A study of Canonicity for bi-implicative algebras

This thesis is an analytical study of canonicity for logics with a language consisting of constants and implications. More specifically, logics associated with certain distinguished sub-quasivarieties of (bi-)implicative algebras, the best known of which are the varieties of (bi-)Hilbert algebras and (bi-)Tarski algebras. The axiomatization of bi-implicative algebras essentially says that the binary relation $\leq$ defined on A equivalently as:

$$a \to b = \top \quad \text{iff} \quad a \leq b \quad \text{iff} \quad a \leftarrow b = \bot$$

is a partial order and $(A, \leq, \top, \bot)$ is a bounded poset.

This is a case study for a program which proposes bringing inspiration from the field of abstract algebraic logic (AAL) to the theory of canonical extensions in the form of suggesting logical filters, or S-filters for the appropriate associated logic S, as the "right" choice when no all-purpose choice of filters and ideals is available. We have chosen to move away from the lattice setting, where the logical and the mathematically optimal choices overlap, to be able to clearly see the power and limitations of a logic-based choice. The subtraction operator is added to give a clear notion of logical ideal.

Although the AAL perspective was applied in the setting of canonical extensions of Hilbert algebras in [17], the constuction took a detour before applying the parametrised method of canonical extensions for posets (of [16]) with $\mathscr{F}$ and $\mathscr{I}$ taken as the down-directed upsets and up-directed downsets respectively. This gives a canonical extension which is not symmetric in its relationship to the logical filters and ideals; moreover, it is an ad-hoc construction which really utilises the powerful axiomatization of Hilbert algebras and is not applicable in more general contexts.

Our goal is to apply the parametrised method directly, with logically motivated choices of $\mathscr{F}$ and $\mathscr{I}$. To define the $(\mathscr{F}, \mathscr{I})$-extension on a bi-implicative algebra, the algebra must also satisfy three additional conditions (and their duals). The first, (Det), (which relates logically to the detachment theorem) is needed to ensure that the algebra can be embedded in its extension, and other two, (OR_1) and (OP_2), (which express that $\to$ is order reversing in the first coordinate and order preserving in the second respectively) allow $\to$ to be extended. The class of algebras which satisfy these axioms, however, is still a proper extension of the the class of bi-Hilbert algebras. Furthermore, this generalised setting allows us to study canonicity in a more modular way than, for example in [17]. Amongst our results we show canonicity of the bi-implicative axioms in the presense of the conditions $\top \leftarrow a = \top$ and $a \to \bot = \bot$ where intuitively, the first expresses that all non-bottom states are consistent and the second that all non-top states are not tautological (ie. they are informative).