A study of Canonicity for Bi-Implicative Algebras

MSc Thesis (*Afstudeerscriptie*)

written by

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Introduction

The aim of this thesis is an analytical study of algebraic canonicity of a family of logics whose language consists of constants and implications. More specifically, the logics to which this study applies are those associated with certain distinguished sub-quasivarieties of (bi-)implicative algebras, the best known of which are the varieties of (bi-)Hilbert algebras and (bi-)Tarski algebras, Hilbert and Tarski algebras being the pure $\to$ reducts of intuitionistic and classical propositional logics respectively.

There are two reasons for this line of research. The first is that the recent developments in the algebraic theory of canonical extensions have greatly expanded the range of logics to which it is applicable, far beyond the best known family of lattice-based logics such as classical modal logics, and so it is natural to try and explore its potential and limitations in interesting case studies that are detached from the lattice setting, but to which the methodology and results obtained via canonical extensions potentially apply. The second reason is rooted in the content of these recent developments, that are briefly described in what follows.

Canonical extensions were first introduced by Jónsson and Tarski in the 1950s for Boolean algebras with operators in their papers [25] and [26]. In this Boolean setting, canonical extensions provide an algebraic formulation of the Stone representation theorem for Boolean algebras which is alternatively achieved via Stone duality. In [18, 23] and [24] Jónsson and Gehrke showed that topological duality for distributive lattices can similarly be captured algebraically. Since then, significant applications of the theory of canonical extensions to modal logic have been discovered both in the classical and distributive cases as witnessed in [19]. In [15] the theory of canonical extensions was extended to the case of non-distributive lattices. This has proven particularly useful as the completion obtioned is much simpler to work with than the available duality theories.

The first step of any theory of canonical extensions is the existence theorem.

In the Boolean and distributive settings the existence of the canonical extension can be proven via the Stone (resp. Priestley) duality, however existence can also be proven constructively (and therefore independently from a duality). Indeed, in the lattice case, existence is shown by means of a construction that involves the families $\mathsf{Fi}(P)$ and $\mathsf{Id}(P)$ of (proper) filters and ideals of P. The proof of the existence of canonical extensions for posets can be based on the same construction, but in this greatly generalised setting there is no default or established choice as to what the right notion of filter and ideal of a poset P should be.

The first appearance of the theory of canonical extensions for posets was in [11], where it was applied to a fragment of the full λ-calculus consisting of a binary, associative and completely join preserving fusion operator and its two residuals. The filters and ideals in the construction of the extension in this paper were chosen to optimise the mathematical properties of the extension rather than for any logical reasons.

In [16] a parametrised theory of canonical extensions is developed. For every set of upsets containing the principal upsets $\mathscr{F}$, and every set of downsets containing the principal downsets $\mathscr{I}$, the existence of a unique $(\mathscr{F}, \mathscr{I})$-canonical extension constructed from $\mathscr{F}$ and $\mathscr{I}$ is shown. It is also shown how different choices of $\mathscr{F}$ and $\mathscr{I}$ (endowed with certain properties) guarantee different features of the corresponding extensions (such as commutativity with products or order-dualisation) each of which is desirable for the development of a smooth theory. (The theory of canonical extensions for posets has also been developed in a different direction in [22] where the canonical extension is viewed as one in a hierarchy of completions.) Of course, the choice between alternative notions of filters and ideals can be based on their satisfying or not satisfying certain properties of particular interest in special contexts, but what is important to remark here is that an ultimate, all-purpose notion of filter or ideal does not emerge from this analysis.

In [17] the first attempt was made at linking canonical extensions with AAL. The underlying motivation for connecting these two, up to now, separate algebraic methods in logic is that in all the successful applications of the theory of canonical extensions (to, for example, the canonicity issues of certain classes of modal logics) the chosen filters are uniformly identified by means of a fundamental notion in AAL, that of logical filter or S-filter for the appropriate associated logic S. This suggests that, when an all-purpose choice of filters and ideals is not available, the "right" choice could depend, in different settings, on the different logical systems at hand, and on their properties.

Unfortunately the definition for canonical extensions of Hilbert algebras presented in [17] has certain drawbacks. A detour was taken through the meet semi-lattice of finitely generated S-filters before applying the parametrised theory with $\mathscr{F}$ and $\mathscr{I}$ as the sets of down-directed upsets and up-directed downsets respectively. This gives a canonical extension which is not symmetric in its relationship to the logical filters and ideals; moreover, it is an ad-hoc construction which really utilises the powerful axiomatisation of Hilbert algebras and is not applicable in more general contexts. Furthermore, the canonicity result obtained is not analytical, as all four Hilbert axioms are lifted together in one step.

One goal in this thesis is to apply the parametrised method directly with logically motivated choices of $\mathscr{F}$ and $\mathscr{I}$. As we briefly mentioned above, in order to be able to clearly see the power and limitations of the logic-based choice we need to get away from the lattice setting, since there the logical and the mathematically optimal choices overlap. This justifies our choice of a signature with only constants and implications.

All previous work on canonical extensions for the signature $\{\top, \to\}$ has focused on Hilbert (and Tarski) algebras. In order to attempt to apply to the widest possible setting, an extended class of algebras, the implicative algebras, will be considered. Another reason to generalise to the level of implicative algebras is to facilitate a more modular investigation into the canonicity of the Hilbert axioms taken individually rather than all together, as has been done in previous work.

The natural logical choice of filters for the signature $\{\top, \to\}$ is that of implicative filters, upsets closed under $\top$ and $\to$. In order to allow a similarly logical choice of ideals, we choose a signature which also contains a binary subtraction operator $\leftarrow$. The set of ideals can then be taken to be those downsets which are closed under $\bot$ and $\leftarrow$.

In any complete, distributive, dually algebraic lattice, a subtraction operator can always be defined as the left residual of the join (just as implication is the right residual of the meet in a heyting algebra) by letting:

$$a \leftarrow b = \bigwedge\{c : a \leq c \vee b\}$$

As discussed in [30], the subtraction operator gives rise to a second negation defined by $\ x = \top \leftarrow x$ in addition to the usual $\neg x = x \to \bot$. Interestingly, both of these negations seem to occur naturally in ordinary English with different meanings, as: "It is not false that p" and "It is false that not p". Note that the concept of subtraction has particular intuitive relevance in a (semi-)

boolean algebra (of sets), where it is just the very familiar set difference operator (see [12]).

The natural starting point of this project is then that of bi-implicative algebras. The axiomatization of bi-implicative algebras essentially says that the binary relation $\leq$ defined on A equivalently as

$$a \to b = \top \quad \text{iff} \quad a \leq b \quad \text{iff} \quad a \leftarrow b = \bot$$

is a partial order and $(A, \leq, \top, \bot)$ is a bounded poset. Since these algebras are special posets, the theory of canonical extensions for posets can be applied to them. Further, the same order has two different but equivalent algebraic descriptions in terms of the two arrows. This is very similar to what happens in lattices with the algebraic operations $\wedge$ and $\vee$ giving two different but equivalent descriptions of the lattice order as

$$a \wedge b = a \quad \text{iff} \quad a \leq b \quad \text{iff} \quad a \vee b = b.$$

One major restriction to the applicability of the theory of canonical extensions in all settings is that the basic operations of the algebras to be considered either preserve or reverse the order in each coordinate. Since in any bi-Hilbert algebra $\to$ is order reversing in the first coordinate and order preserving in the second and $\leftarrow$ is order preserving in the first coordinate and reversing in the second, these are the most natural assumptions to make in the more general setting so that the maps $\to$ and $\leftarrow$ can be extended. A positive consequence of basing the canonical extension contruction on the implicative filters and subtractive ideals is that the operation commutes with both products and duals. This makes it possible to define the π and σ extensions of $\to$ and $\leftarrow$ coordinate-wise and this is done in the same way as for the boolean, distributive, and non-distributive lattice cases. In general, $\to$ and $\leftarrow$ are not smooth (see [20]), so a choice between π and σ needs to be made for each. The choice has been made to take the π-extension for $\to$ and the σ-extension for $\leftarrow$ because in the case of a residuated lattice, the extensions will then be residuated with the meet and join respectively in the extension.

The opening chapter of this thesis gives some preliminaries from the field of abstract algebraic logic. In chapter 2, the details of the construction of parametrised canonical extensions for posets are outlayed and some characterists of filters and ideals which induce important properties in the extensions are given. Following this, in chapter 3, background is given on π and σ-extentions of maps that are order preserving or reversing in each coordinate. In chapter 4, bi-implicative algebras are defined and their relationship to bounded posets is explained. Following this, a number of important extensions (including bi-Hilbert algebras) are introduced and the relationship between the additional axioms is investigated. The chapter is rounded off with some examples. In chapter 5, our choices of filters $\mathscr{F}$ and ideals $\mathscr{I}$ are discussed along with some additional assumptions that are required to define the $(\mathscr{F}, \mathscr{I})$-extension in the setting of bi-implicative algebras, namely the axioms (Det), (OP_2), (OR_1) and their subtractive counterparts. The fact that extensions commute with products and duals is shown and the final definitions of the extensions of the operations are given. In chapter 6, we present our canonicity results. In the final chapter, 7, conclusions are made and some ideas for further work explained.

Chapter 1

Introduction to Abstract Algebraic Logic

1.1 Closure Systems/Operators

Definition Given a set A, a closure operator on A is a function $cl : \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$ such that for every $X, Y \subseteq A$:

1. $X \subseteq cl(X)$
2. If $X \subseteq Y$, then $cl(X) \subseteq cl(Y)$
3. $cl(cl(X)) = cl(X)$

Call a set $X \subseteq A$ closed if $cl(X) = X$.

A closure system C on a set A (also referred to as a topped intersection structure in [8]) is a set of subsets of A that includes A and is closed under arbitrary intersection.

Proposition 1.1.1. *If cl is a closure operator on A, then the set of all closed subsets of A, C, is a closure system.*

Proof. $A \subseteq cl(A) \subseteq A$ implies A is closed and hence $A \in C$. Let X_i for $i \in I$ be closed subsets of A. Then for each i, $cl(\bigcap_i X_i) \subseteq cl(X_i) = X_i$, so $cl(\bigcap_i X_i) \subseteq \bigcap_i X$. But we also know that $\bigcap_i X_i \subseteq cl(\bigcap_i X_i)$ so $cl(\bigcap_i X_i) = \bigcap_i X_i$ and hence $\bigcap_i X_i \in C$. □

Proposition 1.1.2. *If C is a closure system on A, then cl defined for all $X \subseteq A$ by $cl(X) = \bigcap\{Y \in C : X \subseteq Y\}$ is a closure operator on A.*

Proof.

1. $X \subseteq Z$ for each $Z \in \{Y \in C : X \subseteq Y\}$, implies $X \subseteq \bigcap\{Y \in C : X \subseteq Y\} = cl(X)$
2. Let $X \subseteq Z$. Take any $Y \in C$ such that $Z \subseteq Y$. Then $X \subseteq Y$ also, so $cl(X) \subseteq Y$. Hence $cl(X) \subseteq \bigcap\{Y \in C : Z \subseteq Y\} = cl(Z)$.
3. If $Y = cl(X)$. Then $Y \in C$, and thus, since $Y \subseteq Y$, $cl(Y) \subseteq Y$. Then since by 1 $Y \subseteq cl(Y)$ so $cl(Y) = Y$ ie. $cl(cl(X)) = cl(X)$.

□

1.2 Connecting algebra and logic

Let L be a propositional language, that is, a set of connectives and Fm_L consists of all terms over L made up from a denumerable set of variables V. A **logic** over L is then a pair $\langle Fm_L, \vdash\rangle$ where $\vdash$ is a consequence relation on Fm_L, meaning that $\vdash \subseteq \mathcal{P}(Fm_L) \times Fm_L$ which satisfies for all $\phi \in Fm_L$, and $\Gamma, \Delta \in \mathcal{P}(Fm_L)$:

1. $\phi \in \Gamma$ implies $\Gamma \vdash \phi$
2. If $\Delta \vdash \phi$ and $\Gamma \vdash \delta \quad \forall \delta \in \Delta$, then $\Gamma \vdash \phi$

Alternatively L can be viewed as a set of function symbols and Fm_L as the term algebra of L over V, the absolutely free L-algebra over V. A consequence relation is then a closure operator $cl_\vdash : \mathcal{P}(Fm_L) \to \mathcal{P}(Fm_L)$ defined by $\phi \in cl_\vdash(\Gamma)$ iff $\Gamma \vdash \phi$ which is invariant under substitutions ie. for every $\sigma \in Aut(Fm_L)$:

$$\sigma[cl_\vdash(\Gamma)] \subseteq cl_\vdash(\sigma[\Gamma])$$

The following is a property of consequence relations that we will find useful later on. It holds for any consequence relation $\vdash \subseteq \mathcal{P}(Fm_L) \times Fm_L$, $\phi \in Fm_L$ and $\Gamma, \Gamma' \in \mathcal{P}(Fm_L)$.

Lemma 1.2.1. *If $\Gamma \subseteq \Gamma'$ and $\Gamma \vdash \phi$, then $\Gamma' \vdash \phi$.*

Proof. Assume $\Gamma \subseteq \Gamma'$. Then by the first condition in the definition of consequence relation, $\Gamma' \vdash \gamma$ for all $\gamma \in \Gamma$. If, in addition, $\Gamma \vdash \phi$ then the second defining clause for consequence relations can be applied to give $\Gamma' \vdash \phi$. □

Note that if a consequence relation $\vdash$ only contains pairs (Γ, ϕ) such that $\Gamma = \{\gamma\}$ is a singleton, we will call it a restricted consequence relation and will drop the use of the brackets, saying $\gamma \vdash \phi$ rather than $\{\gamma\} \vdash \phi$.

1.2.1 The Detachment Theorem

The detachment theorem is a property held by many well-known logics. Here we introduce two different versions, D_1 and D_2, and show their equivalence.

Definition Given a logic $\langle Fm_L, \vdash \rangle$, D_1 holds in $\langle Fm_L, \vdash \rangle$ if for all formulas $\phi, \psi \in Fm_L$ and sets of formulas $\Gamma \in \mathcal{P}(Fm_L)$:

$$\Gamma \vdash \phi \to \psi \text{ implies } \Gamma, \phi \vdash \psi \qquad (D_1)$$

D_2 holds in $\langle Fm_L, \vdash \rangle$ if for all formulas $\phi, \psi \in Fm_L$ and sets of formulas $\Gamma \in \mathcal{P}(Fm_L)$:

$$\Gamma \vdash \phi \to \psi \text{ and } \Gamma \vdash \phi, \text{ implies } \Gamma \vdash \psi \qquad (D_2)$$

Proposition 1.2.2. *D_1 iff D_2*

Proof. ($\Rightarrow$). Fix Γ, ϕ and ψ and assume that $\Gamma \vdash \phi \to \psi$ and $\Gamma \vdash \phi$. Let $\Delta = \Gamma \cup \{\phi\}$. Then by lemma 1.2.1, $\Delta \vdash \phi \to \psi$ so by D_1, $\Delta, \phi \vdash \psi$. So since $\Delta \cup \{\phi\} = \Delta$, $\Delta \vdash \psi$ and (by the first condition for consequence relations) for all $\delta \in \Delta$, $\Gamma \vdash \delta$ so the second condition for consequence relations can be applied to give $\Gamma \vdash \psi$, as required.
($\Leftarrow$). Again fix Γ, ϕ and ψ. Assume that $\Gamma \vdash \phi \to \psi$. Let $\Delta = \Gamma \cup \{\phi\}$. Then by the lemma and the first consequence relation condition, $\Delta \vdash \phi \to \psi$ and $\Delta \vdash \phi$. So by D_2, $\Delta \vdash \psi$, ie. $\Gamma, \phi \vdash \psi$. □

D_2 is equivalent to D_1 in the most general setting but it has the advantage that it is expressable using only sequents of the form $\Gamma \Rightarrow \phi$.

1.2.2 Logics of implicative algebras

In this section we define a restricted consequence relation with particular relevance to this thesis. Let $L = \{\to, \top\}$ and let A be an implicative algebra (defined in section 4.1). Define $\vdash_A \subseteq Fm_L \times Fm_L$ by:

$$\phi \vdash_A \psi \text{ iff } \forall h \in Hom(Fm_L, A),\ h(\phi) \leq h(\psi)$$

where $\leq$ is the order associated with A (defined in section 4.1).

The next proposition shows that $\langle Fm_L, \vdash_A \rangle$ is a logic.
Proposition 1.2.3. *$\vdash_A$ is a restricted consequence relation.*

Proof. 1. Since $\leq$ is reflexive, for all $h \in Hom(Fm_L, A)$ and all $\phi \in Fm_L$, $h(\phi) \leq h(\phi)$, hence $\phi \vdash_A \phi$.

2. Assume $\delta \vdash_A \phi$ and $\gamma \vdash_A \delta$. Fix some $h \in Hom(Fm_L, A)$, then by assumption $h(\delta) \leq h(\phi)$ and $h(\gamma) \leq h(\delta)$. So by the transitivity of $\leq$, $h(\gamma) \leq h(\phi)$. Hence $\gamma \vdash_A \phi$.

□

Finally, we show the relationship between the detachment theorem and a condition on implicative algebras:

$$a \leq b \to c \text{ and } a \leq b \Rightarrow a \leq c \tag{Det}$$

Proposition 1.2.4. *In any implicative algebra A, if A satisfies (Det) then $\vdash_A$ satisfies D_2 for $\Gamma = \{\gamma\}$*

Proof. Take some $\gamma, \phi, \psi \in Fm_L$ and assume $\gamma \vdash_A \phi \to \psi$ and $\gamma \vdash_A \phi$. Fix some $h \in Hom(Fm_L, A)$. By assumption $h(\gamma) \leq h(\phi \to \psi) = h(\phi) \to h(\psi)$ and $h(\gamma) \leq h(\phi)$. Then by Det, $h(\gamma) \leq h(\psi)$. So $\gamma \vdash_A \psi$, as required. □

Chapter 2

Preliminaries on Canonical Extensions

2.1 Parameterised Canonical extensions of posets

In this chapter the parametrised method for generating canonical extensions of posets is outlined. Unlike in the boolean and distributive lattice case, where existence is given by a topological duality, the existence of canonical extensions of posets must be shown explicitly. In the lattice case this construction is based on the sets of (proper) filters and ideals of the lattice and a similar construction can also be done for posets. The reason for parametrising is that while in the lattice case there is a single natural choice of filters and ideals, at the level of posets it is unclear what that choice should be. This section shows that we can in fact use any set of upsets $\mathscr{F}$ and downsets $\mathscr{I}$ which contain the principal upsets and downsets respectively. The canonical extension constructed using $\mathscr{F}$ and $\mathscr{I}$ is then called the $(\mathscr{F}, \mathscr{I})$-completion. The first section includes an abstract definition of $(\mathscr{F}, \mathscr{I})$-completion while the second proves existence. The third shows uniqueness of the extension for fixed $\mathscr{F}$ and $\mathscr{I}$, and the final section outlines certain properties of the choice of filters and ideals give rise to important properties of the extension.

2.2 Defining the $(\mathscr{F}, \mathscr{I})$-extension

Definition An *extension* of a poset $\mathbb{P}$ is a pair $(e, \mathbb{Q})$ where $\mathbb{Q}$ is a poset and $e : \mathbb{P} \to \mathbb{Q}$ is an order embedding ie. for all $a, b \in P$, $a \le b$ iff $e(a) \le e(b)$. A *completion* of a poset $\mathbb{P}$ is an extension $(e, \mathbb{Q})$ where $\mathbb{Q}$ is a complete lattice.

An *upset* in a partial order $\mathbb{P}$ is a set $X \subseteq P$ such that $x \in X$ and $x \le y$ implies $y \in X$. Dually a *downset* of $\mathbb{P}$ is a set $X \subseteq \mathbb{P}$ such that $x \in X$ and $y \le x$ implies that $y \in X$. For each $p \in P$, let $\uparrow p = \{a \in P : a \ge p\}$ and $\downarrow p = \{a \in P : a \le p\}$ denote the principal upset and downset of p respectively.

Let $\mathscr{F}$ be a collection of upsets of $\mathbb{P}$ and $\mathscr{I}$ be a collection of downsets. We say that an extension $(e, \mathbb{Q})$ of a poset $\mathbb{P}$ is $(\mathscr{F}, \mathscr{I})$-*compatible* iff:

1. For every $F \in \mathscr{F}$ there exists an $a \in \mathbb{Q}$ such that $\bigwedge e[F] = a$ and $F = \mathbf{F_a}$ where $\mathbf{F_a} = \{p \in \mathbb{P} \,|\, e(p) \ge a\}$ and
2. For every $I \in \mathscr{I}$ there exists a $b \in \mathbb{Q}$ such that $\bigvee e[I] = b$ and $I = \mathbf{I_a}$ where $\mathbf{I_a} = \{p \in \mathbb{P} \,|\, e(p) \le a\}$.

This means that for each $F \in \mathscr{F}$, $\bigwedge e[F]$ exists in $\mathbb{Q}$ and identifies F as exactly the elements of P whose images lie above $\bigwedge e[F]$. Similarly for each $I \in \mathscr{I}$, $\bigvee e[I]$ exists in $\mathbb{Q}$ and identifies I as exactly the elements of P whose images lie below $\bigvee e[I]$.

We say that an $(\mathscr{F}, \mathscr{I})$-compatible extension $(e, \mathbb{Q})$ of a poset $\mathbb{P}$ is $(\mathscr{F}, \mathscr{I})$-*compact* iff for each $F \in \mathscr{F}$ and $I \in \mathscr{I}$, if $\bigwedge e[F] \le \bigvee e[I]$ then $F \cap I \neq \varnothing$.

In an $(\mathscr{F}, \mathscr{I})$-compatible extension of a poset $\mathbb{P}$, we call an element $\mathscr{F}$-*closed* if $a = \bigwedge e[F]$ for some $F \in \mathscr{F}$ or $\mathscr{I}$-*open* if $a = \bigvee e[I]$ for some $I \in \mathscr{I}$. The collection of $\mathscr{F}$-closed and $\mathscr{I}$-open elements of $(e, \mathbb{Q})$ will be denoted by $K_e^{\mathscr{F}}(\mathbb{Q})$ and $O_e^{\mathscr{I}}(\mathbb{Q})$ respectively.

We say that an $(\mathscr{F}, \mathscr{I})$-compatible extension $(e, \mathbb{Q})$ is $(\mathscr{F}, \mathscr{I})$-*dense* iff every element $q \in \mathbb{Q}$ is both the join of a set of $\mathscr{F}$-closed elements and the meet of a set of $\mathscr{I}$-open elements.

We call an $(\mathscr{F}, \mathscr{I})$-compatible extension an $(\mathscr{F}, \mathscr{I})$-*extension* if it is both $(\mathscr{F}, \mathscr{I})$-compact and $(\mathscr{F}, \mathscr{I})$-dense. Finally, an $(\mathscr{F}, \mathscr{I})$-*completion* is a complete $(\mathscr{F}, \mathscr{I})$-extension.

2.3 Existence

In this section, we take a set of upsets ordered by reverse inclusion $\mathbb{F} = (\mathscr{F}, \supseteq)$ and a set of downsets ordered by inclusion $\mathbb{I} = (\mathscr{I}, \subseteq)$ and show the existence of an $(\mathscr{F}, \mathscr{I})$-extension $\mathbb{F} \oplus \mathbb{I}$ under the additional assumptions that $\uparrow p \in \mathscr{F}$ and $\downarrow p \in \mathscr{I}$ for every $p \in \mathbb{P}$. We will then see that by taking its Macneille completion, $\overline{\mathbb{F} \oplus \mathbb{I}}$, we get an $(\mathscr{F}, \mathscr{I})$-completion.

2.3.1 The Amalgamation

First we define a binary relation $\leq^+$ on $\mathscr{F} \uplus \mathscr{I}$. Let:
$F \leq^+ F'$ iff $F' \subseteq F$
$I \leq^+ I'$ iff $I \subseteq I'$
$F \leq^+ I$ iff $F \cap I \neq \varnothing$
$I \leq^+ F$ iff for every $i \in I, f \in F$ $i \leq f$.

Lemma 2.3.1. *$\leq^+$ is a preorder on $\mathscr{F} \uplus \mathscr{I}$*

Proof. Clearly $\leq^+$ is reflexive. For transitivity take $F, F' \in \mathscr{F}$ and $I, I' \in \mathscr{I}$. Two cases will be shown, the others are straight forward to check.
(1.) If $F \leq^+ I \leq^+ F'$ then there exists an $x \in F \cap I$. Also $x \leq y$ for all $y \in F'$, so $\uparrow x \supseteq F'$. But since F is an upset $\uparrow x \subseteq F$ so $F' \subseteq \uparrow x \subseteq F$. Hence $F \leq^+ F'$.
(2.) $I \leq^+ F \leq^+ I'$. Since $F \leq^+ I'$ there must exist an $x \in F \cap I'$. Together with the fact that $I \leq^+ F$ this tells us that $i \leq x$ for all $i \in I$. Hence $I \subseteq \downarrow x \subseteq I'$. So $I \leq^+ I'$.

□

Let $\equiv$ be the equivalence relation induced by $\leq^+$ (i.e. $G \equiv H$ iff $G \leq^+ H$ and $H \leq^+ G$). Then clearly $\leq$, defined as $\equiv / \leq^+$, is a partial order such that for all $G, H \in \mathscr{F} \uplus \mathscr{I}$, $[G] \leq [H]$ iff $G \leq^+ H$.

The next lemma illustrates the crucial role played by the principal upsets and downsets in ensuring that $\mathbb{P}$ embeds into $\mathbb{F} \oplus \mathbb{I}$. From here on we will assume that $\uparrow p = \{a \in P : a \geq p\} \in \mathscr{F}$ and $\downarrow p = \{a \in P : a \leq p\} \in \mathscr{I}$, for each $p \in \mathbb{P}$.

Lemma 2.3.2.

1. *For all* $p \in P$, $[\uparrow p] = [\downarrow p]$.
2. *These are the only non-equal elements identified by* $\equiv$.
3. $[\uparrow] : \mathbb{P} \to \mathbb{F} \oplus \mathbb{I}$ *is an order embedding.*

Proof.
(1.) For all $q \in \downarrow p$ and $r \in \uparrow p$, $q \leq p \leq r$ so $\downarrow p \leq^+ \uparrow p$. Also $p \in \uparrow p \cap \downarrow p$ so $\downarrow p \leq^+ \uparrow p$.
(2.) Let $G, H \in \mathscr{F} \uplus \mathscr{I}$ and assume $[G] = [H]$. If $G, H \in \mathscr{F}$ or $G, H \in \mathscr{I}$ then clearly $G = H$ so let $G \in \mathscr{F}$ and $H \in \mathscr{I}$. Then there is some $x \in G \cap H$ and for all $g \in G, g \geq x$ and for all $h \in H, h \leq x$. Therefore since G is an upset containing x, $G = \uparrow x$ and since H is a downset containing x, $H = \downarrow x$.
(3.) $p \leq q$ iff $\uparrow q \subseteq \uparrow p$ iff $\uparrow p \leq^+ \uparrow q$ iff $[\uparrow p] \leq [\uparrow q]$ □

Finally, we define the amalgamation $\mathbb{F} \oplus \mathbb{I}$ as $(\mathscr{F} \uplus \mathscr{I} / \equiv, \leq)$. The next three subsections will show that $([\uparrow], \mathbb{F} \oplus \mathbb{I})$ is an $(\mathscr{F}, \mathscr{I})$-extension of $\mathbb{P}$.

2.3.2 Compactness

This lemma essentially shows that every filter is the meet of its elements, and every ideal is the join of its elements.

Lemma 2.3.3. *For every* $F \in \mathscr{F}, I \in \mathscr{I}$:

1. $\bigwedge\{[\uparrow p] | p \in F\} = [F]$
2. $\bigvee\{[\uparrow p] | p \in I\} = [I]$

Proof.
(1.) For every $p \in F$, $\uparrow p \subseteq F \Rightarrow [F] \leq [\uparrow p]$. Now assume there is some $[G] \leq [\uparrow p]$ for every $p \in F$. If $G \in \mathscr{F}$ then $F \subseteq G$ and hence $[G] \leq [F]$. Otherwise, if $G \in \mathscr{I}$, then for every $g \in G$, $g \leq p$ for every $p \in F$. Hence $[G] \leq [F]$.
(2.) Can be proven exactly dually to (1) since $[\uparrow p] = [\downarrow p]$ for each $p \in P$. □

Proposition 2.3.4. $\mathbb{F} \oplus \mathbb{I}$ *is* $(\mathscr{F}, \mathscr{I})$*-compact.*

Proof. For every $F \in \mathscr{F}, I \in \mathscr{I}$:
$\bigwedge[\uparrow][F] = \bigwedge\{[\uparrow p] | p \in F\} = [F]$ and $\bigvee[\uparrow][I] = \bigvee\{[\uparrow p] | p \in I\} = [I]$ by the lemma. So $\bigwedge[\uparrow][F] \leq \bigwedge[\uparrow][F] \Rightarrow [F] \leq [I]$ and therefore $F \cap I \neq \varnothing$. □

2.3.3 $(\mathscr{F},\mathscr{I})$-compatibility

This lemma shows that the members of $\mathbb{P}$ lie above a filter F (below an ideal I) in the amalgamation, if and only if they belong to F (belong to I).

Lemma 2.3.5. *For every $F \in \mathscr{F}, I \in \mathscr{I}$:*

1. $F_{[F]} = F$.
2. $I_{[I]} = I$.

Proof. (1.) For every $p \in P$, $p \in F_{[F]}$ iff $\uparrow p \geq [F]$ iff $\uparrow p \subseteq F$ iff $p \in F$
(2.) Can be proven exactly dually to (1). □

Proposition 2.3.6. $\mathbb{F} \oplus \mathbb{I}$ *is an $(\mathscr{F},\mathscr{I})$-compatible extension of* $\mathbb{P}$.

Proof. Follows directly from lemmas 2.3.3 and 2.3.5. □

2.3.4 Denseness

Proposition 2.3.7. $\mathbb{F} \oplus \mathbb{I}$ *is $(\mathscr{F},\mathscr{I})$-dense.*

Proof. For $x \in \mathscr{F} \uplus \mathscr{I}$, there are two cases:
(1.) If $x = [F]$ for some $F \in \mathscr{F}$, then clearly $x = \bigvee\{[F]\}$ and as seen in lemma 2.3.3 $x = \bigwedge e[F] = \bigwedge\{[\downarrow p] : p \in F\}$.
(2.) Similarly, if $x = [I]$ for some $I \in \mathscr{I}$, then $x = \bigwedge\{[I]\}$ and as seen in lemma 2.3.3 $x = \bigvee e[I] = \bigvee\{[\uparrow p] : p \in I\}$. □

2.3.5 Taking the MacNeille completion

Definition Let $\langle P, \leq\rangle$ be a partially ordered set. For any $A \subseteq P$ let $A^u = \{p \in P : p \geq a \text{ for all } a \in A\}$ and $A^l = \{p \in P : p \leq a \text{ for all } a \in A\}$. The MacNeille completion of $\mathbb{P}$, denoted by $\overline{\mathbb{P}}$, consists of the collection $\{A \in \mathcal{P}(P) : (A^u)^l = A\}$ with the inclusion ordering and the embedding $\downarrow$ which sends $a \in P$ to $\downarrow a$.

The special defining characteristics of the MacNeille completion, (given as Theorems 7.40 and 7.41 in [8]) are the following:

- Taking the MacNeille Completion of a poset preserves all its (existing) joins and meets
- A poset $\mathbb{P}$ is both join dense and meet dense in its MacNeille completion

- If $(i, \mathbb{L})$ is a completion of $\mathbb{P}$ such that $\mathbb{P}$ is both join dense and meet dense in $\mathbb{L}$, then $\mathbb{L}$ is isomorphic to the MacNeille completion of $\mathbb{P}$ via an order isomorphism fixing $\mathbb{P}$.

Let $\mathbb{P}$ be a poset, let $(e, \mathbb{Q})$ be an $(\mathscr{F}, \mathscr{I})$-compatible extension of $\mathbb{P}$ and let $(\overline{e}, \overline{\mathbb{Q}})$ be the completion of $\mathbb{P}$ consisting of MacNeille completion of $\mathbb{Q}$ together with the embedding $\overline{e} = \uparrow \circ e$.

The following lemma shows that since MacNeille completions preserve existing meets and joins, closed elements in $\mathbb{Q}$ remain closed in $\overline{\mathbb{Q}}$ and open elements in $\mathbb{Q}$ remain open in $\overline{\mathbb{Q}}$.

Lemma 2.3.8.

1. $K_e^{\mathscr{F}}(\mathbb{Q}) \subseteq K_{\overline{e}}^{\mathscr{F}}(\overline{\mathbb{Q}})$
2. $O_e^{\mathscr{I}}(\mathbb{Q}) \subseteq O_{\overline{e}}^{\mathscr{I}}(\overline{\mathbb{Q}})$

Proof.
(1.) If a is closed in $\mathbb{Q}$, then $a = \bigwedge_{\mathbb{Q}} e[F]$ for some $F \in \mathscr{F}$. Then since MacNeille completions preserve meets, $a = \bigwedge_{\overline{\mathbb{Q}}} \overline{e}[F]$, so a is closed in $\overline{\mathbb{Q}}$.
(2.) Dual to 1. □

The next proposition shows the crucial facts that Macneille completions preserve the properties of $(\mathscr{F}, \mathscr{I})$-compatibility, $(\mathscr{F}, \mathscr{I})$-compactness and $(\mathscr{F}, \mathscr{I})$-denseness.

Proposition 2.3.9. *For every poset $\mathbb{P}$ and $(\mathscr{F}, \mathscr{I})$-compatible extension $(e, \mathbb{Q})$:*

1. *$(\overline{e}, \overline{\mathbb{Q}})$ is an $(\mathscr{F}, \mathscr{I})$-compatible extension of $\mathbb{P}$.*
2. *If $(e, \mathbb{Q})$ is $(\mathscr{F}, \mathscr{I})$-compact, then $(\overline{e}, \overline{\mathbb{Q}})$ is $(\mathscr{F}, \mathscr{I})$-compact.*
3. *If $(e, \mathbb{Q})$ is $(\mathscr{F}, \mathscr{I})$-dense, then $(\overline{e}, \overline{\mathbb{Q}})$ is $(\mathscr{F}, \mathscr{I})$-dense.*
4. *If $(e, \mathbb{Q})$ is an $(\mathscr{F}, \mathscr{I})$-extension of $\mathbb{P}$, then $(\overline{e}, \overline{\mathbb{Q}})$ is an $(\mathscr{F}, \mathscr{I})$-completion of $\mathbb{P}$.*

Proof.
(1.) Since $(e, \mathbb{Q})$ is an $(\mathscr{F}, \mathscr{I})$-compatible extension, for every $F \in \mathscr{F}$ there exists an $a \in \mathbb{Q}$ such that $\bigwedge_{\mathbb{Q}} e[F] = a$ and $F = \mathbf{F_a}$. Then $\uparrow a \in \mathbb{Q}$ and since Macneille completions preserve meets, $\bigwedge_{\overline{\mathbb{Q}}} \overline{e}[F] = \uparrow a$. Then simply because $\uparrow$ is an embedding, we see that $p \in F$ iff $e(p) \leq a$ iff $\overline{e}(p) \leq \uparrow a$. The proof for $I \in \mathscr{I}$ is dual.
(2.) Assume $\bigwedge_{\overline{\mathbb{Q}}} e[F] \leq \bigvee_{\overline{\mathbb{Q}}} e[I]$ for some $F \in \mathscr{F}$ and $I \in \mathscr{I}$. Since $(e, \mathbb{Q})$ is an

$(\mathscr{F},\mathscr{I})$-compatible extension, this meet and join already exist in $\mathbb{Q}$, and are preserved (along with their relationship) in the MacNeille completion. So by denseness of $(e,\mathbb{Q})$, $F \cap I \neq \varnothing$.
(3.) By the lemma, it is enough to show that for each $x \in \overline{Q}$, $x = \bigwedge A = \bigvee B$ for $A \subseteq K(\mathbb{Q})$ and $B \subseteq O(\mathbb{Q})$. Since $\mathbb{Q}$ is meet and join dense in $\mathbb{Q}$, $x = \bigwedge Y = \bigvee Z$ for $Y, Z \subseteq \mathbb{Q}$. Then as $\mathbb{Q}$ is dense, for each $y \in Y$, $z \in Z$, $y = \bigwedge A_y$ and $z = \bigvee B_z$ for $A_y \subseteq K(Q), B_z \subseteq O(Q)$. Then $A = \bigcup\{A_y : y \in Y\} \subseteq K(Q)$ and $B = \{B_z : z \in Z\} \subseteq O(Q)$ and $x = \bigwedge A = \bigvee B$.
(4.) By 1-3. □

Corollary 2.3.10. *$(\overline{e}, \overline{\mathbb{F} \oplus \mathbb{I}})$ is an $(\mathscr{F},\mathscr{I})$-completion of $\mathbb{P}$.*

2.4 Uniqueness

In this section we will assume that $(e,\mathbb{Q})$ is an $(\mathscr{F},\mathscr{I})$-completion of $\mathbb{P}$ and show that it therefore must be isomorphic to $(\overline{e}, \overline{\mathbb{F} \oplus \mathbb{I}})$. Hence proving the uniqueness (up to isomorphism) of $(\mathscr{F},\mathscr{I})$-completions.

Lemma 2.4.1. *Let $(e,\mathbb{Q})$ be an $(\mathscr{F},\mathscr{I})$-compatible extension of $\mathbb{P}$. Then for every $a \in K(\mathbb{Q})$ and $b \in O(\mathbb{Q})$:*

1. *$a = \bigwedge e[F_a]$ and $F_a \in \mathscr{F}$*
2. *$b = \bigvee e[I_b]$ and $I_b \in \mathscr{I}$*

Proof. (1.) Let $a = \bigwedge e[F]$ for some $F \in \mathscr{F}$. Then by $(\mathscr{F},\mathscr{I})$-compatibility there exists an $a' \in \mathbb{Q}$ such that $a' = \bigwedge e[F]$ and $F_{a'} = F$. So $a = a'$ and hence $F_a = F$. Therefore $a = \bigwedge e[F_a]$ and $F_a = F \in \mathscr{F}$.
(2.) Dual to 1. □

The next lemma and the following corollary show that the map $(K(\mathbb{Q}) \cup O(\mathbb{Q}), \leq_{\mathbb{Q}}) \longrightarrow \mathbb{F} \oplus \mathbb{I}$ defined for $a \in K(\mathbb{Q})$ by $a \longmapsto F_a$ and for $b \in O(\mathbb{Q})$ by $b \longmapsto I_b$ is both well-defined and an order embedding.

Lemma 2.4.2. *For all $a, a' \in K(\mathbb{Q})$ and $b, b' \in O(\mathbb{Q})$:*

- *$a \leq a'$ iff $F_a \leq F'_a$*
- *$b \leq b'$ iff $I_b \leq I'_b$*
- *$a \leq b$ iff $F_a \leq I_b$*
- *$b \leq a$ iff $I_b \leq F_a$*

Proof.
(1.) Since $a' = \bigwedge e[F_{a'}]$, we have $a \leq a'$ iff $\forall p \in \mathbb{P}(e(p) \geq a' \Rightarrow e(p) \geq a)$ iff $F'_a \subseteq F_a$ iff $F_a \leq F'_a$.
(2.) Dual to 1.
(3.) $a \leq b$ implies $\bigwedge e[F_a] \leq \bigvee e[I_b]$. So by compactness there exists $c \in F_a \cap I_b$. (Implies $a \leq e(c) \leq b$.) Implies $F_a \leq I_b$. Conversely, $F_a \leq I_b$ implies $\exists c \in \mathbb{P}$ such that $a \leq e(c) \leq b$ and hence $a \leq b$.
(4.) $b \leq a$ iff $\bigvee e[I_b] \leq \bigwedge e[F_a]$ by the lemma
iff $(\forall p \in I_b)(\forall q \in F_a)(e(p) \leq e(q))$ by definition of $\bigvee$ and $\bigwedge$
iff $(\forall p \in I_b)(\forall q \in F_a)(p \leq q)$ since e is an order embedding
iff $I_b \leq F_a$. □

Corollary 2.4.3. *If $a \in K(\mathbb{Q}) \cap O(\mathbb{Q})$, then $[F_a] = [I_a]$*

Then $(\mathscr{F}, \mathscr{I})$-compatibility shows that it must also be onto. Consequently, $K(\mathbb{Q}) \cup O(\mathbb{Q}) \cong \mathbb{F} \oplus \mathbb{I}$.

Theorem 2.4.4. $(e, \mathbb{Q}) \cong \overline{K(\mathbb{Q}) \cup O(\mathbb{Q})} \cong \overline{\mathbb{F} \oplus \mathbb{I}}$

Proof. Denseness of $(e, \mathbb{Q})$, means that $K(\mathbb{Q}) \cup O(\mathbb{Q})$ is both join and meet dense in $(e, \mathbb{Q})$. But we saw above that the MacNeille completion $\overline{K(\mathbb{Q}) \cup O(\mathbb{Q})}$ must then be isomorphic to $(e, \mathbb{Q})$. So it only remains to put this together with the above result that $K(\mathbb{Q}) \cup O(\mathbb{Q}) \cong \mathbb{F} \oplus \mathbb{I}$, which clearly implies that $\overline{K(\mathbb{Q}) \cup O(\mathbb{Q})} \cong \overline{\mathbb{F} \oplus \mathbb{I}}$. □

2.5 Properties of the $(\mathscr{F}, \mathscr{I})$-completion

In this section we give conditions which, if satisfied, guarantee that taking canonical extensions commutes with both duals and products. We also see conditions which ensure that the canonical extension is join generated by its completely join irreducibles and meet generated by its completely meet irreducibles.

Throughout this section, because we will be referring to multiple posets, the notation $\mathscr{F}(\mathbb{P})$ will be used to refer to the chosen set of filters of each poset $\mathbb{P}$. Similarly $\mathscr{I}(\mathbb{P})$ will refer to the chosen ideals of $\mathbb{P}$. It will be assumed that the principal filters and ideals are included in $\mathscr{F}(\mathbb{P})$ and $\mathscr{I}(\mathbb{P})$ respectively, so that the $(\mathscr{F}(\mathbb{P}), \mathscr{I}(\mathbb{P}))$-completion, which we will call $\mathbb{P}^\sigma$, is always defined. Further, we will assume that $\mathbb{P}$ is included in (rather than just embeddable in) its canonical extension, and reference to the filters and ideals when talking about closed and open elements will be dropped; they will be denoted by $K(\mathbb{P}^\sigma)$ and $O(\mathbb{P}^\sigma)$ respectively.

2.5.1 Commuting with duals

For any partial order $\mathbb{P}$, let $\mathbb{P}^\delta$ denote its order dual.

Proposition 2.5.1. *For every partial order* $\mathbb{P}$*, if* $\mathscr{F}(\mathbb{P}^\delta) = \mathscr{I}(\mathbb{P})$ *and* $\mathscr{I}(\mathbb{P}^\delta) = \mathscr{F}(\mathbb{P})$ *then:*

1. $K((\mathbb{P}^\delta)^\sigma) = O(\mathbb{P}^\sigma)^\delta$
2. $O((\mathbb{P}^\delta)^\sigma) = K(\mathbb{P}^\sigma)^\delta$
3. $(\mathbb{P}^\delta)^\sigma = (\mathbb{P}^\sigma)^\delta$

Proof.
(1.) For every $a \in K((\mathbb{P}^\delta)^\sigma)$, $a = \bigwedge^\delta F = \bigvee F$ for some $F \in \mathscr{F}(\mathbb{P}^\delta) = \mathscr{I}(\mathbb{P})$. Hence $a \in O(\mathbb{P}^\sigma)$. Conversely, for every $b \in O(\mathbb{P}^\sigma)$, $b = \bigvee I = \bigwedge^\delta I$ for some $I \in \mathscr{I}(\mathbb{P}) = \mathscr{F}(\mathbb{P}^\delta)$, so $b \in K((\mathbb{P}^\delta)^\sigma)$. So, as sets, $K((\mathbb{P}^\delta)^\sigma) = O(\mathbb{P}^\sigma)$. But the order on $O(\mathbb{P}^\sigma)$ is that of $\subseteq$, whereas the order on $K((\mathbb{P}^\delta)^\sigma)$ is that of $\supseteq$, so, as ordered sets, $K((\mathbb{P}^\delta)^\sigma) = O(\mathbb{P}^\sigma)^\delta$.
(2.) Dual to 1.
(3.) To see this we need to show that $K((\mathbb{P}^\delta)^\sigma) \cup O((\mathbb{P}^\delta)^\sigma)$ is isomorphic to $O(\mathbb{P}^\sigma) \cup K(\mathbb{P}^\sigma)$ as it sits inside $(\mathbb{P}^\sigma)^\delta$, that is, to $(O(\mathbb{P}^\sigma) \cup K(\mathbb{P}^\sigma))^\delta$. To this end just notice that the conditions that uniquely determine the order on the union, see lemma 2.4.2, are self-dual in the sense that one gets the exact same conditions if closed is replaced by open, open by closed, and the order is reversed. □

2.5.2 Commuting with products

Proposition 2.5.2. *For every pair of partial orders* $\mathbb{P}_1$ *and* $\mathbb{P}_2$*, if*
-for every $F \in \mathscr{F}(\mathbb{P}_1 \times \mathbb{P}_2)$ $\pi_i[F] \in \mathscr{F}(\mathbb{P}_i)$ *for* $i = 1, 2$ *and for every* $F_i \in \mathscr{F}(\mathbb{P}_i)$, $F_1 \times F_2 \in \mathscr{F}(\mathbb{P}_1 \times \mathbb{P}_2)$
-for every $I \in \mathscr{I}(\mathbb{P}_1 \times \mathbb{P}_2)$ $\pi_i[I] \in \mathscr{I}(\mathbb{P}_i)$ *for* $i = 1, 2$ *and for every* $I_i \in \mathscr{I}(\mathbb{P}_i)$, $I_1 \times I_2 \in \mathscr{I}(\mathbb{P}_1 \times \mathbb{P}_2)$
then:

1. $K((\mathbb{P}_1 \times \mathbb{P}_2)^\sigma) = K(\mathbb{P}_1^\sigma) \times K(\mathbb{P}_2^\sigma)$
2. $O((\mathbb{P}_1 \times \mathbb{P}_2)^\sigma) = O(\mathbb{P}_1^\sigma) \times O(\mathbb{P}_2^\sigma)$

3. $(\mathbb{P}_1 \times \mathbb{P}_2)^\sigma = \mathbb{P}_1^\sigma \times \mathbb{P}_2^\sigma$

Proof.
(1.) ($\subseteq$) Let $\overline{u} = (u_1, u_2) \in K((\mathbb{P}_1 \times \mathbb{P}_2)^\sigma)$. Then $\overline{u} = \bigwedge F$, for some $F \in \mathscr{F}(\mathbb{P}_1 \times \mathbb{P}_2)$, so $\pi_i[F] \in \mathscr{F}(\mathbb{P}_i)$, $i = 1, 2$. Hence $\bigwedge \pi_i[F] \in K(\mathbb{P}_i^\sigma)$, so it is enough to show that $\overline{u} = (\bigwedge \pi_1[F], \bigwedge \pi_2[F])$: As $\overline{u} \leq \overline{v}$ for every $\overline{v} \in F$, then $u_i \leq \bigwedge \pi_i[F]$. Conversely, as $\bigwedge \pi_i[F] \leq v_i$ for every $v_i \in \pi_i[F]$, then $(\bigwedge \pi_1[F], \bigwedge \pi_2[F]) \leq \overline{v}$ for every $\overline{v} \in F$, and so $(\bigwedge \pi_1[F], \bigwedge \pi_2[F]) \leq \bigwedge F = \overline{u}$.

($\supseteq$) Let $\overline{u} = (u_1, u_2) \in K(\mathbb{P}_1, \mathbb{P}_1^\sigma) \times K(\mathbb{P}_2, \mathbb{P}_2^\sigma)$. Then $u_i = \bigwedge F_{u_i}$ and $\varnothing \neq F_{u_i} \in \mathscr{F}(\mathbb{P}_i)$, so $\varnothing \neq F = F_{u_1} \times F_{u_2} \in \mathscr{F}(\mathbb{P}_1 \times \mathbb{P}_2)$. Let us show that $\overline{u} = \bigwedge F$: As $u_i = \bigwedge F_{u_i}$, then $u_i \leq p_i$ for every $p_i \in F_{u_i}$, hence $\overline{u} = (u_1, u_2) \leq (p_i, p_2)$ for every $(p_1, p_2) \in F$. If $(v_1, v_2) \leq (p_1, p_2)$ for every $(p_1, p_2) \in F$, then $v_i \leq \bigwedge \pi_i[F] = \bigwedge F_{u_i} = u_i$, hence $(v_1, v_2) \leq (u_1, u_2)$. $\square$

2.5.3 Completely join irreducibles

Definition Let $\mathbb{C}$ be a complete lattice. Then $j \in \mathbb{C}$ is completely join irreducible iff for every $A \subseteq C$: $j = \bigvee A$ implies $j \in A$. An element $m \in C$ is completely meet irreducible iff: $m = \bigwedge A$ implies $m \in A$. Let $J(\mathbb{C})$ and $M(\mathbb{C})$ refer to the sets of completely join irreducible and completely meet irreducible elements of $\mathbb{C}$ respectively.

The following gives a condition on $\mathscr{F}$ under which $\mathbb{P}^\sigma$ is join generated by its completely join irreducibles and a condition on $\mathscr{I}$ under which $\mathbb{P}^\sigma$ is meet generated by its completely meet irreducibles.

Proposition 2.5.3. *For every partial order $\mathbb{P}$,*

1. *if $\mathscr{F}(\mathbb{P})$ is closed under unions of $\subseteq$-chains, then*
   (a) *for every $a \in K(\mathbb{P}^\sigma)$, $b \in O(\mathbb{P}^\sigma)$, if $a \nleq b$, then $j \leq a$ and $j \nleq b$ for some $j \in J(\mathbb{P}^\sigma)$, and so*
   (b) *for every $x \in \mathbb{P}^\sigma$, $x = \bigvee\{j \in J(\mathbb{P}^\sigma) \mid j \leq x\}$.*
2. *if $\mathscr{I}(\mathbb{P})$ is closed under unions of $\subseteq$-chains, then*
   (a) *for every $a \in K(\mathbb{P}^\sigma)$, $b \in O(\mathbb{P}^\sigma)$, if $a \nleq b$, then $m \ngeq a$ and $m \geq b$ for some $m \in M(\mathbb{P}^\sigma)$, and so*
   (b) *for every $x \in \mathbb{P}^\sigma$, $x = \bigwedge\{m \in m(\mathbb{P}^\sigma) \mid x \leq m\}$.*

Proof. (1.) Let $a \in K(\mathbb{P}^\sigma)$, $b \in O(\mathbb{P}^\sigma)$. If $\bigwedge F_a = a \nleq b = \bigvee I_b$, then $F_a \cap I_b = F_a\!\uparrow \cap I_b\!\downarrow = \varnothing$. As $\mathscr{F}(\mathbb{P})$ is closed under unions of $\subseteq$-chains, then by Zorn's lemma there exists $F \in \mathscr{F}$ that is maximal among the elements in $\mathscr{F}$ that are disjoint from I_b. Let $j = \bigwedge F$. As $F \cap I_b = \varnothing$, then $j \nleq b$, for if $\bigwedge F = j \leq b = \bigvee I_b$, then by compactness $F \cap I_b \neq \varnothing$. Moreover, $F = F_j = \{p \in P \mid j \leq p\}$: indeed, as $j = \bigwedge F$, then $F \subseteq F_j$. If $F \subset F_j$, then by maximality, $F_j \cap I_b \neq \varnothing$, hence there exists $p \in P$ such that $j \leq p \leq b$, contradiction.
Let us show that j is join irreducible: By denseness, it is enough to show that if $A \subseteq K(\mathbb{P}^\sigma)$ such that $j = \bigvee A$, then $j \in A$. Suppose that $j \neq a'$ for every $a' \in A$. Hence, as $j = \bigvee A$ implies that $a' \leq j$ for every $a' \in A$, we get $a' < j$, and so $F = F_j \subset F_{a'}$, which implies, by maximality, that $F_{a'} \cap I_b \neq \varnothing$, hence for every $a' \in A$ there exists $p_{a'} \in I_b$ such that $a' \leq p_{a'}$. Then $\bigwedge F = a = \bigvee A \leq \bigvee I_b$ and so, by compactness, $F \cap I_b \neq \varnothing$, contradiction.

(2.) Let $x \in \mathbb{P}^\sigma$. By denseness $x = \bigvee\{a \in K(\mathbb{P}^\sigma) \mid a \leq x\}$, and so by associativity of the join, it is enough to show that $a = \bigvee\{j \in J(\mathbb{P}^\sigma) \mid j \leq a\}$ for every $a \in K(\mathbb{P}^\sigma)$. Let $a \in K(\mathbb{P}^\sigma)$, $y = \bigvee\{j \in J(\mathbb{P}^\sigma) \mid a \leq j\}$, and suppose that $a \neq y$. Then, as $y = \bigvee\{j \in J(\mathbb{P}^\sigma) \mid a \leq j\} \leq a$, we get that $a \nleq y$. So in order to get to a contradiction, it is enough to show that $a \nleq y$ implies that there exists $j_0 \in J(\mathbb{P}^\sigma)$ such that $j_0 \leq a$ (hence $j_0 \leq \bigvee\{j \in J(\mathbb{P}^\sigma) \mid a \leq j\} = y$) and $j_0 \nleq y$. By denseness, if $a \nleq y = \bigwedge\{b \in O(\mathbb{P}^\sigma) \mid y \leq b\}$, then there exists $b \in O(\mathbb{P}^\sigma)$ such that $y \leq b$ and $a \nleq b$, but then, by item 1 of this proposition, $j \leq a$ and $j \nleq b$ for some $j \in J(\mathbb{P}^\sigma)$, hence $j \nleq y$ (for if not, then $j \leq y \leq b$). □

Chapter 3

Canonical Extensions of monotone maps

3.1 Defining f^π and f^σ for monotone f

Monotone maps are maps that are order preserving or reversing in each coordinate. In this chapter, canonical extensions are defined for monotone maps (as in [11]) in the case that the operation of taking canonical extensions commutes both with products and duals. For each monotone map f, two alternatives of extension are defined, f^π and f^σ.

Definition A monotone map (over P) is an order preserving function $f : \prod_{i \le n} P_i \longrightarrow P$ where P is a poset and for each $1 \le i \le n$, either $P_i = P$ or $P_i = P^\delta$.

Let f be a monotone map over a poset P and let $P^\sigma = \overline{\mathbb{F} \oplus \mathbb{I}}$ for a choice of $\mathscr{F}$ and $\mathscr{I}$ that satisfy the pre-conditions of propositions 2.5.1 and 2.5.2.

Since each closed element F in P^σ is the meet of elements in P, and each open element I in P^σ is the join of elements in P, the most obvious way to extend f to closed and open elements of P^σ is as follows:

$$f(F) = \bigwedge\{f(p) : F \le p \in P\}$$

$$f(I) = \bigvee\{f(p) : I \ge p \in P\}$$

Then since each element of the extension is both a meet of open elements and a join of closed elements, there are two natural ways f^π and f^σ to extend f to each $u \in P^\sigma$:

$$f^{\pi}(u) = \bigwedge\{f(I) : u \leq I \in O(P^{\sigma})\}$$
$$f^{\sigma}(u) = \bigvee\{f(F) : u \geq F \in K(P^{\sigma})\}$$

In summary we define:

$$f^{\pi}(u) = \bigwedge\{\bigvee\{f(p) : I \geq p \in P\} : u \leq I \in O(P^{\sigma})\}$$
$$f^{\sigma}(u) = \bigvee\{\bigwedge\{f(p) : F \leq p \in P\} : u \geq F \in K(P^{\sigma})\}$$

Note that this is also how monotone maps are extended for canonical extensions in lattice settings (see [31]).

3.2 Simplifying the definitions for two special cases

We now turn our attention to the special cases of two binary maps, f and g such that f is order reversing in the first coordinate and order preserving in the second and g is order preserving in the first coordinate and order reversing in the second, and we show how we can simplify the definitions given above. Clearly f and g can be viewed as monotone maps such that $f : P^{\delta} \times P \longrightarrow P$ and $g : P \times P^{\delta} \longrightarrow P$. We now show how we can simplify the definitions of f^{π} and g^{σ}.

Let $\bar{u} = (u_1, u_2) \in P^{\delta} \times P$. By the above definition (using $\bar{i}$ in place of I):

$$f^{\pi}(\bar{u}) = \bigwedge\{\bigvee\{f(\bar{p}) : \bar{i} \geq \bar{p} \in P^{\delta} \times P\} : \bar{u} \leq \bar{i} \in O((P^{\delta} \times P)^{\sigma})\}$$

By propositions 2.5.2 and 2.5.2:

$$O((P^{\delta} \times P)^{\sigma}) = O((P^{\delta})^{\sigma}) \times O(P^{\sigma}) = (K(P^{\sigma}))^{\delta} \times O(P^{\sigma})$$

So $\bar{i} \in O((P^{\delta} \times P)^{\sigma})$ iff $\bar{i} = (c, o)$ for some $c \in (K(P^{\sigma}))^{\delta}$ and $o \in O(P^{\sigma})$.

Then $\bar{i} \geq \bar{p}$ iff $c \leq p_1$ and $o \geq p_2$, and $\bar{u} \leq \bar{i}$ iff $u_1 \geq c$ and $u_2 \leq o$.

Hence we can rewrite f^{π} as:

$$f^{\pi}(u_1, u_2) = \bigwedge\{\bigvee\{f(p_1, p_2) : p_1, p_2 \in P, p_1 \geq c, p_2 \leq o\} : c \in K, o \in O, c \leq u_1, o \geq u_2\}$$

As for g^{σ}, we start with (replacing F with $\bar{f}$):

$$g^{\sigma}(\bar{u}) = \bigvee\{\bigwedge\{g(\bar{p}) : \bar{f} \leq \bar{p} \in P \times P^{\delta}\} : \bar{u} \geq \bar{f} \in K((P \times P^{\delta})^{\sigma})\}$$

Then propositions 2.5.2 and 2.5.2 tell us that:

$$K((P \times P^\delta)^\sigma) = K(P^\sigma) \times K((P^\delta)^\sigma) = K(P^\sigma) \times O((P^\sigma))^\delta$$

So $\bar{f} \in K((P \times P^\delta)^\sigma)$ iff $\bar{f} = (c, o)$ for some $c \in (K(P^\sigma))$ and $o \in O(P^\sigma)^\delta$.

Then $\bar{f} \geq \bar{p}$ iff $c \leq p_1$ and $o \geq p_2$, and $\bar{u} \leq \bar{f}$ iff $u_1 \geq c$ and $u_2 \leq o$.

So we can rewrite g^σ as:

$$g^\sigma(u_1, u_2) = \bigvee\{\bigwedge\{g(p_1, p_2) : p_1, p_2 \in P, p_1 \geq c, p_2 \leq o\} : c \in K, b \in O, c \leq u_1, o \geq u_2\}$$

Chapter 4

Bi-implicative Algebras and extensions

In this chapter implicative (and bi-implicative) algebras are introduced and their relationship to topped (bounded) partially ordered sets is explained. In particular, every implicative algebra gives rise to a poset and every poset to a normal implicative algebra. In fact the category of normal implicative algebras with injective homomorphisms is isomorphic to the category of topped posets with top-preserving order embeddings. Following this, a number of important extensions to the axiomatization of implicative algebras are discussed and their relationship to one another is investigated. The first of these conditions is an algebraic modus tollens which is shown in section 5.1 to be important for ensuring that an algebra can be embedded in its canonical extension. Next are conditions expressing that the implication operator is order reversing in the first coordinate and order preserving in the second, and the subtraction operator is order preserving in the first and order reversing in the second coordinate, which are necessary for defining the extensions of these operators. Then we look at Hilbert (and bi-Hilbert) algebras, a well-known class of algebras. Finally, $(\top)$ and $(\bot)$ are two conditions such that if they hold in a bi-implicative algebra, its extension will be again a bi-implicative algebra.

4.1 Implicative algebras and their relationship to partially ordered sets

Implicative algebras are the weakest algebras of the signature $(\to, \top)$ such that the relation defined by $a \leq b$ iff $a \to b = \top$ is a poset with top $\top$. It is shown below that the class of implicative algebras properly contains the more well-known class of Hilbert algebras. Note that the class of implicative algebras is not equationally definable. For a more comprehensive expose of the theory of implicative algebras see [29].

Definition A set A together with a binary operation $\to$ and constant $\top$ is an implicative algebra iff the following conditions are satisfied for all $a, b \in A$:

$$a \to a = \top \tag{I_1}$$

$$\text{If } a \to b = \top \text{ and } b \to c = \top, \text{ then } a \to c = \top \tag{I_2}$$

$$\text{If } a \to b = \top \text{ and } b \to a = \top, \text{ then } a = b \tag{I_3}$$

$$a \to \top = \top \tag{I_4}$$

Given an implicative algebra $\mathbb{A}$, define a relation on A by:

$$a \leq b \text{ iff } a \to b = \top$$

Then (I_1),(I_2) and (I_3) imply that $\leq$ is reflexive, transitive and anti-symmetric respectively and (I_4) implies that $\top$ is a largest element in A. So $\langle A, \leq, \top\rangle$ which we shall call $\mathbb{A}_*$ is a topped poset.

Alternatively, if we begin with a partially ordered set $\mathbb{P} = \langle P, \leq\rangle$ with a greatest element $\top$, we can define $\mathbb{P}^*$ with an implication $\to$ on P by letting

$$p \to q = \begin{cases} \top & \text{if } p \leq q \\ q & \text{otherwise} \end{cases} \tag{N}$$

Then the following proposition shows that this definition always yields an implicative algebra.

Proposition 4.1.1. *If $\mathbb{P} = \langle P, \leq, \top\rangle$ is a topped partially ordered set, then $\mathbb{P}^* = \langle P, \to, \top\rangle$ (where $\to$ is given by (N)) is an implicative algebra.*

Proof. Note that if $p = \top$ then $q \leq p$ and we are in the first case, so $q \leq p$ iff $q \to p = \top$. There are four axioms to verify.

1. $q \leq q \Rightarrow q \to q = \top$.
2. $q \to p = \top$ and $p \to r = \top \Rightarrow q \leq p$ and $p \leq r \Rightarrow q \leq r \Rightarrow q \to r = \top$.
3. $q \to p = \top$ and $p \to q = \top \Rightarrow q \leq p$ and $p \leq q \Rightarrow q = p$.
4. For all q, $q \leq \top$ so $q \to \top = \top$.

□

Note that not every implicative algebra can be defined in this way (see example just above 4.3.3), and let those that are be called normal. Note also that an implicative algebra $\mathbb{A}$ is normal iff $a \to b \in \{\top, b\}$ for all $a, b \in \mathbb{A}$. In the remainder of this section, the relationships between the implicative algebras, the normal implicative algebras and topped posets is investigated. Let Imp denote the category of implicative algebras with homomorphisms, Imp* denote the category of normal implicative algebras with injective homomorphisms, and TPos denote the category of topped posets with top-preserving order isomorphisms. Earlier in this section it was shown how one can obtain a topped poset from any implicative algebra. Let us extend this to a functor $(\_)_* : Imp \longrightarrow TPos$ by letting $(A)_*$ be defined as before for each object A and $(f)_* = f$ for each morphism f. The following proposition shows that this is well defined. It also clearly preserves identity morphisms and composition of morphisms, so is in fact a functor.

Proposition 4.1.2. *Let $f : \mathbb{A} \to \mathbb{B}$ be an injective homomorphism ($A, B \in Imp$). Then $f : \mathbb{A}_* \to \mathbb{B}_*$, is an order embedding that preserves $\top$.*

Proof. Immediately we have that f preserves top. Take any $a, b \in A$. If $a \leq b$ then $f(a \to b) = f(\top) = \top$. So $f(a) \to f(b) = \top$ and hence $f(a) \leq f(b)$. If $a \nleq b$ then $a \to b \neq \top$ so by injectivity of f, $f(a \to b) \neq f(\top) = \top$ so $f(a) \to f(b) \neq \top$ and hence $f(a) \nleq f(b)$. □

We also saw a way to define a normal implicative algebra P^* from any topped poset P. Let us now also extend $(\_)^*$ to a functor $(\_)^* : TPos \longrightarrow Imp^*$, defining $(f)^* = f$ for each morphism f. The following proposition shows that $(\_)^*$ is also well-defined. Clearly it also preserves identity morphisms and composition of morphisms.

Proposition 4.1.3. *Let $f : \mathbb{P} \to \mathbb{Q}$ be a top-preserving order embedding. Then $f : \mathbb{P}^* \to \mathbb{Q}^*$, is an injective homomorphism.*

Proof. To show that f is a homomorphism we take $p, q \in P$: If $p \leq q \Rightarrow p \to q = \top$ and $f(p) \leq f(q) \Rightarrow f(p \to q) = f(\top) = \top = f(p) \to f(q)$. Otherwise

$p \to q = q$ and $f(p) \nleq f(q) \Rightarrow f(p \to q) = f(q)$ and $f(p) \to f(q) = f(q)$. So in either case $f(p \to q) = f(p) \to f(q)$. We also know that f preserves top. To see that f is injective, let $f(p) = f(q)$. So $f(p) \leq f(q)$ and $f(q) \leq f(p)$ which implies that $p \leq q$ and $q \leq p$. Hence $p = q$. □

Note that there also exists the inclusion functor $i : Imp^* \longrightarrow Imp$. The compositions of $(\_)^*$, $(\_)_*$ and i. The compositions of these functors then give three more functors $(\_)_* \circ i : Imp^* \longrightarrow TPos$, $(\_)^* \circ (\_)_* : Imp \longrightarrow Imp^*$ and $i \circ (\_)^* : TPos \longrightarrow Imp$ shown in the following diagrams.

$$\begin{tikzcd} & Imp^* \arrow[dl, "i"'] \arrow[dr, "(\_)_* \circ i"] & \\ Imp \arrow[rr, "(\_)_*"'] & & TPos \end{tikzcd}$$

$$\begin{tikzcd} & Imp \arrow[dl, "(\_)_*"'] \arrow[dr, "(\_)^* \circ (\_)_*"] & \\ TPos \arrow[rr, "(\_)^*"'] & & Imp^* \end{tikzcd}$$

$$\begin{tikzcd} & TPos \arrow[dl, "(\_)^*"'] \arrow[dr, "i \circ (\_)^*"] & \\ Imp^* \arrow[rr, "i"'] & & Imp \end{tikzcd}$$

This gives us the following pairs of functors between TPos, Imp and Imp*.

$$\begin{tikzcd} Imp^* \arrow[r, bend left, "(\_)_* \circ i"] & TPos \arrow[l, bend left, "(\_)^*"] \end{tikzcd}$$

$$\begin{tikzcd} Imp \arrow[r, bend left, "(\_)^* \circ (\_)_*"] & Imp^* \arrow[l, bend left, "i"] \end{tikzcd}$$

$$\begin{tikzcd} Imp \arrow[r, bend left, "(\_)_*"] & TPos \arrow[l, bend left, "i \circ (\_)^*"] \end{tikzcd}$$

The following lemma shows that $()_* \circ i$ and $()^*$ give an isomorphism between Imp* and TPos.

Lemma 4.1.4.

1. *For all* $A \in Imp^*$, $(\mathbb{A}^*)_* = \mathbb{A}$
2. *For all* $P \in TPos$, $(\mathbb{P}_*)^* = \mathbb{P}$
3. $(f^*)_* = f$ *for each morphism* $f \in TPos$
4. $(f_*)^* = f$ *for each morphism* $f \in Imp*$

Proof.

1. $dom(A) = dom(A_*) = dom((A_*)^*)$ and for all $a, b \in A$, $a \to b = \top$ in A iff $a \le b$ in A_* iff $a \to b = \top$ in $(A_*)^*$
2. $dom(P) = dom(P^*) = dom((P^*)_*)$ and for all $p, q \in P$, $p \le q$ in P iff $p \to q = \top$ in P^* iff $p \le q$ in $(P^*)_*$
3. and 4. are trivial since $(f)^* = f_* = f$.

□

4.2 Bi-implicative algebras

A bi-implicative algebra is an implicative algebra $(A, \to, \top)$ augmented with a binary subtraction operator $\leftarrow$ and constant $\bot$ such that the algebraic dual of $(A, \leftarrow, \bot)$ is an implicative algebra and for all $a, b \in A$,

$$a \to b = \top \text{ iff } a \le b \text{ iff } a \leftarrow b = \bot$$

where $\le$ is the partial order associated with $(A, \to, \top)$. The axioms for the subtraction connective are as follows:

$$a \leftarrow a = \bot \tag{$I_1^{\leftarrow}$}$$

$$\text{If } a \leftarrow b = \bot \text{ and } b \leftarrow c = \bot, \text{ then } a \leftarrow c = \bot \tag{$I_2^{\leftarrow}$}$$

$$\text{If } a \leftarrow b = \bot \text{ and } b \leftarrow a = \bot, \text{ then } a = b \tag{$I_3^{\leftarrow}$}$$

$$\bot \leftarrow a = \bot \tag{$I_4^{\leftarrow}$}$$

Now starting from a bounded poset $\langle P, \le, \top, \bot \rangle$ with topped poset reduct $\mathbb{P}$, we can define a bi-implicative algebra $\langle P, \to, \leftarrow, \top, \bot \rangle$ where $\langle P, \to, \top \rangle = \mathbb{P}^*$ and $\leftarrow$ given by

$$p \leftarrow q = \begin{cases} \bot & \text{if } p \le q \\ p & \text{otherwise} \end{cases} \tag{$N^{\leftarrow}$}$$

We call all bi-implicative algebras which can be defined in this way normal. See section 4.4.2 for an example of a non-normal bi-Hilbert algebra. Let Bi-Imp* be the category of normal bi-implicative algebras with injective homomorphisms. Let Bi-Imp denote the category of bi-implicative algebras with injective homomorphisms, Bi-Imp* denote the category of normal bi-implicative algebras with injective homomorphisms, and BPos denote the category of bounded posets with order isomorphisms that preserve these bounds. As was done in the previous section for implicative algebras, $(\_)_*$ and $(\_)^*$ can be extended to give functors in the bi-implicative setting. Define $(\_)_* :$ Bi-Imp $\longrightarrow$ BPos on morphisms f by letting $f_* = f$. The following lemma shows that this definition is well defined. Again obviously $(\_)_*$ preserves identities and commutes with composition.

Proposition 4.2.1. *Let $f : \mathbb{A} \to \mathbb{B}$ be an injective homomorphism ($A, B \in Bi-Imp$). Then $f : \mathbb{A}_* \to \mathbb{B}_*$, is an order embedding that preserves $\top$ and $\bot$.*

Proof. f is an injective homomorphism from A to B implies that f is an injective homomorphism from the implicative reduct of A to the implicative reduct of B. So by Proposition 4.1.2, f is an order embedding that preserves $\top$. Since f is also a homomorphism between bi-implicative algebras, $f(\bot) = \bot$, so f preserves $\bot$. □

$(\_)^* :$ BPos $\to$ Bi-Imp* is also defined on morphisms f by $f^* = f$. The following proposition shows that this is well defined. Then again it is obviously a functor.

Let $f : P \to Q$ be an order embedding that preserves top and bottom.

Proposition 4.2.2. *$f : P^* \to Q^*$ is an injective homomorphism.*

Proof. Since f is an order embedding that preserves top, Proposition 4.1.3 says that $f(\top) = \top$, for all $p, q \in P$, $f(p \to q) = f(p) \to f(q)$ and also that f is injective. By assumption it is also clear that $f(\bot) = \bot$ so it only remains to show that for each $p, q \in P$, $f(p \leftarrow q) = f(p) \leftarrow f(q)$. If $p \leq q$ then $p \leftarrow q = \bot$ and $f(p) \leq f(q) \Rightarrow f(p \leftarrow q) = f(\bot) = \bot = f(p) \leftarrow f(q)$. Otherwise $p \leftarrow q = q$ and $f(p) \nleq f(q) \Rightarrow f(p \leftarrow q) = f(q)$ and $f(p) \leftarrow f(q) = f(q)$. □

Again, there also exists the inclusion functor $i :$ Bi-Imp* $\longrightarrow$ Bi-Imp and the compositions of the three functors, give three more functors $(\_)_* \circ i :$ Bi-Imp* $\longrightarrow$ BPos, $(\_)^* \circ (\_)_* :$ Bi-Imp $\longrightarrow$ Bi-Imp* and $i \circ (\_)^* :$ BPos $\longrightarrow$ Bi-Imp shown in the following diagrams.

$$\begin{tikzcd}
& Bi-Imp^* \arrow[dl, "i"'] \arrow[dr, "(_)_* \circ i"] & \\
Bi-Imp \arrow[rr, "(_)_*"'] & & BPos
\end{tikzcd}$$

$$\begin{tikzcd}
& Bi-Imp \arrow[dl, "(_)_*"'] \arrow[dr, "(_)^* \circ (_)_*"] & \\
BPos \arrow[rr, "(_)^*"'] & & Bi-Imp^*
\end{tikzcd}$$

$$\begin{tikzcd}
& BPos \arrow[dl, "(_)^*"'] \arrow[dr, "i \circ (_)^*"] & \\
Bi-Imp^* \arrow[rr, "i"'] & & Bi-Imp
\end{tikzcd}$$

Therefore there are the following pairs of functors between BPos, Bi-Imp and Bi-Imp*.

$$\begin{tikzcd}
Bi-Imp^* \arrow[r, shift left, "(_)_* \circ i"] & BPos \arrow[l, shift left, "(_)^*"]
\end{tikzcd}$$

$$\begin{tikzcd}
Bi-Imp \arrow[r, shift left, "(_)^* \circ (_)_*"] & Bi-Imp^* \arrow[l, shift left, "i"]
\end{tikzcd}$$

$$\begin{tikzcd}
Bi-Imp \arrow[r, shift left, "(_)_*"] & BPos \arrow[l, shift left, "i \circ (_)^*"]
\end{tikzcd}$$

The following lemma shows that $()_* \circ i$ and $()^*$ give an isomorphism between Bi-Imp* and BPos.

Lemma 4.2.3.

1. *For all* $A \in$ *Bi-Imp**, $(\mathbb{A}^*)_* = \mathbb{A}$
2. *For all* $P \in$ *BPos*, $(\mathbb{P}_*)^* = \mathbb{P}$
3. $(f^*)_* = f$ *for each morphism* $f \in BPos$
4. $(f_*)^* = f$ *for each morphism* $f \in Bi-Imp*$

Proof.

1. $dom(A) = dom(A_*) = dom((A_*)^*)$ and for all $a, b \in A$, $a \to b = \top$ in A iff $a \leq b$ in A_* iff $a \to b = \top$ in $(A_*)^*$ and $a \leftarrow b = \bot$ in A iff $a \leq b$ in A_* iff $a \leftarrow b = \bot$ in $(A_*)^*$
2. $dom(P) = dom(P^*) = dom((P^*)_*)$ and for all $p, q \in P$, $p \leq q$ in P iff $p \to q = \top$ in P^* iff $p \leq q$ in $(P^*)_*$ and $p \leq q$ in P iff $p \leftarrow q = \bot$ in P^* iff $p \leq q$ in $(P^*)_*$
3. and 4. are trivial since $(f)^* = f_* = f$.

□

4.3 Additional axioms

4.3.1 Bi-Hilbert algebras

Hilbert algebras were introduced by Monteiro in [28]. Diego also made a fundamental contribution to the development of the theory of Hilbert algebras and implicative filters (see [9] and [10]) and in 1960 showed that they are equationally definable. In the same year, L. Iturrioz proved that (H) and (F) are independent. Further developments of the theory were introduced by Busneag, in [4] and [5].

Definition A Hilbert algebra is an implicative algebra in which the following two extra conditions hold:

$$a \to (b \to a) = \top \tag{H}$$

$$(a \to (b \to c)) \to ((a \to b) \to (a \to c)) = \top \tag{F}$$

The dual conditions which are added to define a bi-Hilbert algebra are:

$$(a \leftarrow b) \leftarrow a = \bot \tag{H$^{\leftarrow}$}$$

$$((a \leftarrow c) \leftarrow (b \leftarrow c)) \leftarrow ((a \leftarrow b) \leftarrow c) = \bot \tag{F$^{\leftarrow}$}$$

Lemma 4.3.1. *In every Hilbert algebra* $\top \to a = a$

Proof. Since $(\top \to a) \to (\top \to a) = \top$, an application of F reveals that $((\top \to a) \to \top) \to ((\top \to a) \to a) = \top$. But $(\top \to a) \to \top = \top$ so we have $\top \to ((\top \to a) \to a) = \top$. Therefore $(\top \to a) \to a = \top$ so $\top \to a \leq a$. By H $\top \to a \geq a$ so putting this together, $\top \to a = a$. □

To generate examples of Hilbert and bi-Hilbert algebras we make use of the complex algebra construction which generates a Heyting algebra from any partially ordered set. Then we take its $\langle \rightarrow, \top \rangle$ reduct, which is a Hilbert algebra, and define a subtraction operator to be the right residual of the union, just as the implication operator is the left residual of the intersection.

Given a partially ordered set $P = \langle X, \leq \rangle$, recall its complex heyting algebra is defined as $\langle \mathcal{P}^i(X), \cap, \cup, \longmapsto, X, \varnothing \rangle$ where $\mathcal{P}^i(X)$ is the set of subsets of X which are also upsets and for any $U, V \in \mathcal{P}^i(X)$, $U \longmapsto V = \bigcup\{Z \in \mathcal{P}^i(X) : Z \cap U \subseteq V\} = \{x \in X : \uparrow x \cap U \subseteq V\}$.

Note that by this definition, $Z \cap U \subseteq V$ iff $Z \subseteq U \longmapsto V$ ie. $\longmapsto$ is the left residual of $\cap$. We define the subtraction operator by

$$U \longleftarrow\!\!\dashv V = \bigcap\{W \in \mathcal{P}^i(X) : U \subseteq V \cup W\}$$

so that $U \subseteq V \cup W$ iff $U \longleftarrow\!\!\dashv V \subseteq W$. We shall call $P^+ = \langle \mathcal{P}^i(X), X, \varnothing, \longmapsto, \longleftarrow\!\!\dashv \rangle$ the complex bi-Hilbert algebra of P, and its implicative reduct, the complex Hilbert algebra of P.

This lemma gives a more workable definition of $\longleftarrow\!\!\dashv$, and the following theorem shows that P^+ is in fact a bi-Hilbert algebra.

Lemma 4.3.2. *For any $U, V \in \mathcal{P}^i(X)$, $\bigcap\{W \in \mathcal{P}^i(X) : U \subseteq V \cup W\} = \{x \in X : U \nsubseteq V \cup X \backslash \downarrow \{x\}\}$*

Proof. Each direction will be shown contrapositively.

- ($\subseteq$) If $U \subseteq V \cup X \backslash \downarrow x$, then $X \backslash \downarrow x$ is a $W \in \mathcal{P}^i(X)$ such that $U \subseteq V \cup W$ and $x \notin X \backslash \downarrow x$ so $x \notin \bigcap\{W \in \mathcal{P}^i(X) : U \subseteq V \cup W\}$.
- ($\supseteq$) If $x \notin W$ for some $W \in \mathcal{P}^i(X)$ such that $U \subseteq V \cup W\}$, then $\downarrow x \cap W = \varnothing$ so $W \subseteq X \backslash \downarrow x$. Therefore $V \cup W \subseteq V \cup X \backslash \downarrow x$, so $U \subseteq V \cup X \backslash \downarrow x$ and hence $x \notin \{x \in X : U \nsubseteq V \cup X \backslash \downarrow \{x\}\}$

□

Proposition 4.3.3. *Given a partially ordered set $P = \langle X, \leq \rangle$, $P^+ = \langle \mathcal{P}^i(X), X, \varnothing, \longmapsto, \longleftarrow\!\!\dashv \rangle$ is a bi-Hilbert algebra.*

Proof. The fact that $\longleftarrow\!\!\dashv$ is residuated with $\bigcup$ shows that $U \longleftarrow\!\!\dashv V = \varnothing$ iff $U \subseteq V \cup \varnothing = V$, so P^+ is a bi-implicative algebra. To see ($\mathrm{H}^{\leftarrow}$), let $x \in U \longleftarrow\!\!\dashv V$, so $U \nsubseteq V \cup X \backslash \downarrow x$. Hence $U \nsubseteq X \backslash \downarrow x$, so there is some $u \in U$ such that $u \notin X \backslash \downarrow x$ ie. $u \leq x$. Then as U is an upset, $x \in U$. So $U \longleftarrow\!\!\dashv V \subseteq U$.

To prove ($F^{\leftarrow}$) we make extensive use of the residuation which shows that:
$(U \longleftarrow\!\!| \, W) \longleftarrow\!\!| \, (V \longleftarrow\!\!| \, W) \subseteq (U \longleftarrow\!\!| \, V) \longleftarrow\!\!| \, W$
iff $(U \longleftarrow\!\!| \, W) \subseteq (V \longleftarrow\!\!| \, W) \cup (U \longleftarrow\!\!| \, V) \longleftarrow\!\!| \, W$
iff $U \subseteq W \cup (V \longleftarrow\!\!| \, W) \cup (U \longleftarrow\!\!| \, V) \longleftarrow\!\!| \, W$.
Then we take some $x \in U$. Assume $x \notin W$ and $x \notin V \longleftarrow\!\!| \, W$. Then $W \subseteq X\backslash{\downarrow}x$ and $V \subseteq W \cup X\backslash{\downarrow}x = X\backslash{\downarrow}x$. So $V \cup X\backslash{\downarrow}x = X\backslash{\downarrow}x$. Then since $x \in U$ and $x \notin X\backslash{\downarrow}x$, $U \nsubseteq V \cup X\backslash{\downarrow}x$. Again using the residuation we get $U \longleftarrow\!\!| \, V \nsubseteq X\backslash{\downarrow}x$. So $U \longleftarrow\!\!| \, V \nsubseteq W \cup X\backslash{\downarrow}x$ and hence $x \in (U \longleftarrow\!\!| \, V) \longleftarrow\!\!| \, W$. □

4.3.2 Normality and the Hilbert axioms

The following theorem shows that every normal implicative algebra is a Hilbert algebra.

Proposition 4.3.4. *Let A be a normal implicative algebra, then A is a Hilbert algebra.*

Proof.

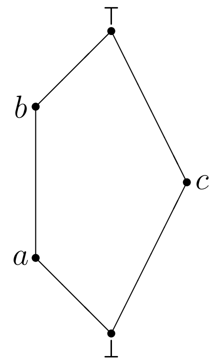
- (H) For all $a, b \in A$, $b \to a = a$ or $b \to a = \top$ so clearly $a \leq b \to a$ and hence $a \to (b \to a) = \top$.
- (F) Let $a, b, c \in A$. If $(a \to b) \to (a \to c) = \top$ then we are done, so assume otherwise. Then by proposition 4.3.7, we know $b \nleq c$. Hence $a \to (b \to c) = a \to c = (a \to b) \to (a \to c)$.

□

This result can be extended to show that every normal bi-implicative algebra is a bi-Hilbert algebra.

To see that this inclusion is proper, we take an example from the class defined above.

Example Let P be the poset given below, and let P^{+} be its complex Hilbert algebra. Clearly $\uparrow c$ and $\uparrow b$ belong to P^{+} and $\uparrow c \longmapsto \uparrow b = \{x \in X : \uparrow x \cap \uparrow c \subseteq \uparrow b\} = \{a, b, \top\}$ which is neither equal to X, nor $\uparrow b$. So P^{+} is a Hilbert algebra that is



not normal.

4.3.3 An algebraic version of the detachment theorem

As is shown in section 5.2, the following condition is required to ensure that each algebra can be embedded in its canonical extension.

$$\text{If } a \leq b \to c\text{, then for all } d \text{ such that } d \leq a \text{ and } d \leq b\ ,\ d \leq c \qquad (\text{Det}')$$

We call it (Det′) because it relates to the Detachment theorem of logic (see proposition 1.2.4 for details). An alternative version of this axiom used by Rasiowa which we shall also use is:

$$a \leq b \to c \text{ and } a \leq b \Rightarrow a \leq c \qquad (\text{Det})$$

Proposition 4.3.5. *(Det′) and (Det) are equivalent.*

Proof. ($\Rightarrow$) Replace d with a in (Det′). Then since $a \leq a$ trivially holds it can be removed as an antecedent, leaving (Det).
($\Leftarrow$) Assume $a \leq b \to c$ and $d \leq a$ and $d \leq b$ for some d. Then $d \leq b \to c$ and $d \leq b$ so (Det) can be applied to get $d \leq c$. □

The dual condition of (Det)

$$a \leftarrow b \leq c \text{ and } b \leq c \Rightarrow a \leq c \qquad (\text{Det}^{\leftarrow})$$

.

Proposition 4.3.6. *(F) implies (Det) and ($F^{\leftarrow}$) implies ($Det^{\leftarrow}$).*

Proof. Assume $a \leq b \to c$ and $a \leq b$ for some $a, b \in A$. Then $(a \to (b \to c)) = \top$, so by (F) $\top \to ((a \to b) \to (a \to c)) = \top$ and hence $(a \to b) \to (a \to c) = \top$. Therefore $a \to b = \top$ implies $a \to c = \top$ i.e. $a \leq c$.
The rest is proved dually. □

4.3.4 Order preserving and reversing properties of $\to$ and $\leftarrow$

The following conditions express that $\to$ is order preserving in the second coordinate and order reversing in the first.

$$a \leq b \text{ implies } c \to a \leq c \to b \tag{OP$_2$}$$

$$a \leq b \text{ implies } b \to c \leq a \to c \tag{OR$_1$}$$

The duals of these conditions express that $\leftarrow$ is order preserving in the first coordinate and order reversing in the second.

$$a \leq b \text{ implies } a \leftarrow c \leq b \leftarrow c \tag{OP$_1^{\leftarrow}$}$$

$$a \leq b \text{ implies } c \leftarrow b \leq c \leftarrow a \tag{OR$_2^{\leftarrow}$}$$

These assumptions are required in order to be able to define the π and σ-extensions of $\to$ and $\leftarrow$ in the canonical extension (see chapter 5). The following lemmas show that they hold in any bi-Hilbert algebra.

Proposition 4.3.7. *(F) implies (OP_2)*

Proof. $a \leq b$ implies $a \to b = \top$. So for any c, $c \to (a \to b) = \top$. Hence by (F) $(c \to a) \to (c \to b) = \top$. So $c \to a \leq c \to b$. □

Dually ($F^{\leftarrow}$) implies ($OP_1^{\leftarrow}$).

Proposition 4.3.8. *(OR_1) holds in any Hilbert algebra.*

Proof. Assume $a \leq b$. By (H) $b \to c \leq a \to (b \to c)$ and by (F) $a \to (b \to c) \leq (a \to b) \to (a \to c)$. Since $a \to b = \top$ we can apply lemma 4.3.1 to get $(a \to b) \to (a \to c) = a \to c$. So $b \to c \leq a \to c$. □

A dual proof shows that ($\mathrm{OR}_2^{\leftarrow}$) holds in an bi-Hilbert algebra.

4.3.5 ($\top$) and ($\bot$)

There are two more axioms that are of interest to us in this work:

$$x \neq \top \Rightarrow \top \leftarrow x = \top \tag{$\top$}$$

$$y \neq \bot \Rightarrow y \to \bot = \bot \tag{$\bot$}$$

Intuitively the first expresses that all non-bottom states are consistent and the second that all non-top states are not tautological(ie. they are informative).

The next proposition shows that ($\top$) and bottom ($\bot$) are satisfied in every normal bi-implicative algebra.

Proposition 4.3.9. *(N) implies* ($\bot$) *and ($N^{\leftarrow}$) implies* ($\top$)

Proof. For any $a \in A$, $a \neq \bot$ implies $a \nleq \bot$ implies $a \to \bot = \bot$. Similarly $a \neq \top$ implies $\top \nleq a$ implies $\top \leftarrow a = \top$. □

The next proposition gives a characterisation of those posets whose complex Hilbert algebras satisfy ($\bot$).

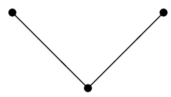
Proposition 4.3.10. *For any partial order $P = (X, \leq)$, P^+ satisfies* ($\bot$) *iff P is updirected*

Proof. Assume P is updirected and let U be a non-empty increasing subset of X, so there is some $u \in U$. For every $x \in X$, there is some y such $y \geq x$ and $y \geq u$. So $\uparrow x \cap U \neq \varnothing$ and hence $x \notin U \longmapsto \varnothing$. Therefore $U \longmapsto \varnothing = \varnothing$. On the other hand, if P is not updirected then there exists $x, y \in X$ such that $\uparrow x \cap \uparrow y = \varnothing$. So $y \in \uparrow x \longmapsto \varnothing$ and therefore ($\bot$) doesn't hold. □

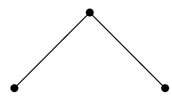
A consequence of this proposition is that one can easily obtain an example of a Hilbert algebra that doesn't satisfy ($\bot$) (and hence show that $(H) + (F) \nRightarrow (\bot)$) by starting with any partial order that is not updirected.

Example P_1 below is an example of a partial order that is not updirected and P_2 is a partial order that is not downdirected, which is shown below to imply that its complex Hilbert algebra does not satisfy $(\top)$.

P_1



P_2



Note that, since N implies $(\bot)$, the complex Hilbert algebra obtained from this partial order is also not normal. The dual correlation between posets being downdirected and their complex bi-Hilbert algebras satisfying $(\top)$ also holds.

Proposition 4.3.11. *For any partial order $P = (X, \leq)$, P^+ satisfies $(\top)$ iff P is downdirected*

Proof. Assume P is downdirected and let $U \neq X$ be an increasing subset of X, so there is some $y \in X$ such that $y \notin U$. For every $x \in X$, there is some x_y such that $x_y \leq x$ and $x_y \leq y$. x_y cannot belong to U as U is upwards closed, and $x_y \in \downarrow x$ implies that $x_y \notin X \backslash \downarrow x$. Hence $x_y \notin U \cup X \backslash \downarrow x$. So $X \nsubseteq U \cup X \backslash \downarrow x$ and therefore by lemma 4.3.2, $x \in X \longmapsfrom U$. As this holds for every $x \in X$, $X \longmapsfrom U = X$. Now assume P is not downdirected. Then there exist $x, y \in X$ such that $\downarrow x \cap \downarrow y = \varnothing$. But then $X \subseteq X \backslash \downarrow x \cap \downarrow y = X \backslash \downarrow x \cup X \backslash \downarrow y$. So $y \notin \{z \in X : X \nsubseteq X \backslash \downarrow x \cup X \backslash \downarrow y\} = X \longmapsfrom X \backslash \downarrow x$ and $X \backslash \downarrow x \in \mathcal{P}^i(X)$. $\square$

Now we give a characterisation of those partial orders whose complex Hilbert algebras are normal. First we must define the characterising order theoretic condition.

Definition Call a partial order P prime if it satisfies the following for every $p, q \in P$ and $V \in \mathcal{P}^i(P)$: If $\forall r(p \leq r$ and $q \leq r$ implies $r \in V)$, then $p \in V$ or $q \in V$.

The name comes from the fact that if p and q have a join $p \vee q$, then the antecedent reduces to $p \vee q \in V$ and so the condition gives the usual definition of primeness of V, ie. $p \vee q \in V$ implies $p \in V$ or $q \in V$.

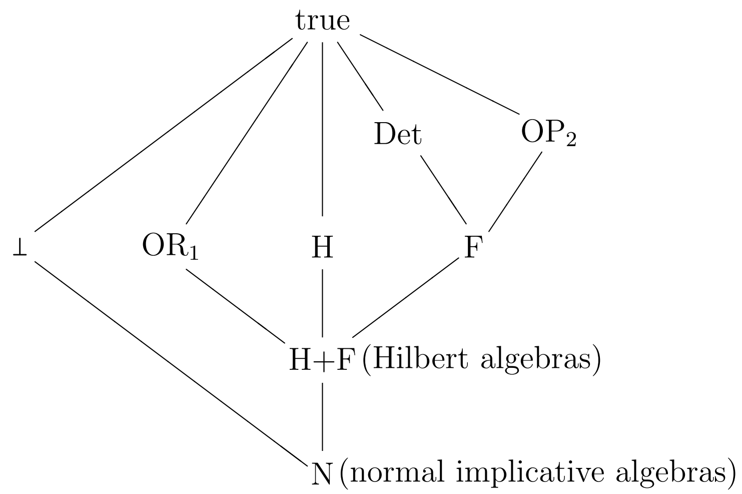
Proposition 4.3.12. *For any partial order $P = (X, \leq)$, P^+ satisfies N iff P is prime*

Proof. Assume P is prime and take $U, V \in P^+$. If $U \subseteq V$ then clearly for all $x \in X$, $\uparrow x \cap U \subseteq V$, so $U \longmapsto V = X$. On the other hand, if $U \nsubseteq V$, then there is some $u \in U - V$. Since P^+ is a Hilbert algebra, we already know that $V \subseteq U \longmapsto V$ so it only remains to show that $U \longmapsto V \subseteq V$. Let $x \in U \longmapsto V$. So $\uparrow x \cap U \subseteq V$. This means that $y \geq x$ and $y \geq u$ implies $y \in V$, so by primeness and the fact that $u \notin V$ we can conclude that $x \in V$. Hence $U \longmapsto V = V$.
Now assume that P is not prime, so there exist some u, x and V such that $y \geq u$ and $y \geq x$ implies $y \in V$ but $u \notin V$ and $x \notin V$. Then clearly $\uparrow x \nsubseteq V$. But the anticedent also tells us that $\uparrow u \cap \uparrow x \subseteq V$. So $u \in \uparrow x \longmapsto V$ and hence $\uparrow x \longmapsto V \nsubseteq V$. □

This proof gives rise to many more examples of Hilbert algebras that are not normal, by starting with partial orders which are not prime. Combining this with the previous propositions we can also manufacture examples of Hilbert algebras which satisfy ($\perp$) but not (N). In fact, P_1 above is prime, so it gives rise to a normal implicative algebra. P_2 on the other hand is not prime and so gives rise to an example of a Hilbert algebra which is not normal but still satisfies ($\perp$).

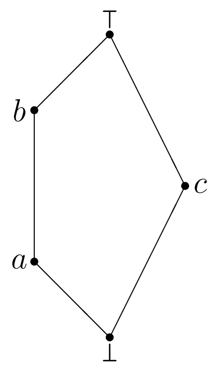
4.4 Relationships between the axioms

The following Hasse diagram shows the axioms introduced in this chapter and summarises the known dependencies between them. The relationships represented in the diagram exist in the context of the axioms for implicative algebras, so $\phi \leq \psi$ means $((\mathrm{I}_1), (\mathrm{I}_2), (\mathrm{I}_3), (\mathrm{I}_4)$ and $\phi)$ implies ψ, or equivalently, in every implicative algebra A, $A \vDash \phi$ implies $A \vDash \psi$. The examples of Hilbert algebras given earlier in this chapter together with the non-Hilbert algebras given in the following section prove that the ordering given in the diagram is anti-symmetric.



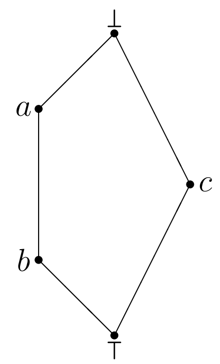
4.4.1 Examples

In this section some important examples of non-Hilbert implicative algebras are given. They will all relate to the non-distributive lattice:



To shed some light on the reason why we only need to consider this restricted class of algebras, we do some sums. This lattice has 5 elements, and hence 25 pairs of elements. 13 of these pairs are of the form (x, y) where $x \leq y$ and thus in any implicative (bi-implicative) algebra relating to this lattice $x \to y$ (and $x \leftarrow y$) have the fixed value $\top$ (and $\perp$) for these x and y. For each of the other 12 pairs, the only restrictions on $x \to y$ (and $x \leftarrow y$) is that they cannot be $\top$ (or $\perp$) and thus there are 4 remaining options for each. Therefore there are 4^{12} implicative and 4^{24} bi-implicative algebras which relate to this lattice. Note that, in each of these groups of algebras, only 1 is normal. This shows just how special the normal algebras are. Other calculations show that 4^8 of the implicative algebras satisfy ($\bot$) and $2^5.3^3 = 864$ of them satisfy (H). Note that this gives an upper bound on the number of Hilbert algebras in this group, showing that they are a tiny proportion of all implicative algebras.

Each example below is of an implicative algebra $\mathbb{A} = (A, \to, \top)$ with a bottom element $\bot$, that satisfies and falsifies certain conditions. Generating a bi-implicative algebra from each that further satisfies the duals of the conditions satisfied in $\mathbb{A}$ and falsifies the duals of the conditions falsified in $\mathbb{A}$ is easy. First take the dual structure of $\mathbb{A}$ (see lemma 5.4.1). This structure then immediately satisfies and falsifies the right conditions. The problem with using this definition of $\leftarrow$ directly is that it relates to the dual of the lattice given above (see lemma 5.4.2), namely:



But it is easy to see that the two lattices are isomorphic, so it is only necessary to apply this isomorphism to the dual of $\mathbb{A}$ to get a definition of $\leftarrow$ that is compatible with $\to$. As the satisfaction and falsification of axioms is preserved under isomorphism, it is now clear that $(A, \to, \leftarrow, \top, \bot)$ is a bi-implicative algebra with the properties described above.

4.4.2 Non-Hilbert algebras with canonical extensions

The significance of the next six examples is that they show that we have succeeded in this thesis to apply the theory of canonical extensions to a wider class of algebras than has been done previously, as they each satisfy the axioms required to define their canonical extensions in our framework but do not belong to the class of Hilbert algebras.

The first two examples are implicative algebras that satisfy (Det),(OP_2), (OR_1) and (F) but do not satisfy (H). The one of the left additionally satisfies ($\bot$) while the one on the right does not. To see that they do not satisfy (H), notice that in each: $b \to a = c \not\geq a$.

$\to$	$\top$	c	b	a	$\bot$
$\top$	$\top$	c	$\bot$	$\bot$	$\bot$
c	$\top$	$\top$	$\bot$	$\bot$	$\bot$
b	$\top$	c	$\top$	c	c
a	$\top$	c	$\top$	$\top$	c
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$

$\to$	$\top$	c	b	a	$\bot$
$\top$	$\top$	c	$\bot$	$\bot$	$\bot$
c	$\top$	$\top$	$\bot$	$\bot$	$\bot$
b	$\top$	c	$\top$	c	$\bot$
a	$\top$	c	$\top$	$\top$	$\bot$
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$

The next two implicative algebras satisfy (Det), (OP_2),(OR_1) and neither F nor H. The one of the left additionally satisfies ($\bot$) while the one on the right does not. To see that they do not satisfy (F), notice that in each: $a \to (b \to c) = a \to c = c \not\leq \bot = \top \to c = (a \to b) \to (a \to c)$
To see that they do not satisfy (H), notice that in each: $\top \to c \not\geq c$.

$\to$	$\top$	c	b	a	$\bot$
$\top$	$\top$	$\bot$	b	a	$\bot$
c	$\top$	$\top$	b	a	$\bot$
b	$\top$	c	$\top$	a	$\bot$
a	$\top$	c	$\top$	$\top$	$\bot$
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$

$\to$	$\top$	c	b	a	$\bot$
$\top$	$\top$	$\bot$	b	a	$\bot$
c	$\top$	$\top$	b	a	a
b	$\top$	c	$\top$	a	$\bot$
a	$\top$	c	$\top$	$\top$	$\bot$
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$

The last two implicative algebras is this subsection satisfy (Det), (OP_2),(OR_1) and (H) but not (F). Again, the one of the left additionally satisfies ($\bot$) while the one on the right does not. To see that they do not satisfy (F), notice that in each: $b \to (c \to a) = b \to b = \top \not\leq b = c \to a = (b \to c) \to (b \to a)$

$\to$	$\top$	c	b	a	$\bot$
$\top$	$\top$	c	b	a	$\bot$
c	$\top$	$\top$	b	b	$\bot$
b	$\top$	c	$\top$	a	$\bot$
a	$\top$	c	$\top$	$\top$	$\bot$
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$

$\to$	$\top$	c	b	a	$\bot$
$\top$	$\top$	c	b	a	$\bot$
c	$\top$	$\top$	b	b	b
b	$\top$	c	$\top$	a	$\bot$
a	$\top$	c	$\top$	$\top$	$\bot$
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$

Finally, we give an example of an implicative algebra that does not satisfy any of the additional axioms we have discussed. A proof that each of these axioms is not satisfied can be given as follows:

- (Det) $a \leq b \rightarrow c$ and $a \leq b$ but $a \nleq c$.
- (OP_2) $a \leq b$ but $c \rightarrow a = c \nleq b = c \rightarrow b$.
- (OR_1) $a \leq b$ but $b \rightarrow c = a \nleq c = a \rightarrow c$.
- (H) $b \rightarrow c = a \ngeq c$.
- ($\perp$) $c \rightarrow \perp = c \neq \perp$.
- (F) implied by the fact that (Det) is not satisfied.

$\rightarrow$	$\top$	c	b	a	$\perp$
$\top$	$\top$	c	b	a	$\perp$
c	$\top$	$\top$	b	c	c
b	$\top$	a	$\top$	a	b
a	$\top$	c	$\top$	$\top$	a
$\perp$	$\top$	$\top$	$\top$	$\top$	$\top$

Chapter 5

Defining $\mathbb{A}^\sigma$

In this chapter we define the canonical extension $\mathbb{A}^\sigma$ of every basic bi-implicative algebra $\mathbb{A}$, where the basic bi-implicative algebras are those that satisfy certain minimal conditions which make it possible to define their canonical extensions (in our chosen framework). We first introduce our logically motivated choices of filters and ideals. We then show that (Det) and (Det$^{\leftarrow}$) characterise exactly when the prinicipal upsets and downsets are contained in our sets of filters and ideals respectively. This allows us to define the domain of $\mathbb{A}^\sigma$. Following this we show that, under our definition, duals and products commute with extensions which allows for a straightforward definition of the extensions of $\rightarrow$ and $\leftarrow$, under the assumptions (OR_1),(OP_2),($\mathrm{OP}_1^{\leftarrow}$) and ($\mathrm{OR}_2^{\leftarrow}$). Motivation is then given for choosing the π-extension for $\rightarrow$ and the σ-extension for $\leftarrow$. Finally we show that top and bottom elements are preserved in the extension, and define $\mathbb{A}^\sigma$. We conclude the chapter by showing that the extension is join generated by its completely join irreducibles and meet generated by its completely meet irreducibles. For definitions of duals and products see [3].

5.1 Choosing $\mathscr{F}$ and $\mathscr{I}$

Definition Given a bi-implicative algebra $\mathbb{A} = \langle A, \rightarrow, \leftarrow, \top, \bot, \rangle$, $F \subseteq A$ is an implicative filter of A if:
$\top \in F$ and for each $a, b \in A$, $a \in F$ and $a \rightarrow b \in F \Rightarrow b \in F$.
Similarly, $I \subseteq A$ is a subtractive ideal of A if:
$\bot \in I$ and for each $a, b \in A$, $a \in I$ and $b \leftarrow a \in I \Rightarrow b \in I$.

Let $\mathscr{F}$ be the set of implicative filters and $\mathscr{I}$ be the set of subtractive ideals of a bi-implicative algebra.

Proposition 5.1.1. *$\mathscr{F}$ is a closure system.*

Proof. It is always the case that $A \in \mathscr{F}$. To show that $\mathscr{F}$ is closed under intersection, take $\{F_i\}_{i\in I} \subseteq \mathscr{F}$. Clearly $\top \in \bigcap_{i\in I} F_i$ since $\top \in F_i$ for each $i \in I$. To show that $\bigcap_{i\in I} F_i$ is closed under $\to$, assume $a, a \to b \in \bigcap_{i\in I} F_i$. Then $a, a \to b \in F_i$ for each $i \in I$. Hence $b \in F_i$ for each $i \in I$ and therefore $b \in \bigcap_{i\in I} F_i$. □

Similarly $\mathscr{I}$ is also a closure system. $\mathscr{F}$ and $\mathscr{I}$ therefore have associated closure operators which we shall call $cl^{\to}$ and $cl^{\leftarrow}$ respectively. In any bi-implicative algebra $\mathbb{A}$, it is very well-known that the implicative filters are the S-filters of the implicative reduct of $\mathbb{A}$ and the subtractive ideals are the S-filters of the implicative reduct of the dual of $\mathbb{A}$.

Proposition 5.1.2. *Each $F \in \mathscr{F}$ is an upset.*

Proof. Let $a \in F$ and take $b \geq a$. Then $a \to b = \top$ and so belongs to F. Hence by definition of implicative filter, $b \in F$. □

Dually, each $I \in \mathscr{I}$ is a downset.

5.2 Existence of the canonical embedding

In chapter 2, we saw the need to have the principal upsets included in $\mathscr{F}$ and the principal downsets included in $\mathscr{I}$ to ensure that A could be embedded into its $(\mathscr{F}, \mathscr{I})$-extension. This is not in general true for the sets of implicative filters and subtractive ideals. In this section, the conditions (Det) and (Det$^{\leftarrow}$) are shown to characterise exactly when the principal upsets and downsets are included in the sets of implicative filters and subtractive ideals respectively.

Theorem 5.2.1. *The following are equivalent:*

1. *For all $a \in A$, $\uparrow a \in \mathscr{F}$*

2. *If $p \leq a \to b$ then $\forall c(c \leq p$ and $c \leq a \Rightarrow c \leq b)$*

3. *$a \leq b \to c$ and $a \leq b \Rightarrow a \leq c$*

Proof. ($1 \Rightarrow 2$): Take p, a and b such that $p \leq a \to b$. Take some $c \in A$ and assume that $c \leq p$ and $c \leq a$. Then by transitivity $c \leq a \to b$, so $a \to b \in \uparrow c$. Then as $\uparrow c$ is an implicative filter, $a \in \uparrow c$ and $a \to b \in \uparrow c$ implies that $b \in \uparrow c$. Hence $c \leq b$.
($2 \Rightarrow 1$): $q \leq \top$, so $\top \in \uparrow q$. Assume $a \in \uparrow q$ and $a \to b \in \uparrow q$. Then as $q \leq a \to b$, Det states that for all $c \in A$, $c \leq q$ and $c \leq a \Rightarrow c \leq b$. In particular for $c = q$ we get that $q \leq q$ and $q \leq a \Rightarrow q \leq b$. Clearly $q \leq q$ and $a \in \uparrow q \Rightarrow q \leq a$, so this implies that $q \leq b$ as required.
($2 \Leftrightarrow 3$): given by proposition 4.3.5. □

Dually, ($\text{Det}^{\leftarrow}$) is satisfied in a bi-implicative algebra $\mathbb{A}$ iff $\downarrow a \in \mathscr{I}$ for all $a \in \mathbb{A}$. This means that those bi-implicative algebras $\mathbb{A}$ that satisfy (Det) and ($\text{Det}^{\leftarrow}$), are exactly those for which the canonical embedding can be defined. We shall call such algebras *detachable*. We are now in a position to define the domain of our canonical extension for $\mathbb{A}$.

Definition For every detachable bi-implicative algebra $\mathbb{A}$, let A^σ be the $(\mathscr{F}, \mathscr{I})$-completion of $\mathbb{A}_*$ where $\mathscr{F}$ is the set of implicative filters of $\mathbb{A}$ and $\mathscr{I}$ is the set of subtractive ideals of $\mathbb{A}$.

The following property of A^σ will be useful later on.

Lemma 5.2.2. *For any set $C \subseteq A$, $\bigvee C = cl^{\leftarrow}(\{\bot\} \cup C) \in \mathscr{I}$.*

Proof. Let $\bigvee_{\oplus} C$ be the join of C in $\mathscr{F} \oplus \mathscr{I}$ and let $X = cl^{\leftarrow}(\{\bot\} \cup C)$. Clearly $c \leq X$ for each $c \in C$. If there is some other $I' \in \mathscr{F} \oplus \mathscr{I}$ such that $c \leq I'$ for all $c \in C$ then clearly $I' \in \mathscr{I}$, so $\{\bot\} \subseteq I'$ and I' must be closed under $\leftarrow$. So $X \subseteq I'$ and hence $X \leq I'$. Then, since MacNeille completions preserve all existing joins, we know that $\bigvee C = \bigvee_{\oplus} C$ and so we are done. □

5.3 Monotonicity of $\to$ and $\leftarrow$

In chapter 3, extensions were defined only for maps that preserve or reverse the order in each coordinate. This is a problem for us because maps in bi-implicative algebras are not in general monotone (for a counter-example see section 4.4.2). In chapter 4, it was shown that in a bi-Hilbert algebra, $\to$ is order reversing in the first coordinate and order preserving in the second, whereas $\leftarrow$ is order preserving in the first coordinate and order reversing in the second, but this is not the case in an arbitrary bi-implicative algebra. Since we need our maps to be monotone and also so that the definition of canonical extension (where defined) is uniform over bi-implicative and bi-Hilbert algebras, it is natural to focus on bi-implicative algebras that preserve and reverse the order in the same way as in a bi-Hilbert algebra, i.e. those that satisfy (OR_1),(OP_2),($\mathrm{OP}_1^{\leftarrow}$) and ($\mathrm{OR}_2^{\leftarrow}$).

Definition Call a bi-implicative algebra *basic* if it is detachable (i.e. satisfies (Det) and ($\mathrm{Det}^{\leftarrow}$)) and further satisfies (OR_1),(OP_2),($\mathrm{OP}_1^{\leftarrow}$) and ($\mathrm{OR}_2^{\leftarrow}$).

We will now show that A^σ has the properties required for a straightforward defintion of the extensions of the maps: first that it comutes with duals, then that it commutes with products.

5.4 Duals

For any bi-implicative algebra $\mathbb{A}$, let $\mathbb{A}^\delta$ denote its algebraic dual. Similarly for any poset $\mathbb{P}$ let $\mathbb{P}^\delta$ denote its order dual.

The first proposition shows that the dual of a bi-implicative algebra is again a bi-implicative algebra.

Proposition 5.4.1. *For any bi-implicative algebra $\mathbb{A}$, $\mathbb{A}^\delta$ is a bi-implicative algebra. Note: $\mathbb{A}$ satisfies the $\rightarrow$ axioms iff $\mathbb{A}^\delta$ satisfies the $\leftarrow$ conditions (further, one by one)*

Proof.

1. $a \rightarrow a = \top$ in $\mathbb{A}$ iff $a \leftarrow a = \bot$ in $\mathbb{A}^\delta$
2. $a \rightarrow b = \top$ and $b \rightarrow c = \top$ in $\mathbb{A}^\delta$
   $\Rightarrow b \leftarrow a = \bot$ and $c \leftarrow b = \bot$ in $\mathbb{A}$
   $\Rightarrow c \leftarrow a = \bot$ in $\mathbb{A}$
   $\Rightarrow a \rightarrow c = \top$ in $\mathbb{A}^\delta$.
3. $a \rightarrow b = \top$ and $b \rightarrow a = \top$ in $\mathbb{A}^\delta$
   $\Rightarrow b \leftarrow a = \bot$ and $a \leftarrow b = \bot$ in $\mathbb{A}$
   $\Rightarrow a = b$ in $\mathbb{A}$
   $\Rightarrow a = b$ in $\mathbb{A}^\delta$.
4. $a \rightarrow \top = \top$ in A' iff $\bot \rightarrow a = \bot$ in $\mathbb{A}$.

□

It should also be noted here that since ($\mathrm{Det}^{\leftarrow}$), ($\mathrm{OP}_1^{\leftarrow}$) and ($\mathrm{OR}_2^{\leftarrow}$) are the duals of (Det), (OP_2) and (OR_1) respectively, that if $\mathbb{A}$ is a detached (or basic) bi-implicative algebra, then its dual $\mathbb{A}^\delta$ is also detached (or basic). This is important because it ensures that for each detached bi-implicative $\mathbb{A}$, $(\mathbb{A}^\delta)^\sigma$ is defined.

The next lemma shows that for any bi-implicative algebra $\mathbb{A}$, the order related to the dual of $\mathbb{A}$ is the order dual of the order related to $\mathbb{A}$.

Lemma 5.4.2. *For any bi-implicative algebra $\mathbb{A}$, $(\mathbb{A}_*)\delta = (\mathbb{A}^\delta)_*$*

Proof. $a \leq b$ in $(\mathbb{A}_*)^\delta$ iff $b \leq a$ in $\mathbb{A}_*$ iff $b \to a = \top$ in $\mathbb{A}$ iff $a \leftarrow b = \bot$ in $\mathbb{A}^\delta$ iff $a \leq b$ in $(\mathbb{A}^\delta)_*$. □

The next proposition when combined with proposition 2.5.1, shows that duals commute with extensions i.e. for each detached $\mathbb{A}$, $(A^\sigma)^\delta = (\mathbb{A}^\delta)^\sigma$.

Proposition 5.4.3. *For any $X \subseteq A$:*

1. *X is a filter in $\mathbb{A}$ iff X is an ideal in $\mathbb{A}^\delta$*
2. *X is an ideal in $\mathbb{A}$ iff X is a filter in $\mathbb{A}^\delta$*

Proof. 1. First note that $\top \in X$ in A iff $\bot \in X$ in A^δ. Assume X is a filter in A and let $a, b \leftarrow a \in X$ in A^δ. Then $a, a \to b \in X$ in A, so $b \in X$. Hence X is closed under $\leftarrow$ in A^δ. Similarly, if $a, a \to b \in X$ in A and X is an ideal in A^δ, then $a, b \leftarrow a \in X$ in A^δ implies $b \in X$.
2. Dual to 1. □

5.5 Products

For detached bi-implicative algebras $\mathbb{A}_1$ and $\mathbb{A}_2$, let $\mathbb{A}_1 \times \mathbb{A}_2$ denote their algebraic product. First we show that the class of detached bi-implicative algebras is closed under taking products.

First note that for all $a, b \in \mathbb{A}_1$ and $a', b' \in \mathbb{A}_2$:
$(a_1, a_1') \leq (a_2, a_2')$
iff $(a_1, a_1') \to (a_2, a_2') = \top$
iff $(a_1 \to a_2, a_1' \to a_2') = (\top, \top')$
iff $a_1 \leq a_2$ and $a_1' \leq a_2'$.
In particular, this implies that $(\mathbb{A}_1 \times \mathbb{A}_2)_* = (\mathbb{A}_1)_* \times (\mathbb{A}_2)_*$.

Fact 5.5.1. $\mathbb{A}_1 \times \mathbb{A}_2$ *is a detached bi-implicative algebra.*

Proof. That $\mathbb{A}_1 \times \mathbb{A}_2$ is a bi-implicative algebra follows easily from the definition of product. To see (Det), take $a, b, c \in \mathbb{A}_1$ and $a', b', c' \in \mathbb{A}_2$ such that $(a, a') \leq (b, b')$ and $(a, a') \leq (b, b') \to (c, c') = (b \to c, b' \to c')$. Then $a \leq b$, $a' \leq b'$, $a \leq b \to c$ and $a' \leq b' \to c'$. So since (Det) holds in $\mathbb{A}_1$ and $\mathbb{A}_2$, $a \leq c$ and $a' \leq c'$ which implies that $(a, a') \leq (c, c')$ as required. □

Lemma 5.5.2. *Each* $F \in \mathscr{F}(\mathbb{A}_1 \times \mathbb{A}_2)$ *is an upset.*

Proof. Take $(a_1, a_1') \in F$ and $(a_2, a_2') \geq (a_1, a_1')$. Then by the above note $a_1 \to a_2 = \top$ and $a_1' \to a_2' = \top'$. So since $\top = (\top, \top') \in F$, $(a_1 \to a_2, a_1' \to a_2') = (a_1, a_1') \to (a_2, a_2') \in F$. Therefore since F is closed under $\to$, $(a_2, a_2') \in F$. □

We now show that taking products commutes with taking canonical extensions, by way of proposition 2.5.2. Below, each of the conditions required for the application of this proposition are shown to hold in our setting.

Proposition 5.5.3. *For each* $F \in \mathscr{F}(\mathbb{A}_1 \times \mathbb{A}_2)$ $\pi_i(F) \in \mathscr{F}(\mathbb{A}_i)$ *for* $i \in \{1, 2\}$.

Proof. $(i = 1)$ Clearly since $(\top, \top') \in F$, $\top \in \pi_1(F)$. Let $a_1, a_1 \to a_2 \in \pi_1(F)$. Then there exist a_1' and a_2' such that $(a_1, a_1'), (a_1 \to a_2, a_2') \in F$. By the lemma this implies that $(a_1 \to a_2, \top') = (a_1 \to a_2, \top' \to \top') = (a_1, \top') \to (a_2, \top') \in F$. Also by the lemma, $(a_1, \top') \in F$. So since F is closed under $\to$, $(a_2, \top') \in F$. Hence $a_2 \in \pi_1(F)$. □

Proposition 5.5.4. *For each* $F_1 \in \mathscr{F}(\mathbb{A}_1)$ *and* $F_2 \in \mathscr{F}(\mathbb{A}_2)$, $F_1 \times F_2 \in \mathscr{F}(\mathbb{A}_1 \times \mathbb{A}_2)$.

Proof. Clearly since $\top \in F_1$ and $\top' \in F_2$, $(\top, \top') \in F_1 \times F_2$. Let $(a_1, a_1'), (a_1, a_1') \to (a_2, a_2') \in F_1 \times F_2$. So $a_1, a_1 \to a_2 \in F_1$ and $a_1', a_1' \to a_2' \in F_2$, which implies that $a_2 \in F_1$ and $a_2' \in F_2$. Hence $(a_2, a_2') \in F_1 \times F_2$. □

Similarly, for each $I \in \mathscr{I}(\mathbb{A}_1 \times \mathbb{A}_2)$, $\pi_i(I) \in \mathscr{I}(\mathbb{A}_i)$ for $i \in \{1, 2\}$ and for each $I_1 \in \mathscr{I}(\mathbb{A}_1)$ and $I_2 \in \mathscr{I}(\mathbb{A}_2)$, $I_1 \times I_2 \in \mathscr{I}(\mathbb{A}_1 \times \mathbb{A}_2)$.

Corollary 5.5.5. $(\mathbb{A}_1 \times \mathbb{A}_2)^\sigma = \mathbb{A}_1^\sigma \times \mathbb{A}_1^\sigma$.

5.6 Choosing π or σ

Let $\mathbb{A}$ be a basic bi-implicative algebra. Since we have shown that canonical extensions commute with products and duals, we can now treat $\to$ and $\leftarrow$ as unary, order preserving maps $\to: \mathbb{A}^\delta \times \mathbb{A} \longrightarrow \mathbb{A}$ and $\leftarrow: \mathbb{A} \times \mathbb{A}^\delta \longrightarrow \mathbb{A}$. We are therefore almost ready to define the extensions of our implications, but one decision remains; whether to choose the π-extension or the σ-extension in each case. The following proposition, related to residuation, sheds some light on this question.

Proposition 5.6.1.

1. *If f is the right residual of j, then f^π is the right residual of j^σ.*
2. *If k is the left residual of g, then k^π is the left residual of g^σ.*

Proof. Following directly from Proposition 3.6 of [11]. □

Then we consider the fact that in a bi-Heyting algebra, $\rightarrow$ is the right residual of $\wedge$ and $\vee$ is the left residual of $\leftarrow$.

We also notice that by the asymmetry of the definitions, $\wedge^\sigma$ coincides with the $\wedge$ in A^σ and $\vee^\pi$ coincides with the $\vee$ in A^σ (but in general $\wedge^\pi$ does not coincide with the $\wedge$ in A^σ and $\vee^\sigma$ does not coincide with the $\vee$ in A^σ). Therefore, it is natural to choose the π-extension for $\rightarrow$ and the σ-extension for $\leftarrow$, because then in the case where $\rightarrow$ and $\leftarrow$ are residuated in $\mathbb{A}$, $\rightarrow^\pi$ and $\leftarrow^\sigma$ will be residuated in A^σ. This way our definition of canonical extension can uniformly extend to the bi-Heyting case.

Finally, we apply the simplified definitions (for f and g) given in chapter 3 and this gives us the following definitions for the extensions of our maps $\rightarrow$ and $\leftarrow$. For all $u, v \in A^\sigma$ let:

$$u \rightarrow^\pi v = \bigwedge\{\bigvee\{p \rightarrow q : p, q \in P, p \geq a, q \leq b\} : a \in K, b \in O, a \leq u, b \geq v\}$$

$$u \leftarrow^\sigma v = \bigvee\{\bigwedge\{p \leftarrow q : p, q \in P, p \geq a, q \leq b\} : a \in K, b \in O, a \leq u, b \geq v\}$$

5.7 $\top$ and $\bot$

Fact 5.7.1. *$\top \in A$ is the top element in A^σ.*

Proof. We know that for any $u \in A^\sigma$, $u = \bigvee B$ for some $B \subseteq O$. Then since $\top = [{\downarrow}\top] = [A]$, we have that for each $I \in B$, $[I] \leq \top$ and hence $\bigvee B \leq \top$. □

Fact 5.7.2. *$\bot \in A$ is also the bottom element in A^σ.*

5.8 Defining $\mathbb{A}^\sigma$

Finally we define the canonical extension of $\mathbb{A}$, which we call $\mathbb{A}^\sigma$ in the complete signature $\{\to, \top, \leftarrow, \bot\}$, for each basic bi-implicative algebra $\mathbb{A}$.

Definition Let $\mathbb{A} = (A, \top, \bot, \to, \leftarrow)$ be a basic bi-implicative algebra. Then let its canonical extension $\mathbb{A}^\sigma$ be defined as $\mathbb{A}^\sigma = (A^\sigma, \top, \bot, \to^\pi, \leftarrow^\sigma)$.

5.9 Completely join irreducibles

Here we show that A^σ is join generated by its completely join irreducibles and meet generated by its completely meet irreducibles. The significance of this condition is that it guarantees that you have enough points to construct an "ultrafilter frame", that is, allowing for a "discrete duality" between these extensions and their dual relational structures. This paves the way for the development of relational semantics and correspondence theory.

The result follows directly from proposition 2.5.3 and the following proposition.

Proposition 5.9.1. *$\mathscr{F}$ and $\mathscr{I}$ are closed under unions of $\subseteq$-chains*

Proof. That $\mathscr{F}$ is closed under unions of chains is given by Theorem 1.3 of [29]. The proof that $\mathscr{I}$ is closed under unions of chains can be given in exactly the same way. □

Chapter 6

Canonicity

Throughout this chapter, assume that $\mathbb{A}$ is a basic bi-implicative algebra and $\mathbb{A}^\sigma$ is its canonical extension as defined in chapter 5. To say an axiom ϕ is canonical means that $\mathbb{A} \vDash \phi$ implies $\mathbb{A}^\sigma \vDash \phi$. To say ϕ is canonical in the presence of ψ means: if $\mathbb{A} \vDash \phi$ and $\mathbb{A} \vDash \psi$ then $\mathbb{A}^\sigma \vDash \phi$. In order to simplify the notation in this chapter, the following notational conventions have been adopted. The lowercase letters $\{p, q, r, s\}$ will be used exclusively in reference to members of $\mathbb{A}$, while F and F' will refer to closed elements of $\mathbb{A}^\sigma$ and I and I' will refer to open elements. $\{u, v, w, x, y\}$ will be used for arbitrary elements of $\mathbb{A}^\sigma$. Note that using F and F' for closed elements in $\mathbb{A}^\sigma$ is actually shorthand for $[F]$ and $[F']$ where $F, F' \in \mathscr{F}$. F and F' shall also be used in this sence to refer to the underlying filters. Similarly I and I' shall also be used to refer to the underlying members of $\mathscr{I}$.

6.1 $\mathbf{I}_1$ and $\mathbf{I}_1^{\leftarrow}$ are canonical

Proposition 6.1.1. *For all* $u \in \mathbb{A}^\sigma$, $u \to^\pi u = \top$.

Proof. $u \to u = \bigwedge\{F \to^\pi I : F \leq u, I \geq u\}$. But $F \leq u$ and $I \geq u$ implies that $F \leq I$ and hence that $F \to^\pi I = \top$. Hence $u \to^\pi u = \top$. $\square$

Dually, for all $u \in \mathbb{A}^\sigma$, $u \leftarrow^\sigma u = \bot$

6.2 Connecting the order with the operations

The first result relating the order with the $\to$ is that $(u \leq v \Rightarrow u \to v = \top)$ is canonical. We show this first for closed u and open v, and then it is extended to arbitrary u and v.

Proposition 6.2.1. $[F] \leq [I]$ *implies* $[F] \to^{\pi} [I] = \top$.

Proof. $[F] \leq [I]$ implies $F \cap I \neq \varnothing$ implies $\exists r \in A : r \geq [F]$ and $r \leq [I]$
$\Rightarrow r \to r = \top \leq [F] \to^{\pi} [I] \Rightarrow [F] \to^{\pi} [I] = \top$. □

Corollary 6.2.2. $u \leq v$ *implies* $u \to^{\pi} v = \top$ *for all* $u, v \in \mathbb{A}^{\sigma}$.

Proof. $a \leq u \leq v \leq b$ implies $a \to b = \top$ and hence $u \to^{\pi} v = \wedge\{\top\} = \top$ □

It is also the case that $(u \leq v \Rightarrow u \leftarrow v = \bot)$ is canonical. Again we show this first for closed u and open v, and then extend to arbitrary u and v.

Proposition 6.2.3. $[F] \leq [I]$ *implies* $[F] \leftarrow^{\sigma} [I] = \bot$.

Proof. $[F] \leq [I]$ implies $F \cap I \neq \varnothing$ implies $\exists r \in A : r \geq [F]$ and $r \leq [I]$
$\Rightarrow r \leftarrow r = \bot \geq [F] \leftarrow^{\sigma} [I] \Rightarrow [F] \leftarrow^{\sigma} [I] = \bot$. □

Corollary 6.2.4. $u \leq v$ *implies* $u \leftarrow^{\sigma} v = \bot$ *for all* $u, v \in \mathbb{A}^{\sigma}$.

The next proposition shows that, in the presence of $(\top)$, $(a \to b = \top \Rightarrow a \leq b)$ is canonical.

Proposition 6.2.5. *If for all* $x \neq \top \in \mathbb{A}$, $\top \leftarrow x = \top$, *then for every closed* F *and open* I, $F \to I = \top \Rightarrow F \leq I$ *and hence for every* $u, v \in \mathbb{A}^{\sigma}$, $u \to v = \top \Rightarrow u \leq v$.

Proof. Let $F \to I = \bigvee\{p \to q : p \in F, q \in I\} = \top$. Then $\top \in X_n$ for some n where $X_{n+1} = \{x :$ there is some $y \in X_n$ such that $x \leftarrow y \in X_n\}$ and $X_0 = \{p \to q : p \in F, q \in I\}$. If $\top \in X_n$ then $\top \leftarrow y, y \in X_{n-1}$ and either $y = \top$ or $\top \leftarrow y = \top$ so $\top \in X_{n-1}$. Therefore by induction, $\top = p \to q \in X_0$. Then $p \leq q$ implies that $q \in F$ and hence $F \cap I \neq \varnothing$.
For the general case assume $u \to v = \bigwedge\{F \to I : F \leq u, I \geq v\} = \top$. So $F \to I = \top$ and hence $F \leq I$ for each $F \leq u, I \geq v$. Recall that $u = \bigvee\{F : F \leq u\}$ and $v = \bigwedge\{I : I \geq v\}$. Take some $I \geq v$. For each $F \leq u$, $F \leq I$ so $u \leq I$. Therefore, as this is for any $I \geq v$, $u \leq v$. □

Dually, in the presence of $(\bot)$, $(a \leftarrow b = \bot \Rightarrow a \leq b)$ is canonical.

Putting the results of this section together tells us that, if $\mathbb{A}$ is a bi-implicative algebra which satisfies $(\bot)$ and $(\top)$ then $\mathbb{A}^\sigma$ is a bi-implicative algebra.

Corollary 6.2.6. *If for each $x \neq \top, y \neq \bot \in \mathbb{A}$, $\top \leftarrow x = \top$ and $y \rightarrow \bot = \bot$, then for all $u, v \in \mathbb{A}^\sigma$, $u \rightarrow v = \top$ iff $u \leq v$ iff $u \leftarrow v = \bot$ and hence, since $\leq$ is a partial order, $\mathbb{A}^\sigma$ is a bi-implicative algebra.*

6.3 $(\top)$ and $(\bot)$ are canonical

The next two theorems show that these conditions are themselves canonical.

Proposition 6.3.1. *If for each $x \neq \top \in \mathbb{A}$, $\top \leftarrow x = \top$ then for all $v \neq \top \in \mathbb{A}^\sigma$, $\top \leftarrow^\sigma v = \top$.*

Proof. First take any $I \neq \top \in K$. Then $T \leftarrow^\sigma I = \bigwedge\{\top \leftarrow q : q \in I\}$. Since $I \neq \top$, $\top \notin I$. So for all $q \in I$, $\top \leftarrow q = \top$ and hence $T \leftarrow^\sigma I = \top$.
Now take any $v \neq \top \in \mathbb{A}^\sigma$ and recall that $\top \leftarrow^\sigma v = \bigvee\{a \leftarrow^\sigma b : a \leq \top, b \geq v\}$. Since $v = \bigwedge\{I : I \geq v\}$, there must be some $I \geq v$ such that $I \neq \top$. Then since $\top \geq \top$ we have $\top \leftarrow^\sigma v \geq \top \leftarrow^\sigma I = \top$. □

Proposition 6.3.2. *If for each $x \neq \bot \in \mathbb{A}$, $x \rightarrow \bot = \bot$ then for all $v \neq \bot \in \mathbb{A}^\sigma$, $v \rightarrow^\pi \bot = \bot$.*

6.4 (OP_2) and (OR_1) are canonical

The first lemma in this section shows that $\rightarrow^\pi$ actually extends $\rightarrow$.

Lemma 6.4.1. *For $p, q \in P$, $p \rightarrow^\pi q = p \rightarrow q$*

Proof. Let $a \in K$,$b \in O$ such that $a \leq p$ and $b \geq q$. Then $p \rightarrow q \in \{p' \rightarrow q' : p', q' \in P, p' \geq a, q' \leq b\}$ so $p \rightarrow q \leq \bigvee\{p' \rightarrow q' : p', q' \in P, p' \geq a, q' \leq b\}$. Now assume there is some $x \leq \bigvee\{p' \rightarrow q' : p', q' \in P, p' \geq a, q' \leq b\}$ for each closed a and open b with $a \leq p$ and $b \geq q$. Then in particular, since p is closed and q is open, $x \leq \bigvee\{p' \rightarrow q' : p', q' \in P, p' \geq p, q' \leq q\}$. Then by the assumptions on $\rightarrow$, $p \leq p'$ and $q' \leq q$ implies $p' \rightarrow q' \leq p \rightarrow q$. So since $p \rightarrow q$ also belongs to $\{p' \rightarrow q' : p', q' \in P, p' \geq p, q' \leq q\}$, it must be that $p \rightarrow q = \bigvee\{p' \rightarrow q' : p', q' \in P, p' \geq p, q' \leq q\}$ and hence $x \leq p \rightarrow q$ which completes the proof that $p \rightarrow q = \bigwedge\{\bigvee\{p' \rightarrow q' : p', q' \in P, p' \geq a, q' \leq b\} : a \in K, b \in O, a \leq p, b \geq q\}$ □

Dually, it can be shown that $\leftarrow^\sigma$ extends $\leftarrow$, i.e. $p \leftarrow^\sigma q = p \leftarrow q$.

The next lemma, is a very useful fact. It gives a definition of $u \rightarrow^\pi v$, for closed u and open v, as a join of elements in $\mathbb{A}$.

Lemma 6.4.2. $[F] \rightarrow^\pi [I] = \bigvee\{p \rightarrow q : p \geq [F], q \leq [I]\}$

Proof. Let $a = [G]$ be closed, $b = [H]$ be open. Then $a \leq [F]$ and $b \geq [I]$ implies that $F \subseteq G$ and $I \subseteq H$. Hence $\{p \rightarrow q : p \geq [F], q \leq [I]\} \subseteq \{p \rightarrow q : p \geq a, q \leq b\}$ and hence $\bigvee\{p \rightarrow q : p \geq [F], q \leq [I]\} \leq \bigvee\{p \rightarrow q : p \geq a, q \leq b\}$. Therefore since $[F]$ and $[I]$ could also be such a and b, we have that $\bigwedge\{\bigvee\{p \rightarrow q : p \geq a, q \leq b\} a \in K, b \in O, a \leq [F], b \geq [I]\} = \bigvee\{p \rightarrow q : p \geq [F], q \leq [I]\}$. □

This in turn allows for a definition of $u \rightarrow^\pi v$, for arbitrary u and v, as the meet of implications between closed and open elements.

Corollary 6.4.3. *$u \rightarrow^\pi v = \bigwedge\{a \rightarrow^\pi b : a \in K, b \in O, a \leq u, b \geq v\}$ for all $u, v \in \mathbb{A}^\sigma$.*

Similarly $u \leftarrow^\sigma v$, for closed u and open v, can be defined as a meet of elements in $\mathbb{A}$ and hence for arbitrary u and v, as the join of implications between closed and open elements.

Proposition 6.4.4. $[F] \leftarrow^\sigma [I] = \bigwedge\{p \leftarrow q : p \geq [F], q \leq [I]\}$

Corollary 6.4.5. *$u \leftarrow^\sigma v = \bigvee\{a \leftarrow^\sigma b : a \in K, b \in O, a \leq u, b \geq v\}$ for all $u, v \in \mathbb{A}^\sigma$.*

Finally we arrive at the main results of this section, which show that (OP_2) and (OR_1) (as well as their duals) are canonical.

Theorem 6.4.6. *$\rightarrow^\pi$ is order reversing in the first co-ordinate and order preserving in the second.*

Proof. Assume $u \leq u'$ for some $u, u' \in \mathbb{A}^\sigma$. Then
$\{a \rightarrow^\pi b : a \in K, b \in O, a \leq u, b \geq v\} \subseteq \{a \rightarrow^\pi b : a \in K, b \in O, a \leq u', b \geq v\}$ and
$\{a \rightarrow^\pi b : a \in K, b \in O, a \leq v, b \geq u'\} \subseteq \{a \rightarrow^\pi b : a \in K, b \in O, a \leq v, b \geq u\}$ so
$\bigwedge\{a \rightarrow^\pi b : a \in K, b \in O, a \leq u', b \geq v\} \leq \bigwedge\{a \rightarrow^\pi b : a \in K, b \in O, a \leq u, b \geq v\}$ &
$\bigwedge\{a \rightarrow^\pi b : a \in K, b \in O, a \leq v, b \geq u\} \leq \bigwedge\{a \rightarrow^\pi b : a \in K, b \in O, a \leq v, b \geq u'\}$.
Hence $u' \rightarrow^\pi v \leq u \rightarrow^\pi v$ and $v \rightarrow^\pi u \leq v \rightarrow^\pi u'$. □

Dually, $\leftarrow^\sigma$ is order preserving in the first co-ordinate and order reversing in the second.

6.5 (H) and (H$^{\leftarrow}$) are canonical

Theorem 6.5.1. *If for every $p, q \in \mathbb{A}$, $p \to (q \to p) = \top$ then for every $u, v \in \mathbb{A}^\sigma$, $u \to (v \to u) = \top$.*

Proof. By corollary 3.24, it is enought to show that $u \le v \to u = \bigwedge\{F \to^\pi I : F \in \mathscr{F}, I \in \mathscr{I}, F \le v, I \ge u\}$, that is for each $F \in \mathscr{F}$ and each $I \in \mathscr{I}$ such that $I \ge u$, $u \le F \to^\pi I$ where $F \to^\pi I = \bigvee\{p \to q : p \in F, q \in I\}$.

- Case 1:$u \in \mathscr{F}$. Then $u \le I$ implies there is some $r \in \mathbb{A}$ such that $r \in u \cap I$. Since $r \to (\top \to r) = \top \in u$ and $r \in u$ and u is an implicative filter, $\top \to r \in u$. But $\top \to r$ also belongs to $\{p \to q : p \in F, q \in I\} \subseteq F \to^\pi I$ so $u \cap F \to^\pi I \neq \varnothing$ and hence $u \le F \to^\pi I$.

- Case 2:$u \in \mathscr{I}$. Since $F \to^\pi I \in \mathscr{I}$, we must show $u \subseteq F \to^\pi I$ so take $r \in u$. $u \le I$ implies $u \subseteq I$ so $\top \to r \in F \to^\pi I$. Also $r \to (\top \to r) = \top$ implies that $r \le \top \to r$ so, since $F \to^\pi I$ is downward closed, $r \in F \to^\pi I$.

- Case 3: By case 2 we now know that $I \to^\pi (F \to^\pi I) = \top$ and hence that $I \le F \to^\pi I$. So for each $I \ge u$, $F \to^\pi I \ge u$ as required.

□

6.6 Non-canonicity of Normality

In this section, we provide a counter-example to the canonicity of (N), that is, a normal bi-implicative algebra $\mathbb{A}$ such that $\mathbb{A}^\sigma$ is not normal. The first lemma shows that every upset is a filter in a normal bi-implicative algebra.

Lemma 6.6.1. *Let $\mathbb{A}$ be a normal Bi-Hilbert Algebra. Then for every non-empty subset $X \subseteq A$, X is an implicative filter iff X is an upset.*

Proof. We saw in chapter 5 that every implicative filter is an upset, so it only remains to show that each non-empty downset $X \subseteq A$ is an implicative filter. Let $x, x \to y \in X$. If $x \le y$, then $y \in X$ as X is an upset. Otherwise $x \to y = y$, so again $y \in X$. □

Similarly every non-empty subset $X \subseteq A$ is a subtractive ideal iff X is a downset.

The next lemma shows that $F \to^\pi I$ satisfies the normality condition for closed F and open I.

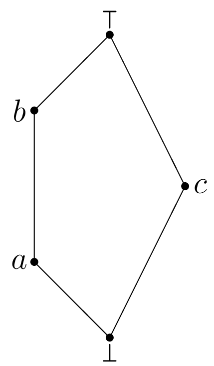
Lemma 6.6.2. *For every closed F, and open I, $F \to I = \begin{cases} \top & \text{if } F \leq I \\ I & \text{otherwise} \end{cases}$*

Proof. By corollary 6.2.2, $F \leq I$ gives $F \to I = \top$. So assume $F \nleq I$. Then $F \cap I = \varnothing$. This implies that for any $p \in F$ and $q \in I$, $p \nleq q$ (because otherwise q would also belong to F) so $p \to q = q$. Therefore $F \to I = \bigvee\{q : q \in I\} = I$. □

This lemma just gives a simplification of the definition of $\to^\pi$ amongst closed elements, which is convenient for our specific purpose.

Lemma 6.6.3. *For closed F_1 and F_2, $F_1 \to^\pi F_2 = \bigwedge\{F_1 \to^\pi I : I \geq F_2\}$*

Proof. For every I, $F \leq F_1$ implies that $F_1 \to I \leq F \to I$. So $\bigwedge\{F_1 \to I : I \geq F_2\}$ is a lower bound on $\{F \to I : F \leq F_1, I \geq F_2\}$. Assume K is another lower bound. Then as $F_1 \leq F_1$, for each $I \geq F_2$, $K \leq F_1 \to I$. Hence $K \leq \bigwedge\{F_1 \to I : I \geq F_2\}$. □



Proposition 6.6.4. *(N) is not canonical.*

Proof. For an example of a normal Bi-hilbert algebra with non-normal extension, take the above non-distributive lattice $\mathbb{P}$. Then let $\mathbb{A} = \mathbb{P}^*$ be its associated normal Bi-hilbert Algebra. Let $F_1 = \uparrow c$, $F_2 = \uparrow\{b, c\}$. Then $F_2 \notin F_1$ implies $F_1 \nleq F_2$. $I \geq F_2$ iff $I \cap F_2 \neq \varnothing$ iff $I \in \{\downarrow b, \downarrow c, \downarrow\{b, c\}, \downarrow\top\}$. Each I besides $\downarrow b$ has a non-empty intersection with F_1. So $F_1 \to \downarrow c = F_1 \to \downarrow\{b, c\} = F_1 \to \downarrow\top = \top$ and $F_1 \to \downarrow b = \downarrow b = b$. Therefore $F_1 \to F_2 = \bigwedge\{F_1 \to I : I \geq F_2\} = \bigwedge\{b, \top, \top, \top\} = b \neq F_2$. □

Conclusion

Let us briefly summarise the results obtained so far. We showed that in order to define the $(\mathscr{F}, \mathscr{I})$-extension on a bi-implicative algebra, where $\mathscr{F}$ is the set of implicative filters and $\mathscr{I}$ the set of subtractive ideals, the algebra must also satisfy three additional conditions (and their duals). The first of these, (Det), (which relates logically to the detachment theorem) is needed to ensure that the algebra can be embedded in its extension. The others, (OP_2) and (OR_1), we assume in order to define the extensions of the implication operation $\rightarrow$. These requirements define our basic setting and we have shown that the class of (bi-)implicative algebras axiomatized by these conditions, here called the basic (bi-)implicative algebras, is a strict superclass of the class of (bi-)Hilbert algebras.

Further, our logically inspired choice of filters and ideals was shown to be advantageous in that it allowed for a straightforward definition of the π and σ extensions of the operations. This was due to the fact that our extensions commute with products and duals. Moreover, the fact that they are closed under unions of chains ensures the possibility of extracting relational duals to these extensions. The π extension was chosen for $\rightarrow$ and the σ for $\leftarrow$ because they then preserve the property of residuation whenever $\rightarrow$ and $\leftarrow$ happen to be residuated and therefore allow our setting to extend uniformly to the case of canonical extensions for (bi-)Heyting algebras.

We showed that under these definitions, (OP_2),(OR_1),(I_1), (H) and their corresponding dual axioms, are canonical. Further the implications $u \leq v \Rightarrow u \rightarrow v = \top$ and $u \leq v \Rightarrow u \leftarrow v = \bot$ are canonical, and their converses are also canonical in the presence of $(\top)$ and $(\bot)$, two conditions we defined which turned out to be both logically meaningful and themselves canonical. We also showed that Normality is not canonical.

Further Work

Representation theorems Representation theorems for hilbert algebras have been extensively studied in [9, 10, 29] and more recently in [6] and [7]. A natural question to ask is how to relate these representations with the canonical extensions for bi-Hilbert algebras developed here.

Relational semantics and correspondence As has been done with modal logic ([25, 26]) and substructural logic ([11]) the results given here pave the way for the development of a uniform relational semantics for the appropriate family of logics and for research into correspondence theory. As correspondence theory has been a major factor in the success of modal logic, it would certainly be desirable to develop correspondence in this setting.

Formal Concept Analysis In section 4.4, a partial picture of the dependencies between the axioms was given. With further investigation, using the methods of formal concept analysis, a complete picture could be developed.

Open canonicity questions Amongst the canonicity results presented in chapter 6 it was shown that ($\top$) is sufficient to ensure the canonicity of $u \to v = \top \Rightarrow u \leq v$, but it is not clear whether it is in fact necessary. Also, it was shown in [17] that Hilbert algebras are canonical (with respect to a different definition of canonical extension) but the canonicity of (F) by itself has yet to be shown in any setting. We intend to investigate this further, first showing canonicity of (F) in the presence of (N).

Interaction between $\to$ and $\leftarrow$ The only link between the two implications in our setting has been the order, but there are also other possible interactions, for example, axiomatic interactions (analogous to distributivity in the lattice case, which connects the $\wedge$ and $\vee$). Because all the axioms we treated only involved $\to$ and $\top$ or only involved $\leftarrow$ and $\bot$, rather than both sides together, we took filters related to $\to$ and $\top$ and ideals related to the $\leftarrow$ and $\bot$. This logically amounts to taking two copies of the implicative side of the logic (one rightside up and the other upside down). An alternative would be to consider the S-filters generated by the entire signature $(\to, \top, \leftarrow, \bot)$.

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