Decomposition Theorem for Abstract Elementary Classes

MSc Thesis (*Afstudeerscriptie*)

written by

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Introduction

Classical model theory deals essentially with elementary classes, namely, the classes that consist of models of a given complete first-order theory. Yet, many natural mathematical classes are non-elementary; examples include the class of well-ordered sets and the class of Archimedean ordered fields. The concept of abstract elementary classes (AEC) was introduced by Shelah in [12], as a way to lift classical results from elementary classes to classes which, despite being non-elementary, share properties with elementary ones. In [7], Rami Grossberg and Olivier Lessmann proposed a number of axioms in order to lift and generalize the decomposition theorem, first proved by Shelah in [11], to the AEC setting. The decomposition theorem was generated to prove part of the main gap theorem, one of Shelah's most famous results. Informally, the main gap theorem states that for any first-order theory T, the function $I(T,\kappa)$ –that is, the number of non-isomorphic models of T of cardinality κ– takes either its maximum value 2^κ or every model of T can be decomposed as a tree of small models; in this case, the number of such trees gives an upper bound to $I(T,\kappa)$ below 2^κ. The decomposition theorem deals precisely with assigning such a tree to every model.

This thesis has two objectives. The first and key objective is to provide a detailed proof of the abstract version of the decomposition theorem in the spirit of [7]. This detailed proof is provided because, although the results in [7] are correct, some of the proofs contain mistakes and missing details[1]. In addition, the axiomatic framework outlined here varies slightly from [7], and many proofs differ completely in their approach[2]. The second objective is to present an application of the abstract version of the decomposition theorem for the class of $(D,\aleph_0)$-models of a totally transcendental good diagram D. It will be shown that any two models of cardinality λ of a totally transcendental good diagram which are $L_{\infty,\lambda}$-equivalent, are isomorphic (for a large enough λ). This application is an extension of a theorem proved by Shelah for the first-order case (see [12], chapter XII).

The text is divided as follows. Section 1 addresses the preliminaries. In subsection 1.1, notation and basic concepts are outlined. Three topics which

[1]See for example the proofs of lemma 1.12, lemma 1.22 and lemma 1.24.

[2]I would like to thanks both Prof. Jouko Väänänen and Dr. Tapani Hyttinen for their help and support in this respect.

deserve a special treatment are discussed in subsections 1.2–1.4: trees, infinitary languages and pregeometries. Proofs are presented only for trees given their import to the entire thesis, while for infinitary languages and pregeometries results will be stated with references to proof sources. Section 2 contains the core argumentation and has two parts. First, in subsection 2.1, a brief introduction to abstract elementary classes is presented, bringing in Galois types and the monster model convention. In subsection 2.2, the axiomatic framework for the decomposition theorem is presented together with its revised proof. Finally, in section 3, totally transcendental diagrams are introduced in subsection 3.1 and the above-mentioned application regarding $L_{\kappa,\lambda}$-equivalence as an invariant is proved in subsection 3.2.

1 Preliminaries

1.1 Notation and Conventions

Set Theory: Let A be a set. By $|A|$ we denote the cardinality of A. Cardinal numbers will be denoted by κ and μ. Arbitrary ordinals will be denoted by α, β, δ and finite numbers are denoted by n, m and k. We use λ either as a cardinal number or as an arbitrary limit ordinal. The sequence $(a_0, \ldots, a_{n-1}) \in A^n$ is denoted by a and its length by $ln(a) = n$. We often write $a \in A$ meaning that $a \in A^n$ for some n, if no ambiguity arises. For a function $f : A \to B$, by $f(a)$ we denote the sequence $(f(a_0), \ldots, f(a_{n-1}))$. For a set B, we use $B \cup a$ as an abbreviation for $B \cup \{a_0, \ldots, a_{n-1}\}$. Given a set A and a cardinal λ we denote by $[A]^{<\lambda}$ the set of all subsets of A of cardinality less than λ.

Languages, theories, maps and models: First-order languages are denoted by L, L', L_0, etc. The cardinality of L, in symbols $|L|$, is the number of non-logical symbols in L. For a set A, L_A is the expansion of L with a new constant for every element in A. We say a formula ϕ has parameters from A, if it is an L_A-formula. Letters M and N will be restricted for models (with all the possible subscripts and superscripts like M_0, N^*, etc.). We allow a handy ambiguity using M both for the structure and its universe. We let $\mathrm{Aut}(M)$ be the automorphism group of M. For a language L, an L-theory is a set of L-sentences. As usual, we denote by $M \models T$ that M is a model of T. A class $\mathcal{K}$ of models is an elementary class if $\mathcal{K} = Mod(T)$ for some T. If M is an L-model and $A \subseteq M$, we let $\mathrm{Th}_A(M) = \{\phi \in L_A : M_A \models \phi\}$, where M_A denotes the expansion of M to L_A in which each new constant is interpreted in the obvious way. The unique substructure generated by A in M is denoted by $\langle A \rangle_M$. A map $f : A \to N$ is an elementary map if for all $a \in A$ and L-formulas ϕ we have that

$$M \models \phi(a) \Leftrightarrow N \models \phi(f(a)).$$

A map $f : A \to N$ is a partial isomorphism between M and N if f can be extended to an isomorphism $\pi : \langle \mathrm{dom} f \rangle_M \to \langle \mathrm{ran} f \rangle_N$. The set of all partial isomorphism between M and N is denoted by $\mathrm{Part}(M, N)$.

Types: For $x = (x_0, \ldots, x_{n-1})$, an n-type p over A is set of L_A-formulas in free variables $\{x_0, \ldots, x_{n-1}\}$ such that $p \cup \mathrm{Th}_A(M)$ has a model. A n-

type p over A is complete if $\phi(x) \in p$ or $\neg\phi(x) \in p$ for all $\phi(x)$ in L_A. The set of all complete n-types over A is be denoted by $S_n^M(A)$. We let $S^M(A) = \bigcup_{n<\omega} S_n^M(A)$. For a complete theory T, $S(T)$ denotes the set of all complete types over the emptyset (i.e., $S^M(\emptyset)$ for M a model of T). A type p is realized in M if there is $a \in M$ such that $M \models \phi(a)$ for all $\phi \in p$; otherwise we say that the type is omitted in M. That a realizes p is denoted by $a \models p$ and we let $p(M)$ be the set of all $a \in M$ realizing p. For $a \in M$, the type of a over A is the set $\{\phi(x) \in L_A : M \models \phi(a)\}$, and we denote it by $t_M(a/A)$. Notice that $t_M(a/A)$ is always a complete type. Moreover, $t_M(a/A) = t_M(b/A)$ if and only if there is an elementary extension N of M and $f \in \mathrm{Aut}(N)$ such that f fixes A pointwise and $f(a) = b$ (a proof of this last fact can be found in [8] p. 117-118).

1.2 Trees

Let $P = (A, \leq)$ be a partial order. We say $a, b \in A$ are comparable if either $a < b$ or $b < a$. For $a \in A$, we denote by $a_< = \{b \in A : b < a\}$, the set of all elements below a. It is clear from the notation what we mean by $a_\leq$ and $a_>$. A subset $B \subseteq A$ is a chain if B is linearly ordered by $<$ (i.e., every two elements in B are comparable). B is an anti-chain if every two elements in B are incomparable. We say B is downward closed if whenever $a \in B$ and $b < a$ for $b \in T$, we also have that $b \in B$. Finally, P is well-founded if it does not contain infinite descending sequences. We will focus on some properties of trees, a particular kind of well-founded partial orders.

Definition 1.1. *A partial order $(T, \leq)$ is a tree if for all $a \in T$, the set $a_<$ is well-ordered with respect to $<$.*

For notational simplicity, we will often use T for the pair $(T, \leq)$ when its meaning can be easily inferred from the context. The pair $(T', \leq')$ is a subtree of $(T, \leq)$ if T' is a downward closed subset of T and $\leq'$ is the restriction of $\leq$ to T'. Thus, for an arbitrary $A \subseteq T$, the downward closure of A (that is, $\{b \in T : b \leq a, a \in A\}$) forms a subtree of T containing A. Notice that if T is a tree, then for all $a \in T$ both $a_<$ and $a_\leq$ are subtrees of T. Moreover, if $a \in T$ is a maximal element (also called a leaf), then $T - \{a\}$ is a subtree of T. We denote by a^- the predecessor of a (if it has one).

We say a tree is rooted if it has a minimum (i.e., $\exists x \forall y (x \leq y)$) called the root of the tree. The well-foundedness of trees allows us to define a rank function for its elements which is traditionally called the *height* of an element. For a tree T and $a \in T$, the height of a, denoted by $ht(a)$, is the ordinal type of $a_<$. The height of the tree, denoted by $ht(T)$, is $\sup\{ht(a) : a \in T\}$. We say T is an ω-tree if $ht(T) \leq \omega$. Notice that if T is rooted and a is the root, we have by definition that $ht(a) = 0$.

An important kind of tree used in section 2.2 corresponds to the set of sequences of a given ordinal ordered by initial segment. We present some special notation for this case:

Example 1.2. Let λ be an ordinal. The set ${}^{\leq k}\lambda$ is the set of all sequences η of λ such that $ln(\eta) \leq k$. We let ${}^{<\omega}\lambda$ for union of all ${}^{\leq k}\lambda$ for all $k < \omega$. Given two elements $\eta, \nu \in {}^{\leq k}\lambda$ we let $\eta \leq \nu$ if and only if η is an initial segment of ν. It is easy to check that for a subset $I \subseteq {}^{<\omega}\lambda$ which is closed under initial segment, the pair $(I, \leq)$ is a rooted tree (the root being the empty sequence). Notice that such I is an ω-tree. For $\eta \in {}^{<\omega}\lambda$ and $\alpha < \lambda$, the sequence $\eta^\frown\alpha$ corresponds to the concatenation of η and the one element sequence $\langle\alpha\rangle$. In this case we have that $(\eta^\frown\alpha)^- = \eta$. Here the height of an element corresponds simply to the its length.

Now let us consider unions of trees. The following lemma shows that the union of a chain of subtrees is again a tree:

Lemma 1.3. *Let $((T_i, \leq_i) : i < \lambda)$ be an increasing sequence of trees, that is, $(T_i, \leq_i)$ is a subtree of $(T_j, \leq_j)$ whenever $i \leq j$. Then $(T, \leq) = (\bigcup_{i<\lambda} T_i, \bigcup_{i<\lambda} \leq_i)$ is a tree.*

Proof: It is easy to check that $(T, \leq)$ is a partial order. We show that $a_<$ is well-ordered by $<$. Let $a \in T$ and $i < \lambda$ such that $a \in T_i$. Then, by assumption, $a_{<_i}$ is well-ordered by $<_i$. It is enough to show that $a_< = a_{<_i}$. By definition we have that $a_{<_i} \subseteq a_<$. For the converse, take $b \in T$ such that $b < a$. Then $b <_j a$ for some $j < \lambda$. If $j \leq i$, then $<_i$ extends $<_j$, so $b <_i a$. If $i < j$, since $(T_i, \leq_i)$ is a subtree of $(T_j, \leq_j)$, then T_i is downward closed with respect to $\leq_j$. Thus, $a \in T_i$ and $b <_j a$ imply that $b \in T_i$. Again, by the definition of subtree we conclude $b <_i a$. $\square$

Corollary 1.4. *Let $((T_i, \leq_i) : i < \lambda)$ be an increasing sequence of ω-trees. Then $(T, \leq) = (\bigcup_{i<\lambda} T_i, \bigcup_{i<\lambda} \leq_i)$ is an ω-tree.*

By the previous lemma, it remains to show that the $ht(T) \leq \omega$. Assume towards a contradiction that $ht(T) > \omega$. Then $\sup\{ht(a) : a \in T\} > \omega$, so there must be $a \in T$ such that $ht_T(a) > \omega$ (where $ht_T(a)$ means the height of a with respect to $(T, \leq)$). But then $a \in T_i$ for some $i < \lambda$, and since $a_{<_i} = a_<$ (as showed in the previous proof), then $ht_{T_i}(a) > \omega$ contradicting the fact that T_i is an ω-tree. $\square$

The following theorem is one of the key facts about trees used in section 2.2. A general proof for well-founded partial orders which appears in [3] is presented.

Theorem 1.5. *Every well-founded partial order can be extended to a well-order*

Proof: Let $P = (A, \leq)$ be a well-founded partial order. We recursively build an ordinal δ and a sequence $(C_i : i < \delta)$ of anti-chains of P as follows:

(1) $C_0 = \{x : \neg\exists y(y < x)\}$ (i.e., all the minimal elements in P).

(2) $C_i = \{x : x \in A - \bigcup_{j<i} C_j \text{ and } \neg\exists y(y \in A - \bigcup_{j<i} C_j \wedge (y < x))\}$ (i.e., all the minimal elements in $A - \bigcup_{j<i} C_j$).

We let δ be the smallest ordinal such that $A - \bigcup_{i<\delta} C_i = \emptyset$ (which must exists since A is a set). Clearly, every C_i is an anti-chain, since if $x, y \in C_i$ and $x < y$, one of them is not minimal. Let $<_i$ be a well-order for C_i. Consider the order on A defined by the lexicographic sum of $(<_i : i < \delta)$:

$$x \lhd y \Leftrightarrow \begin{cases} x <_i y & \text{if } x, y \in C_i \\ i < j & \text{if } x \in C_i, y \in C_j \end{cases}$$

Since each $<_i$ is a well-order, it is easy to see that $\lhd$ is a well-order. It remains to show that it extends $\leq$. Let $x, y \in A$ such that $x < y$. Assume $x \in C_i$ and $y \in C_j$. Since $x < y$ we must have that $i \neq j$. If $i < j$, we have by definition that $x \lhd y$, and we are done. Assume towards a contradiction that $j < i$. Then by definition $x \in A - \bigcup_{k<i} C_k$ and $y \in A - \bigcup_{k<j} C_k$, but $x < y$ which contradicts the minimality of y. $\square$

Trees are by definition well-founded partial orders. The previous theorem will often be used as follows: let T be a tree and $(\xi_i : i < \delta)$ be an enumeration of T such that $\xi_i <_T \xi_j$ implies $i < j$ (where $<_T$ corresponds to the order of the tree).

1.3 Infinitary Logic

Infinitary logic studies expressions of infinite length. The most common example is $L_{\omega_1,\omega}$ which allows countable conjunctions and disjunctions of formulas. For $\lambda > \omega$, $L_{\kappa,\lambda}$ also permits quantification over infinite subsets of variables of cardinality less than λ. The formal definition for these languages is the following:

Definition 1.6. *Let $\kappa \geq \lambda$ be cardinals and L a language. An expression of $L_{\kappa,\lambda}$ is defined recursively as follows:*

- *Every L-atomic formula is an $L_{\kappa,\lambda}$-expression.*
- *If ϕ is an $L_{\kappa,\lambda}$-expression, so is $\neg\phi$.*
- *If $\mu < \kappa$ and $(\phi_i : i < \mu)$ is a sequence of $L_{\kappa,\lambda}$-expressions, then $\bigwedge_{i<\mu} \phi_i$ is an $L_{\kappa,\lambda}$-expression.*
- *If $\delta < \lambda$, ϕ is an $L_{\kappa,\lambda}$-expression and $(v_i : i < \delta)$ is a sequence of variables then $\exists_{i<\delta} v_i \phi$ is an $L_{\kappa,\lambda}$-expression.*

A $L_{\kappa,\lambda}$-formula is a $L_{\kappa,\lambda}$-expression with less than λ free variables.

The restriction in the definition of $L_{\kappa,\lambda}$-formulas is due to preserve the first-order fact that every formula might be turned into a sentence by quantifying its free variables. Once the formulas in $L_{\kappa,\lambda}$ are defined, we define the formulas in $L_{\infty,\lambda}$ simply as the union of the formulas in $L_{\kappa,\lambda}$ for all $\kappa \geq \lambda$. Briefly speaking, the formulas in $L_{\infty,\lambda}$ might contain arbitrary large conjunctions and disjunctions, and quantifications over subsets of variables of cardinality less than λ.

Our next step is to present a characterization of $L_{\infty,\lambda}$-equivalence between models. Two models M, N are $L_{\infty,\lambda}$-equivalent if they satisfy the same $L_{\infty,\lambda}$-sentences, and we denote this fact by $M \equiv_{\infty\lambda} N$. The characterization corresponds to the infinite-counterpart of back-and-forth sets.

Definition 1.7. *Let M and N be L-structures. A λ-back-and-forth set for M and N is any non-empty set $P \subseteq Part(M, N)$ such that*

(1) $\forall f \in P \forall A \in [M]^{<\lambda} \exists g \in P(f \subseteq g$ *and* $A \subseteq dom(g))$

(2) $\forall f \in P \forall B \in [N]^{<\lambda} \exists g \in P(f \subseteq g$ *and* $B \subseteq ran(g))$

The structures M and N are said to be λ-partially isomorphic, in symbols $M \simeq_p^\lambda N$, if there is a λ-back-and-forth set for them.

The expected characterization is resumed in the following theorem (for a proof see [14], or [4]).

Theorem 1.8. *Let M and N be L-structures. The following conditions are equivalent:*

1. $M \equiv_{\infty,\lambda} N$.
2. $M \simeq_p^\lambda N$.

Among other characterizations that have been introduced in recent years, one of the most successful ones corresponds to a counterpart of the Ehrenfeucht-Fraïssé game for first-order logic. Informally speaking, the Ehrenfeucht-Fraïssé game for $L_{\infty,\lambda}$ proceeds as the first-order game but the players, instead of choosing elements from the models, they choose sequences of length less than λ of elements. As in the first-order case, $M \equiv_{\infty,\lambda} N$ if and only if Player II has a winning strategy. For a more detailed exposition of this characterization see [14].

1.4 Pregeometries

Pregeometries (or combinatorial geometries) are a big area of study in mathematics. Even if its most important theorems deal with finite pregeometries (or matroids), the general concept has many applications. We start with the definition of a closure operation:

Definition 1.9. *Let X be a set and cl an operation on $\mathcal{P}(X)$. We say that cl is a closure operation if it satisfies the following conditions for all A, B subsets of X:*

1. $A \subseteq cl(A) = cl(cl(A))$.
2. $A \subseteq B \Rightarrow cl(A) \subseteq cl(B)$.

Having a closure operation on a set we define a pregeometry as follows:

Definition 1.10. *Let X be a set and cl a closure operation on X. The pair (X, cl) is a pregeometry if it satisfies:*

1. ***(Finite character)*** *For all $a \in X$, $a \in cl(A)$ implies that there is a finite $A_0 \subseteq A$ such that $a \in cl(A_0)$.*

2. ***(Exchange Property)*** *for all $a, b \in X$ and $A \subseteq X$, if $a \in cl(A, b) - cl(A)$, then $b \in cl(A, a)$.*

Let (X, cl) be a pregeometry and $A, B \subseteq X$. We say that A is cl-independent over B if $a \notin cl((B \cup A) - \{a\})$ for all $a \in A$. Furthermore, A is a basis of $Y \subseteq X$ over B if A is independent over B and $cl(A) = cl(Y)$. Notice that bases correspond to maximal independent subsets (since the closure of a maximal independent subset of Y must be equal to the closure of Y). The standard (and most important) result about pregeometries is stated in the following theorem (for a proof see [8], p. 210-211):

Theorem 1.11. *Let (X, cl) be a pregeometry and $Y, B \subseteq X$. Then Y has a basis over B. Moreover, If A_1, A_2 are bases for Y over B, then $|A_1| = |A_2|$.*

As in vector spaces, the previous theorem grants permission to define the dimension concept. Given a pregeometry (X, cl) and $Y, B \subseteq X$, the dimension of Y over B is the cardinality of the bases (or maximal independent sets) of Y over B. This result will be used in the last section of this thesis.

2 Decomposition Theorem for Abstract Elementary Classes

Abstract elementary classes (AEC) were introduced by Shelah to generate a common framework to treat those classes of models that, in spite of being non-elementary, behave similarly to elementary classes. Part of the motivation was the study of classes of models for theories in infinitary languages. As stated in the introduction, Rami Grossberg and Olivier Lessmann presented an axiomatic framework which generalizes the decomposition theorem to AEC in [7]. This theorem states that for a class of models $\mathcal{K}$ satisfying the postulated axioms, every $M \in \mathcal{K}$ can be decomposed in a tree of small submodels such that M is prime and minimal over their union. This section is devoted to prove that theorem.

There are two differences between [7] and the approach here outlined which are important to mention. Firstly, the choice of axioms is slightly different. In [7], the authors present three kinds of axioms postulating: the existence of an independence relation which is well-behaved over models; the existence—over certain sets—of a special kind of prime models, called primary, which are unique modulo isomorphism and behave well with respect to the independence relation; and finally, the existence of certain types, called regular, which also behave well with respect to the independence relation. The axioms proposed below will still postulate the existence of prime models over certain sets, but instead of assuming as an axiom the uniqueness of primary or prime models, an alternative axiom—related to types—will be posited. The motivation for this new axiom will be clear from the proofs.[3] Secondly, an additional condition is added to the definition of decomposition. Although this additional condition does not change the main idea of the proposed decomposition theorem proof, it simplifies some of the argumentation in section 3.

[3]This axiom was suggested to me by Tapani Hyttinen. When primary models are not *bona fide*(see [7], p. 6), the new axiom is used to prove their uniqueness, so our choice for it still complies with the axiomatic framework suggested in [7]. It is important to notice that to relativize all the axioms from primary models to prime models makes essentially no changes to the proofs.

Subsection 2.1 constitutes a brief introduction to abstract elementary classes. Both the corresponding concept of types —called Galois types— and the monster model convention are presented. In subsection 2.2 the decomposition theorem is proved.

2.1 Abstract Elementary Classes

We start with the definition of an abstract elementary class.

Definition 2.1. *Let L be a language. We say $(\mathcal{K}, \prec)$ is an* abstract elementary class (AEC) *if $\mathcal{K}$ is a class of L-structures partially ordered by $\prec$ satisfying the following conditions:*

(C1) $\mathcal{K}$ is closed under isomorphisms. Moreover, isomorphisms preserve $\prec$, i.e., if $M \prec N$ and $f : N \to N'$ is an isomorphism, then $f(M) \prec N'$.

(C2) $M \prec N \Rightarrow M \leq N$ (M is a substructure of N).

(C3) If $M \leq N$ and there is N^* such that $N \prec N^*$ and $M \prec N^*$, then $M \prec N$.

(C4) There is a cardinal $LS(\mathcal{K})$ such that for all $A \subseteq M$ there is N containing A such that $N \prec M$, and $|N| \leq |A| + LS(\mathcal{K})$.

(C5) Let λ be an ordinal and $(M_i : i < \lambda)$ a be a $\prec$-increasing and continuous chain of structures in $\mathcal{K}$. Then

$$M = \bigcup_{i<\lambda} M_i \in \mathcal{K}$$

Moreover, $M_i \prec M$ for all $i < \lambda$ and, if for $N \in \mathcal{K}$ we have that $M_i \prec N$ for all $i < \lambda$, then $M \prec N$.

Condition $(C5)$ is also named in the literature as $\mathcal{K}$ being closed under Tarski-Vaught chains. It is sufficient to state in $(C5)$ that $M_0 \prec M$ since $M_i \prec M$ follows by induction using $(C3)$. As expected, the class of models of a first-order theory, taking $\prec$ as the elementary substructure relation, forms an AEC. In other words, traditional elementary classes are also AEC.

We turn our attention to embeddings. Instead of working with arbitrary embeddings, we restrict ourselves to embeddings that preserve $\prec$, i.e.:

Definition 2.2. *Let $M, N \in \mathcal{K}$. We say that an embedding $f : M \to N$ is a $\mathcal{K}$-embedding if in addition $f(M) \prec N$.*

The second part of $(C1)$ guarantees that the composition of $\mathcal{K}$-embeddings is again a $\mathcal{K}$-embedding. We will omit several proofs for basic results about AEC. For a more detailed treatment the reader should look at [1]. Still, in order to get the flavour of those arguments at least once, a simple fact which uses almost all conditions listed in definition 2.1 is showed:

Lemma 2.3. *Let $(M_i : i < \lambda)$ and $(N_i : i < \lambda)$ be $\prec$-increasing and continuous sequences of structures in $\mathcal{K}$ and $(f_i : i < \lambda)$ an increasing and continuous sequence of $\mathcal{K}$-embeddings such that $f_i : M_i \to N_i$ for all $i < \lambda$. Then*

$$\bigcup_{i<\lambda} f_i : \bigcup_{i<\lambda} M_i \to \bigcup_{i<\lambda} N_i$$

is a $\mathcal{K}$-embedding.

Proof: Let $M = \bigcup_{i<\lambda} M_i$, $N = \bigcup_{i<\lambda} N_i$ and $f = \bigcup_{i<\lambda} f_i$. Since $i \leq j$ implies $f_i \subseteq f_j$, it is a rutinary exercise to check that f is an embedding. It remains to show that $f(M) \prec N$. First, we notice that

$$f(M) = \bigcup_{i<\lambda} f_i(M_i)$$

Secondly, we show that $(f_i(M_i) : i < \lambda)$ is a $\prec$-increasing and continuous sequence of $\mathcal{K}$ structures. Each $f_i(M_i)$ is isomorphic to M_i, hence by $(C1)$ $f_i(M_i) \in \mathcal{K}$. Suppose that $i < j$. On the one hand, since $M_i \prec M_j$, by $(C2)$ we have that M_i is a substructure of M_j, hence since $f_i \subseteq f_j$, $f_i(M_i)$ is a substructure of $f_j(M_j)$. Moreover, since each f_i is a $\mathcal{K}$-embedding, we have $f_i(M_i) \prec N_i$ and $f_j(M_j) \prec N_j$. Since $N_i \prec N_j$, by transitivity we have that $f_i(M_i) \prec N_j$. But then by $(C3)$ we can conclude that $f_i(M_i) \prec f_j(M_j)$. The continuity of the sequence follows simply from the continuity of our initial sequences. Then, since $f_i(M_i) \prec N_i \prec N$ for all $i < \lambda$ (by condition $(C5)$) and we showed $(f_i(M_i) : i < \lambda)$ is a $\prec$-increasing and continuous sequence of $\mathcal{K}$ structures, $(C5)$ implies that $f(M) = \bigcup_{i<\lambda} f_i(M_i) \prec N$. $\Box$

Now we focus on amalgamation. Since the framework of AEC is formula-free, we need to find a counterpart for then notion of type. Amalgamation allows us to define a well-behaved counterpart for this notion.

Definition 2.4. *An AEC* $(\mathcal{K}, \prec)$ *has* amalgamation *if it satisfies:*

(1) Joint Embedding Property (JEP): for all $M_0, M_1 \in \mathcal{K}$ *there is* $N \in \mathcal{K}$ *and* $\mathcal{K}$*-embeddings* $f_i : M_i \to N$ *for* $i = 0, 1$.

(2) Amalgamation Property (AP): For all $M_0, M_1, M_2 \in \mathcal{K}$*,* $\mathcal{K}$*-embeddings* $f_i : M_0 \to M_i$ *for* $i = 1, 2$*, there is* $N \in \mathcal{K}$ *and* $\mathcal{K}$*-embeddings* $g_i : M_i \to N$ *such that* $g_1 \circ f_1 = g_2 \circ f_2$.

$$\begin{tikzcd} M_1 \arrow[r, "g_1"] & N \\ M_0 \arrow[u, "f_1"] \arrow[r, "f_2"] & M_2 \arrow[u, "g_2"] \end{tikzcd}$$

In the literature, the name *amalgamation* has been given to the conjunction of these two properties given that in many cases JEP is implied by AP. For instance, in the elementary case where $\mathcal{K} = Mod(T)$ for some first-order theory T, if T is complete, then $\mathcal{K}$ has the JEP. In addition, any class $\mathcal{K}$ that satisfies AP and has a prime model also satisfies the JEP (and if $\mathcal{K} = Mod(T)$, then T is also complete). We define now our abstract notion of type. Remember that a, b, c denote finite sequences.

Definition 2.5. *Let* $(\mathcal{K}, \prec)$ *be an AEC and* $\sim$ *be a relation on triples* (a, M, N) *such that* $M \prec N$ *and* $a \in N$ *defined by*

$$(a_0, M, N_0) \sim (a_1, M, N_1)$$

if and only if $ln(a_0) = ln(a_1)$ *and there are* $N^* \in \mathcal{K}$ *and* $\mathcal{K}$*-embeddings* $f_i : N_i \to N^*$ *for* $i = 0, 1$ *such that* $f_i \restriction M = id$ *and* $f_0(a_0) = f_1(a_1)$.

Lemma 2.6. *Let* $(\mathcal{K}, \prec)$ *be an AEC with amalgamation. Then* $\sim$ *is an equivalence relation.*

Proof: Reflexivity and symmetry are trivial. Let $A, B, C \in \mathcal{K}$ such that $(a, M, A) \sim (b, M, B)$ and $(b, M, B) \sim (c, M, C)$. Thus, by definition, there are $N_1, N_2 \in \mathcal{K}$ and $\mathcal{K}$-embeddings f_i and g_i for $i = 0, 1$ such that the following diagrams commute

$$\begin{tikzcd} A \arrow[r, "f_1"] & N_1 \\ M \arrow[u, "id"] \arrow[r, "id"] & B \arrow[u, "f_2"] \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "g_1"] & N_2 \\ M \arrow[u, "id"] \arrow[r, "id"] & C \arrow[u, "g_2"] \end{tikzcd}$$

and, moreover, $f_1(a) = f_2(b)$ and $g_1(b) = g_2(c)$. By AP, let $N_3 \in \mathcal{K}$ and h_1, h_2 be $\mathcal{K}$-embeddings such that the following diagram commutes:

$$\begin{array}{ccccc} A & \xrightarrow{f_1} & N_1 & \xrightarrow{h_1} & N_3 \\ & & \uparrow{\scriptstyle f_2} & & \uparrow{\scriptstyle h_2} \\ & & B & \xrightarrow{g_1} & N_2 \\ & & & & \uparrow{\scriptstyle g_2} \\ & & & & C \end{array}$$

The $\mathcal{K}$-embeddings $h_1 f_1$ and $h_2 g_2$ satisfy what we want since:

$$h_1 f_1(a) \overbrace{=}^{f_1(a)=f_2(b)} h_1 f_2(b) \overbrace{=}^{AP} h_2 g_1(b) \overbrace{=}^{g_1(b)=g_2(c)} h_2 g_2(c)$$

Hence $(a, M, A) \sim (c, M, C)$. $\square$

The previous lemma allows us to have the following definition:

Definition 2.7. *Let $(\mathcal{K}, \prec)$ be an AEC with amalgamation. The* Galois type of a over M in N *is the equivalence class $(a, M, N)/\sim$, denoted by $gt(a/M, N)$. For $M \in \mathcal{K}$ we denote by $S(M)$ the set of all Galois types over M, i.e.,*

$$S(M) = \{gt(a/M, N) : a \in N \textit{ for some } N \in \mathcal{K}\}$$

We say that N_0 realizes $gt(a/M, N)$ if $M \prec N_0$ and there exists $a_0 \in N_0$ such that $gt(a_0/M, N_0) = gt(a/M, N)$. The type notation introduced in the preliminaries will be also used for Galois types. Thus, letters like p, q and r will denote Galois types. For $p \in S(M)$ and $a \in N$ the expression $a \vDash p$ (or in words, that a realizes p) means then $p = gt(a/M, N)$. We say that $p \in S(M)$ is realized in N if there is some $a \in N$ such that $p = gt(a/M, N)$ (remember though that Galois types are not sets of formulas).

Remark 2.8. Let f be a $\mathcal{K}$-embedding such that $\mathrm{dom}(f) = N$ and $f \upharpoonright M$=id. Then $gt(a/M, N) = gt(f(a)/M, f(N))$. A simple picture shows it:

$$\begin{array}{ccc} N & \xrightarrow{f} & f(N) \\ \uparrow{\scriptstyle id} & & \uparrow{\scriptstyle id} \\ M & \xrightarrow{id} & f(N) \end{array}$$

Unfortunately this semantic approach to types does not capture the whole syntactic type concept. Nevertheless it tackles the notion of a type over a model. The next proposition shows this relation:

Proposition 2.9. *Let T be a complete theory with amalgamation, $\mathcal{K}$=$Mod(T)$ and $\prec$ denote the elementary substructure relation. Let M, N_1, N_2 be models of T such that $M \prec N_i$ for $i = 1, 2$ and $a \in N_1$, $b \in N_2$. Then*

$$t_{N_1}(a/M) = t_{N_2}(b/M) \Leftrightarrow gt(a/M, N_1) = gt(b/M, N_2)$$

Proof:
($\Rightarrow$) By amalgamation there is $N_3 \in \mathcal{K}$ such that $N_2 \prec N_3$ and there is an elementary embedding $g : N_1 \to N_3$ fixing M pointwise. By elementary properties

$$t_{N_3}(g(a)/M) = t_{N_1}(a/M) = t_{N_2}(b/M) = t_{N_3}(b/M)$$

this implies that there is an elementary extension N_4 of N_3 and $\pi \in \mathrm{Aut}(N_4)$ fixing M pointwise such that $\pi(g(a)) = b$. Trivially we also have that g is an embedding from N_1 to N_4 and $N_2 \prec N_4$. Notice that in this context our $\mathcal{K}$-embeddings are just elementary embeddings. We claim $f_1 = \pi g$ and $f_2 = id$ prove our claim. Trivially both fix M. Moreover, $f_1(a) = \pi(g(a)) = b = f_2(b)$, which shows $(a, M, N_1) \sim (b, M, N_2)$.

($\Leftarrow$) By assumption there is $N_3 \in \mathcal{K}$ and elementary embeddings $f_i : N_i \to N_3$ for $i = 1, 2$ such that they fix M and $f_1(a) = f_2(b)$. Let $\phi(x, y)$ be an L-formula and $m \in M$.

$$\begin{aligned} \phi(x, m) \in t_{N_1}(a/M) &\Leftrightarrow N_1 \models \phi(a, m) \\ &\Leftrightarrow N_3 \models \phi(f_1(a), f_1(m)) \\ &\Leftrightarrow N_3 \models \phi(b, m) \\ &\Leftrightarrow f_2(N_2) \models \phi(b, m) \\ &\Leftrightarrow N_2 \models \phi(b, m) \Leftrightarrow \phi(x, m) \in t_{N_2}(b/M) \end{aligned}$$

Hence $t_{N_1}(a/M) = t_{N_2}(b/M)$. $\Box$

We define the corresponding notion of saturation for Galois types:

Definition 2.10. *Let $N \in \mathcal{K}$. We say N is λ-Galois saturated if N realizes every $p \in S(M)$ for all $M \prec N$ such that $|M| < \lambda$.*

It turns out that the existence of Galois saturated structures is related to amalgamation. The link arises through Jónsson's proof of the existence of universal and strongly model-homogeneous structures for a class satisfying JEP and AP. His proof could be taken as the first proof according to the spirit of AEC. We will state the definition of universality and strong model-homogeneity and assume the existence of such structures without going through the proof (a good source for it is [2], p. 202-213):

Definition 2.11. *Let $(\mathcal{K}, \prec)$ be an AEC, $N \in \mathcal{K}$ and $\lambda > LS(\mathcal{K})$. N is λ-universal if for every $M \in \mathcal{K}$ such that $|M| < \lambda$, there is a $\mathcal{K}$-embedding $f : M \to N$. N is λ-model-homogeneous if for every $M \prec M'$ such that $|M'| < \lambda$, if $M \prec N$ then there is a $\mathcal{K}$-embedding $f : M' \to N$, such that $f \upharpoonright M = id$. N is strongly λ-model-homogeneous if, in addition, for every $M_1, M_2 \in \mathcal{K}$ such that $|M_i| < \lambda$ and $M_i \prec N$ for $i = 1, 2$, every isomorphism between M_1 and M_2 extends to an automorphism of N.*

Theorem 2.12. ***(Jónsson)*** *Let $(\mathcal{K}; \prec)$ be an AEC with the amalgamation property and arbitrary large models. Let $\lambda \geq LS(\mathcal{K})$ be a regular cardinal. Then there is a λ-universal and strongly λ-model-homogeneous structure $N \in \mathcal{K}$.*

A notable result of Shelah (in [13]) shows that for an AEC satisfying amalgamation, Galois-saturation and model-homogeneity are equivalent. By Jónsson's theorem, this implies the existence of Galois-saturated models. Since for our purposes there is no need to show the equivalence, we will only prove that model-homogeneity implies Galois-saturation. Proofs for the converse can be found in [13] and [5].

Theorem 2.13. ***(Shelah)*** *Let $(\mathcal{K}; \prec)$ be an AEC with the amalgamation property and arbitrary large models. Let $\lambda > LS(\mathcal{K})$ be a regular cardinal. Then $N \in \mathcal{K}$ is λ-Galois saturated if and only if N is λ-homogeneous.*

Proof: Let $M \in \mathcal{K}$ such that $|M| < \lambda$ and $M \prec N$. Let $p \in S(M)$ such that $p = gt(a/M, N_1)$. Then by $(C4)$, there is $M' \prec N$ containing $M \cup a$ such that $|M'| < \lambda$. By $(C3)$ we have that $M \prec M'$. This shows (using the identity function) that

$$gt(a/M, N_1) = gt(a/M, M')$$

By λ-model homogeneity there is a $\mathcal{K}$-embedding $f : M' \to N$ fixing M pointwise. Hence, by remark 2.8 we have that

$$gt(a/M, M') = gt(f(a)/M, f(M'))$$

Finally, the identity function also shows

$$gt(f(a)/M, f(M')) = gt(f(a)/M, N)$$

Hence by transitivity $gt(a/M, N_1) = gt(f(a)/M, N)$, which shows that p is realized in N. $\Box$

The previous two theorems motivate the following framework convention:

Convention 2.14 (Monster Model). *Let $(\mathcal{K}, \prec)$ be an AEC with the amalgamation property and arbitrary large models. For a big enough cardinal $\bar{\kappa}$, we fix a $\bar{\kappa}$-universal and strongly $\bar{\kappa}$-model-homogeneous (hence $\bar{\kappa}$-Galois saturated) model $\mathfrak{C} \in \mathcal{K}$. Every set and structure is assumed to be respectively a subset and a substructure of $\mathfrak{C}$ of cardinality less than $\bar{\kappa}$. The model $\mathfrak{C}$ is called the monster model.*

Working inside a monster model simplifies some arguments and definitions without loosing generality. This is because for every $M \in \mathcal{K}$ such that $|M| < \bar{\kappa}$, by $\bar{\kappa}$-universality there is $N \prec \mathfrak{C}$ such that M and N are isomorphic. Hence, modulo isomorphism, we are working with all the models in $\mathcal{K}$ of cardinality less that $\bar{\kappa}$. Since in addition $\bar{\kappa}$ can be taken as large as we want, we can always find a suitable $\bar{\kappa}$ that includes every model we are dealing with.

Galois types constitute an example where a simplification of both arguments and definitions arises. By Shelah's theorem 2.13, for $M \prec \mathfrak{C}$ such that $|M| < \bar{\kappa}$, every type in $S(M)$ is realized in $\mathfrak{C}$, hence, every $p \in S(M)$ has the form $p = gt(a/M, \mathfrak{C})$. This allows us to write $gt(a/M)$ instead of $gt(a/M, \mathfrak{C})$. Moreover, Galois types can be viewed as orbits in $\mathrm{Aut}(\mathfrak{C})$, that is, we take $gt(a/M)$ as the set of all $b \in \mathfrak{C}$ such that there is $f \in \mathrm{Aut}(\mathfrak{C})$ fixing M pointwise and $f(a) = b$. This is the content of the following proposition:

Proposition 2.15. *$gt(a/M) = gt(b/M)$ if and only if there is $f \in Aut(\mathfrak{C})$ such that $f \restriction M = id$ and $f(a) = b$.*

Proof: The right to left direction follows by the definition of Galois types. For the converse, assume that $gt(a/M) = gt(b/M)$. Then, by $(C4)$ there are $N_i \prec \mathfrak{C}$ such that $|N_i| < \bar{\kappa}$ for $i = 1, 2$ and $gt(a/M, N_1) = gt(b/M, N_2)$. This implies by definition that there is $N \in \mathcal{K}$ and $\mathcal{K}$-embeddings $f_i : N_1 \to N$ for $i = 1, 2$ such that $f_i \restriction M = \mathrm{id}$ and $f_1(a) = f_2(b)$. Without loss of generality we can assume that $f_2 = \mathrm{id}$, so $f_1(a) = b$. Again by $(C4)$ we may assume that $|N| < \bar{\kappa}$. By model-homogeneity, there is $g : N \to \mathfrak{C}$ such that $g \restriction N_2 = \mathrm{id}$. Consider $g \circ f = h : N_1 \to \mathfrak{C}$. We have in particular that h is an isomorphism between N_1 and $h(N_1)$ such that $h \restriction M = \mathrm{id}$ and $h(a) = g(f(a)) = g(b) = b$. Then, using strong model-homogeneity, there is $\pi \in \mathrm{Aut}(\mathfrak{C})$ extending h, which is what we wanted. $\square$

The previous proposition gives us an alternative definition of Galois types in terms of the monster model automorphisms. This shows that a monster model convention constitutes an alternative way of acquiring the benefits of amalgamation. Moreover, such definition motivates an easy extension of Galois types for infinite sequences. Fixing enumerations for every subset of $\mathfrak{C}$, we define a relation $\sim$ between pairs (A, M) as follows. For sets A, B with respective enumerations $(a_i)_{i<\lambda}$ and $(b_i)_{i<\lambda}$, and for M a model, we let

$$(A, M) \sim (B, M)$$

if and only if there is $f \in \mathrm{Aut}(\mathfrak{C})$ such that f fixes M pointwise and $f(a_i) = b_i$ for all $i < \lambda$. That $\sim$ is an equivalence relation follows now by simple properties of automorphisms (for instance, transitivity follows by composing automorphisms). As before, we denote by $gt(A/M)$ the equivalence class $(A, M)/\sim$. It is easy to see that if $gt(A/M) = gt(B/M)$ and $D \subseteq \lambda$, then for $A_0 = \{a_i : i \in D\}$ and $B_0 = \{b_i : i \in D\}$, we have that $gt(A_0, M) = gt(B_0, M)$. For notational simplicity, we will often omit the enumeration and write $f(A) = B$ instead of $f(a_i) = b_i$ for all $i < \lambda$. It is important to notice that the definition of $S(M)$ will remain restricted to the set of Galois types of the form $gt(a/M)$ where a is a finite sequence (and the same goes for the Galois-saturation of our monster model, i.e., $\mathfrak{C}$ realizes every Galois type in $S(M)$ and not necessarily every Galois type of the form $gt(A/M)$).

2.2 The Decomposition Theorem

In this section we present a set of axioms from which the decomposition theorem is derived. We start with an axiom that defines an independence relation as a relation between triplets of subsets of $\mathfrak{C}$. We denote this relation by $A \underset{C}{\downarrow} B$ (in words: "A is free from B over M" or "A is independent from B over C").

Axiom 1 (Independence).

- [def] ***(Definition)*** $A \underset{C}{\downarrow} B \Leftrightarrow A \underset{C}{\downarrow} B \cup C$
- [tri] ***(Triviality)*** *If* $A \not\subseteq M$ *then* $A \underset{M}{\not\downarrow} A$
- [fin] ***(Finite Character)*** $A \underset{C}{\downarrow} B$ *if and only if* $A_0 \underset{C}{\downarrow} B_0$ *for all finite* $A_0 \subseteq A$ *and* $B_0 \subseteq B$.
- [mon] ***(Monotonicity)*** *Let* $C \subseteq C' \subseteq B$ *and* $B' \subseteq B$ *such that* $A \underset{C}{\downarrow} B$. *Then* $A \underset{C'}{\downarrow} B'$.
- [loc] ***(Local Character)*** *Let* $(M_\alpha : \alpha < \lambda)$ *be an* $\prec$*-increasing and continuous sequence of models such that* $\bigcup_{\alpha<\lambda} M_\alpha = M$. *Then, for every* a *there is* $\alpha < \lambda$ *such that* $a \underset{M_\alpha}{\downarrow} M$.
- [tra] ***(Transitivity)*** *Let* $M_0 \subseteq M_1 \subseteq N$. *Then* $A \underset{M_1}{\downarrow} N$ *and* $A \underset{M_0}{\downarrow} M_1 \Leftrightarrow A \underset{M_0}{\downarrow} N$.
- [sym] ***(Symmetry)*** $A \underset{M}{\downarrow} B \Leftrightarrow B \underset{M}{\downarrow} A$
- [inv] ***(Invariance)*** *Let* f *be an embedding with* $A \cup B \cup C \subseteq dom(f)$. *Then* $A \underset{C}{\downarrow} B \Leftrightarrow f(A) \underset{f(C)}{\downarrow} f(B)$

Having an independence relation allows us to define the concept of an independent set over a model analogously to the concept of an independent set of vectors in linear algebra:

Definition 2.16. *We say* $\{B_i : \alpha < \lambda\}$ *is an independent set (or iset) over* M *if for all* $i < \lambda$:

$$B_i \underset{M}{\downarrow} \bigcup \{B_j : j \neq i, j < \lambda\}$$

We extend this definition to cover a similar concept for trees of models. This concept is usually known as an independent system or (in Shelah's terminology) as a system in complete amalgamation ([11]). We will often use the expression *independent tree*, or simply *itree*.

Definition 2.17. *Let T be a rooted tree. We say $\langle M_\eta : \eta \in T\rangle$ is an independent tree (itree), if $M_\sigma \subseteq M_\tau$ whenever $\sigma \leq \tau$ and for all $\eta \in T$:*

$$M_\eta \underset{M_{\eta^-}}{\downarrow} \bigcup_{\eta \not\leq \sigma} M_\sigma$$

Let T be a tree. We say that $\langle M_\eta : \eta \in T\rangle$ is over N if $\bigcup_{\eta\in T} M_\eta \subseteq N$. For $U \subseteq T$, we let $M_U = \bigcup_{\eta\in U} M_\eta$. Notice that in this case M_U is not necessarily a model. In general, all the trees here considered will be rooted trees.

Another concept that will play a determinant role here is the concept of prime model. Actually, prime models and itrees are closely related. We first define what a prime model is.

Definition 2.18. *A structure $M \in \mathcal{K}$ is prime over A if for every $N \in \mathcal{K}$ containing A there is a $\mathcal{K}$-embedding $f : M \to N$ such that $f \upharpoonright A = id$.*

Remark 2.19. Notice that if M is prime over A and $f : M \to M'$ is an isomorphism fixing A pointwise, then M' is also prime over A. This is just because if N contains A and g is a $\mathcal{K}$-embedding from M to N fixing A pointwise, then $g \circ f^{-1} : M' \to N$ is a $\mathcal{K}$-embedding fixing A pointwise.

In our setting there are alternative ways to define the concept of prime model. For instance, one can say that a structure $M \in \mathcal{K}$ is prime over A if for every $f \in \mathrm{Aut}(\mathfrak{C})$ and $N \in \mathcal{K}$ containing $f(A)$ there is $g \in \mathrm{Aut}(\mathfrak{C})$ such that $f \upharpoonright A \subseteq g \upharpoonright M$ and $g(M) \subseteq N$. This is the content of the following proposition:

Proposition 2.20. *Let $M \in \mathcal{K}$ and $A \subseteq M$. The following are equivalent:*

(1) For every $N \in \mathcal{K}$ containing A there is a $\mathcal{K}$-embedding $f : M \to N$ such that $f \upharpoonright A = id$.

(2) For every $f \in Aut(\mathfrak{C})$ and $N \in \mathcal{K}$ containing $f(A)$ there is $g \in Aut(\mathfrak{C})$ such that $f \upharpoonright A \subseteq g \upharpoonright M$ and $g(M) \subseteq N$.

Proof: We show (1) implies (2). Let $f \in \text{Aut}(\mathfrak{C})$ and $N \in \mathcal{K}$ such that $f(A) \subseteq N$. By the previous remark, $f(M)$ is prime over $f(A)$ (as in definition 2.18). Thus there is a $\mathcal{K}$-embedding $g : f(M) \to N$ such that $g \restriction f(A) = \text{id}$. Thus $g \restriction f(M)$ is an isomorphism, so it extends to $h \in \text{Aut}(\mathfrak{C})$. Claerly $h(f(M)) \subseteq N$. Moreover, $h \circ f \restriction A = f \restriction A$, since $g \restriction f(A) = \text{id}$ and h extends g. Therefore, $h \circ f$ satisfies all we need. For the converse, let $N \in \mathcal{K}$ containing A. Then $\text{id} \in \text{Aut}(\mathfrak{C})$ and $\text{id}(A) \subseteq N$. Hence, by (2), there is $g \in \text{Aut}(\mathfrak{C})$ such that $\text{id} \restriction A \subseteq g \restriction M$ and $g(M) \subseteq N$. Then $g \restriction M$ is a $\mathcal{K}$-embedding such that $g \restriction A = \text{id}$. $\square$

If we deal with formulas, prime models can be defined in terms of elementary maps, i.e., maps that preserve formulas. Formally, for $A \subseteq M$, a map $f : A \to N$ is elementary if $M \vDash \phi(a) \Leftrightarrow N \vDash \phi(f(a))$ for all $a \in A$. A structure M is said to be prime over A if for every elementary map $f : A \to N$ there is an elementary embedding $f' : M \to N$ such that $f \subseteq f'$. All three definitions are equivalent in our framework. This will be used tacitly in section 3. We use the previous proposition to show the equivalence.

Proposition 2.21. *Let $M \in \mathcal{K}$ and $A \subseteq M$, where $\mathcal{K} = Mod(T)$ for some complete theory T. The following are equivalent:*

(1) For every $f \in Aut(\mathfrak{C})$ and $N \in \mathcal{K}$ containing $f(A)$ there is $g \in Aut(\mathfrak{C})$ such that $f \restriction A \subseteq g \restriction M$ and $g(M) \subseteq N$.

(2) For every elementary map $f : A \to N$ there is an elementary embedding $g : M \to N$ such that $f \subseteq g$.

Proof: Assume (1) and let f be an elementary map $f : A \to N$. Let $f' \supseteq f$ defined in the obvious way such that $f' : \langle A \rangle_M \to N$ is an elementary embedding. By the monster model convention, there is $g \in \text{Aut}(\mathfrak{C})$ such that $f' \subseteq g$. Then by (2), there is $h \in \text{Aut}(\mathfrak{C})$ such that $g \restriction A \subseteq h \restriction M$ and $h(M) \subseteq N$. Clearly h is an elementary embedding and $f \subseteq h$ and we are done. For the converse, let $f \in \text{Aut}(\mathfrak{C})$ and $N \in \mathcal{K}$ containing $f(A)$. Then $f \restriction A$ is an elementary map, so by (2), there there is an elementary embedding $g : M \to N$ such that $f \subseteq g$. Hence g extends to $h \in \text{Aut}(\mathfrak{C})$. Trivially $f \restriction A \subseteq h \restriction M$ and $h(M) \subseteq N$.$\square$

There are two properties we would like prime models to have. The first one corresponds to their existence for some special cases and the second one tackles the relation between prime models and the independence relation. Both are respectively expressed in the following two axioms.

Axiom 2 (Existence of Prime Models).

1. *There is a prime model over the $\emptyset$.*
2. *For $a \in N - M$, there is $M' \prec N$ prime over $M \cup a$ (usually also denote by $M(a)$).*
3. *If $\langle M_\eta : \eta \in T\rangle$ is an itree over N, then there is $M' \prec N$ prime over $\bigcup_{\eta \in T} M_\eta$.*

Axiom 3 (Dominance). [dom] *If $A \underset{M}{\downarrow} B$ and M_1 is prime over $M \cup B$ then $A \underset{M}{\downarrow} M_1$.*

Up to this point we have chosen almost exactly the same axioms as in Grossberg and Lessmann's paper [7] modulo the relativization to the subclass of primary models. As pointed our before, Grossberg and Lessmann have an axiom stating that primary models are unique up to isomorphism. Instead of that axiom, the following axiom will be added.[4]

Axiom 4 (Uniformity). *If $gt(A/M) = gt(B/M)$ and both $A \underset{M}{\downarrow} N$ and $B \underset{M}{\downarrow} N$, then $gt(A/N) = gt(B/N)$.*

We show some consequences of the above listed axioms. The following lemma is an extended version of the transitivity of independence.

Lemma 2.22 (Extended Transitivity). *Let $M \subseteq M'$ models. Then*

$$A \underset{M'}{\downarrow} B \wedge A \underset{M}{\downarrow} M' \Rightarrow A \underset{M}{\downarrow} B$$

Proof: By finite character we can assume B is finite, say b. Let $M(b)$ be prime over $M' \cup b$. By dominance we get $A \underset{M'}{\downarrow} M(b)$. Then, since $M \subseteq M' \subseteq M(b)$, by transitivity, we have that $A \underset{M}{\downarrow} M(b)$, hence by monotonicity $A \underset{M}{\downarrow} b$, what we wanted.[5] $\Box$

[4]In [10] the property expressed in the axiom is also called uniqueness.

[5]This is different from the axiom since we do not require B to be a model.

Hereafter, we will use the term transitivity ambiguously referring either to the axiom or to this lemma. A useful application of it is the following (for simplicity, we use the abbreviation ab for $a \cup b$):

Lemma 2.23. *Let $M \prec N$ and assume $ab \underset{M}{\downarrow} N$. Then $a \underset{M}{\downarrow} b \Leftrightarrow a \underset{N}{\downarrow} b$.*

Proof: For the left to right direction, let $M' \prec M''$ by such that M' is a prime model over $M \cup b$ and M'' is a prime model over $M \cup ab$. This is made possible by the following argument. Since M' contains $M \cup b$, there is a $\mathcal{K}$-embedding f from M' to M'' fixing $M \cup b$ pointwise. Then f is an isomorphism between M' and $f(M')$ fixing $M \cup b$ pointwise, so $f(M')$ is prime over $M \cup b$. Then $f(M')$ is the wanted prime model since trivially $f(M') \prec M_1$. For the left-to-right direction consider the following:

$$ab \underset{M}{\downarrow} N \overbrace{\Rightarrow}^{sym} N \underset{M}{\downarrow} ab \overbrace{\Rightarrow}^{dom} N \underset{M}{\downarrow} M'' \overbrace{\Rightarrow}^{mon} N \underset{M'}{\downarrow} a \overbrace{\Rightarrow}^{sym+def+mon} a \underset{M'}{\downarrow} Nb \tag{1}$$

Assuming $a \underset{M}{\downarrow} b$, by dominance we have that $a \underset{M}{\downarrow} M'$. Then by transitivity with 1, we get $a \underset{M}{\downarrow} Nb$, and the result follows by two applications of monotonicity. For the converse, by monotonicity and symmetry $ab \underset{M}{\downarrow} N$ implies $a \underset{M}{\downarrow} N$. Hence, assuming $a \underset{N}{\downarrow} b$, the result follows by transitivity. □

The next lemma will be often used. It corresponds to a kind of exchange property, and following [7] we call it concatenation.

Lemma 2.24 (Concatenation). *If $A \underset{M}{\downarrow} BC$ and $C \underset{M}{\downarrow} B$, then $C \underset{M}{\downarrow} BA$.*

Proof: By finite character we may assume B and C are finite. Let M_0 be prime over $M \cup B$ and M_1 prime over $M \cup B \cup C$ such that $M_0 \prec M_1$. Then we have the following chain of arguments:

$$A \underset{M}{\downarrow} BC \overbrace{\Rightarrow}^{dom} A \underset{M}{\downarrow} M_1 \overbrace{\Rightarrow}^{mon} A \underset{M_0}{\downarrow} C \overbrace{\Rightarrow}^{sym} C \underset{M_0}{\downarrow} A \overbrace{\Rightarrow}^{def} C \underset{M_0}{\downarrow} AM_0$$

On the other hand by dominance we have that $C \underset{M}{\downarrow} M_0$. Hence by transitivity we get $C \underset{M}{\downarrow} AM_0$, which implies by monotonicity $C \underset{M}{\downarrow} AB$. □

The last set of axioms is intended to capture the relation between types and the independence relation. This requires the extension of our definition of independence to types. To some degree, this defines a new 'independence' relation for types, called orthogonality, derived from the original independence relation.

Definition 2.25. *Let $p \in S(M)$.*

(1) Let $N \prec M$. We say p is independent from M over N, denoted by $p \underset{N}{\downarrow} M$, if $a \underset{N}{\downarrow} M$ for all a realizing p.

(2) p is stationary if for all N such that $M \prec N$, there is a unique extension $p_N \in S(N)$ of p such that $p_N \underset{M}{\downarrow} N$. We say p_N is a free extension of p.

(3) Let $q \in S(N)$. We say p is orthogonal to q, denoted by $p \perp q$, if $a \underset{M_1}{\downarrow} b$ for all M_1 containing $M \cup N$ and all $a \vDash p_{M_1}$ and $b \vDash q_{M_1}$.

(4) p is orthogonal to N, denoted by $p \perp N$, if $p \perp q$ for all $q \in S(N)$.

(5) If $M_0 \prec M_1, M_2$, we write $gt(M_1/M_0) \perp M_2$ if and only if $gt(a/M_0) \perp M_2$ for all $a \in M_1 - M_0$.

(6) Assume p is stationary. Then, p is regular if for all N containing M and $q \in S(N)$ extending p, either $q = p_N$ or $q \perp p$.

The previous definition contains a lot of tacit information. First, notice that in (1) it is equivalent to say *for all* or *for some*. This follows readily from invariance of independence. Second, if $M \subseteq N$ and $A \underset{M}{\downarrow} N$, then $gt(A/N)$ is the unique extension of $gt(A/M)$ such that $B \underset{M}{\downarrow} N$ for some B realizing it. By assumption, $gt(A/N)$ is such an extension. For the uniqueness, assume that there is another extension satisfying the property, say $gt(B/N)$. Then we have that $gt(A/N) \neq gt(B/N)$ but $gt(A/M) = gt(B/M)$ since $gt(B/N)$ extends $gt(A/M)$. Since $A \underset{M}{\downarrow} N$ and $B \underset{M}{\downarrow} N$, we have that $gt(A/N) = gt(B/N)$ by the uniformity axiom. Hence $gt(A/N)$ is unique. This allows us to use the same notation as in part (2) of the previous definition and write $gt(A/M)_N$ in case $A \underset{M}{\downarrow} N$. Notice that the same applies for finite sequences: if $M \subseteq N$ and $a \underset{M}{\downarrow} N$, then a realizes $gt(a/M)_N$ (or equivalently, $gt(a/N) = gt(a/M)_N$).

For the remaining axioms, which capture the desired behavior of our types, we use the same axioms as in [7]:

Axiom 5 (Existence of Stationary Types). *Let M be a model. Then $p \in S(M)$ is stationary.*

Axiom 6 (Existence of Regular Types). *If $M \subseteq N$ and $M \neq N$, then there exists a regular type $p \in S(M)$ realized in $N - M$.*

Axiom 7 (Perpendicularity). *Let $M \prec N$ and $p \in S(N)$ be regular. Then $p \perp M$ if and only if $p \perp q$ for all regular types $q \in S(M)$. Moreover $gt(M_1/M_0) \perp M_2$ if and only if for all regular types $p \in S(M_0)$ realized in M_1 we have that $p \perp M_2$.*

Axiom 8 (Equivalence). *Let $p, q \in S(M)$ be regular and $a \notin M$ realize p. Then q is realized in $M(a) - M$ if and only if $p \not\perp q$.*

Lemma 2.26 (Parallelism). *(i) Let $p \in S(M)$ and $q \in S(N)$. Then, $p \perp q$ if and only if $q \perp p$.*

(ii) Let $p \in S(M)$, $q \in S(N)$ and assume $M \prec N$. Then, $p \perp q$ if and only if $p_N \perp q$.

(iii) Let $p, q \in S(M)$ and $M \prec N$. Then, $p \perp q$ if and only if $p_N \perp q_N$.

Proof: (i) follows directly from symmetry. For (ii), assume first that $p \perp q$. Since $M \cup N = N$, we take $M_1 \supseteq N$ and a, b such that $a \models (p_N)_{M_1}$ and $b \models q_{M_1}$. Since $p_{M_1} = (p_N)_{M_1}$, by definition of $p \perp q$ we get $a \underset{M_1}{\downarrow} b$. For the converse, assume $p_N \perp q$ and let $M_1 \supseteq N$ and a, b such that $a \models p_{M_1}$ and $b \models q_{M_1}$. Again, using the fact that $p_{M_1} = (p_N)_{M_1}$, by definition of $p_N \perp q$ we get $a \underset{M_1}{\downarrow} b$. Finally, for (iii), the left to right direction follows from an analogous argument than in (2). For the converse, assume towards a contradiction $p_N \perp q_N$ and that there are $M_1 \supseteq M$ and a, b such that $a \models p_{M_1}$ and $b \models q_{M_1}$ but $a \underset{M_1}{\not\downarrow} b$. Let $N_1 \in \mathcal{K}$ containing $M_1 \cup N$. Consider the type $r = gt(ab/M_1)$. Let be r_{N_1} its unique free extension over M_1 and cd realize r_{N_1}. This implies in particular that $cd \models gt(ab/M_1)$, which implies that there is $f \in \mathrm{Aut}(\mathfrak{C})$ fixing M_1 pointwise and sending ab to cd. Hence, by invariance, $a \underset{M_1}{\not\downarrow} b$ implies $c \underset{M_1}{\not\downarrow} d$. But since r_{N_1} is free over M_1, by definition we have that $cd \underset{M_1}{\downarrow} N_1$. Hence by lemma 2.23 we get $a' \underset{N_1}{\not\downarrow} b'$, which contradicts $p_N \perp q_N$. $\square$

The next lemma shows that regularity is preserved over free extensions and over restrictions for independent subsets:

Lemma 2.27. *Let $p \in S(M)$ be a regular type and $M_0 \subseteq M \subseteq N$. Then*

(i) p_N is a regular type.

(ii) If $p \underset{M_0}{\downarrow} M$, then $p \upharpoonright M_0$ is a regular type.

Proof: For (i), let $M' \supseteq N \supseteq M$ and $q \in S(M')$ such that $q \neq (p_N)_{M'}$. Since $(p_N)_{M'} = p_{M'}$, by the regularity of p we have that $p \perp q$, so by parallelism $p_N \perp q$, and we are done. For (ii), assume towards a contradiction that $p \upharpoonright M_0$ is not regular. Therefore, there is $M' \supseteq M_0$ and $q \in S(M')$ such that $p' = (p \upharpoonright M_0)_{M'} \neq q$ and $p \not\perp q$. Then there is $M^* \supseteq M'$ and a, b such that

$$a \models p'_{M^*} \qquad b \models q_{M^*} \qquad a \underset{M^*}{\not\downarrow} b \tag{2}$$

Let M'' be a model containing M and M^*. Consider the type $gt(ab/M^*)$ and let $cd \models gt(ab/M^*)_{M''}$. By definition we have that

$$cd \underset{M^*}{\downarrow} M'' \tag{3}$$

Moreover, since $cd \models gt(ab/M^*)$, by definition and invariance of independence we also have that:

$$c \models p'_{M^*} \qquad d \models q_{M^*} \qquad c \underset{M^*}{\not\downarrow} d \tag{4}$$

Claim 1: $c \underset{M''}{\not\downarrow} d$. Assume towards a contradiction that $c \underset{M''}{\downarrow} d$. By (3), we have that $c \underset{M^*}{\downarrow} M''$, so by transitivity we get that $c \underset{M^*}{\downarrow} b$, contradicting (4).

Claim 2: $p'_{M''} = p_{M''}$. Again, by (3) we have that $c \underset{M^*}{\downarrow} M''$ and by (4) we have that $c \underset{M'}{\downarrow} M^*$. Then by transitivity we get $c \underset{M'}{\downarrow} M''$. Moreover, since

$c \vDash p'$, we also have that $c \underset{M_0}{\downarrow} M'$, so again by transitivity we have $c \underset{M_0}{\downarrow} M''$. By monotonicity this implies that $c \underset{M_0}{\downarrow} M$. Therefore we have that $c \vDash (p \restriction M_0)_M$. Since by assumption $p \underset{M_0}{\downarrow} M$, we have that $p = (p \restriction M_0)_M$, hence $c \vDash p$. Notice that from $c \underset{M_0}{\downarrow} M''$ we also get that $c \underset{M}{\downarrow} M''$ by monotonicity, so $c \vDash p_{M''}$. Hence $p'_{M''} = gt(c/M'') = p_{M''}$.

Claim 3: $q_{M''} \neq p_{M''}$. Assume that $q_{M''} = p_{M''}$. By claim 2, this is $q_{M''} = p'_{M''}$. Then we must have that $q_{M''} \restriction M' = p'_{M''} \restriction M'$, which contradicts our assumption, since $p'_{M''} \restriction M' = p'$ and $q_{M''} \restriction M' = q$.

Claims 1-3 complete the proof since they explicitly contradict the regularity of p. $\square$

The next lemma shows that under the equivalence axiom the relation $p \not\perp q$ is an equivalence relation between regular types over the same model.

Lemma 2.28. *$\not\perp$ is an equivalence relation between regular types in $S(M)$*

Proof: Let $p, q, r \in S(M)$ be regular types. Reflexivity follows from triviality of independence and symmetry follows from parallelism (lemma 2.26 part (i)). For transitivity, assume $p \not\perp q$ and $q \not\perp r$. Let $a \in M$ realize p. By the Equivalence axiom it is sufficient to prove that r is realized in $M(a) - M$. Since $p \not\perp q$, by Equivalence q is realized in $M(a) - M$, so let $b \in M(a) - M$ realize q. Since $b \notin M$, by equivalence and the fact that $q \not\perp r$, we have that r is realized in $M(b) - M$. But since $b \in M(a)$, $M(b) \subseteq M(a)$, hence r is realized in $M(a) - M$. $\square$

The following is a desired consequence of having $\not\perp$ as an equivalence relation:

Lemma 2.29. *Let $M_0 \subseteq M_1 \subseteq M_2$, $p \in S(M_2)$, $q \in S(M_1)$ and $r \in S(M_0)$ be regular types such that $p \not\perp q$. Then, $p \perp r \Rightarrow q \perp r$.*

Proof: By lemma 2.27, both q_{M_2} and r_{M_2} are regular. By parallelism, $p \not\perp q$ implies $p \not\perp q_{M_2}$. We prove the claim by contraposition, so assume that $q \not\perp r$. Again, parallelism implies $q_{M_2} \not\perp r_{M_2}$. But the previous lemma says $\not\perp$ is an equivalence relation, so by transitivity, $p \not\perp q_{M_2}$ and $q_{M_2} \not\perp r_{M_2}$ imply $p \not\perp r_{M_2}$, which by parallelism implies $p \not\perp r$. $\square$

Lemma 2.30. *Let $p = gt(a/M)$ be a regular type such that $p \perp N$ for $N \subseteq M$. Then $gt(M(a)/M) \perp N$.*

Proof: Remember $gt(M(a)/M) \perp N$ if $q \perp N$ for all $q \in S(M)$ which are realized in $M(a) - M$. By the perpendicularity axiom, it is sufficient to show that $q \perp r$ for all regular $r \in S(N)$. Assume towards a contradiction that there is $r \in S(N)$ such that $q \not\perp r$. First, parallelism implies that $q \not\perp r_M$. By Equivalence, since q is realized in $M(a) - M$, we have that $q \not\perp p$. Hence by transitivity of $\not\perp$ we have that $p \not\perp r_M$. Thus, again by parallelism we have $p \not\perp r$, contradicting the fact that $p \perp N$. $\square$

Lemma 2.31. *Let $M_0 \subseteq M$ and assume $gt(A_i/M) \perp M_0$ for $i = 1, 2$, $A_1 \underset{M}{\downarrow} A_2$ and $B \underset{M_0}{\downarrow} M$. Then $A_1A_2 \underset{M}{\downarrow} B$.*

Proof: By finite character, assume A_1, A_2 and B are finite, say a_1, a_2 and b. Notice that by uniqueness of free extensions, our assumption $b \underset{M_0}{\downarrow} M$ implies that $gt(b/M_0)_M = gt(b/M)$, or in another notation $b \vDash gt(b/M_0)_M$. Let M' be prime over $M \cup a_2$. Consider $gt(b/M_0)$. First, $gt(a_i/M) \perp M_0$ implies by definition $gt(a_i/M) \perp gt(b/M_0)$, and again by definition we get $a_2 \underset{M}{\downarrow} b$ (since $M_0 \subseteq M$)). By symmetry and dominance we get $b \underset{M}{\downarrow} M'$. By uniqueness, the latter implies that $b \vDash gt(b/M)_{M'}$. Second, $a_1 \underset{M}{\downarrow} a_2$ implies $a_1 \underset{M}{\downarrow} M'$ by dominance, so $a_1 \vDash gt(a_1/M)_{M'}$. By assumption, $gt(a_1/M) \perp gt(b/M_0)$, so $a_1 \underset{M'}{\downarrow} b$ (since $M_0 \subseteq M \subseteq M'$). Then by definition and monotonicity $a_1 \underset{M'}{\downarrow} ba_2$. Using $a_1 \underset{M}{\downarrow} M'$, transitivity implies $a_1 \underset{M}{\downarrow} ba_2$, and the desired result follows by concatenation and symmetry. $\square$

Corollary 2.32. *Let $M_0 \subseteq M \subseteq N_i$ for $i = 1, 2$. Assume that $gt(N_i/M) \perp M_0$ for $i = 1, 2$, and that $N_1 \underset{M}{\downarrow} N_2$. Then $gt(N_1N_2/M) \perp M_0$.*

Proof: Suppose not. Then let $q \in S(M_0)$ and $M_1 \supseteq M$ such that $ab \vDash gt(n_1n_2/M)_{M_1}$ (for n_1n_2 a finite sequence from N_1N_2), $c \vDash q_{M_1}$ but $ab \underset{M_1}{\not\downarrow} c$. In particular we have that $gt(ab/M) = gt(n_1n_2/M)$ and since $N_1 \underset{M}{\downarrow} N_2$, by finite character we also have that $n_1 \underset{M}{\downarrow} n_2$, which by invariance implies

$a \underset{M}{\downarrow} b$. Moreover, we have that $ab \underset{M}{\downarrow} M_1$, so by concatenation we get $a \underset{M}{\downarrow} bM_1$, and by two applications of monotonicity $a \underset{M_1}{\downarrow} b$. On the other hand, since $a \vDash gt(n_1/M)_{M_1}$ and $b \vDash gt(n_2/M)_{M_1}$, we have by parallelism that $gt(a/M_1) \perp M_0$ and $gt(b/M_1) \perp M_0$. Since, $c \vDash q_{M_1}$ we have $c \underset{M}{\downarrow} M_1$. Hence, by the previous lemma we have that $ab \underset{M_1}{\downarrow} c$, which is a contradiction. $\square$

Corollary 2.33. *Let $M_0 \subseteq M \subseteq N_i$ for $i \leq n$. Assume that $gt(N_i/M) \perp M_0$ for $i \leq n$, and that $(N_i : i \leq n)$ is an iset over M. Then $gt(\bigcup_{i \leq n} N_i/M) \perp M_0$.*

Proof: By induction on n. If $n = 0$ there is nothing to prove. Assume the claim for all $k < n$. So by induction hypothesis we have that $gt(\bigcup_{i \leq k} N_i/M) \perp M_0$ and by assumption that $gt(N_n/M) \perp M_0$. Also by assumption, we have that $N_i \underset{M}{\downarrow} \bigcup_{i \leq k} N_i$, so the result follows by the previous corollary. $\square$

Lemma 2.34. *Let $\langle M_\eta : \eta \in T \rangle$ be a tree satisfying:*

1. *$(M_\sigma : \sigma^- = \eta, \sigma \in T)$ is independent over M_η for all $\eta \in T$*
2. *$gt(M_\eta/M_{\eta^-}) \perp M_{\eta^{--}}$ for all η such that η^{--} exists.*

then $\langle M_\eta : \eta \in T \rangle$ is an itree.

Proof: By finite character we may assume T is finite. We proceed by induction on $|T|$. We fix $\eta \in T$ and we prove

$$M_\eta \underset{M_{\eta^-}}{\downarrow} \bigcup_{\eta \not\leq \sigma} M_\sigma$$

Notice that if $ht(\eta) \leq 1$ for all $\eta \in T$, the claim follows from condition (1). This covers the base case of the induction. For the successor, notice that we can assume that η is a leaf, since if it is not, the result follows by induction hypothesis for a tree $T' = T - \{\sigma\}$ where σ has maximal length and $\eta < \sigma$. So we split in cases assuming η is a leaf.

• **Case 1:** Assume that for all $\sigma > \eta^-$, $\sigma^- = \eta^-$ (i.e., every element above η^- has the same height). Notice that if η^{--} does not exists this case reduces again to the case where the height of every element is less than 1, which follows by condition (1). Let $U = \{\rho : \rho^- = \eta^-, \rho \neq \eta\}$. By condition (1) we have that

$$M_\eta \underset{M_{\eta^-}}{\downarrow} M_U$$

By condition (2) and corollary 2.33 we have both

$$gt(M_\eta / M_{\eta^-}) \perp M_{\eta^{--}} \text{ and } gt(M_U / M_{\eta^-}) \perp M_{\eta^{--}}$$

Moreover by induction hypothesis we have that

$$M_{\eta^-} \underset{M_{\eta^{--}}}{\downarrow} \bigcup_{\eta^- \not\leq \sigma} M_\sigma$$

Hence by lemma 2.31, we have that

$$M_\eta M_U \underset{M_{\eta^{--}}}{\downarrow} \bigcup_{\eta^- \not\leq \sigma} M_\sigma$$

and the result follows by concatenation, symmetry and monotonicity.

• **Case 2:** Assume there is $\sigma > \eta^-$ such that $\sigma^- \neq \eta^-$ (notice that this implies the existence of σ^{--}). Take such a σ of maximal length. Let $U = \{\rho : \rho \neq \eta, \rho \neq \sigma\}$. By induction hypothesis we have that

$$M_\eta \underset{M_{\eta^-}}{\downarrow} M_U \text{ and } M_\sigma \underset{M_{\sigma^-}}{\downarrow} M_U \tag{$*$}$$

Let $\overline{M_U}$ be a prime model over M_U (which exits by induction hypothesis and the existence axiom). Then by dominance we have both

$$M_\eta \underset{M_{\eta^-}}{\downarrow} \overline{M_U} \text{ and } M_\sigma \underset{M_{\sigma^-}}{\downarrow} \overline{M_U}$$

By condition (2) we have that $gt(M_\sigma / M_{\sigma^-}) \perp M_{\sigma^{--}}$, which implies by stationarity and parallelism that $gt(M_\sigma / \overline{M_U}) \perp M_\sigma^{--}$. Moreover since $\sigma^{--} \in U$ and $\sigma^{--} \geq \eta^-$, by monotonicity we also have that

$$M_\eta \underset{M_{\sigma^{--}}}{\downarrow} \overline{M_U}$$

This shows that $gt(M_\eta / \overline{M_U}) = gt(M_\eta / M_{\sigma^{--}})_{\overline{M_U}}$, so by the definition of orthogonality we can conclude that

$$M_\sigma \underset{\overline{M_U}}{\downarrow} M_\eta$$

Finally, by two applications of monotonicity and symmetry we get

$$M_\eta \underset{M_U}{\downarrow} M_\sigma M_U$$

and the result follows by transitivity. $\square$

The following definition will play a crucial role in the decomposition theorem. It was introduced by Shelah in [11].

Definition 2.35. *$\mathcal{K}$ has the NDOP (non-dimensional order property) if for every $M_0, M_1, M_2 \in \mathcal{K}$ such that $M_1 \underset{M_0}{\downarrow} M_2$ the following holds: for all M' prime over $M_1 \cup M_2$, and for every regular type $p \in S(M')$ either $p \not\perp M_1$ or $p \not\perp M_2$.*

Before proving the main consequence of the NDOP we need the following lemma, which uses heavily the uniformity axiom:

Lemma 2.36. *Let $\langle M_\eta : \eta \in T\rangle$ be an independent system and $(\xi_i : i \leq \alpha)$ be an enumeration of T such that $\xi_i < \xi_j$ implies $i < j$. Then there is a sequence of models $(N_i : i \leq \alpha)$ with the following properties:*

(i) $N_0 = M_r$, *where r is the root of T.*

(ii) N_{i+1} *is prime over* $N_i \cup M_{\xi_i}$

(iii) $N_\delta = \bigcup_{i<\delta} N_i$

(iv) N_i *is prime over* $\bigcup_{j<i} M_{\xi_j}$

Proof: We construct the sequence by induction. Set $N_0 = M_r$. Assume that we have defined the sequence up to i. Then, since $\langle M_\eta : \eta \in T\rangle$ is an independent system we have that

$$M_{\xi_i} \underset{M_{\xi_i^-}}{\downarrow} \bigcup_{\xi_i \not\leq \sigma} M_\sigma$$

Since $\bigcup_{j<i} M_{\xi_j} \subseteq \bigcup_{\xi_i \not\leq \sigma} M_\sigma$ by monotonicity we get

$$M_{\xi_i} \underset{M_{\xi_i^-}}{\downarrow} \bigcup_{j<i} M_{\xi_j}$$

So by induction hypothesis and dominance we have that

$$M_{\xi_i} \underset{M_{\xi_i^-}}{\downarrow} N_i$$

Then the triple $(M_{\xi_i}, N_i, M_{\xi_i^-})$ is an independent system, so by the existence of prime models axiom, we let N_{i+1} be prime over $N_i \cup M_{\xi_i}$. For i a limit ordinal we define N_i as the union of the previous N_i exactly as in condition (iii). We prove now condition (iv) for the successor step. We actually prove a stronger result in order to let the induction follow for the limit case. Let $M \supseteq \bigcup_{j<i+1} M_{\xi_j}$, and by induction hypothesis let $G \in \mathrm{Aut}(\mathfrak{C})$ such that $G \restriction \bigcup_{j<i} M_{\xi_j} = \mathrm{id}$ and $G(N_i) \subseteq M$. We show that for any such G there is $F \in \mathrm{Aut}(\mathfrak{C})$ such that

$$G \restriction N_i = F \restriction N_i \qquad F \restriction \bigcup_{j<i+1} M_{\xi_j} = \mathrm{id} \qquad G(N_{i+1}) \subseteq M$$

First, since G fixes M_{ξ_i}, and $M_{\xi_i^-} \subseteq M_{\xi_i}$ (notice that ξ_i^- corresponds to the predecessor of ξ_i in the order of T), then

$$gt(M_{\xi_i}/M_{\xi_i^-}) = gt(G(M_{\xi_i})/M_{\xi_i^-})$$

Moreover, by induction hypothesis N_i is prime over $\bigcup_{j<i} M_{\xi_j}$ and G fixes $\bigcup_{j<i} M_{\xi_j}$, we have that $G(N_i)$ is prime over $\bigcup_{j<i} M_{\xi_j}$. Then, by assumption and monotonicity we have that

$$M_{\xi_i} \underset{M_{\xi_i^-}}{\downarrow} \bigcup_{j<i} M_{\xi_j}$$

First by dominance we have that

$$M_{\xi_i} \underset{M_{\xi_i^-}}{\downarrow} G(N_i)$$

Also by dominance with N_i and invariance it follows that

$$G(M_{\xi_i}) \underset{M_{\xi_i^-}}{\downarrow} G(N_i)$$

Hence by the uniformity axiom we may conclude that

$$gt(M_{\xi_i}/G(N_i)) = gt(G(M_{\xi_i})/G(N_i))$$

This implies that there is $G' \in \mathrm{Aut}(\mathfrak{C})$ such that $G \restriction G(N_i) = \mathrm{id}$ and $G'(M_{\xi_i}) = G(M_{\xi_i})$. Consider $G^* = G'^{-1} \circ G$. First notice that since N_{i+1} is prime over $N_i \cup M_{\xi_i}$, then $G^*(N_{i+1})$ is prime over $G^*(N_i \cup M_{\xi_i}) = G(N_i) \cup M_{\xi_i}$. Then, since $G(N_i) \cup M_{\xi_i} \subseteq M$, let $G'' \in \mathrm{Aut}(\mathfrak{C})$ such that

$$G'' \restriction G(N_i) \cup M_{\xi_i} = \mathrm{id} \qquad\qquad G''(G^*(N_{i+1})) \subseteq M$$

We claim $F = G'' \circ G^*$ satisfies the desired properties. First, we have that

$$\begin{aligned} F \restriction \bigcup_{j<i+1} M_{\xi_j} &= G'' \circ G'^{-1} \circ G \restriction \bigcup_{j<i+1} M_{\xi_j} \\ &= G'' \circ G'^{-1} \restriction G(M_{\xi_i}) \cup \bigcup_{j<i} M_{\xi_j} \\ &= G'' \restriction M_{\xi_i} \cup \bigcup_{j<i} M_{\xi_j} = \mathrm{id} \end{aligned}$$

Also,

$$F \restriction N_i = G'' \circ G'^{-1} \circ G \restriction N_i = G'' \circ G'^{-1} \restriction G(N_i) = G'' \restriction G(N_i) = G \restriction N_i$$

Finally,

$$F(N_{i+1}) = G''(G^*(N_{i+1})) \subseteq M$$

What remains is to show condition (iv) holds for the limit case. Assume δ is a limit ordinal, and let $M \supseteq \bigcup_{j<\delta} M_{\xi_j}$. Then by the stronger result proved in the successor case, we may assume that we have functions $F_i \in \mathrm{Aut}(\mathfrak{C})$ such that

(a) If $i < j < \delta$, then $F_i \restriction N_i = F_j \restriction N_i$.

(b) $F_i \restriction \bigcup_{j<i} M_{\xi_j} = \mathrm{id}$.

(c) $F_i(N_i) \subseteq M$.

Properties (a)-(c) show that $F = \bigcup_{i<\delta} F_i \restriction N_i$ is well defined and that it is an isomorphism from N_δ to $\bigcup_{i<\delta} F(N_i)$. By strong model-homogeneity, let $F' \in \mathrm{Aut}(\mathfrak{C})$ extending F. Clearly, F' satisfies all the needed properties. □

Theorem 2.37. *Assume $\mathcal{K}$ has the NDOP. Let $(M_i : i \in T)$ be an independent system and $M \in \mathcal{K}$ prime over M_T. Let $p = gt(a/M)$ be regular. Then there is $\eta \in T$ such that $p \not\perp M_\eta$.*

Proof: Let $(\xi_i)_{i\leq\alpha}$ be an enumeration of T and $(N_i)_{i\leq\alpha}$ its corresponding chain of models satisfying all the properties listed in lemma 2.36. Since N_α contains M_T, and M is prime over M_T, without loss of generality we can assume that $N_\alpha \subseteq M$ (this is because there is $G \in \mathrm{Aut}(\mathfrak{C})$ such that $G \restriction M_T = \mathrm{id}$ and $G(N_\alpha) \subseteq M$, so the chain $(G(N_i))_{i\leq\alpha}$ also satisfies all the properties listed in lemma 2.36). By parallelism it suffices to show that $p_{N_\alpha} \not\perp M_\eta$ for some $\eta \in T$. Let $p' = p_{N_\alpha}$. Let $i \leq \alpha$ be the least i such that $p' \not\perp N_i$ (there is at least one, namely α, by triviality of independence). Let $q \in S(N_i)$ be a regular type such that $p' \not\perp q$. We first show that i cannot be a limit ordinal. Assume towards a contradiction it is. Then, by local character we have that $q \underset{N_j}{\downarrow} N_i$ for some $j < i$, so there is $q' \in S(N_j)$ such that $q'_{N_i} = q$. Hence, by parallelism we have that $p' \not\perp q'$, which contradicts the minimality of i. Then we have that $i = k+1$. By assumption and monotonicity we have that

$$M_{\xi_k} \underset{M_{\xi_k^-}}{\downarrow} \bigcup_{j<k} M_{\xi_j}$$

By lemma 2.36, N_i is prime over $\bigcup_{j<k} M_{\xi_j}$, so by dominance

$$M_{\xi_k} \underset{M_{\xi_k^-}}{\downarrow} N_k$$

But we also have that N_i is prime over $M_{\xi_k} \cup N_k$, so by the NDOP we have either

$$q \not\perp M_{\xi_k} \text{ or } q \not\perp N_k$$

If the latter is true, since $\not\perp$ is an equivalence relation and parallelism, we have that $p' \not\perp N_k$ contradicting the minimality of i. If the former is true, by the same reason we get $p' \not\perp M_{\xi_k}$ which is what we wanted. $\square$

Below is the central concept of this section, namely, the definition of a decomposition for a model.

Definition 2.38. *We say that that $\langle M_\eta, a_\eta : \eta \in T\rangle$ is a decomposition of M if it satisfies the following properties*

(1) $ht(T) \leq \omega$.

(2) $\langle M_\eta : \eta \in T\rangle$ is a tree over M.

(3) *If η^{--} exists, then $gt(M_\eta/M_{\eta^-}) \perp M_{\eta^{--}}$.*

(4) *For all $\eta \in T$, $(M_\sigma : \sigma^- = \eta, \sigma \in T)$ is independent over M_η.*

(5) *Let r be the root of T. Then M_r is prime over $\emptyset$. Moreover, M_η is prime over $M_{\eta^-} \cup a_\eta$.*

(6) *If η^- exists, $gt(a_\eta/M_{\eta^-})$ is a regular type.*

(7) *For all $\eta \in T$, $\rho^- = \eta$ and $\sigma^- = \eta$, either $gt(a_\rho/M_\eta) = gt(a_\sigma/M_\eta)$ or $gt(a_\rho/M_\eta) \perp gt(a_\sigma/M_\eta)$.*

Condition (7) corresponds to the condition added to the original definition of decomposition in [7]. Notice that for r the root of T, the element a_r does not play any role in our definition (in Shelah's words: "a_r is immaterial", [11], p. 565). Following the strategy adopted in [7], the proof of the decomposition theorem uses an application of Zorn's lemma with respect to a partial ordering for decompositions of a fixed model. We will first introduce this partial order and prove that it is closed under chains, setting the requirement for the application of Zorn's lemma. Fix a model $M \in \mathcal{K}$. Let $\mathcal{D}$ be the set of all decompositions of M. Consider the following order on $\mathcal{D}$:

$$\langle M_\eta, a_\eta : \eta \in T_1\rangle \leq_{\mathcal{D}} \langle N_\sigma, b_\sigma : \sigma \in T_2\rangle$$

if and only if T_1 is a subtree of T_2 and $M_\eta = N_\eta, a_\eta = b_\eta$ and for all $\eta \in T_1$.

Lemma 2.39. $(\mathcal{D}, \leq_{\mathcal{D}})$ *is not empty and is closed under chains.*

Proof: Let M_r be the prime model over $\emptyset$ (which exists by existence of prime models). As a_r does not play any role, it follows readily that for any $a \in \mathfrak{C}$, $\langle M_r, a\rangle$ is a decomposition over M. This shows that $\mathcal{D} \neq \emptyset$. Now, let $(\mathcal{S}_i : i < \delta)$ be a $\leq_{\mathcal{D}}$-chain of decompositions of M (for δ a limit ordinal), where $\mathcal{S}_i = \langle M^i_\eta, a^i_\eta : \eta \in T_i\rangle$. First it follows by lemma 1.3 that $T = \bigcup_{i<\alpha} T_i$ is a tree. Consider $\langle M_\eta, a_\eta : \eta \in T\rangle$ defined by $M_\eta = M^i_\eta$ and $a_\eta = a^i_\eta$ if $\eta \in T_i$ (which is well defined by the definition of $\leq_{\mathcal{D}}$ and the assumption that $(\mathcal{S}_i : i < \delta)$ is a chain). It remains to show that $\langle M_\eta, a_\eta : \eta \in T\rangle$ is a decomposition of M. By corollary 1.4 we have that $ht(T) \leq \omega$. Assume that $\langle M_\eta : \eta \in T\rangle$ is not a tree over M, then there will be some $\langle M^i_\eta : \eta \in T^i\rangle$ that would not be a tree over M, which is a contradiction. An analogous argument shows properties (3), (5), (6) and (7). To show (4), assume there is $\sigma \in T$ such that the set $\langle M_\sigma : \eta^- = \sigma, \eta \in T \rangle$ is not independent over M_σ. Then by finite character, there is a finite subset $B \subseteq T$ such that $\langle M_\sigma : \eta^- = \sigma, \eta \in B \rangle$ is not independent over M_σ. But then, there must be T^i containing both σ and B, which contradicts that $\mathcal{S}_i$ is a decomposition over M. $\square$

Before proving the decomposition theorem we define what it means for a model M to be minimal over A.

Definition 2.40. *A model M is minimal over A if prime models exist over A and whenever $N \subseteq M$ is prime over A, then $N = M$.*

We finally have all the ingredients to prove the main result of this section:

Theorem 2.41 (Decomposition Theorem). *Suppose $\mathcal{K}$ has the NDOP and satisfies axioms 1-8. Then for every $M \in \mathcal{K}$ there is a decomposition $\langle M_\eta, a_\eta : \eta \in T \rangle$ of M such that M is prime and minimal over $\bigcup_{\eta \in T} M_\eta$.*

Proof: Fix $M \in \mathcal{K}$. By the previous lemma and Zorn's lemma applied to $(\mathcal{D}, \leq_\mathcal{D})$, let $\langle M_\eta, a_\eta : \eta \in T \rangle$ be a maximal decomposition of M. By lemma 2.34 $\langle M_\eta : \eta \in T \rangle$ is an itree, so by the existence of prime models axiom, let $M' \subseteq M$ be a prime model over $\bigcup_{\eta \in T} M_\eta$. We show that $M = M'$. Suppose not. Then, by the axiom of existence of regular types, let $a \in M - M'$ such that $p = gt(a/M')$ is a regular type. By theorem 2.37, there is $\eta \in T$ such that $p \not\perp M_\eta$. We choose η of minimal length. Hence by the perpendicularity axiom, there is a regular type $q \in S(M_\eta)$ such that $p \not\perp q$. Without loss of generality we can assume either that $q = gt(a_\rho/M_\eta)$ for some ρ such that $\rho^- = \eta$ or that $q \perp gt(a_\rho/M_\eta)$ for all ρ such that $\rho^- = \eta$. This is because if $p \not\perp gt(a_\rho/M_\eta)$ for some ρ such that $\rho^- = \eta$, then we take $q = gt(a_\rho/M_\eta)$; if $p \perp gt(a_\rho/M_\eta)$ for all ρ such that $\rho^- = \eta$, then for q such that $p \not\perp q$ we also have that $q \perp gt(a_\rho/M_\eta)$ for all such ρ by lemma 2.29. Let $q_{M'} \in S(M')$ be the unique extension of q free over M_η, that is, $q_{M'} \underset{M_\eta}{\downarrow} M'$. By equivalence, there is $b \in M - M'$ such that b realizes $q_{M'}$. Now, let $M(b) \subseteq M$ be the prime model over $M_\eta \cup b$. Let T' be $T \cup \{\eta'\}$, where η' is a single new successor of η. We claim that $\langle M_\eta, a_\eta : \eta \in T' \rangle$ is a decomposition of M, where $M_{\eta'} = M(b)$ and $a_{\eta'} = b$. Conditions (1),(2), (5) and (6) are trivially satisfied. Condition (7) is satisfied by the choice of q. For condition (3), assume that η^- exists. By the minimal choice of η, we have that $p \perp M_{\eta^-}$. Since $p \not\perp gt(b/M_\eta)$, by lemma 2.29 and perpendicularity we have that $gt(b/M_\eta) \perp M_{\eta^-}$. Finally, by lemma 2.30 the latter implies that $gt(M(b)/M_\eta) \perp M_{\eta^-}$, which is what we wanted. For condition (4), since $q_{M'} \underset{M_\eta}{\downarrow} M'$, we have that $b \underset{M_\eta}{\downarrow} M'$. Hence, by dominance, we have $M(b) \underset{M_\eta}{\downarrow} M'$. Monotonicity implies then

$$M(b) \underset{M_\eta}{\downarrow} \bigcup \{M_\nu : \nu^- = \eta, \nu \in T\}$$

But, since $\{M_\nu : \nu^- = \eta, \nu \in T\}$ is independent over M_η (by the definition of decomposition), concatenation implies that $\{M_\nu, M(b) : \nu^- = \eta, \nu \in T\}$ is independent over M_η, which establishes condition (4). This contradicts the maximality of our initial decomposition. □

Notice that the proof of the previous theorem reveals more than what the theorem states. We showed the existence of a decomposition for M such that M was prime and minimal over it, but also this is shown to be true for any maximal decomposition for M; it was Zorn's lemma that guaranteed the existence of such a maximal decomposition. We state this result as a corollary.

Corollary 2.42. *For every maximal decomposition $\langle M_\eta, a_\eta : \eta \in T \rangle$ of M, M is prime and minimal over $\bigcup_{\eta \in T} M_\eta$.*

Proof: The same proof as in the previous theorem. □

3 An Application to Finite Diagrams

In his PhD thesis [9], Olivier Lessman extended a number of results in stability theory to different non-elementary classes. The development and improvement of those results are part of a sequence of papers [10], [6], [7] (some of them co-authored with his PhD thesis supervisor Rami Grossberg), reaching a proof of the main gap theorem for the class of $(D, \aleph_0)$-homogeneous models of a totally transcendental good diagram D (also in [7]). The second part of this thesis is intended as a small contribution to that work. In a section of [11] called "For Thomas the Doubter", Shelah proved that any two models of cardinality λ of a superstable first-order theory with the NDOP which are $L_{\infty,\lambda}$-equivalent, are isomorphic (for a suitable big λ). The objective here is to show that the same result can be lifted to the class of $(D, \aleph_0)$-homogeneous models of a totally transcendental good diagram D.

In section 3.1, diagrams are introduced and the main properties of the class $\mathcal{K}$ of $(D, \aleph_0)$-homogeneous models of a totally transcendental diagram D are stated. The satisfaction of the axioms presented in section 2.2 for $\mathcal{K}$ will not be proved since this corresponds essentially to what Lessmann and Grossberg achieved in [10] and the second part of [7]. Finally, in section 3.2 a proof of the above-mentioned theorem will be outlined for $\mathcal{K}$.

3.1 Finite Diagrams

Given a first-order theory T and a model M of T, the finite diagram of M is the set of complete types over the empty set realized in M. In general, a finite diagram D is a subset $D \subseteq S(T)$. For a given finite diagram D, we will be interested in the class of models of T such that their finite diagram is a subset of D. In other words, the class of models that omits all types which are not in D. The following is a formal definition:

Definition 3.1. *Let T be a complete first-order theory in a language L. Let M a model of T, $A \subseteq M$ and $D \subseteq S(T)$. Then*

1. *The finite diagram of A is the set $D(A) = \{t(a/\emptyset) : a \in A\}$*
2. *The set A is a D-set if $D(A) \subseteq D$. Equivalently M is a D-model if $D(M) \subseteq D$.*

3. A type $p \in S_n(A)$ is a D-type over A if and only if $A \cup a$ is a D-set for some a realizing p.

4. $S_D(A)$ is the set of all D-types over A.

Thus, given a finite diagram D, we are interested in the class of D-models. The next step is to introduce homogeneity. Remember that a model M is λ-homogeneous if, whenever $A \subseteq M$ with $|A| < \lambda$, $f : A \to M$ is an elementary map and $a \in M$, there is an elementary map $f' : A \cup a \to M$ such that $f \subseteq f'$.

Definition 3.2. *M is a (D, λ)-homogeneous model if M is a is λ-homogeneous D-model and it realizes every type in $S_D(A)$ for all $A \subseteq M$ such that $|A| < \lambda$.*

The definition of (D, λ)-homogeneous models captures both classical homogeneity and saturation for the corresponding class of types. A (D, λ)-model corresponds to a D-model that realizes every type in $S_D(A)$ for all $A \subseteq M$ of cardinality less than λ. In our previous section, amalgamation was used to settle our work in a homogeneous and saturated monster model. In finite diagrams this corresponds to the meaning of "good". Formally:

Definition 3.3. *A class $\mathcal{K}$ of finite diagrams is good if, for a large enough cardinal $\bar{\kappa}$, there are $(D, \bar{\kappa})$-homogeneous models $\mathfrak{C} \in \mathcal{K}$.*

Hence, as before, we will work with a monster model convention as 2.14. It remains for us to explain the meaning of "totally transcendental". As in classical stability theory, this is related to a concept of rank. Here we use a rank definition as in [10]:

Definition 3.4. *For a set of formulas $p(x, b)$ with parameters b, and A such that $b \subseteq A \subseteq \mathfrak{C}$, we define the relation $R_A[p] \geq \alpha$ by induction on α as follows*

(1) $R_A[p] \geq 0$ is p is realized in $\mathfrak{C}$.

(2) $R_A[p] \geq \delta$, for δ a limit ordinal, if $R_A[p] \geq \alpha$ for all $\alpha < \delta$.

(3) $R_A[p] \geq \alpha + 1$ if the following two conditions hold:

- *(a) There is $a \in A$ and a formula $\phi(x, y)$ such that $R_A[p \cup \phi(x, a)] \geq \alpha$ and $R_A[p \cup \neg\phi(x, a)] \geq \alpha$*
- *(b) For every $a \in A$ there is a set of formulas $q(x, y)$ such that $R_A[p \cup q(x, a)] \geq \alpha$*

The rank $R_A[p]$ is defined as an ordinal, -1 or ∞ (where we have the usual ordering $-1 < \alpha < \infty$) as follows:

$R_A[p] = -10$ *if p is not realized in $\mathfrak{C}$.*

$R_A[p] = \alpha$ *if $R_A[p] \geq \alpha$ but is not the case that $R_A[p] \geq \alpha + 1$.*

$R_A[p] = \infty$ *if $R_A[p] \geq \alpha$ for all α.*

For a set of formulas $p(x)$ over $A \subseteq \mathfrak{C}$ we let

$$R_A[p] = \min\{R_A[q] : q \subseteq p \restriction B, B \subseteq dom(p), B \textit{ finite } \}$$

$R[p]$ stands for $R_{\mathfrak{C}}[p]$.

A diagram is totally transcendental if the rank is defined everywhere, formally:

Definition 3.5. *A diagram D is totally transcendental if $R_A[p] < \infty$ for all $A \subseteq \mathfrak{C}$ and all $p \in S_D(A)$.*

All the concepts being defined, we can introduce the class we will work with, namely, the class of $(D, \aleph_0)$-homogeneous models of totally transcendental good diagram D. Hereafter we will work with the following convention:

Convention 3.6. *$\mathcal{K}$ is the class of $(D, \aleph_0)$-models of a totally transcendental good diagram D.*

The next step is to define the independence relation in our monster model. In analogy to the classical case, here the independence relation is afforded by rank as follows:

$$A \underset{C}{\downarrow} B \text{ if and only if } R[t(a/B)] = R[t(a/B \cup C) \text{ for every } a \in A$$

Finally we will mention (without proving them) some important properties of this class. First, $\mathcal{K}$ satisfies all axioms 1-8 from section 2.2. Proofs for axioms $1, 2, 4, 5$ and 6 proofs can be found in [10]. Notice that in this case the axioms should be relativized to a subclass of prime models called *primary models*[6]. Existence is proved for this subclass in [10] but furthermore,

[6] I believe this is partly why in [7] they decided to present the axiomatization with respect to this subclass.

uniqueness up to isomorphism (i.e., two primary models over the same subset are isomorphic). The uniqueness of primary models will play a crucial role for the present application. However, this relativization does not harm any of the arguments provided in section 2.2; even remark 2.19 has an analogous version for primary models. Axioms $3, 7$ and 8 are proved in [7]. This enables us to have both the decomposition theorem and corollary 2.42 for $\mathcal{K}$. In addition, in [10], Lessmann proved that the dependence relation defines a pregeometry in the set of elements realizing a regular type, formally:

Theorem 3.7. ***(Lessmann)*** *Let $M \in \mathcal{K}$, $B \subseteq M$ and $p \in S_D(B)$ realized in M. If p is regular, then (W, cl) is a pregeometry, where $W = p(M) - B$ and cl is defined by*

$$a \in cl(C) \Leftrightarrow a \underset{B}{\not\downarrow} C \text{ for } a \in W \text{ and } C \subseteq W.$$

Hence, for $N \prec M$ and every regular type $p \in S_D(N)$ we have a pregeometry in $W = p(M) - N$ defined by the dependence relation, which implies (by the properties of pregeometries, 1.11), that maximal independent sets in W have the same cardinality. This will allow us to have a well-defined notion of dimension (the cardinality of a maximal independent set over another set) which will be used in the following section. In particular, for a set $W = p(M) - N$ and $A \subseteq M$, the dimension of W over (A, N) is the cardinality of a maximal independent subset of W over A.

3.2 $L_{\infty,\lambda}$-equivalence as an Invariant

In chapter XIII of [11], Shelah presents the following application of the decomposition theorem:

Theorem 3.8. ***(Shelah)*** *Suppose $\lambda > |T| + 2^{\aleph_0}$, T is superstable with the NDOP and has prime (primary) models over independent trees. Then any two $L_{\infty,\lambda}$-equivalent models of T of power λ are isomorphic.*

In this section we lift this theorem to our class $\mathcal{K}$. We first prove some lemmas relying on special properties of $\mathcal{K}$, such as the uniqueness of primary models.

Lemma 3.9. *Given $(a_i : i \leq n)$, there are M_i prime (primary) over $M \cup a_i$ such that $M_i \prec N$ for all $i \leq n$, where N is prime (primary) over $M \cup \bigcup_{i \leq n} a_i$.*

Proof: Fix $i \leq n$. Let $M(a_i)$ be prime over $M \cup a_i$. Since N contains $M \cup a_i$, there is a $\mathcal{K}$-embedding $f : M(a_i) \to N$ fixing $M \cup a_i$ pointwise. Let $f(M(a_i)) = M_i$. Then, by remark 2.19 M_i is prime over $f(M \cup a_i) = M \cup a_i$, and $M_i \prec N$, which is what we wanted. $\square$

Lemma 3.10. *Assume that prime (primary) models are unique. Then, if $(a_i : i < \alpha)$ is independent over M, $(M(a_i) : i < \alpha)$ is also independent over M, where $M(a_i)$ is the prime (primary) model over $M \cup a_i$.*

Proof: By finite character we may assume that $\alpha = n < \omega$. Let N be prime over $M \cup \{a_j : j \leq n, j \neq i\}$. By the previous lemma and the uniqueness of prime (primary) models we may assume that $M(a_j) \prec N$ for all $j \leq n$ and $j \neq i$. Then we have that

$$a_i \underset{M}{\downarrow} \bigcup_{i \neq j} a_j \overbrace{\Rightarrow}^{dom} a_i \underset{M}{\downarrow} N \overbrace{\Rightarrow}^{sym+dom} N \underset{M}{\downarrow} M(a_i) \overbrace{\Rightarrow}^{mon+sym} M(a_i) \underset{M}{\downarrow} \bigcup_{i \neq j} M(a_j) \square$$

Now we are ready to prove the main theorem of this section.

Theorem 3.11. *Let $M_1, M_2 \in \mathcal{K}$ of cardinality $\lambda > LS(\mathcal{K}) + 2^\omega$. Then if M_1 and M_2 are $L_{\infty,\lambda}$ equivalent, they are isomorphic.*

Proof: We build a sequence of decompositions $\mathcal{S}^l_n = \langle N^l_\eta, a^l_\eta : \eta \in T_n \rangle$ for M_l where $l = 1, 2$ satisfying the following properties:

(1) T_n is a subtree of ${}^{n\geq}\lambda$.

(2) $\mathcal{S}^l_m \leq_{\mathcal{D}} \mathcal{S}^l_n$ for $m < n$ (where $\leq_{\mathcal{D}}$ is the decomposition order defined in the previous section).

(3) $\mathcal{S}^l_n$ is a maximal decomposition up to height n, i.e., there is no $\eta \in T_n$ such that $ht(\eta) < n$ and for some $\nu = \eta^\frown\alpha$, $\nu \notin T_n$, there are N^l_ν and a^l_ν such that $\langle N^l_\rho, a^l_\rho : \rho \in T_n \cup \{\nu\} \rangle$ is a decomposition for M_l.

(4) F_n is an elementary embedding from $\bigcup_{\eta \in T_n} N^1_\eta$ to M_2, mapping N^1_η onto N^2_η.

(5) $(M_1, c)_{c \in N^1_\eta} \equiv_{\infty,\lambda} (M_2, F_n(c))_{c \in N^1_\eta}$.

(6) $F_m \subseteq F_n$ for $m < n$.

Assume first that such sequences exist. Let $T = \bigcup_{i<\omega} T_i$ and $\mathcal{S}_l = \langle N^l_\eta, a^l_\eta : \eta \in T\rangle$ defined as in lemma 2.39. By the same lemma, we have that $\mathcal{S}_l$ is a decomposition for M_l. We now show that it is a maximal decomposition. Suppose not. Then there is $\eta \in T$ such that for some $\nu = \eta^\frown\alpha$, $\nu \notin T$, there are N^l_ν and a^l_ν such that $\langle N^l_\rho, a^l_\rho : \rho \in T \cup \{\nu\}\rangle$ is a decomposition for M_l. Let $ht(\eta) = n$. Then, we have that $\langle N^l_\rho, a^l_\rho : \rho \in T_{n+1} \cup \{\nu\}\rangle$ is a decomposition for M_l extending T_{n+1}, which contradicts (3). This shows that both $\mathcal{S}_l$ are maximal. Hence, by corollary 2.42, we know that M_l is prime and minimal over $\bigcup_{\eta\in T} N^l_\eta$. In addition, we have that $F = \bigcup_{n<\omega} F_n$ is an elementary map with domain $\bigcup_{\eta\in T} N^1_\eta$. Hence, there is an elementary embedding $F' : M_1 \to M_2$ extending F. By the minimality of M_2, this embedding must be onto, hence M_1 and M_2 are isomorphic. It remains to show how to build sequences satisfying (1)-(5).

We build the sequences by induction on n. For $n = 0$, let N^1_0 be the prime model over the $\emptyset$ and $a^1_0 \in M_1$ (as in lemma 2.39 we can pick any element). We have that $|N^1_0| = LS(\mathcal{K}) < \lambda$ by condition (C4) of AEC. Since $M_1 \equiv_{\infty,\lambda} M_2$, there is a back-and-forth set for M_1 and M_2. Hence, there is a partial isomorphism g with domain N^1_0. We let N^2_0 be the image of N^1_0 under g. Trivially this implies that

$$(M_1, c)_{c\in N^1_0} \equiv_{\infty,\lambda} (M_2, g(c))_{c\in N^1_\eta}$$

It remains to show that N^2_0 is prime over $\emptyset$, but this is also satisfied by remark 2.19. Let a^2_0 be any element in M_2. Then, we let $T_0 = \{0\}$, $\mathcal{S}^l_0 = \langle N^l_0, a^l_0\rangle$ and $F_n = g$. Clearly we have that $\mathcal{S}^l_0$ is a decomposition for M_l and they satisfy condition (3) since there is just one decomposition of height 1 (by the definition of decomposition). By the choice of F_0 we have both conditions (4) and (5). This completes the base case.

Assume $\mathcal{S}^l_n$ and F_n have been defined. We proceed to define $\mathcal{S}^l_{n+1}$ and F_{n+1}. Notice that the new elements in T_{n+1} must be above those $\eta \in T_n$ such that $ht(\eta) = n$, otherwise condition (3) is contradicted. Thus, let be $\eta \in T_n$ such that $ht(\eta) = n$. First, we notice that it is enough to have an ordinal α and a sequence $(a_\beta : \beta < \alpha)$ such that:

(a) $(a^l_\beta : \beta < \alpha)$ is independent over N^l_η.

(b) $t(a^l_\beta/N^l_\eta)$ is regular and $t(a^l_\beta/N^l_\eta) \perp N^l_{\eta^-}$ if η^- exists.

(c) For $\beta < \beta' < \alpha$, either $t(a^l_\beta/N^l_\eta) = t(a^l_{\beta'}/N^l_\eta)$ or $t(a^l_\beta/N^l_\eta) \perp t(a^l_{\beta'}/N^l_\eta)$.

(d) $(a^l_\beta : \beta < \alpha)$ is maximal with respect to (a), (b) and (c).

(e) There is an isomorphism $F_\beta : N^1_\beta \to N^2_\beta$, where N^l_β is the prime model over $N^l_\eta \cup a^l_\beta$, such that $F_\beta \restriction N^1_\beta = F_n \restriction N^1_\beta$.

(f) $(M_1, c)_{c \in N^1_\beta} \equiv_{\infty,\lambda} (M_2, F_\beta(c))_{c \in N^1_\beta}$.

Assume that for each $\eta \in T_n$ of height n we can find an ordinal α_η satisfying the above conditions. Then, we let

$$T_{n+1} = T_n \cup \{\eta^\frown\beta : \beta < \alpha_\eta, \eta \in T_n, ht(\eta) = n\}$$

$$\mathcal{S}^l_{n+1} = \langle N^l_\rho, a^l_\rho : \rho \in T_{n+1}\rangle$$

where $N^l_{\eta^\frown\beta} = N^l_\beta$ and $a^l_{\eta^\frown\beta} = a^l_\beta$. Condition (1) is trivially satisfied. Condition (2) follows from (a)-(c) and lemma 3.10. Condition (3) follows from (d). We define F_{n+1} as follows:

$$F_{n+1} = \bigcup\{\bigcup_{\beta<\alpha_\eta} F_\beta : \eta \in T_n, ht(\eta) = n\}$$

We fist show this function is well-defined. Let $\beta < \alpha_\eta$ and $\gamma < \alpha_{\eta'}$. Let $x \in N^1_{\eta^\frown\beta} \cap N^1_{\eta'^\frown\gamma}$. Since $\mathcal{S}^1_{n+1}$ is a decomposition, we must have that $x \in N_\nu$ for $\nu < \eta$ and $\nu < \eta'$ (otherwise, we contradict triviality of independence). Hence, by (e) we have that $F_\beta \restriction N^1_\nu = F_n \restriction N^1_\nu = F_\gamma \restriction N^1_\nu$, and it is well defined in this case. Now assume towards a contradiction that there is $x \in M_1 - (N^1_{\eta^\frown\beta} \cap N^1_{\eta'^\frown\gamma})$ such that $x \in \mathrm{dom}(F_\beta) \cap \mathrm{dom}(F_\gamma)$. Then, again, since $\mathcal{S}^l_{n+1}$ is a decomposition we have that

$$N^1_{\eta^\frown\beta} \underset{N_\nu}{\downarrow} N^1_{\eta'^\frown\gamma}$$

And by monotonicity we have that

$$x \underset{N_\nu}{\downarrow} x$$

which again contradicts triviality of independence. Therefore the function is well-defined. Finally, properties (e) and (f) imply conditions (4)-(6). We proceed then to find α and $(a^l_\beta : \beta < \alpha)$ for $\eta \in T_n$ of height n.

We let

$$I_l = \{a \in M_l : t(a/N^l_\eta) \text{ is regular}, t(a/N^l_\eta) \perp N^l_{\eta^-} \text{ if } \eta^- \text{ exists}\}$$

If $I_l = \emptyset$, we set $\alpha = 0$ and we do not add any elements above η. Hence assume that $I_l \neq \emptyset$. We consider a family $\mathcal{J}_l$ of subsets of $J \subseteq I_l$ defined by

$$J \in \mathcal{J}_l \Leftrightarrow \text{ for all } a, a' \in J \text{ either } t(a/N^l_\eta) = t(a'/N^l_\eta) \text{ or } t(a/N^l_\eta) \perp t(a'/N^l_\eta)$$

It is easy to see that $(\mathcal{J}_l, \subseteq)$ is non-empty and closed under unions of chains (if two elements of the union realize different non-orthogonal types, there is a subset in the chain to which they belong, which is a contradiction). Hence, by Zorn's lemma, we let J_l be maximal elements in $\mathcal{J}_l$ for $l = 1, 2$. Without loss of generality, we can assume that the types being realized by elements in J_1 and J_2 are the same (here we employ a back-and-forth argument using condition (5) of the induction hypothesis). Now we build by induction on γ an ordinal γ^* and a sequence of subsets $J^\gamma_l \subseteq J_l$ such that:

- the elements in J^γ_l satisfy the same $L_{\infty,\lambda}$-type.
- the dimension of J^γ_l over $(\bigcup_{\beta<\gamma} J^\beta_l, N_\eta)$ is less than λ.
- γ^* is the first ordinal where we cannot continue this sequence.

Since all the types realized in J_l are orthogonal, by definition of γ^*, the dimension of J_l over $(\bigcup_{\beta<\gamma^*} J^\beta_l, N^l_\eta)$ is either λ or zero. Assume it is λ and let $(b^l_i : i < \lambda)$ be a maximal independent set over $(\bigcup_{\beta<\gamma^*} J^\beta_l, N^l_\eta)$. Then we define $(a^l_\beta : \beta < \lambda)$ by induction on $\beta < \lambda$ as follows:

(i) $(a^l_\xi : \xi \leq \beta)$ is independent over $(\bigcup_{\beta<\gamma^*} I^\beta_l, N^l_\eta)$.

(ii) $(M_1, c, a^1_\beta)_{c \in N^1_\eta} \equiv_{\infty,\lambda} (M_1, F_n(c), a^2_\beta)_{c \in N^l_\eta}$

(iii) $b^1_\beta \in (a^1_\xi : \xi \leq 2\beta)$

(iv) $b^2_\beta \in (a^2_\xi : \xi \leq 2\beta + 1)$

Assume that $(a^l_\xi : \xi < 2\gamma)$ has been defined satisfying the previous conditions. For notational simplicity assume $2\gamma = \beta$. Take $a^1_\beta = b^1_\beta$. If there is $\xi < \beta$ such that $a^1_\xi = b_\beta$, then we define $a^2_\beta = a^2_\xi$. If not, then let b^2_ζ for $\zeta \geq \beta$, such that $t(b^1_\beta/N^1_\eta) = t(b^2_\zeta/N^2_\eta)$. The fact that they satisfy the same $L_{\infty,\lambda}$-type over N^l_η guarantees condition (ii). That there is such ζ is granted by the following argument. Suppose there is no such b^2_ξ with the same type. Then, using the back-and-forth set from condition (5) of our induction hypothesis there is $c \in J_2$ such that $t(b^1_\delta/N^1_\eta) = t(c/N^2_\eta)$. Since $t(c/N^2_\eta) \neq t(b^2_\beta/N^2_\eta)$ for all $\beta < \lambda$, by the definition of J_2, the type $t(c/N^2_\eta)$ is ortogonal to $t(b^2_i/N^2_\eta)$ for all $i < \lambda$. Hence $(b^2_i : i < \lambda) \cup \{c\}$ is still independent over $(\bigcup_{\beta<\gamma^*} J^\beta_l, N^l_\eta)$, which is a contradiction. This implies there is such a ζ. For $\beta = 2\gamma + 1$, we start setting $a^2_\beta = b^2_\beta$, and proceed to define a^1_β in an analogous way. The maximality of J_l and the sequence $(b_i : i < \lambda)$ guarantee properties (a)-(d). If the dimension is zero, we do not add any elements to the tree and this completes the construction. Finally, condition (ii) implies that the map $F_n \cup \{a^1_\beta, a^2_\beta\}$ is an elementary map. Hence there is an embedding from $F_\beta : N^1_\beta \to N^2_\beta$. But then $F_\beta(N^1_\beta)$ is also primary over $N^2_\eta \cup a^2_\beta$, so by the uniqueness of primary models we have that F_β is an isomorphism. This completes the proof of conditions (e) and (f), and the construction. $\square$

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