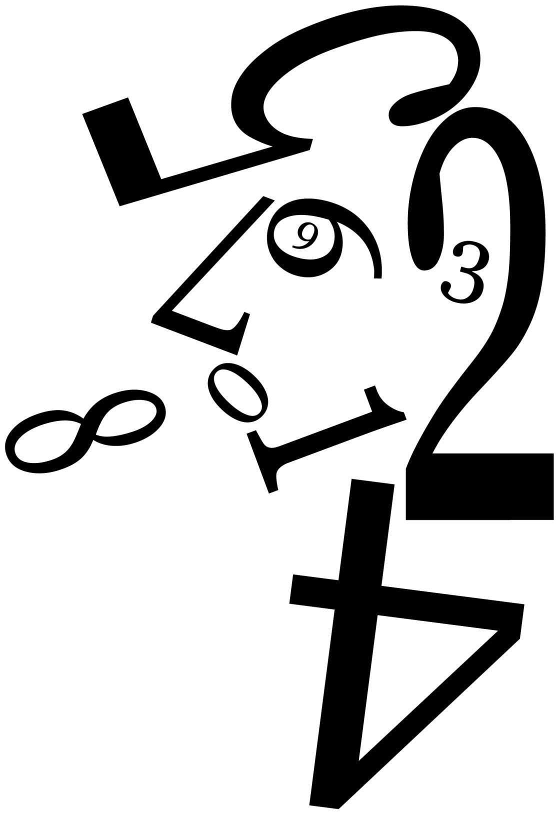
Autistic Number Learning

What Autism Can Tell Us About the Acquisition of Number Concepts

MSc Thesis (Afstudeerscriptie)



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Abstract

Autists count differently when compared with typically developing individuals. Autists differ from typically developing individuals in their counting skills by a slower reaction time when naming quantities, a later development of sequencing skills and recalling positions and no benefit from recognizing a canonical placement of dots. In this thesis the typical development of number knowledge, especially counting skills, and the model of working memory is discussed and adapted to explain the autistic number learning. The three major theories about autism: disabilities in theory of mind, executive dysfunction and weak central coherence; and their influence on the autistic counting skills are discussed. A lack of visual-spatial working memory in autists is the most direct explanation for their impaired skills. This weak visual-spatial working memory combined with a strong rote memory could lead autists to develop an alternative number knowledge strategy based on memory.

Acknowledgement

I would like to thank my parents, Hanny and Tim, for their patience and for all their advice I at first disregarded, but which I eventually acted on. I am sure they still have a lot more.

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Cover art: self-portrait of the author.

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Abbreviations used in this thesis:

CC	- Central Coherence	ToM	- Theory of Mind
CCN	- Piaget's *The Childs Conception of Number*	ToMM	- Theory of Mind Mechanism
DEF	- Disabled Executive Functioning	WCC	- Weak Central Coherence
EF	- Executive Functioning	WM	- Working Memory

Introduction

We can count and track multiple objects simultaneously. We have instant and precise feelings about small quantities and can easily approximate large quantities. Eventually most of us learn to count. We easily remember different aspects of number information: cardinal facts (number value), ordinal facts (place in an order) and nominal facts (numbers used as names).

Not everybody learns to use numbers to the full extent. Mental disabilities can disrupt our development. Among autists different levels of disability occur. Some autists never learn to count. A significant amount of autists develop a high level of intelligence including counting. A small portion of autists is capable of extraordinary mathematical feats. Even though autists can count and perform difficult mathematical tasks, they have problems. They have less social interaction and have abnormal verbal and non-verbal communication. They show repetitive and stereotyped behaviour. And autists have problems with global reasoning. Autists think differently.

But most high functioning autists can handle number situations just fine. How can autists have problem with so many situations, but not with number situations? To answer this question we look at what problems autists do have with numbers, at how typical developing children learn to deal with numbers and at how autism is explained.

We find that autists lack visual-spatial working memory to model number situations, but that they eventually can learn to mimic a typical level of number knowledge by using their verbal memory. Developing this memory strategy takes more time than typical developing children need to learn numbers and this strategy is less flexible. For some autists the strategy is however efficient enough to deal with day to day number situations.

Chapter 1

Autistic Counting

Not all counting procedures are handled in the same way by autistic and TD (typically developing) children. Because of these differences also later mathematical skills, most of which build on the initial counting skills, will differ. In this chapter we will present four ways in which autistic and typical counting differ. We will try to explain these difference in chapter 3. First in chapter 2 we will look at theories about number development. These theories will hopefully allow us to explain the differences between autists and TD children in counting and mathematical skills in chapter 4.

The following differences between autistic and TD children are specific to number knowledge: Autists do not benefit much from the canonical placing of dots. Autists have a different curve in reaction time when naming quantities. Autists are late to develop their sequencing skills. And autists have problems with recalling positions, which is not a counting skill in itself but is very likely to have an influence.

1.1 Counting Speeds

Jarrold and Russell investigated how well autistic children, children with moderate learning disorder and TD children count randomly and canonically placed dots.[1] The autistic children in their study were between 6 and 18 years, with a mean age of 12 years and 6 months. They were matched on the basis of their verbal mental age, which ranged between 4 years and 9 months and 9 years and 6 months, with a mean age of 7 years round. In the experiment a computer was used to present a white screen with black dots which were either placed randomly or canonically, as seen on dice. Together with the randomly distributed dots there are also twice as many distractor stimuli (white squares) displayed.

The children were told that they simply had to count the number of dots as quickly as possible and tell the experimenter how many dots they saw. If the child was correct the experimenter would press a button which both timed the response and initiated the next trail. First twelve canonical placed dots were shown and second twelve distributed dots. In both sets 3, 4, 5 and 6 dots were shown three times. In the canonical set each picture was shown three times and in the distributed (random) set each picture was different.

Overall the autists were not as quick as the other groups to react. When however reaction times were compared within each group between the canonical and distributed stimuli, autist seemed to benefit least of all from the canonical form in which the first set was presented. For the numbers 5 and 6 they benefit significantly less.

In a similar experiment Gagnon et al.[2] found autists to have a different response-time curve when counting small numbers. Autists in this study are all males, between 10 and 21 years, mean age 15 years. They are matched with children with similar verbal and non-verbal IQ's of 14,5 years old. Randomly placed white squares on a black background were checked for canonical patterns before they were used in the experiment. The children were asked with equal emphasis on speed and accuracy to tell how many white squares they saw on the screen.

Two different trials were run, one where the children were close to the screen and one where the children were further away. This resulted in the stimuli taking up a visual angle of 8 degrees or 2 degrees. The squares were being presented for 600ms after which children had 4500ms to respond, then 500ms later the next stimulus was shown. Children were tested in 4 batches of 80 trials each. The responses were recorded and timed. Trials with interfering sounds were discarded from the results. The experimenter transcribed the recorded session. The responses and stimuli were compared later. Accuracy judgements were determined by the number of errors for each numerosity, (incorrect answers / total number of answers).

The study showed no difference in results for the different angles and the data collected in the different angles were merged. The autist group was producing slightly more accurate results, but not significantly better. Reaction times were similar, but differing in one respect. Autist were significantly slower in recognizing the number 4. Differences between the reaction times of 3 to 5 were linear. Compared to the TD children this is striking because they have virtually no difference in reaction time when naming 3 or 4 and a sharp increase in reaction time between 4 and 5. TD children can name 4 a lot faster than autistic children, about 100 ms. Only 3 out of 14 of the tested autistic children showed an angle in their reaction-time curve between

1 C. Jarrold and Russell J., Counting Abilities in Autism: Possible Implications for Central Coherence Theory, *Journal of Autism and Developmental Disorders*, vol.27 no.1 p.25-37, 1997.

2 L. Gagnon, Mottron L., Bherer L. and Joanette Y., Quantification Judgement in High Functioning Autism: Superior or Different?, *Journal of Autism and Developmental Disorders*, vol.34 no.6 p.679-689, 2004.

3 and 5. Where 13 out of 14 typically developing children showed this angle.

1.2 Sequencing Shapes

McGonigle-Chalmers et al.[3] tested children with autism and Asperger's syndrome on their sequencing skills. The autistic and Asperger's syndrome children between 6 years and 10 months and 13 years and 10 months, mean age 10 years round, were matched for age with typically developing children, mean age 9 years and 6 months. The children were tested using a sequencing 'game' on a touchscreen which displayed a star shape in different sizes. Sizes ranged from 5 to 38cm with a minimal 3cm interval. Children were instructed to touch the stars in an ascending or descending order. This order was cued by using different colours, either blue or pink.

Children were given four warm up trials with only five shapes to get acquainted with the game. The real game started with 9 shapes and allowed for children to improve to more shapes or 'crashback' to less shapes. The maximum number of shapes was 12. Lower levels consisted of smaller subsets of the sizes, for example level 9 consisted of shape sizes: 1, 2, 4, 5, 7, 8, 10, 11and 12. When children got 4 out of 6 trials right they would reach criterion for that quantity and proceed to a greater quantity. If they failed 18 times they would go down a level. Children who failed the starting level (9) were directly taken back to the 'crashback' level 7 and could regain level 9 by training.

Two autistic children failed to reach criterion on all levels. Most autistic children, 12 out of the remaining 18 crashed back to level 7, 6 succeeded on level 9 immediately. The opposite was the case for TD children, where 6 crashed back and 14 succeeded. The following diagram depicts for each autistic and TD child the highest level on which they reached the criterion.

	Crashback Levels		Entry Level	Higher Levels		
Number of items in set	7	8	9 -> -> 9	10	11	12
Children with Autism or Asperger's (n=18)	7	5	2 -> -> 0	1	1	2
TD Children (n=20)	3	0	5 -> -> 3	0	3	6

Diagram 1: Children reaching criterion on the sequencing task by McGonigle-Chalmers et al.

The top number in the entry level column states the children that reach the criterion on this level straight away, the bottom number gives the children that reached the criterion after walking trough both crashback levels.

When comparing the older and younger children of both groups, McGonigle-Chalmers et al. conclude that the older autistic children are on par with the younger TD children. So in this respect there is a developmental difference of about 2 years. They found no difference in performance between the autism and Asperger's syndrome subgroups. Duration of the task was not significantly different between the groups. TD children had a mean of 36,5 trials and autists a mean of 43 trials.

According to Steele et al.[4] autists also have problems with recalling positions of objects, which is likely related to sequencing problems. In their study Steele et al. tested 29 high functioning autist and 29 TD individuals matched for verbal and performance IQ, age and socioeconomic status. Age ranged from 8 to 29 years in both test groups, with the autist group having a mean age of 14 years and 10 months and the TD group having a mean age of 16 years and eleven months.

Participants were introduced to a touchscreen showing a group of boxes. Hidden beneath one of these boxes was a token and they were asked to search for it. Touching a box would reveal whether a token was

3 M. McGonigle-Chalmers, Bodner K., Fox-Pitt A. and Nicholson L., Size Sequencing as a Window on Executive Control in Children with Autism and Asperger’s Syndrome, *Journal of Autism and Developmental Disorders*, vol.38 no.7 p.1382-1390, 2008.

4 S.D. Steele, Minshew N.J., Luna B. and Sweeney J.A., Spatial Working Memory Deficits in Autism, *Journal of Autism and Developmental Disorders*, vol.37 no.4 p.605-612, 2007.

beneath it. When a token was found, the same set of boxes was hiding a new token. Each box would only hide a token once. This was told to the participants. In one set of trials each box would hide a token once. When all tokens had been found boxes changed colour and position to show a new set of trials had started. Participants started with a warm-up trial with 3 boxes. After finding all tokens participants progressed to the next level with 4 boxes. After this they progressed to 6 boxes and to 8 boxes. Each level was played 4 times before going the the next, making the total tokens to be found during the experiment 72.

Steele et al. distinguished two types of errors: the within trial error, choosing one box twice before finding a new token and between trial error, choosing a box which had been hiding a token on previous trials. By calculating the possible number of errors on each set of trials, a comparison was possible between sets with a different number of boxes. They also evaluated the possibility of participants to adopt a search pattern. Participants could follow a certain search path on each trial and by cleverly computing the possibilities of these paths Steele et al. could give a number between 8 and 56 indexing the use of strategy.

There was no significant difference between groups on within trial errors. There was a general increase on between trial errors (choosing a box which had been used) when the number of boxes increased. Autists made more between trial errors than TD participants. The increase in mistakes was practically linear in TD participants. In the autist group going from 4 to 6 boxes had a steep increase, whilst going from 6 to 8 boxes the relative number of mistakes remained almost equal. There were significant strategy score differences between the two groups. Autistic participants were less consistent in their sequential search strategy than TD participants. When participants did not use a sequential search strategy in a trail, a significant correlation could be found with between trial errors. The number of between search errors and strategy scores could be correlated with performance IQ but not with verbal IQ for the autism group but nor for the TD group. No correlation was found between search errors or strategy scores and age.

Chapter 2

Understanding and Developing Numbers

In this chapter we will look at how we understand numbers. We will look at an early developmental theory by Piaget[5] and at later developmental theory by Bloom[6] and others. Because of chronology we will start with Piaget theory and move on to later theories.

2.1 Early Developmental Theory

Piaget was one of the first researchers in the field of developmental psychology. Together with Vygotsky, Piaget was one of the first to put concept formation at the centre of our development. Piaget was puzzled by how we come to deal with abstract knowledge. He approached human development theoretically as well as empirically. Number knowledge was important to Piaget because it is generally seen as an abstract form of knowledge.

2.1.1 Piaget's Developmental Theory

Piaget formed a developmental theory. He distinguished four developmental stages. In each stage we develop differently. We start with the sensorimotor stage, which lasts from birth to our 2nd year. In this stage we learn to feel, give meaning to and get acquainted with these feelings. We discover we can move our body. We continue to develop our motor skills in the preoperational stage, from our 2nd to 6th year. Before going into the third stage we have complete control over our body. In our 6th or 7th year we start to think logically about actions and events. Piaget calls this the concrete operational state, which lasts until our 11th year. In the final formal operational stage we become fully developed abstract thinkers. According to Piaget there is no standard duration for development and ages are only an indication.

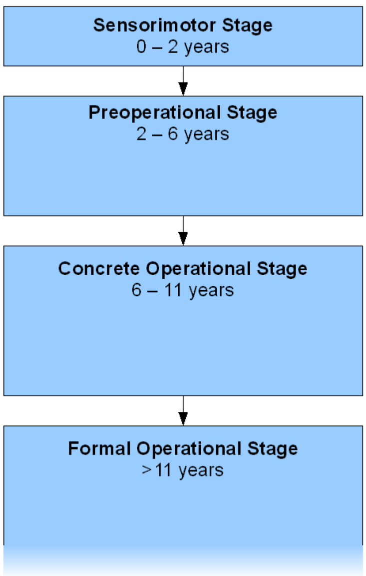

Diagram 2: Piaget's Four Stages

In each of these stages we have a different way of reasoning, this starts out with our sense and develops into abstract reasoning. Each type of reasoning builds on the previous forms of reasoning. According to Piaget the stages can only take place in this necessary order. During each stage our understanding and control increases. Sufficient increase in understanding and control accommodates the next stage and novel way of understanding.

In the first stage we learn what feelings are like and in what way we can move our body. In the second stage we move on to the manipulation of objects. Manipulating objects in turn drives and necessitates the comparison of objects. Finally comparing objects makes it possible to compare ideas. We learn to transform

5 J. Piaget, *The Childs Conception of Number*, Routledge & Kegan Paul Limited, 1952.
6 P. Bloom, *How Children Learn the Meanings of Words*, The MIT Press, Cambridge, Massachusetts, 2000.

our initially laboriously manufactured feelings and concepts into piecemeal concepts that we can use more easily. These set concepts become the building blocks of our reasoning. Each stage can be divided into substages describing smaller changes; the forming of a concept, combining concepts and making a concept out of a routine.

Combining Theoretical and Empirical Insight

Piaget combined theoretical with empirical insight and this caused controversy. Hamlyn would not accept the mixing of philosophical ideas and empirical facts.[7] He thought that the philosophical questions about understanding could not be answered by looking and studying actual development. According to Hamlyn researching individuals does not answer any epistemological questions. Hamlyn's point of critique being that Piaget's necessary order is not falsifiable, even Martian babies need to develop sensorimoter substage 4 before 5. We need to know 'means' as such before we can connect those means to ends. According to Hamlyn describing the only possible and thus the obviously necessary order in which new developments arise, it not a theory. To Hamlyn Piaget's theory gives no new insights at all, but is merely some facts and some obvious logical truths put together.

Boden[8] defends Piaget saying that (these) successive stages could also appear simultaneously, instead of one after the other. Piaget's empirical facts augment his ideas by filling in the contingent possibilities, such as two developmental steps being separate or coinciding. Reasoning about the order of development alone, can never give us a complete picture of development. In this way Piaget's empirical observations combined with philosophical and theoretical ideas results in an interesting epistemological theory. Piaget constructed a valuable developmental theory with theoretical and empirical insights.

2.1.2 Epistemological Considerations

Piaget placed himself in-between the 'tabula rasa' and the 'a priori' knowledge theory by saying that a rational mind can only come to exist by developing from innate structures. Piaget did not believe our mind is completely empty at the start, instead he believes that our biological make-up makes us act in certain ways. A newborn tries to suck on everything it encounters, this being the way we tackle the world until we have developed further. When we are young we do not automatically know how to handle numbers. Only by repeatedly being in certain situations we understand such situations. And only after understanding certain simple situations can we understand more complex situations. When we never encounter certain situations we will not learn how to deal with the concepts involved in these situations. Not developing number concepts because one does not encounter them is however not likely. Number concepts recur in a large group of very basic skills and it would be impossible not to encounter any of these situations.

Piaget places himself opposite to Descartes by saying rationality can only develop in living and moving creatures. Descartes accepted the possibility of rationality to exist without a physical substance. On the contrary, Piaget's developmental theory builds up in such a way that having intelligence presupposes a physical, living and moving entity. Because, according to Piaget, active and conscious movement needs to be experienced before we can develop simple preoperational cognition etc. up to abstract thought.

According to Piaget, psychology is important for our epistemology, it is as important as logic. Piaget gives two reasons.[9] First, human knowledge is itself a combination of psychological and rational/logical ideas. Therefore the theory describing our knowledge should agree with logic and psychological theory. And second, logic is based on and constructed with our human ways of reasoning. Thus our rigid logic relies partly on our less rigorous psychological make-up.

Individual Learning

According to Rotman, Piaget's developmental theory does not mention social context or social knowledge and when failing to do so Piaget conveniently ignores the social context of knowledge.[10] A common view about mathematical proof is that it is constructed in a social context and such a proof can be regarded as socially acquired or constructed knowledge. Piaget fails to give an adequate account of how mathematical proof is constructed. Piaget concentrates on the development of the individual and not on how the development of the individual relates to the development of others. His focus on individual development is reflected in his view of constructing concepts. Piaget thinks that objective truth can only be obtained by individuals forming the ideal concept for objective truth. He claims that even though humans are very social and learn in a social context they develop their thinking-structures independently. Therefore we would also develop the structure which represents 'objective' knowledge alone.

7 p.83-84 of M. A. Boden, *Piaget*, Fontana Press, 1979, with new foreword from 1994.
8 M. A. Boden, *Piaget*, Fontana Press, 1979, with new foreword from 1994.
9 p.86 of M. A. Boden, *Piaget*, Fontana Press, 1979, with new foreword from 1994.
10 Chapter 7 of B. Rotman, *Jean Piaget: psychologist of the real*, Ithaca NY, Cornell University Press, 1977.

Opposite to Piaget's view of individual learning is the view that mathematical truth is constructed in a social context. By discussing ways to construct mathematical objects, proofs, an intersubjective truth is created within (a part of) the mathematical social context. This truth can then be called objective. Piaget seems inclined to some form of *a priori* truth. Piaget thinks each human is capable of concluding this truth from the world without the help of any social context. Of course one would only come to develop such an ideal concept for objective truth presupposing an ideal situation and an ideal being. Surprisingly Piaget does think that humans are fallible and that development is not always successful.

Piaget based his theory on psychologically qualitative investigations instead of a scientifically statistical research. A basis which is not always appreciated. Piaget started his career with a psychoanalytic approach where the doctor examines the patient. Over the course of his life he gradually conformed to doing statistical research. For hypotheses generation such observation of individuals can reveal very salient facts. However statistical analyses based on a broad population is generally preferred for testing hypotheses. So when Piaget uses empirical facts to back up parts of his theory, a statistically significant number from a broad investigation would be more convincing than a few psychoanalytic examinations. Later researchers have duplicated a lot of Piaget's trials and did so with statistical analyses. We will reference these new trials when possible.

Another shortcoming in Piaget's epistemic theory was the minor role of language, which is hardly discussed in his trials. Later some of his analyses turned out to have very different results when children were asked more specific questions or when questions had a different focus.[11] (More about this in 2.1.4.)

Despite not dealing with social context and language and basing his analyses on qualitative investigations, we will look at Piaget's book *The Child's Conception of Number* in which he analyses children's development of numbers. Piaget's overall theory is still very complete and compelling. We are interested in comparing Piaget's views with those of later researchers, because Piaget has an interesting view of number knowledge and influenced many researchers in developmental psychology.

After looking at what Piaget said about number learning, we will see that later researchers filled some of his omissions (see 2.2). Whether social knowledge, social learning and the social context of knowledge is really important in the case of learning numbers might be answered by comparing how autists and non-autists learn (see 2.3 and 3).

2.1.3 Physical Development Before Conception of Numbers

The most interesting changes in our concept of number take place at the end of the preoperational stage when children go into the operational stage. In those stages of our development some basic number concepts are constructed every time a situation involving numbers is encountered. These concepts evolve into ready ideas. But before we discuss Piaget's number concepts we have to look at the developments in the sensorimotor and preoperational stage.

According to Piaget the organization of thinking takes shape long before children start talking. This can be seen in early childhood when children interact with their environment. The first steps of cognitive development take place in the sensory and motor skills. Piaget distinguishes six steps of development in the sensorimotor stage: 1) reflexes and spontaneous movement, 2) repetitive habits, 3) active pursuit of interesting sights 4) co-ordination (the discovery) of causality between means and ends, 5) discovery of means, which means belong to which ends and 6) start of representational thought, which enables the instant comprehension of means and causal relations between events in new situations.

11 M. Donaldson, Conservation: What is the question?, *British Journal of Psychology*, Vol.73 p. 199-207, 1982.

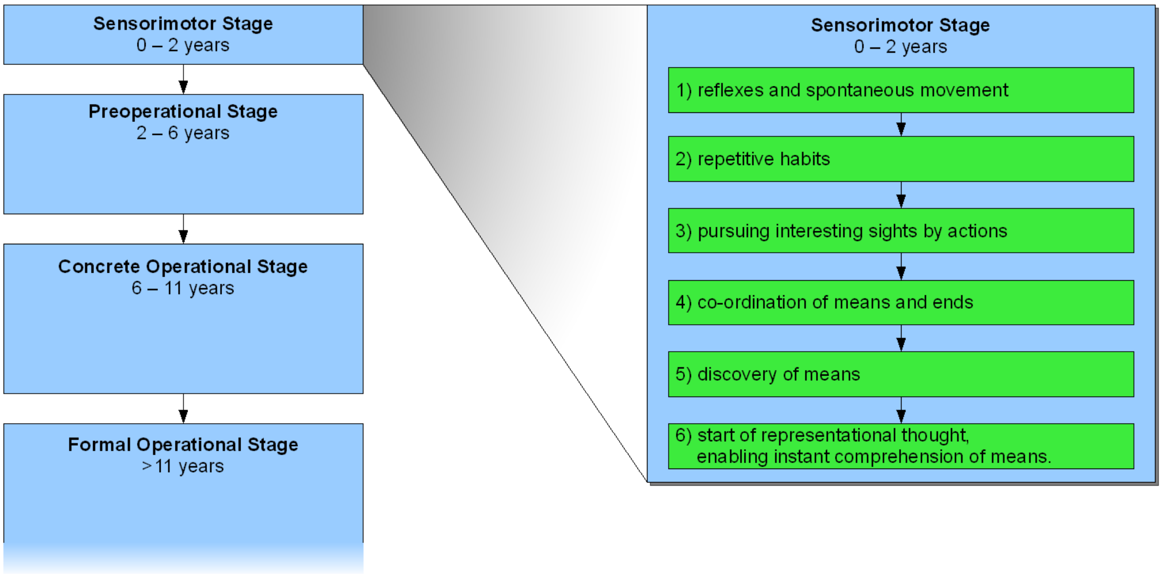

Diagram 3: The Sensorimotor Stage

Our rapid development of skills shows that we are born with some innate structures for reasoning. Not having any innate structures would never allow such a speedy development of coordinated movement. According to Piaget our development can not be accelerated significantly by education. We need time to develop our motor skills and more time to develop our social and logical skills. Of course Piaget acknowledges that western education has some effect on the speed of our development, nonetheless he predicts that the four stages are universal and also occur (in cultures) without formal education. Piaget puts stress on the order of our development and not on its speed.

At the end of the sensorimotor stage we learn to see our first 'means and ends' and we learn to understand the world as causally connected during the pre-operational stage. This development is accelerated by our interaction with the external world. By interacting a general understanding of time and space is developed. We learn and need to make more precise movements for more precise manipulation of the world. This happens approximately between the 2nd and 4th year of life. At this age we learn to use words with actions, but according to Piaget these early words are not linked to abstract concepts yet. Intelligence at this age can not be measured by the use of language, since we do not yet use abstract concepts in our thinking. Subsequently we do not use abstract concepts in our speech. At this age we do use words that adults use for abstract concepts, but attribute a different meaning to these terms. We have different concepts because we lack the experience to construct the complete concepts. Piaget says that this is why we cannot yet make proper use of class words such as all, some and equals and why we use words as 'same' and 'sort of' differently from our 7th to 9th year.[12]

12 p.37 & p.41. of M. A. Boden, *Piaget*, Fontana Press, 1979, with new foreword from 1994.

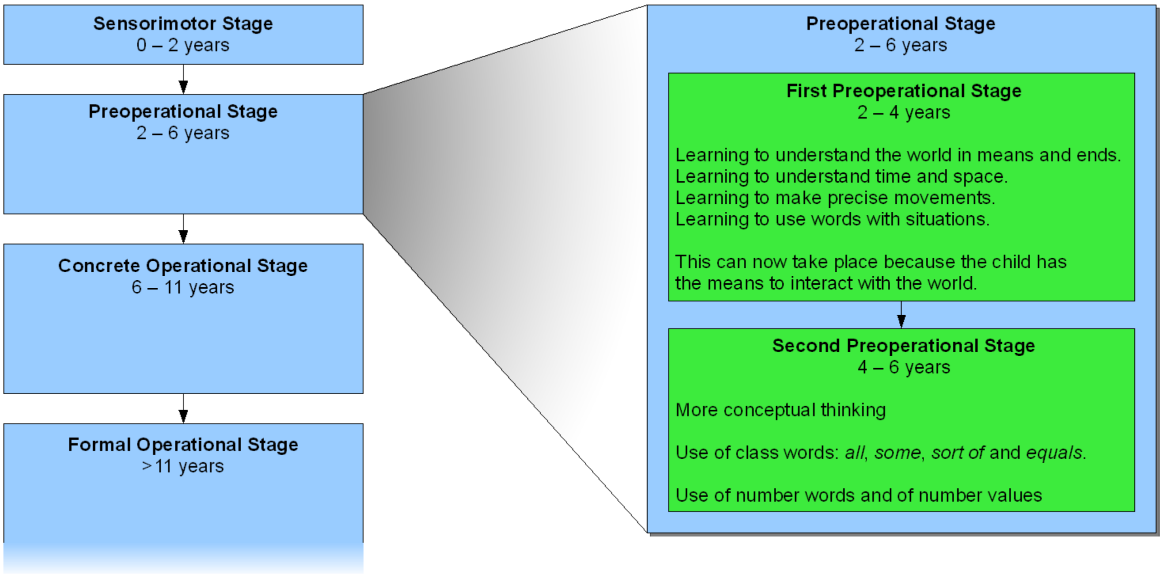

Diagram 4: First and Second Preoperational Stage

According to Piaget language helps us develop mathematical skills, but language does not cause us to develop mathematical skills. Number knowledge is something that develops alongside language. New research in animal number skills supports this view. An understanding of quantity skills in animals does not go hand in hand with (an advanced) language skill.

Beran found quantity conservation in capuchin monkeys.[13] [14] This means the monkeys were able to retain number information without seeing the objects whose quantity they had counted. Beran concludes that the monkeys must have a quantity skill. The monkeys capability for conservation is very similar to human conservation. Beran quotes writers who found such quantity skills in lions, dolphins, gorilla's, baboons, orang-outangs, salamander, ants, rats, fish, pigeons, dogs and squirrel monkeys. Not all these animals have 'counting-like' skills and different strategies to successfully deal with number situations are found. Associative learning strategies were found in smaller monkeys and rule learning strategies in larger apes. Nonetheless, the list of species gives us some reason to believe that number cognition arises separate from language.

The second half of the preoperational stage between the 4th and the 7th year is when we turn into more conceptual thinkers. In this stage we start to use numbers words as representing a number value. The use of class words however remains problematic. The understanding of quantities develops mostly around our 6th year. This development marks the transition from the preoperational to the the concrete operational stage. In The Childs Conception of Number (CCN) Piaget confronts children with different situations in which they have to deal with quantity, ordering, numbers and counting.[15] We will look at the analyses Piaget made of these situations and give a short overview of his theory.

With the investigation of quantity, ordering, numbers and counting Piaget distinguishes three degrees of comprehension: 1) not understanding, 2) understanding but being confused or overwhelmed by complexity and 3) full understanding. By analysing these three degrees for each concept Piaget gives form to his theory. The child develops all the different concepts simultaneously. We try to give a diagram which shows the simultaneous development of the different skills.

We will discuss the number concepts in the same order as Piaget in CCN. Piaget starts with conservation by looking at continuous and discontinuous quantities. He continues with correspondences, seriation and ordination and cardination.

13 M.J. Beran, Capuchin monkeys Cebus apella succeed in a test of quantity conservation, *Animal Cognition*, vol.11 no.1, 2008.

14 The capuchin monkeys in Beran's trials performed well on a conservation trial. The monkeys were trained to choose from the more numerous of two arrays of squares. On certain trials one of the arrays would be changed; a) a few squares were added or subtracted b) the squares moved closer together or further apart or c) a combination of a) and b). After the change the monkeys were allowed to choose between the arrays for a second time and most often they chose the more numerous array.

15 J. Piaget, *The Childs Conception of Number*, Routledge & Kegan Paul Limited, 1952.

2.1.4 Number Concept: Conservation

According to Piaget conservation is a presupposition of rationality and mathematical rationality. We can only form concepts if our ideas or notions are constant. If a concept would change in its 'definition', what would we be thinking about? Piaget describes this conservation as the principle of identity. He says that by the continuing experience of an object we presuppose its continuity. This form of conservation is likely one of the first constructions we form.[16] Granted that conservation is a necessary condition of experience, it does not explain all possible forms of conservation. Some forms of conservation we develop later on, among them conservation of number value and conservation of quantity.

Piaget claims that number concepts can be acquired based on this principle of identity. Conservation does not precede any number activities and is not completely a priori. This is at least what Piaget concludes from his analyses of how children experience quantities of water (continuous) or beads (discontinuous) when poured into smaller containers. Piaget says that we do not start out respecting conservation, but that we gradually construct it.

In one conservation trial Piaget showed a child a glass of water. This glass was subsequently poured into two or more smaller glasses equal in size. The child was then asked whether there was more or less water. These two steps were repeated, the water was divided over more and smaller glasses of equal size. After each transformation the more or less question was asked again.

Children reacted in three distinct ways, roughly divided by age. The youngest group thought that either bigger glasses or more glasses resulted in more water. In their experience this change occurs when the water is poured into the other glasses, thus within seconds. The oldest group understood that the starting amount could not be changed by mere division. Children in the middle group however thought the amount stays the same after the first division, but that the quantity changes after further divisions. Or they think the amount of water stays equal only when the subsequent glasses are relatively similar (in height or width) to the first glass.

The middle group is the most interesting, because these children are in the transition from one thinking system into another. According to Piaget this transition is most often found in children aged between 5 and 7 years. If differences in the magnitude of the glasses is too great these children will fail to see conservation of quantity. From what Piaget shows us of his trials, the children in this transition are clearly straining to construct their opinion about the amount.

> Pie (5;0): 'Is there the same amount here (A1) and there (A2)?–(He tested the levels.) *Yes*.–(A1 was poured into B1+B2). Is there the same amount to drink in these two together as in the other one?–(He examined the levels in B1 and B2, which were higher than in A1.) *There's more here*.–Why?–*Oh yes*, *it's the same*.–And if you pour the two glasses (B1 and B2) into these three (C1+C2+C3), is it the same?–*There's more in the 3.*–And if I pour it back into the 2?–*Then there'll be the same* (B1+B2) *as there* (A2).'[17]

According to Piaget the development of understanding of these children is due to them starting to co-ordinate, 'a multiplication of logical relations'.[18] At this age we get a better grasp of the situation and learn to simultaneously look at the different (spatial) qualities of the water. We learn to tackle more information simultaneously and we no longer look only at the width of the glasses or the height of the liquid, but combine these dimensions to estimate the volume of the liquids. This development allows us to see that the amount of water remains equal.

Motivation in Development

When the situation gets too complicated, we cannot cope with the workload and we return to a simpler representation of the situation. This fall back can be seen in these transitional phases. Meadows remarks that this lack of precision could be seen as a lack of interest.[19] When a child is not interested it chooses the path of least resistance. Meadows witnessed this in middle school children. Because of its lower workload children fall back to an earlier but less precise strategy, instead of constructing a precise answer with a new strategy. We learn to construct a new complexer strategy eventually. Training concepts, repeated experience of complex concepts leads to developing a better understanding of these concepts and to a lower workload when reasoning with these concepts. So with practice we give more precise answers even in complex situations. Nonetheless personal interest is a dimension which influences the speed of development.

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16 It is unclear whether Piaget takes this principle to be innate or not. By calling it a principle he makes it distinct from a concept and thus a more structural element in his theory.

17 p.14 of J. Piaget, *The Childs Conception of Number*, Routledge & Kegan Paul Limited, 1952.

18 Terms used by Piaget to describe the forming of concepts, not actual multiplications.

19 p.35-49 of S. Meadows, *The child as thinker: the development and acquisition of cognition in childhood*, Routledge, London, 1993.

Conservation of Magnitudes
In the preoperational stage we do not experience the conservation of the quantity of water. This conservation is not realized until we fully understands how to combine the different dimensions of containers and their contained liquid. When we get to the concrete operational stage we know how to combine the dimensions and how to compare different containers. At this point in our development we compare volumes and realize that a given quantity cannot change.

According to Piaget when we learn to consider quantities or magnitudes we also learn to see partitions. When we compare two quantities we see the difference between these quantities. We develop this skill into comparing differences between quantities. The partitioning of magnitudes starts with knowing how to value magnitudes and when seeing different magnitudes to consider the difference in these quantities. (We mentally divide a glass into equal 'layers' or 'columns' of water. We compare the quantities of these partitions.)

In another conservation trial Piaget replaced the water with beads. These beads are taken from the glasses and are used to make necklaces, which allows the children to asses how many beads there are from the length of the necklaces. They can then answer the experimenter by comparing the necklaces from two glasses to decide whether the glasses contain the same or a different quantity of beads. The children participating in this trial can again be divided into three groups; no understanding, transitional and understanding. Most interesting is the middle or transitional group of children. In the next quote the A1 and A2 glasses are filled with an equal amount of red and green beads. In addition to voluminous quantity the child is asked whether the necklaces have the same length. The researcher and the child then fill the glasses together with an equal amount of beads.

> Ari (5;6): '*They're the same* (A1 and A2).–And if we make two necklaces, etc?–*The same length*.–And if we pour these (A2) into that (L)?–*There'll be more there* (L).–Why?–*Because it's higher*.–And if we make it like this (A1 into 4 E)?–*There'll be more there* (4 E).–And if we make a necklace?–*It'll be longer*.[20]

Only when the children in this transition consider the beads as a linear quantity do they see the quantities are equal. With the beads in the glasses the quantities are treated as a continuous quantity, as water. In this transitional phase thinking about the beads as a line in their 'necklace-state' or as a volume changes the evaluation of the children.

The Order of Understanding Conservation
Similar conservation trials have generally verified Piaget's results, but show that overall children conserve numbers better and earlier than Piaget expected.[21] Piaget already treated the different forms of conservation in separate paragraphs in CCN and now Jamison found a statistically significant relation between discontinuous and continuous conservation in TD children. Good performance on a discontinuous trial predicts a good performance on a continuous trail.

Gruen and Vore compared the conservation of number, magnitude and weight of TD and familialy retarded children.[22] [23] They compared 30 familialy retarded children with two groups of 30 TD children. The retarded group was equally divided into the mental ages 5, 7 and 9 years. The retarded group was matched with a same mental age TD group and a same corporeal age TD group. All children were tested individually. Gruen and Vore asked the children whether two quantities were equal or not, which of the two was bigger and why it was bigger. These questions were repeated after a transformation of the quantities. The objects under investigation were poker chips (discontinuous), water in beakers (continuous) and play-doh (for weight).

The answers the child gave to the question of equality and size were matched with the reason for the change or conservation of the situation. Answers and reasons which did not match were marked ambiguous and counted as non-conservation. Only 6,8% of all questions was mark ambiguous. All three retarded age groups conserved best with numbers (discontinuous quantity), followed by magnitude (continuous quantity) and least best on weight. The same results can be found in TD children aged 5. Gruen and Vore conclude that the development of the familialy retarded children is lagging behind. Eventually they do reach a normal understanding of conservation.

Both Jamison and Gruen and Vore conclude that children first learn to conserve discontinuous quantities and

20 p.30 of J. Piaget, *The Childs Conception of Number*, Routledge & Kegan Paul Limited, 1952.
21 W. Jamison, Knowledge of Number Conservation and the Acquisition of Quantity Conservation in First Graders, *Journal of Psychology*, vol.112 no.2 p.237, 1982.
22 Familial retardation is not causes by physical disability, but is mostly due to their family circumstances. This includes social background, educational levels of the parents, prenatal care, nutrition etc.
23 G. E. Gruen and Vore D.A., Development of Conservation in Normal and Retarded Children, *Developmental Psychology*, vol.6 no.1, 1972.

then learn to match continuous quantities (and then weight). Of course these developmental stages partially overlap, but the eventual understanding of the different stages has this set order. Both Gruen, Vore and Jamison verify the order of the concept formation of conservation as postulated by Piaget.

A Child's Interpretation of Trials

Piaget did not consider how children look at the purpose of the actions performed in his trials. Donaldson shows us that children have to learn that rearranging the position of objects, like the transformations in Piaget's experiment, is something which is not relevant for the conservation question.[24] She tells us that young children tend to interpret the question and the context as a whole.[25] According to Donaldson it is due to this interpretation that children think that they are supposed to choose the set that is 'bigger'. Donaldson and McGarrigle adapted Piaget's experiment by having a 'naughty teddy' rearrange the objects in the trial.[26] This change to the trial caused more children to give conserving responses. The transformation being made by the teddy bear instead of the experimenter changed the rearranging of objects to an incidental act instead of an intentional act.

Donaldson also adapted Piaget's continuous conservation trial. Donaldson told children they were going to play a game with the experimenter. Both players were given the same amount of pasta shells, collected in a beaker. This beaker is then remarked to have a potentially dangerous chip. The pasta shells are transferred from the chipped beaker to a replacement beaker which is found in the same room. This new beaker has a different shape and thus children are unsuspectingly tested on their conservation skills. In another version of the trial replacement of the beakers is intentional and only 5% of the children in this trial conserved, whereas 70% of the children in the chipped beaker trial conserved.

Donaldson acknowledges that transformations in her trials still have an implicit meaning, but her transformations have a different force from Piaget transformation. The naughty teddy transformation is almost explicitly implying it is not important to the situation. The replacement of the chipped beaker has a more neutral implicit meaning and is an example of the relevance of context that Piaget missed.

Piaget ignored context in his trials. Adapting trials could prevent children from misinterpreting the experimenters motives, but Donaldson thinks that adaptation can only do a part. Children change their focus all the time, sometimes this focus is directed to impersonal, physical features and sometimes to interpersonal features. Disregarding this interpersonal stance can mess up the result of a conservation trail.

According to Dehaene Piaget's conservation trials are not about number conservation but about the resisting of distraction.[27] Children get confused by Piaget because he asks which of the two options is bigger, whilst they are equal. Thereby the experimenter downplays the possibility that the two options are equal, this naturally leads to the children choosing this option less often.

Behavioural Trials

Dehaene criticized[28] Piaget on the way he tested children. According to Dehaene we need to devise tests in which we can read children's reactions indirectly, because they are not yet capable of direct communication. Dehaene proposes a way of 'reading' the child by using methods borrowed from animal research. This behavioural testing allows us to test younger children and it prevents some possible miscommunication. Piaget's psychoanalytic testing uses direct communication, but speech is something young children cannot yet do and something that somewhat older children do not do in the exact way we expect them to. This method does not use language, but looks at how children react to quantity with their behaviour.

The different trials on conservation show how difficult it is to design a in which only conservation is tested. A part of the behavioural approach can be found in a replication of Piaget's number conservation trial by Mehler and Bever.[29] Dehaene concludes from this trial that children have spectacular capabilities very early on in life. Rothenberg and Courtney reproduced the Mehler and Bever research and show again that a slight

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24 M. Donaldson, Conservation: What is the question?, *British Journal of Psychology*, vol.73 p.199-207, 1982.

25 Donaldson puts this opposite to an adult way of thinking of language. Adults mostly think of language as having what Grice dubbed 'timeless meaning'; an absolute meaning of sentences. Note that this is how adults **think** of language, not how they actually use it.
p.202 of M. Donaldson, Conservation: What is the question?, *British Journal of Psychology*, vol.73 p.199-207, 1982.

26 M. Donaldson and McGarrigle J., Some clues to the nature of semantic development, *Journal of Child Language*, vol.1 p.185-194, 1974.

27 p.47 of Dehaene, *The Number Sense: How the mind creates mathematics*, OUP, 1997.

28 p.44 of Dehaene, *The Number Sense: How the mind creates mathematics*, OUP, 1997.

29 J. Mehler and T.G. Bever, Cognitive capacity of very young children, *Science*, vol.158 p.141-142, 1967.

change in context changes the child's interpretation.[30]

Mehler and Bever replicated one of Piaget's conservation trials with M&Ms. Children were presented with two rows of M&Ms and were asked to take the one row that they wanted to eat. This was done in comparison with a test in which clay pellets were used and the children were asked to take the row with 'more'. Children do pick the most numerous M&Ms row. Mehler and Bever were surprised to find a 6 month period around the age of 4 in which children lose their capability to chose the more numerous pellets row. Mehler and Bever think this gap has to do with an overdependence on perception, which that makes the longer row is seen as bigger. According to Mehler and Bever this dependence is however overcome when the children are tested with candy instead of clay pellets. In this case the dependence on perception is overcome by the understanding that more candy units is better.

Rothenberg and Courtney did a slightly different version of the trial and did not find a dip around 4 years of age. Rothenberg and Courtney still doubted whether the younger children could understand the question and changed it from 'which row has more?' to 'which row do you want too eat?'. And they did see an overall better conservation performance on the M&Ms compared to the clay pellets. Contrary to the word 'more', 'eat' is understood correctly by all children and therefore trials with the eat-question get a better conservation score and look like a better way to judge the children's capacity for number conservation.

Rothenberg and Courtney consider the possibility that the children in Mehler and Bever's trial chose the more numerous row because the experimenter transformed this row and it was therefore marked. According to Rothenberg and Courtney children in the 6 month dip probably interpreted 'more' as 'takes up more space'. Children after the 6 month dip understand 'more' as 'more numerous' and chose the right row. This misunderstanding in language would explain the 6 month dip in the research done by Mehler and Bever, but does not tell us anything about the conservation skills of these children.

These conservation trials show that very young children can already conserve quantities despite transformations and that this skills is present long before we are 5 years old. Piaget did show we can name and express ourselves correctly at that age about conservation.

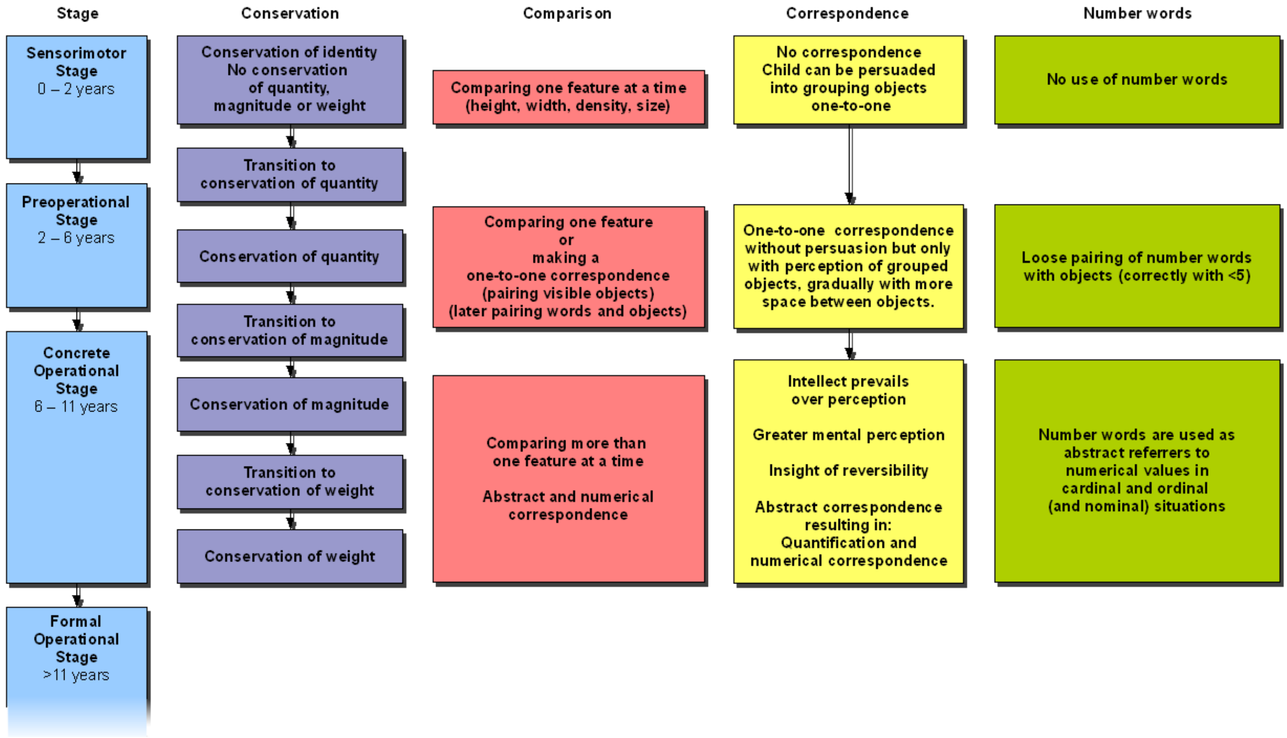

Diagram 5: Stage, Conservation, Comparison, Correspondence and Number words

(large version in Appendix B)

2.1.5 Number Concept: One-to-one Correspondence

According to Piaget an understanding of the concepts of value and the skill of lasting conservation numerical of quantity (remembering numbers) are developed simultaneously. These skills develop simultaneously

30 B.B. Rothenberg and Courtney R.G., Cognitive capacity of very young children: a replication of and comparison with Mehler and Bevers study, *The Journal of Psychology*, Vol. 70, p. 205-212, 1968.

according to Piaget because we are only capable of these two new insights after we develop abstract correspondence. And we develop abstract correspondence only after the simpler one-to-one correspondence. In Piaget's developmental theory of number knowledge the transition from correspondence to number proficiency is supported only slightly by empirical evidence. Piaget might be right or wrong, the empirical evidence is at least not speaking against him.

The second number concept Piaget investigates in CCN is correspondence, starting with one-to-one correspondence. In the most basic one-to-one correspondence the relation between two sets can be seen as a correspondence between the number of elements in two sets or between the size of the volumes of two sets. Note that number and volume of a set need not be equal. When adults make a comparison between two sets they often make a one-to-one correspondence between the number of elements within these sets. This is a cardinal one-to-one correspondence. In a correspondence the quantity of the objects is not necessarily understood in symbolic terms, it may simply be paired one-to-one: one object from the first set with one from the second set. We are not innately capable of doing a correspondence, let alone a cardinal correspondence. We start by looking at each set as one whole and make no one-to-one correspondence between sets, but we compare them globally. Piaget states that: 'the child should begin by considering non-analysed wholes, without feeling the need to decompose them as long as experience does not compel him to do so, is perfectly consistent with what we know of the psychology of thought at this level.'[31]

Piaget talks about two situations in which we develop one-to-one correspondence: 1) pairing of objects that complement each other and 2) making a one-to-one correspondence of two set of the same kind of objects. The second situation is the more advanced and the one we develop later. Examples of 1) pairing an egg and an egg-cup, pairing people and drinks, paring plates and cutlery. Example of 2) comparing two amounts of candy or marbles.

In his first correspondence trials Piaget asks children to make correspondences between different objects and sets of the same sort of object. In the easier 'two-object' situation the children are asked to correspond 6 glasses and 6 bottles. The bottles contain water and to make things easier the children can pour the water from the bottles into the glasses. In a later correspondence trial the children are asked to get a similar set of buttons (pennies, sweets or other tokens) as the experimenter from a box of buttons. The reactions to these correspondences can again be divided in three degrees of understanding. The youngest children do not grasp one-to-one correspondences, children in the preoperational stage at first understand but fail to when the situation is rearranged and the oldest children understand the situation even when objects are rearranged. The youngest can be persuaded into pairing different objects one by one. The youngest children do not do this spontaneously, but only when they are instructed to do so.[32] According to Piaget the experience gained from the instruction does help them to learn to make correspondences. Here is an example from the transitional group, where some understanding of pairing objects can be seen:

> Fu (5;9) poured the contents of the six bottles into 6 glasses and put the glasses in front of the empty bottles. 'Is there the same number of glasses and bottles?–*Yes.*–(The bottles were grouped together in front of the glasses.) Are they the same?–*No.*–Where are there more?–*There are more glasses.*–(The reverse process then took place.) And now?–*There are more bottles.*–What must we do to have the same number?–*We must spread out the glasses like this, no, we'll need some more glasses.*'[33]
>
> Per (5;7) had no difficulty in making a row of 6 sweets corresponding to the model. The model was then closed up: '*I've got more*.–Why?–*Because it's a longer line*.–(The process was reversed.)–*Now there are more there, because it is a big line*.' But a moment later Per said the opposite: 'Are there more here (spaced out)?–*No*.–Why not?–*Because it's long.*–And there (closed up)?–*There are more there, because there is a little bundle* (= close together).–Then are there more in a little bundle than in a big line?–*Yes*.' After this Per went back using length as the criterion, made the two rows the same length again, and said: '*Now they're both the same*.'[34]

From Perception to Abstraction with Reversibility
In trial similar to his correspondence trials Piaget tries to find out whether children understand reversibility. Whether children understand that paired objects which are split can again be joined.

> Sim (5;7) put one flower into each vase. They were taken out and bunched together: 'Is there the same number of flowers and vases? –*No*.–Why?–*There are more vases*.–Are there enough flowers for the vases?–*Yes*.–They're both the same amount then?–*No, here* (vases) *there are more, because they're spread out*.'

31 p.86 of J. Piaget, *The Childs Conception of Number*, Routledge & Kegan Paul Limited, 1952.
32 p.51 of J. Piaget, *The Childs Conception of Number*, Routledge & Kegan Paul Limited, 1952.
33 p.45 of J. Piaget, *The Childs Conception of Number*, Routledge & Kegan Paul Limited, 1952.
34 p.79 of J. Piaget, *The Childs Conception of Number*, Routledge & Kegan Paul Limited, 1952.

> Sim (5;7) took six eggs to correspond to six egg-cups and put the eggs in. They were taken out and placed further apart: 'Is there the same number of eggs and egg-cups?–*No*.–Where are there more?–*Here* (eggs).–If we wanted to put one egg back in each egg-cup would there still be the right number?–*Yes ... I don't know*.'[35]

In the preoperational stage we are only capable of making the one-to-one correspondences when we can actually perceive the items to be grouped one by one. We cannot yet bring the different dimensions of the situation together. Information about the space that objects occupy is not combined with the number of objects. Piaget remarks that children at this age are capable of counting up to ten, but these children do not yet quantify objects. When objects are however perceived in a one-to-one relation with another set of objects the preoperational children can see that the two sets have the same amount of elements.

During the preoperational stage we develop a better understanding of one-to-one correspondence. We learn to see correspondence when objects are separated by space and we learn that situations can be reversed. Only after understanding this reversibility, do we understand the situation by operation rather than by perception. During this stage we learn by practice that spatial manipulations of objects are reversible. When we move on to imagining these reversible manipulations, we learn to equate two quantities without perceptual help. In this way we develop a more abstract correspondence from the perceptual one-to-one correspondence. We manipulate two sets in our imagination to see whether they can be matched one-to-one. This is when we enter the operational stage with respect to correspondence. According to Piaget the ability to see abstract correspondence allows us to conceive of lasting conservation of quantities. Eventually knowing that a given quantity can not change, enables us to learn a concept for a quantity size, a number concept.

Pairing Objects with Number Words

According to Piaget we do not have an abstract understanding of numbers in the preoperational stage, but we only pair the number words to objects in the same way we pair 'unrelated' objects like bottles and flowers. In these situations an adult could easily mistake the child's simpler 'pairing' of a number word with an object for the more advanced abstract number word. Pairing objects with number words at the end of the preoperational stage is a development which will later enable the understanding of the number concept of ordinality.

In CCN Piaget also briefly describes correspondence trials where children also counted out loud. When these children simultaneously count out loud and exchange objects one by one, this does not help them to count better or to develop a correspondence skill in comparison to children which do not count out loud.[36]

Cardinal Understanding and Decomposition

To see how children deal with cardinal values Piaget trialled children by asking them to take the same amount of counters from a box as a figure build up from counters (individual similar objects, e.g. matches). In this trial the children can again be divided in three degrees of understanding. The youngest children did not take the right amount out of the box. They were satisfied when they had created a form looking similar to the demonstrated figure. Children in the transitional stage had a better understanding and made correct partial correspondences between the demonstrated figure and their own. The oldest children had no problem picking the right amount of counters and some of these children did not even recreate the figure.[37]

According to Piaget the youngest children assumed no conservation, but were satisfied with a global evaluation. He thinks that at first children treat all quantities as continuous quantities and later learn to properly decompose forms. Piaget briefly talks about the need for decomposition and the capability to use decomposition.

When a child in the first stage of understanding is asked to pick out 'as many' counters it does not interpret this as a question about exact quantity. It does however roughly copy the form. And in making this rough copy, the child does decompose the form. The form is decomposed into lines and shapes. Very simple forms are even copied correctly. Lines are however copied by length not by the quantity of their elements. Piaget thinks that children at this stage miss the coordination to properly decompose the forms. Every comparison remains vague.

At the end of the preoperational stage children decompose correctly, although they still focus on a single element and not on the complete form. At this age children make a better copy, however they are not yet capable of making a lasting correspondence between the two complete forms. At this age the children are still confused when the form is changed and they are asked to adapt their copy.

35 p.52 of J. Piaget, *The Childs Conception of Number*, Routledge & Kegan Paul Limited, 1952.
36 p.56-64 of J. Piaget, *The Childs Conception of Number*, Routledge & Kegan Paul Limited, 1952.
37 p.83 of J. Piaget, *The Childs Conception of Number*, Routledge & Kegan Paul Limited, 1952.

Intuitive and Operational correspondence
According to Piaget children gradually develop a lasting correspondence skill after the preoperational stage. The children become less and less dependent on the perception of the objects. The children develop a qualitative correspondence that becomes independent from the perception of objects. This qualitative correspondence then develops into a quantitative correspondence. The children understand that the counters (or other objects) are interchangeable, making the numerical value of the counters the salient factor.

Piaget distinguished intuitive correspondence, which is based on perception, from operational correspondence, which is separate from perception. A qualitative correspondence can be both intuitive and operational, but numerical correspondence can only be operational, except maybe the smallest numbers. We first develop a simple qualitative and intuitive understanding. Then we develop a more precise qualitative understanding. Piaget thinks that the perceptual basis of the intuitive correspondence is stopping us to grasp lasting equivalence, but when we develop an operational understanding and we are no longer limited by our perception.

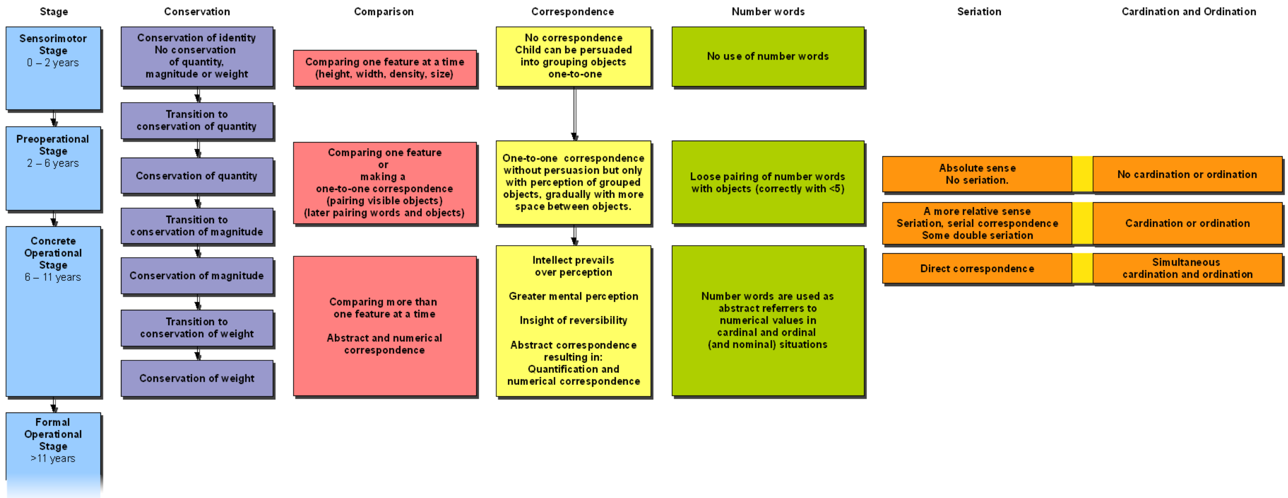

Diagram 6: Seriation, Ordination and Cardination

(large version in Appendix B)

2.1.6 Number Concept: Seriation

Piaget distinguishes three forms of seriation: making a simple qualitative series, qualitative correspondence between two series and ordinal numerical correspondence between two series. In the first part of a seriation trial designed by Piaget children are asked to arrange dolls of different sizes. This will show whether children possess simple qualitative seriation skills. If a child is capable of making such series it is asked to arrange the corresponding walking-sticks of these dolls. One of the sets is rearranged after which the experimenter tells that one of the dolls is going for a walk. The children are asked to present this dolls walking stick. Either the dolls or the sticks have been spread out or put closer together. If children pick out the right stick with the chosen doll, this shows these children possess qualitative correspondence skills. In the second part of the trial one of the sets order is reversed, this is done to test whether children can make an ordinal numerical correspondence. Due to this reversal children have to count the objects in the rows in order to find the right stick with the doll.

According to Piaget children at the beginning of the preoperational stage can not make a correspondence between series. Older children use three correspondence strategies which Piaget called double seriation, simple serial correspondence and direct correspondence. We learn these strategies in the transitional phase towards the operational stage. At first we either use 1a) double seriation: making a series of the dolls and making a series of the sticks, which we can consequently put together or 1b) simple serial correspondence: making one series and then putting the other elements with the ones they correspond to. In the operational stage we also use 2) direct correspondence: putting elements together directly or simultaneously with the construction of the series. Before the preoperational stage we are not capable of making a series at once, let alone devising a tactic for a problem containing two series. We deal with each element in isolation. Before the preoperational stage we are only capable of making one qualitative comparison at a time.

In the operational stage we put elements in the right order a lot faster by comparing more elements simultaneously. We develop a relative sense of measure besides the absolute sense we know from the sensorimotor stage. According to Piaget we understand in the operational stage that there is (always) an object that can go in between a smaller and bigger object. This relativity is however an intuitive relativity, one that is found when handling with these objects. It is not yet an operational relativity. This intuitive relativity enables us to correctly use serial correspondence and double seriation. When constructing series in the preoperational stage we still make the same mistakes that occur in the sensorimotor stage. We skip objects and do not yet compare objects with the complete set. Overview of the situation is not yet present. When we understand relativity of quantity/size in an operational manner we have also developed the three strategies for seriation. These seriation strategies are now equally simple to perform. We understand that when we put an element in a series of elements we have to compare it to all other elements.

In the second part of a seriation trial the elements are rearranged: the position of the walking-sticks which correspond to the dolls is reversed so that doll 1 is opposite to stick 10 and doll 10 opposite to stick 1. After rearranging the experimenter points to a doll and asks the children which stick belongs to this doll. All children in the experiment can see that doll 1 and stick 1 go together, as well as doll 10 and stick 10. For all in between dolls the children in the preoperational stage pick at random.

In the operational stage children develop an understand the ordinality of the two series. Piaget shows that children in this development can find some more of the corresponding walking-sticks for the dolls. According to Piaget when children find more walking sticks (but are still often off by one position) this shows they have developed a sense of ordinality, but have not yet connected the ordinal places with number values. When the experimenter asks which stick belongs to the fifth doll, these children count the four dolls in front of the fifth but subsequently point to the fourth stick. These children do not give a cardinal value to the sets and their evaluation of the quantity of sets is not constant. They change their evaluation when the sets is rearranged. They have developed a dissociation between qualitative and quantitative values, but have not yet fully developed this skill. They do know that reversing a rearrangement makes the sets equal again.

According to Piaget after the preoperational stage children develop an operational understanding, which comes to exist next to the intuitive understanding. Operational understanding is 'co-ordination', with this development we get a broader view of situations. In the operational stage ordinal understanding is grasped and during the stage it gets connected with cardinal understanding. Then children see ordination and cardination as one. At this point the *n*th doll is number *n* in a cardinal sense as well as in an ordinal sense.

In the second part of a seriation trial the dolls and sticks are disarranged. The children are told a number of the smaller dolls go to bed and the children are asked to place their sticks in the cupboard. The youngest children try but make a lot of mistakes, merely putting a few sticks in the cupboard. The children developing operational skills have a sense of cardinality and make series form the dolls and sticks. They understand that the number of sticks and dolls need to correspond. These children do however still make some mistakes with seriation and counting. So only if both seriation and counting goes right, they give the right answer. When children reach the operational stage they master both cardination and ordination at the same time and they chose the right sticks to put into the cupboard

2.1.7 Number Concept: Cardination and Ordination

To investigate cardinal and ordinal understanding Piaget designed another trial. Children in this trials were asked to order sticks according to length, after which they were presented with some 'forgotten' sticks. These forgotten sticks needed to be inserted with the ordered sticks. Then the children were asked to count the sticks. The amount of sticks that they could easily count remained on the table and all sticks above that number were removed from the experiment. The length of the sticks was explained to be an indication for the height of a staircase which a doll would ascend. A stick was pointed out and the child was asked to say how many steps the doll should take to get to that height. To answer this question correctly the children needed to transform an ordinal value into a cardinal number. The stick has a place in the order of the sticks that had just been laid out. According to Piaget the children needs to understand that this place in the order corresponds to a number value to answer this question.

With these results Piaget theorizes about how we construct our cardinal and ordinal values.[38] According to Piaget when we start to development seriation we have no ordination or cardination skills. We value everything on its perceptual and qualitative impression. When we develop an understanding of seriation, our understanding of cardination is based on our understanding of ordination. We use one-to-one correspondence to connect the perceptible place with a number value. When we do not yet have a lasting understanding of ordinality, we do not have lasting cardinality and we need to rely on our perception. For example: When a child perceives 8 objects it could use an ordinal measure to get to the number 8. The child will however not continue to 'see' the value 8 when the 8 objects are divided into a group of 5 and 3 elements.

38 p.121-147 of J. Piaget, *The Childs Conception of Number*, Routledge & Kegan Paul Limited, 1952.

According to Piaget we develop our understanding of ordination and cardination simultaneously. When we make series operationally we know that objects in a series relate to each other according to a certain principle. Each object has at least one relation with the other objects, being either smaller (lighter, lower, darker, etc.) or larger (heavier, higher, lighter, etc.) or both with respect to other objects.

At the beginning of the operational stage we can evaluate more than one characteristic at a time. We can for example see a doll, Bert, as part of the set of dolls on the table consisting of Anne, Bert and Chris. And we can see the doll Bert as smaller than Chris and bigger than Anne. We are seeing Bert at once as equivalent and as non-equivalent with respect to Anne and Chris. Piaget takes equivalence to lead to class and non-equivalence to lead to relations.

According to Piaget we understand 'numbers' as both a class and a relation. When we have ordination and cardination skills we have developed the ability to view objects in different ways at once: as having a relation with other objects and as being part of a (number) class. Number is a class in the sense that all numbers are part of the same set. Number is a relation in the sense of being a place in a series. Cardinality is understood as a set of units, ordinality as a series of which the elements have a set order. The understanding of numbers as both cardinal and ordinal at the same time, is understanding that numbers can be either classes put into a series or series put into classes. Cardinal units can be ordered into series and ordinal elements can be grouped into classes.

2.1.8 Overview of Piaget's number theory

According to Piaget number theory starts with a logically given, the necessary conservation of identity of ideas. This start is a logically given because thinking without this conservation of ideas is impossible. According to Piaget the conservation of identity of ideas necessarily results in a conservation of identity of objects. Objects can only be paired with ideas that stay the same ideas. Pairing an object with an idea that does not stay the same does not develop our understanding of the world.

According to Piaget the conservation of identity of objects is constrained by our experience of the world, memory workload and co-ordination workload. The distinctive qualities of a individual determine which combinations of objects and ideas are conserved. How (well) can the individual perceive? What and how much has it already perceived and how did it perceive these things? The power of the individual's memory and of its ability to coordinate all this information prepare it for dealing with future situations.

At a certain moment in our development we starts to understand that processes are reversible. Objects can be manipulated, but these manipulations can be undone. This enables us to see situations free from their place in time and space. The situations themselves can become a sort of quality of the objects.

According to Piaget experience of the world leads to our developments. Identifying objects is learned due to experience and over time we learn to identify objects with a certain ease. We starts to group objects, which can be seen as a first step towards counting. We start with only grouping discontinuous matters (separate objects) and not with continuous substances like water. Grouping objects is different from use of categories such as 'food', 'non-food' or 'living things'. By grouping objects Piaget means groupings of specific objects for a certain purpose. For example grouping of toy cups and toy saucers, or giving one cookie to each family member.

Apart from grouping different objects, we learn to group equal objects. With these equal object groups we start the conservation of small quantities in objects. The conservation of small quantities is constrained by the individual's memory and it's capacity to track object simultaneously (in Piaget's words: coordination).

Before we are 5 years old we learn some number words and start to pair number words with objects. At first this is done just like the grouping of objects. We seem to experience the relation between word and object as an arbitrary relation. After some dealing with groups and series the understanding of number words expands. By grouping objects we come to understand cardinal values. We develop an understand of ordinal values by making series. Later on we develop an understanding of the relation between cardinal and ordinal values, which we treat like classes of values and respectively relations between values. This last development ties together two important views of numerosity and makes our understanding of the number words more abstract.

Once we have identified the small quantities, we start to understand the differences between larger quantities. According to Piaget we compare larger quantities by making one-to-one correspondences between them. We can make correspondences between the larger quantities before we can name them. When we make a one-to-one correspondences between grouped objects we construct of their equal quantity. By repeatedly making one-to-one correspondences we start to deduce the conservation of number. Once we make such elaborate correspondences we know the numerosity of a group will stay the same even when the objects are put closer together or spread out. According to Piaget this is the way in which conservation of quantity is achieved for discontinuous objects.

Over time we develop our memory and by training with number situations we get better at tracking number information. According to Piaget this helps us to easily understand the conservation of all discontinuous number quantities. Continuous quantities (magnitudes or 3D-quantities) are learned only a little later. We first learns to see 'blocks' in the continuous matter. With these partitions we can workout that conservation of quantity also holds for the magnitude of these continuous quantities.

At this stage in our development we have a number sense which we can use in different situations and at different times. Piaget goes on to discuss our development of numbers, but we will stop here as there is enough to discuss already.

2.2 How we can understand numbers

Our number knowledge determines what we can do with numbers. Piaget has interpreted number knowledge as being a set of skills or operations that we learn to use, which involve numbers and numerosity. This is however not the only way of looking at this knowledge. Among others Bloom has been investigating language and numerosity skills from a word, object and concept perspective. These different views of understanding numerosities lead to somewhat distinct views of learning numbers.

According to Piaget number knowledge consists of our skills to deal with numerosity. These skills are formed by the operations we learn to master. These operational skills are not factual, but more habitual or behavioural. We develop our operations gradually from simple to more complex. Language is one of these habits that we essentially develop separately from our number skills.

According to Bloom advanced number knowledge inherently contains our skills to communicate about numerosity, because most learning is mediated by language skills. The role that Bloom gives to language makes it an essential part of number knowledge and in fact of any other advanced knowledge. By learning concepts and words simultaneously we create an interwoven understanding of our physical and social surroundings. Concepts are the building stones of our understanding. To learn these concepts we have an innate dependency on our language skills.

Piaget and Bloom give us two views of number knowledge. We will name these scopes isolated and integral number knowledge. Piaget describes the most compact form of number knowledge, therefore isolated number knowledge. Bloom and others develop our number basis by adding language. In this view number knowledge is not isolated but incorporated with language, therefore we will call this integral number knowledge.

What arguments for and against having language as an integral part of number knowledge. Piaget does not use language in his theory and Bloom thinks that our use of language in learning numbers cannot be replaced by another mechanism. Piaget's view is understandable if we do not consider communicating about numbers a part of number knowledge.

If number knowledge is dependent on language it follows that number knowledge is part of our social skills, since human language is very dependent on our social skills. Whether number knowledge is an interwoven part of our social skills is an interesting question. Being an interwoven part would make number knowledge depend on our Theory of Mind (ToM) skills and this is exactly one of the skills autists are said to lack. We will discuss what ToM is in 2.2.1. After that we will try to discover whether the learning of number knowledge is depending on our understanding of theory of mind, when we discuss the learning of language in 2.2.2.

On some developmental factors Piaget and Bloom agree, for example in what general order we learn to use numbers. We start out with some innate skills and we learn a lot of skills by experiencing and interacting with our surroundings. We also need to be in a physical condition which allows us to experience the world, including a working operational system.

When we develop (number) knowledge we start out with some innate skills. These skills do not need any training. We have these skills straight away form birth. Some of our innate skills allow us to deal with quantity and ratios. Of course we need to develop physically before we can start to learn certain things. Our brain matures and so does our body, e.g. being capable of turning our own head. Due to these physical developments and by combining our innate skills we can learn to do more advanced operations. And by combining these basic and advanced skills with language we can communicate about them. We will discuss the development of the innate, primary and secondary number skills in 2.2.2.

To use our innate skills we need some operational skills that allow us to physically and cognitively do things. We need to be able to recognize and remember things, plan actions and think about the world around us. All these things and more combined form our operational apparatus. This operational apparatus is important because it mediates between our cognitive skills and the world around us. We understand the world through these skills. If our operational apparatus is not working like it should be, that could result in severe deficits. In the next paragraph we will look at our operational apparatus. Later on we will see how problems with our operational apparatus can lead to problems with number knowledge.

2.2.1 Operational Apparatus

We learn to count via our operational apparatus. Our body and brain limit the our view of and interaction with the world. These limits of our body are sometimes were clear, for example legs determine whether we can walk. But some limits are only understood by comparing TD and disabled individuals. For example reasons for why some people can count while others cannot, are not so obvious. To understand our number knowledge we will have to look at these hidden workings of our body. We use mental operations to access and create our knowledge.

Our mental operations are divided over different mental modules. These modules work relatively independent, they do not influence each other. The mental module model is widely accepted and makes it possible to discover a structure in our mental operations. We have to find out how they limit, relate are influence each other and which ones we need to develop our number knowledge.

The mental part of our operational apparatus roughly comprises of three parts: our executive functions, our short and long term memory and our working memory. We use these three skills in almost all of our conscious actions. Doing without them is virtually impossible. One other skill we will look at in this paragraph is the Theory of Mind (ToM), which might be just as important as the other operational functions, but which has a more specific domain; other humans.

The hypothesis of the executive functions has at least been around since the 1950 and has become more popular at the start of this century. Psychologists have grouped a few mental modules under this umbrella name. With these operations we can consciously give direction to our actions. Most important functions are: planning, inhibition, set shifting, generativity and self monitoring.

Planning our actions is important for obvious reasons. Being able to actively plan our actions is actually quite remarkable. But this function would never work as well as it does without two other functions namely inhibition and self-monitoring. Our inhibition helps us to keep our focus on the required information and not to be distracted unnecessary. Inhibiting distractions keeps us focussed on the task we were planning to do. Self-monitoring instrumentalizes ourselves in our planning process. This is important because we need to see ourselves separately from our surroundings.

Set-shifting and generativity help us to come up with solutions and alternative routes to achieving our goals. Set-shifting enables us to let go of a certain view of a situations and adopt a different perspective. Looking differently at certain situations can open up different possibilities and perspectives. And looking at a situation with two different perspectives can simply be enriching. Generativity, the skill to generate ideas and solutions, is important to eventually solve problems. Without generating different ideas we cannot select a potentially successful solution.

Together the executive functions allow us to actually choose and do what we want. But without putting our choices in the perspective of our previous choices and their outcome this would not be much help. Our memory helps us put things into perspective. In most theories our memory get divided into long term memory, short term memory and working memory. When knowledge becomes part of the long term memory it is imprinted in the brain so deeply that we can hardly forget it. The long term memory is in some sense who we are. Facts that remain in the short term memory are important but not as important as those in the long term memory. If these facts need to be remembered often enough they become more permanent and become part of the long term memory.

The working memory is not really a memory but more like the actual state of thoughts. This is where we interpret the space and sounds around us. We will look at Baddeley's model of working memory to understand this part of the operational apparatus. Baddeley divides our working memory into three parts, the episodic working memory of episodic buffer, the visuospatial sketchpad and the phonological loop. The first is the one in which we hold our ideas, the second we use to map the space around us and in the third we memorize what we have just heard. These working memories are directed by what Baddeley calls the Central Executive, which is very similar to the executive functions. It directs our attention in the working memory to the right place, enabling us to use to power of our working memory to pursue our goals.

The different parts of our working memory are connected and our working memory can be used to access our short and long term memory. Baddeley emphasis the difference between the two by calling the working memory a fluid system and the long term memory a crystallized system. This model can mediate both our visual and auditive surrounding so we can count them both within the episodic buffer.

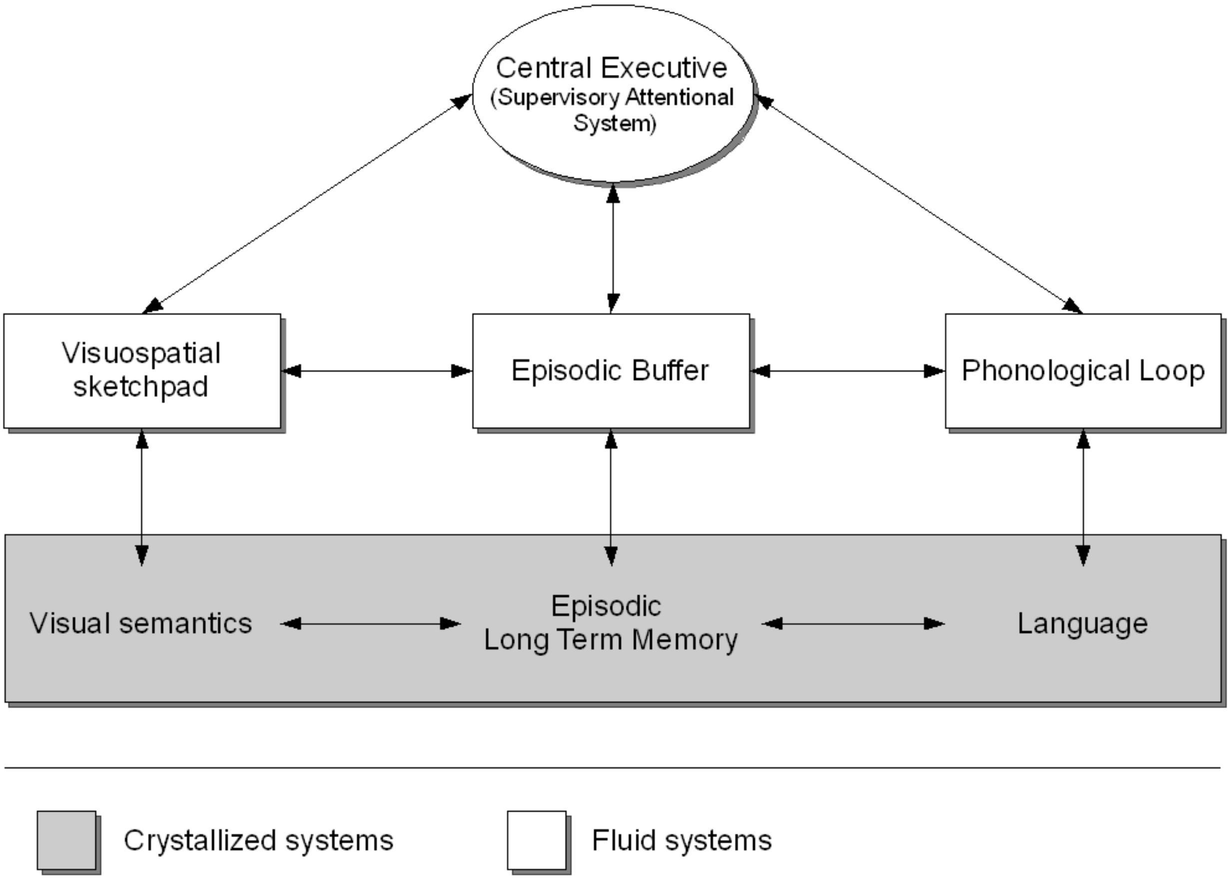

Diagram 7: Baddeley's Model of Working Memory

With the executive functions, memory and working memory we can understand and act in a lot of situations. But to understand the difference between object and agents we need a Theory of Mind. It is not a given that we can understand the difference between these two. Luckily we do not need to consciously deduce this, but we have a developed innate sense to do this. This ToM develops from innate gaze following, which we do from 6 months of age and joint attention, which has been shown to start between 9 and 12 months of age. These early forms develop into (proto-declarative) pointing to indicate what one wants from for example the parent. And they develop into actually seeing other people as agents with their own agenda's and goals.

The understanding of others starts as understanding they having feelings. This build up to understanding they have an independent awareness of situations, which can actually be different form ones own understanding. Eventually understanding that someone else can have a mistaken understanding of a situation. Which leads to the understanding that we can ourselves have mistaken understandings and that our understanding of our surroundings is itself only an interpretation. This development is called the Theory of Mind. What is generally meant when the term ToM is used is the understanding of others as agents.

Environment can have an influence on ToM. Sabbagh, Xu and Carlson studied preschoolers and compared U.S. and Chinese children. This study implicates there is a relation between the number of siblings and the development of ToM, having (more) siblings leads to an earlier understanding of ToM.[39]

The ToM sums up the our list of mental modules which together form the operational apparatus. Now we can move on to talk about the mental modules that we use and develop to handle situations with numerosities.

2.2.2 Development of Number knowledge

The understanding of number knowledge via our operational apparatus goes through several stages. We will start with the innate number senses, some of which we share with animals. We will then look at how we learn to count from these innate number skills. An important step in this development will be concept and/or word mapping. We will first look at Blooms integral number knowledge and second at Piaget's isolate number

39 M. A. Sabbagh, F. Xu, S. M. Carlson, L. J. Moses and K. Lee, The Development of Executive Functioning and Theory of Mind: A Comparison of Chinese and U.S. Preschoolers, *Psychological Science*, vol.17 no.1 p.74-81, 2006.

knowledge. Both theories however demand an explanation of the actual mapping and we will look at how concepts and innate number sense can be connected at the end of the paragraph. Eventually a developed counting skill will allow us find many more uses of the natural numbers. Developing to count is however quite a task in itself.

Innate Number Sense

The innate number sense we share with animals and which has been extensively researched is the accumulator. In rats this counting mechanism has been investigated at a neuronal level. Dehaene describes this mechanism in his book The Number Sense.[40] The accumulator seem to work like an electrical condenser, it gathers energy which can then be measured. The rat can 'count' by storing similar amounts of energy in the accumulator every time he 'adds' a unit. When he is done counting he can assess the height of the energy stored in the accumulator and this will give him his 'number'.

The 'analogue' accumulator is not very precise. When the rats are chemically brought into an aroused state they count too rapid, because their energy levels are higher than normal. They keep counting with the same speed, causing more energy to end up in the accumulator. The aroused rat's accumulator is indicating two or three units when it should only be indicating one. The rat counts up to a higher number, than it has actually seen or heard. And visa-versa when these rats are chemically slowed down, they count too slow and expect smaller numbers than they have in fact experienced.

Dehaene explains this higher or lower expectation by using a Robinson Crusoe metaphor[41]. Crusoe got to the island but got a blow on his head and lost all capacity to count. He invented a water accumulator which he positioned next to a stream. When a group of hostile tribesmen visits the island Robinson pours some water into the accumulator for each tribesman. This pouring is done by leading some water via a hollow stem into the accumulator. The stream is however not always running at the same speed and thus pouring water into the hollow stem does not always lead to the same measure. The speed of the stream eventually decides how well Robinson counts his adversaries. Robinson can subtract from this accumulator every time a tribesman leaves by taking out some water. When his accumulator is empty he knows he is safe to come out of hiding.

The Crusoe story shows us the limitations of this innate number understanding. It goes well for small quantities but gets less accurate with bigger numbers. Continued adding and subtracting with the same number leads to inaccuracy. The discrete numbers that go into the accumulator come out as a single continuous level of energy. Neither does the accumulator allow a very precise reading by the animal. Animals can only experience quantity as fluctuating and do not get a precise grip[42]. Evolution selects the more advanced accumulator. Animals with inaccurate accumulators get eaten by predators they accidentally subtracted or starve by waiting for a prey to come out of hiding, where all prey has already run away. That we must have this accumulator follows from the research of Mehler and Bever and Rothenberg and Courtney quoted in chapter one.

Besides the accumulator we have other innate mental modules that help us understand numbers. We categorize objects pretty much instantly from a very young age. This kind of categorizing is called domain specific[43] learning. Its great advantage is that it does not require complex reasoning, because it only says something about a small subdomain of the world. Opposite to Piaget's domain general learning, it is quick and simple. Domain specific learning enables us to learn things quite automatically, without the danger of loosing these skills once we learn something else. This is a potential risk in domain general learning, where everything we learn is interconnected.

An example of domain specific learning is our ability to instantly categorize over food and tools.[44] As Santos Hauser and Spelke show in their research, we share this domain specific knowledge with our direct

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40 Dehaene, *The Number Sense: How the mind creates mathematics*, OUP, 1997.

41 p. 28 of Dehaene, *The Number Sense: How the mind creates mathematics*, OUP, 1997.

42 A lot of research is done with respect to animal counting. It turns out that our more distant relatives such as rats have counting skills. Our closest relatives all seem to have a more developed sense for counting skills. Some of them even conserve quantity.
M.J. Beran, Capuchin monkeys Cebus apella succeed in a test of quantity conservation, *Animal Cognition*, vol.11 no.1, January 2008.

43 Domain specificity is a reaction to an empiricist view of learning, which states that there are only a few ways of experiencing the world. In contrast to this view domain-specific learning is done in a subdomain of the world. A subdomain which we experience but that may not be there as such. For example experiencing spelke-objects and creating a theory of mind.

44 L.R. Santos, Hauser M.D. and Spelke E.S., Chapter 27: Domain-Specific Knowledge in Human Children and Non-Human Primates: Artifact and Food kinds, from *The Cognitive Animal: Empirical and theoretical perspectives on animal cognition*, edited by M. Bekoff, Allen C. and Burghardt G.M., MIT Press, 2002.

evolutionary relatives. Children aged 2 years and 6 months generalize differently when told something is food from when told something is an artefact. These children generalized artefacts according to shape and food according to colour. The fact that they make this generalisation instantly, without previous experience with an object shows us that some learning mechanism is at work. Rhesus monkeys probably learn the distinction in a similar way, by seeing what is edible when other monkeys eat or don't eat stuff. So what is edible can be learned by observation from the social environment, resulting in generalisation over foods by colour but not by form.

Could there be a number domain knowledge working by the same principle? Numbers form their own domain, Kant even gave number knowledge its own category. But if we have a special capacity to understand numbers, then isn't it weird that we learn to count so slowly. Not if we consider that a part of number knowledge rests in language or symbols and in the number system. We first need to master this system and we need to master language. Two major reasons why it is so hard to learn numbers. You cannot really tell someone what 5 is, you need to sense it. And you need to learn the number word five. Acquiring language takes a long time and is certainly not instantaneous. This means we have no domain specific knowledge about numbers.

But we do have domain specific knowledge about size and some limited specific knowledge about quantities. We have the ability to subitize[45], which means we can immediately 'see' the numerosity of a small number of objects and know this numerosity without counting, being able to say the corresponding number word. This describes perceptual-verbal subitizing, where a number value is directly transformed into a number word. This is also the way subitizing was initially considered to work; as an 'explicit quantification operator'.

Subitizing was later found to be purely perceptual and perceptual-preverbal. These notions of subitizing come close to the mental module described as the accumulator operator. The two mental modules work differently as subitizing is instantaneous and only works accurately with small numbers (4 and under) whereas the accumulator works over time by 'adding' and 'subtracting' and can work with large numbers, but is not very accurate.[46] Because it is instantaneous and only about the number domain, subitizing is domain specific knowledge.

Benoit et al. researched whether children develop their number knowledge via subitizing or via counting.[47] They looked at three groups of children of around 3, 4 an 5 years of age and found a gradual shift from subitizing and precision up to 3 to a slower counting mechanism and a precision up to 6. Between 1 and 6 dots were shown to the children either simultaneously for 800ms or consecutive (each dot) for 800ms. In the simultaneous showing the children were asked to name the number of dots and in the consecutive showing they were asked to count the dots (counting after each appearance) and at the end of a sequence to 'say how many there were'. The dots were placed in canonical and random patterns. Of course when children grow older, they become better at these number tests. Surprising is that children do well on a simultaneous counting below 3, but not with quantities between 4 and 6. The children from the second group, aged around 4, had a strong increase in right answers in the counting test. And only at age 5 do children get better at assessing cardinality.[48]

Benoit et al. concluded from their research that to our initial subitizing we add the more or less separate counting mechanism. Other research has shown that subitizing and counting occur at the same location in the brain[49]. Benoit et al. also concluded that canonical configurations of dots can be learned, as these configurations lead to better results.

Hauser and Spelke grouped the skills from which we develop our other knowledge as core knowledge[50]. The

45 Benoit, Lehalle and Jouen tell about the origin of the notion of subitizing: it originates form a 1946 research by Kaufman, Lord, Reese and Volkmann.
L. Benoit, Lehalle H., and Jouen F., Do young children acquire number words through subitizing or counting?, *Cognitive Development*, no.19 p.291-307, 2004.

46 The subitizing could also be a seen as a simple and automated use of the accumulator. In the use of the accumulator with very low quantities we are likely to make very little mistakes. We can also track different objects simultaneously from a very young age. Combining our tracking 'module' with the accumulator, gives us a flawless subitizing capability.

47 L. Benoit, Lehalle H., and Jouen F., Do young children acquire number words through subitizing or counting?, *Cognitive Development*, no.19 p.291-307, 2004.

48 This research is in correspondence with Piaget's findings, placing cardinality after seriation. Piaget has not however said anything about subitizing.

49 M. Piazza, A. Mechelli, B. Butterworth and C. J. Price, Are Subitizing and Counting Implemented as Separate or Functionally Overlapping Processes?, *NeuroImage*, vol.15 p.435-446, 2002.

50 M. D. Hauser and Spelke E., Evolutionary and developmental foundations of human knowledge: a case study of mathematics, Chapter 61 in *The Cognitive Neurosciences III*, edited by M. Gazzaniga,

modules of our core knowledge have according to Hauser and Spelke four characteristics; they are domain-specific, task-specific, relatively encapsulated and relatively automatic. A few modules of this core knowledge are useful for understanding numerosity. Our capability to subitize is one of them. This is domain-specific because it only deals with numerosity. It is task-specific because its representation only deals with the question how many. It is encapsulated because it only uses shape or rhythm (not colour/texture or pitch/timbre) for input and only number-knowledge as output. And finally it is automatic, we do not need to actively engage in subitizing.

According to Hauser and Spelke two core knowledge skills lead us to construct natural numbers. An exact small number skill and an approximation skill of large numerical magnitudes. Two modules which could also be called the subitizing module and the accumulator module, which we have already discussed.

An effect known as Weber's Law could be a side effect of how the accumulator or large approximate number skill works. When rats and pigeons were tested for timing, a large number approximation skills was shown to exist. Their timing was tested by having them press a lever for a certain duration. When they were tested to press the lever for a longer time their margin of error also became proportionally larger. Weber's Law takes place; the minimal difference perceived in quantity change is proportional to the total quantity. When holding a weight, the minimal perceivable difference doubles when the weight is doubled. The results of Weber's Law can be explained with the workings of the accumulator module. Since our number system starts out with the accumulator this law influences us as well as animals.

When human infants compare large numbers it is the ratio which decides how difficult this comparison is. Comparing 8 and 16 is equally difficult to comparing 10 and 20, as it also has the 1 : 2 ratio. Comparing ratios with a smaller difference is harder (e.g. 2 : 3, 8 and 12 or 10 and 15) and comparing ratios with a greater difference is easier (e.g. 1 : 3, 8 and 24 or 10 and 30). At 6 months of age we can already discriminate between an arrays which contains 8 and one that contains 16 dots without training[51]. We do not see the difference between 6 and 12 dots, but we do see the difference between 16 and 32 and 16 and 24[52]. The manner in which ratio is connected to success rate of comparisons is also in accordance with Weber's Law and with the accumulator module. Similar results are found when children listen to beeps which were spread out over time, instead of spots which could be seen all at once.

Because of these similar results in both the visual and auditory domain Hauser and Spelke think that the counting module can abstract over these different dimensions. It works independent from visuals or auditory modules and works for both spatial arrays or temporal sequences. It is not linked to objects. Regarding this statement by Hauser and Spelke, we think it is very likely that our numbers skills are not inside either the visuospatial sketchpad or the phonological loop, but are episodes in the episodic buffer.

As could be expected ans as Hauser and Spelke assumed we can do approximate additions from a young age with our accumulator/large number module. This hypothesis was tested by Barth et. al.[53] and they found that approximate addition and subtraction can be done without the use of symbolic arithmetic. We have got a feeling for quantity. Our precise large number arithmetic has a basis in non-symbolic arithmetic.

Hauser and Spelke also talk about the small precise number system or subitizing. They take this skill to have four characteristics; a set size limit of 3 or 4, elements are spatially distinct and not superimposed or overlapping, elements are not linked by lines but separated by space and finally it does not work when elements appear and disappear discontinuously, but does work when elements are motionless, move together, or continuously move and are only hidden from view for brief moments.

Similar to the large number system the small number system can also be found in animals. The small number system differs from the large number system in two ways. Subitizing skills do not grow when we grow up and when subitizing Weber's Law does not hold.

Language

We develop a grasp of numbers more precise than this instinctual accumulator. To deal with natural numbers we need skills that are more precise than the small and large number systems. It is generally accepted that we need to combine our skills to develop more sophisticated forms of reasoning.

Conservation is important in this developmental step because it enables us to deal with quantity in a very

p.853-864, Cambridge, MIT Press, 2004

51 F. Xu and Spelke E.S., Large number discrimination in 6-month old infants, *Cognition*, vol.74, B1-B11, 2000.

52 F. Xu, Spelke E.S. and Goddard S., Number sense in human infants, *Developmental Science*, vol.8 no.1 p.88-101, 2005.

53 H. Barth, La Mont K., Lipton J., Dehaene S., Kanwisher N. and Spelke E., Non-Symbolic arithmetic in adults and young children, *Cognition*, vol.88 p.199-222, 2006.

precise manner over time and regardless of transformations. To further our precise grip on numbers we first need to develop some concepts and/or understand language.

According to Hauser and Spelke language plays an important role in bridging this gap. They think however that even though we count with words the small and large number skills remain present just under the surface. According to Hauser and Spelke our language skills distinguish us from our evolutionary relatives, but our instinctual core number knowledge is the mechanism dealing with the numbers.

> Even after humans acquire the capacity to enumerate sets by counting, however, they retain the two core systems and use them during all quantitative reasoning tasks.[54]

Hauser and Spelke quote neuro researches that tested adults and impaired adults and found that large approximate core number systems underpins our human number knowledge.[55] [56] Even though they think Language is important for representation, Hauser and Spelke assume that because of the automatic nature of core knowledge, number knowledge is not susceptible to explicit beliefs. We can only see, or better still; we can not not see what amount something is when we subitize or when we use our large approximate system. Therefore Hauser and Spelke think that application of core number knowledge is independent of the theory of mind (or less coordinated beliefs and desires[57]).

Bloom also thinks we need language to take the developmental step towards the understanding of the natural numbers, but he thinks language plays an indispensable role. Language gives us a developmental leap towards the understanding of numbers, because it enables us to conserve quantity in an easy non demanding way. We can easily learn words and connect these to meaning. Now according to Bloom if we map some of these words to quantity we can later map number words to quantities. Quantities that are in fact outside of our comprehensible scope. Our two core knowledge number systems do not allow us to fully comprehend the meaning of precise large numbers.[58] Bloom states that:

> [...] following Dehaene [...] the ability to reason about the larger numbers –to understand, for instance, that if you remove two objects from 20 objects, 18 will remain– is impossible without the possession of a natural language. This makes a prediction about acquisition: only people who have learned a generative number system can reason about these larger numbers. But a stronger version of this theory makes the prediction that only people who can *access* the language of numbers can reason about them. Without language, all that remains is the approximate accumulator mechanism that humans share with rats and other animals.[59]

According to Bloom when we learn language, we connect inner language[60] to our spoken language. We connect concepts to words. To share the combination of concepts and words with others we need two things: a common basis of understanding and the understanding that others exist. A common basis can be found in our innate skills. We will shortly look at Spelke-objects, because they play an obvious part in sharing our views of the world. The understanding that our world is inhibited by other thinking entities is often called Theory of Mind (ToM), which we have talked about at the end of 2.2.1.

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54 From paragraph 3. Putting the systems together of M.D. Hauser and Spelke E., Evolutionary and developmental foundations of human knowledge: a case study of mathematics, Chapter 61 in *The Cognitive Neurosciences III*, edited by M. Gazzaniga, p.853-864, Cambridge, MIT Press, 2004.

55 Dehaene, *The Number Sense: How the mind creates mathematics*, OUP, 1997.

56 C. Lemer, Dehaene S., Spelke E. and Cohen L., Approximate quantities and exact number words: dissociable systems, *Neuropsychologia*, vol.41 no.14 p.1942-1958, 2003.

57 With less coordinate beliefs and desires Hauser and Spelke mean those beliefs and desires animals can have without need for a Theory of Mind.

58 You can understand the number 6873 and even calculate its difference with 429, but you do not comprehend these numbers like the quantity of 1 or 3 or as a notion of magnitude (6873 cabbages is a lot, 6873 rice grains is enough for two, but 6873 is just a number).

59 p. 250 of P. Bloom, *How Children Learn the Meanings of Words*, The MIT Press, Cambridge, Massachusetts, 2000.

60 The idea of inner language does not need to be taken so literally. R. Bartsch created an viable theory explaining how thinking can work as a language with her conceptual semantics. This theory explains how we can form concepts from raw data we gain by experiencing situations. This data is grouped in such a way that utterances and their respective satisfaction situations can be ordered into sets according to their similarity. This structure gives us general concepts.
This theory does suppose we have an innate skill to deal with data, which is available before any concepts have been constructed. This innate data skill allows us to deal with space-time causality and with simple similarities.
R. Bartsch, *Dynamic Conceptual Semantics: a logico-philosophical investigation into concept formation and understanding*, CSLI Publications, 1998.

Our innate knowledge about objects has been researched by Spelke. She found that we automatically define objects as objects under certain conditions. We do so when objects are unified, bounded and persistent through time. These objects were dubbed Spelke-objects. That we do have this automatic skill is very convenient when we want to share concepts (via language) with others. We do not need to define boundaries and can be relatively certain how someone else is going to understand an object we show them.

With a common ground to stand on and recognition of the other as a thinking entity we can start to learn words. We first only learn to understand words, receptive use. This shows in young children by reactions to words and looking at named objects. According to Bloom ToM is already important at this stage and he quotes Morales et al.[61] who have concluded a correlation between gaze following at 6 months of age and the receptive vocabulary at 12 months of age. Bloom takes gaze following as a precursor to understanding referential intent, how a word can refer to a certain intention. At 8 and 10 month-old children supposedly have a median understanding of respectively 15 and 35 words. This receptive skill is not just recognition of words but understanding of the meaning. Active word use generally starts between 10 and 14 months. According to Bloom the actual start of word learning is in part due to the development of our ToM. The ToM enables us to understand the intentions of people, which enables us to align the words we hear and intentions we sense

Word learning is said to happen in two phases, first by association and after that by mapping. But according to Bloom it is a misconception that language is learned in different phases. He thinks there is only one way of learning new words and the difference in word learning speed does not show us a radical break in word learning strategies. We are just not experienced at the beginning and we pick up speed later.

Supposedly at first we would learn to use approximately 50 words by association. And only later between 16 and 19 months would we start fast mapping words. Fast mapping words would be build on word learned by association. Association cannot build on any other form of understanding and would thus be very slow. Fast mapping supposedly causes such a change in the learning of words that this period has been called the word spurt.

However, Bloom does not think there is a spurt or a change in the way of learning words. He thinks there is a gradual growth in the speed of learning words. Words are always learned via the same mechanism, this mechanism is just evolving to take place more and more efficient.[62] The 13-month-old children simply lack experience in word learning. Other factors also disturb our success at word learning at this age. The (fast) mapping before about 18 months can be frustrated by; easy distraction and short attention span, need for repetition or getting used to before getting to terms with new situations, limited memory etc.

Word learning speeds up noticeably between 16 and 19 months, but in absolute numbers words are actually learned much faster between 10-17 years.[63] Bloom is right to remark that a change from knowing 50 to 55 words might be more noticeable than a change from knowing 60.000 to 60.600 words. Yet a change being noticeable doesn't mean something special is happening.

At about 18 months children still make a lot of mistakes in word learning; words are wrong because of errors in speech, errors in word retrieval from memory, overextension of meaning, by giving the wrong meaning to words, etc. After about 18 months words get learned at a higher pace because we get more experienced. We learn syntax gives cues about the meaning of words. This syntactic knowledge is not present at the start of word learning. Syntactic knowledge could in this way cause some increase in the speed of word learning. We

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61 M. Morales, Mundy P. and Rojas J., Following the direction of gaze and language development in 6-month-olds, *Infant behavior & development*, vol.21 no.2 p.373-377, 1998.

62 Bloom quotes a research by Woodward et al. in which 13 month-old children learn words by fast mapping and can point them out a day later above chance levels.
A.L. Woodward, Markman E.M. and Fitzsimmons C.M., Rapid Word Learning in 13- and 18-Month-Olds, *Developmental Psychology*, vol.30 no.4 p.553-566, 1994.

63 Bloom combines information from research (done by Fenson et al. 1994 and Anglin 1993) which suggests the following word learning rates:

1;00	to 1;04	0,3 words per day	37	words learned in this period	87 total
1;04	to 1;11	0,8	170		257
1;11	to 2;06	1,6	340		597
2;06	to 6;00	3,6	657		1254
6;00	to 8;00	6,6	4818		6072
8;00	to 10;00	12,1	8833		14905

Considering estimates that adults know about 60.000 to 80.000 words, between 10 and 18 we learn an additional 45.000 words. An average of 15,4 words a day between 10 and 18.
p. 44 of P. Bloom, *How Children Learn the Meanings of Words*, The MIT Press, Cambridge, Massachusetts, 2000.

have acquired an increased memory capacity, which makes remembering words easier.[64] At this time we appropriate a correct understanding of kinds and individuals. None of these new abilities causes an extraordinary increase in the speed of word learning. They all add a little to the increase in the speed of word learning. In general the gained experience seems to account for the increase.

By experience we learn syntactic cues (most notably lexical constrains) and referential intent. Of course which words get learned depends on what the environment has to offer. But this also depends on the interests of the individual. Nelson[65] explains that 'style' can differ as some individuals seem to have more interest in objects where others have more interest in 'expressive strings of words'. That is what Nelson found out with her systematic research of vocabularies of children between 10 and 25 months, with a follow-up at 30 months of age. Most of these children are in between the extremes[66]: the object style and the personal-social style.

Primary Number Skills

Our first number words are learned by connecting words to the quantity concepts we grasp innately; small and big, little and much. Only later de we connect words to quantities. There is no number 'style' like those Nelson found; an expressive or referential styles. In the first 50 actively used words learned by children number words seldom show up. Nelson did look for number words and letter words, but found that numbers and letters together only add up to 1 percent of the first fifty words. Numbers and letters are probably too abstract for us at this age. Because of the abstraction needed to understand number, a personal interest in numbers does not seem likely to be present from the start of development. A number interest or style can only develop after a basic grasp of language and social interaction.

When we supposedly start fast mapping, at about 12 months, numbers are not picked up quickly either. Even though numbers are encountered in all possible places and brought to our attention in a lot of different ways, we do not fast map them. Something which is quite extraordinary. Numbers are marked by class, they often have a lexically marked position and they statistically occur very frequent. All languages treat number as a separate class. Words like 'double' and 'triple' (in some languages even larger amounts) are used as adjectives. For example 'double bed', 'double doors' or 'dubbele boterham'. These adjectives can be understood by very young children to signify a certain relation and thus might help in understanding numerical values. But such words often have a very local meaning and they are not abstract but relate to objects. Nonetheless all these marked number forms should help us to fast map numbers. Surprisingly we do not fast map number words.

We learn the smallest numbers 1 and 2 quite quickly. In comparison we do not map the larger numbers very fast, even though they frequently pop up in our environment. The appearance of number words in our spoken and written corpus has been investigated by Dehaene.[67] His statistics show that we hear the smaller numbers more often than the larger numbers and that we also hear round numbers more often that their surrounding numbers. Local increases can be found around 10, 12, 15, 20, 50 and 100. Dehaene and Mehler collected number frequencies for both spoken and written French, Japanese, English, Dutch, Catalan, Spanish and Kannada (a Dravidian language from similar to Tamil). In all these languages they found the same results; frequency dropped as numbers got bigger. Exceptions to this drop were 0, which frequency is very low. The round numbers form local peaks in the graph. From a learning perspective these peaks, a statistical division, give us an easy way to distinguish between approximation and sharp numbers. This distinction is a social aspect of our linguistic use of number words and can only be determined in a situation. Besides the smaller numbers being simpler, they are also most present in our speech. This could be the main reasons for us to learn these numbers before the larger numbers. It is commonly accepted that learning starts with simple concepts and builds up to more complex concepts.[68]

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64 Research by Woodward et al. showed that children at 10 months can learn a word after it has been repeated for 9 times in a 5 minute time span.
A.L. Woodward, Markman E.M. and Fitzsimmons C.M., Rapid Word Learning in 13- and 18-Month-Olds, *Developmental Psychology*, vol.30 no.4 p.553-566, 1994.

65 K. Nelson, Structure and strategy in learning to talk, from the series: *Monographs of the society for Research in Child Development*, 38 [serial no. 149], The University of Chicago Press, 1973.

66 The 10 most referential oriented children in Nelson's research had 75% nominal words in their first 50 words, whilst the 8 most expressive children had only 52% nominal words in their first 50. The most expressive children did have the double amount of personal-social categorized words and had an average of 8 function words, were referential children had 1.

67 S. Dehaene and Mehler J., Cross-linguistic regularities in the frequency of number words, *Cognition*, vol.43 p.1-29, 1992.

68 There are no number theories that start with the understanding of large numbers. This is simply counter intuitive.

That numbers are no objects is not what makes it impossible for children to map them. Other non-object features for example colour and material are fast mapped. And also abstract non-physical features like the relation of ownership get mapped. Numbers however seem to be different. Numbers do not get mapped like objects, features or relationships.

That larger numbers do not get fast mapped could mean that there is no inner language equivalent for the larger numbers. Where we can easily understand concepts like objects, colour and personal relationships, we might not have an inner language for numbers. Whether we first need to develop an inner language for numbers or whether we use number words to form our inner language, remains unclear. Either way we need language, internal or spoken language, to understand numbers.

Carey tells us that we start to connect the number word one with quantity concept 1at about 2 years of age, two and 2 at about 2 years and 6 months and three and 3 at about 2 years and 8 months.[69] Again some months later we can count. So about 3 years we can do some counting, but we still need training before we can count fluently. The words are not the problem, before we count we have already learned a lot of number words. We learn these words as meaning some quantity not knowing exactly which. At first we learn the number sequence as we would learn a rhyme. Since we do not yet understand the number concepts, a rhyme is all it is. Before we become 2 we are aware that number words have a quantity value. Semantic cues give number words a marked position in language, which helps us to distinguish the number words from other words. The number sequence can be learned without understanding which value corresponds to which word.

Connecting Number Words and Numbers Concepts

According to Bloom and Carey we connect the first few number concepts with the first words of the sequence and learn to use these words (and/or concepts) as placeholders for precise quantity value. By developing our knowledge of these placeholder for number value concepts, we slowly learn to grasp the first few number values.

We use a sequence we have already learned, to connect our number concepts to. This is convenient, because the words can help us find the right number concept. Tracking a series of objects without mapping (or corresponding) them to something we know, a memorized placeholder, would be straining our working memory. If we did not learn to memorize the number sequence, the next easiest way to keep track of arbitrary objects is to map them with another (non-number) memorized sequence. Children find it easy to keep track of objects of dissimilar sizes if they are structured as a family: father, mother and child. Since the family nucleus is a structure all humans are familiar with, this is a very successful placeholder.[70] Tracking a number of objects with a metaphor is a lot easier than tracking objects without metaphor. Without placeholder or metaphor we would need to map straight to number value. With 1, 2 and 3 this is doable, up to 6 this seems somehow possible. When we get to twenty, it is highly unlikely we can manage the workload. To reduce the workload we learn number words before number value concepts. When we count the elements and we only need to remember one number word, which is far easier.

To learn the natural numbers Carey thinks we use an uniquely human capability, bootstrapping.[71] Bootstrapping is the process of creating new concepts from existing concepts. To explain the origin of any concept Carey states that we need three things; first we need to know our innate 'concepts', second we need to specify how our target concept differs from the ones we know and third we need to know how we can learn this goal concept. Carey thinks innate knowledge is necessary for us to be capable of learning anything. We cannot bootstrap without any knowledge.

With respect to numbers we know our innate concepts. Among others we have discussed Hauser and Spelke's core knowledge, concepts with which Carey will agree. What we want to learn is an understanding of the positive integers which enables us to count. This differs from our small precise number system and large approximate number systems. What we want is to be precise for the large numbers.

Carey distinguishes two bootstrapping mechanisms, semantic and syntactic bootstrapping, which we need both when learning language.[72] The semantic bootstrapping we use to identify which words connect to which objects or adjectives. The syntactic bootstrapping mechanism we use to identify certain words constructions with certain concepts. An example of syntactic bootstrapping is finding out that when there is a gift, there is also a giver and receiver. According to Carey these two forms of bootstrapping follow the same pattern and

69 S. Carey, Bootstrapping and the Origin of Concepts, *Deadalus*, p.59-68, 2004.
Chapter 1 of S. Carey, *The Origin of Concepts*, New York, Oxford University Press, in press.

70 U. Goswami, Transitive Relational Mappings in Three- and Four-Year-Olds: The Analogy of Goldilocks and the Three Bears, *Child Development*, vol.66 no.3 p.877-892, 1995.

71 S. Carey, Bootstrapping and the Origin of Concepts, *Deadalus*, p.59-68, 2004.

72 Chapter 1 of S. Carey, *The Origin of Concepts*, New York, Oxford University Press, in press.

are in fact the same mechanism applied to two different kinds of concepts.[73]

With respect to numbers Carey notes that we get help from natural language quantifiers. Before we count we handle the some words as quantifiers; for example distinguishing between plural and singular, 'some' and 'a'. Research shows this holds for English speaking children. Other languages have other grammar concerning the singular-plural distinction. Japanese does not have such a distinction, where Russian has distinct grammatical forms for groups up to and smaller than 4 and groups greater than 4. This difference in language leads Japanese children be relatively late 'two-knower's' and Russian children to make a difference between the smaller and larger sets long before they can count.

Carey thinks that the moment that we construct the positive integers is after we have connected the first few number words with our innate precise number concepts. According to Carey knowing integers is a qualitative change in the child's way of representing the world and we can make this qualitative change by bootstrapping. When we bootstrap and understand the positive integers we create a new mental symbol (concept) for a set of relations. We learn a new concept. Bootstrapping is the capability to create and use new concepts. Bootstrapping is the transition from a sets of concepts and the relations among these concepts and a new concept which contains or encompasses the previous concepts and relations. By creating a concept (and a word) for an initial set of concepts and relations, we can understand the things we see and retrieve them later by recalling the concept (or word) that represents them.[74]

Understanding the positive integers is understanding the 'formal' rule that the number n in the number sequence is the previous number k plus one: $n = k+1$. Concluding this rule seems arbitrary to Rips et. al.[75], as we do not need to conclude a linear number system from 1, 2, 3 & 4. According to Rips we could just as well (with a little help from a diabolic parent) conclude a circular system, where 11, 21 etc. would again be 1. This system would be similar to the numbers on a clock. Rips prefers an axiomatic approach to learning numbers. Children could better learn the axioms described by Dedekind in which case they would have no need to bootstrap. The writers of this critique however blatantly ignore Carey's description of the innate approximate large number system. This system is very likely to block a circular number system or any other possible number system. Margolis and Laurence[76] remark that children also have to perceive a difference of exactly 1 to use the counting sequence. Perceiving this difference of 1 will also stop them from constructing a circular system, as the difference between 10 and 1 is not 1.

When children learn the quantity value of numbers, they know the quantities of 1, 2, 3 and 4 with their small number skill. These first four quantities are derived from our innate number representations. All other precise numbers are derived with a bootstrapping procedure. According to Carey these numbers can only be derived by bootstrapping. We cannot be told what 2 or 5 is, we can only discover it. Other knowledge might be connected to words, for number knowledge we miss the relevant concepts. These concepts, how quantity works, we can only discover by ourselves.

From innate knowledge we develop understanding of integers. What exactly happens when we bootstrap is probably best described by Bartsch' conceptual semantics. We use our experience of the world and the concepts we already have to fabricate new concepts. During Carey's bootstrapping process we must consider all our number knowledge at once. If we do not consider all our knowledge we might end up with a wrong or contradictory system. Luckily a contradictory system might be very convenient to construct the proper system a little later on.

To help us do the bootstrap we have our innate number systems and the number words 1, 2, 3 and 4, we have our linguistic number knowledge, like natural language quantifiers and more number words and we have all the situations with numbers and number words we experienced. Plus we have to know the fact that

73 Werker and Yeung even claim that a bootstrapping process starts before birth, leading to our phonetic preferences. Babies prefer their mothers voice and their mothers tongue. At first the bootstrapping process is build on statistical learning, but after 9-10 months social interaction becomes more important. We can learn word fast because we bootstrap to words from syllables and to syllables from speech and to speech from sounds. This process in effect eliminates all sounds, letters, syllables and words that are not used in the language we learn. In the same way we learn to prefer stress patterns, grammatical rules and more.
J.F. Werker and Yeung H., Infant speech perception bootstraps word learning, *TRENDS in Cognitive Sciences*, vol.9 no.11 p.519-527, 2005.

74 This system of building on concepts is, though less complicated nor related to experiences, similar to that of Bartsch.

75 L.J. Rips, Asmuth J. and Bloomfield A., Giving the boot to bootstrap: How not to learn the natural numbers, *Cognition*, vol.101 no.3 p.51-60, 2006.

76 E. Margolis and Laurence S., How to learn the natural numbers: Inductive inference and the acquisition of number concepts, *Cognition*, [Electronic publication ahead of print], 2007.

the difference between subsequent numbers can only be exactly one. To perceive this difference we need adequate conservation and correspondence skills. All this knowledge must help and subsequently become a part of the positive integer concept that we construct.

All this knowledge comes naturally to us. We do not need to actively consider all of it. So when we simultaneously understand that we can use number words as placeholders to keep track of quantities beyond 4 and that we need to differentiate between these quantities with a difference of exactly one, then we (more or less) understand the principle of natural numbers, integers. Fully realizing this may still take some time and practise. But the concept of the positive integers will allow us to do great stuff.

Combining the set sequence of the number words and the sense of magnitude from our large number system allows us to see numbers as ordinals and cardinals simultaneously. This allows us to substitute one type of ordering for the other, depending which type we find more convenient for the task at hand. After some training we can use our counting skill fluently and we can expand our counting skill with adding numbers, subtracting numbers, multiplying, dividing and so on.

2.2.3 Two Views of Number Knowledge

The two different views of number knowledge are led by Piaget and Bloom. In Piaget's view number knowledge is not dependent on language skills, but is centred around the operations involving numbers. Language is only used to communicate about numbers and language does not help us to understand numbers. We will call Piaget's theory the isolated number knowledge theory. Opposing this theory is Bloom who thinks that language does play an important and indispensable role in the understanding of numbers. Therefore we will call this view integral number knowledge.

When we consider numbers as integral knowledge and we think about numbers, then number words are mentally used as symbols or a kind of placeholders. We know these number words refer to a cardinal or ordinal value, but most often we do not need to actively construct this value or place. Especially the larger number words are used as placeholder for which we can construct meaning, if the need arises. Because language is such an important part of number knowledge in the integral number knowledge view, ToM is also more important. ToM seems to be an indispensable part of language and thus of the integral number knowledge theory.

In the isolated view skills like conservation, comparison, correspondence and seriation tell the story of how we get to understand the numbers. To Piaget our development of skills shows when we make correct use of number words and quantities. Isolated number knowledge is not dependent on either language nor ToM. To Piaget using language to communicate with the participants of these trials is not a problem, as language simply develops along side the other skills and solely facilitates our communication needs. Piaget focusses on the number information we are dealing with to learn the number operations. Language is just another problem. Piaget hints at an alternative for using words as placeholders; more use of our conservation of identity. Piaget treats the conservation of identity as one of the prerequisites to reasoning. 'Only if we can hold on to ideas, can we construct concepts.' seems to be his way of reasoning. According to Piaget our skill to conserve identity also enables us to conserve number identities without language. Strikingly Piaget thinks that we learn the words after and separately from the number concepts. It seems an inconvenient use of our energy to do this twice, but in Piaget's reasoning he is just keeping things simple.

Nevertheless when number knowledge would be isolated from language, to become fully developed adults we need to learn to communicate and we need to somehow connect number words and number values. So even when the development of number knowledge is isolated, this would only be a relative isolation since we learn the number words almost simultaneously. However isolated number knowledge would allow us to develop number knowledge despite language or ToM deficits.

To some extent Carey agrees with Piaget, since she says that small number knowledge has to be derived individually. You cannot learn what 1, 2, 3 or 4 is from someone else. You have to grasp the feeling of these numbers by yourself. For the larger numbers however Carey presupposes language and the knowledge of number words.

Hauser and Spelke are more inclined to the isolated number view, as they think ToM plays no part in small or large number skills. They state that the modules of core knowledge work separately from other modules. And they state that our innate number systems remain the backbone of all further number knowledge throughout our lives. That according to Hauser and Spelke our number systems works independent form the beliefs generated by our ToM, does not mean that ToM and language are not vital to the development of our number knowledge. So in the end Hauser and Spelke are eventually not sharing the isolated number knowledge view with Piaget. The general agreement about the integration of language and number knowledge however allows us to combine the separate parts of the theories into one.

Complementing views
Piaget's focus on operations and current researchers focus on factual knowledge could also complement each other in a number knowledge theory. The role of language is however a problem when we want to combine these views.[77] The other developmental steps in these two theories could actually complement each other. We will first give a diagram representing Piaget's isolated number knowledge view and second give a diagram representing the integral number knowledge view. With these two schemes we will talk about how these two theories can be combined.

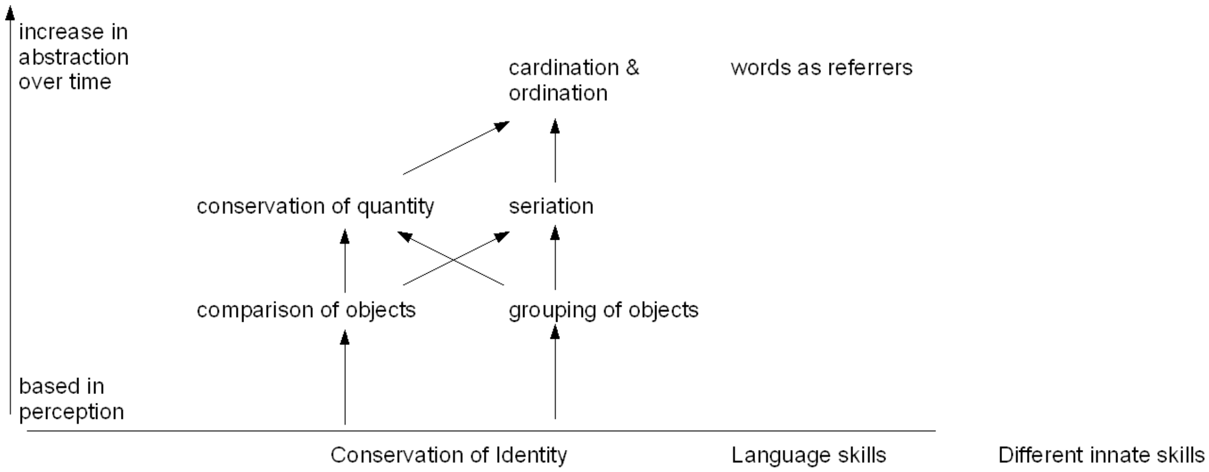

Diagram 8: Isolated Number Knowledge Theory

The diagram of isolated number knowledge shows that there is no influence of the language skill on our final counting skills. The conservation of identity helps us to compare objects and to group these objects. These skills help to actually conserve the quantity of objects and to start making series. Seriation and our development of this skill, correspondence, double seriation etc. together with the conservation of identity eventually help us to grasp a cardinal and ordinal understanding of numbers. Language skills are not having an influence on the counting skills and develop to par with the number skills horizontally.

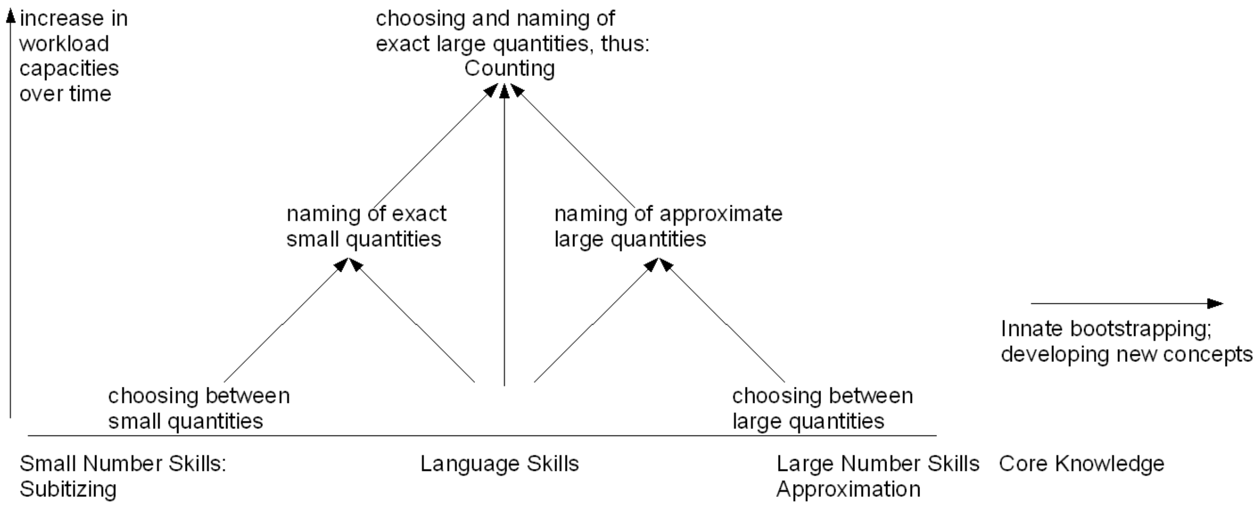

Diagram 9: Integral Number Knowledge Theory

The diagram of Integral number knowledge shows that language plays an important role in the learning of numbers. Our language skills are first combined with both the small and large number skills. This allows us to name the precise small numbers and the approximate large numbers. This knowledge combined with our knowledge of the number words then develops into the skill of naming the exact large quantities and counting.

The integral number knowledge view does not talk about the conservation of identity. Comparing and

77 We will not be talking about the factual details of Piaget's theory. For example: we will not look the age at which children know conservation. As Piaget thinks the conservation skill appears a lot later than current research has shown. We will just look at the fact that they do know it and its place in the order of Piaget's development.

grouping are not really discussed in current research and conservation of quantity is not put forward as important. Current research does not talk about the role of seriation. In Piaget's isolated number theory there is no talk about subitizing nor about the approximate large number skills. It does not talk about learning number words or about how we actually construct new concepts. In this respect the theories complement each other and we can construct the following diagram.

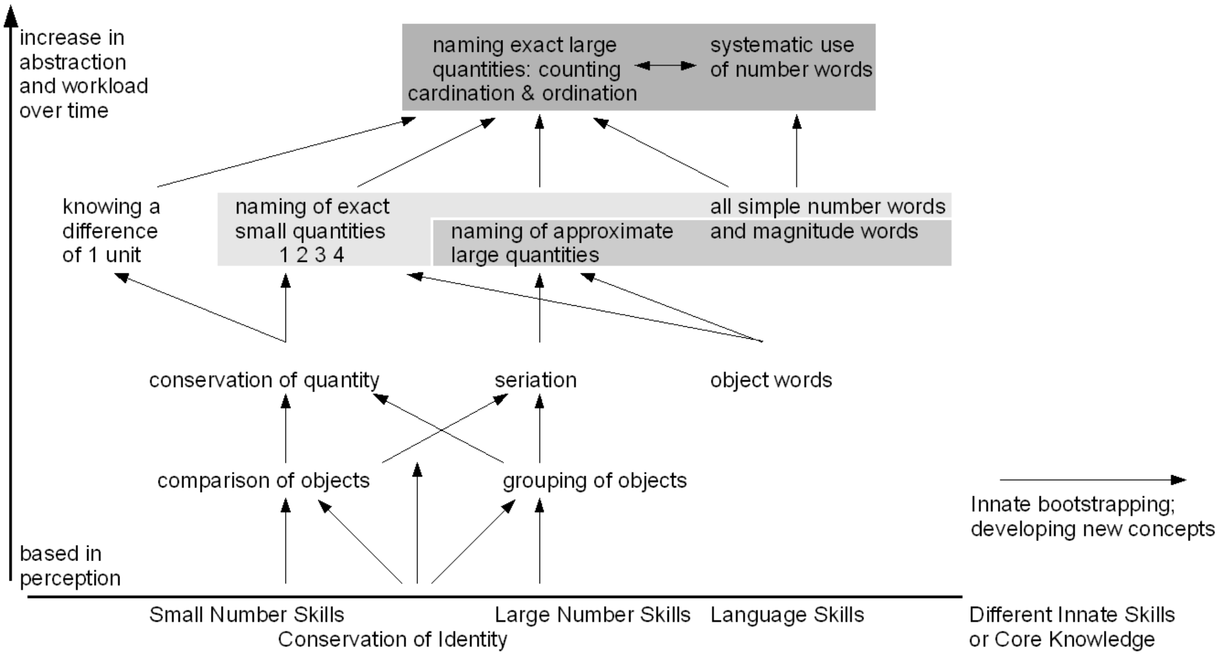

Diagram 10: Combined Development of Number Knowledge Theory

In this diagram we have combined the theory behind Piaget's isolated number knowledge view and the different theories behind the integral number knowledge theory. Combining both views can bring some of the development of the number operations back to the developmental discussion. This diagram gives a more complete picture of how we learn numbers. If you favour the isolated number knowledge view, then interpret the arrows from the language skill colon more horizontally.

The small and large number skills and the innate conservation of identity have a general influences to all later developed concepts and understanding. Knowing a difference of one unit is added to this diagram for more detailed. The different grey areas show us how number concepts and number words get connected. Knowing a difference of one (developed from conservation and correspondence skills), naming of the smaller and larger number words and our linguistic number skills brings us to the develop our counting skill.

Chapter 3

What would an autistic counting theory look like?

In chapter 1 we saw several impairments in autistic counting. In this chapter we will look at how these impairments can be explained with theories about autism. This will help us to form a developmental theory about autistic counting skills.

We identified 4 problems in autistic counting compared to counting in TD children. 1) There is not much benefit from the canonical placing of dots. 2) Quantity 4 is named slower compared to TD children – which shows autists do not subitize 4 but count instead. 3) Autists are late to develop their sequencing skills. 4) And autists have problems with recalling positions.

To explain these four differences we will put forward a theory about the autistic development of number knowledge. This theory will show how autists develop counting skills and how building on this development autists possibly deal with other number knowledge.

What to Expect
An explanation of the autistic development of counting skills will look a lot like the typical development thereof, but it will differ in some respects which will lead to noticeable impairments. Forming this theory as a developmental scheme will reflect that disabilities in the autism spectrum are part of a developmental syndrome. This means a disability in a central system causes something to go wrong with the development of the individual, which can cause various degrees of severity. The severity of the spectrum goes from mild and moderate up to profound and severe. Pervasive developmental disorder, not otherwise specified (PDD-NOS) forms a large mild group in this spectrum. Autism is found in at least one in thousand children and Asperger syndrome is found in about one in three thousand children.[78]

To be able to explain something about autistic counting we will limit ourselves to the skills of high functioning autists. This subgroup is more developed and is better capable to take part in trials.[79] The general idea is that the impairments present in high functioning autists also appear in low functioning autists. The impairment of individuals in the autistic spectrum is very divers as some are close to normal and others have no language skills. We will try to theorize about the development of high functioning autists and not TD individuals with autistic characteristics.

3.1 What is Autism?

Individuals in the Autistic Spectrum have a collection of symptoms. These symptoms are divided into three traditional symptom groups; A) a disruption of social interaction, B) abnormalities in verbal and non-verbal communication and C) limited, repetitive and stereotyped behaviour. A problem D) with global processing (supposedly enhancing local processing) has only been recognized recently. Research on this symptom has really only started this century. There are other more individual symptoms which are not essential for diagnosis, such as abnormal sleeping or eating habits, phobias, temper tantrums and self-directed aggression.

There are several theories about what causes the symptoms of autism; 1) Disabilities in Theory of Mind, 2) Disabled Executive Functioning Theory with 3) a few other reasons for operational dysfunctioning in autism and 4) Weak Central Coherence Theory.

Disabilities in ToM primarily explain A) disruption of social interaction, but these problems could lead to not developing normal communication skills and normal behaviour. The executive dysfunction, situated in the frontal lobes, predicts frustration of our acting, inhibition, creative thinking, self-monitoring and set-shifting. A WCC predicts problems integrating information. And research of working memory can predict some problems due to a different balance in visual-spatial and verbal working memory.

We will look at the theories explaining autism and pay special attention to the implications for our counting skills. We will look at the three main theories and at our working memory and finally at how the differences between typical and autistic counting can be explained by these theories. In the appendix we describe how the theories explaining autism relate to each other.

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78 E.M.D. Fombonne, Epidemiology of Autistic Disorder and Other Pervasive Developmental Disorder, *The Journal of Clinical Psychiatry*, vol.66 suppl.10 p.3-8, 2005.

79 There is no scientific definition of high functioning autism. High functioning mostly indicates individuals that have a relatively high IQ (>80) and that are capable of communication and day-to-day activities. About 30% of autists have no language skills.

3.1.1 Disabilities in Theory of Mind

Autists have problems with social interaction, verbal and non-verbal communication which are likely due to problems with ToM. Autists have difficulty with reasoning about mental states of others. This causes problems with social interaction and direct communication. Not being able to attribute states shows in adolescents and adult autists, but problems start with the precursors of ToM being impaired in autistic children. In a typical development of ToM we go through a few stages; from gaze following to full awareness of other individual's mental states. Impairment of ToM can happen during any of these stages and in the worst case can even stop development altogether.

Leslie thinks that the ToM starts out as a mechanism rather than a theory.[80] [81] He works out the different stages we go through before we come to think in a theoretical manner about other peoples reasoning. According to Leslie we have an innate ToMM (Theory of Mind Mechanism) similar to our Spelke-object bias, which we develop into a ToM. With our ToMM we go through a few stages and changes before we form a real Theory of Mind and start 'theorizing' about others. According to Leslie the ToMM permits, promotes and directs our representations of peoples thoughts, beliefs and desires and he thinks the mechanism must be innate.

It is these representations of thoughts of others, called metarepresentations, that Leslie and Happé think autists might not be able to make.[82] They would not be fully capable of representing their own thoughts nor those of others. This problem with representation can explain the problems autists have in the false belief task, yet it leaves space for some ToM or ToMM. If autists have problems with these metarepresentations, then this part of the ToMM does not start the module which is supposed to look at what others mean by what they are doing. The disability in ToMM is a disability in the development of the autistic individual as it stops the individual from developing social and communicative skills.

Disabilities of the ToMM can cause ToM problems of several degrees of severity. The most fundamental way in which ToMM can go wrong is when an individual does not understand himself as a separate thinking entity apart from the world. This causes severe retardation because no causal relations between action and reaction arise, which causes the world not to be experienced as something that can be manipulated. This ToMM problem would probably not lead to any development in number knowledge.

A little less severe effect of ToMM problems is not recognizing others as thinking entities. The autist develops its own reasoning, but is living in a world of its own. Nobody is recognized as another thinking person so reasoning is not developed into a social skill and becomes very particular. Some moments in which others align their attention with the autistic individual might still create situations in which the individual can learn. Number knowledge might possible develop in this scenario, but will most likely not be very elaborate.

Still less severe would be problems with the disentangling of thoughts and knowledge of oneself and another. Not being able to disentangle different desires, would explain why autists have problems with the false belief task. But problems with disentangling would allow for joint attention initiated by the autist.

ToM and Learning Language

According to Bloom and the integral number knowledge theory we formulated, we need language to learn how to count. To learn language we depend on ToM and we only fully develop our language if we fully develop ToM. When we learn (language) from others, we will need to understand what these others are attending to. In language we give a semantic meaning to words with our intentions. These intentions, which are inherent to any meaningful language, make language a social act. When we learn numbers and number words (or numbers in any other form of communication) the intentional stance is very important, because only with an intention do words get an actual meaning. To understand someone's intentions we use ToM.

When we learn words we unite a worldly fact and a social convention. ToM is a necessary skill to understand this social convention, to pick the right meaning from a context. According to Bloom an (arbitrary) word that is learned is a symbol for the thoughts of other humans. Of course such symbols could be learned by association of situations and words, but with our ToM we simply know that people think. Therefore we actively seek out other peoples thoughts to grasp the meaning of their words.

The learning of words, let alone number words, brings us philosophical problems. How do we know that the word 'rabbit' refers to an actual rabbit and not to its tail or its white colour?[83] It could be a form of association;

80 A.M. Leslie, Friedman O. and German T.P., Core mechanisms in theory of mind, *TRENDS in Cognitive Sciences*, vol.8 no.12, December 2004.

81 A.M. Leslie, Developmental parallels in understanding minds and bodies, *TRENDS in Cognitive Sciences*, vol.9 no.10, October 2005.

82 A.M. Leslie and Happé F., Autism and ostensive communication: the relevance of metarepresentation, *Developmental and Psychopathology*, vol.1 p.205-212, 1989.

83 The mentioned Spelke-object bias helps us to start our focus on whole independent objects.

simple association by covariation (Mill, Hume, Locke), association by reinforcement and punishment (Skinner), association through a sensory network or a network sensitive to statistical regularities (Plunkett). Or to choose the right object form a situation we learn to sense what the situation is attending to. Like gaze following and joint attention we develop this social skill.

Most of the traditional theories suggest that we learn words when we are looking at objects. But only in about 30 percent of the cases do word learning children actually look at an object when it is named.[84] If looking and learning would always coincide we should be making mapping mistakes quite frequently. But we do not confuse the meaning of the word the situation is attending to for what we are looking at. We simply learn the meanings and words of objects from a situation. A strict associationist would have a hard time explaining how we learn object words and an even harder time doing the same for non-physicals, such as numbers.[85]

Disabled ToM and Learning Numbers

The disabled ToM influences the ability of autists to learn to count, because according to Bloom our ToM determines how we learn words. Learning to count depends on language and learning language is depending on a well developed ToM. According to Bloom the words learned by autists do not have the same meaning as the words learned by TD children.[86] Word learning with a disabled ToM leads to a disabled learning of language and an impaired understanding of words.[87] Autists learn language at a lower pace, they use less words and thus have a less thorough understanding of words. This lack in word use also makes the autistic understanding of words and concepts less interconnected. Autists learn words disconnected from a social context. Therefore it is likely that an autistic understanding of words will be very factual.

The problems with learning words are also present when leaning number words. The lower word-use of autists will make the understanding of number words local and factual. This causes them to only have limited number knowledge when compared to TD children. According to Bloom normally the grammatical structure of number words is used to connect these words to number values. Autists lack this understanding of number structure. Learning the number words without a social context will make the autistic use of number words even more factual. Besides number words the understanding of number values also remain relatively isolated. The local and factual understanding of the number words will make it harder for autists to develop a connected and full understanding of number knowledge and to succeed in number situations.

These difficulties will at least make autists develop their number skills slower. And due to a lack of factual number knowledge and a lack of understanding of number values, autists might not be able to bootstrap to the counting skill at all. A spread in the severity of problems caused by ToM (and ToMM) can be linked to a spread in the severity of problems with number skills. According to Leslie only autist with sufficiently working ToM skills will learn to use numbers. Due to a disabled ToM autists could develop a less complete, less connected and less active number knowledge. They will be structurally behind on counting and being behind on counting also leads to less developed mathematical skills.

Counting without ToM

Let's assume that we do not need ToM or language to learn how to count. When assuming the isolated number knowledge theory instead of the integral number knowledge theory, can autists learn to count without ToM? We can try to explain two different outcomes: no, they need and have some ToM; or yes, learning to count can be done without a ToM. The first answer will lead us back to what we have already discussed

84 In the case of objects names this is most probable. And the 30-50 percent is only achieved in the most supportive family environment, which is not available for all children. Bloom got this data from:
G.M. Collins, Visual co-orientation and maternal speech, in H.R. Schaffer (Ed.), *Studies in mother infant interaction*, London: Academic Press, 1977.
M. Harris, Jones D. and Grant, J., The non-verbal context of mothers' speech to infants, *First Language*, 1983.
Object names are most probably learned in overheard speech:
Lieven E., in Galloway C. and Richards B., *Input and Interaction in Language Acquisition*, p.56-74, Cambridge University Press, 1994.
E.L. Schieffelin, Performance and the Cultural Construction of Reality, *American Ethnologist*, vol.12 no.4 p.707-724, 1985.

85 Children with autism more often associate the object they are looking with the word they hear. These children have a hard time unlearning these wrong associations.
S. Baron-Cohen, Baldwin D.A. and Crowson M., Do Children with Autism Use the Speaker's Direction of Gaze Strategy to Crack the Code of Language?, *Child Development*, vol.68 no.1 p.48-57, 1997.

86 p.78-81 of P. Bloom, *How Children Learn the Meanings of Words*, The MIT Press, Cambridge, Massachusetts, 2000.

87 The different degrees in which autism is found could explain that some autists do learn language, but that in general autists might have an impaired ToM.

about ToM. The second answer we have yet to discuss.

If we assume that counting can be learned without ToM, autists will still have general difficulties due to ToM with learning. They still have problems with finding the focus of a situation, what others are attending to, but this does not mean they will never learn anything in such a situation. Others could help them to overcome this problem. Nelson[88] says learning is dependent on parents looking at what their children are attending to. If only in some of the situations others are looking out for autists, then they could benefit and learn. But these situation could still imply some autistic knowledge of ToM.

Could autists learn themselves to count, without the help of others and thus without a ToM? This is hard to say. To make counting possible autists would need to be able to learn (and develop advanced reasoning) without help from others.[89] To learn without ToM, some voids need to be filled. The first function autists need to replace is to choose what to learn. ToM helps TD children to pick out what is important and what they exactly need to learn. Without ToM a picking strategy will result in very personal (as opposed to socially) driven choices of what is important. This picking process could be aided by a statistical or associative learning strategy (alternatives also available to TD individuals). Of course the lack of ToM makes autists pick out very different things to learn.

Giving values to actions is a second function of ToM in our learning process, which needs to be filled by autists. In TD individuals actions become meaningful by interaction with others. Autists do not experience the social value of actions and will most likely not fill this gap with alternative values. This will make different actions neutral when compared by autists. Nonetheless some value will be given to actions. The action of counting small quantities is likely to receive some meaning for autists without the need for ToM. Relating these small quantities with words (or other symbols) and learning to see a difference of 1 between quantities both seems possible without ToM. Relating all quantities to set a string of (number) words without someone initiating this relation becomes less likely, but could still be possible without ToM.[90]

Learning to understand the larger numbers and counting is already described as the process of bootstrapping, a skill one cannot learn but only find out. Due to ToM TD children feel pressure to make the bootstrap to counting. Autists without ToM will be less likely to feel the need to make a bootstrap or even to feel that they are missing out on quantity information.

Concluding Disabilities in ToM
ToM has a major influence on our general development. Autists develop slower due to their disabled ToM. Whether ToM is needed or not to learn counting, we can imagine ways in which autists are able to learn how to count. Autists most likely have some ToM and having more ToM will benefit their possibilities to learn and to learn counting. Whether ToM is needed to learn and thus to learn counting is hard to say; to us learning seems possible without ToM. If ToM is not needed to learn number skills, the autists understanding of ToM (or ToMM) will still influence the speed at which number skills are developed. Autists give a more factual meaning to number words and values due to their disabled ToM.

3.1.2 Disabled Executive Functioning Theory

The executive dysfunction theory explains the symptoms of autism being caused by problems with Executive Functioning (EF). The EF is part of the operational system and is not separate as a mental module, hence Disabled Executive Functioning (DEF) has an effect on all our actions.

Hill wrote an extensive evaluation of executive dysfunction in autism and in children with moderate learning difficulties.[91] Executive dysfunction is associated with deficits and damage to the frontal lobes. She describes EF as those actions that require us to disengage from our direct surroundings. According to Hill autists are impaired in executing their planning, mental flexibility, inhibition, generativity and self-monitoring.

Hill reports some problems in executive dysfunction research because of poor distinctions between autists and people with moderate learning difficulties. Hill is sceptical about older research, because autists might not have been recognized as such and could have 'contaminated' the moderate learning difficulties control

88 K. Nelson, Constraints on word meaning? *Cognitive Development*, vol.3 p.221-246, 1988.
89 If however autists can develop advanced reasoning without ToMM, these autists might be instructed to reason about other peoples feelings. This would allow an autist to form an alternative ToM and enable the autist to learn in a more or less typical way. The individual must be extremely accurate and persevering to reach this stage, but would be able to learn everything a TD individual can learn.
90 The relation between quantities and words could possibly happen without ToM, but through statistical or associative learning. That autists do not see other peoples as thinking entities does not make them insensitive to their environment.
91 E.L. Hill, Evaluating the theory of executive dysfunction in autism, *Developmental Review*, vol.24 p.189-233, 2004.

groups. This contaminations and the small number of subjects would lead to ambiguous results.

Planning
DEF is most noticeable in our planning skills. Planning actions to pursue a goal requires a lot of our capacities. We need to conceptualize a sequence of changes; from our current situation to a goal situation. To execute a plan we need to monitor what we do, re-evaluate our status and keep this up to date. Simultaneously we need to explore alternative routes and adapt our plan and possibly even alter our goal.

According to Hill planning is complex and dynamic. Planning is usually tested with a tower task. In this task participants are given a set of disks stacked on three pegs of different heights. They are asked to copy some examples of such a set in a minimum number of moves, moving one disk at a time. Autists (children, adolescents and adults) have a deficit in comparison with normally developed subjects, but not in comparison with subjects with a learning disorder.[92]

Another version of such a tower task by Hughes et al. was divided in easy and difficult questions.[93] Easy questions could be solved in two or three moves and difficult questions in four or five. Autists were shown to have more problems with the difficult questions in comparison to the participants with moderate learning disabilities. To Hughes et al. these results showed that general ability has influence on planning and that at least high functioning autists seem capable of some planning skills. According to Hill we need to be careful in our conclusions and consider that autists may also have moderate learning disabilities.[94] Combined autism and learning disabilities could have an added effect, which would be what Hughes et al. found.

According to Hill and others planning is difficult to measure as such, as most planning also involves other skills, most often memory and inhibition skills. In the tower task a lot of skills are required besides planning. The task has physical requirements: keeping track of pegs and discs. The task requires understanding of the question of coping the example in a minimum number of steps. Answering the question requires some ToM and some understanding of numbers. It also requires another EF skills: generativity is needed to come up with possible solutions.

We cannot consider counting to be as complex as the tower task. One of the requirements of the tower task already is number related; keeping the steps to a minimum. It is not probable that the autistic counting is limited by a dysfunctioning planning skill.

Self-monitoring
Self-monitoring is the capability to see ones own actions as means. This enables us to think about our actions and compare between actions in an planning situation. Autists find it hard to disengage from their own behaviour and actions. Matched control groups with learning disabilities show the same problems with self-monitoring. More research is needed to determine whether this is a developmental delay or specific to autism. Whether we accept self-monitoring as a part of EF depends on how we theorize about EF and ToM. We will not discuss these views as EF and ToM can both explain each other (see appendix).

Self monitoring does not seem important to counting directly, but is important in general to our development towards our number knowledge.

Set-shifting
Hill describes mental flexibility or set-shifting as the skill with which we can look at objects in their different modalities. Poor mental flexibility results in repetitive and stereotyped behaviour. The cause of rigid thinking in autism has been identified as a lack of mental flexibility, not being able to change your point of view. According to Hill repetitive thinking is also related to verbal IQ, but this relation does not help us in explaining rigid thinking. Problems with inhibition might add to the explanation why autists do not change their point of view. Due to problems with inhibition autists might rule out options that are wrongly identified as distractions. These over-inhibition mistakes could cause a rigid focus on only one view. When we use over-inhibition as an explanation for rigid thinking, it resembles the set-shifting explanation for rigid thinking.

In their study Hughes et al. compared autism with other cognitive disabilities on a set-shifting task.[95] In this

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92 Hill based this on different versions of the so called Tower task: Bennetto et al., 1996; Ozonoff & Jensen, 1999; Ozonoff et al., 1991; Ozonoff & McEvoy, 1994. And on a connect the dots trial by Rumsey & Hamburger, 1988.

93 C. Hughes, Executive function in preschoolers: Links with theory of mind and verbal ability, *The British journal of developmental psychology*, vol.16 no.2 p.233-254, 1998.

94 When comparing high and low IQ children with autism and children with other learning disorders suggests that planning is more related to general IQ rather than to autism.
p. 196 of E.L. Hill, Evaluating the theory of executive dysfunction in autism, *Developmental Review*, vol.24 p.189-233, 2004.

95 C. Hughes, Russell J. and Robbins T.W., Evidence for executive dysfunction in autism,

task the participants had to find rules for choosing between shapes with a computer showing them whether they had chosen right or wrong by displaying this on a screen. The rule that the participants have to find changes in two ways: intradimensional, colour was determining what was the right choice and the colour now changes from pink to white; extradimensional, colour was determining what was the right choice, this now changes to shape determining what is right. The study had one extradimensional rule shift at the end at which autist scored far below TD participants and far below moderate learning disabled participants. According to Hill the study by Hughes et al. suggests that autists get stuck in a set, rather than having a rigid non set-shifting thinking style.

Impaired set-shifting could cause problems with sequencing (see 1.2). Not being able to shift focus from a subset to the whole set when comparing sizes in a subset could lead to problems with sequencing the whole set. This could be a clarification for problems with shifting between local and global scope. Autists tend to get 'stuck' in one scope which prevents them from getting to the answer, because they do not consider all elements.

A possible developmental problem that set-shifting causes with respect to autistic number knowledge is autists not being able to see both the cardinal and ordinal value of a number. According to Piaget this simultaneous understanding of both these modalities is important for a full development of number knowledge. At least at one point in time we need to understand both simultaneously to construct the connection and to be able to translate cardinal values into ordinal values and vice-versa. Understanding both modalities simultaneously is necessary to learn to understand numbers. Not shifting easily between these views can limit the usefulness of numbers for autists.

Another problem which could arise due to a lack of set-shifting is a confusion over number use. When different kinds of number dimensions are used, cardinal or ordinal opposed to nominal[96], this could lead to confusion. Especially nominal numbers are used differently from cardinal or ordinal numbers and could be very confusing if interpreted rigidly. Saying that one lives at number 22 for example could then be confusing.

When numbers are used in their nominal dimension this produces a very literal use of those numbers. Some autist give the nominal number meaning a personal touch; not liking or being fond of a certain number, associating a number with certain characteristics. This personification or association-with can help autists with their number knowledge, but it can also work against them.[97]

The shifting between the different number dimensions might be evaded by treating the uses of numbers as facts. Such a strategy will make it easier to seemingly combine different dimensions of numbers, but this will cause a lack in understanding.

In our later (sometimes called secondary) mathematical skills set-shifting will be used more frequently. In this stage set-shifting can help us to develop and test alternative arithmetical strategies, for example when children start with multiplication.[98] The mathematical skills beyond counting may therefore be more difficult for autists. In these processes inventing different strategies also requires the problematic generativity.

Generativity

Spontaneously creating words, ideas, solutions and taking initiative is called generativity. A lack thereof is another symptom of autism. Hill takes impaired generativity in autists to be related to repetitive behaviour and the dislike of change. The lack of generativity makes it difficult to react to change or abnormalities. Autists perform below the norm when asked to come up with as many as possible words starting with a certain letter, things to do with a newspaper or design that can be made with some shapes within a set time.

This part of the EF does not play a direct role in counting, but maybe important in the development of our number skills. According to Bloom the generative part of language helps us a great deal to understand numbers.[99] Most of the higher number words are 'generated' when the need arises. The generative feature of language is our ability to 'creatively' construct an infinite amount of sentences with a limited number of words. With a few strict rules this same generative principle allows us to track or characterize infinity in a quantitative manner.

Autists also often have impaired linguistic generativity, which in turn could lead to problems with generating

Neuropsychologia, vol.32 Issue 4 p.477-492, 1994.

96 A number used as a name, e.g. “take the 5 to Amstelveen” where 5 refers to tram line 5.

97 D. Park and Youderain P., Light and number Ordering principles in the world of an autistic child, *Journal of Autism and Childhood Schizophrenia*, vol.4 no.4, 1974.

98 P. Lemaire and Siegler R.S., Four Aspects of Strategic Change: Contributions to Children's Learning of Multiplication, *Journal of experimental psychology*, vol.124 no.1 p.83-96, 1995.

99 p. 235 of P. Bloom, *How Children Learn the Meanings of Words*, The MIT Press, Cambridge, Massachusetts, 2000.

numbers. High functioning autists can find rules, but compared to TD children they do not perform well. Autists could have problems understanding and generating numbers due to an impaired generativity.

Inhibition
Not all problems with inhibiting are symptoms of autism. According to Hill certain inhibition problems may only be present in autistic children with learning disabilities, other inhibition problems only in retarded autistic children. Nevertheless Hill consistently found the inhibition of a prepotent response to be problematic.

No specific inhibition problems are known regarding number knowledge. General problems with inhibition could however cause problems with counting. By inhibiting distraction from our surroundings we can focus on the task at hand. Our working memory can be fully dedicated to this task. With an impaired inhibition a part of the working memory gets occupied by other unimportant information and we simply have less resources to dedicate to the task at hand. When dealing with quantity or numerosity situations a lack of inhibition could have an impact on the autistic precision. A correlation between inhibition skills and mathematical skills that works in this way is found by Bull.[100] Remarkably the children in Bulls trials remember less relevant information and conversely remember more irrelevant information. From this Bull concludes that attention of only a limited amount or resources can be directed by our EF, for better or worse.

DEF and Learning Numbers
When autists execute their counting skills, things can go wrong. Impaired set-shifting and inhibition cause problems with finding the right focus for autists. Impaired set-shifting causes problems with the execution of sequencing. Impaired inhibition could prevent autists from counting objects due to over-inhibition or distraction by other things due to a lack of inhibition. Not shifting easily between cardinal and ordinal values can limit the usefulness of numbers for autists. And impaired set-shifting could cause problems with understanding nominally used number words.

Some EF skills do not seem to cause problems for autistic counting. High functioning autists seem to have sufficient planning skills. But DEF could also cause a few developmental problems. A general developmental effect could be caused by self-monitoring. While self-monitoring does not seem important to counting directly, it could have an important role in our general development towards our number knowledge. A specific developmental effect could be due to impaired set-shifting. This could effect the mathematical skills beyond counting, because in later mathematical tasks set-shifting is a more prominent used skill. Another developmental deficit is due to the impaired linguistic generativity. This generativity plays an important role in number knowledge, but autists could have problems understanding and generating numbers due to the impairment of their linguistic generativity.

3.1.3 Working Memory in Autism

Related to the disabilities in the EF are the autists problems with working memory. Working memory also determines how functions are executed, but working memory fulfils a more basic function in our reasoning when compared to the functions described in the EF theory. The EF however ties in nicely with Baddeley's working memory theory and overlaps with what Baddeley calls the central executive (see 2.2.1). Problems with working memory are linked to DEF but need to be treated separately.

Correlation between Working Memory and Mathematical Skills
Bull et al. found that in TD children at preschool ages (between age 4 and 5) the visuospatial sketchpad is a specific indicator for mathematical skills in the first years of primary school (between 7 and 8 years).[101] Bull tested 121 TD children on a battery of tasks including mathematical and reading performance tests (PIPS), an executive task testing for inhibition called shape school, tower (of London) task and a spatial visual (Corsi blocks) and digit span memory task. Reversed order pointing to remembered object locations and reversed naming of remembered digits, was used to test visual and verbal working memory span. In these tasks reversing the order adds an action to the short term memory task, which necessitates the use of the working memory.

Bull et al. used the PIPS, Performance Indicators in Primary Schools, to measure development of the children. As the PIPS includes age specific tests the raw scores were used to indicate the children's development over three different time points during the research period. The PIPS tests were conducted at the beginning and end of the first year and at the end of the third year of primary school.

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100 R. Bull, Espy K.A. and Wiebe S.A., Short-Term Memory, Working Memory, and Executive Functioning in Preschoolers: Longitudinal Predictors of Mathematical Achievement at Age 7 Years, *Developmental Neuropsychology*, vol.33 no.3 p.205-228, 2008.

101 R. Bull, Espy K.A. and Wiebe S.A., Short-Term Memory, Working Memory, and Executive Functioning in Preschoolers: Longitudinal Predictors of Mathematical Achievement at Age 7 Years, *Developmental Neuropsychology*, vol.33 no.3 p.205-228, 2008.

A good digit span provided children with a head start in mathematical and reading skills. Better digit span related to a better PIPS score at the beginning of primary school. Retaining one digit more resulted in a 2 point higher PIPS score. Remembering 1 more location in the visual memory task resulted in an increase of 2.39 on the PIPS score. When a child could do one level more on the tower task they measured a 0.53 higher PIPS score.

More striking was the correlation between visual-spatial inhibition and mathematical and reading skills. In the shape school test children were familiarized with 15 objects which represented children in a class. These 'children' were named by their colour. The participating children were asked to select those shapes which had supposedly finished their work and could have a break form school. This was depicted by the shapes having a smiley face (opposed to a sad face). The testing for inhibition occurred by placing hats on some of the objects which were not to be named by their colour but by their shape. Only about half of the objects got a hat, so participants had to pay attention to whether and how to name an object. Bull et al. found a whopping 4.91 increase in PIPS score when children scored one unit above mean in the shape school test.

Bull et al. also looked at the relation of these skill with the PIPS score at age 7 and found that short term visual or digit memory span are not significantly related to mathematical skills. The executive measures, working memory, inhibition, shifting and planning, were found significantly correlated with mathematical skills.

Bull et al. conclude that all skills they tested had some correlation with and influence on mathematical skills. Thus these skills should best be regarded as general indicators of a capacity to learn. Only a few of these skills can be regarded as indicators of either a mathematical or reading skill. In the first year of primary school, our visual-spatial short term memory is an indicator of mathematical skill. And in a lesser degree verbal short term memory, inhibition and planning are indicators in this year. By 7 to 8 years of age, at the third year of primary school, visual-spatial working memory is the most important indicator.

With the skills which provide general learning capacity a head start can be maintained the first three years of primary school. Bull et al. are aware that some of the tested skills may only get an important use in mathematics later on in development. Set-shifting which is tested in the shape school is probably more important in later mathematical skills. The visual-spatial sketchpad is regarded as a possible foundation of representing abstract problems in a sort of concrete form. The sketchpad provides a workspace which supports links between informal concrete knowledge and abstract language and symbols used in primary school. At the beginning of the article Bull et al. already state the following about visual-spatial skills:

> Visual-spatial skills may impact math at various levels—number inversions and reversal, misalignment of column digits, problems in visual attention and monitoring such as ignoring signs or changing operation part-way through completion of problem, and acquiring concepts of borrowing and carrying. The visual-spatial system also supports other aspects of non-verbal numerical processing such as number magnitude, estimation, and representing information in a spatial form, as in a mental number line... [102]

Bull et al. associate the visuospatial sketchpad with non-verbal mathematical skills such as a feeling for magnitude, estimation, spatial information representation and the number line. A number of researchers is cited (among which Dehaene, Spelke and Geary) that found that children with poorer mathematical skills also have poorer inhibition skills.

The Bull at al. research is important because autists have poor visual-spatial memory. A deficit in spatial memory is found in autists by Steele et al. and Williams et al..[103] [104] Autistic individuals of all ages are not as good in recalling positions of objects compared to TD individuals. These results together indicate why autists could have problems with numbers in this period of their life.

Visual-spatial and Verbal Working Memory

Williams et al. tested the same autistic individuals on both visual-spatial and verbal working memory and found no deficits in verbal working memory. The tested autists even had a greater verbal memory than TD individuals.

Williams et al. trialled 31 adults and 24 children, all high functioning autists and compared them with 25 TD adults and 44 TD children. Children were between 8 and 16 and adults were between 17 and 48. Williams et

102 p.208 of R. Bull, Espy K.A. and Wiebe S.A., Short-Term Memory, Working Memory, and Executive Functioning in Preschoolers: Longitudinal Predictors of Mathematical Achievement at Age 7 Years, *Developmental Neuropsychology*, vol.33 no.3 p.205-228, 2008.

103 S.D. Steele, Minshew N.J., Luna B. and Sweeney J.A., Spatial Working Memory Deficits in Autism, *Journal of Autism and Developmental Disorders*, vol.37 no.4 p.605-612, 2007.

104 D.L. Williams, Goldstein G., Carpenter P.A. and Minshew N.J., Verbal and Spatial Working Memory in Autism, *Journal of Autism and Developmental Disorders*, vol.35 no.6 p.747-756, 2005.

al. tested both children and adults on the N-back letter trail. Participants were presented with letters and were asked to respond when this letter was the same as a certain letter. In one trail they were asked to look for the baseline letter – the first letter in a sequence. In a second trail they were asked to look for the letter presented before the target letter. In a third trail they were asked to look for the same letter as the letter presented two letters before the target letter. This different trials were called 0-back, 1-back and 2-back. Number of correct ' hits' and reaction times were measured. The number of misses and false alarms was too small for statistical analysis. No significant difference was found between autists and controls.

Williams et al. also used the Wechsler Memory Scale (WMS-III) Letter-Number Sequencing Subtest for the adults and the Wide Range Assessment of Memory and Learning (WRAML) Number/Letter Memory Subtest for the adolescents and children. In these procedures participants hear a sequence of numbers and letters. The children and adolescents were asked to simply repeat these series and adults were asked to first repeat in alphabetical order only the letters and then in ascending order only the numbers. Measures were adjusted for age to compare all participants. Adults are also given a forward and backward WMS-III Spatial Span Subtest, in which they are asked to repeat a sequence of taps on ten cubes fixed on a board. Sequences range from 2 to 9 taps in length. In the forward condition participants repeat the sequence in the backward condition they repeat the sequence in reverse order. The children are tested with the WRAML Finger Windows Subtest in which they copy a sequence of windows presented by the examiner.

Both in the adult and child group no significant difference was found between autists and controls on the verbal tasks. On the spatial tasks however both the autistic adults and children group performed significantly less well. Difference between adult autists and controls was -3,48 digits less and difference between autist and control children was -2,18 digits.

Autists can in some cases use their verbal skill to compensate for a lack of spatial working memory. They could transcode spatial tasks into verbal tasks. According to Williams et al. such transcoding could have taken place in some spatial tasks from older research. Steele et al. state that the spatial working memory deficit is more articulated in a demanding situation. Presumably autists have sufficient visual-spatial working memory to deal with simple situations, but might have lesser number skills due to a somewhat impaired visual-spatial working memory.

Significance of Working Memory

Holmes and Adams find that the visual-spatial working memory is more important in predicting mathematical skills in younger children and that the verbal short term memory is more important predicting skills in older children.[105] These predictions reflect the development of our counting skills. At first problems are visualized to solve them. Training leads to verbalization after which solutions can be retrieved from long term memory, where number knowledge is stored verbally.

Holmes and Adams can prove this by a correlation analysis of a National Curriculum test for England[106] and their own tests of visual and phonological working memory and central executive skills. The National Curriculum test was analysed to look for specific mathematical skills. In this test Holmes and Adams were capable of clustering: A) number algebra and mental arithmetic and B) shapes, space and measure handling and C) easy and D) difficult questions.[107] The visuospatial sketchpad and central executive could be intercorrelated and were significant for both A) and B). Only the central executive could be significantly related to C), the phonological loop could almost be significantly related to C). The visuospatial sketchpad could be almost related significantly to both C) and D).[108]

Especially mathematical skills, which develop later on, can be correlated to the phonological loop. At that time the visual-spatial skills become less important. According to Holmes and Adams this reflects the mastery of symbolic-linguistic arithmetic and mature solution strategies like direct retrieval of solutions from long term memory, a strategy which relies on verbal code. The visuospatial sketchpad predicts a small but significant amount of variance in both age groups. In D) the more difficult questions the visual-spatial memory plays a more important role than in C) the easy questions, which predicts that the visuospatial sketchpad is still important in more demanding questions. With these more difficult questions the children fall back to their previous strategy.

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105 J. Holmes and Adams J.W., Working Memory and Children's Mathematical Skills: Implications for mathematical development and mathematics curricula, *Educational Psychology*, vol.26 Issue 3 June p.339-366, 2006.

106 Tests are produced under responsibility of the Qualifications and Curriculum Authority and are taken by every child at 3 supposed key stages in their development; at 7, 11 and 14 years.

107 A) and B) were found in the second key stage test. C) and D) were found in the third key stage test.

108 p.356 of J. Holmes and Adams J.W., Working Memory and Children's Mathematical Skills: Implications for mathematical development and mathematics curricula, *Educational Psychology*, vol.26 Issue 3 June p.339-366, 2006.

On a developmental note Holmes and Adams refer to research that has shown that children with mathematical difficulties have a weak, even incomplete network of number facts in long term memory. Studies have also shown that these children have problems with their phonological loop. Problems with the loop can lead to poor number fact knowledge.

WM and Learning Numbers
Both visual-spatial and verbal working memory influences our development. Autists have a less developed visual-spatial working memory, but at least a typical developed verbal memory. Visual-spatial working memory is more important in the early development of our number knowledge. Verbal working memory is more important later on. This might lead autists to develop their number skills later than TD children and to develop their verbal memory more than their visual-spatial memory.

These differences in working memory could lead autists to develop an alternative strategy for dealing with number knowledge. Perhaps autists develop a strategy which focusses more on verbal processing than on visual-spatial processing.

3.1.4 Weak Central Coherence

In 1989 Frith proposed Central Coherence (CC), which is supposed weak in autists.[109] The theory presupposes mental modules and presents CC as a principle which coordinates information between these modules. TD individuals spontaneously create an overall picture of a situation. This picture is created to be maximally coherent with respect to all available information. With this picture the CC effects our linguistic and perceptual information. Visual or perceptual coherence is an easier task than linguistic coherence, because visual information does not require as much interpretation and memory as linguistic information does. Frith and Happé call coherence problems involving language and memory higher-level coherence deficits.[110]

CC has a direct influence on our counting skills because CC increases or decreases our visual interpretation or understanding of a situation. The understanding of a situation determines how well we can count objects or elements in this situation. The linguistic coherence could be of importance later in our development and have an influence on our later mathematical skills.

Focus on Details or Lack of Coherence
It is known that autists do not spontaneously integrate information, but instead focus on the individual details of a situation. From this knowledge Frith and Happé concluded that autists have a Weak Central Coherence (WCC) by comparing autistic and TD children on recognizing figures in a larger object (Embedded Figures test) and on recognizing block designs (the Block Design test). In both these tests the children that recognize details get high scores. The result was that autists got high scores, as they seem to focus on details.

Jolliffe and Baron-Cohen however think that autists can reason coherently, but they only look for coherence when instructed to do so or when they themselves consciously decide to use a contextual strategy.[111] Jolliffe and Baron-Cohen conclude that a perceived weakness in central coherence is due to a non-conscious preferred reasoning strategy found in autism and Asperger syndrome. Jolliffe and Baron-Cohen's results contradict that a lack in global conceptual coherence is filled in by local perceptual skills.[112] This enhanced skill is concluded by Frith and Happé and Jarrold and Russell on the basis of the Embedded Figures test and the Block Design test.

In their trials Jolliffe and Baron-Cohen ask children to count dots, some of which are arranged canonically (like on a dice), others in a random pattern. Autists count the canonically grouped numbers slower than typically developed children and children with a moderate learning difficulties. Where Jarrold and Russell conclude that this is due to enhanced local processing, Jolliffe and Baron-Cohen suggest these autistic children really counted the numbers, because that is what they were asked to do.

When Jolliffe and Baron-Cohen compared high-functioning autists and people with Asperger syndrome with a matched control group no increased local perceptual skills are found. In their trials they had subjects describe line drawings of objects (all but one belonging to a certain situation) and line drawings of complete situations. Subjects were also asked to choose or find the object which didn't match the situation. They timed the reactions and gave scores for these descriptions and for the reason given why the odd object did not

109 U. Frith and Happé F., Autism: beyond “theory of mind”, *Cognition*, vol.50 p.115-132, 1994.
110 p.128 of U. Frith and Happé F., Autism: beyond “theory of mind”, *Cognition*, vol.50 p.115-132, 1994.
111 T. Jolliffe and Baron-Cohen S., A Test of Central Coherence Theory: Can Adults with High-Functioning Autism or Asperger Syndrome Integrate Fragments of an Object?, *Cognitive Neuropsychiatry*, vol.6 no.3 p.193-216, 2001.
112 p.93 of T. Jolliffe and Baron-Cohen S., A Test of Central Coherence Theory: Can Adults with High-Functioning Autism or Asperger Syndrome Integrate Fragments of an Object?, *Cognitive Neuropsychiatry*, vol.6 no.3 p.193-216, 2001.

match the situation.

They did find problems with contextualizing after they had ruled out problems with motivation, with impulsiveness, problems with memory and problems with locating objects or other perceptual problems. Motivational problems were ruled out because no impairment was found on the control tasks and participants acted motivationally similar to the control group. Most notably subjects fervently tried to find other suspicious objects when they could not find the object not belonging to the situation. Impulsive problems were ruled out because subjects responded faster when they found the odd object and slower when they did not, just like the control group. Memory problems were ruled out because the material remained in view and control tasks required a similar use of memory. No locating problems were found as subjects were able to find named objects. Perceptual problems were ruled out because of the known strength on perceptual tasks.[113]

Jolliffe and Baron-Cohen designed their trial in such a way that the participants needed to make inferences about context to solve it. The autists had problems with solving the trials. They also found giving descriptions about the depicted situations demanding. Jolliffe and Baron-Cohen think a strategy for dealing with meaning could clarify why normal individuals search for objects in places where they expect objects to be. The autistic individuals did not always follow this strategy of dealing with meaning.

Decentral Coherence

López et al. compared perceptual and linguistic coherence in autistic and TD children, but they did not find the expected relation between the perceptual and the linguistic coherence skills.[114] López et al. found an inverse correlation with a semantic memory test, the block design test and a face perception test. They used the face recognition test alongside the block design test to specifically test for the use of holistic information rather than whether holistic information is ignored. Their results are striking because the weakness in perceptual coherence was supposed to be related to the weakness in conceptual coherence – Frith and Happé consider conceptual coherence a higher level.

In the facial recognition test the 15 autistic and 16 TD participants were shown a composition of a complete face for 500ms and after a 500ms delay they were asked to choose the same face from two faces one of which was slightly altered. In the second part of the trial only one feature of the original face was shown after the delay (e.g. 2 mouths, 2 noses or 2 pairs of eyes).

In the visual semantic memory test the same participants were shown four sets of 12 pictures. The pictures in these sets were either related (all animals or all vehicles) or unrelated (different categories matched for difficulty). The participants were asked to name all objects they remembered right after they were shown.

The results of these tests showed no significant difference between autistic and TD participants. In the facial test the relative difference in recognition of complete faces and single facial features was a little better in TD participants than in autistic participants. In the visual semantic test the difference between remembering related and unrelated pictures was measured. Here the relative difference between autists and TD participants was found to be even smaller.

López et al. conclude that the weakness in central coherence might not be so central after all. They did not even find a correlation between perceptual and conceptual coherence in the TD children. Most of the autistic participants had difficulty with either the visual or the semantic test, but not with both or none. López et al. do not deny anomalies with the perceptual and conceptual tasks in autism, but question whether these anomalies have a common source.

WCC and Learning Numbers

If a WCC or a decentral weak coherence has an influence on autistic counting skills, then it is via perception. Weak coherent perception makes it harder to see canonical groups and to see other groupings of elements. Weak verbal coherence might make it harder for autists to form a complete network of number facts. Coherence influences the general development of number skills, both counting and mathematical skills.

3.2 Autistic Number Learning

The four differences between typical and autistic counting can be explained by each of the four theories explaining autism.

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113 p.95 of T. Jolliffe and Baron-Cohen S., A Test of Central Coherence Theory: Can Adults with High-Functioning Autism or Asperger Syndrome Integrate Fragments of an Object?, *Cognitive Neuropsychiatry*, vol.6 no.3 p.193-216, 2001.

114 B. López, Leekam S.R. and Arts G.R.J., How central is central coherence? Preliminary evidence on the link between conceptual and perceptual processing in children with autism, *Autism*, vol.12 no.2 p.159-171, 2008.

Impairments in Autistic Counting: / Theories explaining autism:	Not much benefit from canonical placing of dots	Quantity 4 is named slower	Later development of sequencing skills	Recalling positions
Disabled Theory of Mind	a general retardation due to disabilities in ToM causes autists not to learn the canonically placed dots as quantities or problems with verbal interaction	a general retardation due to disabilities in ToM causes autists not to learn the canonically placed dots as quantities or problems with verbal interaction	a general retardation due to disabilities in ToM causes autists not to develop sequencing skills as fast as TD children or problems with verbal interaction	a general retardation due to disabilities in ToM causes autists not to have problems with verbal interaction, which causes problems with memory recall
Disabled Executive Functioning	impaired set shifting and inhibition cause autists not to see or find canonical patterns of dots	autists default to a counting strategy due to their rigid and stereotyped nature and stick to it due to impaired set shifting and counting is a little slower than subitizing	impaired planning causes autists to have problems when conducting complicated tasks such as sequencing	impaired inhibition and set shifting causes autists to make the wrong choices, which make it seem like they have problems with recalling positions
Disabled Working Memory	the impaired visuospatial sketchpad causes autists not to see or find canonical patterns of dots	the impaired visuospatial sketchpad causes autists not to subitize quantities greater than 4 quickly and autists resort to counting	the impaired visuospatial sketchpad causes problems with overseeing a complex situation and tracking all elements, which causes problems with sequencing	the impaired visuospatial sketchpad causes problems with tracking all locations and their information simultaneously
Weak Central Coherence	due to impaired coherence autists cannot see the global grouping of canonically placed dots	due to impaired coherence autists cannot subitize above 3 and resort to counting	the impaired coherence causes problems with the understanding of a complex situation, which causes problems with sequencing	the impaired coherence causes problems with the understanding of a complex situation, which causes problems with recalling information

Diagram 11: Autistic counting explained by four different causes

(large version in Appendix B)

These explanations are however not the most interesting about the theories that explain autism. How these different theories effect the development of number knowledge is more interesting, because the influence of these theories on the developmental model of how we learn numbers can possibly explain much more. How these theories influence the model could not only explain why some autists do not have certain skills, but also why not or how they might expand their skills.

Every separate function of the EF theory and working memory model influence both how we operate and develop our number skills. This may be less evident but is also true for ToM and CC. Autistic counting and it's developmental model will in general resemble TD counting and the typical developmental model. We will take a quick look at how the theories explaining autism influence the TD model.

Disabled ToM and Number Learning
We have innate number skills, but our number knowledge is part of a social system. Therefore we need to develop ToM and language to become fully developed number knowledgeable. We looked at this in 3.1.1 and found that autists have a disrupted ToM, but count relatively well. This seems contradictory and makes us wonder whether ToM is needed to learn how to count. We conclude that (high functioning) autists are likely to have some ToM, but also that they could learn numbers with strategies that do not require ToM. That there may be alternative learning strategies is a new perspective on the developmental model.

We found two other differences that may influence the autistic counting. The disabled ToM makes number knowledge, words and values, very factual and unconnected. It also causes autists to have a general slower development, which also hampers number knowledge.

WCC and Number Learning
A lack of coherence will be most noticeable in the autistic counting skill via perception. It is harder to see (canonical) groupings of elements. A lack of verbal coherence might make it harder for autists to form a complete network of number facts.

DEF, Disabled Working Memory and Number Learning
EF divides into separate functions which can be dysfunctioning. Impaired self-monitoring could cause problems in our general development towards our number knowledge similar to disabled ToM. According to Bloom autists may lack generative skills that are needed to understand the linguistic infinity of numbers. Impaired set-shifting skills limits the use of the different dimensions of number words, which slows down our development of number words. We can group impaired set-shifting and inhibition together as they cause problems with focussing on the task at hand. Working memory also influences our focus on the task at hand. The effect of impaired working memory (or DEF) resembles the effect of a lack of perceptual coherence.

The different development of working memory might lead autists to develop their verbal memory more than their visual-spatial memory. This could lead autists to develop an alternative strategy for dealing with number knowledge.

Alternative Learning Strategy?
Counting could be regarded as the first formal system we use. How and how well we deal with this formalisation is dependent on how we have developed our number skills. A different perception of numbers

could lead to an alternative view of numbers and possibly an alternative strategy or logic.

We will construct a possible alternative counting strategy on the premises that autists have a poor visuospatial sketchpad. This visual-spatial working memory does not help them as much to develop number knowledge as it does in TD children and the size of the autistic working memory undermines the understanding of the counting procedure. The impaired visual-spatial working memory of autists will limit their ability to mentally reconstruct counting problems and procedures. This limitation slows down the initial number learning and could even stop the development of number skills.

The phonological loop however is in perfect condition and could enable autists to quickly step up to a number fact retrieval strategy. With our phonological loop we transfer the learned information into long term memory, where the information is stored in a verbal form. In general we also store the number words and the most frequently used number facts in long term memory.

By memorizing the counting sequence and the right responses to certain situations autists could mimic a counting skills which works by constructing information. Maybe all that is needed is a little perseverance to remember a lot of possible situations and an autist is able to learn how to 'count'. If autists would apply this strategy they would initially not have a full understanding of their nonetheless correct answers. The understanding of number knowledge would be developed via a conditioning process. Conditioning could take place without an elaborate understanding of quantity. Whether autists can learn how to count without any visual-spatial memory seems doubtful. At least some visual-spatial skill is needed to learn some small numbers before a switch to a retrieval strategy seems possible.

A result of this number learning strategy could be a very literal and nominal use of numbers. Numbers are considered less part of a system, but are regarded as individual entities or objects.

Chapter 4

Discussion

4.1 Discussing Autistic Counting

In chapter 1 we have looked at four problems autists might have with number knowledge. In chapter 2 we have looked at how number knowledge is typically developed and at different theories explaining autism and at how these relate. In chapter 3 we have looked at what an autistic counting theory might look like.

With all this information we will now look in depth at how we can best explain the differences between autistic and typical counting and we will form a theory about autistic number knowledge. From there we will try to explain how autists develop their counting skills and what this development could mean for the mathematical skills that build on the counting skill.

4.1.1 Discussing Counting Speeds and Sequencing

Jarrold and Russell's trials showed us that autistic children take more time when they count and do not benefit in counting speed from canonically placed dots. In this result we can recognize the preference for local processing, which is explained directly by WCC or DEF.[115]

According to Jarrold and Russell enhanced local processing makes that autists count randomly placed dots relatively faster than canonically placed dots. Jolliffe and Baron-Cohen found out that a perceived enhanced local processing is rather a lack of global processing. According to Snyder et. al. this lack of global processing is a distraction autists do not have to deal with when counting randomly placed dots.[116] Jarrold and Russell acknowledge that there might be other reasons for the autistic children in their research to not count canonically placed dots faster. Jolliffe and Baron-Cohen think that these autistic children were not faster because they were asked to count and did so quite literally. If autistic children have problems with understanding such question than we also need to consider that disabilities in ToM could lead to this effect.

Jarrold and Russell also remark that counting canonically placed dots might require higher level reasoning. They probably refer to global reasoning, similar to Snyder et. al. who theorize that it is easier for autists to estimate or count dots because they are not bothered by higher level 'interference'. Literal interpretation of an array, just dots, is easier for autists compared to typical individuals. TD individuals have interference from their global level interpretations, which make TD individuals see constellations or other meaningful arrays, not just dots.

Snyder et al. even think that this disconnection of meaning can account for the astonishing counting skills of savants. They found proof for this by inducing this 'savant skill' in normal people by a magnetic bombardment of a specific brain region. The estimation skills of 9 out of 12 people where much better shortly after a specific magnetic procedure and returned to normal about one hour later. Whether the literal interpretation is caused by a preference for local processing or vice-versa that a lack of global processing causes the literal interpretation, cannot be concluded from this research. With either the literal or local processing explanation our estimating skills are helped.

Unexpectedly a lack of certain skills can influence our counting and number knowledge. The theoretical lack of CC or EF causes autists to count with a very local style.

> [Frith] predicted that autistic subjects would be relatively good at tasks where attention to local information - relatively piece-meal processing - is advantageous, but poor at tasks requiring the recognition of global meaning.[117]

Similar to Jarrold and Russell's results Gagnon et al. found that high functioning autists are slower when counting the small quantity 4. Their results lead Gagnon et al. to believe that autists do not subitize but instead count small numbers.[118] Reaction times when enumerating small quantities were slower for 4, but slightly faster for 5 compared to TD participants.

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115 Due to a weak coherence autists focus their attention to local processing. In DEF the combination of problems with set-shifting and inhibition supposedly locks an autist into local processing.

116 p. 842 of A. Snyder, Bahramali H., Hawker T. and Mitchell D.J., Savant-like numerosity skills revealed in normal people by magnetic pulses, *Perception*, no.35 p.837-845, 2006.

117 p. 121 of U. Frith and Happé F., Autism: beyond “theory of mind”, *Cognition*, vol.50 p.115-132, 1994.

118 L. Gagnon, Mottron L., Bherer L. and Joanette Y., Quantification Judgement in High Functioning Autism: Superior or Different?, *Journal of Autism and Developmental Disorders*, vol.34 no.6 p.679-689, 2004.

Geary researches disabilities in mathematical learning.[119] [120] He also reports counting rather than subitizing and thinks that mathematically disabled children lack exactly the subitizing skills but have no further problems with EF or CC.[121] These children develop normal social skills and language skills. According to Geary poor subitizing leads to very poor secondary mathematical skills. There is a possibility that mathematical learning disabilities frequently occur simultaneously with autism. However some autists do eventually learn to count. Even if this co-occurrence would be the case, the syndromes do not always coincide.

The mathematical learning disabled children in Geary's trials subitize the quantities 1 and 2, but for 3 they resort to counting. In the trials by McGonigle-Chalmers et al. the autistic children are not slower compared to TD children on naming 3. Even if autists cannot subitize they count fast enough to be on par with TD children up to 3. Jarrold and Russell give no reason for us to believe that autists cannot subitize. That autists do not subitize quantity 4 does not mean they can not subitize. Autists might not (like to) subitize due to a lack of visual-spatial working memory. Such a lack was found by Steele et al. and by Williams et al. and this is likely to have an impact on autistic subitizing.[122]

Subitizing might be seen as a form of global reasoning, a form of reasoning which autists dislike. The longer reaction times in autistic counting could then be due to a general lack or dislike of global reasoning. The most likely mental module where global reasoning takes place is however in the working memory. Since subitizing is based in perception, the visual-spatial working memory would be used. We have established that autists have a problem with this part of the working memory. So the lack of visual-spatial working memory might also cause problems with perceptual global reasoning.

As reported by Steele et al. and Williams et al. autists are not as good as TD children, adolescents and adults in recalling positions. McGonigle-Chalmers et al. report problems with sequencing tasks. Visual-spatial memory plays a role in both these tasks. Autists have some visual-spatial memory and can solve tasks up to a certain degree of difficulty. When tasks are more demanding, the autistic impairment is more prominent.

The impaired visual-spatial working memory might not be the only reason for problems with sequencing. When answering a sequencing task a TD individual makes use of the large approximate number system. To what extend autists develop a large approximate number system is unclear. The problems with the sequencing task shows that not all autists use a global view when sequencing. We cannot check the importance of the large approximate number system, because there is no theory about how the large system works and whether it makes use of visual-spatial working memory when mapping a situation. So a lack of visual-spatial memory might not be a problem, but the approximate system using working memory is a quite logical and plausible culprit.

Of course the autist might not make use of a global view because of problems shifting to this view. Because autists cannot shift their view to oversee the situation, a problem with set shifting could lead to problems with sequencing. If set shifting is the only problem, by chance half of the autists should start with the global view and have no problems with the task. This is however not the case. This is probably due to the autistic preference for local processing. The combination of a lack of set shifting and a preference for local processing could lead to a under development of the large approximate number system.

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119 On p.498-499 Geary talks about subitizing.
D.C. Geary, An Evolutionary Perspective on Learning Disability in Mathematics, *Developmental Neuropsychology*, vol.32 no.1 p.471-519, 2007.

120 Geary investigated the evolutionary history of our mathematical skills and makes a distinction between primary and secondary abilities. According to Geary evolution has pushed our intelligence towards anticipation of problems. In order to do that we need to be able to inhibit our automated responses. In this evolutionary process Geary predicts the development of other primary skills. He sees ADHD as an adaptation to a fierce environment which requires fast responses. As people with ADHD have problems inhibiting their responses in our modern day society, their skills set could be considered quite competitive in a less technological environment. Autism could be a remnant from an evolutionary adaptation.
Geary's primary mathematical abilities coincide with our innate core modules. According to Geary numerosity, ordinality, counting, simple arithmetic, estimation and some geometry are our primary mathematical abilities. These abilities together with our general intelligence allow us to develop our secondary mathematical skills. The secondary abilities are build on the primary abilities. The secondary skills enable us to process mathematical information. We can adapt these later skills suit our mathematical needs.

121 Geary reports counting rather than subitizing by mathematically learning disabled children in a study by Koontz and Berch from 1996 and in his own Number Sets Test from 2007.

122 The study by Steele et. al. is discussed in 1.1.2 and the study by D.L. Williams et al. in 2.4.4.

Sequencing tasks could simply be difficult for the same reason that counting speeds are low: a lack of visual working memory disables autists in comparison with TD individuals. According to Williams et al. autists with visual working memory problems have however no deficit in verbal working memory.[123] Therefore autists might use their verbal working memory to globally assess situations. This verbal skill could be used operationally to compensate for a lack of spatial working memory. By transcoding visual information in to auditory information. A transcoding step would not help in every sequencing task, since it cannot help to transcode e.g. size information.

How much influence the impaired visual-spatial working memory has on development remains to be seen. Williams et al. do not think that the impaired visual-spatial working memory is responsible for problems with problem solving and planning in autists. They think these problems are due to other executive deficits.

Memory Strategy towards Number Knowledge

The impaired visual-spatial working memory is probably the first and biggest problem autists encounter when they develop their number knowledge. However most often this impairment does not keep autists from developing number knowledge. We predict that autists develop a number and counting (and possibly later mathematical) mechanism largely based on verbally acquired facts. This means autistic number knowledge is based in long term memory. In TD children of about 9 or 10 years a transition takes place from the construction of number value to the number-fact retrieval from long term memory.[124] Autists partially skip the initial constructing of numbers and simply learn to retrieve the number facts.

According to Dehaene and Cohen we learn or remember different facts of number knowledge with our memory.[125] The two most important forms of rote number memory are 1) parity knowledge or number characteristics; e.g. knowing (for a fact) that 6 = 2x3 and that 2 is even and 2) nominal number knowledge; e.g. that 28 is the number of normal human teeth, not including the third molars (wisdom teeth) or that 28 was the year the Frisians negotiated a treaty with the Romans at the River Rhine to avoid conquest.

Autists can probably compensate for their lack of visual-spatial working memory by retrieving number facts. TD individuals use the visual-spatial working memory to construct quantity facts and to solve mathematical tasks. Individuals lacking the visual-spatial working memory to construct quantity facts would lag behind with the basic number skills. Autists can compensate their lack with learned number facts. Compensating in this way will lead them to make certain mistakes and have certain reaction times. Autists would count relatively slow, but answer to calculations they know relatively fast.

Not everything from the TD social world is understandable for autists, but number facts are constant (just as names of peoples and places). Therefore these facts may be picked up faster than more complicated (and constantly changing) social facts. Autists are likely to learn number facts. Remembering facts may however only developed after learning sufficient language or after the sufficient development of another symbolic system. Typically language helps us to remember events and facts.

We think autists understand numbers (words and values) nominally rather than as having mathematical characteristics.[126] The mathematical characteristics that a TD individual might learn are not as likely to have meaning for an autist, since a lack of visual-spatial working memory causes a lack of quantity understanding. So even a number fact like 6 = 2x3 will be understood more like a individual and almost personal characteristic of the number six. This could mean that autists can only calculate as good as their rote memory allows them to remember number facts. It would mean that autists learn little from repeated questioning as they do not learn by quantity calculations. With a memory strategy the autists does not learn new quantity strategies, but only new facts. Having a limited visual-spatial working memory limits the later learning strategy in autists when compared to TD children.

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123 Williams et al. do suggest on p.754 that other tests which make use of verbal elements and did find problems with verbal memory are actually showing impairment as predicted by WCC.
D.L. Williams, Goldstein G., Carpenter P.A. and Minshew N.J., Verbal and Spatial Working Memory in Autism, *Journal of Autism and Developmental Disorders*, vol.35 no.6 p.747-756, 2005.

124 J. Holmes and Adams J.W., Working Memory and Children's Mathematical Skills: Implications for mathematical development and mathematics curricula, *Educational Psychology*, vol.26 Issue 3 June p.339-366, 2006.

125 According to Dehaene and Cohen we have a separate rote memory for number knowledge. They concluded this by examining patient with lesions. One of these patients was able to retrieve number facts (multiplication tables) from memory, but could not make simple calculations.
S. Dehaene and Cohen L., Cerebral pathways for calculation: Double dissociation between rote verbal and quantitative knowledge of arithmetic, *Cortex*, vol.33 p.219–250 1995.

126 Autists have been reported to have obsessive likes or dislikes of certain numbers. Where they even use mathematical operations as giving meaning to these numbers. Giving meaning to numbers in this way looks like a very particular form of numerology, a confusion of the meanings of number.

There is evidence for a basis of rote memory in autistic savant calendar calculation skills instead of a algorithmic strategy.[127] Mottron et al. note that this non-hierarchical recall of facts is documented in savants and autists in other domains; graphic recall, memory for proper names and list recall. Mottron et al. even suppose that in autistic savants perceptual system are rededicated to the processing of symbolic information.[128]

This alternative memory strategy combined with a good memory should allow for a surprising display of number fact retrieval. With an autistic brain this strategy could be advantageous. Not getting distracted by global or social reasoning might help in pursuing the memory strategy to knowledge. The preference for local reasoning is helpful in this strategy, because global reasoning occupies resources which can now be dedicated to memory. Automatic grammatical like rule learning might even allow autists to develop a mathematical structure to go with their learned number facts. Investigating autistic rule learning and forming could bring new insight to the field.

About ten percent of autists has been found to have savant skills, this ten percent could be the autists who can successfully combine a memory strategy with a good memory and if lucky develop some rule forming skills.

Developing number knowledge seems possible even for autists if they have at least some basic skills. The basic skills they need are: a visual-spatial working memory in order to grasp the very basic reasoning behind counting and numbers, symbolic/language skill to use place facts in (long term) memory and a good memory.

4.1.2 Discussing Combined Number Knowledge

In chapter two we have constructed the start of a very complete theory on the development of number knowledge. This theory describes the development in TD individuals. Lets now look at how autists would develop their number knowledge in comparison. A slightly different working of operational apparatus causes a change in how autists deal with number knowledge. The development of number concepts can however not change that much. A basic understanding of numbers remains necessary to get any number understanding.

The combined number knowledge theory we formed at the end of chapter 2 does not yet show us how the different parts of our operational apparatus influence our development. In a TD individual the language skills will make use of the phonological loop and the number skills and conservation of identity will make use of the visuospatial sketchpad. When a TD individual can name quantities, small precise or large approximate, the individual will use both forms of working memory simultaneously. Both forms of working memory will most certainly be used when the TD individual starts counting. When discriminating two quantities with a difference of 1 we do not need to use of the phonological loop, since we do not need to communicate this fact, we only need to know it.

127 L. Mottron, Lemmens K., Gagnon L. and Seron X., Non-Algorithmic Access to Calendar Information in a Calendar Calculator with Autism, *Journal of Autism and Developmental Disorders*, vol.36 no.2 p.239-247, 2006.

128 A likely reason a lot of savants can make calendar calculation is because these facts do not change like many other daily facts do.

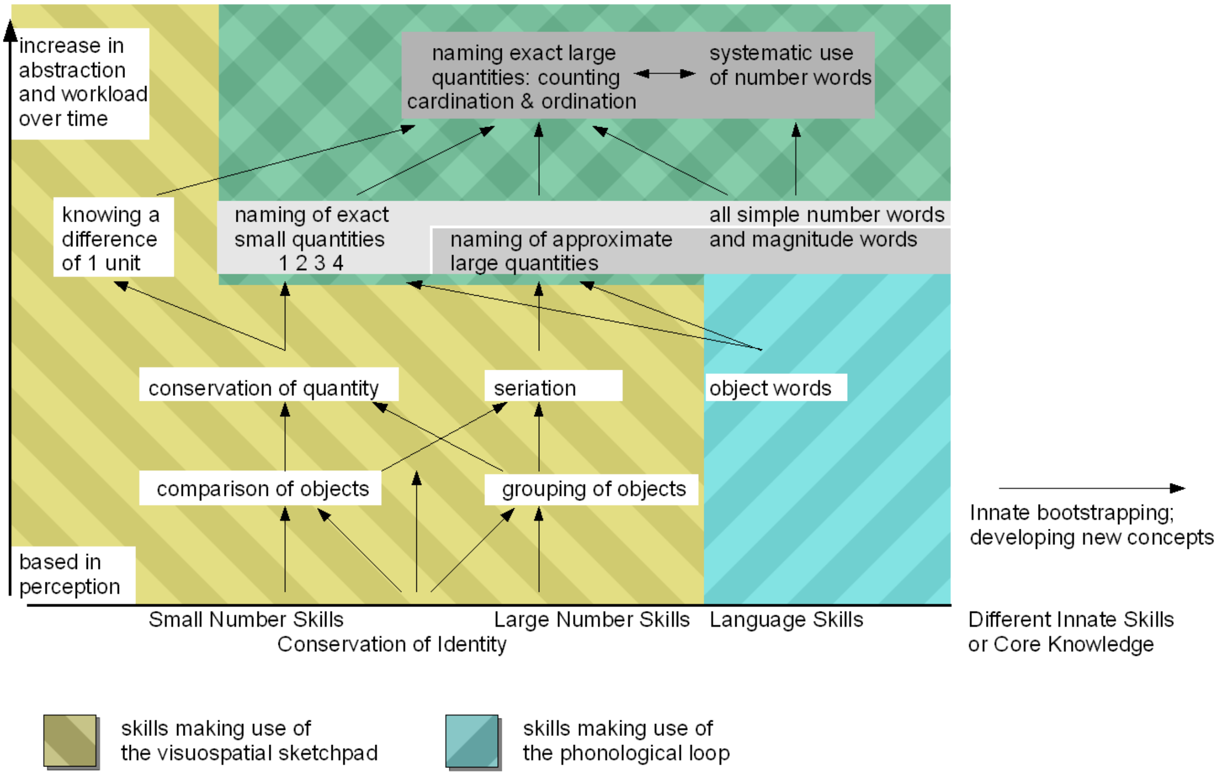

Diagram 12: Working Memory and Number Knowledge Theory in Typical Development

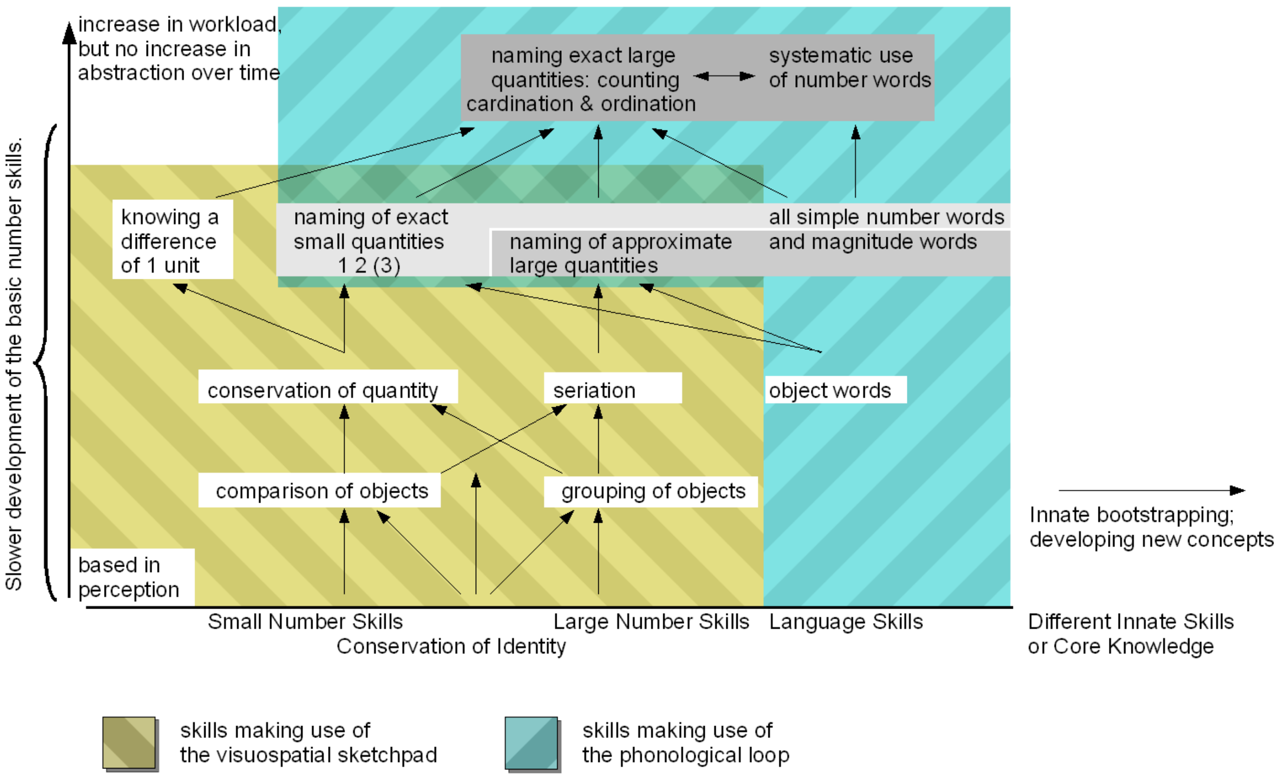

Diagram 13: Working Memory and Number Knowledge Theory in Autism

When we consider the autistic working memory in the development of number knowledge we will see only a few changes. No number skills develop if the lack of visual-spatial working memory is too great or if there is a general lack of intelligence. Most of the development of number knowledge as it takes place in TD individuals also takes place in a high functioning autists. The autistic development differs in a few respects: less advanced subitizing skills and number concept naming skills. This is due to s smaller than typical visual-spatial working memory.

As our number knowledge partially depends on the proficiency of our working memory, the speed of our development also depends on this proficiency. Since autists have a less effective visual-spatial working memory, the number skill development will be behind compared to TD children. Whether this lag is due to the smaller effective visual-spatial memory or due to other causes cannot be said in a straightforward way.

Bootstrapping
The three number skills autists need to bootstrap to the counting skill are: 1) discriminating two quantities with a difference of one, 2) subitizing 1, 2 and 3 and 3) have some understanding of large approximate quantities.

Discriminating a difference of exactly 1 between two quantities is probably only possible for autists when the involved quantities are relatively small. A limited visual-spatial working memory will only be capable of tracking a small number of objects simultaneously. Subitizing of small quantities can possibly be reduced to 3. Subitizing 3 is probably still sufficient to be able to bootstrap to an understanding of the integers. To what extend a large approximate number system needs to be developed in order to execute the autistic counting strategy based upon memory cannot be predicted. Some large approximation is bound to take place and the most crude form of making a distinction between quantities is probably sufficient to initiate the understanding of the integers in the autistic brain.

Missing one of these three skills will lead to an impoverished autistic counting system, where the autists is not capable of understanding how number quantities work. A successful bootstrap to the integers becomes difficult. Without 1) discrimination of a difference of one or 3) a large approximate number system it will be hard for autists to develop a linear model of the number line. These autists fall prey to the problems that Rips et al. predict, they might not conclude a linear system, but a circular. Besides that they will have problems with understanding relations between two quantities. Without 2) the subitizing of 1, 2 and 3 it seems impossible to bootstrap to understand the integers.

Missing one of these basic number skills and consequently not making a bootstrap might leave autists capable of producing bare number facts and mimicking number knowledge. Alternative strategies might allow autists to be make the right choices in number situations and actually be successful when using numbers. This understanding of numbers is however different form a typical understanding.

Integral Number Knowledge
The autistic number strategy based on memory will not work without a language to place and retrieve number facts from long term memory. This autistic number strategy can only work within the integral number knowledge theory, by developing number and language knowledge simultaneously. The understanding of number mechanisms is however minimal in the autistic number strategy. Autistic knowledge is mostly factual knowledge and not constructed knowledge. Though the effective behaviour of autists with such a factual knowledge strategy might be sufficient to work for day to day number situations, it is not likely to be a very flexible and profound understanding.

It may nonetheless be possible for autists to get a more typical understanding of number knowledge by rule learning. Though this may only be possible for talented autists as in general autists are not very skillful rule learners. Autists can grasp mathematical rules just like they can grasp e.g. grammatical rules. Such a rule based understanding of mathematics differs from a typical understanding of rules because it is not backed by fully developed innate number skills.

4.2 Conclusion

Do autists count? And if yes, How do autists count? And how does this counting skill impact later mathematical skills? To answer these question we need to know a few things. We need a theory about number knowledge and about development of number knowledge. And we need to explain autism, which is unfortunately not very straightforward, but we have gained enough insight to answer our initial question.

Number Knowledge & Developmental Theory
Hard to answer, but very important is the ontological question: what really is counting and mathematics? Depending on how we answer this question, we talk, think and theorize about how we need to investigate numbers. Two views have been given in answer to this question. One view says mathematical or number knowledge is how we deal with quantities and this knowledge is primarily concerned with operations. This is how Piaget is generally dealing with numbers and mathematics. Bloom however seems more interested in factual knowledge. In this other view number knowledge is primarily dealing with number facts and communication thereof. We shall not give a definite answer to the ontological question, but we can say that we need to combine these dimensions, as both are important in our day to day activities.

We have introduced isolated and integral number knowledge. With these views of number knowledge we can make a distinction between the two ontological views. Since these number knowledge views both concentrate on only one dimension of number knowledge we might combine them. Isolated and integral number knowledge also describe different developments of number knowledge and of how we develop concepts and how we develop skills. Combining these developments can best describe our actual development.

Single steps within our development of number knowledge, concepts and skills, can be explained by research and theories from developmental psychology, philosophy, neurology and research of disabilities. By combining theories we constructed a combined theory of our development of number knowledge. This combined theory gives an overview of what concepts and skills are used when we are counting and an explanation of how we developed them.

The developmental explanation of number knowledge implies our existence as a physical entity. We have a physical development before we can start our development of number knowledge: understanding and using our senses, manipulating objects and making sound and symbols. This physical development has an effect on our cognitive development of number knowledge. We have described our operational apparatus based on Baddeley's model of working memory. We have looked at different number concepts and skills we develop: comparing, approximating, grouping, making series, conserving quantity, understanding a difference of 1, subitizing, naming approximations, learning number words and finally counting.

The later mathematical skills build on our counting skill. The different mathematical operations we learn to perform when we can count can be (and most often are) defined by a simple or repeated and, or inverse counting operation. This shows that counting is a key skill in the developmental of mathematical knowledge.

Autism

We have discussed four symptoms of autism and three different theories explaining autism. The symptoms are; a disruption of social interaction, abnormalities in verbal and non-verbal communication, limited, repetitive and stereotyped behaviour and problems with global processing. These do not overlap one on one with the different theories; disabilities with Theory of Mind, Disabled Executive Functioning and Weak Central Coherence.

When autists have problems with ToM it is likely this will lead to problems with number knowledge. The Executive Functioning, Central Coherence and other parts of the operational apparatus are also important for the development of number knowledge. Problems in these systems can also lead to problems with number knowledge.

Which of these three theories explains the most fundamental problem for developmental we cannot say. It is hard to prefer one of these theories over the others because these theories can explain each other[129]. And the trials used to research these theories often involve the different skills related to all three theories simultaneously. Because we cannot choose between these theories will look for the most direct explanation of the deficits we found in the autistic counting.

The most direct explanation for the described autistic counting disabilities, a lack of counting speed and sequencing skills, is an impaired visual-spatial working memory. The expected effect an impaired visual-spatial working memory has on the development of number knowledge is fitting. Visual-spatial working memory is fundamental to our number development and this impairment logically influences our later mathematical skills.

ToM and Counting

ToM is needed to learn how to count. How do autists learn counting with an impaired ToM? We answered this question in chapter 3. Autists can learn how to count with an impaired ToM, but an impaired ToM effects the speed of their learning and the way in which they understand numbers.

According to integral number knowledge theory we need a ToM to learn to count. In order to learn words and their meaning we need ToM to understand what others attend to when they use words. Autists do not have access to this strategy to learn words, because of their impaired ToM. Autists often learn the meaning of words correctly when their elders or peers are attending to their point of view (and not the other way round as is most often the case with TD children). This impairment of ToM does however not stop the autist from learning words.

We only learn (number) words correctly after we have grasped the meaning of (quantity) concepts they attend to. We can then connect the concept to the word we pick up by observing how others attend to this concept. With an impaired ToM connecting concepts and words will take more time. Autists will likely be late to connect number words with quantity concepts, even later than connecting other words and concepts, because the construction of the quantity concepts will take them longer.

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129 More about these theories explaining each other in appendix A.

When TD children connect quantity concepts to words they simultaneously incorporates this knowledge into their social knowledge. To become fully developed number knowledgeable we need to develop ToM and language to connect quantities to our social system. Even though autists grasp quantity concepts to some extent, they are not successful in connecting these concepts to a social system, which leaves their number knowledge somewhat impaired socially. Their understanding of numbers is probably very factual and local.

Autists benefit from situations where others are looking after their interests.[130] Such situations enable them to learn more. Eventually learning things about a preferred subject helps the autist to develop and become interested in other subjects. The fully developed ToM of other non-autistic individuals can in this way help autists.

Counting Speeds and Sequencing

A lack of visual-spatial working memory is the most direct explanation for a lower counting speed and impaired sequencing skills. This lack seems to influence the development of subitizing skills and it might influence the use and development of the large approximate number skill. Because these basic skills of autists are cruder, their development of number proficiency lags behind when compared to that of TD children.

Whether the impaired counting and sequencing skills are solely caused by a lack of visual-spatial working memory cannot be said for certain. A lack of set shifting can also play a role in both counting and sequencing. The impairment in set shifting might keep autists from looking at the global situation. This prevents the autists from evaluating all elements that need to be counted or that need to be sequenced. The other way round, a lack of visual-spatial working memory might also cause problems with global reasoning. Since global reasoning, especially perceptual global reasoning, is likely to take place in this part of the working memory.

A lack of global reasoning or a preference for local reasoning can be explained by the limited visual-spatial working memory. Global reasoning, at least perceptual global reasoning likely takes place in the visual-spatial working memory. With the working memory we can track different elements and search for patterns or relations between all elements. A lack of visual-spatial working memory would make it hard to track a lot of elements and find these relations.

Alternative Number Knowledge Strategy

Some autists compensate their problems with visual-spatial working memory by developing an alternative strategy to deal with number knowledge. This strategy is based on memory. By memorizing mathematical facts autists can mimic number knowledge. This does not however allow them to be very flexible with number information, since number facts cannot be treated as fluidly as number concepts.

Higher functioning autists will probably have a little more visual-spatial working memory which will allow for better mathematical understanding. Lower functioning autists will probably have a greater lack of visual-spatial working memory and consequently they will develop hardly any mathematical skill. Autists with better memory skill will be better equipped to mimic the TD number knowledge with a memory strategy. Autistic savants can do great on any subject that they like, because they have a very efficient memory.

130 R.L. Koegel. Dyer K. and Bell L.K., The influence of child-preferred activities on autistic children's social behavior, *Journal of Applied Behavior Analysis*, vol.20 no.3 p.243-252, 1987.

Appendix A

A.1 Autistic Symptoms Explained

As we have seen in chapter 3 not all of the theories about autism, explain the four symptoms associated with autism. Only the executive dysfunction can say something about all four symptoms directly and not via a general disruption. We will now look at each characteristic symptom of autism and at which theory best explains this symptom.

Disrupted social interaction A) is most obviously and best explained with disabilities in ToM. The metarepresentation that is so essential in its precursor, the ToM mechanisms, can however be associated with our executive functioning. An important part of our executive functioning is our ability to monitor ourselves. This monitoring can only take place if we can dissociated our selves from our surroundings. Seeing ourselves as separate entities seems a small step from understanding others as entities like ourselves. If these two skills are both innate (not developmentally connected), their structures are likely related. In that case the executive problems with recognizing ourselves would be related to recognizing others. A WCC cannot explain disrupted social interaction as such, apart form this problems with coherence leading to problems with ToM.

Abnormalities in verbal and non-verbal communication B) are explained by all three theories to some extent. Disabilities in ToM disrupts communication and this results in a disruption of development, which causes abnormal behaviour. Not being able to understand others causes the autists abnormal communication. Executive dysfunctioning explains the abnormal communication by a generally disrupted behaviour of autists. In the same manner the WCC theory can explain the abnormal communication is due to a incomplete understanding of the autist of his surroundings and of what is demanded from him in communication.

Limited and repetitive behaviour C) can be explained by a disabled ToM when the behaviour is due to a general lack of development. WCC explains this behaviour because having only limited coherence leads to a limited set of responses. A lack of coherence would however not make it repetitive but would rather make it erratic. A dysfunctioning executive system can explain why behaviour is stereotyped and repetitious. Behaviour is limited and repetitious due to a lack of generativity and set-shifting, this keeps autists doing the same things over and over. It is stereotyped because behavioural patterns (of others or through time) get copied and are not generated as new behaviour. Problems with inhibition also result in the same prepotent responses to certain triggers and the inability not to react. With all these executive dysfunctions explaining specific parts of these responses, the executive dysfunctioning theory is more convincing than the other two.

A supposed preference for local processing and lack of global processing D) can not be explained by disabilities in ToM directly (but can be explained by ToM via the other theories). Weakness in coherence explains the focus on local processing as separate processing. An object is not understood in its relation with other stuff, but is evaluated as separate object. This theory leaves autist incapable of understanding anything global or too complex. In the dysfunctioning executive system a lack of set-shifting keeps the autist focussed on the local dimension of an object. So here the autist can understand the global view, but simply does not engage in looking at the situation in this manner. They fail to shift attention or inhibit the global view.

A.2 Relating between theories explaining Autism

Martin and McDonald investigated how these three theories can explain pragmatic language use in different syndromes and symptoms.[131] Like Martin and McDonald we will compare the theories by looking at the predictions they make about our behaviour. They found that comparing weak central coherence, executive dysfunction and ToM deficits is problematic because these theories predict similar pragmatic deficits. Different factors lead to the same deficits, giving no reason to choose one theory above the other.[132] To effectively compare these theories we have to look at out how exactly these theories predict we will behave with respect to numbers.[133]

131 I. Martin and McDonald S., Weak coherence, no theory of mind, or executive dysfunction? Solving the puzzle of pragmatic language disorders, *Brain and Language*, vol.85 p.451-466, 2003.

132 According to Martin and McDonald a pragmatic impairment in understanding sarcasm is predicted by weak central coherence, executive functioning and by a disrupted social inference, respectively due to a difficulty using context to derive non-literal meaning, rigid and concrete information processing and a difficulty using speaker perspective to derive non-literal meaning. p. 462.

133 Martin and McDonald also warn not to concentrate on language production at the expense of language comprehension or vice versa. They warn because researchers often conclude a WCC from impaired

The three theories seemingly predict the same pragmatic outcome for language as well as numbers. The ability to use language to engage socially is impaired. This means that our operational capabilities to count are (somewhat) impaired. The disability to use language and engage socially also lead to problems with our general development. So these disabilities also lead to problems with the development of our number knowledge. In this respect the big picture of each theory leads to the same results. Of course the causal reasons for these difficulties differ between the possible explanations. And looking in depth will allow us to say more about these theories. We will not just try to compare these theories in order to choose one, but we will instead compare and match the different theories. We will try to mix and match the different reasoning within them in order to find the best explanation of the autistic counting mechanism.

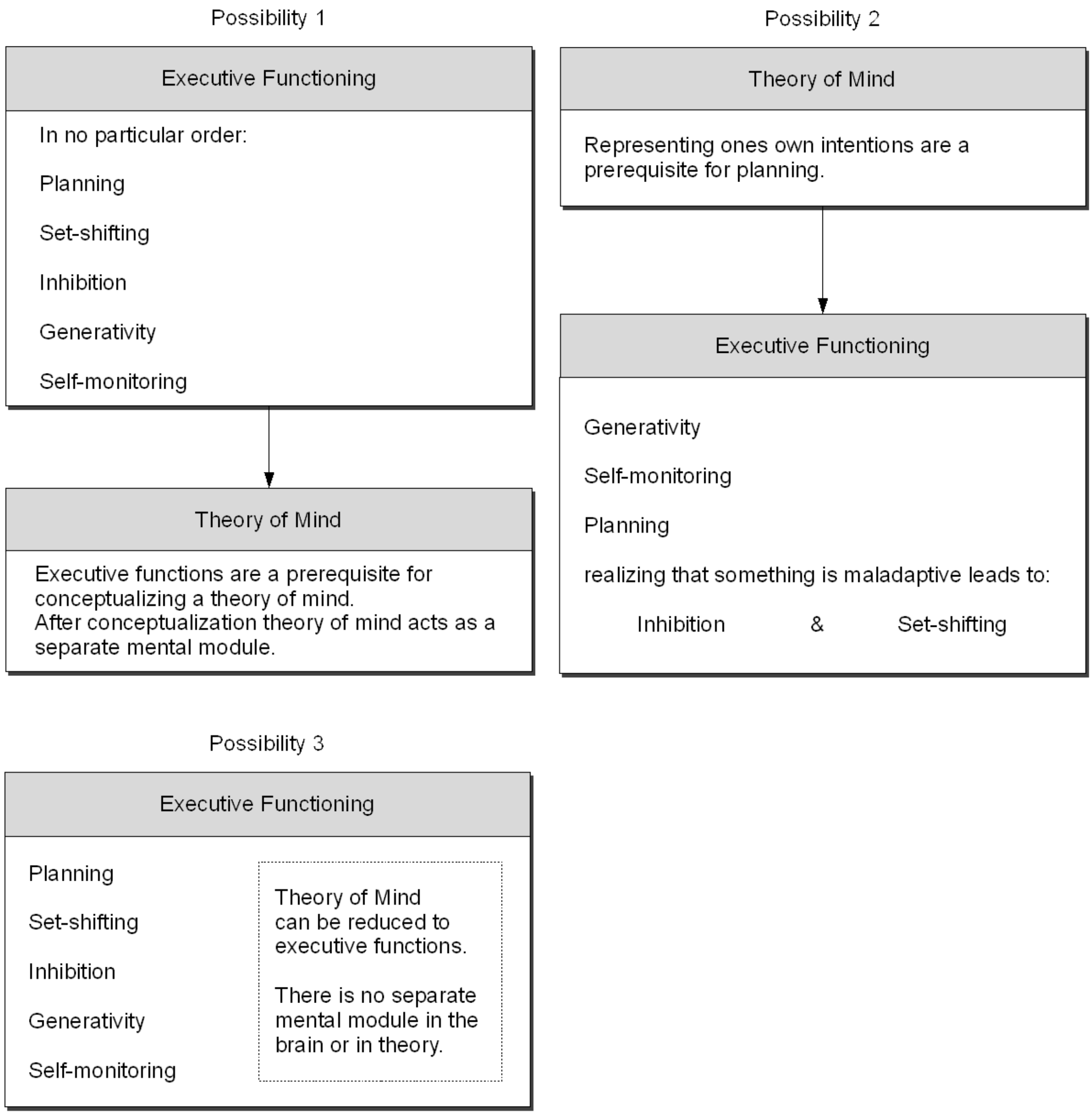

Diagram 14: Hill's 3 Relations between Executive Functioning and Theory of Mind

In matching these theories we will look to combine disabilities in ToM with WCC and DEF. We will look whether WCC and DEF can be combined and whether all three theories complement each other. Others have already tried to relate these theories. Leslie and Happé distinguish three possible relations between the

information handling and an executive dysfunction from a lack of expressing oneself. Luckily trials with numerosity often require both comprehension and production of numbers and number words. A focus on either comprehension or production will not put us off balance.

affective and cognitive disorders.[134] Hill discussed three possible relations between EF and ToM.[135]

ToM and EF

Leslie and Happé distinguish three possible relations between the affective and cognitive disorders. The affective disorder causes cognitive impairments. Affective and cognitive disorders exists independently. The cognitive disorders cause the affective disorder or the affective disorder can be explained in terms of the cognitive. Hill discussed three similar possible relations between ToM and EF. Either EF allow us to develop a ToM, the ToM allows us to develop EF or ToM can be reduced to EF. In effect this leaves us with four options: 1) EF enables ToM, 2) ToM enables EF, 3) ToM = EF (one can be reduced to the other, no separate mental modules) and 4) EF and ToM exist separately.

Hill cites research by Hughes which seems in favour of option 1) EF determines the development of ToM. In one of her studies, Hughes establishes a correlation between inhibition, mental flexibility and deceit.[136] A second study by Hughes shows that this relation only works one way; performance on EF tests has predictive value over performance on ToM tests but not the other way round.[137]

Leslie and Happé believe 2), that our development depends upon our ability to understand the actions of others. They think that the ToM and especially our metarepresentation of others lies at the hart of our development. In autism something is wrong with this metarepresentation and that has its repercussions on our development. So they place ToM before EF or CC.

View 3) and 4) are less popular. Reducing ToM to EF, in other words our reasoning about others can be translated into the functions of our execution without leaving any parts specific to ToM. This would mean ToM would disappear at the same time as EF. This is however not the case. Individuals with severe lesions in the brain at the place where the EF is thought to reside, still have ToM. Seeing ToM and EF as two completely separate modules is not likely because relations between the two have been found to exist.

CC and EF or ToM

Frith and Happé never thought that a WCC was a cause for ToM deficits. They see no correlation between the two. They relate a WCC to social deficits independent of ToM.[138] The coherence weakness generates a preference for low-level bottom-up processing and social situations require high-level processing. According to Jolliffe and Baron-Cohen the theory of WCC is not in conflict with the ToM and executive dysfunction accounts. They think the three theories can shade into each others areas.

Jolliffe and Baron-Cohen explain WCC as a dysfunctioning strategy for contextual meaning, rather than a general lack of coherence. Jolliffe and Baron-Cohen see CC as the ability to integrate meaning to make inferences. When autists are asked to describe a scene, as in Jolliffe and Baron-Cohen's trials, they do give the most coherent description. This could be due to the autists not integrating meaning or correctly processing meaning.

The Jolliffe and Baron-Cohen approach of WCC can be combined with DEF or disabled ToM. A WCC is likely to exert some influence on to the operations involved in ToM and the development of ToM. Jolliffe and Baron-Cohen suppose for example that WCC could cause a delay in our mentalizing ability resulting in operational and developmental problems with ToM. Another example is that WCC could cause problems with the integration of information that normally enables us to recognize others as thinking entities.

Jolliffe and Baron-Cohen suppose that WCC could cause problems in EF and so change the DEF actually being due to WCC. If context is not interpreted right due to WCC then a default response is chosen. This learned default response can be incorrect when not applied in the context in which it was learned. A wrong application of responses is due to WCC rather than problems with EF. Difficulty with integrating information inhibits the making of plans.

ToM and EF and CC

The three theories have some overlap. Practical problems also prevent us from making a certain choice

134 A.M. Leslie and Happé F., Autism and ostensive communication: the relevance of metarepresentation, *Developmental and Psychopathology*, vol.1 p.205-212, 1989.

135 E.L. Hill, Evaluating the theory of executive dysfunction in autism, *Developmental Review*, vol.24 p.189-233, 2004.

136 C. Hughes, Executive function in preschoolers: Links with theory of mind and verbal ability, *The British journal of developmental psychology*, vol.16 no.2 p.233-254, 1998.

137 C. Hughes, Finding your marbles: Does preschoolers' strategic behavior predict later understanding of mind, *Developmental Psychology*, vol.34 no.6 p.1326-1339, 1998.

138 On p. 26 Jarrold and Russell interpret Frith and Happé.
C. Jarrold and Russell J., Counting Abilities in Autism: Possible Implications for Central Coherence Theory, *Journal of Autism and Developmental Disorders*, vol.27 no.1 p.25-37, 1997.

between them. Tests used to determine the presence of theory of mind are very complex and do not single out the testing of theory of mind. These tests also involve executive functioning and memory functions and we would need at least some coherence to solve any problem. So according to Hill the studies she discusses do not lead to any straightforward conclusions about the relations between executive functioning and theory of mind. Jolliffe and Baron-Cohen also have some reservations about the WCC theory. They think the theory might suffer from over-extension.

Eventually each theory can be used to explain the other two. And no real contradiction between these theories results from their combination.

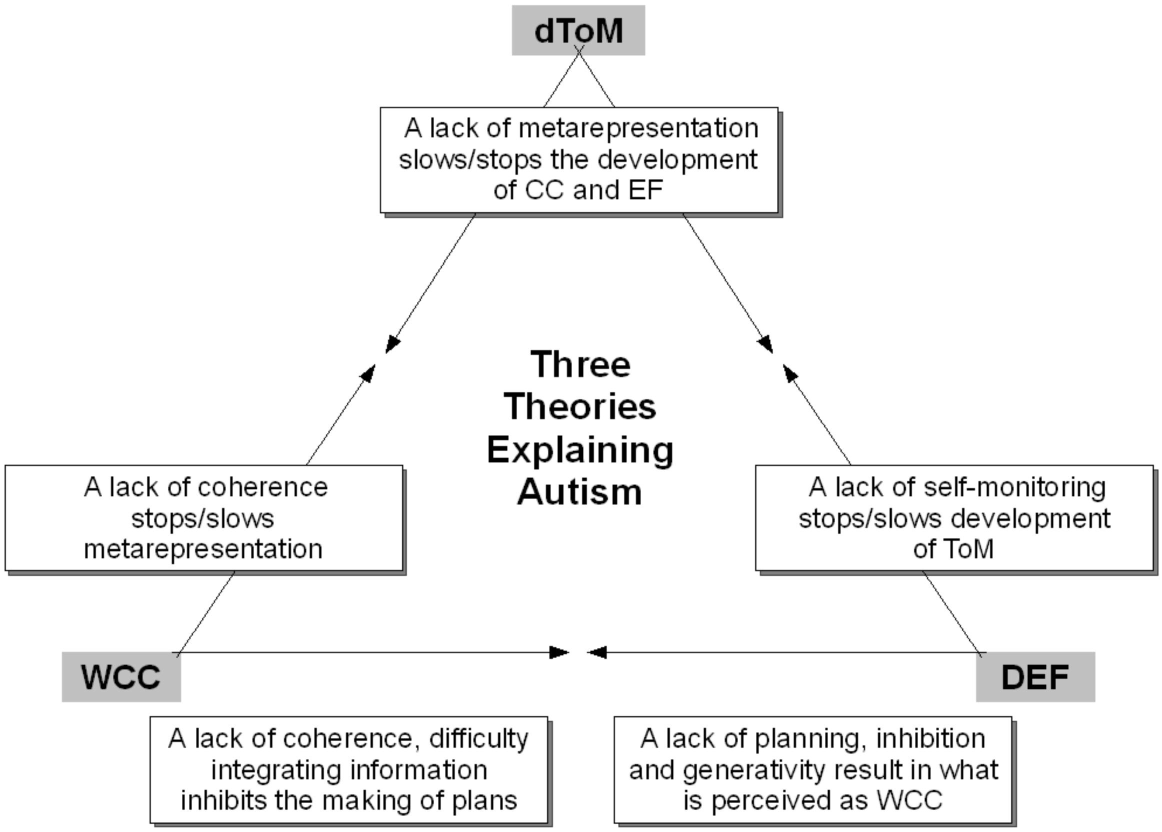

Diagram 15: Theories Explaining Autism and Explaining Each Other

Appendix B

Diagram 16: Piaget's stages

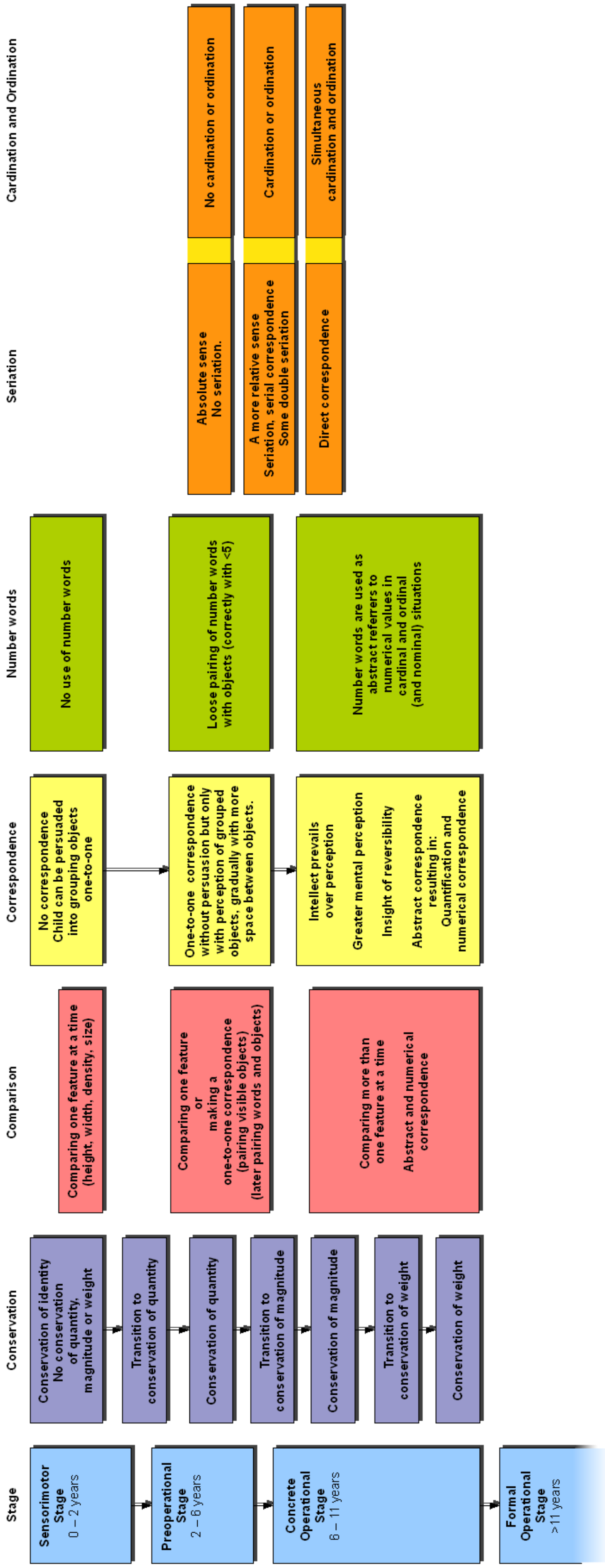

Diagram 17: Autistic counting explained

Impairments in Autistic Counting: / Theories explaining autism:	Not much benefit from canonical placing of dots	Quantity 4 is named slower	Later development of sequencing skills	Recalling positions
Disabled Theory of Mind	a general retardation due to disabilities in ToM causes autists not to learn the canonically placed dots as quantities or problems with verbal interaction	a general retardation due to disabilities in ToM causes autists not to learn the canonically placed dots as quantities or problems with verbal interaction	a general retardation due to disabilities in ToM causes autists not to develop sequencing skills as fast as TD children or problems with verbal interaction	a general retardation due to disabilities in ToM causes autists not to have problems with verbal interaction, which causes problems with memory recall
Disabled Executive Functioning	impaired set shifting and inhibition cause autists not to see or find canonical patterns of dots	autists default to a counting strategy due to their rigid and stereotyped nature and stick to it due to impaired set shifting and counting is a little slower than subitizing	impaired planning causes autists to have problems when conducting complicated tasks such as sequencing	impaired inhibition and set shifting causes autists to make the wrong choices, which make it seem like they have problems with recalling positions
Disabled Working Memory	the impaired visuospatial sketchpad causes autists not to see or find canonical patterns of dots	the impaired visuospatial sketchpad causes autists not to subitize quantities greater than 4 quickly and autists resort to counting	the impaired visuospatial sketchpad causes problems with overseeing a complex situation and tracking all elements, which causes problems with sequencing	the impaired visuospatial sketchpad causes problems with tracking all locations and their information simultaneously
Weak Central Coherence	due to impaired coherence autists cannot see the global grouping of canonically placed dots	due to impaired coherence autists cannot subitize above 3 and resort to counting	the impaired coherence causes problems with the understanding of a complex situation, which causes problems with sequencing	the impaired coherence causes problems with the understanding of a complex situation, which causes problems with recalling information

List of Diagrams

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