ANALYSIS OF KNOWLEDGE, ASSERTION, VERIFICATION

MSc Thesis (Afstudeerscriptie)

written by

Daniele Chiffi
(born November 19th, 1979 in Taranto, Italy)

under the supervision of **Prof.dr Martin Stokhof**, and submitted to the Board of Examiners in partial fulfillment of the requirements for the degree of

MSc in Logic

at the *Universiteit van Amsterdam*.

Date of the public defense:
August 28th, 2009

Members of the Thesis Committee:
Prof. dr Frank Veltman
Prof. dr Martin Stokhof
Prof. dr Paul Dekker

For Chiara

TABLE OF CONTENTS

§.1. *Introduction*

> One is often bewitched by a word.
> For example, by the word "know".
>
> Wittgenstein (1969, §435).

Since Plato's *Theatetus*, propositional knowledge has been a perennial topic in philosophy. Even though the concept of knowledge and its analysis present such a long tradition, there is still the presence of "an extraordinary range of existing disagreements concerning conditions of knowing that should figure in an analysis of knowing" (Shope 2002, p. 25).

According to the Platonic 'tripartite analysis of knowledge', *knowledge* is *justified true belief*. In this definition, the term truth reminds us of a realistic concept, that is, a mind-independent concept, while the justification of a belief reminds us of a mind-dependent concept[1]. The relation between mind and world is also a basic feature of the concept of assertion. This is so, since every assertion is based on an act of judgement which has to acknowledge the truth of a proposition, which speaks in itself of the world. Thus, the primary role is to linguistically express our judgements about the external world. Hence both knowledge and assertion regard the interplay between mind and world.

[1] A very fundamental distinction in epistemology (which I will not analyse further in the present work) corresponds with the dichotomy *a priori* and *a posteriori*. According to the *received view* in epistemology, a-priori knowledge is based on analytic judgements, while a-posteriori knowledge is based on synthetic judgments. In any case, Kant observes that synthetic a-priori judgements can exist and Kripke observes that a-posteriori analytic judgements can exist as well.

On the one hand, one could claim that knowledge and assertion are independent concepts, but I am very sceptical of this, since they are both propositional attitudes, belong to the same linguistic category. On the other hand, one could maintain that knowledge and assertion are concepts of the same linguistic category, a view that I agree with, since they both aim to the truth of determined propositions. The same propositions express our thoughts on the world, and they are analysed in terms of beliefs and judgements, which are mind-dependent concepts.

In the following sections I will try:

i) to clarify how knowledge can be analysed in a fallibilist and probabilistic setting so that it can be connected to the concept of assertion in order to overcome the counterexamples that any previous analysis of knowledge have presented, ii) to determine the constitutive rule(s) of the act of assertion iii) to establish the consequences of the concepts thus analysed of assertion and knowledge for the verificationist programs in (constructive) mathematics and theory of meaning (notably in the dispute between Dummett and Hintikka on the *correct* logic of verificationism).

Usually, the topic i) is mainly considered to belong to epistemology, the topic ii) to philosophy of language and iii) to constructive and/or epistemic logics. I hope that my unified view can open new horizons on these nested concepts.

Notice that, differently from the proposals of a descriptive (or naturalized) epistemology, the present work has been written having in mind a normative framework for epistemology, within which it is possible to introduce criteria of justification in order to get a rational reconstruction about the concepts of knowledge and assertion. Namely, I am interested in presenting an *explication* of these concepts in order to make sense of the paradoxes that turn out to be connected with knowledge and assertion. Thus, I will not focus too much on the common use of these terms from a descriptive (and cognitive) point of view. Nevertheless, their rational reconstruction offers a proper linguistic treatment which can clarify the ambiguities (and paradoxes) of their use in natural language. Of course, different approaches to knowledge and assertion will determine a variety of interpretations and theories connected with these notions. Only after the assumption of a possible initial framework within which analysing a notion, one can apply a determined theory that turns out to be coherent with respect to the initial framework. In this sense, every theory implies some (partially hidden) philosophical and methodological assumptions, due also to external factors, that lead and determine the object of the research[2]. If so, then there exists the problem of comparing different approaches towards similar phenomena. My proposal indicates that only a rational reconstruction of a notion can handle the minimal features that every interpretation of that notion requires. Once the rational reconstruction has been fixed as a criterion of material adequacy, one can apply a particular theory (with its philosophical and empirical assumptions) which saves the phenomena *explicated* in the rational reconstruction. Of course, there can exists cases in which there is no agreement on the minimal features of the rational reconstructions of a notion, so, only in this case the requirement of the rational reconstruction can be overcome.

Section 2 explores the problems of the analysis of knowledge and indicates my probabilistic treatment of the issue, while in Section 3 I show the validity and the limits of Williamson's account of assertion and I claim that assertions are governed by two rules of assertions, namely the *knowledge rule* and the *warrant rule*. Moreover, I show the connection between these two rules and two different tendencies in the verificationist program, that I have called *epistemic verificationism* and *pragmatic verificationism*. In case of mathematical knowledge these two tendencies require different formalisms, as it follows in the analysis of the Dummett-Hintikka dispute. In Section 4, I will be back to the dichotomy between normative and descriptive epistemology, in order to reconsider the initial claims of the present work concerning the analysis of knowledge and assertion.

§.2.0 *Knowledge and Gettier Problems*

According to the standard Platonic epistemology, we can define every "justified true belief" as "knowledge". In 1963, Gettier showed some counterexamples for this standard tripartite analysis of knowledge. I propose a probabilistic reformulation of the standard definition of knowledge in which Gettier's counterexamples do not hold anymore.

[2] In Stokhof (2006) this issue is defined as the problem of the 'choice of invariants'.

As I have indicated above, the tripartite definition of knowledge has the following structure:

Def. 1

The subject S knows φ iff

(a) φ is true.

(b) S believes that φ.

(c) S is justified in believing that φ.

The big disagreement on what knowledge is does not facilitate a complete understanding of this concept and the many counterexamples to the tripartite analysis that one can find in the literature have suggested to many philosophers to adopt one of the following ways to redefine knowledge: (i) strengthening the conditions of justification of the belief; (ii) adding a fourth condition to the analysis of knowledge; or (iii) refuting the paradigm of a possible analysis of knowledge by assuming knowledge as a primitive notion (as in Williamson 2000). For the most part, philosophers have assumed that the tripartite analysis is not sufficient to warrant knowledge, but it is necessary, hence there should be at least an extra condition (case ii)). Notice that all the attempts to analyse knowledge have incurred some counterexamples, and ironically this is the only feature that all attempts to analyse knowledge have in common. But the fact that a correct analysis has not yet proposed does not imply that it is *in principle* impossible.

Some examples of analysis of knowledge (which I consider interesting because, although they turn out to be incorrect, they are on the good track) are:

- *causal theory of knowledge*. It states that there must exist a causal relation between the fact expressed by a proposition and the beliefs of a subject (Goldman 1967). But this analysis fails in handling mathematical knowledge, since causality does not play any role in logical and mathematical knowledge.
- *conclusive reason account of knowledge*. This view states that if a proposition is false, then a subject does not present conclusive reasons for believing the truth of the proposition (cf. Dretske 1971). Conversely, this means that if one has

conclusive reasons for believing that *p*, then *p* is true. But this fact is not materially adequate, since an act expressed by a sentence can be justified by different (conclusive) reasons, which might not conclude to the truth of the sentence. Thus, the concept of 'reason' needs to be *explicated* in a rigorous way, otherwise we end up analysing knowledge with a vaguer and more difficult concept to grasp than knowledge itself. (For some counterexamples to this view see Pappas & Swain (1973)).

conditional account of knowledge. This view states that if a proposition *p* were not true, the subject would not believe that *p*, while if *p* were true, the subject would believe that *p* (Nozick 1981). But Kripke presented a convincing counterexample to the conditional account of knowledge: "Peg is looking at a red barn, but not all barns are real. Nevertheless, red barns cannot be fake, while other colours can. According to Nozick analysis of knowledge, Peg knows that there is a red barn, but she does not know that there is a barn, since if there was no barn, Peg would not believe there was. She would believe of a white fake barn that it was a barn". Even if one can claim that Nozick's conditions are relativized to particular methods or reasons, then Nozick's analysis turns out to be a sophisticated extension of Dreske's account of knowledge, even if it puts knowledge in a more dynamic framework.

In accounts of knowledge, words like "methods", "reasons", etc. occur. But these concepts cannot be easily *explicated* in a rigorous manner and new counterexamples to these analyses can be presented. I will show that the concepts of proof, verification and probability can provide a more rigorous definition of knowledge, which can overcome the epistemic counterexamples. Furthermore, these concepts connect knowing and asserting (see §3).

In continuation, I will introduce my analysis of knowledge. In my view, the key point in the tripartite definition of knowledge is the concept of "justification" in (c). I define the concept of justification of a belief as a *proof* of the truth of the propositional content of a belief[3]. Thus, we can define (c) in Def.1 in the following way:

DEF. 2

(*i*) *S* is justified in believing that φ is true iff *S* is justified in asserting φ, namely $\vdash_s(\varphi)$ is justified.

(*ii*) $\vdash_s(\varphi)$ is justified iff *S* has a *conclusive* proof that φ is true[4].

So, the subject *S* is justified in believing the truth of a proposition only if *S* has a proof of the truth of that proposition. The concept of proof requires a further analysis at this point. In fact, the concept of proof can be logical or empirical, conclusive or non-conclusive, direct or indirect and its nature depends on the particular discipline, e.g., a proof in law is not equivalent with a mathematical proof.

Logical proofs are always conclusive (with the possible exception of the family of the nonmonotonic logics), while empirical proofs can be either conclusive (verification) or non-conclusive (degrees of confirmation). A conclusive (certain) proof of φ provides the truth of φ, while a non-conclusive proof of φ is uncertain, allowing further revisions which do not conclude to the truth of the proposition, but rather presents a probabilistic basis (or confirmation degree) of φ with respect to the available evidence. Thus, a non-conclusive proof of φ is consistent with the falsity of φ.

We can distinguish two different types of justification based on proofs, notably: a *strong* justification, based on the conclusive notion of proof and a *weak* justification, based on the notion of the non-conclusive proof. There are good reasons to state that a

---

[3] Gödel's theorems do not pose a problem, since it is possible to state that there exists a *proof* that the *undecidable* sentence G is not *provable*.

[4] "$\vdash_s$" means the assertion expressed by the subject *S* obtained by a proof. Notice that the concept of justification depends on the concept of proof in my framework. I distinguish an *externalist* justification of an assertion, and the internal justification of a judgment. Both Frege and Dummett assume that the internal counterpart of an act of assertion is a judgment, that is to say the acknowledgment of the truth of a proposition.

belief can provide knowledge only if it is justified in the *strong* sense. In fact, this distinction handles the classical dichotomy between *episteme* and *doxa*. On the one hand, in the DEF. 2 I define the belief of φ in terms of the assertion of φ and this is a type of strong justification since the assertion of φ is justified on the basis of a conclusive proof of the truth of φ. On the other hand, there are also good reasons to state that the strong justification of a belief is undue and we can substitute it with the weak form of justification. In fact, apart from the analytical sentences and some synthetic phenomenal sentences (e.g. "I have a sensation of red here and now"), no synthetic sentence can be handled by a conclusive proof in an austere account of knowledge. The opposition between the holders of conclusive proofs and the holders of non-conclusive views in epistemology can respectively be traced back to Wittgenstein in (Waismann (1967)) and Carnap (1936).

According to Wittgenstein a proof can only be conclusive (verification), while according to Carnap no conclusive verification of synthetic sentences is possible, just a *degree of empirical confirmation*[5]. This raises another question though, if verification requires a conclusive proof of a proposition, then no (or hardly any) synthetic sentence can be verifiable. Thus, if one maintains a strong view on the epistemic justification, then no synthetic sentence can be verified in any case, i.e. no synthetic sentence might be known *in principle* and this is absurd. Moreover, if the concept of justification involved in Def. (1) is the strong one, then the clause (a) turns out to be redundant since a conclusive proof of φ entails the truth of φ (assuming the soundness of the proof procedures). On the other hand, if one assumes that the justification of knowledge can be based on the non-conclusive proofs, then Gettier's counterexamples hold. *These counterexamples show that the notion of weak justification allows a justified true belief*

[5] Skorupski (1997) observes that Wittgenstein's verificationism is "an operational kind of verificationism", since Wittgenstein underlines the specific *method* adopted in the verification. Namely, the knowledge of the specific method determines the *sense* of the verified sentence (different methods will determine different senses), while the Neopositivist verificationism regards the acknowledgment of the existence of the logical (or empirical) verification, independently from the specific operation followed. Hence, Wittgenstein's verificationism is connected to the knowledge of a particular verification procedure for the subject, while this is not the case for the Neopositivist verificationism.

which is not knowledge. This means that the conditions (a), (b) and (c) in Def. 1 are not jointly sufficient to convert a belief into knowledge.

The first of Gettier's counterexamples is the following one:

(1) Assume that Jones and Smith have applied for a job and Smith has high evidence that Jones will get the job. Assume also that Smith has a high level of evidence that Jones has ten coins in his pocket, then Smith believes that:

(d) Jones is the man who will get the job, and Jones has ten coins in his pocket.

But if Smith believes (d) he will also believe

(e) the man who will get the job has ten coins in his pocket.

But imagine, further, that not known to Smith, he himself, not Jones, will get the job and that Smith himself has ten coins in his pocket, but he does not know that. In this case, the following conditions hold for Smith.

(a) the sentence (e) is true
(b) Smith believes the sentence (e)
(c) Smith is justified in believing that the sentence (e) is true.

Note that (e) is true for Smith, while (d) – the sentence from which (e) is inferred is false – but Smith does not know that (e) is true in virtue of the number of coins in Smith's pocket, because Smith does not know how many coins there are in his pocket, and he bases his belief in (e) on the number of coins in Jones's pocket, whom he falsely believes to be the man who will get the job.

(2) the second of Gettier's counterexamples is the following. Assume that Smith has strong evidence that

(f) Jones owns a Ford.

From (f) Smith can infer the following statements even if he does not know where Brown is, notably:

(g) Either Jones owns a Ford, or Brown is in Boston
(h) Either Jones owns a Ford, or Brown is in Barcelona
(i) Either Jones owns a Ford, or Brown is in Brest-Litovsk.

Now, suppose that Jones drives a rented car and that by coincidence Brown is in Barcelona. Therefore, Smith does not know that (h) is true although

(a) the sentence (h) is true
(b) Smith does believe that the sentence (h) is true
(c) Smith is justified in believing that (h).

The structure of Gettier's counterexamples can be handled in the following way: assume that *S* is justified (in the weak sense) in believing that φ, on the basis of the global available evidence *E*. Suppose that S can deduce ψ from φ. But if ψ has been deduced from φ and if S is justified in believing that φ, then S is justified in believing that ψ. Assume now that φ is false (and this is possible if one maintains the weak justification, i.e. one can be justified in believing a false proposition) and ψ is true. In this case *S* has a justified true belief in ψ which does not count as knowledge (Musgrave 1993, chapter I). Gettier's specific counterexamples have the above structure: counterexample 1 is obtained by substituting in the above argument φ with P(t) and ψ with $\exists x\, P(x)$, while counterexample 2 is obtained by replacing ψ with $(\varphi \vee \psi)$.

Notice that Gettier's counterexamples make sense only if one assumes a weak sense of justification, since a conclusive proof of φ entails the truth of φ. Hence, if one holds a strong epistemic justification, then it is not possible that one can infer the falsity of φ from a conclusive proof of φ. But if one holds a weak epistemic justification, the probability of the conclusions of Gettier's cases turns out to be equal or greater than the probability of their premises.

The value of the tripartite analysis of knowledge also lies in the idea that skepticism attacks some (or all) constitutive elements of knowledge such as truth, belief, and justification that make knowledge impossible. Namely, one can be sceptical about the *source* of our *beliefs* (hypothesis of a dream, etc..), but the most important form of skepticism regards the *justification* of our beliefs (academic skepticism)[6], which can take place when some information is warranted (or defined) by even more primitive information. Nonetheless, such primitive meaningful data have to be explained and justified. Thus, there is a *regressus ad infinitum*, while every procedure of justification has to be a finite procedure. No infinite sequence of reasons in an argument can be considered to be a justification, since a justification is materially adequate if it can be controlled. Infinite sequences cannot be epistemically surveyable.

In an axiomatic system, the meaning of the primitive notions is explicated by a system of axioms which implicitly define them. If the axiomatic system is an empirical theory, then there will be some rules of correspondence between observation statements outside the theory and some expressions of the system, in order to partially interpret the system. Given this structure for empirical theories, in the case of empirical knowledge, it is more convenient to adhere to a *coherentist* and *probabilistic* view rather than to a foundationalist one, since the material adequacy of the theory with respect to the external world is obtained by the rules of correspondence.

§. 2.1 *Analysis of Knowledge, Probability and Proofs*

My attempt to solve Gettier's problem is based on the strong sense of justification and conclusive proof. As we just saw, a "classical" epistemology based only on a strong justification will end up placing all synthetic sentences outside the limit of knowledge. In

---

[6] Another ancient version of skepticism is "Pyrrhonian Skepticism", which states that nobody can come to know anything, namely knowledge is an empty concept. This is an issue of *global* skepticism.

contrast, what I suggest is a *probabilistic account of knowledge based on conclusive proofs*. I also indicate an *explication of the concept of non-conclusive proof of a proposition* φ *in terms of a conclusive proof of the higher level probabilistic statement built on* φ*,* $(pr(\varphi \mid E)) = r$, which expresses that the probability (pr) of φ, relatively to the available global evidence E, is r (a real number in the close interval [0,1])[7]. Although φ cannot be conclusively proven, a conclusive proof of the probabilistic sentence $(pr(\varphi|E)) = r$ is provided by Bayes' Theorem.

$$\mathrm{pr}(\varphi \mid E) = \frac{\mathrm{pr}(\varphi \mid \top)\, \mathrm{pr}(E \mid \varphi)}{\mathrm{pr}(E \mid \top)}$$

with the condition that $pr(\varphi \mid \top)$, $\mathrm{pr}(E \mid \varphi)$ and $pr(E \mid \top)$ are determined (where $\top$ stands for a tautology, in order to consider $\mathrm{pr}(\varphi \mid \top)$ and $pr(E \mid \top)$ as the *a priori* probability which one can assign to φ and E [8].

In this way it is possible to rephrase, in probabilistic terms, the standard definition of propositional knowledge in the following way:

Def. 3

S knows that $(pr(\varphi|E)) = r$ iff

(a) $(pr(\varphi|E)) = r$ is true

(b) S believes that $(pr(\varphi|E)) = r$

(c) S is justified (in the strong sense) in believing that $(pr(\varphi|E)) = r$

---

[7] In the final pages of Russell (1984), one can find a similar argument.

[8] The determination of the *a-priori* probability is still a very open issue, since both the objectivist and the subjectivist interpretation of probability have to face this problem. In any case, the mathematical validity of Bayes theorem is not controversial.

Of course, Def. 3 does not define the knowledge of φ, but only the knowledge of the probability of φ, once it is given the evidence *E* of *S*. From a philosophical point of view, this is meant to support the thesis that to possess the non-conclusively justified knowledge of the truth of φ is equivalent to possessing the conclusively justified knowledge of the proposition which assigns a determined probability to φ, relatively to the available global evidence. But this change of the object of knowledge entails a redefinition of the standard account of knowledge in terms of a more extensive probabilistic account of it like in Def. 3. Moreover, the knowledge of φ can be obtained from Def. 3 as a limit case. This case can be rephrased in the probabilistic language asserting that the probability of φ, relative to $\top$, is 1. Notably, from Def. 3 we get:

Def. 3'

S knows that φ iff

(a) $pr(\varphi \mid \top) = 1$ is true,

(b) *S* believes that $pr(\varphi \mid \top) = 1$,

(c) *S* is justified (in the strong sense) in believing that $pr(\varphi \mid \top) = 1$.

One can see that Def 3' is completely equivalent to (1). As I have stressed above, the condition (a) in Def 2 and Def. 3 can be ruled out because it is redundant (i.e., the concept of proof involved is conclusive). Let's consider Gettier's counterxamples again. It is easy to see that Gettier's counterexamples do not hold in Def. 3 and Def. 3', since the probability of their conclusions is equal or greater than the probability of the premises, e.g. the counterexample (1), rephrased in the probabilistic language (and applying the axioms of the probabilities):

$$\vdash_s ((P(t) \mid E) = r)$$

:

:

$$\vdash_s ((\exists x\, P(x) | E) \geq r)$$

While the counterexample (2) has the following structure:

$$\vdash_s (pr(\varphi \mid E) = r)$$

$$\vdots$$

$$\vdash_s ((pr(\varphi \vee \psi) | E) \geq r)$$

If ψ is the sentence in (2) expressing the real fact that Brown is in Barcelona, than we can state that $\vdash (pr(\psi \mid E) = 1)$. Notice that the assertion sign is used here impersonally, since it does not refer to the epistemic subject *S*. Hence, from $\vdash (pr(\psi \mid E) = 1)$, we cannot derive $\vdash_s (pr(\psi \mid E) = 1)$[9]. This step allows *S* not to derive $\vdash_s (pr(\varphi \vee \psi) | E) = 1)$. An analogous argument can be easily presented for counterexample (2).

A similar treatment of knowledge was presented by Keynes (1921, chap. II), who assigned probabilities to propositions instead of the events and considers the concept of probability as a logical and objective concept which can be formalized as a relation between the premises of an argument and the conclusion. If the inference is certain then the probabilistic relation is a logical consequence (probability 1), otherwise we can get a lower degree of probabilistic inference. Thus, no event has a probability itself, but only the sentences expressing the rational belief in the events in relation with the premises of our arguments[10]. *E.g.*, if one knows the evidence *E* and knows that $pr(\varphi \mid E) = r$, then *it is*

[9] Notice that it is very plausible to assume that the assertions of the subject *S* are only a proper subset of the possible valid assertions, i.e. there does not exist any knowing subject which knows all the valid assertions (intended as sentences proven to be truth and certain).

[10] It can be interesting to compare Keynes' point of view on probability with some propositions in Wittgenstein's *Tractatus*: "The truth of tautology is certain, of propositions possible, of contradiction impossible. (Certain, possible, impossible: here we have an indication of that gradation which we need in

rational to believe the sentence φ with the degree of probability *r*. Note that the *rational belief* in φ is justified even if φ is false, but *E* is certain and true (this is a case very similar to those presented in Gettier's counterexamples). Moreover, it is possible not to believe φ, even if φ is true but *E* is not true and certain. Thus, the knowledge of the evidence is a type of object-level knowledge and it does not assert any (non trivial) probabilistic relation, while the analysis of knowledge requires metalinguistic features depending on probabilistic inferences, i.e., this is the case when a sentence asserts the existence of a probabilistic relation[11].

If we adopt a subjectivist interpretation of probabilities we also get some interesting results. Notice that if $pr(\varphi \mid E) > pr(\neg \varphi \mid E)$ in (Def. 2), then it is rational to bet on φ and the heightening of the degree of probability of φ (given *E*) will increase our "confidence" in φ, *but this confidence in* φ *cannot lead us to the knowledge of* φ, because the knowledge of the truth of a proposition entails the assertion of the *conclusive proof* of the proposition. Nevertheless, through Bayes' Theorem we can compute the betting reasons of a proposition expressing an event and its negation evaluated according to the same evidence *E*. We can apply Bayes' Theorem to the probability of the proposition φ expressing a determined event and to the probability of $\neg\varphi$.

$$pr(\varphi \mid E) = \frac{pr(E \mid \varphi)\, pr(\varphi)}{pr(E)} \qquad\qquad pr(\neg\varphi \mid E) = \frac{pr(E \mid \neg\varphi)\, pr(\neg\varphi)}{pr(E)}$$

If we divide the respective members of the two above expressions we obtain the following expression where $pr(E)$ does not occur anymore:

---

the theory of probability" (4.464) and "[…] if *p* follows from *q*, the proposition *q* gives to the proposition *p* the probability I. The certainty of logical conclusion is a limiting case of probability. (Application to tautology and contradiction)" (5.152).

[11] See Russell (1948, section V, chap. V).

$$\frac{pr(\varphi \mid E)}{pr(\neg\varphi \mid E)} = \frac{pr(E \mid \varphi)\,pr(\varphi)}{pr(E \mid \neg\varphi)\,pr(\neg\varphi)}$$

If we call $\frac{pr(\varphi \mid E)}{pr(\neg\varphi \mid E)}$ the betting reasons (odds) (R) in favour of φ, we can write the last result in this way:

$$R(\varphi \mid E) = \frac{pr(E \mid \varphi)}{pr(E \mid \neg\varphi)} R(\varphi)$$ [12].

Despite the belief that the probability of an event is equal to 0 on the basis of the available evidence *E*, it does not entail that the subject can come to know the event after a change of the evidence (or the method of proof). Nevertheless, a low level of the empirical confirmation degree enables *S* to deny the possibility of a complete verification of the proposition expressing the event with respect to the evidence *E*. Hence, it will be rational for *S* not to believe in the proposition expressing that the event is the case, because of its low level of empirical confirmation[13].

Notice that a modification of the degree of belief in a Bayesian framework can fail if two subjects S_1 and S_2 assign (to the same evidence) a degree of confirmation equal to 0 or greater than 0. In fact, from the definition of conditional probability we have:

$$Pr(\varphi \mid E) = \frac{pr(\varphi \wedge E)}{pr(E)}$$

with the condition that $pr(E) > 0$. But now according to S_1 his degree of confirmation cannot be modified in the Bayesian manner, since he considers *E* impossible, while *E* is considered possible by S_2. But if S_2 does not *convert* S_1 in believing that $pr(E) > 0$, then

[12] $R(\varphi)$ stands for $pr(\varphi) / pr(\neg\varphi)$.

[13] We say that the evidence *E confirms* the hypothesis *h*, iff $pr(h|E) > pr(h)$; *E disconfirms h* iff $pr(h|E) < pr(h)$; *E* is *neutral* with respect to *h* iff $pr(h|E) = pr(h)$.

no modification of the belief can take place (even if we add new data) in order to achieve an agreement between them. Thus, a *conversion* is different from a simple *modification of the belief* (see Hacking 2000, §. 21)[14]. So, we can apply probabilities in epistemology only within the same framework of possibilities. A good heuristic principle in epistemology is to assign a probability different from 0 to propositions describing the events which are at least *physically* possible, and to assign a value less than 1 even to propositions describing events which turn out to be considered intuitively true in our life. In this way, the Bayesian constraints on knowledge also acquires a good philosophical justification, which leads to a fallibilist epistemology, viz. there are no secure foundations of our empirical knowledge, without leading into skepticism.

This heuristic principle also finds a mathematical explanation if we consider the principle called *Jeffrey conditionalization*. Let's assume that we assign probability to events as well. Imagine that one wants to know the value of confirmation of a hypothesis after an observation. By using a generalization of the simple rule of conditional probability, we have that the probability pr_{old} (probability of the global evidence before the observated event e) of an empirical hypothesis h is $0 < pr_{old}(h) < 1$ can be computed by Jeffrey's conditionalization (as it is presented in Williamson 2000), that is expressed in the following manner:

$(i)\ Pr_{old}(h) = pr_{old}(e)\, pr_{old}(h \mid e) + pr_{old}(\neg e)\, pr_{old}(h \mid \neg e)$

$(ii)\ Pr_{new}(h) = pr_{new}(e)\, pr_{new}(h \mid e) + pr_{new}(\neg e)\, pr_{new}(h \mid \neg e)$

Note that $pr_{new}(h)$ indicates the probability of the hypothesis h given the global evidence after the event e took place. Hence, $pr_{old}(h)$ and $pr_{new}(h)$ do not indicate respectively the a priori probability and the a posteriori one on a fixed range of possibility, but they express the empirical confirmation degree of a hypothesis with respect to the occurrence or not of the event e. Namely, we apply conditional probabilities within the same space of possible conditions by the standard rule of Bayesian conditionalization, but if the occurrence (or

[14] In any case, the treatment of knowledge in a multi-agent framework goes beyond the scope of this work. Note that in a multi-agent framework one needs a probabilistic calculus *plus* the dynamic change of information as in the Monthy Hall Problem.

not) of an event can change the space of conditions of possibilities of an empirical hypothesis, then we can appeal to Jeffrey conditionalization.

Assume that $\{e_1, e_2, \ldots, e_n\}$ is a partition such that $pr(e_i) > 0$ for each i ($1 \leq i \leq n$). Then we can compute $pr_{new}(h)$ from $pr_{old}(h)$ with respect to $\{e_1, e_2, \ldots, e_n\}$ iff h satisfies Jeffrey's conditionalization (JC), namely:

$$\text{(JC): } pr_{new}(h) = \sum_{1 \leq i \leq n} pr_{new}(e_i)\, pr_{old}(h \mid e_i).$$

If we restrict our analysis to a partition $\{e, \neg e\}$ with $pr_{new}(h) = 1$, then we get the Bayesian conditionalization as a limit case of JC. Note that if $Pr_{old}(h) = 1$ also $Pr_{new}(h) = 1$, thus JC does not work as a heuristic principle in this case.

Hence, the standard Bayesian approach assumes that we learn something from a certain set of truths, so we apply Bayes' Theorem and finally we come to know other truths, i.e. we update only our a posteriori probabilities. But in our common way of reasoning we are not always sure of our global evidence, since we tend to change opinion on the validity of our evidence, especially when we become aware of new facts that conflict or modify our system of beliefs regarding the available evidence. In this case, we cannot apply Bayes' Theorem, but we can apply JC.

There can also be the case that we know new evidence E and we want to revisit the degree of belief in the a priori probability of different hypotheses h_1, h_2.. The probability of E with respect to every particular hypothesis is called *likelihood*. Thus, $pr(E \mid h_1)$ is the likelihood of the new evidence E on h_1, while $pr(E \mid h_2)$ is the likelihood of E on h_2, etc.. The likelihoods on a partition of different hypothesis are not additive, while only the probabilities of the elements of the partition are. This lack of additivity seems to be one of the main problems of the probabilistic account of knowledge in the case of the *acceptability* of an empirical hypothesis but this fact does not affect the Bayesian epistemology intended in its dimension of *empirical confirmability*.

Hence, for all these problems on the empirical confirmation of events with probability equal to 0 and 1, it is reasonable to avoid assigning these values to the object of our empirical knowledge. If no data has probability 1, then there exists no secure foundation of the empirical knowledge. Moreover, if we explain the concept of verification as the assignment of a probability 1, then no verification (conclusive proof expressed by an assertion) of any empirical statement can take place. That is why I suggest to transform the probability of an assertion (verification) into the assertion of a (non-analytical) proposition of higher level expressing its degree of probability

Another problem regarding the applicability of the probability calculus to knowledge concerns the concept of "infinite". As a matter of fact, if an event is considered possible (namely it has a probability different from zero), then it can be proven as actually occurring if the domain of the discourse is considered to be infinite and the events of a sequence are mutually independent. If a man types randomly on a computer and we assign the probability, 0.00002 that he can write the Divine Comedy in an infinite lapse of time, then the event of writing the Divine Comedy would occur with probability 1. This fact seems paradoxical. Note that in this case, even if we assign a very low level of probability to the event of "writing the Divine Comedy", there is still a paradoxical result. Note that the concept of infinite does not conform to the concept of physical possibility, within which we can assign any value of probability different from 0. The probability calculus is a mathematical theory which needs some methodological arrangements in order to be applied to empirical knowledge. One of these arrangements consists in avoiding the use of concept of infinite in physics, which can lead, otherwise, to the Divine Comedy paradox. Once we have distinguished the physically possible events from the impossible ones, we can assign the probability in the open interval (0,1) to any event, which can be expressed by a metalinguistic sentence that 'speaks' of the (epistemic probability of an) object (or event) of our knowledge. If the object of our knowledge is itself the probability (as frequency) of an event, then our epistemic probability of it turns out to be an (epistemic) probability of a probability (expressed by a frequency) (Hacking 2000, chap. 11).

My analysis of knowledge turns out to be connected with the issue regarding the constitutive features of assertions, which can be handled by my distinction between conclusive and non-conclusive proofs and by a probabilistic setting. The following section is dedicated to such treatment of assertions.

§ 3.0. *Assertion*

> "The constant assertion of belief
> is an indication of fear".
>
> J. Krishnamurti

Although the concept of assertion has a long tradition in the history of logic (notably in the form of apophantic judgements), the modern meaning of assertion essentially traces back its origins in Frege's works. In the *Begriffsschrift*, Frege analysed the assertion sign '⊢' as consisting of two parts: the horizontal stroke, a sign showing that the content is judgeable, and the vertical stroke, a sign showing that the content is asserted. In *Begriffsschrift*, § 3, Frege considers the assertion sign as being the common predicate for all judgements, while under the footnote 7 in *Function and Object* he points out that the assertion sign has no semantic content, and this is Frege's definitive view on assertion. Writes Frege:

> "The assertion sign cannot be used to construct a functional expression; for it does not serve, in conjunction with other signs, to designate an object. '⊢2+3=5' does not designate anything; it asserts something". (Frege 1984, p. 149).

Hence, for Frege's definitive view on assertion, the assertion sign does not present any semantic role, since it cannot modify a thought, but it can only express a judgement, intended as the acknowledgement o f the truth of a thought.

As observed by Green (2002), Frege's assertion sign has an inferential significance, since in the *Begriffsschrift*, an inference occurs between assertions, not between propositions[15]. *E.g. modus ponens* (MP) assumes the following features when it happens to be analysed in terms of assertions.

---

[15] Martin-Löf was inspired by this Fregean distinction in his intuitionistic type theory.

(MP) (a) $\vdash p$

(b) $\vdash (p \rightarrow q)$

$\therefore$(c) $\vdash q$

Note that p and q are asserted respectively in (a) and (c), while in (b) they are used, respectively, as the antecedent and the consequent of a conditional. Only the whole conditional in (b) is asserted, while p and q in (b) are unasserted.

Russell (1903) observed that there is something odd in the standard account of (MP), namely in the inference: p, $p \rightarrow q$; therefore q. If 'q' means the same thing in the second premise as it does in the first premise, then the premises would seem already to contain the conclusion, while if 'q' means something different in the premise and in the conclusion, then we could incur a *fallacy of equivocation*. This problem is also known as the 'embedded problem'[16].

Frege's account of (MP) does not incur the problems raised by Russell, since assertions do not modify the semantic content, but they express the fact that the content has been judged true.

Only with Reichenbach (1947), it was clarified that the assertion sign works in pragmatic capacity, since it expresses the use of a sentence. This Frege-Reichenbach tradition on assertion met the Oxonian tradition of the philosophy of ordinary language of Austin, who translated Frege's *Foundations of Arithmetic* and was very well acquainted with Frege's works. Using Austin's terminology we can consider assertions as illocutory acts, since they regard the linguistic use of the sentence by a subject, i.e. assertions claim that a sentence holds. The type of illocutory act is determined by what Frege and Dummett call *force*, which shows the use of the semantic content, e.g. a force can indicate that a propositional content is asserted, ordered, etc.., without considering the context in which it takes place[17].

---

[16] In metaethics there exists an analogous problem regarding the possibility of moral inferences called Frege-Geach problem. See (Geach 1965).

[17] For a detailed analysis and the genesis of the concept of assertion, see Pagin (2007).

§3.1 *Constitutive Rules, Regulative Rules and Assertions*

Questions regarding the rules are a very important topic in the philosophy of law and in the philosophy of language (and epistemology). Rules can be differentiated in many ways, e.g. by their ontological status, their logical level[18], their role, targets etc.. My analysis will mainly focus on the ontological function of a rule. There is a long tradition in philosophy concerning the distinction between constitutive norms and regulative norms (for the origins of this distinction see Kant 1781/7, Quine 1948, Rawls 1955, Ross 1968, Searle 1969). Traditionally, this distinction has been viewed as asserting that constitutive norms define an object or activity which does not pre-exist to the rule itself, while regulative norms govern an activity which does exist prior to the rule. E.g. the rules of chess define the game of chess, thus they are constitutive, while the rules of traffic law govern the phenomenon of traffic, which pre-exists to any traffic law. The constitutive function of norms turns out to be equivalent to the role of constructive definitions in mathematics. In fact, according to the Platonist point of view, the definition has the role of determining a mathematical object which exists before the act of definition. So in Platonism, all definitions are non-constitutive. In contrast, a constructive definition institutes a new mathematical object, which does not exist before its definition. Hence, definitions and norms[19] can be viewed by considering their constitutive dimension which is very important from an ontological point of view.

A concept very connected with the concept of constitutive norm is that of *convention*. The concept of norm necessary entails a sanction, but this is not the case for the concept of *convention*. A sanction is externally defined and imposed, while a *convention* is something internal and contingent.

I want to analyse now the concept of convention, which will turn out to be important for the analysis of Williamson's constitutive norms of assertion. David Lewis,

[18] *E.g.* it is important to distinguish norms from metanorms in normative systems, in order to avoid juridical paradoxes.

[19] See Von Wright (1963) and Ross (1968) for an analysis of all types and functions of norms.

in his doctoral dissertation, has proposed a treatment of the concept of convention[20]. He has also offered many examples which explain how a convention can be instituted. Usually, a convention is determined by coordination games or by phenomena that pose coordination problems.

Imagine two guys that row a boat. The best way to row the boat is that of synchronizing their movements, which is a social convention, since all prefer to follow that practice if the others follow it, too. The conformity to a certain practice is an *equilibrium* of coordination in this context[21].

According to Lewis, a definition of convention is the following:

> "a regularity R in the behaviour of members of a population P when they are agents in a recurrent situation S is a convention if and only if it is true that, and it is common knowledge in P that, in almost any instance of S among members of P,
> (1) almost everyone conforms to R;
> (2) almost everyone expects almost everyone else to conform to R;
> (3) almost everyone has approximately the same preferences regarding all possible combinations of actions;
> (4) almost everyone prefers that any one more conform to R, on condition that almost everyone conforms to R;
> (5) almost everyone would prefer that any one more conform to R', on condition that almost everyone conform to R', where R' is some possible regularity in the behaviour of members of P in S, such that almost no one in almost any instance of S among members of P could conform both to R' and to R" (Lewis 1969, p. 78).

Once a convention is assimilated in a population, its members will go on to solve the problems of coordination and then they will keep following such conventions. If a shock occurs in the common state of affairs of the members of a population, the convention can

[20] See Lewis (1969).

[21] This example is due to Hume in his essay, *A Treatise of Human Nature*.

be changed[22]. *E.g.* if one wants to understand how a conventional aspect of language can come to be instituted, the following condition, which is an adaptation of Lewis' example, can be imagined. Consider two individuals A and B. They do not share a language. A utters "XYZ" when he meets B. If they want to meet again after one week the easiest thing to do is to go to the same place, because the best place to go for A is the place where B will go and, conversely, the best place to go for B is the place where A will go. If A succeeds to meet B, then also B succeeds to meet A. This would be the best strategy both for A and B. If other members of the population started following this convention, probably, "XYZ" would end up meaning something like "see you here again". In this way a linguistic convention can be abstracted from an individual expression. Of course, this is a probabilistic process and some empirical difficulties can occur. I am just interested in the fact that a convention could occur *in principle* by a uniformity of behaviour.

Let us now consider Williamson's views on conventions and constitutive rules. Williamson states that a convention must not be confused with a constitutive rule:

> "constitutive rules are not conventions. If it is a convention that one must φ; conventions are arbitrary and can be replaced by alternative conventions. In contrast, if it is a constitutive rule that one must φ, then it is necessary that one must φ" (Williamson 2000, p. 239).

This distinction is important for the connection between the concepts of knowledge and assertion (see Williamson (2000), notably chapter 11). Williamson (2000) claims that i) knowledge is a primitive notion, that cannot analysed in other more primitive constituents

[22] Note that not all the problems of coordination can be solved referring to conventions. Imagine a situation where there are two individuals. We promise them that they can get a great sum of money if they choose the same number, but they cannot communicate with each other. In 40% of the cases, both individuals choose the number 1 (see Schelling 1960). Of course, this choice does not depend on the rationality of the individuals, but rather on aesthetic, accidental, or social issues (known as *focal points*) which can give reasons for simple collective choices. If a focal point becomes common knowledge, namely the probability of choosing the number 1 becomes close to 1, this fact can suffice to convert a focal point into a convention.

and ii) "only knowledge warrants assertion". Williamson's account of assertion has been criticized by many philosophers and many of them have tried to show the inadequacy of his analysis[23]. In my view, Williamson's account of assertion needs to be integrated with another constitutive rule in order to overcome all the objections raised against Williamson's analysis. Namely, due to the fact that he does not give a conclusive argument for the idea that there exists one and only one constitutive rule for assertion.

My probabilistic interpretation of the tripartite analysis of knowledge solves Gettier's type problems; so that knowledge can be viewed as something complex, composed of other notions like truth, belief, etc.. Thus, one is not obliged to consider knowledge as a primitive notion. The possibility to deal with knowledge as a complex term is more adequate, since it correlates other notions like truth and belief which can be defined without any reference to the concept of knowledge. *E.g.* the concept of validity of a formula requires the concept of truth (given the assumption of a semantic explication of validity), but the acknowledgement of the validity is an epistemic issue. Consider the rule that states that from $\varphi \rightarrow \psi$ one can infer $\neg\psi \rightarrow \neg\varphi$. The validity of such logical rule can be determined by truth-tables, hence the concept of truth has a primitive role for the concept of validity of a formula. Nevertheless, if one *knows* the meaning of $\varphi \rightarrow \psi$, then it does not follow that $\neg\psi \rightarrow \neg\varphi$ is inferable. In fact, one can not know this logical rule. Hence, the set of valid formulae that one knows is a subset of the valid ones[24]. So, it is more adequate to consider the concept of truth as primitive, and on the other side, the concept of knowledge as a derivate and more complex notion. The only case when knowledge has to be considered as something primitive is when we talk about the evidence of our epistemic (and probabilistic) arguments. If we did otherwise we could fall in a *regressus ad infinitum* when trying to justify the evidence with other evidence, etc..

[23] In any case, a knowledge rule for assertion is also supported by DeRose and Unger. Similar claims can be found in Moore too.

[24] This fact depends on the *explication* of "meaning": in an inferentialistic semantics, knowing the meaning *is* knowing which inferences it validates. Cf. Brandom (1994, chap. 2).

Let us consider the concept of assertion, for which I present a similar argument. The notion of assertion I refer to in this work is the one that does not require a contextual dimension. The distinction between assertions in a context and the *constitutive norms* of assertions is clarified in (Stalnaker 2006)[25]:

> "There are two ways of approaching the task of giving an account of a speech act such as assertion, both of which have their roots in J. L. Austin's work on speech acts. Speech acts obviously alter the situation in which they take place, and one might try to explain what it is to make an assertion by saying how it changes, or is intended to change, the context. Alternatively, one might characterize assertions in terms of the way they are assessed. Speech acts are generally assumed to be moves in a rulegoverned institutional practice, and one might focus on the constitutive norms that constrain the practice. A speech act might be successful in the sense that it succeeds in changing the context in the way that assertions are intended to change the context, but still be defective in some way– still be an assertion that failed to meet some standard or norm that assertions are supposed to meet".

I consider the concept of assertion as being connected with the concepts of proof or verification (or empirical confirmation). My claim is that the concept of knowledge can turn out to be connected to the concept of assertion. Namely, if a subject *S* asserts φ, this means that *S* has a proof (or a degree of empirical confirmation) that φ holds. But *S* can still believe that the method of verification (or empirical confirmation) is not tenable or adequate for concluding to the truth of φ. Thus, even if φ is true and one has a proof (or degree of empirical confirmation) for φ, one can still believe that the method of verification (or empirical confirmation) is insufficient since it does not warrant the knowledge of φ for *S*.

[25] A classical paper on the contexualist point of view about assertions is Stalnaker (1978).

§.3.2 *Rules of assertion*

In this section I maintain that assertions are governed by two constitutive rules: the *warrant rule*, which says that one can perform an assertion if one has a proof (or verification or empirical confirmation) of the content of the assertion and the *knowledge rule* which says that one can perform an assertion if one knows the content of the assertion. Williamson (2000) maintains that assertion is governed only by the knowledge rule. This is Williamson's thesis (WT). But many recent papers have showed that some objections can be given to (WT). Thus, I will try to replace (WT) with the following thesis that I name two-rule thesis (2RT): *the warrant rule* and the *knowledge rule* are jointly the necessary and sufficient conditions for a complete *explication* of the concept of assertion.

Many works on assertion are based on Williamson's analysis presented in Williamson (2000), a turning point for the linguistic and philosophical treatment of assertion. (WT) poses some problems for a complete analysis of assertion and the aim of this section is to introduce a new analysis that can overcome these problems. First of all, I introduce Williamson's point of view on assertion and, secondly, I present (2RT) in order to show the validity and the limits of Williamson's analysis, by indicating a proper treatment for the *explication* of the concept of assertion

Williamson states five features of assertion:

1) a constitutive rule is essential to the constituted act.
2) the constitutive rule for assertion takes the form of a C-rule: One must: Assert *p* only if *p* has C.

There are five possible C-rules for assertion:

2a) *Truth rule* (T): One must: assert *p* only if *p* is true.
2b) *Warrant rule* (WR): One must: assert *p* only if one has warrant to assert *p*.
2c) *Knowledge rule* (KRA): One must: assert *p* only if one knows *p*.

2d) (*BK) rule*: One must: assert *p* only if one believes that one knows *p*.

2e) (*RBK)* rule: One must: assert *p* only if one rationally believes that one knows that *p*.

3) the default use of declarative sentences is to make assertions.
4) when one breaks a rule of assertion, one does not thereby fail to make an assertion.
5) "in mastering the speech act of assertion, one implicitly grasps the C-rule, in whatever sense on implicitly grasps the rules of a game in mastering it" (p. 241)[26].

Williamson considers (KRA) as being the *only* correct constitutive rule for assertion. So, (WT) is based on (KRA) and it indicates that the *necessary* and *sufficient* condition for an assertion is given by the knowledge of *p*, while (2RT) assumes that the necessary and sufficient conditions for an explication of the concept of assertion are jointly given by (WR) and (KRA)[27].

The justification of (KRA) is the main problem that I want to consider at the moment. The justification of a norm can be *formal* or *material* (Bobbio 1958). A rule is *formally justified* if it is *valid* with respect to the normative inferences of a system of norms (namely, if the consequences of a norm do not lead to any contradiction with respect to the other valid norms of the normative system), while a norm is *materially justified* if it is adequate with respect to some basic (intuitive) principles that we want to handle by that norm within a normative system. So, the relation of formal justification is about norms, while the material justification is about norms and (intuitive) principles. For Williamson, the justification of (KRA) is only material, since there are no other

[26] I will explain all these issues in the next pages.

[27] Brown (2008) observes that Williamson seems to support a weaker reading of (KRA) in other parts of the same book. Namely, (KRA) is assumed to be only a necessary condition or only a sufficient condition for the assertion.

constitutive norms which govern assertion[28]. According to Williamson, (KRA) merely determines the possible set of admissible assertions. In the case of (2RT), we need not only a material justification but also a formal justification for both rules, since (WR) and (KRA) needs not to conflict in a system of norms for assertion.

Williamson assumes that (KRA) is materially justified, since it handles the problem of epistemic Moore's paradox sentences and the knowledge in the context of lotteries (I will explain such a context later on), but he does not explain why the C-rule is the *only* rule of assertion (Brown 2008).

I will argue that (KRA) is not the only rule (with constitutive function) which we need in order to make sense of the epistemic versions of Moore's paradox and knowledge in the cases of lotteries. Namely, (KRA) has to be integrated with (WR). An epistemic version of Moore's sentences is the following:

(EM) I assert p, but I do not know p.

The sentence (EM) is very important in the analysis of assertion. (EM) is also named as the epistemic Moore's paradox. The term 'paradox' implies the idea that (EM) can turn out to be odd in some circumstances, namely it can conflict with a common belief without leading to a logical contradiction. Instead, if a sentence and its negation logically hold and lead to a contradiction, then we face an antinomy.

In order to justify (KRA), Williamson also considers the case of the lotteries, in which there is only one winning ticket among 1000 tickets. If one buys a ticket, one cannot assert anything of his ticket, before the result of the lottery is publically given. Hence, if p stands for winning at the lottery, and $\neg p$ for losing at the lottery, then one can assert conclusively p or one can assert $\neg p$ only after one knows the result of the lottery and this fact should confirm (KRA). Despite this fact, I argue that one can conclusively assert the sentence $pr(p \mid E) = r$, with r equal to the probability 1 on 1000 for

[28] Notice that the normativity expressed by the constitutive function of (KRA) "is not moral or teleological", but it simply states the conditions of possibility for the existence of an activity. Notably, the constitutive function of (KRA) is directed towards *theoretical aims*.

winning at the lottery. Moreover, one can conjecture p, since the assertion $\neg p$ is not justified[29] (and one can also conjecture $\neg p$, since p is unjustified). If one is lucky at the lottery, the simple conjecture p can be transformed into the assertion of p, since a verification of p exists.

Another key concept for my analysis is the concept of non-conclusive assertion. An assertion p is non-conclusive if it is based on a non-conclusive proof such that $pr(p \mid E) > ½$ and $\neq 1$, namely a non-conclusive assertion of p does not conclude to the truth of p. In any case, it is important not to confuse a non-conclusive assertion with a conjecture, since they have different conditions of justifications.

Williamson holds that (KRA) is a *constitutive rule*, while I prefer to use the expression "constitutive function of a rule", in order to distinguish the type of rules from their aims[30]. He writes that "constitutive rules do not lay down necessary conditions for performing the constituted act" (Williamson 2000, p. 240), because even when one breaks a rule of assertion, one does not fail to make an assertion. Moreover, Williamson clarifies that a violation of a constitutive rule does not depend on any grammatical mistake. Notice that in a formal language the constitutive function of a rule determines the necessary and sufficient conditions for the assertion of a sentence, while in natural language or in a game there is "some sensitivity to the difference between conforming the rule and breaking it" (*Ibidem*).

Let us consider the first rule for assertion, namely the *truth rule* (T):

(T) One must: (assert p only if p is true).

In my view, (T) cannot be the only norm of assertion. (T) is not sufficient to warrant assertion, since there can exist Gettier's contexts, in which a condition turns out to be true by chance, not by a proof (or verification). Nevertheless, (T) is a necessary requirement for the validity of a conclusive assertion.

[29] In fact, the conjecture of p is justified if and only if the assertion of its negation is unjustified, namely there is no proof of $\neg p$. See (Bellin & Biasi (2004)).

[30] Salmon (1963) made the same distinction in case of definitions.

Williamson also considers the following possible rule for assertion, which he names Warrant Rule (WR)

(WR) One must: assert *p* only if one has warrant to assert *p*.

According to Williamson, (WR) handles an antirealist account of assertion, since it collapses the concept of truth with the concept of warranted assertion. I consider (WR) as being ambiguous, since it can be interpreted in two different ways:

(WR1) One must: assert *p* if there exists a conclusive proof (or verification) of *p*.

(WR2) One must: assert *p* if and only if there exists a (conclusive or non-conclusive) proof (or verification) of *p*.

Notice that only (WR1) implies (T), since a conclusive proof of a proposition implies the truth of the same proposition in any case, while (T) does not follow from (WR2), since a non-conclusive proof might not conclude to the truth of the proposition. Note that from (WR)[31] it is possible to construct a new Moore-type sentence (WM).

(WM): I assert *p*, but the assertion *p* is not warranted.

If we assume (WR1), then (WM) is contradictory. In fact, "I assert *p* (conclusively), but the assertion *p* is not warranted" is contradictory. This is not the case, if we apply (WR2) to (WM). Namely, "I assert *p* (non-conclusively), and the assertion *p* is not warranted" is a plausible sentence, since a non-conclusive assertion of *p* might not conclude to the truth of *p*.

Let us consider again (KRA). We saw that if we apply (KRA) to (EM), we get a contradiction, since it is not possible to assert *p*, without knowing *p*. But if we interpret (EM) by (WR) (both WR1 or WR2), then we do not get a contradiction, since a subject can assert *p* (having a proof or verification of *p*) without having knowledge that *p*.

[31] From now on, when I refer to (WR) I mean both the interpretation (WR1) and (WR2).

Namely, the subject can imagine that the proof is not adequate or does not suffice to achieve an optimal level of epistemic acceptability. *Thus, the knowledge rule of assertion* (KRA) *does not suffice to be the only rule which constitutes the assertions. This is so since we also need the warrant rule of assertion* (WR), *in order to make sense of all the Moore-type sentences available by the concepts of assertion and knowledge.*

In this sense, the concept of proof is more primitive than the concept of knowledge and a true belief can be considered justified (tripartite analysis of knowledge) only if a subject *S* has a proof of the truth of the propositional content of the belief. This fact explains why (WR) has a primary role in (2RT). By applying (WR) to (EM), we assume that what is known has to be justified by a proof. By contrast, Williamson assumes that knowledge is a primitive notion that does not need to be justified by a proof. For this reason (WT) is based only on (KRA), while applying (WR) to an epistemic sentence means to consider knowledge as being justified by a proof. In this way, (WR) can explain why there are contexts in which one can assert *p*, without knowing *p*, since *S* can assert *p* only if he has a proof of the truth of *p*, but *S* cannot know *p*, since he does not believe *p*. If so, *S* does not accept the belief condition for the tripartite analysis of knowledge. Hence, the tripartite definition of knowledge and the problem regarding the norms of assertion are two nested issues, which can receive a proper treatment in (2RT), but not in (WT), because (WT) cannot determine all these important epistemic and pragmatic distinctions.

It is possible to construct other Moore-type sentences in which the first conjunct is known, while the second in not warranted, e.g. “I know *p*, but I do not have warrant to assert *p*”. If we apply (KRA) to this Moore-type sentence, we get “I assert *p*, but I do not have warrant to assert *p*”, which is (WM).

It is also possible to consider only conclusive assertions, assuming (WR2) as a derivate rule if we enrich our language with probabilities; i.e. when we interpret (WM) by (WR2), it follows that from a non-conclusive assertion of *p*, it is not possible to get the warranted (conclusive) assertion of *p*. In fact, if we transform a non-conclusive assertion of *p* into a conclusive assertion of a higher-level sentence expressing the probability *r* of *p* (given the evidence *E*) different from 1, then we can assert conclusively $pr(p \mid E) = r$, but we cannot conclusively assert *p*. From a philosophical point of view, I prefer to handle Moore-type sentences with (WR1) and probabilities, since it is harder to justify a mere non-conclusive assertion, while the justification of every conclusive assertion (on a probabilistic statement) is more directly determined by a proof. Nevertheless, from a technical point of view, the two alternatives are equivalent.

The tripartite analysis of knowledge as justified true belief suggests the concept of non-conclusive assertion, otherwise the condition of truth would be redundant, due to the fact that the (strong) justification of a sentence by a conclusive proof implies the truth of that sentence. Williamson accepts (KRA), since he considers knowledge as a primitive concept, while the standard tripartite analysis of knowledge requires (KRA) and (WR1) *plus* a probabilistic setting (if we allow only conclusive assertions on probabilistic sentences) or (KRA) *plus* (WR2) in case we do not want to use probabilities and we want to generically speak of non-conclusive assertions. These subtle distinctions are capable of making sense of the objections moved against (WT). There are many contexts in which the constitutive norm of assertion is not (KRA), i.e. when one can assert *p*, without knowing *p* (see Weiner 2005, Lackey 2007). (2RT) explains why these contexts can exist by assuming both (WR) and (KRA) for a proper treatment of all assertion-contexts. I show now how (2RT) can deal with the contexts in which one asserts a proposition without knowing that proposition. E.g., the creationist teacher's example in Lackey (2007) can be understood by my analysis of knowledge and assertion. Lackey's example runs in this way:

> "Stella is a devoutly Christian fourth-grade teacher, and her religious beliefs are grounded in a deep faith that she has had since she was a very young child. Part of this faith includes a belief in the truth of creationism and, accordingly, a belief in the falsity of evolutionary theory. Despite this, Stella fully recognizes that there is an overwhelming amount of scientific evidence against both of these beliefs. Indeed, she readily admits that she is not basing her own commitment to creationism on evidence at all but, rather, on the personal faith that she has in an all-powerful Creator. Because of this, Stella does not think that religion is something that she should impose on those around her, and this is especially true with respect to her fourth-grade students. Instead, she regards her duty as a teacher to include presenting material that is best supported by the available

> evidence, which clearly includes the truth of evolutionary theory. As a result, while presenting her biology lesson today, Stella asserts to her students, "Modernday Homo sapiens evolved from Homo erectus," though she herself neither believes nor knows this proposition" (p. 599)[32].

This example can be associated to my analysis of (EM) by (WR1). Notably, the scientific proof of evolutionism cannot overcome Stella's personal certainties. Hence, Stella's certainties are pre-epistemic and concern the conditions of possibility for the justification of her judgements. In Stella's framework of beliefs, the possibility that there exists negative evidence for her non-evolutionistic point of view is ruled out. But, as Wittgenstein pointed out, knowledge is connected with its opposite counterparts like doubt, negative evidence. There is no knowledge if there is no possibility (in principle) that the negation of our beliefs does not make sense at all. Hence, Stella's beliefs are not epistemic, since their negations do not make sense for her, even when there exists a scientific proof that shows the contrary.

Another example is the following (Weiner 2005, p. 231):

> "Sherlock Holmes and Doctor Watson are brought to a crime scene. Holmes scans the scene and says (truthfully, as it turns out) "This is the work of Professor Moriarty! It has the mark of his fiendish genius".
> Holmes, at this point, has not found any evidence (in the criminal rather than epistemological sense) incriminating Professor Moriarty, but he is sticking his neck out based on his sense of what Moriarty's crimes are like. Intuitively, […] Holmes does not know what he asserts, even if his assertion turns out to be true".

Sherlock Holmes' example turns out to be plausible if we interpret (EM) with (WR2). Namely, Sherlock Holmes might only make a non-conclusive assertion, which can turn out to be either true or false. If the assertion is recognized to be true after an adequate proof

---

[32] A similar example was presented by Wittgenstein. Imagine a railroad announcer, who says: "Train no .. will arrive at .. o'clock. Personally I don't believe it" (Wittgenstein 1974, §486).

(e.g. a DNA-test), we are in a Gettier-style context. We cannot apply (KRA) in such contexts; or else we would fall into a contradiction. Despite the fact that the contexts indicate which rules have to be applied, my analysis is not essentially contextualist. As a matter of fact, a complete understanding of a particular context is not required, just a merely sufficient one in order to acknowledge if an assertion is conclusive or not in a determined context. This is useful in order to predict the linguistic behaviour of the concepts of assertion and knowledge[33].

Williamson considers also the following possible constitutive rules that govern assertions.

(BK) One must: assert *p* only if one believes that one knows *p*.

(RBK) One must: assert *p* only if one rationally believes that one knows *p*.

I will show that both (BK) and (RBK) can be handled by (KRA) and (WR), i.e. (BK) and (RBK) are complex rules, which make sense in my analysis too. As a matter of fact, if (KRA) is applied to the first conjunct in (BK), we get "Assert *p* only if one believes that one knows *p*". If we apply (KRA) to the second conjunct of this Moore-type sentence we get: "Assert *p* only if one believes that one asserts *p*"[34]. From the previous sentence, we can obtain a Moore-type sentence

[33] For a contexualist analysis of the rule of assertion, see DeRose (2002). For important critical remarks to DeRose's contextualism, see Bach (2008).

[34] Notice that the concept of assertion involved in the second conjunct does not stand for its pragmatic illocutory use, but it is a semantic description of an act of assertion. That is why it can stand for the semantic content of a belief. So, in (BK) and (RBK) there are no illocutory assertions as in the other rules of assertions but descriptive propositions expressing acts of assertions. The same distinction occurs in philosophy of law between the logic of descriptive propositions about norms (which can be only true or false as in deontic logic) and the logic of norms itself (in which norms are not true or false but justified or unjustified). Thus, the 'assertion of a belief' is about the pragmatic illocutory act of assertion and its propositional content is a proposition expressing a belief, while a 'belief of an assertion' corresponds to a descriptive statement in which there is no illocutory function for the assertion. Furthermore, notice that the belief of a (conclusive) assertion *p* means that I believe that I have a proof that *p* is true, while the belief of

(BK-WM) "I believe that I assert *p*, but the assertion *p* is not warranted".

If we apply (WR1) to (BK-WM), we get: "I believe that I assert *p* (conclusively), but the assertion *p* is not warranted". This is a plausible sentence, since a belief can turn out to be false, but the subject can still hold the belief. Moreover, if we apply (WR2) to (BK-WM), we get "I believe that I assert *p* (non-conclusively), but *p* is not warranted". Also in this case, the resulting sentence has a plausible meaning. Thus, (BK) does not depend too much on the type of proof expressed in the assertion, and does not imply (T), which is necessary for the analysis of conclusive assertions. Hence, (BK) can be analysed by (KRA) and (WR), but cannot be considered to have a constitutive function, since (BK) is a more complex rule with respect to (KRA) and (WR). If one does not want to apply (WR2) to (BK-WM), we can apply (WR1) to (BK-WM) *plus* a probabilistic framework in order to handle non-conclusive assertions. In this case the belief turns out to be *rational*, since the subject believes that $pr(p \mid E) \leq 1$ in the first conjunct of (BK-WM), and in the second conjunct $pr(p \mid E) < 1$, while, in case of a conclusive assertion for the first conjunct, we have that the belief turns out to be *non rational*, since the subject believes that $pr(p \mid E) = 1$, while in the second conjunct $pr(p \mid E) < 1$.

The last rule considered by Williamson is the (RBK) rule:

(RBK): One must: assert *p* only if one rationally believes that one knows that *p*.

If we apply (KRA) to (RBK) we get:

"Assert *p* only if one rationally believes the assertion of *p*". From the previous sentence I can introduce a new Moore-type sentence:

---

p means that I believe *p* without having a proof of *p*. In any case, one can be in a Gettier context, in which the content of the believe is true by chance, not by a proof. Thus in this Gettier context, the belief of a (conclusive) assertion *p* is not justified since we do not possess a proof of *p*, while the mere belief of *p* can be justified, since a proof of *p* is not required in that case.

(RBK-WM) “I rationally believe that the assertion of p, but the assertion p is not warranted”.

I state that the assertion p can be rationally believed if $pr(p \mid E) > pr(\neg p \mid E)$. If we handle (RBK-WM) with (WR1), we would fall into a contradiction, since “I rationally believe the assertion of p (conclusively), but the assertion p is not warranted” is paradoxical. Given that the probability of a conclusive assertion is 1, then I have to rationally believe p, but this is contradictory with the second conjunct of (RBK-WM). If we interpret (RBK-WM) by (WR2), then we get a plausible sentence, namely, “I rationally believe that I assert p (non-conclusively), and the assertion p is not warranted”. If $pr(p \mid E) > pr(\neg p \mid E)$, then the assertion p can be rationally believed, even if this fact cannot warrant p. Hence, even (RBK) can be handled by (KRA) and (WR).

Williamson observes about the nature of (BK) and (RBK):

> “Suppose that I rationally believe myself to know that there is snow outside; in fact, there is no snow outside. On the (BK) and (RBK) accounts, my assertion ‘There is snow outside’ satisfies the rule of assertion. Yet something is wrong with my assertion, neither the (BK) nor the (RBK) account implies that it is. They can allow that something is wrong with my belief that I know that there is snow outside, for it is false, but that is another matter. The (BK) and (RBK) accounts lack the resources to explain what we regard the false assertion itself, not just the asserter, as faulty”[35].

[35] (Williamson 2000, p. 262). In my analysis, all the objections for the possibility of having rules with a constitutive function for assertion are overcome if one assumes a normative point of view on knowledge and assertion. *By contrast*, let us consider one that holds a non-normative view. Cappelen argues that:

1) “*it does not exist at all what philosophers call “assertion”*. This no-assertion view is based on the idea that assertions are not illocutory acts, and he presents (only in some points of his article) assertions as being simple locutory acts. In my view, assertions are illocutory acts, since their use is to acknowledge the truth of a proposition. A simple locution (an utterance) is connected with one of the many possible illocutory acts.

2) “*we do not play the game of assertion (as intended by Williamson), since linguistic rules and the rules of a game behave differently in counterfactual contexts*”. Namely, we could say that the game of tennis would

(2RT) shows why (BK) and (RBK) lack these resources. The distinctive feature of (2RT) is that it overcomes all the objections moved toward (WT) and allows to handle (BK) and (RBK) as well. Hence, (BK) and (RBK) are not constitutive rules, since they can be handled by more simple rules like (WR) and (KRA). Moreover, (T) is considered in (2RT) as a necessary but not sufficient condition for the explication of the concept of assertion, while in both (BK) and (RBK) the rule (T) is not even necessary and (2RT) can be considered as a proper treatment of assertions, since (2RT) handles the lottery contexts and Moore-type sentences as well as (WT)[36], but (2RT) does not incur the objections that (WT) raises. For all these reasons (2RT) is more adequate in providing an explication of the concept of assertion than (WT) does.

---

cease to exist if we changed some basic rules, while this is not the case if one accepts a different constitutive rule for assertion. But this fact can be viewed as an objection to Williamson, not to the existence of assertions. Dummett (2006, chapter I) also observes that different languages can be translated one other, in a way that games cannot.

3) "*it is not possible to break the rule of assertion*". In any case, the assertion of a sentence which involves the performance of a physically impossible act cannot be justified. *E.g.*, if one asserts from our planet: "I saw the dark side of the moon in the sky with my eyes", then we get the idea that there is something odd in the assertion. As a matter of fact, an assertion claims that an act X holds, but if X is physically impossible, then also the assertion about X is unjustified. So if one asserts X, then one breaks a rule of assertion.

4) "*assertions are not so frequently attributed, and there is no word in natural language whose default use is to vehicle assertions*". But from this, it is not clear to me how one can hold that there is not such a thing as assertion. Not all the components of natural language have a lexical counterpart. Only a subtle analysis of the concept of assertion can *explicate* the role that assertions have in natural language.

[36] In any case, (2RT) can handle many different versions of the Moore-type sentences as I have already showed, while (WT) can only handle the Moore-type sentence (EM).

Remark 1

In my analysis of assertion it is possible to distinguish a *belief* from a *rational belief*. Even the concept of certainty turns out to be twofold: there exists the *rational certainty* (expressed by a conclusive verification and based on the probability calculus) and the *non-epistemic certainty* (as in the case of the creationist teacher), which is connected with the (prejudgemental) *beliefs* of a subject. These individual beliefs can turn out to be either rational or irrational and resemble Wittgenstein's account of certainties[37], since they are *presupposed* when we make assertions (Wittgenstein 1969). The latter type of certainties does not belong to the same category of the epistemic-judgement certainties (Wittgenstein 1969, §308; van der Schaar 2003), since these certainties are not propositional and are conceived "as lying beyond being justified or unjustified", namely "they are something animal" (Wittgenstein 1969, §. 359) and they do not need to be justified by a proof. The possibility to make a distinction between epistemic and non-epistemic certainties is one of the fundamental results of my analysis, which connects epistemology and philosophy of language.

§3.3. *Assertion and Knowability*

A possible objection to my analysis of assertions concerns the use of the term "proof". Writes Williamson (2000):

> "How untypical are mathematical assertions? Proofs are often supposed to warrant them in a way inapplicable to most of all empirical assertions: proofs, it is said, are conclusive, whilst empirical warrants are not. However, the nature of the contrast is unclear" (p. 265) .

---

[37] One could say that the non-epistemic certainties determine the framework for judging our epistemic beliefs.

But my probabilistic reformulation of the tripartite analysis of knowledge clarifies this point. It is true that mathematical and logical proofs are conclusive, while empirical proofs, verifications (or empirical confirmations) are not, but it is possible to convert the assertion given by a non-conclusive proof of φ into an assertion on a conclusive proof of a probabilistic sentence on φ. This sentence speaks about the empirical confirmation (or verification) of φ, given the available evidence *E*. In this way, the structure of the empirical assertions and the logical-mathematical assertions are *explicated* in the same manner, even if this is not the case shown in Williamson (2000). Namely, in my view, logical-mathematical assertions are a limit case of the general case of all assertions, since they are certainly true (probability = 1). The formal unification of empirical and logical-mathematical assertions is something methodologically intriguing. This connection between the concept of assertion and the concepts of proof and verification was done by Dummett. He observes:

> "Such a [mathematical] theory of meaning generalizes readily to the non-mathematical case. Proof is the sole means which exists in mathematics for establishing a statement as true: the required general notion is, therefore, that of verification. On this account, an understanding of a statement consists in a capacity to recognize whatever is counted as verifying it, i.e. as conclusively establishing it as true. It is not necessary that we should have any means of deciding the truth or falsity of the statement, only that we be capable of recognizing when its truth has been established. The advantage of this conception is that the condition for a statement's being verified, unlike the condition for its truth under the assumption of bivalence, is one which we must be credited with the capacity for effectively recognizing when it obtains; hence there is no difficulty in stating what an implicit knowledge of such a condition consists in – once again, it is directly displayed by our linguistic practice" (Dummett 1993, pp. 70-71).

It is important to distinguish the concept of *formal proof* (namely a proof in Peano Arithmetic **PA**) from the concept of *informal proof* (intended as any valid argument).

The concept of *formal proof* is handled by the modal system G that presents the following characteristic axiom, called *Gödel-Löb axiom*:

$$\Box(\Box\varphi \rightarrow \varphi) \rightarrow \Box\varphi.$$

If this axiom is added to the modal system K4, we obtain the modal system G. Such system was formalized by Solovay in 1976 and it is complete with respect to the conversely well-founded frames. The following *reflection principle* does not hold:

$$\Box\varphi \rightarrow \varphi.$$

In fact, if we interpret $\Box$ with the meaning of "provable in the same formal system", then we will incur the limitations of the second theorem of incompleteness. Notably, in 1936, Hilbert and Bernays in their proof of the second theorem of incompleteness stated the following conditions for the concept of provability in **PA**:

1. If PA $\vdash \varphi$, then PA $\vdash Prov(\ulcorner\varphi\urcorner)$;
2. PA $\vdash Prov(\ulcorner\varphi \rightarrow \psi\urcorner) \rightarrow (Prov(\ulcorner\varphi\urcorner) \rightarrow Prov(\ulcorner\psi\urcorner))$;
3. PA $\vdash Prov(\ulcorner\varphi\urcorner) \rightarrow Prov(\ulcorner Prov(\ulcorner\varphi\urcorner)\urcorner)$[38].

By contrast, the concept of proof involved in intuitionistic logic is *informal*. By this informal account of proof, the intuitionistic constants are explicated. The method of *explication* of the meaning of the intuitionstic constants is provided by the following BHK (Brouwer – Heyting – Kolmogorov) interpretation:

a. A proof of $p \wedge q$ consists in a proof of p and a proof of q *plus* the conclusion $p \wedge q$.

b. A proof of $p \vee q$ consists in a proof of p or a proof of q *plus* the conclusion $p \vee q$.

c. A proof of $p \rightarrow q$ consists in a method which converts a proof of p into a proof of q.

---

[38] Note that $\ulcorner\varphi\urcorner$ is Gödel's number. In 1955, Löb proved that from the Hilbert-Bernays conditions (nowadays called Löb conditions) *plus* Gödel's lemma of diagonalization – which says that for any arithmetical formula C(x) there is an arithmetical formula ψ such that PA$\vdash \psi \leftrightarrow C(\ulcorner\psi\urcorner)$ – suffice to prove that from PA $\vdash Prov(\ulcorner\varphi\urcorner) \rightarrow \varphi$, we can deduce that PA $\vdash \varphi$.

d. No proof of $\bot$ (mathematical absurd) exists[39].

e. A proof of $\exists x P(x)$ consists in a name d for an object constructed in the intended domain of discourse *plus* a proof of $P(d)$ and the conclusion $\exists x P(x)$.

f. A proof of $\forall x P(x)$ consists in a method that for every object d, constructed in the intended domain of discourse, provides a proof of $P(d)$.

In any case, there exists a system of logic presented in (Dalla Pozza & Garola, 1995) in which it is possible to handle both classical (*formal*) proofs and intuitionistic (*informal*) proofs, since some elements of the meta-language are reflected in the language object through the assertion sign. This fact allows not to incur the limitations of the second theorem of incompleteness. Moreover, it is also possible to distinguish an intuitionistic fragment from a classical one. For further formal details, see Dalla Pozza & Garola's article. What is important is the idea that it is possible to provide a unification between these two different accounts of proof (and their corresponding assertions) in order to be able to make sense of the different views on verificationism and verificabilism.

I want to point out that the two rules (KRA) and (WR) show the connection between epistemology and verificationism. Nevertheless, there can exist forms of verificationism which do not appeal to the concept of knowledge as well as forms of epistemology which do not appeal to the concept of verification (or assertion) (Skorupski 1997).

The epistemic version of verificability is based on (KRA) and states: (EV) "*What is true can be known*", while the mere pragmatic verificabilism (PV) is based on (WR) and states: "*What is true can be proven*". Hence, it is important not to confuse (PV) and (EV).

[39] If we consider a proof as being an empirical verification, then d. does not make sense, since the negation of an empirical contingent sentence p is a contingent sentence as well. So, the negation of p does not imply any absurdity. Hence, in the case of empirical verification it is better to assume intuitionistic negation as a primitive sign of the language.

In mathematics, there exist *constructive* (or intuitionistic) formal systems, which assume mainly (WR), while there are systems of *epistemic mathematics* which mainly appeal to (KRA) (Shapiro 1985). Nevertheless, there exists an interplay between *epistemic* and *constructive* mathematical knowledge too (Sundholm 1997). In case of mathematical knowledge (EV) implies (PV), since a mathematical object (in a constructive system) can be known only by showing a proof of it. In case of empirical knowledge, one could think that (EV) does not imply (PV), since if an empirical verification of a sentence *p* cannot be conclusive, then one cannot know *p*. In any case, if we transform a non-conclusive verification of *p* into a conclusive verification of the sentence which expresses the probability of *p* given the available evidence, then (EV) implies (PV) even in the framework of empirical knowledge. In this way, on the one hand, it is possible to save the probabilistic and fallibilist features of empirical knowledge, and, on the other hand, it is possible to provide a unified framework for mathematical and empirical knowledge.

More problematic is the concept of verificabilism (possibility of verification), since it can lead to the *paradox of knowability* in epistemic logic, when one tries to know (EM)[40]. Hence, (EM) turns out to be a very important sentence not only in philosophy, but also in logic. If one assumes the (epistemic) verificabilist view, then one has to accept this principle:

(3) $\varphi \rightarrow \Diamond K\varphi$ (if a sentence is true, then it can be known).

" $\Diamond$ " indicates the sign of possibility, while " K " indicates "it is known by someone at some time that ..". Hence, K refers to an empirical account of knowledge in this case.
It is very plausible to assume that there exists at least one truth that we do not know:

(EM) $p \wedge \neg Kp$. This formula is the epistemic Moore paradox.

[40] See Brogaard & Salerno (2004). I will present the paradox in order to underline the connection between the analysis of knowledge and assertion that I have presented and some logical issues concerning the verificabilist program.

We substitute (EM) for p in (3) and get

(4) $(p \wedge \neg Kp) \rightarrow \Diamond K(p \wedge \neg Kp)$.

But from (EM) and (4), by (MP) we can derive:

(5) $\Diamond K(p \wedge \neg Kp)$. In any case, it is possible to get the negation of (5) by using very minimal (epistemic and modal) principles concerning $\Diamond$ and K, namely:

(6) $K(p \wedge q) \rightarrow (Kp \wedge Kq)$ the knowledge of two propositions entails the knowledge of each conjunct.

(7) $Kp \rightarrow p$; the knowledge of a proposition entails the truth of the proposition.

(8) If p is a theorem, then $\Box p$; that is, if p is a theorem, then p is necessary.

(9) $\Box \neg p \Rightarrow \neg \Diamond p$.

Assume the following principle which says that (EM) is known

(10) $K(p \wedge \neg Kp)$

If (6) is applied to (10), we get:

(11) $Kp \wedge K \neg Kp$

If we apply (7) to the second conjunct in (11), we derive a contradiction:

(12) $Kp \wedge \neg Kp$

Therefore, the assumption (10) can be discharged and from (10) and (12) by *reductio ad absurdum*, we get:

(13) $\neg K(p \wedge \neg Kp)$

If (8) is applied to (13), one can derive:

(14) $\Box\ \neg K(p \wedge \neg Kp)$

And if we apply (9) to (14), then we get:

(15) $\neg \Diamond\ K(p \wedge \neg Kp)$.

The formula (15) says that it is not possible to know (EM), and this is contradictory with (5). It has been observed that a similar paradox can be obtained in a non-epistemic account of verificability, which I have called pragmatic verificability (Usberti 1995). Hence, the concept of verificability (both pragmatic and epistemic) can lead to some paradoxes and proves to be problematic. These paradoxes show that a proper analysis of knowledge and assertion cannot be given in a mere verificabilist framework. That is why I decided to present my analysis of knowledge and my conceptual reconstruction of the constitutive rules of assertions (in which Moore-type sentences turn out to be fundamental) by a probabilistic setting in which the concept of proof is still essential.

§. 3.4 *Assertion and Verifiability*

The origins of the verificationist paradigm in philosophy can be traced back to the ideas of the first Vienna Circle philosophers, even if some verificationist ideas were prefigured in Mach's works and in the pragmatist school. The main philosophical slogan of the

Vienna Circle was the following: "the meaning of a proposition is the method of its verification" (Schlick 1936). Schlick, the founder of the Vienna Circle, claims that the meaning of a sentence is not the actual process of verification, but the concept of "possibility of verification" (*verifiability*). The concept of possible involved in the notion of verifiability can be logical or empirical. The *logical verifiability* is the idea that a sentence is meaningful if it follows from logical laws without contradiction. That is, if a sentence can be logically *conceived* (it is not a reference to actual *psychological* conceivability). The *empirical verifiability* (or *testability* in Carnap's terminology) involves the concept of physical possibility for which a sentence is meaningful if it follows from the physical laws without contradiction (and the logical ones, of course).

The neopositivistic point of view about the theory of meaning was conveyed on the basis of a misconception of the main themes of Wittgenstein's *Tractatus*. In fact, the concept of meaning in the *Tractatus* is truth-conditional, e.g., consider sentence 4.431 "The expression of the agreement and disagreement with the truth-possibilities of the elementary propositions expresses the truth-conditions of the proposition. The proposition is the expression of its truth-conditions" and sentence 4.024 "To understand a proposition means to know what is the case, if it is true". After the publication of the *Tractatus*, Wittgenstein stopped his works in philosophy for many years. In 1929, he attended to a lecture of the Dutch mathematician Brouwer, whose ideas made a considerable impression on him[41]. After that lecture, Wittgenstein took up his philosophical work again. The philosophical views that Wittgenstein presents in the 30's can be placed between Wittgenstein of the *Tractatus*, and the so called "second Wittgenstein". In that middle phase, Wittgenstein can be considered as a holder of the (actual) verificationist view on meaning, while, for the first Wittgenstein, meaning is truth-conditional. For the second Wittgenstein, meaning is the use of the sentence. The

[41] For the role of this lecture on Wittgenstein's philosophical work, cf. Wrigley (1989). Wrigley claims that some verificationist aspects were already implicit in the *Tractatus*, but he does not present any analytical explanation of this fact.

development of Wittgenstein's ideas cannot be handled by these sharp distinctions[42], but, in any case, Wittgenstein's views on meaning were of verificationist type in the 30's. In fact, he writes: "every proposition is a signpost for a verification", "the verification is not *one* token of the truth, it is *the* sense of a proposition", "the sense of a proposition is the method of its verification", "where there are different verifications there are also different meanings"[43].

A new reformulation of the verificationist program is mainly due to Michael Dummett, who connects the verificationist program in epistemology and theory of meaning with intuitionistic logic, while the neopositivists developed their research in the framework of classical logic. Dummett maintains that a theory of *meaning* must be a theory of *understanding*. Thus, Dummett is mainly limiting language to its assertive use. He was inspired by Wittgenstein's slogan: "Meaning is the use". He claims, in the spirit of Frege, that a theory of meaning needs a tripartite analysis of the *sense*, the *tone*, and the *force*. Frege assumes that the thought expressed by a sentence is its *sense*, which determines the reference. He indicates that the sense is both the cognitive meaning of a sentence (an epistemological issue) and it expresses the truth conditions (logical issue) of the sentence. Between these two definitions of *sense* in Frege there is a possible contrast. So, the issue is not completely clear as observed by many interpreters of Frege's works.

Let us consider now the second ingredient of Dummett's theory of meaning. The *tone* is everything that cannot occur in the determination of the truth value of the proposition (but it is something that can affect our imagination or fancy), while the *force* is the illocutionary act of *using* the content of the proposition, in order to perform acts like querying, asserting, or defining, etc. According to Dummett, the truth-conditional dimension of meaning cannot count as a theory of understanding, because the central notion of such a theory is the concept of *truth*. That is, there is no way of explaining the method for the understanding a proposition. In short, the truth-conditional theory of meaning cannot explain the role of the force, because there is no appeal to the use of the

---

[42] In fact, there exist some features of continuity in the development of Wittgenstein's thought, but I will not deal with this difficult point at the moment.

[43] Cf. Wittgenstein (1975), pp. 174, 200 and Wittgenstein (1979), pp. 53, 79.

proposition. Dummett argues that if the use must be the central concept of the theory of meaning, then it is necessary to replace the realistic concept of *truth* with the concept of *verification*, since a verification can be a good candidate for the analysis of the phenomenon of *understanding*. *The complete understanding of a sentence implies both the knowledge of the conditions for the utterance and the consequences of the utterance*. There must be a "harmony" between these two groups. As in natural deduction calculus, *the conditions of assertability of a sentence are fixed by the introduction rules (which are analytical), while the elimination rules can be considered as representing the consequences of an utterance (the synthetic part).* One could say that the introduction rules have the constitutive function for the meaning of the logical constants (as observed by Gentzen 1935). In any case note that the rules in natural deduction correspond to derivable propositions in an axiomatic framework, otherwise we would incur the paradoxes presented in (Prior (1960)). There is not such a requirement for other types of rules or norms. Hence, the possibility to base a theory of meaning on the *use* of the natural deduction rules is not so different from a truth-conditional one in an axiomatic framework, because of the peculiarity of the rules in natural deduction.

According to Dummett, *understanding* is partially implicit, therefore we need a systematic way to make it explicit, *viz.* through a verification. In Dummett's opinion, classical logic cannot explain the real process of understanding, due to the principle of bivalence. For instance, bivalence implies a realistic account of meaning, in the sense that every sentence is determinately true or false independently of our knowledge of the proposition (*semantic realism*). The underlying logic of the *realistic* view on meaning is classical logic, since *excluded middle* is a tautology, i.e., $\varphi \vee \neg \varphi$ is always true, without the knowledge of the truth value of φ. In the case of intuitionistic logic *excluded middle* does not hold, since there *excluded middle* means: there is an informal proof (valid mathematical argument or verification) of φ or there is no informal proof of φ. But, since φ can be undecided, *excluded middle* is not an intuitionistic principle.

Because of the requirement of *manifestability* of understanding which makes our implicit knowledge explicit, a simple realistic conception of meaning cannot explain how communication works. In the anti-realistic perspective, the meaning of a sentence depends on the *possession* of a method for making explicit our knowledge of the sentence, namely by an informal proof (a verification). Meaning cannot be something private (a subjective state), but it is something that can be made intersubjective through a process of verification.

In the case of atomic formulae, the informal proof cannot be a valid mathematical argument, just an empirical verification. Because of the importance that the concept of informal proof has in Dummett's program, intuitionistic logic seems to be adequate for the *anti-realistic* theory on meaning (the truth value of a sentence depends on our knowledge of it). Instead, classical logic and the *principle of bivalence* are adequate for a realistic view on meaning, because there exists an independent domain of 'reality' that validates the truth-conditions (Dummett 1976)[44]. Realistic truth is "evidence transcendent", which cannot be the object of a method of testability, while for the anti-realistic theory of meaning, the correct logic is the intuitionistic one, since truth is defined as the possession of an informal proof of a sentence (no place for evidence transcendent truths). Moreover, Dummett argues that in the case of undecidable propositions, their meaning cannot be defined by truth conditions. According to Dummett, the truth conditional account of meaning fails in the analysis of the propositions that are in *principle* undecidable, but also in the analysis of *effectively* undecided propositions such as the ones about the past (or inaccessible places), untestable infinite domains, etc. This is only because the principle of manifestability cannot be applied. This follows from Dummett's interpretation about the realistic point of view on the truth-conditional account of meaning. On the contrary, I maintain that undecidable sentences cannot be handled by a verificationist theory of meaning, since a method for verifying the proposition does not exist *in principle*. In short, Dummett's logic for the verificationist program is intuitionistic logic, since it is adequate for handling the anti-realist features of his theory of meaning. But if verificationism is intended as a semantic program, it turns out to be completely untenable, since there are sentences which express a well defined meaning, even though they are not verifiable (even in principle). If we assume the term verificationism to be a pragmatic (and/or an epistemic) notion (i.e. a relation between a

[44] In any case, it has been possible for the neopositivists to hold a verificationist point of view with classical logic. So, the connection between a system of logic and a philosophical and linguistic thesis is not so direct, contrary to what Dummett claimed.

subject and a sentence), then it turns to be inadequate since there exist true propositions that are not verified. But also a semantic verificabilist view is not adequate, since there exist some meaningful propositions which cannot be verified even in principle. E.g. imagine a machine which can compute the validity of a mathematical statement *p* in a lapse of time greater than the time of the complete life of our universe. Although *p* cannot be verified even in principle, *p* has a definite meaning. Hence, the *verificabilist* program has to face very important problems as well[45] (as I have already indicated in case of the paradox of knowability).

§3.5 *What is the logic of verificationism?*

In this section, I will show a dispute between Dummett and Hintikka regarding the *correct logic* for the verificationist program in mathematics. I will demonstrate that this dispute makes sense if we interpret Dummett and Hintikka's views respectively by the epistemic verificationism (EV) and the pragmatic verificationism (PV). Namely, I will show the interpretative function of (EV) and (PV) in a concrete case determined by a philosophical dispute on the nature and the epistemic features of constructive systems in mathematics.

Jaakko Hintikka criticized the assumption that intuitionistic logic is the *correct logic* for the verificationist program. He claims that the *correct logic* is the logic expressed in Gödel's *Dialectica* Interpretation, which has a suitable semantics in Hintikka's Game-Theoretical Semantics. In the subsequent sections, I will indicate that there is not a proper logic for the verificationist program, since different logics can handle different versions of the verificationist program.

[45] Dalla Pozza (2008)

§3.5.1. *Game-Theoretical Semantics*

Game-Theoretical Semantics (GTS) is a theory developed by Hintikka for logical-linguistic analysis and for the philosophical analysis of meaning. Hintikka was inspired by Wittgenstein's concept of "language game"; he tried to formalize a particular category of this vague concept and GTS is the result of this formalization. In any case, I am very sceptical that Wittgenstein's linguistic games can be formalized.

GTS is a semantics based on formal acts of choice in a game between two ideal players, usually entitled the *verifier* (or *Myself*) and the *falsifier* (or *Nature*), which try, respectively, either to verify or falsify the formulae occurring in the game. As a matter of fact, from the game-theoretical point of view, language is a goal-directed and a rule-governed process and this assumption is fundamental in GTS.

The concept of truth is defined as *the existence of a winning strategy* for the *verifier*, while falsity is defined as the existence of a winning strategy for the *falsifier*. The players' rules are based on the fact that they choose individuals and assign names to them. The first order language L of GTS is an interpreted language, therefore we assign to L a model M with domain a D(M) of individuals on which the non-logical symbols of L are evaluated; i.e., if one adds the names of the individuals of D(M) to L, every atomic formula of the extended language is either true or false. On the basis of that, GTS extends the notion of truth to all non-atomic sentences of L.

The rules (R) of this game, as related to the classical logic, are the following (the definitions follow by induction on the complexity of the formula):

($R\vee$) the game starts with the choice of Myself between α_1 and α_2.

($R\wedge$) the game starts with the choice of Nature between α_1 and α_2

($R\ \exists$) the game starts with the choice of Myself of an individual from the domain. Its name will be "x_1".

($R\forall$) this rule is like the precedent one, only that the choice is made by Nature

($R\alpha_1$) if α_1 is an atomic formula or an identity, if α_1 is true, then Myself wins, while Nature loses. On the contrary, if α_1 is false Myself loses and Nature wins.

($R\neg\alpha_1$) this rule is like the precedent one, but Myself and Nature exchange their strategies of verification and falsification.

A game characterized by the following rules is a game of two players with perfect information and zero sum; a winning strategy for each player can be obtained *independently* from the strategy of the other player. A game is called *determined* if one of the players has a winning strategy. In GTS a game is determined if *excluded middle* and *bivalence* hold.

GTS can be used also for giving a semantics for the verificationist account of meaning. Hintikka claims that the logic for verificationism can be handled by GTS, imposing that all the strategies of the verifier (for proving truth) are recursive, while falsity is defined as the existence for Nature of a recursive strategy, which wins against any strategy of Myself. Therefore, there will be sentences which are neither true nor false in suitable models (Hintikka & Sandu (1997)). In any case, Hintikka claims that the failure of *excluded middle* occurs also in games in which informational independence is allowed, i.e., in those games in which one player chooses the name for a constant without knowing the acts of choice of the other player. The phenomenon of the *independence of information* also occurs in the logic with branching quantifiers. This logic is also known as the logic of "partially ordered quantifiers" (introduced in (Henkin 1959)), which can be expressed by the following two-dimensional graph:

(19) $\forall x \exists y$

$$R\,(x, y, z, u)$$

$\forall z \exists u$

The intuitive significance of (19) is that the variable "*y*" exclusively depends on "*x*", while the variable "*u*" exclusively depends on "*z*". According to Hintikka, (19) can formalize expressions of the natural language, such as:

(20) 'Certain relatives of every inhabitant in the countryside and certain relatives of every inhabitant of the city hate each other'.

which cannot be expressed in classical logic. In fact, the truth conditions of a prenex formula in classical logic are represented by Skolem functions, for which the existentially quantified variables depend on all the universally quantified variables preceding them. Namely, a formula like:

(21) $\forall x \exists y \forall z \exists u \; R(x, y, z, u)$; R is a quantification free matrix

The Skolemian form of (21) is the following second-order formula which represents the truth conditions of (21):

(22) $\exists f \exists g \forall x \forall z \, R(x, f(x), z, g(x, \mathrm{z}))$ with f and g representing function symbols.

Note that the Skolemian form of (19) is

(23) $\exists f \exists g (\forall x)(\forall z) R((x, z, f(x), g(x))$.

Recently, Hintikka has presented a calculus called IF logic[46], in which it is possible to express the independence of information between quantifiers, connectives, modal expressions, etc..

Let us consider the conditional as a primitive sign in the system. So, a conditional like "If φ_1, then φ_2" in GTS means that there is a method which transforms a verification of the truth of φ_1 into a verification of the truth of φ_2[47]. In this sense, a conditional can be considered as two sub-games. Assume that G_0 = G(If φ_1 then φ_2). In the first subgame $G(\varphi_1)$ *Myself* and *Nature* play with reversed roles. If *Myself* wins, *Myself* wins G_0. If Nature wins, the players move to play the subgame G(φ_2) with their normal roles.

[46] IF Logic means "Independence Friendly Logic". Hintikka wrote many articles about this new logic. E.g. Hintikka (2002).

[47] This analysis of the conditional gives a very constructive flavour to GTS.

Notice that Myself has access to Nature's strategy in G(φ_1). The player who wins G(φ_2) wins the complete game.

The intuitive idea is that Myself has access to the functions (strategies) of the Nature. Consider the following two formulae in the Skolemian form:

(24) $F_0 = (\exists f)\ (\forall h)\ F[f, h]$ and

(25) $\mathrm{G}_0 = (\exists g)\ (\forall i)\ G[g, i]$.

The rule for $F_0 \rightarrow \mathrm{G}_0$ in GTS can be given in different ways:

(26a) $(\exists\ \gamma)\ (\exists h)(\forall f)(\forall i)\ (F[f, h] \rightarrow G[\gamma(f), i])$.

(26b) $(\exists\ \eta)\ (\exists g)(\forall f)(\forall i)\ (F[f, \eta(i)] \rightarrow G[g, i])$.

(26c) $(\exists\ \gamma)\ (\exists\ \eta)(\forall f)(\forall i)\ (F[f, \eta(i)] \rightarrow G[\gamma(f), i])$.

(26d) $(\exists\ \gamma)\ (\exists\ \eta)(\forall f)(\forall i)\ (F[f, \eta(i, f)] \rightarrow G[\gamma(f), i])$.

The choice of one of them is determined by the amount of information that *Myself* has to remember when he is playing the second subgame of the conditional. According to Hintikka, the above rules for the conditional can handle different types of anaphoric coreferential sentences, such as the donkey's sentences. In any case, I do not go further in the linguistic applicability of GTS, since I am mainly interested in its role for the verificationist program.

§. 3.5.2 *Epistemic and Pragmatic Verificationism*

I consider the dispute between Hintikka and Dummett once again. Dummett argues that "the comparison between the notion of truth and that of winning a game still seems to me a good one"[48]. Hintikka replies that truth cannot be the act of winning a game, but truth

[48] Dummett (1976), p. 19.

can be replaced by the *existence of a winning strategy for the verifier*, namely the existence of a function for winning the game. The player can follow the winning strategy or not, but what is important is the objective existence of a set of acts of choice for winning the game. As follows, GTS should be something similar to the realistic account of meaning that Dummett pointed out. Hintikka places the players in an abstract game, while Dummett is interested in the concrete behaviour of the players. Dummett also writes that there can be some games in which the aim is not that of winning the game, but something else, i.e., the intentions of the players are also important in a game and, therefore, "truth is a more complicated notion than that of winning"[49]. On the contrary, in Hintikka's abstract games there is no place for the intentions of the players and the aim of the semantic games is defining the concept of truth. If the aim of the game is different, it cannot be a semantic game, but one of another kind. In fact, Hintikka proposed other kinds of games with different targets such as: proof games, interrogative games, etc and bases his verificationist views on a constructive system called *Dialectica Interpretation*[50]. *Dialectica* calculus is an extension of intuitionistic logic since all intuitionistic theorems hold in *Dialectica calculus*, but there are some principles like *Markov's principle*, *Independence of Premises* and *the axiom of choice AC* that only hold in the *Dialectica calculus* but not in intuitionistic logic. Hence, *Dialectica* calculus is a broader system than intuitionism and turns out to be more adequate for handling (PV). With a restriction in the values of all higher-order quantifiers to recursive entities of finite type, the rules of *Dialectica* calculus are formally equivalent with the verificationist games in GTS.

By contrast, Dummett assumes (EV) in his antirealist theory of meaning, since he wants to deny the existence of a Platonic world of proofs which are not available to the antirealist. Writes Dummett:

> "there is no intelligible anti-realist notion of truth for mathematical statements under which a statement is true only if there is a proof of it, but may be true because such a proof exists, even though we do not know it, shall never know it, and have no effective means of discovering it. The reason is evident: we can

[49] *Ibidem*.

[50] See Gödel (1958) and Feferman (1997).

> introduce such a notion only by appeal to some platonistic conception of proofs as existing independently of our knowledge, that is, as abstract objects not brought into being by our thought. But, if we admit such a conception of proofs, we can have no objection to a parallel conception of mathematical objects such as natural numbers, real numbers, metric spaces, etc.; and then we shall have no motivation for abandoning a realistic, that is, platonist, interpretation of mathematical statements in the first place" (Dummett 1993, pp. 258-9).

In contrast, the existence of a winning strategy for the verifier (Hintikka) and the existence of an (abstract and atemporal) canonical proof (Prawitz 1987a)[51] is only coherent with (PV), since the existence of a proof does not depend on the knowledge of it by a subject. So, Hintikka's observation that "it is not clear at this time whether Dummett's claim that intuitionistic logic is the true logic of a verificationist semantics is defensible or not" can be answered. I maintain that the Dummett-Hintikka dispute is based on an ambiguity in the use of the term verificationism. Intuitionistic logic can be adequate for handling (EV), while the logic of *Dialectica* Interpretation can be adequate for handling (PV). At the same time no semantic criterion of verification can be proposed at all because of the paradoxes that it raises.

In any case, the adequacy of a system of logic is something conventional, hence other points of view on verificationism can be handled by different systems of logic, *e.g.* those in which a dynamic change of the context can occur. If so, there exists a family of logics which turn out to be adequate for handling the different versions of verificationism.

[51] See also (Prawits 1977 and 1987).

§4. *Concluding Remarks*

In the previous sections I have presented the validity and the limits of the constructive views on knowledge and assertion. This is a quite abstract analysis of these complex notions and we know that the 'concrete' meaning expressed by an assertion can be grasped if one also takes into consideration the non-linguistic and cultural dimension of communication. So, my rational reconstruction is one of the many steps toward a more general understanding of these notions, which has to connect the mere abstract concepts of knowledge and assertions with our *praxis*. This task fits with a more empirical and computational research about human communication in a descriptive framework, while my framework of analysis has been mainly normative. Descriptive epistemology and normative epistemology are intuitively connected, even if from a normative argument we cannot *logically* infer any descriptive statement about knowledge and *vice versa* as Hume pointed out. Thus, the possibility of such connection does not lie on a logical matter but on an interplay between language and praxis that we experience in our 'world of life'.

In any case, there must exist an objective way that expresses how things are that does not depend on human belief connected with a social or cultural dimension, since it is very common to confuse phenomena with their representations (Boghossian 2006). The objectivity of a system of knowledge can be achieved only if some epistemic norms are satisfied, once that we have fixed (explicated) the meaning of the pre-theoretical invariants of our system of knowledge[52]. The determination of these invariants and the valid transformations within which they hold seems to be one of the main open questions in epistemology[53]. Namely, knowledge has to be conceived under a veil of objectivity through its invariants.

[52] Anyway, it is possible that we first have an elegant structure that we want to apply and subsequently we try to handle the right invariants for its application.

[53] Cf. Nozick (1998) for an analysis of objectivity intended as what remains invariant under various transformations. According to Nozick, an objective fact is: i) "accessible from different angles" ii) "there is or can be intersubjective agreement about it" iii) "it holds independently of people's beliefs, desires, hopes, and observations or measurements". Nozick also admits that there are some problems in determining the

valid transformations of a system of knowledge expressed by a theory. Nevertheless, he does not take into consideration the possibility that the choice of a theory itself can imply the choice of a set of invariants simply due to contingent and external factors.

REFERENCES

Bach, K. (2008), 'Applying Pragmatics to Epistemology', *Philosophical Issues*, Vol. 18, pp. 68-88.

Bellin G. & Biasi C. (2004), 'Towards a Logic for Pragmatics. Assertions and Conjectures', *Journal of Logic and Computation*, Vol. 14, No. 4, pp. 473-506.

Bobbio, N. (1958), *Teoria della norma giuridica*, Giappichelli, Torino.

Boghossian, P.A. (2006), *Fear of Knowledge*, Oxford University Press, Oxford.

Brandom, R.B. (1994), *Making It Explicit: Reasoning, Representing, and Discursive Commitment*, Harvard University Press, Cambridge, MA.

Brogaard, B. & Salerno, J. (2004), 'Fitch's Paradox of Knowability', *The Stanford Encyclopedia of Philosophy*, Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/entries/fitch-paradox/ >.

Brown, J. (2008), 'The Knowledge Norm for Assertion', *Philosophical Issues*, Vol. 18(1), pp. 89-103.

Cappelen, H. (2009), 'Against Assertion', in J. Brown – H. Cappelen (2009), *New Essays on Assertion*, Oxford University Press, Oxford.

Carnap, R. (1936), 'Testability and Meaning', *Philosophy of Science*, Vol. 3, pp. 419-471.

Conte, A.G. (1990), *Filosofia del linguaggio normativo*. I. Studi 1965-1981, Giappichelli, Torino.

Dalla Pozza, C. (2008), *Il problema della demarcazione*, ESE, Lecce.

Dalla Pozza, C. & Garola, C. (1995), ‘A pragmatic interpretation of intuitionistic propositional logic’, *Erkenntnis*, 43, pp. 81-109.

DeRose, K. (2002), ‘Assertion, Knowledge and Context’, *The Philosophical Review*, Vol. 111, No. 2, pp. 167-203.

Dretske, F. (1971), ‘Conclusive Reasons’, *Australasian Journal of Philosophy*, Vol. 49, pp. 1-22.

Dummett, M. (1976), ‘What is a theory of meaning? (II)’, in G. Evans & J. McDowell (eds.), *Truth and Meaning*, Clarendon Press, Oxford, pp. 67-137.

Dummett, M. (1993), *The Seas of Language*, Oxford University Press, Oxford.

Dummett, M. (2006), *Thought and Reality*, Oxford University Press, Oxford.

Feferman, S. (1997), ‘Gödel’s Dialectica interpretation and its two-way stretch’, in G. Gottlob et at. (eds.), *Computational Logic and Proof Theory* (Proc. 3d Kurt Gödel Colloquium, Brno Aug. 1993), LNCS 713, pp. 23-40.

Frege, G. (1879), *Begriffsschrift, eine der aritmetischen nachgebildete Formelsprache des reinen Denkens*, Halle. Translation in J. van Heijenoort (ed.), *Frege and Gödel: Two Fundamental Texts in Mathematical Logic*, Harvard University Press, Cambridge, Mass. 1970.

Frege, G. (1984), *Collected Papers on Mathematical Logic and Philosophy*, B. McGuinness (ed.), Blackwell, Oxford.

Geach, P. (1965), ‘Assertion’, *Philosophical Review*, Vol. 74, pp. 449-65.

Gentzen, G. (1935), ‘Untersuchungen über das logische Schliessen’, *Mathematische Zeithschrift*, 39, pp. 176-210, 405-431.

Gettier, E.L. (1963), ‘Is Justified True Belief Knowledge?’, *Analysis* 23, pp. 121-123.

Gödel, K. (1958), ‘Uber eine bisher noch nicht benützte Erweiterung des finiten Standpunktes’, *Dialectica*, 12, pp. 280-287.

Goldman, A.I. (1967), ‘A Causal Theory of Knowing’, *The Journal of Philosophy*, Vol. 64, pp. 357-72.

Green, M.S. (2002), ‘The Inferential Significance of Frege’s Assertion Sign’, *Facta Philosophica*, Vol. 4, No. 2, pp. 201-209.

Hacking, I. (2000), *An Introduction to Probability and Inductive Logic*, Cambridge University Press, Cambridge.

Henkin, L. (1961), 'Some Remarks on Infinitely Long Formulas', *Infinitistic Methods*, Proceedings of the Symposium on Foundations of Mathematics, Pergamon Press, Oxford, London, pp. 176-183.

Hilbert, D. & Bernays, P. (1939), *Grundlagen der Mathematik*, Vol. II, Springer, Berlin. Second enriched edition in 1970.

Hintikka, J. (1987), 'Game-Theoretical Semantics as a Synthesis of Verificationist and Truth-Conditional Meaning Theories', in E. LePore (ed.), *New Directions in Semantics*, Cognitive Science Series, Academic Press Inc., pp. 235-258.

Hintikka, J. (1996), *The Principles of Mathematics Revisited*, Cambridge University Press, Cambridge.

Hintikka (2002), 'Hyperclassical Logic (a.k.a. IF Logic) and Its Implications for Logical Theory', *Bulletin of Symbolic Logic*, vol. 8, n. 3, pp. 404-423.

Hintikka, J. & Sandu, G. (1987), 'Game-Theoretical Semantics', in J. van Benthem & A. ter Meulen (eds.), *Handbook of Logic and Language*, pp. 361-410.

Kant, I. (1965), *Critique of Pure Reason*, trans. N. Kemp Smith, St. Martin's Press, New York [1781/7].

Keynes, J.M. (1921), *A Treatise On Probability*, MacMillan, New York.

Lackey, J. (2007), 'Norms of assertion', *Noûs*, 41(4), pp. 594-626.

Lewis, D.K. (1969), *Convention: A Philosophical Study*, Harvard University Press, Cambridge (MA).

Löb, M.H. (1955), 'Solution of a problem of Leon Henkin', *Journal of Symbolic Logic*, Vol. 20, pp. 115-118.

Musgrave, A. (1993), *Common Sense, Science and Scepticism*, Cambridge University Press, Cambridge.

Nozick, R. (1981), *Philosophical Explanations*, Harvard University Press, Cambridge, Mass.

Nozick, R. (1998), 'Invariance and Objectivity', *Proceedings and Addresses of the American Philosophical Association,* Vol. 72, No. 2, pp. 21-48

Pagin, P. (2007), 'Assertion', *The Stanford Encyclopedia of Philosophy*, Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/entries/assertion/ >.

Pappas, G.S. & Swain, M. (1973), 'Some Conclusive Reasons against "Conclusive Reasons"', *The Australasian Journal of Philosophy*, 51, pp. 72-76.

Prawitz, D. (1977), 'Meaning and Proofs', *Theoria*, 43, pp. 2-40

Prawitz, D. (1987), 'Some Remarks on Verificationistic Theories of Meaning', *Synthese*, 73, pp. 471-477.

Prawitz, D. (1987a), 'Dummett on a theory of meaning and its impact on logic', in B.M. Taylor (ed), *Michael Dummett, Contributions to Philosophy*, Martinus Nijhoff Publishers, Dordrecht, pp. 117-65.

Prior, A. (1960), 'The runabout inference ticket', *Analysis*, 21, pp. 38-39.

Quine, W. V. O. (1948), 'On What There Is', *Review of Metaphysics*, 2, pp. 21-38.

Rawls, J. (1955) 'Two Concepts of Rules', *The Philosophical Review*, 64, pp. 3-32.

Reichenbach, H. (1947), *Elements of Symbolic Logic*, Free Press, New York.

Ross, A. (1968), *Directives and Norms*, Routledge & Kegan Paul, London.

Russell, B. (1903), *Principles of Mathematics*, Norton, New York.

Russell, B. (1948), *Human knowledge: its scope and limits*, George Allen & Unwin, Ltd, London.

Russell, B. (1984), *The Collected Papers of Bertrand Russell*, Vol. 7: *Theory of Knowledge: The 1913 Manuscript*, Elizabeth Ramsden Eames and Kenneth Blackwell, eds., George Allen & Unwin, Ltd, London.

Salmon, W.C. (1963), *Logic*, Prentice Hall, Englewood Cliffs (N.J.).

van der Schaar, M. (2003), 'Judgement and Certainty', in *Pre-Proceedings of the 26th International Wittgenstein Symposium*. Austrian Ludwig Wittgenstein Society, Kirchberg am Wechsel, pp. 311-313.

Schelling T.C. (1960), *The Strategy of Conflict*, Harvard University Press, Cambridge (MA).

Schlick, M. (1936), 'Meaning and Verification', *The Philosophical Review*, 45, pp. 339-369.

Schlick, M. (1985), *General Theory of Knowledge*, translated by Albert Biumberg, Open Court Pub. Co., La Salle. [1918 revised in 1925].

Searle, J. (1969), *Speech Acts. An Essay in the Philosophy of Language*, Cambridge University Press, Cambridge.

Shapiro, S. (1985), 'Epistemic and Intuitionistic Arithmetic', in S. Shapiro (ed.), *Intensional Mathematics*, North Holland, Amsterdam, pp. 11-46.

Shope, R.K. (2002), 'Conditions and Analyses of Knowing', in R. Audi, M. Nussbaum, A. Plantinga, E. Sosa (eds.), *The Oxford Handbook of Epistemology*, Oxford - New York, pp. 25-70.

Skorupski, J.M. (1997), 'Meaning, Verification, Use', in R. Hale, C. Wright (eds.), *The Blackwell Companion to the Philosophy of Language*, Blackwell Publishing, Oxford, UK, pp. 29-59

Solovay, R.M. (1976), 'Provability interpretations of modal logic', *Israel journal of mathematics*, Vol. 25, pp. 287-304.

Stalnaker, R. (1978), 'Assertion', in P. Cole (ed.), *Syntax and Semantics* 9*,* New York Academic Press, New York, pp. 315-32.

Stalnaker, R. (2006), 'Conditional propositions and conditional assertions', in J. Gajewski, V. Hacquard, B. Nickel, and S. Yalcin (eds.), *New Work on Modality*, MIT Working Papers in Linguistics, Cambridge, Mass.

Stokhof, M. (2006), 'The future of semantics?', *Theoretical Linguistics*, 32, pp. 91-100.

Sundholm, G. (1997), 'Implicit Epistemic Aspects of Constructive Logic', *Journal of Logic, Language, and Information*, 6, pp. 191–212.

Usberti, G. (1995), *Significato e conoscenza*, Guerini, Milano.

Von Wright, G. H. (1963), *Norm and Action*, Routledge & Kegan Paul, London.

Waismann, F. (1967), *Wittgenstein and the Vienna Circle. Conversations Recorded*, ed. by B. McGuiness, transl. by J. Schulte and B. McGuiness, Blackwell, Oxford

Weiner, M. (2005), 'Must we know what we say?', *The Philosophical Review*, 114(2), pp. 227-251.

Williamson, T. (2000), *Knowledge and its Limits*, Oxford University Press, Oxford.

Wittgenstein, L. (1922), *Tractatus Logico-Philosophicus*. The German text of Ludwig Wittgenstein's Logisch-philosophische Abhandlung with English translation by C.K. Ogden, Routledge & Kegan Paul, London.

Wittgenstein, L. (1956), *Remarks on the Foundations of Mathematics*, edited by G.H. von Wright, R. Rhees, G.E.M. Anscombe, Basil Blackwell, Oxford.

Wittgenstein, L. (1969), *On Certainty*, Basil Blackwell, Oxford.

Wittgenstein, L. (1974), *Letters to Russell, Keynes and Moore*, ed. G.H. von Wright, Blackwell, Oxford.

Wittgenstein, L. (1975), *Philosophical Remarks*, R. Rhees (ed.), Blackwell, Oxford.

Wrigley, M. (1989), 'The Origins of Wittgenstein's Verificationism', *Synthese*, Vol. 78, pp. 265-290.

ACKNOWLEDGEMENTS

Foremost, I would like to express my sincere gratitude to prof. Martin Stokhof, who ploughed through several preliminary versions of my text, posing challenging questions and making sense of my vague thoughts. The result of his guidance and supervision has been fundamental for writing my thesis.

My sincere thanks also goes to prof. Carlo Dalla Pozza, Jonathan Zvesper and Gian Maria Greco who helped me to clarify many arguments of this thesis. I presented with Carlo Dalla Pozza a very preliminary version of section 2 at the Sixth National Conference of the Italian Society of Analytic Philosophy.

I would like to thank prof. Dick de Jongh, for our “constructive” discussions.

Many friends helped me during the last year. They contributed with high dosage of moral support, since writing a master thesis and a PhD thesis on two different topics in the same year is a very unique experience. Notably, I thank: Davide Sergio, Silvia Gaio, Giovanni D’oria, Gabriele Biandolino, Alessandro Grecucci, Giacomo Liuzzi, Giacomo Signore, Alessandro De Salve, Alfredo Di Giorgio, Claudia Ampuero, Pier Paolo Tarsi, Dario Vadacca.

A final thanks to Tanja Kassenaar for her excellent administrative work and to my family.