we reached a contradiction.
□

This theorem tells us something about the range of the value function in infinite models.
However, as we saw, the minimax theorem and its consequences do not hold for infinite models; because of this, the calculation of $V(\phi)$ in such models presents additional difficulties - as we will see, the minimax theorem will be of paramount importance for the analysis of game values of next section, as well as for the treatment of (game-theoretical) negation in the last part of this work.

Moreover, the very concept of game value loses much of its interest when the minimax theorem does not hold: indeed, in this case $V(\phi)$ is not the payoff, for Player **II**, of an equilibrium pair of strategies $(\beta; \gamma)$, but just the supremum of the payoffs that Player **II** is able to guarantee by playing behavioral strategies.

Because of these reasons, in the rest of this work I will be mostly concerned with *finite models*.

4 Probabilistic Dependence Logic and its semantics

Let us now consider the logic that associates to every formula ϕ the "*truth value*" $V_{II}(\phi)$.

Somewhat loosely, this can be thought of as a form of *fuzzy logic*, since its truth function takes values in the real interval $[0, 1]$; however, it differs sharply from most forms of fuzzy logic both in its purpose and in its properties.
Indeed, the definition of $V(\phi)$ does not involve any form of approximated reasoning[16], nor do the properties expressible in this logic correspond to imprecise attributes such as hot⟶cold or tall⟶short; rather, the $V(\phi)$ coincide with the winning probabilities of the Verifier in the semantic games $H(\phi)$.

Because of this, I will call this logic *Probabilistic Dependence Logic.*

Now, Hodges' semantics for Slash Logic was of great help in clarifying the nature of the independence relations, and was instrumental for the development of Dependence Logic: thus, I will now attempt to provide a compositional semantics, similar to Hodges', for the computation of the values $V^{\mathcal{M}}_{II,s}(\phi)$ in finite models.

Afterwards, I will try to find out a few results about the values of sentences - in particular, we will be able to observe an analogy between the interpretation of dependence formulas

$$=(t_1 \dots t_n) \equiv \exists z_1 \dots z_{n-1}(\exists z_n \backslash \{z_1 \dots z_{n-1}\})(z_1 = t_1 \wedge \dots \wedge z_n = t_n)$$

and one of the notions of approximate functional dependence used in Database Theory [20].

4.1 The game $H_\mu(\phi)$

As, in order to define $\mathcal{T}$ for Hodges' semantics, we first considered the game $H_X(\phi)$, where X is a team, we will now introduce the game $H_\mu(\phi)$: where μ is a *probabilistic team*

[16] Not intentionally, anyway. . .

Definition 16 (Probabilistic team)
A probabilistic team *μ with domain $dom(\mu) = \{x_1 \dots x_n\}$ is a probability function over the set of all assignments on $\{x_1 \dots x_n\}$, that is, a function*

$$\mu : \{s : dom(s) = \{x_1 \dots x_n\}\} \to [0, 1]$$

such that

$$\sum_{dom(s)=\{x_1 \dots x_n\}} \mu(s) = 1$$

Then, let us define the following game $H_\mu^{\mathcal{M}}$:

Definition 17 ($H_\mu^{\mathcal{M}}(\phi)$)
Let ϕ be a formula in NNF, let $\mathcal{M}$ be a model, and let μ be a probabilistic team. The game $H_X^{\mathcal{M}}(\phi)$ is then played as follows:

1. *First, an assignment $s \in X$ is selected by a third player, which we will call Nature, according to the probability distribution μ;*
2. *Then, the game $H_s^{\mathcal{M}}(\phi)$ is played.*

The definitions of strategy, uniform strategy, behavioral strategy and uniform behavioral strategy are as usual; however, this time a play *will be determined by a triple (s, σ, τ), where s is the initial assignment (chosen according to μ) and σ, τ are pure strategies.*
Thus,

$$P(H_\mu(\phi); \beta; \gamma) = \sum_{dom(s)=\{x_1 \dots x_n\}} \mu(s) P(H_s(\phi); \beta; \gamma)$$

It can be easily verified that

$$P(H_s(\phi); \beta; \gamma) = P(H_{\eta_s}; \beta; \gamma)$$

where η_s is the probabilistic team which chooses s with certainty, that is,

$$\eta_s(s') = \begin{cases} 1 & \text{if } s' = s; \\ 0 & \text{otherwise.} \end{cases}$$

Definition 18 (r-trumps and $\mathcal{T}$)
*A probabilistic team μ is a r-*trump *of a formula ϕ if and only if*

$$\exists \gamma \forall \sigma P(H_\mu(\phi); \sigma; \gamma) \geq r$$

where, as usual, it makes no difference whether Player **I** *can choose behavioral strategies β or if he is limited to pure strategies σ.*

Then, we define

$$\mathcal{T} = \{(\phi, \mu, r) : \mu \text{ is a } r\text{-trump of } \phi\}$$

Clearly we have that

$$V_s(\phi) = \sup\{r : (\phi, \eta_s, r)\} \in \mathcal{T}$$

More in general, we will define $V_\mu(\phi)$ as

$$V_\mu(\phi) = \sup\{r : (\phi, \mu, r)\} \in \mathcal{T}$$

In order to characterize $\mathcal{T}$, we will have to introduce a few operations:

Definition 19 (Linear combination)
If μ_1, μ_2 are probabilistic teams with $dom(\mu_1) = dom(\mu_2) = \{x_1 \dots x_n\}$ and $p \in [0,1]$ we say that

$$\mu' = p\mu_1 + (1-p)\mu_2$$

if and only if

$$\mu'(s) = p\mu_1(s) + (1-p)\mu_2(s), \text{ for all } s \text{ s.t. } dom(s) = \{x_1 \dots x_n\}$$

It is easy to verify that μ' is still a team:

$$\sum_s \mu'(s) = p \sum_s \mu_1(s) + (1-p) \sum_s \mu_2(s) = p + (1-p) = 1$$

Definition 20 (Supplementation)
If μ is a probabilistic team with domain $\{x_1 \dots x_n\}$, F is a function from $\{s : dom(s) = \{x_1 \dots x_n\}\}$ to probability distributions over M, that is, a mapping

$$F : \{s : dom(s) = \{x_1 \dots x_n\}\} \to \mathcal{D}(M)$$

and $y \notin \{x_1 \dots x_n\}$, we define $\mu[F/y]$ as the probabilistic team such that

$$\mu[F/y](s[m/y]) = \mu(s) \cdot F(s)(m)$$

for all s such that $dom(s) = \{x_1 \dots x_n\}$ and for all $m \in M$.

Let us verify that $\mu[F/y]$ is a team over $\{x_1 \dots x_n, y\}$:

$$\begin{aligned} & \sum_{dom(s')=\{x_1 \dots x_n, y\}} \mu[F/y](s') = \\ = & \sum_{dom(s)=\{x_1 \dots x_n\}} \sum_{m \in M} \mu[F/y](s[m/y]) = \\ = & \sum_{dom(s)=\{x_1 \dots x_n\}} \sum_{m \in M} \mu(s) \cdot F(s)(m) = \\ = & \sum_{dom(s)=\{x_1 \dots x_n\}} \mu(s) \sum_{m \in M} F(s)(m) = \\ = & \sum_{dom(s)=\{x_1 \dots x_n\}} \mu(s) = 1 \end{aligned}$$

There is no probabilistic equivalent of the duplication operation $X[M/x]$. Indeed, the whole reason for introducing this operation in the non-probabilistic case was that, if τ is a u.w.s. for $H_{X[M/x]}(\psi)$, then τ is also a u.w.s. for $H_{X[F/x]}(\psi)$, for every $F : X \to M$: thus, no matter how Player **I** chooses the value of x in $H_X(\forall x\psi)$ the strategy τ allows Player **II** to win the game.
In other words, the duplication operation can be used to characterize the semantic of universal quantification because of the following property ([27], page 24):

Proposition 3 (Closure Test for (non-probabilistic) DF-logic) *If $(\phi, X) \in \mathcal{T}$ and $Y \subseteq X$ then $(\phi, Y) \in \mathcal{T}$*

Proof:
This result is a direct consequence of the game semantics for DF-logic: if Player **II** has an u.w.s. for $H_X(\phi)$ and $Y \subseteq X$, it is clear that the same strategy is also winning for $H_Y(\phi)$.
□

However, once we start considering game values, behavioral strategies and probabilistic teams the closure test is no longer applicable: in general, the fact that Player **II** can guarantee a payoff of at least r in $H_\mu(\phi)$ does not tell us anything about the payoffs of the games $H_\xi(\phi)$ for subteams ξ of μ - for example, in the team

$\mu =$

	x	weight
s_1	a	1/2
s_2	b	1/2

the formula $x = a$ has value 1/2, but in the subteams of μ (that is, in the probabilistic teams ξ such that, for some ξ' and p, $\mu = p\xi + (1-p)\xi'$) it assumes values ranging from 1 (in the case of the subteams containing only s_1) to 0 (in the case of the subteams containing only s_2).

Because of this, in general there is no way of building a "duplicated probabilistic team" $\mu[M/x]$ such that, for all γ,

$$\forall\sigma P(H_{\mu[M/x]}(\phi); \sigma; \gamma) \geq r \Leftrightarrow \forall F \forall\sigma P(H_{\mu[F/x]}(\phi); \sigma; \gamma) \geq r$$

This said, let us characterize $\mathcal{T}$:

Theorem 9
If $\mathcal{M}$ is a finite *model and ϕ is a formula in NNF, the following results hold:*

1. *If ϕ is a literal, then $(\phi, \mu, r) \in \mathcal{T}$ if and only if*

$$\sum_{s \models_{FO} \phi} \mu(s) \geq r$$

2. $(\psi \vee \theta, \mu, r) \in \mathcal{T}$ *if and only if* μ *can be written as a linear combination of probabilistic teams*

$$\mu = p\xi_1 + (1-p)\xi_2$$

such that, for some r_1 *and* r_2*, the following conditions hold:*

$$(\psi, \xi_1, r_1) \in \mathcal{T};$$
$$(\theta, \xi_2, r_2) \in \mathcal{T};$$
$$pr_1 + (1-p)r_2 \geq r$$

3. $(\psi \wedge \theta, \mu, r) \in \mathcal{T}$ *if and only if for all* ξ_1, ξ_2, p *such that*

$$\mu = p\xi_1 + (1-p)\xi_2$$

there exist r_1, r_2 *such that*

$$(\psi, \xi_1, r_1), (\theta, \xi_2, r_2) \in \mathcal{T}$$

and

$$pr_1 + (1-p)r_2 \geq r$$

4. $(\exists x\psi, \mu, r) \in \mathcal{T}$ *if and only if there exists a*

$$F : \{s : dom(s) = dom(\mu)\} \to \mathcal{D}(M)$$

such that

$$(\psi, \mu[F/x], r) \in \mathcal{T}$$

5. $(\exists x \backslash \{x_1, \ldots, x_k\}\psi, \mu, r) \in \mathcal{T}$ *if and only if the conditions for the above case hold, and moreover*

$$s(x_1) = s'(x_1), \ldots, s(x_k) = s'(x_k) \Rightarrow F(s) = F(s')$$

for any two s*,* s' *with the same domain of* μ*.*

6. $(\forall x\psi, \mu, r) \in \mathcal{T}$ *if and only if for all*

$$F : \{s : dom(s) = dom(\mu)\} \to \mathcal{D}(M)$$

it holds that

$$(\psi, \mu[F/x], r) \in \mathcal{T}$$

7. $(\forall x \backslash \{x_1 \ldots x_k)\psi, \mu, r) \in \mathcal{T}$ *if and only if the conditions of the previous case hold for all* F *such that*

$$s(x_1) = s'(x_1), \ldots, s(x_k) = s'(x_k) \Rightarrow F(s) = F(s')$$

for every two s*,* s' *with the same domain of* μ*.*

Proof:

1. If ϕ is a literal, there are no strategies available except the trivial ones, and therefore

$$(\phi, \mu, r) \in \mathcal{T} \text{ iff } P(H_\mu(\phi); \emptyset; \emptyset) \geq r \text{ iff } \sum_{s \models_{FO} \phi} \mu(s) \geq r$$

2. Suppose that $(\psi \vee \theta, \mu, r) \in \mathcal{T}$: then, there exists an uniform behavioral strategy γ such that, for all σ,

$$P(H_\mu(\psi \vee \theta); \sigma; \gamma) \geq r$$

Now, for every assignment s, let λ_s be the probability, according to γ, that Player **II** chooses the left disjunct ψ when the initial assignment s is extracted - that is,

$$\lambda_s = (\gamma_1(\psi \vee \theta, s))(\psi, s)$$

Then, the total probability that the left disjunct is selected is

$$p = \sum_s \mu(s)\lambda_s$$

As a consequence, the conditional probability distribution

$$Prob(s \text{ is selected by Nature in } H_\mu(\psi \vee \theta) \mid \text{the next position is } (\psi, s))$$

is given by

$$\xi_1(s) = \frac{\mu(s)\lambda_s}{p} = \frac{\mu(s)\lambda_s}{\sum_s \mu(s)\lambda_s}$$

And, analogously,

$$Prob(s \text{ is selected by Nature in } H_\mu(\psi \vee \theta) \mid \text{the next position is } (\theta, s))$$

is

$$\xi_2(s) = \frac{\mu(s)(1-\lambda_s)}{1-p} = \frac{\mu(s)(1-\lambda_s)}{\sum_s \mu(s)(1-\lambda_s)}$$

Clearly,

$$\mu = p\xi_1 + (1-p)\xi_2$$

Moreover, let γ^L and γ^R be two behavioral strategies for $H(\psi)$ and $H(\theta)$ such that

$$\gamma_i^L((\psi, s) \ldots p_i) = \gamma_{i+1}((\psi \vee \theta, s)(\psi, s) \ldots p_i);$$
$$\gamma_i^R((\theta, s) \ldots p_i) = \gamma_{i+1}((\psi \vee \theta, s)(\theta, s) \ldots p_i).$$

Analogously, for each pure strategy σ for Player **I** let us define σ^L and σ^R such that

$$\sigma_i^L((\psi, s)\dots p_i) = \sigma_{i+1}((\psi \vee \theta, s)(\psi, s)\dots p_i);$$
$$\sigma_i^R((\theta, s)\dots p_i) = \sigma_{i+1}((\psi \vee \theta, s)(\theta, s)\dots p_i).$$

Then we have that

$$P(H_\mu(\psi \vee \theta); \sigma; \gamma) = \sum_s \mu(s) P(H_s(\psi \vee \theta); \sigma; \gamma) =$$
$$= \sum_s \mu(s)\lambda_s P(H_s(\psi); \sigma^L; \gamma^L) + \sum_s \mu(s)(1-\lambda_s) P(H_s(\theta); \sigma^R; \gamma^R) =$$
$$= p\sum_s \xi_1(s) P(H_s(\psi); \sigma^L; \gamma^L) + (1-p)\sum_s \xi_2(s) P(H_s(\theta); \sigma^R; \gamma^R)$$

Now, by hypothesis $P(H_\mu(\psi \vee \theta); \sigma; \gamma) \geq r$; therefore, there exist r_1 and r_2 such that

$$P(H_{\xi_1}(\psi); \sigma; \gamma^L) \geq r_1 \text{ for all } \sigma;$$
$$P(H_{\xi_2}(\theta); \sigma; \gamma^R) \geq r_2 \text{ for all } \sigma;$$
$$pr_1 + (1-p)r_2 \geq r.$$

as required.

Conversely, suppose that

$$\mu = p\xi_1 + (1-p)\xi_2$$

with

$$(\psi, \xi_1, r_1) \in \mathcal{T};$$
$$(\theta, \xi_2, r_2) \in \mathcal{T};$$
$$pr_1 + (1-p)r_2 \geq r$$

Then, by definition of $\mathcal{T}$, there exist behavioral strategies γ^L, γ^R such that

$$P(H_{\xi_1}(\psi); \sigma^L; \gamma^L) \geq r_1 \text{ for all } \sigma^L;$$
$$P(H_{\xi_2}(\theta); \sigma^R; \gamma^R) \geq r_2 \text{ for all } \sigma^R;$$

Then consider the following behavioral strategy γ for Player **II** in $H_\mu(\psi \vee \theta)$: if the assignment s is selected, choose the left disjunct ψ with probability

$$\lambda_s = \frac{p\xi_1(s)}{\mu(s)}$$

that is, let

$$(\gamma_1(\psi \vee \theta, s))(\psi, s) = \lambda_s$$

Then, for the successive moves, let

$$\gamma_{i+1}((\psi \vee \theta, s)(\psi, s), \ldots) = \gamma_i^L((\psi, s), \ldots);$$
$$\gamma_{i+1}((\psi \vee \theta, s)(\theta, s), \ldots) = \gamma_i^R((\theta, s), \ldots).$$

Then, we have that, for all strategies σ,

$$P(H_\mu(\psi \vee \theta); \sigma; \gamma) = \sum_s \mu(s) P(H_s(\psi \vee \theta); \sigma; \gamma) =$$
$$= \sum_s \mu(s)\lambda_s P(H_s(\psi); \sigma^L; \gamma^L) + \sum_s \mu(s)(1-\lambda_s) P(H_s(\theta); \sigma^R; \gamma^R) =$$
$$= p \sum_s \xi_1(s) P(H_s(\psi); \sigma^L; \gamma^L) + (1-p) \sum_s \xi_2(s) P(H_s(\theta); \sigma^R; \gamma^R) =$$
$$= pP(H_{\xi_1}(\psi); \sigma^L; \gamma^L) + (1-p) P(H_s(\theta); \sigma^R; \gamma^R) \geq pr_1 + (1-p)r_2 \geq r$$

where σ^L, σ^R are defined as above and I used the fact that

$$1 - \lambda_s = \frac{\mu(s) - p\xi_1(s)}{\mu(s)} = \frac{(1-p)\xi_2(s)}{\mu(s)}$$

3. Suppose that $(\psi \wedge \theta, \mu, r) \in \mathcal{T}$: then, there exists a behavioral strategy γ for Player **II** such that, no matter which behavioral[17] strategy β Player **I** uses to select ϕ or ψ, the payoff $P(H_\mu(\psi \wedge \theta); \beta; \gamma)$ is greater or equal to r.

   Now, suppose that $\mu = p\xi_1 + (1-p)\xi_2$, and let β^L, β^R be any two behavioral strategies for Player **I** for the games $H(\psi)$ and $H(\theta)$; then, let us define β as

$$(\beta_1(\psi \wedge \theta, s))(\psi, s) = p\xi_1(s_i)/\mu(s_i);$$
$$\beta_{i+1}((\psi \wedge \theta, s)(\psi, s), \ldots) = \beta_i^L((\psi, s), \ldots);$$
$$\beta_{i+1}((\psi \wedge \theta, s)(\theta, s), \ldots) = \beta_i^R((\theta, s), \ldots);$$

   Then, for γ^L and γ^R defined as in the previous case, we have that

$$r \geq P(H_\mu(\psi \wedge \theta; \beta; \gamma) = \sum_s \mu(s) P(H_s(\psi \vee \theta; \beta; \gamma) =$$
$$= p \sum_s \xi_1(s) P(H_s(\psi); \beta^L; \gamma^L) + (1-p) \sum_s \xi_2(s) P(H_s(\theta); \beta^R; \gamma^R) =$$
$$= pP(H_{\xi_1}(\psi); \beta^L; \gamma^L) + (1-p) P(H_{\xi_2}(\theta); \beta^R; \gamma^R)$$

[17] Recall that, if γ guarantees a payoff of r against all pure strategies σ, it also guarantees the same payoff against all behavioral strategies β.

and, therefore, there exist r_1, r_2 such that

$$(\psi, \xi_1, r_1) \in \mathcal{T};$$
$$(\theta, \xi_2, r_2) \in \mathcal{T};$$
$$pr_1 + (1-p)r_2 \geq r$$

as required.

Conversely, suppose that whenever

$$\mu = p\xi_1 + (1-p)\xi_2$$

there are r_1, r_2 such that

$$(\psi, \xi_1, r_1) \in \mathcal{T};$$
$$(\theta, \xi_2, r_2) \in \mathcal{T};$$
$$pr_1 + (1-p)r_2 \geq r$$

Then, let β be any behavioral strategy for Player **I**, and, as usual, let

$$\lambda_s = (\beta_1(\psi \wedge \theta, s))(\psi, s)$$

As usual, let ξ_1 and ξ_2 be the assignment distributions when Player **I** chooses ψ or θ, that is,

$$\xi_1(s_i) = \frac{\mu(s_i)\lambda_i}{\sum_i \mu(s_i)\lambda_i}$$

and

$$\xi_2(s_i) = \frac{\mu(s_i)(1-\lambda_i)}{\sum_i \mu(s_i)(1-\lambda_i)}$$

Then, for $p = \sum_i \mu(s_i)\lambda_i$ we have

$$\mu = p\xi_1 + (1-p)\xi_2$$

Now, by hypothesis, there exist r_1, r_2 such that$(\psi, \xi_1, r_1), (\theta, \xi_2, r_2) \in \mathcal{T}$ and $pr_1 + (1-p)r_2 \geq r$, and thus it is possible to find two behavioral strategies γ^L and γ^R for Player **II** such that

$$P(H_{\xi_1}(\psi); \beta'; \gamma^L) \geq r_1, \text{ for all } \beta';$$
$$P(H_{\xi_2}(\theta); \beta''; \gamma^R) \geq r_2, \text{ for all } \beta''.$$

Now, let the strategy γ for Player **II** in $H(\psi \wedge \theta)$ be defined as

$$\gamma_{i+1}((\psi \wedge \theta, s), (\psi, s), \ldots) = \gamma_i^L((\psi, s), \ldots);$$
$$\gamma_{i+1}((\psi \wedge \theta, s), (\theta, s), \ldots) = \gamma_i^R((\theta, s), \ldots).$$

Then we have that

$$\begin{aligned}
&P(H_\mu(\psi \wedge \theta);\beta;\gamma) = \sum_s \mu(s) P(H_s(\psi \wedge \theta);\beta;\gamma) = \\
&= p \sum_s \xi_1(s) P(H_s(\psi);\beta^L;\gamma^L) + (1-p)\sum_s \xi_2(s) P(H_s(\theta);\beta^R;\gamma^R) \geq \\
&\geq pr_1 + (1-p)r_2 \geq r
\end{aligned}$$

Thus,

$$\forall\beta\exists\gamma P(H_\mu(\psi \wedge \theta);\beta;\gamma) \geq r$$

But then, by the minimax theorem and its corollary,

$$\exists\gamma\forall\beta P(H_\mu(\psi \wedge \theta);\beta;\gamma) \geq r$$

and, in conclusion, $(\psi \wedge \theta, \mu, r) \in \mathcal{T}$.

4. Suppose that $(\exists x\psi, \mu, r) \in \mathcal{T}$: then, there is a behavioral strategy γ such that, for all σ,

$$P(H_\mu(\exists x\psi);\sigma;\gamma) \geq r$$

Then, for all assignments s, let $F(s)$ be defined as

$$F(s)(m) = (\gamma_1(\exists x\psi, s))(\psi, s[m/x]), \text{ for all } m \in \mathbb{M}$$

Moreover, let us define the strategy γ' as

$$\gamma'_i((\psi, s[m/x])\ldots) = \gamma_{i+1}((\exists x\psi, s)(\psi, s[m/x])\ldots)$$

Now, let σ' be any strategy for Player **I** in $H_{\mu[F/x]}(\psi)$, and let us find a strategy σ for $H_\mu(\psi)$ such that

$$(F(s))(m) > 0 \Rightarrow \sigma((\exists x\psi, s), (\psi, s[m/x]), \ldots) = \sigma'((\psi, s[m/x]), \ldots)$$

Then,

$$\begin{aligned}
r \leq P(H_\mu(\exists x\psi);\sigma;\gamma) &= \sum_s \mu(s) P(H_s(\exists x\psi);\sigma;\gamma) = \\
&= \sum_s \mu(s) \sum_m (F(s))(m) P(H_{s[m/x]}(\psi);\sigma';\gamma') = \\
&= P(H_{\mu[F/x]}(\psi);\sigma';\gamma')
\end{aligned}$$

Since this holds for all σ', we can conclude that

$$(\psi, \mu[F/x], r) \in \mathcal{T}$$

as required.

Conversely, suppose that there exists a behavioral strategy γ' such that

$$P(H_{\mu[F/x]}(\psi); \sigma'; \gamma') \geq r \text{ for all } \sigma'$$

Then, let us define the strategy γ for $H_\mu(\exists x\psi)$ as follows:

$$(\gamma_1(\exists x\psi, s))(\psi, s[m/x]) = F(s)(m);$$
$$\gamma_{i+1}((\exists x\psi, s), (\psi, s[m/x]), \ldots) = \gamma'_i((\psi, s[m/x]), \ldots)$$

Now, let σ be any strategy for Player **I** in $H_\mu(\exists x\psi)$, and let σ' be such that

$$\sigma'_i((\psi, s[m/x]) \ldots) = \sigma_{i+1}((\exists x\psi, s)(\psi, s[m/x]), \ldots)$$

Then,

$$P(H_\mu(\exists x\psi); \sigma; \gamma) = \sum_s \mu(s) P(H_s(\exists x\psi); \sigma; \gamma) =$$
$$= \sum_s \sum_m \mu(s)(F(s))(m) P(H_{s[m/x]}(\psi); \sigma'; \gamma') =$$
$$= P(H_{\mu[F/x]}(\psi); \sigma'; \gamma') \geq r$$

as required.

5. Suppose that $(\exists x\backslash V\psi, \mu, r) \in \mathcal{T}$, where V is a set of variables, and let γ be the corresponding uniform behavioral strategy for Player **II** in $H_\mu(\exists x\backslash V\psi)$.
Then, as in the previous case, let us define the function F by

$$(F(s))(m) = (\gamma_1(\exists x\backslash V\psi, s))(\psi, s[m/x])$$

Then it is possible to verify, using exactly the same argument of the previous case, that

$$(\psi, \mu[F/x], r) \in \mathcal{T};$$

Moreover, since γ is uniform we have that

$$s(x_i) = s'(x_i) \text{ for all } x_i \in V \Rightarrow F(s) = F(s')$$

as required.

Conversely, suppose that there exists a γ' such that, for all σ',

$$P(H_{\mu[F/x]}(\psi), \sigma'; \gamma') \geq r$$

where F is such that

$$s(x_i) = s'(x_i) \text{ for all } x_i \in V \Rightarrow F(s) = F(s')$$

Then, as in the previous case, let us define the behavioral strategy γ for $H_\mu(\exists x \backslash V\psi)$ as follows:

$$(\gamma_1(\exists x \backslash V\psi, s))(\psi, s[m/x]) = F(s)(m);$$
$$\gamma_{i+1}((\exists x \backslash V\psi, s), (\psi, s[m/x]), \ldots) = \gamma'_i((\psi, s[m/x]), \ldots)$$

For the same argument used for the non-backslashed existential quantifier, we then have that

$$P(H_\mu(\exists x \backslash V\psi); \sigma; \gamma) \geq r$$

and it only remains to verify that γ is uniform.

Indeed, consider two partial plays $(p_1 \ldots p_i)$ and $(p'_1 \ldots p'_i)$ of $H_\mu(\exists x \backslash V\psi)$, where Player **II** follows γ, p_i and p'_i are of the form $(\exists z \backslash V'\theta, s)$ and $(\exists z \backslash V'\theta, s)$ for the same instance of this subformula, and

$$s(x_i) = s'(x_i) \text{ for all } x \in V'$$

Then I state that

$$\gamma(p_1 \ldots p_i) = \gamma'(p_1 \ldots p_i)$$

Indeed,

- If $i = 1$, our subformula is $\exists x \backslash V\psi$, and

  $$s(x_i) = s'(x_i) \text{ for all } x_i \in V \Rightarrow F(s) = F(s') \Rightarrow$$
  $$\Rightarrow (\gamma_1(\exists x \backslash V\psi, s))(\psi, s[m/x]) = (\gamma_1(\exists x \backslash V\psi, s'))(\psi, s[m/x]), \text{ for all } m \in M$$

  as required.
- If $i > 1$ and $(p_1 \ldots p_i)$, $(p'_1 \ldots p'_i)$ are as above, then $(p_2 \ldots p_i)$ and $(p'_2 \ldots p'_i)$ are plays of $H_{\mu[F/x]}(\psi)$ in which Player **II** follows γ', and since γ' is uniform we have the desired result.

6. Suppose that there exists a behavioral strategy γ for Player **II** such that, for all behavioral strategies β for Player **I**,

$$P(H_\mu(\forall x\psi); \beta; \gamma) \geq r$$

Now, let F be any function

$$F : \{s : dom(s) = dom(\mu)\} \rightarrow \mathcal{D}(M)$$

and let β' be any strategy of Player **I** for $H_{\mu[F/x]}(\psi)$.

Then, let the strategy β for $H_\mu(\forall x\psi)$ be defined by

$$(\beta_1(\forall x\psi, s))(\psi, s[m/x]) = (F(s))(m);$$
$$\beta_{i+1}((\forall x\psi, s)(\psi, s[m/x]), \ldots) = \beta'_i((\psi, s[m/x]) \ldots).$$

By hypothesis,

$$P(H_\mu(\forall x\psi);\beta;\gamma) \geq r;$$

Therefore, if we define the strategy γ' for Player **II** in $H_{\mu[F/x]}(\psi)$ as

$$\gamma'_i((\psi, s[m/x])\ldots) = \gamma_{i+1}((\forall x\psi, s)(\psi, s[m/x])\ldots)$$

we have that

$$r \leq P(H_\mu(\forall x\psi);\beta;\gamma) = \sum_s \mu(s) P(H_s(\forall x\psi);\beta;\gamma) =$$
$$= \sum_s \mu(s) \sum_m (F(s))(m) P(H_{s[m/x]}(\psi);\beta';\gamma') = P(H_{\mu[F/x]}(\psi);\beta';\gamma')$$

and therefore $(\psi, \mu[F/x], r) \in \mathcal{T}$, as required.
Conversely, suppose that for all $F : \{s : dom(s) = dom(\mu)\} \to \mathcal{D}(M)$ as above there exists a strategy γ^F such that

$$P(H_{\mu[F/x]}(\psi);\beta^F;\gamma^F) \geq r$$

for all behavioral strategies β^F of Player **I**.

Then, let β be any behavioral strategy of Player **I** in $H_\mu(\forall x\psi)$, and let F be defined by

$$(F(s))(m) = (\beta_1(\forall x\psi, s))(\psi, s[m/x])$$

Moreover, let β^F be the strategy given by

$$\beta^F_i((\psi, s[m/x])\ldots) = \beta_{i+1}((\forall x\psi, s)(\psi, s[m/x])\ldots)$$

And let γ be defined by

$$\gamma_{i+1}((\forall x\psi, s)(\psi, s[m/x])\ldots) = \gamma^F_i((\psi, s[m/x])\ldots)$$

Then,

$$P(H_\mu(\forall x\psi);\beta;\gamma) = \sum_s \mu(s) P(H_s(\forall x\psi);\beta;\gamma) =$$
$$= \sum_s \mu(s) \sum_m (F(s))(m) P(H_{s[m/x]}(\psi);\beta^F;\gamma^F) = P(H_{\mu[F/x]}(\psi);\beta^F;\gamma^F) \geq r$$

as required.

7. Let γ be such that, for all uniform behavioral strategies β for Player **I**,

$$P(H_\mu(\forall x \backslash V\psi);\beta;\gamma) \geq r$$

and let $F : \{s : dom(s) = dom(\mu)\} \to \mathcal{D}(M)$ be such that

$$s(x_i) = s'(x_i) \text{ for all } x_i \in V \Rightarrow F(s) = F(s')$$

Then, for every uniform behavioral strategy β' for Player **I** in $H_{\mu[F/x]}(\psi)$, let us define β as

$$(\beta_1(\forall x \backslash V\psi, s))(\psi, s[m/x]) = (F(s))(m);$$
$$\beta_{i+1}((\forall x \backslash V\psi, s)(\psi, s[m/x]) \ldots) = \beta'((\psi, s[m/x]) \ldots)$$

This β is uniform, since β' is uniform and since

$$s(x_i) = s'(x_i) \text{ for all } x_i \in V \Rightarrow F(s) = F(s') \Rightarrow \beta_1(\forall x \backslash V\psi, s) = \beta_1(\forall x \backslash V\psi, s')$$

Therefore, $P(H_\mu(\forall x \backslash V\psi); \beta; \gamma) \geq r$; but then, the γ' defined by

$$\gamma'_i((\psi, s[m/x]) \ldots) = \gamma_{i+1}((\forall x \backslash V\psi, s)(\psi, s[m/x]) \ldots)$$

is such that

$$\forall \beta', P(H_{\mu[F/x]}(\psi); \beta'; \gamma') \geq r$$

as required.

Conversely, suppose that for all F which satisfy the dependence condition there exists a γ^F such that

$$\forall \beta', P(H_{\mu[F/x]}(\psi); \beta'; \gamma') \geq r$$

Then, let β be any uniform behavioural strategy for $H_\mu(\forall x \backslash V\psi)$, and as usual let F be given by

$$(F(s))(m) = (\beta_1(\forall x \backslash V\psi, s))(\psi, s[m/x])$$

Since β must be uniform,

$$s(x_i) = s'(x_i) \text{ for all } x_i \in V \Rightarrow \beta_1(\forall x \backslash V\psi, s) = \beta_1(\forall x \backslash V\psi, s') \Rightarrow F(s) = F(s')$$

Therefore, F satisfies the dependence requirement, and if we let β^F be the restriction of β to the subgame $H(\psi)$, as in the case of the non-backslashed universal quantifier, we have that

$$P(H_{\mu[F/x]}(\psi); \beta^F; \gamma^F) \geq r$$

But then, for the γ defined by

$$\gamma_{i+1}((\forall x \backslash V\psi, s)(\psi, s[m/x]) \ldots) = \gamma_i^F((\psi, s[m/x]) \ldots)$$

we have that

$$P(H_\mu(\forall x \backslash V\psi); \beta; \gamma) \geq r$$

Thus,

$$\forall \beta \exists \gamma P(H_\mu(\forall x \backslash V \psi); \beta; \gamma) \geq r$$

and therefore, by the minimax theorem,

$$\exists \gamma \forall \beta P(H_\mu(\forall x \backslash V \psi); \beta; \gamma) \geq r$$

as required.

□

Thus, $\mathcal{T}$ allows us to compute the value of a formula ϕ in terms of the values of its subformulas: thus, we have a compositional semantics, mimicking Hodges', for the calculation of the game values of our logic.

It must be observed that the proofs of the cases for the conjunction and the universal quantifications depend on the minimax theorem, and thus are valid only for finite models.

This is unavoidable, as these equivalencies do not hold for infinite models: for example, let us consider again the formula

$$\forall x (\exists y \backslash \{\})(y > x)$$

in the model $(\mathbb{N}, <)$.

Let us consider any function $F : \{\emptyset\} \rightarrow \mathcal{D}(\mathbb{N})$, i.e., any probability distribution of x over $\mathbb{N}$.
Then, as we already saw, for every $\epsilon > 0$ there exists a $n \in \mathbb{N}$ such that

$$Prob(x \geq n) < \epsilon$$

Then, if γ is the strategy which always selects this very n for y, we have that

$$P(H_{\emptyset[F/x]}(\exists y \backslash \{\}(y > x)); \sigma; \gamma) \geq 1 - \epsilon \text{ for every } \sigma$$

and therefore

$$(\exists x \backslash \{\}(y > x), \mu[F/x], r) \in \mathcal{T} \text{ for every } r < 1$$

But on the other hand, we already saw that

$$(\forall r > 0)\ \forall \gamma \exists \sigma\ P(H_\mu(\forall x (\exists y \backslash \{\})(y > x)); \sigma; \gamma) < r$$

In conclusion, for infinite models it is not true anymore that if for all F $(\psi, \mu[F/x], r) \in \mathcal{T}$ then $(\forall x \psi, \mu, r) \in \mathcal{T}$ too.

Given the definition of the *value* of a formula ϕ as

$$V_\mu(\phi) = \sup\{r : (\phi, \mu, r) \in \mathcal{T}\}$$

and our results about $\mathcal{T}$, it is easy to verify the following properties of the value function:

Corollary 2

1. *If ϕ is a literal,*
$$V_\mu(\phi) = \sum_{s \models_{FO} \phi} \mu(s);$$

2. *If $\phi = \psi \vee \theta$,*
$$V_\mu(\psi \vee \theta) = \sup\{pV_{\xi_1}(\psi) + (1-p)V_{\xi_2}(\theta) : p\xi_1 + (1-p)\xi_2 = \mu\};$$

3. *If $\phi = \psi \wedge \theta$,*
$$V_\mu(\psi \wedge \theta) = \inf\{pV_{\xi_1}(\psi) + (1-p)V_{\xi_2}(\theta) : p\xi_1 + (1-p)\xi_2 = \mu\};$$

4. *If $\phi = \exists x \psi$,*
$$V_\mu(\exists x \psi) = \sup_F V_{\mu[F/x]}(\psi);$$

5. *If $\phi = \exists x \backslash \{x_1 \dots x_k\}\psi$,*
$$V_\mu(\exists x \backslash \{x_1 \dots x_k\}\psi) = \sup\{V_{\mu[F/x]}(\psi) : F \text{ depends only on } x_1, \dots x_k\};$$

6. *If $\phi = \forall x \psi$,*
$$V_\mu(\forall x \psi) = \inf_F V_{\mu[F/x]}(\psi);$$

7. *If $\phi = \forall x \backslash \{x_1 \dots x_k\}\psi$,*
$$V_\mu(\forall x \backslash \{x_1 \dots x_k\}\psi) = \inf\{V_{\mu[F/x]}(\psi) : F \text{ depends only on } x_1, \dots x_k\}.$$

Proof:

1. Obvious.

2. We have that
$$\begin{aligned}
&\sup\{r : (\psi \vee \theta, \mu, r) \in \mathcal{T}\} = \\
&= \sup\{pr_1 + (1-p)r_2 : \mu = p\xi_1 + (1-p)\xi_2, (\psi, \xi_1, r_1) \in \mathcal{T}, (\theta, \xi_2, r_2) \in \mathcal{T}\} = \\
&= \sup\{pr_1 + (1-p)r_2 : \mu = p\xi_1 + (1-p)\xi_2, r_1 < V_{\xi_1}(\psi), r_2 < V_{\xi_2}(\theta)\} = \\
&= \sup\{pV_{\xi_1}(\psi) + (1-p)V_{\xi_2}(\theta) : p\xi_1 + (1-p)\xi_2 = \mu\}
\end{aligned}$$

3. Similarly to the previous case,
$$\begin{aligned}
&\sup\{r : (\psi \wedge \theta, \mu, r) \in \mathcal{T}\} = \\
&= \sup\{r : \mu = p\xi_1 + (1-p)\xi_2 \Rightarrow \exists r_1 r_2 \text{ s.t. } (\psi, \xi_1, r_1) \in \mathcal{T}, \\
&\quad (\theta, \xi_2, r_2) \in \mathcal{T}, r \leq pr_1 + (1-p)r_2\} = \\
&= \sup\{r : \mu = p\xi_1 + (1-p)\xi_2 \Rightarrow r < pV_{\xi_1}(\psi) + (1-p)V_{\xi_2}(\theta)\} = \\
&= \sup\{r : r < \inf\{pV_{\xi_1}(\psi) + (1-p)V_{\xi_2}(\theta) : p\xi_1 + (1-p)\xi_2 = \mu\}\} = \\
&= \inf\{pV_{\xi_1}(r_1) + (1-p)V_{\xi_2}(\theta) : p\xi_1 + (1-p)\xi_2) = \mu\}
\end{aligned}$$

4. For the existential quantifier,

$$\begin{aligned}&\sup\{r : (\exists x\psi, \mu, r) \in \mathcal{T}\} =\\&= \sup\{r : \exists F(\psi, \mu[F/x], r) \in \mathcal{T}\} =\\&= \sup\{r : \exists F \text{ s.t. } r < V_{\mu[F/x]}(\psi)\} =\\&= \sup_F V_{\mu[F/x]}(\psi)\end{aligned}$$

5. The case for the backslashed existential quantifier is similar to that for the non-backslashed one, except that now F must satisfy a dependence condition.

6. For the universal quantifier,

$$\begin{aligned}&\sup\{r : (\forall x\psi, \mu, r) \in \mathcal{T}\} =\\&= \sup\{r : \forall F, (\psi, \mu[F/x], r) \in \mathcal{T}\} =\\&= \sup\{r : \forall F, r < V_{\mu[F/x]}(\psi)\} =\\&= \inf_F V_{\mu[F/x]}(\psi)\end{aligned}$$

7. The case for the backslashed universal quantifier is exactly as that for the non-backslashed one, except that now F must satisfy a dependence condition.

□

4.2 The value of first-order formulas

In this section, I will show that, for first-order formulas ϕ, the value $V_\mu^{\mathcal{M}}(\phi)$ corresponds to the probability, according to the assignment distribution μ, that ϕ holds in $\mathcal{M}$:

Theorem 10

Let ϕ be a first-order formula in Negation Normal Form with $FV(\phi) = \{x_1 \ldots x_n\}$ let $\mathcal{M}$ be a finite model and let μ be a probabilistic team with $dom(\mu) = FV(\phi)$. Then

$$V_\mu^{\mathcal{M}}(\phi) = \sum_{s \models_{FO} \phi} \mu(s)$$

that is, the value of ϕ is the probability, under the distribution μ, that a random assignment satisfies classically ϕ.

Proof:
The proof is by structural induction on ϕ:

- ϕ is a literal:
  In this case, the result has already been proved.

- $\phi = \psi \wedge \theta$: In this case,

$$V_\mu(\phi) = \inf\{pV_{\xi_1}(\psi) + (1-p)V_{\xi_2}(\theta) : p\xi_1 + (1-p)\xi_2 = \mu\} =$$
$$= \inf\{p \sum_{s \models_{FO} \psi} \xi_1(s) + (1-p) \sum_{s \models_{FO} \theta} \xi_2(s) : p\xi_1 + (1-p)\xi_2 = \mu\}$$

Let us find this infimum. For every assignment s, let λ_s be the fraction of the weight $\mu(s)$ which is assigned to ξ_1, that is,

$$\lambda_s = \frac{p\xi_1(s)}{\mu(s)}$$

Then, it is easy to verify that

$$p = \sum_s \mu(s)\lambda_s$$

and that

$$\xi_1(s) = \frac{\lambda_s \mu(s)}{p};$$
$$\xi_2(s) = \frac{(1-\lambda_s)\mu(s)}{1-p}.$$

Then, every decomposition of μ in $p\xi_1 + (1-p)\xi_2$ is determined by the values of the λ_s; and moreover, every family of values $\lambda_s \in [0,1]$ corresponds to an unique linear decomposition.

Thus,

$$V_\mu(\psi \wedge \theta) = \inf\{\sum_{s \models_{FO} \psi} p\xi_1(s) + \sum_{s \models_{FO} \theta} (1-p)\xi_2(s) : p\xi_1 + (1-p)\xi_2 = \mu\} =$$
$$= \inf\{\sum_{s \models_{FO} \psi} \lambda_s \mu(s) + \sum_{s \models_{FO} \theta} (1-\lambda_s)\mu(s) : \lambda_s \in [0,1] \text{ for all } s\}$$

The infimum is then obtained by letting $\lambda_s = 1$ for all s such that $s \not\models_{FO} \psi$ and $\lambda_s = 0$ for all s such that $s \models_{FO} \psi$ but $s \not\models_{FO} \theta$; the choice of λ_s for the remaining s does not make any difference, and

$$V_\mu(\psi \wedge \theta) = \sum_{s \models_{FO} \psi \wedge \theta} \mu(s_i)$$

as required.

- $\phi = \psi \vee \theta$: The proof is very similar to that for the conjunction: the supremum

$$\begin{aligned}
&\sup\{pV_{\xi_1}(\psi) + (1-p)V_{\xi_2}(\theta) : p\xi_1 + (1-p)\xi_2 = \mu\} = \\
&= \sup\{p \sum_{s \models_{FO} \psi} \xi_1(s) + (1-p) \sum_{s \models_{FO} \theta} \xi_2(s) : p\xi_1 + (1-p)\xi_2 = \mu\} = \\
&= \sup\{ \sum_{s \models_{FO} \psi} \lambda_s \mu(s) + \sum_{s \models_{FO} \theta} (1-\lambda_s)\mu(s) : \lambda_s \in [0,1] \text{ for all } s\}
\end{aligned}$$

  is reached by letting $\lambda_s = 1$ for all s such that $s \models_{FO} \psi$, and $\lambda_s = 0$ and all s such that $s \models_{FO} \theta$; as a consequence,

$$V_\mu(\psi \vee \theta) = \sum_{s \models_{FO} \psi \vee \theta} \mu(s)$$

- $\phi = \forall x \psi$: By definition,

$$V_\mu(\forall x \psi) = \inf_F V_{\mu[F/x]}(\psi) = \inf_F \sum_{s[m/x] \models_{FO} \psi} \mu(s) \cdot (F(s))(m)$$

  The infimum can be reached as follows: given an assignment s, if there exists a $c \in M$ such that $s[c/x] \not\models_{FO} \psi$, let F satisfy

$$F(s)(m) = \begin{cases} 1 & \text{if } m = c; \\ 0 & \text{otherwise.} \end{cases}$$

  If instead $s[c/x]$ satisfies ψ for all c, the choice of the distribution $F(s)$ has no importance, since $\sum_{c \in M} \mu(s) \cdot F(s)(m) = \mu(s)$. In conclusion,

$$V_\mu(\forall x \psi) = \sum_{s \models_{FO} \forall x \psi} \mu(s)$$

  as required.

- $\phi = \exists x \psi$: The proof is as for the universal quantifier: we have that

$$V_\mu(\exists x \psi) = \sup_F V_{\mu[F/x]}(\psi) = \sup_F \sum_{s[m/x] \models_{FO} \psi} \mu(s) \cdot F(s)(m)$$

  The supremum is reached as follows: for every s, if there exists a $c \in M$ such that $s[c/x] \models_{FO} \psi$ then let

$$F(s)(m) = \begin{cases} 1 & \text{if } m = c; \\ 0 & \text{otherwise.} \end{cases}$$

  If this is not the case, the choice of $F(s_i)$ is again of no consequence, and

$$V_\mu(\exists x \psi) = \sum_{s \models_{FO} \exists x \psi} s$$

  This concludes the proof.

□

4.3 The value of Dependence formulas

Let us now try to find out a way of assigning a value to dependence formulas $=(t_1 \dots t_n)$, which have been defined as

$$=(t_1 \dots t_n) \equiv \exists y_1 \dots y_{n-1} (\exists y_n \backslash \{y_1 \dots y_{n-1}\}) \bigwedge_{i=1}^{n} y_i = t_i$$

We have the following result:

Theorem 11 (Value of Dependence formulas)
Given terms $t_1 \dots t_n$ and a probabilistic team μ, we have that

$$(=(t_1 \dots t_n), \mu, r) \in \mathcal{T}$$

if and only if there exist p_1, p_2, ξ_1, ξ_2 such that $p_1 \geq r$,

$$\mu = p_1 \xi_1 + p_2 \xi_2$$

and that, in ξ_1, the value of t_n is determined by the values of $t_1 \dots t_{n-1}$ - that is, for every two assignments s, s' with $0 < \xi_1(s), \xi_1(s')$

$$t_1\langle s \rangle = t_1\langle s' \rangle, \dots, t_{n-1}\langle s \rangle = t_{n-1}\langle s' \rangle \Rightarrow t_n\langle s \rangle = t_n\langle s' \rangle$$

Proof:

- $\Leftarrow$:
  Suppose that $\mu = p_1 \xi_1 + p_2 \xi_2$, and that for any two assignments s and s' such that $\xi_1(s), \xi_1(s') > 0$ and

  $$t_1\langle s \rangle = t_1\langle s' \rangle, \dots, t_{n-1}\langle s \rangle = t_{n-1}\langle s' \rangle$$

  we have that

  $$t_n\langle s \rangle = t_n\langle s' \rangle$$

  Then consider the following strategy for Player **II** in

  $$H_\mu(=(t_1 \dots t_n)) = H_\mu(\exists y_1 \dots y_{n-1} (y_n \backslash \{y_1 \dots y_{n-1}\}) \bigwedge_{i=1}^{n} y_i = t_i) :$$

  Let Player **II** choose a probability distribution over $y_1 \dots y_n$ such that, for all s such that $\xi_1(s) > 0$, the tuple $(s(y_1) \dots s(y_n))$ coincides with the tuple $(t_1\langle s \rangle \dots t_n\langle s \rangle)$.
  Since, by hypothesis, t_n depends on $t_1 \dots t_{n-1}$ over ξ_1, it is possible for

Player **II** to do so while playing an uniform behavioral strategy; and moreover, the probability that she wins the play is greater or equal to the probability that an assignment s is selected such that $\xi_1(s) > 0$, and this is greater or equal to p_1.

For example, let us consider the formula $=(x_1, x_2)$ in the probabilistic team μ thus defined:

$\mu =$

Name	Weight	x_1	x_2
s_1	1/6	a	a
s_2	2/6	a	b
s_3	3/6	b	a

Now, we have that $\mu = 5/6\xi_1 + 1/6\xi_2$, where

$\xi_1 =$

Name	Weight	x_1	x_2
s_2	2/5	a	b
s_3	3/5	b	a

and

$\xi_2 =$

Name	Weight	x_1	x_2
s_1	1	a	a

Then let us consider the following strategy in

$$H_\mu(=(x_1, x_2)) = H_\mu(\exists y_1 \exists y_2/\{y_1\}(y_1 = x_1 \wedge y_2 = x_2))$$

First, Player **II** chooses y_1 according to the function F defined by

$$(F(s_i))(m) = \begin{cases} 1 & \text{if } s_i(x) = m; \\ 0 & \text{otherwise.} \end{cases} \quad \text{for } m \in \{a, b\}$$

Then $\mu[F/y_1]$ is as follows:

$\mu[F/y_1] =$

Name	Weight	x_1	x_2	y_1
s'_1	1/6	a	a	a
s'_2	2/6	a	b	a
s'_3	3/6	b	a	b

Then, Player **II** chooses y_2 so that y_2 depends on y_1 and $y_2 = x_2$ over ξ_1: in other words, let G be the function defined by

$$G(s'_1)(m) = G(s'_2)(m) = \begin{cases} 1 & \text{if } m = b; \\ 0 & \text{otherwise.} \end{cases}$$

$$G(s'_3)(m) = \begin{cases} 1 & \text{if } m = a; \\ 0 & \text{otherwise.} \end{cases}$$

Then we have that

$$\mu[F/y_1][G/y_2] = \begin{array}{|l|l|l|l|l|l|} \hline \text{Name} & \text{Weight} & x_1 & x_2 & y_1 & y_2 \\ \hline s_1'' & 1/6 & a & a & a & b \\ s_2'' & 2/6 & a & b & a & b \\ s_3'' & 3/6 & b & a & b & a \\ \hline \end{array}$$

Now, the conditions $y_1 = x_1$, $y_2 = x_2$ hold for the assignments s_2'' and s_3'', that is, for the assignments in ξ_1: so, in conclusion, the winning probability of Player **II** in this play is the total weight of s_2'' and s_3'', that is, 5/6.

- $\Rightarrow$: Suppose instead that Player **II** has a strategy that allows her to win with probability r in the game $H_\mu(=(t_1 \ldots t_n))$.
  Then, by the definition of this game, there exist functions $F_1 \ldots F_{n-1}$, G such that $G(s)$ depends only on $s(y_1), \ldots s(y_{n-1})$, and that in the team

$$\mu' = \mu[F_1/y_1] \ldots [F_{n-1}/y_{n-1}][G/y_n]$$

we have

$$\sum \{\mu'(s') : s'(y_1) = t_1\langle s'\rangle, \ldots, s'(y_n) = t_n\langle s'\rangle\} = w \geq r$$

Then, let ξ_1' be the probabilistic team defined by

$$\xi_1'(s') = \begin{cases} \frac{\mu'(s')}{w} & \text{if } s'(y_i) = t_i\langle s'\rangle, \text{ for } i = 1 \ldots n; \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, let ξ_2' be

$$\xi_2'(s') = \begin{cases} \frac{\mu'(s')}{1-w} & \text{if } \xi_1(s') = 0; \\ 0 & \text{otherwise.} \end{cases}$$

Then, we have that

$$\mu' = w\xi_1' + (1-w)\xi_2'$$

and, moreover, for all s, s' such that $\xi_1(s), \xi_1(s') > 0$ and

$$t_i\langle s\rangle = t_i\langle s'\rangle, \text{ for } i = 1 \ldots n-1$$

we have that

$$s(y_i) = s'(y_i), \text{ for } i = 1 \ldots n-1$$

and therefore, since y_n is determined by $y_1 \ldots y_{n-1}$ and $s(y_n) = t_n\langle s\rangle$,

$$t_n\langle s\rangle = t_n\langle s'\rangle$$

Now, let ξ_1 and ξ_2 be obtained from ξ'_1 and ξ'_2 by disregarding the values of y_i and joining the assignments when necessary, that is,

$$\xi_1(s) = \sum_{m_1 \ldots m_n} \xi'_1(s[m_1/y_1] \ldots [m_n/y_n])$$

and

$$\xi_2(s) = \sum_{m_1 \ldots m_n} \xi'_2(s[m_1/y_1] \ldots [m_n/y_n])$$

Then we have that

$$\mu = w\xi_1 + (1-w)\xi_2$$

and, since $w \geq r$ and t_n is determined by $t_1 \ldots t_{n-1}$ in ξ_1, this concludes the proof.

□

Using this result, it is possible to find out an expression for the value of dependence formulas:

Corollary 3

Let $=(t_1 \ldots t_n)$ be a dependence formula, an let $B_1, B_2, \ldots B_k$ be the maximal sets of assignments compatible with the corresponding dependence condition - that is,

$$s, s' \in B_i, t_i\langle s\rangle = t_i\langle s'\rangle \text{ for } i = 1 \ldots n-1 \Rightarrow t_n\langle s\rangle = t_n\langle s'\rangle$$

Then

$$V_\mu(=(t_1 \ldots t_n)) = \max_{B_i} \sum_{s \in B_i} \mu(s)$$

Proof:

First of all, it is clear that there exist only a finite number of B_i as above: indeed, since we are working in finite models there are only finitely many assignments over our free variables, and thus there are only finitely many sets of assignments.

Now, every B_i defines a decomposition

$$\mu = q_i\chi^{B_i} + (1-q_i)\bar{\chi}^{B_i}$$

where

$$q_i = \sum_{s \in B_i} \mu(s);$$

$$\chi^{B_i}(s) = \begin{cases} \frac{\mu(s)}{q_i} & \text{if } s \in B_i; \\ 0 & \text{otherwise.} \end{cases}$$

$$\bar{\chi}^{B_i}(s) = \begin{cases} \frac{\mu(s)}{1-q_i} & \text{if } s \notin B_i; \\ 0 & \text{otherwise.} \end{cases}$$

Now, by definition, $V_\mu(=(t_1 \dots t_n)) = \sup\{r : (=(t_1 \dots t_n), \mu, r) \in \mathcal{T}$, and by the above theorem we know that $(=(t_1 \dots t_n), \mu, r) \in \mathcal{T}$ if and only if there exists a linear decomposition $\mu = p\xi_1 + (1-p)\xi_2$ such that

$$\{s : \xi_1(s) > 0\} \subseteq B_i \text{ for some } B_i$$

But if this is the case, then $p \leq q_i$: indeed,

$$p \leq \frac{\mu(s)}{\xi_1(s)}, \text{ for all } s$$

Therefore, for all s we have that $\xi_1(s)p \leq \mu(s)$, and thus

$$p = \sum_{\xi_1(s)>0} \xi_1(s)p \leq \sum_{\xi_1(s)>0} \mu(s)$$

Then, since $\xi_1(s) > 0 \Rightarrow s \in B_i$,

$$p \leq \sum_{\xi_1(s)>0} \mu(s) \leq \sum_{s \in B_i} \mu(s) = q_i$$

Because of this, we have that

$$(=(t_1 \dots t_n), \mu, r) \in \mathcal{T} \text{ iff } \mu = q_i \chi^{B_i} + (1-q_i)\bar{\chi}^{B_i} \text{ for some } q_i \geq r$$

and in conclusion

$$V_\mu(=(t_1 \dots t_n)) = \sup\{r : ((= t_1 \dots t_n), \mu, r) \in \mathcal{T}\} = \max_i q_i = \max_{B_i} \sum_{s \in B_i} \mu(s)$$

□

4.4 The values of conjunctions

As we saw, it is easy to compute the value of a first-order formula ϕ or of a dependence formula $=(t_1, \dots t_n)$ with respect to a model $\mathcal{M}$ and a probabilistic team μ.
This also holds for the conjunction of a dependence formula and a first-order formula:

Theorem 12
If ϕ is a first-order formula, μ is a probabilistic team and $=(t_1, \dots t_n)$ is a dependence formula, we have that

$$V_\mu(=(t_1 \dots t_n) \wedge \phi) = \max_{B_j} \left(\sum_{s \in B_j, s \models_{FO} \phi} \mu(s) \right)$$

where $B_1, \dots B_k$ are the maximal sets of assignments compatible with the dependence condition $=(t_1 \dots t_n)$, as in the previous section.

Proof:
As we already know,

$$V_\mu(=(t_1, \dots t_n) \wedge \phi) = \inf_{p\xi_1 + (1-p)\xi_2 = \mu} pV_{\xi_1}(=(t_1, \dots t_n)) + (1-p)V_{\xi_2}(\phi)$$

Moreover,

$$V_{\xi_1}(=(t_1 \dots t_n)) = \max_{B_j} \left(\sum_{s \in B_j} \xi_1(s) \right)$$

and

$$V_{\xi_2}(\phi) = \sum_{s \models_{FO} \phi} \xi_2(s)$$

As usual, let λ_s be the fraction of $\mu(s)$ which is sent into ξ_1 in our decomposition, that is,

$$\lambda_s = \frac{p\xi_1(s)}{\mu(s)}$$

Then, as we already saw,

$$\xi_1(s_i) = \frac{\lambda_i \mu(s_i)}{p}$$

and

$$\xi_2(s_i) = \frac{(1-\lambda_i)\mu(s_i)}{1-p}$$

Now, any linear decomposition of μ in $p\xi_1(s) + (1-p)\xi_2(s)$ is determined by the choice of the values $\lambda_s \in [0, 1]$, and each such family $\{\lambda_s\}_{dom(s) = FV(=(t_1 \dots t_n) \wedge \phi)}$ corresponds to a decomposition: therefore,

$$V_\mu(=(t_1 \dots t_n) \wedge \phi) = \inf_{\{\lambda_s\}_s} \max_{B_j} \left(\sum_{s \in B_j} \lambda_s \mu(s) \right) + \sum_{s \models_{FO} \phi} (1 - \lambda_s)\mu(s)$$

Let us calculate this infimum.

If $s \not\models_{FO} \phi$, it will always be useful to put $\lambda_s = 0$: indeed, in this way their weights $\mu(s)$ will not contribute at all to the value of the expression.
If instead $s \models_{FO} \phi$, the infimum will be reached for $\lambda_s = 1$: in this way, if s is not in the maximum B_j it will not weigh on $V_\mu(=(t_1, \dots t_n) \wedge \phi)$ at all, and if it is in B_j then it would have weighed for the whole $\mu(s)$ no matter what λ_s is.

More formally, once we eliminated all assignments s such that $s \not\models_{FO} \phi$ from our calculation by letting $\lambda_s = 0$, we have that

$$V_\mu(=(t_1 \dots t_n) \wedge \phi) = \inf_{\{\lambda_i\}} \max_{B_j} \left(\sum_{s \in B_j, s \models_{FO} \phi} \lambda_s \mu(s) \right) + \sum_{s \models_{FO} \phi} (1 - \lambda_s)\mu(s)$$

Let now

$$B = \text{argmax}_{B_j}\left(\sum_{s\in B_j, s\models_{FO}\phi} \mu(s)\right)$$

that is, let B be the B_j which maximizes the above sum, and fix a family $\{\lambda_s\}_s$: then, for

$$B' = \text{argmax}_{B_j}\left(\sum_{s\in B_j, s\models_{FO}\phi} \lambda_s\mu(s)\right)$$

we have that

$$\begin{aligned}
&\max_{B_j}\left(\sum_{s\in B_j, s\models_{FO}\phi} \lambda_s\mu(s)\right) + \sum_{s\models_{FO}\phi}(1-\lambda_s)\mu(s) = \\
&= \sum_{s\in B', s\models_{FO}\phi} \lambda_s\mu(s) + \sum_{s\models_{FO}\phi}(1-\lambda_s)\mu(s) \geq \\
&\geq \sum_{s\in B, s\models_{FO}\phi} \lambda_s\mu(s) + \sum_{s\models_{FO}\phi}(1-\lambda_s)\mu(s) = \\
&= \sum_{s\in B, s\models_{FO}\phi} \mu(s) + \sum_{s\notin B, s\models_{FO}\phi}(1-\lambda_s)\mu(s) \geq \\
&\geq \sum_{s\in B, s\models_{FO}\phi} \mu(s) = \max_{B_j}\left(\sum_{s\in B_j, s\models_{FO}\phi} \mu(s)\right)
\end{aligned}$$

Since this is the case for all $\{\lambda_s\}_s$, we can then conclude that

$$V_\mu(=(t_1\dots t_n)\wedge\phi) \geq \max_{B_j}\left(\sum_{s\in B_j, s\models_{FO}\phi} \mu(s)\right)$$

On the other hand,

$$V_\mu(=(t_1\dots t_n)\wedge\phi) = \inf_{\{\lambda_s\}_s}\max_{B_j}\left(\sum_{s\in B_j, s\models_{FO}\phi} \lambda_s\mu(s)\right) + \sum_{s\models_{FO}\phi}(1-\lambda_s)\mu(s) \leq$$

$$\leq \max_{B_j}\left(\sum_{s\in B_j, s\models_{FO}\phi} \mu(s)\right) + \sum_{s\models_{FO}\phi} 0\cdot\mu(s) = \max_{B_j}\left(\sum_{s\in B_j, s\models_{FO}\phi} \mu(s)\right)$$

Thus, in conclusion,

$$V_\mu(=(t_1\dots t_n)\wedge\phi) = \max_{B_j}\sum_{s\in B_j; s\models_{FO}\phi} \mu(s)$$

and this concludes the proof.
□

So far, for every formula ϕ we considered the value $V_\mu(\phi)$ was the relative size of the greatest subteam of μ which satisfies ϕ.
However, this does not hold for conjunctions of dependence formulas: for example, let us examine the formula

$$=\!(x) \wedge =\!(y)$$

in the team

$$\mu = \begin{array}{|c|c|c|l|} \hline & x & y & \text{weight} \\ \hline s_1 & a & a & 1/3 \\ s_2 & a & b & 1/3 \\ s_3 & b & b & 1/3 \\ \hline \end{array}$$

Clearly, the relative size of any subteam of μ such that both x and y are constant is at most 1/3; however, I state that

$$V_\mu(=\!(x) \wedge =\!(y)) \geq 1/2$$

Indeed, x and y assume only two values in μ, and thus for every subteam ξ of μ it is true that

$$V_\xi(=\!(x)), V_\xi(=\!(y)) \geq 1/2$$

Because of this,

$$V_\mu(=\!(x) \wedge =\!(y)) = \inf_{p\xi_1+(1-p)\xi_2=\mu} pV_{\xi_1}(=\!(x)) + (1-p)(=\!(y)) \geq p/2 + (1-p)/2 = 1/2$$

as required.

4.5 Approximate Functional Dependency in Database Theory

The concept of functional dependency is also one of the main tools of database theory [6].

The definition of functional dependence in Database Theory corresponds exactly to Väänänen's interpretation of the dependence atomic formulas:

Definition 21
Given a relation $r \subseteq A_1 \times \ldots \times A_k$, and two attribute sets $X, Y \subseteq \{A_1, \ldots A_k\}$, we say that Y is functionally dependent from X if and only if, for all the tuples $u, v \in r$,

$$\pi_i(u) = \pi_i(v)\ \forall A_i \in X \Rightarrow \pi_j(u) = \pi_j(v)\ \forall A_j \in Y$$

where $\pi_i(u)$ is the i-th element of the tuple u.

In this case, we write that

$$r \models_{DT} X \to Y$$

This dependency relation satisfies the following *Armstrong's Axioms*:

Axiom of reflexivity: If $X \supseteq Y$, then, for all r, $r \models_{DT} X \to Y$;

Axiom of augmentation: If $r \models_{DT} X \to Y$, then, for all Z, $r \models_{DT} X \cup Z \to Y \cup Z$;

Axiom of transitivity: If $r \models_{DT} X \to Y$ and $r \models_{DT} Y \to X$ then $r \models_{DT} X \to Z$.

which are also known to be complete, in the sense that, given a set $\mathcal{F}$ of dependency conditions, the condition $X \to Y$ is derivable from $\mathcal{F}$ if and only if every relation which satisfies all dependencies in $\mathcal{F}$ satisfies $X \to Y$.

Some measures of *Approximate Functional Dependency* have been introduced, one of the most commonly used ones being the g_3 measure of Kivinen and Mannila [20] [22] [18]:

Definition 22 (g_3 measure)
Let $X \to Y$ be a functional dependency, and let r be a relation over the attribute set R.
Then $G_3(X \to Y, r)$ is the minimum number of tuples we have to remove from r in order to obtain a relation s satisfying $X \to Y$, that is,

$$G_3(X \to Y, r) = |r| - \max\{|r'| : r' \subseteq r, r' \models_{DT} X \to Y\}$$

Then, the g_3 measure is defined as

$$g_3(X \to Y, r) = G_3(X \to Y, r)/|r|$$

This definition is quite similar to the semantics of our dependence operator.
Let us formalize this intuition:

Theorem 13 *Let r be a relation over $A_1 \times \ldots \times A_n$, and let μ be the corresponding probabilistic team over $\{x_1 \ldots x_n\}$ that is,*

$$\mu(s) = \begin{cases} 1/|r| & \text{if } \langle s(x_1), \ldots, s(x_n)\rangle \in r; \\ 0 & \text{otherwise.} \end{cases}$$

Then, for all functional dependencies of the form

$$\{A_{i_1} \ldots A_{i_{q-1}}\} \to \{A_q\}$$

we have that

$$g_3(\{A_{i_1} \ldots A_{i_{q-1}}\} \to \{A_q\}, r) = 1 - V_\mu(=\!(x_{i_1}, \ldots x_q))$$

Proof:
As we already know,

$$V_\mu(=\!(x_{i_1} \ldots x_{i_q})) = \max_{B_j} \sum_{s \in B_j} \mu(s)$$

where $\{B_1, B_2, \dots B_k\}$ are all maximal sets of assignments which satisfy the dependency condition $=\!(x_{i_1}, \dots x_{i_q})$.

Therefore,

$$V_\mu(=\!(x_{i_1} \dots x_{i_q})) = \max_{B_j} \sum_{s \in B_j} \mu(s) =$$
$$= \max_{B_j} \sum \{1/|r| : s \in B_j, \langle s(x_1), \dots s(x_n)\rangle \in r\} =$$
$$= 1/|r| \max_{B_j} |\{s \in B_j, \langle s(x_1), \dots s(x_n)\rangle \in r\}| =$$
$$= 1/|r| \max\{|r'| : r' \subseteq r, s \models_{DT} \{A_1 \dots A_{q-1}\} \to \{A_q\}\}$$

where the last equivalence follows from the fact that every subset of r satisfying the dependence condition corresponds to a subset of some B_j.

In conclusion,

$$V_\mu(=\!(x_{i_1} \dots x_{i_q})) = 1 - g_3(\{A_{i_1} \dots A_{i_{q-1}}\} \to \{A_{i_q}\}, r)$$

as required.
□

5 Extensions

5.1 Game-theoretic negation

Until now, we have been only concerned with formulas in negation normal form. However, there is a fairly natural way of representing negation in game-theoretic semantics, as *player switching* - the verifier becomes the falsifier, and vice versa.

In the non-probabilistic case, this was achieved by Hodges in [16] by introducing, apart from the set $\mathcal{T}$ of the "trumps" of formulas, the set of "co-trumps" $\mathcal{C}$:

Definition 23 (Cotrumps and $\mathcal{C}$)
A set of assignments X is a cotrump *of the formula ϕ if and only if Player* **I** *has an uniform winning strategy in $H_X(\phi)$.*

Then, we define

$$\mathcal{C} = \{(\phi, X) : X \textit{ is a cotrump of } \phi\}$$

Now, we clearly have that

$$(\sim \psi, X) \in \mathcal{T} \text{ if and only if } (\psi, X) \in \mathcal{C};$$

Moreover, the following properties hold of $\mathcal{C}$:

Theorem 14

1. *If ϕ is a literal, $(\phi, X) \in \mathcal{C}$ if and only if*

$$s \not\models \phi, \text{ for all } s \in X;$$

2. *$(\psi \vee \theta, X) \in \mathcal{C}$ if and only if $(\psi, X) \in \mathcal{C}$ and $(\theta, X) \in \mathcal{C}$;*

3. *$(\psi \wedge \theta, X) \in \mathcal{C}$ if and only if $X = Y \cup Z$ for two teams Y and Z such that $(\psi, Y) \in \mathcal{C}$ and $(\theta, Z) \in \mathcal{C}$;*

4. *$(\exists x \psi, X) \in \mathcal{C}$ if and only if $(\phi, X(M/x)) \in \mathcal{C}$;*

5. *$(\exists x \backslash V \psi, X) \in \mathcal{C}$ if and only if $(\phi, X(M/x)) \in \mathcal{C}$;*

6. *$(\forall x \psi, X) \in \mathcal{C}$ if and only if $(\phi, X(F/x)) \in \mathcal{C}$ for some $F : X \to M$;*

7. *$(\forall x \psi \backslash V, X) \in \mathcal{C}$ if and only if $(\phi, X(F/x)) \in \mathcal{C}$ for some $F : X \to M$ such that*

$$s(x_i) = s'(x_i) \text{ for all } x_i \in V \Rightarrow F(s) = F(s');$$

8. *$(\sim \phi, X) \in \mathcal{C}$ if and only if $(\phi, X) \in \mathcal{T}$.*

Proof:
Instead of directly proving these results, let us observe that the following equivalencies hold in our logic:

- $\sim (\psi \vee \theta) \equiv (\sim \psi) \wedge (\sim \theta)$;
- $\sim (\psi \wedge \theta) \equiv (\sim \psi) \vee (\sim \theta)$;
- $\sim (\exists x \psi) \equiv \forall x (\sim \psi)$;
- $\sim (\exists x \backslash V \psi) \equiv \forall x \backslash V (\sim \psi)$;
- $\sim (\forall x \psi) \equiv \exists x (\sim \psi)$;
- $\sim (\forall x \backslash V \psi) \equiv \exists x \backslash V (\sim \psi$;
- $\sim (\sim \psi) \equiv \psi$.

All of these properties can be easily verified: for example, the third one holds because the game

1. Players **I** and **II** switch roles;
2. Player **I** chooses an element m for x;
3. Players **I** and **II** switch roles again;
4. The game continues as $H(\psi)$.

is clearly equivalent to the game

1. Player **II** chooses an element m for x;
2. The game continues as $H(\psi)$.

Using these results, verifying the statements of the theorem is a routine task.

For example, by the definition of game-theoretic negation

$$(\psi \vee \theta, X) \in \mathcal{C} \text{ if and only if } (\sim (\psi \vee \theta, X)) \in \mathcal{T}.$$

Now, by the first one of our equivalencies,

$$((\sim \psi \vee \theta), X) \in \mathcal{T} \text{ if and only if } ((\sim \psi) \wedge (\sim \theta), X) \in \mathcal{T}.$$

And, as was already proved,

$$((\sim \psi) \wedge (\sim \theta), X) \in \mathcal{T} \text{ if and only if } (\sim \psi, X), (\sim \theta, X) \in \mathcal{T}.$$

Finally,

$$(\sim \psi, X), (\sim \theta, X) \in \mathcal{T} \text{ if and only if } (\psi, X), (\theta, X) \in \mathcal{C};$$

and, combining all these implications, we obtain that

$$(\psi \vee \theta, X) \in \mathcal{C} \text{ if and only if } (\psi, X), (\theta, X) \in \mathcal{C}.$$

as required.
□

Now, can we use a similar technique for dealing with negation in our compositional semantics for Probabilistic Dependence Logic?
Again, $\sim \phi$ is intended to correspond to the game in which the roles of the two players are switched, and then $H(\phi)$ is played.
This justifies the following definitions:

Definition 24 (r-cotrumps and $\mathcal{C}$)
*A probabilistic team μ is a r-*cotrump *of ϕ if and only if*

$$\exists \beta \forall \tau P_I(H_\mu(\phi); \beta; \tau) \geq r$$

or, since the sum of the payoffs for Players **I** *and* **II** *is one, if and only if*

$$\exists \beta \forall \tau P_{II}(H_\mu(\phi); \beta; \tau) \leq 1 - r$$

Then, we define

$$\mathcal{C} = \{(\phi, \mu, r) : \mu \textit{ is a } r\textit{- cotrump of } \phi\}$$

Then, we have that

$$(\sim \psi, \mu, r) \in \mathcal{T} \text{ if and only if } (\psi, \mu, r) \in \mathcal{C}$$

The following properties then hold for $\mathcal{C}$:

Theorem 15

1. *If ϕ is a literal, $(\phi, \mu, r) \in \mathcal{C}$ if and only if*

$$\sum_{s \not\models_{FO} \phi} \mu(s) \geq r$$

2. *$(\psi \vee \theta, \mu, r) \in \mathcal{C}$ if and only if for every p and for every ξ_1, ξ_2 such that*

$$\mu = p\xi_1 + (1-p)\xi_2$$

there exist r_1 and r_2 such that

$$\begin{aligned} &(\psi, \xi_1, r_1) \in \mathcal{C}; \\ &(\theta, \xi_2, r_2) \in \mathcal{C}; \\ &pr_1 + (1-p)r_2 \geq r. \end{aligned}$$

3. *$(\psi \wedge \theta, \mu, r) \in \mathcal{C}$ if and only if there exist p, ξ_1, ξ_2, r_1, r_2 such that*

$$\begin{aligned} &\mu = p\xi_1 + (1-p)\xi_2; \\ &(\psi, \xi_1, r_1) \in \mathcal{C}; \\ &(\theta, \xi_2, r_2) \in \mathcal{C}; \\ &pr_1 + (1-p)r_2 \geq r. \end{aligned}$$

4. *$(\exists x\psi, \mu, r) \in \mathcal{C}$ if and only if for all functions*

$$F : \{s : dom(s) = dom(\mu)\} \to \mathcal{D}(M)$$

we have that

$$(\psi, \mu[F/x], r) \in \mathcal{C};$$

5. *$(\exists x \backslash V \psi, \mu, r) \in \mathcal{C}$ if and only if for all functions*

$$F : \{s : dom(s) = dom(\mu)\} \to \mathcal{D}(M)$$

we have that

$$(\psi, \mu[F/x], r) \in \mathcal{C};$$

6. *$(\forall x\psi, \mu, r) \in \mathcal{C}$ if and only if there exists a function*

$$F : \{s : dom(s) = dom(\mu)\} \to \mathcal{D}(M)$$

such that

$$(\psi, \mu[F/x], r) \in \mathcal{C};$$

7. $(\forall x \backslash V\psi, \mu, r) \in \mathcal{C}$ *if and only if there exists a function*

$$F : \{s : dom(s) = dom(\mu)\} \to \mathcal{D}(M)$$

such that

$$s(x_i) = s'(x_i) \text{ for all } x_i \in V \Rightarrow F(s) = F(s')$$

and

$$(\psi, \mu[F/x], r) \in \mathcal{C};$$

Proof:
These properties of $\mathcal{C}$ can be reduced to already proved properties of $\mathcal{T}$, exactly as in the non-probabilistic case.

For example, let us verify the last one: we have that

$$\begin{aligned}
&(\forall x \backslash V\psi, \mu, r) \in \mathcal{C} \text{ iff} \\
&\text{iff } (\sim \forall x \backslash V\psi, \mu, r) \in \mathcal{T}, \text{ iff} \\
&\text{iff } (\exists x \backslash V(\sim \psi), \mu, r) \in \mathcal{T}, \text{ iff} \\
&\text{iff } \exists F \text{ s.t. } (\sim \psi, \mu[F/x], r) \in \mathcal{T}, \text{ iff} \\
&\text{iff } \exists F \text{ s.t. } (\psi, \mu[F/x], r) \in \mathcal{C}
\end{aligned}$$

where in the last two formulas F is required to satisfy the dependency condition

$$s(x_i) = s'(x_i) \text{ for all } x_i \in V \Rightarrow F(s) = F(s')$$

It can also worth noticing that the "falsity condition" for backslashed existential quantification coincide with the one for the non-backslashed one.
This is because the same holds for the "truth conditions" for universal quantification: indeed,

$$\begin{aligned}
&(\exists x \backslash V\psi, \mu, r) \in \mathcal{C} \text{ iff} \\
&\text{iff } (\sim \exists x \backslash V\psi, \mu, r) \in \mathcal{T} \text{ iff} \\
&\text{iff } (\forall x \backslash V(\sim \psi), \mu, r) \in \mathcal{T} \text{ iff} \\
&\text{iff } (\forall x(\sim \psi), \mu, r) \in \mathcal{T} \text{ iff} \\
&\text{iff } (\sim \exists x\psi, \mu, r) \in \mathcal{T} \text{ iff} \\
&\text{iff } (\exists x\psi, \mu, r) \in \mathcal{C}
\end{aligned}$$

□

However, there is another way of introducing this negation in Probabilistic

Dependence Logic: indeed,

$$\begin{aligned}&(\sim \phi, \mu, r) \in \mathcal{T} \Leftrightarrow (\phi, \mu, r) \in \mathcal{C} \Leftrightarrow \\ &\Leftrightarrow \exists\beta\forall\tau P_I(H_\mu(\phi); \beta; \tau) \geq r \Leftrightarrow \\ &\Leftrightarrow \exists\beta\forall\tau P_{II}(H_\mu(\phi); \beta; \tau) \leq 1 - r \Leftrightarrow \\ &\Leftrightarrow \forall\gamma\exists\sigma P_{II}(H_\mu(\phi); \sigma; \gamma) \leq 1 - r \Leftrightarrow \\ &\Leftrightarrow \neg\exists\gamma\forall\sigma P_{II}(H_\mu(\phi); \sigma; \gamma) > 1 - r \Leftrightarrow \\ &\Leftrightarrow (\phi, \mu, r') \notin \mathcal{T} \text{ for all } r' > 1 - r\end{aligned}$$

where the passage from the third to the fourth row is justified by the Minimax theorem (or, more precisely, by the first part of its corollary).

Thus, we can dispose with $\mathcal{C}$ and just define

$$(\sim \phi, \mu, r) \in \mathcal{T} \text{ if and only if } r' > 1 - r \Rightarrow (\phi, \mu, r') \notin \mathcal{T}$$

or, by modus tollens,

$$(\sim \phi, \mu, r) \in \mathcal{T} \text{ if and only if } (\phi, \mu, r') \in \mathcal{T} \Rightarrow r' \leq 1 - r$$

5.2 Linear implication

In [2], Väänänen and Abramsky introduce a "linear implication" $\multimap$ as the adjoint of the Dependence Logic disjunction, called "multiplicative" because its team semantics coincides with that of multiplicative *conjunction* $\otimes$ in linear logic[18].

If we identify every formula ϕ of Dependence logic with the downwards-closed set A of the teams which satisfy it, so that for example

$$A \vee B = \{T_1 \cup T_2 : T_1 \in A, T_2 \in B\}$$

this means that

$$A \vee B \subseteq C \Leftrightarrow A \subseteq B \multimap C$$

This allows us to define the $\multimap$ operator as

$$A \multimap B = \{T : \forall U, U \in A \Rightarrow T \cup U \in B\}$$

Or, in the usual formulation of team semantics,

$$(\phi \multimap \psi, X) \in \mathcal{T} \text{ if and only if } (\psi, X \cup Y) \in \mathcal{T} \text{ for all teams } Y \text{ such that } (\phi, Y) \in \mathcal{T};$$

In this section, I will try to find an analogue of $\phi \multimap \psi$ for Probabilistic Dependence Logic.

[18] By definition, $T \models A \otimes B$ if and only if there exist U, V such that $T = U \cup V$, $U \models A$ and $V \models B$.

Just as, in non-probabilistic Dependence Logic, we can identify a formula with the set of its teams, we will now identify a formula ϕ with the fuzzy set of probabilistic teams A such that

$$\mu \in_\epsilon A \Leftrightarrow (\phi, \mu, \epsilon) \in \mathcal{T} \text{ for } \epsilon \in [0,1]$$

Then we have that

$$\mu \in_\epsilon A \vee B \Leftrightarrow \exists p, v_1, v_2, \xi_1, \xi_2 \text{ s.t. } \begin{cases} \epsilon = pv_1 + (1-p)v_2; \\ \mu = p\xi_1 + (1-p)\xi_2; \\ \xi_1 \in_{v_1} A; \\ \xi_2 \in_{v_2} B. \end{cases}$$

and, as we still wish $\multimap$ to be the adjunct of our disjunction,

$$(\mu \in_\epsilon A \vee B \Rightarrow \mu \in_\epsilon C) \text{ iff } (\mu \in_\epsilon A \Rightarrow \mu \in_\epsilon B \multimap C)$$

Then a semantics for the $\multimap$ connective can be found as follows:

Theorem 16
The above condition holds for

$$\xi_1 \in_{v_1} B \multimap C \textit{ iff } p\xi_1 + (1-p)\xi_2 \in_{pv_1+(1-p)v_2} C \textit{ for all } p, v_2 \in [0,1]$$
$$\textit{and for all } \xi_2 \textit{ such that } \xi_2 \in_{v_2} B.$$

Proof:

- $\Rightarrow$: Suppose that, for all μ and ϵ, if $\mu \in_\epsilon A \vee B$ then $\mu \in_\epsilon C$, and moreover suppose that $\xi_1 \in_{v_1} A$.
  Then $\xi_1 \in_{v_1} B \multimap C$, according to our semantics: indeed, for every v_2, every $\xi_2 \in_{v_2} B$ and every p we have that

  $$p\xi_1 + (1-p)\xi_2 \in_{pv_1+(1-p)v_2} A \vee B$$

  and thus

  $$p\xi_1 + (1-p)\xi_2 \in_{p_1 v_1+(1-p)v_2} C$$

  as required.
- $\Leftarrow$: Suppose that, for all ξ_1 and v_1, if $\xi_1 \in_{v_1} A$ then $\xi_1 \in_{v_1} B \multimap C$, and suppose that $\mu \in_\epsilon A \vee B$.
  Then, by definition, there exist $p, v_1, v_2, \xi_1, \xi_2$ such that

  $$\mu = p\xi_1 + (1-p)\xi_2;$$
  $$\epsilon = pv_1 + (1-p)v_2;$$
  $$\xi_1 \in_{v_1} A;$$
  $$\xi_2 \in_{v_2} B.$$

Then, since $\xi_1 \in_{v_1} A$, we have that $\xi_1 \in_{v_1} B \multimap C$; and therefore, as $\xi_2 \in_{v_2} B$, we can conclude that $\mu \in_\epsilon C$.
Thus,

$$\mu \in_\epsilon A \vee B \Rightarrow \mu \in_\epsilon C$$

as required.

□
In our standard formulation, we would say that

- $\langle \phi \multimap \psi, \xi_1, v_1 \rangle \in \mathcal{T}$ if and only if, for all probabilistic teams ξ_2 and all $p, v_2 \in [0, 1]$, if

  $$\langle \phi, \xi_2, v_2 \rangle \in \mathcal{T}$$

  then

  $$\langle \psi, p\xi_1 + (1-p)\xi_2, pv_1 + (1-p)v_2 \rangle \in \mathcal{T}$$

This definition is very similar to the one for linear implication in [2].

5.3 Dynamic Probabilistic Dependence Logic

As we verified, the value of a conjunction of dependence formulas $=(t_1, \ldots t_n) \wedge =(t'_1, \ldots t'_{n'})$ in a team μ is not necessarily the relative size of the biggest subteam of μ which satisfies both dependencies.
Because of this, it is not in general true that

$$V_\mu(\exists x \backslash \{x_1 \ldots x_k\}\phi) = V_\mu(\exists x (=(x_1 \ldots x_k x) \wedge \phi))$$

For example, let our domain M be $\{a, b\}$, and let us consider the probabilistic team

$\mu =$

	x	y	weight
s_1	a	a	1/3
s_2	a	b	1/3
s_3	b	b	1/3

and the two formulas

$$\phi = (\exists z \backslash \{\})(=(y) \wedge x = z);$$
$$\phi' = \exists z (=(z) \wedge =(y) \wedge x = z).$$

Then I state that $V_\mu(\phi) \leq 1/3$, whereas $V_\mu(\phi') \geq 1/2$.
Indeed, in the game $H_\mu(\phi)$ the function F used to choose z must be constant over the assignments in μ, and therefore it defines a probability distribution $(p, 1-p)$ over $\{a, b\}$.

Thus, $\mu[F/z]$ is

$\mu[F/z] =$

	x	y	z	weight
s'_1	a	a	a	$p/3$
s''_1	a	a	b	$(1-p)/3$
s'_2	a	b	a	$p/3$
s''_2	a	b	b	$(1-p)/3$
s'_3	b	b	a	$p/3$
s''_3	b	b	b	$(1-p)/3$

Then, let us consider the strategy for Player **I** which chooses the right conjunct $x = z$ whenever this equality does not hold, that is, on s''_1, s''_2 and s'_3, and the left conjunct $=(y)$ otherwise: then, the value of ϕ is the size of the biggest subteam of $\mu[F/z]$, not containing s''_1, s''_2 and s'_3, in which y is constant.
Thus,

$$V_\mu(\phi) \leq \max\{\mu[F/z](s'_1), \mu[F/z](s'_2)+\mu[F/z](s''_3)\} = \max\{p/3, p/3+(1-p)/3\} = 1/3$$

On the other hand, $V_\mu(\phi') \geq 1/2$: indeed, let us consider the function F defined by

$$F(s)(m) = \begin{cases} 1 & \text{if } s(x) = m; \\ 0 & \text{otherwise.} \end{cases}$$

Then we have that

$\mu[F/z] =$

	x	y	z	weight
s'_1	a	a	a	$1/3$
s'_2	a	b	a	$1/3$
s''_3	b	b	b	$1/3$

and $V_\mu(\phi') \geq V_{\mu[F/x]}(=(z)\wedge =(y) \wedge x = z)$.
Now, it is clear that it is never convenient for Player **I** to choose the last conjunct, as the condition $x = z$ is always satisfied in $\mu[F/z]$; and no matter how he chooses between $=(z)$ and $=(y)$, the average payoff will always be at least $1/2$, since for every subteam ξ of $\mu[F/z]$ we have that $V_\xi(=(z)), V_\xi(=(y)) \geq 1/2$.

Thus, the value $V_\mu(\phi')$ is at least $1/2$, and is strictly greater than $V_\mu(\phi)$: this is somewhat worrying, as it implies that the equivalence between dependence atomic formulas and slash notation is lost when we consider game values.

The culprit for this situation seems to be our semantics for the conjunction. Indeed, in the subformula $=(z)\wedge =(y)$ Player **I** is supposed to choose between verifying the condition $=(z)$ and $=(y)$, but he is not allowed to verify *both* - and because of this, the value of this subformula may be strictly greater than the size of the biggest subteam in which both x and y are constant.

This can be dealt with by considering a *dynamic* (or, better, *sequential*) conjunction $\psi \cdot \theta$, along the lines of [1], which allows Player **I** to verify both conditions if he so wishes.
Its game semantics is easy to define:

Definition 25 (Game semantics for $\psi \cdot \theta$)
If the current position of our game $H(\phi)$ is $(\psi \cdot \theta, s)$, then the next position is (ψ, s).
*If the play reaches a position (χ, s'), where χ is a literal and $s' \models_{FO} \chi$, then the play continues from the position (θ, s'); otherwise, Player **I** wins.*

However, adding this rule requires us to make a slight adjustment to the definition of uniform strategy.
Indeed, let us consider the formula

$$\phi = (x = a \vee x = b) \cdot (\exists y \backslash \{\})(x = y)$$

and the probabilistic team μ with domain $\{x\}$ which corresponds to the uniform distribution over $\{a, b\}$.

Then, the game $H_\mu(\phi)$ is as follows:

1. First, x is assigned the value a or b with equal probability;
2. Then, Player **II** chooses between the first or the second disjunct;
3. If she chooses the first one,
   - If $x \neq a$, Player **I** wins; otherwise,
   - Player **II** chooses an element m for y, *without knowing* x, and wins if $x = y$ $(= a)$.
4. If instead she chooses the second one,
   - If $x \neq b$, Player **I** wins; otherwise,
   - Player **II** chooses an element m for y, *without knowing* x, and wins if $x = y$ $(= b)$.

In other words, ϕ is supposed to be completely equivalent to the game

$$(x = a \wedge (\exists y \backslash \{\})(x = y)) \vee (x = b \wedge (\exists y \backslash \{\})(x = y))$$

and Player **II** can win with probability one by using the disjunction for signalling whether x is a or b, and by letting the value of y depend on which branch of the disjunction has been traversed.

However, with our definition of uniform strategy this is not the case, and no such strategy exists for $H_\mu(\phi)$: indeed, in the formula ϕ there exists only one instance of $(\exists y \backslash \{\})(x = y)$, and thus the uniformity condition requires that Player **II** chooses y according to the same distribution no matter which disjunction she chose before.

This is not the intended meaning: uniform strategies should be allowed to behave differently on partial plays with the same terminal positions but differently branching histories.

This justifies the next definition:

Definition 26 (Uniform strategy – new definition)
A strategy τ for Player **II** *is* uniform *if and only if, for every two partial plays $(p_1 \dots p_i)$ and $(p'_1 \dots p'_i)$ in which* **II** *uses τ, if*

- *For all $k \in 1 \dots i$, the subformula instance corresponding to p_k and the one corresponding to p'_k coincide, that is,*

$$\forall k \forall \phi, \ \exists s \ s.t. \ (\phi, s) = t_k \Leftrightarrow \exists s' \ s.t. \ (\phi, s') = t'_k;$$

  for the same instance of the subformula ϕ.

- *It holds that*

$$p_i = (\exists x \backslash \{x_1 \dots x_k\} \psi, s)$$

  and

$$p'_i = (\exists x \backslash \{x_1 \dots x_k\} \psi, s')$$

  for the same instance *of $\exists x \backslash \{x_1 \dots x_k\} \psi$;*

- *The assignment s and s' coincide over $\{x_1 \dots x_k\}$;*

then τ selects the same $c \in M$ for x in both p_i and $p'_{i'}$.

The definitions of uniform winning strategy for Player **I**, *and for behavioral strategies, are completely analogous.*

With this new definition, Player **II** has an uniform winning strategy τ for $H_\mu(\phi)$, defined by

$$\tau_2((\phi, s), (x = a \vee x = b, s)) = \begin{cases} (x = a, s) & \text{if } s(x) = a; \\ (x = b, s) & \text{otherwise;} \end{cases}$$

$$\tau_3((\phi, s), (x = a \vee x = b, s), (x = a, s), (\exists y \backslash \{\} x = y, s)) = (x = y, s[y/a]);$$

$$\tau_3((\phi, s), (x = a \vee x = b, s), (x = b, s), (\exists y \backslash \{\} x = y, s)) = (x = y, s[y/b]).$$

It is easy to see that this strategy is winning.
However, it would not be uniform, according to the old definition: indeed, τ_3 chooses two different values for y in two partial plays ending in indistinguishable (for Player II) positions.
But with the new definition, Player II may always recall which branch of the disjunction was traversed[19], and as the two rules for τ_3 apply to partial plays which differ in the formula instances corresponding to their third steps they do not contradict the new uniformity condition, as required.

[19] Although, of course, she does not necessarily remember the values the variables assumed at the previous stages of the game.

The above-presented semantics for the sequential conjunction is somewhat different from the other ones of our game, as it presupposes a *non-local* effect of the subformula $\psi \cdot \theta$ on the winning conditions of the game.
However, this can be avoided if we modify our definition of position, for example by considering positions of the form $(\phi_0, s \mid \phi_1 \ldots \phi_n)$, where $\phi_1 \ldots \phi_n$ is a sequence of formulas that Player **II** needs to verify, after verifying ϕ_0, in order to win the game.
Thus, if in the above position ϕ_0 is a literal and $s \models_{FO} \phi_0$, the next position is $(\phi_1, s \mid \phi_2, \ldots \phi_n)$, and Player **II** wins if and only if she reaches a position of the form $(\phi_n, s' \mid)$, where $s' \models_{FO} \phi_n$.

The other rules for $H_\mu(\phi)$ can be easily adapted to this variant, and the rule for $\psi \cdot \theta$ becomes simply

- If the current position is $(\psi{\cdot}\theta, s \mid \phi_1, \ldots \phi_n)$, the next position is $(\psi, s \mid \theta, \phi_1, \ldots \phi_n)$.

The following properties hold for this sequential conjunction:

Proposition 4

1. *If ψ is a first-order formula in which no quantifiers occur, then for any θ*

$$V_\mu(\psi \cdot \theta) = V_\mu(\psi \wedge \theta)$$

for all probabilistic teams μ.

2. $V_\mu((\psi \vee \theta) \cdot \chi) = V_\mu((\psi \cdot \chi) \vee (\theta \cdot \chi))$*, for all μ.*
3. $V_\mu((\psi \wedge \theta) \cdot \chi) = V_\mu((\psi \cdot \chi) \wedge (\theta \cdot \chi))$*, for all μ.*
4. *For any (either backslashed or non-backslashed) quantifier Q and any two formulas ψ and θ,*

$$V_\mu((Qx\psi) \cdot \theta) = V_\mu(Qx(\psi \cdot \theta))$$

for all μ.

The proofs of these results are a simple matter of unfolding the definitions, and will be omitted.

The last equivalence is particularly significant, as it highlights how, if we admit sequential conjunctions, the binding scope of a quantifier may extend well further than its syntactic scope: this, for example, allows us to express the celebrated *Donkey Sentence* ("A farmer owns a donkey. He beats it.") as

$$\exists x \exists y(\text{Farmer}(x) \wedge \text{Donkey}(y) \wedge \text{Owns}(x, y)) \cdot \text{Beats}(x, y)$$

where the last occurrences of x and y are bound by the corresponding existential quantifers.

Thus, the $\cdot$ operator can be seen as corresponding to the dynamic conjunction of [10].

It is a consequence of the above results that the introduction of this new connective does not increase the expressive power of our logic: any formula in which the operator $\cdot$ occurs is equivalent to one of (non-dynamic) Probabilistic Dependence Logic.
However, the next two theorems show that the sequential conjunction allows us to recover the translation between slashed quantifiers and dependence formulas, and moreover it allows us to compute a generalization of the g_3 measure for functional dependencies:

Theorem 17
It is always the case that

$$V_\mu(\exists x \backslash \{x_1 \dots x_k\}\psi) = V_\mu(\exists x (=(x_1 \dots x_k, x) \cdot \psi))$$

Proof:
By the previous results,

$$\begin{aligned}
&V_\mu(\exists x (=(x_1 \dots x_k, x) \cdot \psi)) = \\
&= V_\mu(\exists x \exists y_1 \dots y_k (\exists y \backslash \{y_1 \dots y_k\}(\bigwedge_{i=1}^{k}(x_i = y_i) \wedge x = y)) \cdot \psi) = \\
&= V_\mu(\exists x \exists y_1 \dots y_k (\exists y \backslash \{y_1 \dots y_k\}(\bigwedge_{i=1}^{k}(x_i = y_i) \wedge x = y) \cdot \psi)) = \\
&= V_\mu(\exists x \exists y_1 \dots y_k (\exists y \backslash \{y_1 \dots y_k\}(\bigwedge_{i=1}^{k}(x_i = y_i) \wedge x = y \wedge \psi)) = \\
&= V_\mu((\exists x \backslash \{x_1 \dots x_k\}\psi)
\end{aligned}$$

where the last equivalence follows from the fact that now every assignment in which the value of x differs from that of y is losing for Player **II**.
□

Finally, we have that

Theorem 18
Let r be a relation over $A_1 \times \dots \times A_n$, and let μ be the corresponding probabilistic team with domain $\{x_1 \dots x_n\}$.
Then,

$$V_\mu(=(x_{i_{1,1}} \dots x_{i_{1,q_1}}) \cdot \dots \cdot =(x_{i_{t,1}} \dots x_{i_{t,q_t}}))$$

is equal to

$$\max\{|r'| : r' \subseteq r, r' \models_{DT} \{A_{i_{1,1}} \dots A_{i_{1,q_1-1}}\} \to \{A_{i_{1,q_1}}\},$$
$$r' \models_{DT} \{A_{i_{2,1}} \dots A_{i_{2,q_2-1}}\} \to \{A_{2_{1,q_2}}\},$$
$$\dots,$$
$$r' \models_{DT} \{A_{i_{t,1}} \dots A_{i_{t,q_t-1}}\} \to \{A_{i_{t,q_t}}\}\}.$$

In particular,

$$1 - V_\mu(=(x_{i_1} \dots x_{i_q}, x_{j_1}) \cdot \dots \cdot =(x_{i_1} \dots x_{i_q}, x_{j_{q'}})) = g_3(\{A_{i_1} \dots A_{i_q}\} \to \{A_{j_1} \dots A_{j_{q'}}\}, r)$$

Proof:
By the previous results,

$$V_\mu(=(x_{i_{1,1}} \dots x_{i_{1,q_1}}) \cdot \dots \cdot =(x_{i_{t,1}} \dots x_{i_{t,q_t}})) =$$
$$= V_\mu(\exists y_{i_{1,1}} \dots \exists y_{i_{1,q_1-1}} \dots \exists y_{i_{t,1}} \dots \exists y_{i_{t,q_t-1}} (\exists y_{i_{1,q_1}} \backslash (y_{i_{1,1}} \dots y_{i_{1,q_1-1}})) \dots$$
$$\dots (\exists y_{i_{t,q_t}} \backslash (y_{i_{t,1}} \dots y_{i_{t,q_t-1}})) \bigwedge_{l,l'} (x_{i_{l,l'}} = y_{i_{l,l'}})) =$$
$$= V_\mu((\exists y_{i_{1,q_1}} \backslash (x_{i_{1,1}} \dots x_{i_{1,q_1-1}})) \dots (\exists y_{i_{t,q_t}} \backslash (x_{i_{t,1}} \dots x_{i_{t,q_t-1}})) \bigwedge_{l} (x_{i_{l,q_l}} = y_{i_{l,q_l}}))$$

Then, let us calculate this value: for the same reasons considered for the case of a single dependence formula. the optimal strategy for Player **II** will be finding the set B of assignments compatible with all the dependency relations and maximizing $\sum_{s \in B} \mu(s)$, and then selecting the $y_{i_{l,q_l}}$ in such a way that they coincide with the $x_{i_{l,q_l}}$ over B.

Once this is known, the result follows from the definition of the g_3 measure.

□

By symmetry, we can also consider a *sequential disjunction* $\psi : \theta$, with the following interpretation

Definition 27 (Game semantics for $\psi : \theta$)
If the current position of our game $H(\phi)$ is $(\psi : \theta, s)$, then the next position is (ψ, s).
*If the play reaches a position (χ, s'), where χ is a literal and $s' \not\models_{FO} \chi$, then the play continues from the position (θ, s'); otherwise, Player **II** wins.*

Again, we can simplify this definition if we define the game positions as tuples

$$(\phi_0, s \mid \phi_1, \dots \phi_n \mid \phi'_1 \dots \phi'_{n'})$$

As before, if ϕ_0 is a literal and $s \models \phi_0$, the next position is

$$(\phi_1, s, \mid \phi_2, \dots \phi_n \mid \phi'_1 \dots \phi'_{n'})$$

and if n was 0, Player **II** wins; moreover, if instead $s \not\models \phi_0$, the next position is

$$(\phi'_1, s \mid \phi_1, \dots \phi_n \mid \phi'_2 \dots \phi'_{n'})$$

and if n' was 0, Player **II** wins.

Then the semantics for the sequential disjunction becomes

- If the current position is

$$(\psi : \theta, s, \mid \phi_1, \dots \phi_n \mid \phi'_1 \dots \phi'_n)$$

  then the next position is

$$(\psi, s \mid \phi_1, \dots \phi_n \mid \theta, \phi'_1 \dots \phi'_n)$$

As in the case of the sequential conjunction, this new connective does not increase the expressive power of our logic: this can be seen either directly or by observing that

$$\psi : \theta \equiv \sim ((\sim \psi) \cdot (\sim \theta))$$

where $\sim$ is the game-theoretic negation.
Then, we can obtain a Dependence Logic formula for $\psi : \theta$ by turning $\sim \psi$ and $\sim \theta$ in Negation Normal Form, disposing of the $\cdot$ operator by using its properties and then negating the result.

Let us define formally this new form of the semantic game:

Definition 28 ($H_\mu(\phi_0 \mid \phi_1 \dots \phi_n \mid \phi'_1 \dots \phi'_{n'})$)
Let ϕ_0, $\phi_1 \dots \phi_n$, $\phi'_1 \dots \phi'_{n'}$ be formulas in NNF.

Then, the positions of the game $H_\mu(\phi_0 \mid \phi_1 \dots \phi_n \mid \phi'_1 \dots \phi'_{n'})$ are tuples $(\psi_0, s \mid \psi_1 \dots \psi_m \mid \psi'_1 \dots \psi'_{m'})$; the first position is $(\phi_0, s \mid \phi_1 \dots \phi_n \mid \phi'_1 \dots \phi'_{n'})$, where s is selected according to the distribution μ; and finally, the rules are as follows:

1. *If ψ_0 is a literal, $s \models_{FO} \psi_0$ and the current position is $(\psi_0, s \mid \psi_1 \dots \psi_m \mid \psi'_1 \dots \psi'_{m'})$, the new position is $(\psi_1, s \mid \psi_2 \dots \psi_m \mid \psi'_1 \dots \psi'_{m'})$; if instead the current position is $(\psi_0, s \mid \ \mid \psi'_1 \dots \psi'_{m'})$, Player **II** wins.*

   *Analogously, if $s \not\models_{FO} \psi_0$ and the current position is $(\psi_0, s \mid \psi_1 \dots \psi_m \mid \psi'_1 \dots \psi'_{m'})$, the new position is $(\psi'_1, s \mid \psi_1 \dots \psi_m \mid \psi'_2 \dots \psi'_{m'})$; if instead the current position is $(\psi_0, s \mid \psi_1 \dots \psi'_m \mid \ )$, Player **I** wins.*

2. *If the current position is $(\psi \vee \theta, s \mid \psi_1 \dots \psi_m \mid \psi'_1 \dots \psi'_{m'})$, Player **II** chooses whether the next position is $(\psi, s \mid \psi_1 \dots \psi_m \mid \psi'_1 \dots \psi'_{m'})$ or $(\theta, s \mid \psi_1 \dots \psi_m \mid \psi'_1 \dots \psi'_{m'})$.*

3. *If the current position is* $(\psi \wedge \theta, s \mid \psi_1 \dots \psi_m \mid \psi'_1 \dots \psi'_{m'})$*, Player* **I** *chooses whether the next position is* $(\psi, s \mid \psi_1 \dots \psi_m \mid \psi'_1 \dots \psi'_{m'})$ *or* $(\theta, s \mid \psi_1 \dots \psi_m \mid \psi'_1 \dots \psi'_{m'})$.

4. *If the current position is*

$$(\exists x\psi, s \mid \psi_1 \dots \psi_m \mid \psi'_1 \dots \psi'_{m'})$$

or

$$(\exists x \backslash V\psi, s \mid \psi_1 \dots \psi_m \mid \psi'_1 \dots \psi'_{m'})$$

Player **II** *chooses an element c of our domain, and the next position is* $(\psi, s[c/x] \mid \psi_1 \dots \psi_m \mid \psi'_1 \dots \psi'_{m'})$.

5. *If the current position is*

$$(\forall x\psi, s \mid \psi_1 \dots \psi_m \mid \psi'_1 \dots \psi'_{m'})$$

or

$$(\forall x \backslash V\psi, s \mid \psi_1 \dots \psi_m \mid \psi'_1 \dots \psi'_{m'})$$

Player **I** *chooses an element c of our domain, and the next position is* $(\psi, s[c/x] \mid \psi_1 \dots \psi_m \mid \psi'_1 \dots \psi'_{m'})$.

6. *If the current position is* $(\psi \cdot \theta, s \mid \psi_1 \dots \psi_m \mid \psi'_1 \dots \psi'_{m'})$*, the next position is* $(\psi, s \mid \theta, \psi_1 \dots \psi_m \mid \psi'_1 \dots \psi'_{m'})$.

7. *If the current position is* $(\psi : \theta, s \mid \psi_1 \dots \psi_m \mid \psi'_1 \dots \psi'_{m'})$*, the next position is* $(\psi, s \mid \psi_1 \dots \psi_m \mid \theta, \psi'_1 \dots \psi'_{m'})$.

The definitions of play, payoff, strategy, uniform strategy and behavioral strategy are as in the previous cases.

It is possible to adapt Hodges' semantics to this game, and thus to the dynamic connectives:

Definition 29 (r-trumps and $\mathcal{T}$ for Dynamic Dependence Logic)
Let ϕ_0, $\phi_1 \dots \phi_n$, $\phi'_1 \dots \phi'_{n'}$ *be formulas in NNF, and let* $r \in [0, 1]$.
Then a probabilistic team μ *is a* r*-trump of* $(\phi_0 \mid \phi_1 \dots \phi_n \mid \phi'_1 \dots \phi'_{n'})$ *if and only if there exists a uniform behavioral strategy* γ *for Player* **II** *such that, for all uniform strategies* σ *of Player* **I**,

$$P_{II}(H_\mu(\phi_0 \mid \phi_1 \dots \phi_n \mid \phi'_1 \dots \phi'_{n'}); \sigma; \gamma) \geq r$$

The set $\mathcal{T}$ *is then defined as*

$$\mathcal{T} = \{(\phi_0 \mid \phi_1 \dots \phi_n \mid \phi'_1 \dots \phi'_{n'}, \mu, r) : \mu \text{ is a } r\text{-trump of } (\phi_0 \mid \phi_1 \dots \phi_n \mid \phi'_1 \dots \phi'_{n'})\}$$

The value of a formula ϕ *in a probabilistic team* μ *is then given by*

$$V_\mu(\phi) = \sup\{r : (\phi \mid \ \mid, \mu, r) \in \mathcal{T}\}$$

The next results characterize $\mathcal{T}$:

Theorem 19
If $\mathcal{M}$ is a finite model and ϕ, $\phi_1 \dots \phi_n$, $\phi'_1 \dots \phi'_{n'}$ are formulas in NNF, the following results hold:

1. *If ϕ is a literal, then $(\phi \mid \phi_1 \dots \phi_n \mid \phi'_1 \dots \phi'_{n'}, \mu, r) \in \mathcal{T}$ if and only if there exist r_1, r_2 such that*

$$\begin{aligned}&(\phi_1 \mid \phi_2 \dots \phi_n \mid \phi'_1 \dots \phi'_{n'}, \xi_1, r_1) \in \mathcal{T};\\&(\phi'_1 \mid \phi_1 \dots \phi_n \mid \phi'_2 \dots \phi'_{n'}, \xi_2, r_1) \in \mathcal{T};\\&pr_1 + (1-p)r_2 \geq r\end{aligned}$$

where p, ξ_1, ξ_2 are defined by

$$p = \sum_{s \models \phi} \mu(s);$$

$$\xi_1(s) = \begin{cases} \mu(s)/p & \text{if } s \models_{FO} \phi;\\ 0 & \text{otherwise.}\end{cases}$$

$$\xi_2(s) = \begin{cases} \mu(s)/(1-p) & \text{if } s \not\models_{FO} \phi;\\ 0 & \text{otherwise.}\end{cases}$$

2. *If ϕ is a literal, then $(\phi, \mid \phi_1 \dots \phi_n \mid \ \mid, \mu, r) \in \mathcal{T}$ if and only if there exists a r_1 such that*

$$\begin{aligned}&(\phi_1 \mid \phi_2 \dots \phi_n \mid \ \mid, \xi_1, r_1) \in \mathcal{T};\\&pr_1 \geq r.\end{aligned}$$

where p, ξ_1, ξ_2 are defined as above.

3. *If ϕ is a literal, then $(\phi, \mid \ \mid \phi'_1 \dots \phi'_{n'}, \mu, r) \in \mathcal{T}$ if and only if there exists a r_2 such that*

$$\begin{aligned}&(\phi'_1 \mid \ \mid \phi'_2 \dots \phi'_{n'}, \xi_2, r_2) \in \mathcal{T};\\&p + (1-p)r_2 \geq r.\end{aligned}$$

where p, ξ_1, ξ_2 are defined as above.

4. *If ϕ is a literal, then $(\phi \mid \ \mid, \mu, r) \in \mathcal{T}$ if and only if*

$$\sum_{s \models_{FO} \phi} \mu(s) \geq r$$

5. *$(\psi \vee \theta \mid \phi_1 \dots \phi_n \mid \phi'_1 \dots \phi'_{n'}, \mu, r) \in \mathcal{T}$ if and only if μ can be written as a linear combination of probabilistic teams*

$$\mu = p\xi_1 + (1-p)\xi_2$$

such that, for some r_1 and r_2, the following conditions hold:

$$(\psi \mid \phi_1 \dots \phi_n \mid \phi'_1 \dots \phi'_{n'}, \xi_1, r_1) \in \mathcal{T};$$
$$(\theta \mid \phi_1 \dots \phi_n \mid \phi'_1 \dots \phi'_{n'}, \xi_2, r_2) \in \mathcal{T};$$
$$pr_1 + (1-p)r_2 \geq r$$

6. $(\psi \wedge \theta \mid \phi_1 \dots \phi_n \mid \phi'_1 \dots \phi'_{n'}, \mu, r) \in \mathcal{T}$ *if and only if for all ξ_1, ξ_2, p such that*

$$\mu = p\xi_1 + (1-p)\xi_2$$

there exist r_1, r_2 such that

$$(\psi \mid \phi_1 \dots \phi_n \mid \phi'_1 \dots \phi'_{n'}, \xi_1, r_1), (\theta \mid \phi_1 \dots \phi_n \mid \phi'_1 \dots \phi'_{n'}, \xi_2, r_2) \in \mathcal{T}$$

and

$$pr_1 + (1-p)r_2 \geq r$$

7. $(\exists x \psi \mid \phi_1 \dots \phi_n \mid \phi'_1 \dots \phi'_{n'}, \mu, r) \in \mathcal{T}$ *if and only if there exists a*

$$F : \{s : dom(s) = dom(\mu)\} \to \mathcal{D}(M)$$

such that

$$(\psi \mid \phi_1 \dots \phi_n \mid \phi'_1 \dots \phi'_{n'}, \mu[F/x], r) \in \mathcal{T}$$

8. $(\exists x \backslash \{x_1, \dots, x_k\} \psi \mid \phi'_1 \dots \phi'_{n'}, \mu, r) \in \mathcal{T}$ *if and only if the conditions for the above case hold, and moreover*

$$s(x_1) = s'(x_1), \dots, s(x_k) = s'(x_k) \Rightarrow F(s) = F(s')$$

for any two s, s' with the same domain of μ.

9. $(\forall x \psi \mid \phi_1 \dots \phi_n \mid \phi'_1 \dots \phi'_{n'}, \mu, r) \in \mathcal{T}$ *if and only if for all*

$$F : \{s : dom(s) = dom(\mu)\} \to \mathcal{D}(M)$$

it holds that

$$(\psi \mid \phi_1 \dots \phi_n \mid \phi'_1 \dots \phi'_{n'}, \mu[F/x], r) \in \mathcal{T}$$

10. $(\forall x \backslash \{x_1 \dots x_k\} \psi \mid \phi_1 \dots \phi_n \mid \phi'_1 \dots \phi'_n, \mu, r) \in \mathcal{T}$ *if and only if the conditions of the previous case hold for all F such that*

$$s(x_1) = s'(x_1), \dots, s(x_k) = s'(x_k) \Rightarrow F(s) = F(s')$$

for every two s, s' with the same domain of μ.

11. $(\psi \cdot \theta \mid \phi_1 \dots \phi_n \mid \phi'_1 \dots \phi'_{n'}, \mu, r) \in \mathcal{T}$ *if and only if*

$$(\psi \mid \theta, \phi_1 \dots \phi_n \mid \phi'_1 \dots \phi'_{n'}, \mu, r) \in \mathcal{T}$$

12. $(\psi : \theta \mid \phi_1 \ldots \phi_n \mid \phi'_1 \ldots \phi'_{n'}, \mu, r) \in \mathcal{T}$ *if and only if*

$$(\psi \mid \phi_1 \ldots \phi_n \mid \theta, \phi'_1 \ldots \phi'_{n'}, \mu, r) \in \mathcal{T}$$

Proof:
Most of these results are proved exactly as in the non-dynamic case.
However, this is not the case for the first three:

1. Suppose that the uniform behavioral strategy γ is such that, for all uniform pure strategies σ of Player **I**,

$$P(H_\mu(\phi \mid \phi_1 \ldots \mid \phi'_1 \ldots); \sigma; \gamma) \geq r$$

But in this game, an assignment s is selected, and the next position is $(\phi_1, s \mid \phi_2 \ldots \mid \phi'_1 \ldots)$ if $s \models_{FO} \phi$, and $(\phi'_1, s \mid \phi_1 \ldots \mid \phi'_1 \ldots)$ otherwise; therefore, the above payoff is given by

$$\sum_{s \models_{FO} \phi} \mu(s) P(H_s(\phi_1 \mid \phi_2 \ldots \mid \phi'_1 \ldots); \sigma'; \gamma') +$$
$$+ \sum_{s \not\models_{FO} \phi} \mu(s) P(H_s(\phi'_1 \mid \phi_1 \ldots \mid \phi'_2 \ldots); \sigma''; \gamma'') =$$
$$= p \sum_s \xi_1(s) P(H_s(\phi_1 \mid \phi_2 \ldots \mid \phi'_1 \ldots); \sigma'; \gamma') +$$
$$+ (1-p) \sum_s \xi_2(s) P(H_s(\phi'_1 \mid \phi_1 \ldots \mid \phi'_2 \ldots); \sigma''; \gamma'') =$$
$$= pP(H_{\xi_1}(\phi_1 \mid \phi_2 \ldots \mid \phi'_1 \ldots); \sigma'; \gamma') + (1-p)P(H_{\xi_2}(\phi'_1 \mid \phi_1 \ldots \mid \phi'_2 \ldots); \sigma''; \gamma'')$$

where p, ξ_1 and ξ_2 are as in the statement of the result, and σ', σ'', γ' and γ'' are the restrictions of σ and γ to the two subgames - for example,

$$\gamma'_i((\phi_1, s \mid \phi_2 \ldots \mid \phi'_1 \ldots) \ldots) = \gamma_{i+1}((\phi, s \mid \phi_1 \ldots \mid \phi'_1 \ldots)(\phi_1, s \mid \phi_2 \ldots \mid \phi'_1 \ldots) \ldots)$$

and so on.
Then, as these substrategies are still uniform for the respective games, we have that

$$P(H_{\xi_1}(\phi_1 \mid \phi_2 \ldots \mid \phi'_1 \ldots); \sigma'; \gamma') \geq r_1;$$
$$P(H_{\xi_2}(\phi'_1 \mid \phi_1 \ldots \mid \phi'_2 \ldots); \sigma''; \gamma'') \geq r_2;$$
$$pr_1 + (1-p)r_2 \geq r$$

as required.

Conversely, suppose that there exist γ', γ'', σ', σ'', r_1, r_2 as above, and let us define γ and σ as

$$\gamma_{i+1}(p_0 p_1 \ldots) = \begin{cases} \gamma'_i(p_1 \ldots) & \text{if } p_1 = (\phi_1, s \mid \ldots \mid \ldots); \\ \gamma''_i(p_1 \ldots) & \text{if } p_1 = (\phi'_1, s \mid \ldots \mid \ldots). \end{cases}$$

and

$$\sigma_{i+1}(p_0 p_1 \ldots) = \begin{cases} \sigma'_i(p_1 \ldots) & \text{if } p_1 = (\phi_1, s \mid \ldots \mid \ldots); \\ \sigma''_i(p_1 \ldots) & \text{if } p_1 = (\phi'_1, s \mid \ldots \mid \ldots). \end{cases}$$

(by our rules, the first position p_0 has a single possible successor, and thus there is no need to specify γ_0 and σ_0.)

Then, we have that

$$\begin{aligned}
&P(H_\mu(\phi \mid \phi_1 \ldots \mid \phi'_1 \ldots); \sigma; \gamma) = \\
&= \sum_s P(H_s(\phi \mid \phi_1 \ldots \mid \phi'_1 \ldots); \sigma; \gamma) = \\
&= \sum_{s \models_{FO} \phi} P(H_s(\phi_1 \mid \phi_2 \ldots \mid \phi'_1 \ldots); \sigma'; \gamma') + \\
&+ \sum_{s \not\models FO} P(H_s(\phi'_1 \mid \phi_1 \ldots \mid \phi'_2 \ldots); \sigma''; \gamma'') = \\
&= pP(H_{\xi_1}(\phi_1 \mid \phi_2 \ldots \mid \phi'_1 \ldots); \sigma'; \gamma') + \\
&+ (1-p)P(H_{\xi_2}(\phi'_1 \mid \phi_1 \ldots \mid \phi'_2 \ldots); \sigma''; \gamma'') = \\
&= pr_1 + (1-p)r_2 \geq r
\end{aligned}$$

as required.

2. This case is exactly as the first one, except that now every assignent in ξ_2 is winning for Player **I**.

3. This case is exactly as the first one except that now every assignment in ξ_1 is winning for Player **II**.

The last two results, instead, follow immediately from the semantics of the sequential conjunction $\psi \cdot \theta$ and the sequential disjunction $\psi : \theta$.
□

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