The Interrogative Model of Inquiry meets Dynamic Epistemic Logics

MSc Thesis (*Afstudeerscriptie*)

written by

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Contents

Overview of the thesis

The present thesis explores possible interactions between the *Interrogative Model of Inquiry (IMI)* and *Dynamic Epistemic Logics (DELs).* On one hand, the IMI is a model of scientific inquiry developed by Hintikka which represents inquiry as an *information-seeking process* by means of *asking questions* and *drawing inferences.* On the other hand, DELs is a family of logics concerned with reasoning about *information change.* Thus, we think that these two programs can benefit from interacting with each other, in particular for the following reasons:

- Inquiry, as an information-seeking process, is an important aspect of rational agency, present both in science and our daily life. Thus, inquiry deserves to be investigated within the general program of dynamic logics, seen as a general theory of rational agency, and more specifically from the perspective of DELs. To this end, the IMI offers a framework to investigate inquiry, whose the two main components are *questions* and *inferences*, linking thereby the study of inquiry with two recent trends in dynamic logics: *dynamic logics of questions* and *dynamic logics of inferences.*

- DELs propose new logical tools to formalize reasoning with information and to represent operations of information acquisition. Thus, DELs offer new possibilities to formalize aspects of the IMI and to represent precisely the epistemic effects of getting information through asking questions and drawing inferences on the informational state of the inquiring agent.

In this thesis, we will explore three main axes at the intersection between the IMI and DELs. We now describe the general structure of the thesis.

Chapter 1: Introduction. In this chapter, we will introduce the interrogative model of inquiry along with the tools from DELs that we will use in the thesis. We will first describe the general structure of the IMI, before presenting three main themes investigated by Hintikka within the interrogative model: the strategic aspects of inquiry, the decomposition of questions and the limitations of inquiry. We also add some remarks on the historical roots of the interrogative method of reasoning. Then, we successively present the bases of epistemic logic, dynamic epistemic logic and more precisely public announcement logic, probabilistic epistemic logic and probabilistic dynamic epistemic logic.

Chapter 2: The IMI as a dynamic logic of questions and inferences. In this chapter, we propose a formalization of the IMI under the form of a dynamic logic of questions and inferences, seeing questions and inferences as *actions* modifying the informational state of the inquiring agent. To this end, we will take inspiration from the *interrogative logic* developed by Hintikka and colleagues which is based on *Hintikka's theory of questions* to represent *questions* and on the *tableau method* to represent *inferences.* Thus, we will first develop separately a *dynamic logic of questions* based on Hintikka's theory of questions, and a *dynamic logic of inferences* based on the tableau method. Then, we will merge these two logical systems into a *dynamic logic of questions and inferences.* We end with a discussion on how to represent the intricate relation between questions and inferences in the inquiry process.

Chapter 3: Entailment, informativity and relevance. In this chapter, we argue that certain works in the study of the semantics and pragmatics of questions and answers in natural language are relevant to the study of interrogative inquiry, and we show how to import them in our framework. More precisely, we show that the *partition-based analysis of questions* can be adopted in our approach, allowing us to define a notion of *entailment* for questions *à la* Groenendijk and Stokhof. Besides, we show that the work of van Rooij on the possibility to define notions of *informativity* and *relevance* using tools from information theory and decision theory can also be exploited to investigate notions of informativity and relevance for questions and propositions with respect to a given inquiry. Our general thesis is that defining measures of epistemic value for questions and investigating the strategic aspects of inquiry are two inter-related issues. Thus, we think that these particular works on the semantics and pragmatics of questions and answers constitute already a step toward a better understanding of the strategic aspects of inquiry.

Chapter 4: Decomposing interrogative inquiry. One important activity of an inquirer is to design *inquiry plans*, i.e., to propose a way to *decompose* her inquiry into 'small' steps. In this chapter, we propose to investigate formally the idea of *decomposition of interrogative inquiries.* To this end, we will propose several definitions aiming to capture this notion. Besides, we will present the work of Wiśniewski on *erotetic search scenarios* which also aims to capture the notion of inquiry plan within the framework of *erotetic logic.* We will then compare Wiśniewski's approach with the one that we propose.

Chapter 1

Introduction

1.1 The interrogative model of inquiry

The *interrogative model of inquiry (IMI)* is a model of scientific inquiry, developed by Jaakko Hintikka in the 1980's, which describes scientific inquiry as an *information-seeking process* by means of *asking questions* and *drawing inferences*.[1] Although Hintikka developed specific aspects of his model in several articles, no uniform presentation of the model and its underlying logic is available today. Thus, we choose to present the IMI by selecting what appears to be the main themes in Hintikka's presentation of the interrogative model.

In this section, we will begin by presenting the general structure of the IMI. Then, we will develop the following themes: the strategic aspects of inquiry, the notion of decomposition of questions, the limitations of inquiry and the historical roots of the interrogative method of reasoning.

1.1.1 The general structure of the interrogative model

In most of his articles on the interrogative model, Hintikka introduces the general structure of the IMI using the terminology of game theory, interpreting scientific inquiry as a *game* between the *Inquirer* and *Nature*:

> The [interrogative] model can be described in game-theoretical terms. The model takes the form of a game which an idealized scientist, called the *Inquirer*, plays against *Nature* on a fixed model (universe of discourse). This model or "world" in practice is usually our actual world or some part of it. (Such parts as can serve as models of theories are often called in physics independent or isolated systems.) The game starts from a theoretical premise T. The Inquirer is trying to derive a preset conclusion C from T. At each stage, the Inquirer has a choice between a *deductive move*, in which a logical conclusion is drawn from what the Inquirer has reached already, and an *interrogative move*, in which the Inquirer puts a question to Nature and registers the answer, when forthcoming, as an additional premise. Speaking of such questions is what presupposes that a model of the combined language of T and C is given to which the questions pertain. Nature's answers are assumed to be true in this model. [...] "questions put to Nature" are typically intended to be observations and experiments. [22, pp. 161-162]

[1]The two main references on the IMI are the volume 5 of Hintikka's selected papers entitled *Inquiry as Inquiry: a Logic of Scientific Inquiry* [22] and Hintikka's recent book (2007) entitled *Socratic Epistemology, Explorations of Knowledge-Seeking by Questioning* [25].

Thus, the general conceptual structure of the IMI is pretty simple: the inquirer aims to establish a certain conclusion C by making *deductive* and *interrogative moves.*[2]

Now, if one wishes to pursue the investigation of the IMI, one needs to provide a framework defining what are deductive and interrogative moves. If we think of deductive moves as *inferences* and interrogative moves as *questions*, then the task amounts to provide a logical theory of inferences and questions. This is exactly what Hintikka and colleagues have done by developing the so-called *interrogative logic* in [26]. This interrogative logic is based on the tableau method for representing inferences, and on Hintikka's own theory of questions and answers for representing questions:

> One kind of logic we clearly need [. . .] is the logic of questions and answers. [. . .] In my theory of questions and answers, I offer an analysis of this crucial question-answer relation - both for conclusive (complete) answers and partial ones. [. . .] It is in any case only one ingredient in the logic of science on my interrogative model.
>
> Another [. . .] is obtained by considering what book-keeping methods one might use in one's questioning game against nature. [. . .] In such circumstances, the natural book-keeping method is to use a mild extension of Beth's semantical *tableaux.* In the initial situation, T is put into the left column of the *tableau* and C in its right column. In making a move, I have a choice between carrying out one step of *tableau* construction according to the usual rules and putting a question to nature and entering the answer in the left column of the *tableau* in question. [22, pp. 120-121]

Thus, Hintikka represents deductive moves as tableau construction steps and interrogative moves as adding new axioms or premisses to the background theory T. In the second chapter of this thesis, we will provide a more detailed presentation of Hintikka's interrogative logic.

Finally, we shall make the following remark regarding the analogy between scientific inquiry and games. In this work, we will consider the use of the game-theoretical terminology as a convenient mean to expose the general structure of the IMI, but not as a way to provide a formal framework to represent deductive and interrogative moves. On this point, we follow Wiśniewski in considering that Hintikka's use of the game-theoretical terminology has mainly for objective to emphasize on the importance of investigating the *strategic aspects* of inquiry:

> The choice between moves as well as the choice between admissible questions is a matter of strategy; interrogative games are called games not in order to use the mathematical results of game theory, but to do justice to the importance of research strategies, modeled in IMI by different questioning strategies. [44, p. 390]

Thus, we will rather talk of *deductive* and *interrogative steps.* Notice that we do not claim that game theory has nothing to contribute to our understanding of the IMI. What we say is that, according to us, a formalization of the IMI *begins* with a representational framework for inferences and questions. Then, as soon as such a framework is defined, one might consider to investigate the strategic aspects of inquiry using eventually tools from game theory.

In the second chapter of this thesis, we will propose a formalization of the IMI predicated on *dynamic epistemic logics*. This formalization will take the form of a *dynamic logic of questions and inferences* inspired by Hintikka's interrogative logic.

[2]Inquiry goals might also have the form of questions: the goal of the inquirer is then to determine the answer to a particular question.

1.1.2 The strategic aspects of inquiry

According to Hintikka, the *strategic aspects* of inquiry, and more generally of reasoning, is one of the most important issue to be investigated within the IMI. In this section, we will explain what Hintikka means by *strategic aspects* and why, according to him, they deserve to be studied.

In order to explain the notion of *strategic aspects* of inquiry and reasoning, Hintikka introduces the distinction between *definitory* and *strategic rules* and uses again the analogy with games (in particular with chess in the following quote):

> The metatheory of logic has been developed in a way that is not focused on excellence in reasoning. In order to gain an overview on the situation, it is useful to make a distinction between *definitory* rules and *strategic* rules of any goal-directed activity that can be conceptualized as a game. For instance, the definitory rules of chess tell you how chessmen may be moved on the board, what counts as checking and checkmating, etc. These rules define the game of chess. [...] In contrast, the strategic rules (or principles) of chess tell you how to make the moves, in the sense of telling which of the numerous admissible moves in a given situation it is advisable to make. They tell you what is good play and what is bad play, if not absolutely, then at least relatively.
>
> Indeed, if you only know the definitory rules of chess, you cannot say that you are a chessplayer. No one would deign to play a game of chess against you. You cannot even say that you know how to play chess. For the purpose, you must have some grasp of the strategic principles of chess. [22, p. 2]

Thus, the *definitory rules* of a game tell you the moves that you are *allowed* to make, whereas the *strategic rules* tell you the moves that you *should* make in order to perform well in the game. According to Hintikka, this distinction also applies to deductive logic: the so-called *rules of inference* are definitory rules in the sense that they tell the reasoner which inferences are correct or admissible, but they do not say anything regarding the *best* inferences to draw given a particular epistemic goal. This view is expressed by Hintikka in the following quote:

> [...] what does a so-called rule of inference have to do with the actual drawing of inferences? If you are given twenty-one potential premises, do the "rules of inference" tell you which conclusions you should draw from them? What conclusions a rational person would draw? To what conclusions would "the laws of thought" lead you from these premises? Or, descriptively, what conclusions do people usually draw from them? The right answer is: None of the above. Logic texts' "rules of inference" only tell you which inferences you may draw from the given premises without making a mistake. They are not rules either in the descriptive sense or in the prescriptive sense. They are merely permissive. They are guidelines for avoiding fallacies. Recently, some philosophers have been talking about "virtue epistemology." But in practice, the virtues that most epistemologists admire in this day and age are in fact Victorian rather than Greek. They are not concerned with true epistemological virtue in the sense of epistemological excellence, but only with how not to commit logical sins, how, so to speak, to preserve one's logical or epistemological virtue. Logical excellence—virtue in the sense that is the first cousin of virtuosity—means being able to draw informative conclusions, not just safe ones. [25, p. 2]

Indeed, this distinction between definitory and strategic rules also applies in the more general setting of the interrogative model, where the inquirer can draw inferences but also *ask questions*. In the case of questions, the definitory rules for questioning say what are the questions that the inquirer is *allowed* to ask, whereas the strategic rules say what are the *best* questions to ask in order for the inquirer to reach her inquiry goal. This speaks for an investigation of *questioning strategies* within the IMI:

> The same distinction between definitory rules and strategic rules as was discussed above in connection with deduction applies to interrogative games with a vengeance. In other words, one of the main new types of studies which the interrogative model opens for us is to strategies of questioning, that is, strategies of information seeking by means of the different choices of questions to be asked and of the order in which they are asked. It is not much of an exaggeration to say that here we have the most important new opportunity which the interrogative model facilitates. [22, p. 34]

Thus, this introduces three possible directions to investigate the strategic aspects of inquiry and reasoning: studying inference drawing strategies, questioning strategies, and strategies with both inferences and questions.

In the third chapter of the thesis, we will propose a way to approach the investigation of questioning strategies by defining different *measures of value* for questions. More precisely, we will define for questions the notions of *entailment*, *informativity* and *relevance*. The main idea being that each of these measures of value yields a ranking of the questions that the agent is allowed to ask, providing the agent with a way to choose the best questions to ask in a given epistemic situation.

1.1.3 Decomposition of questions

An important theme in Hintikka's writings on the IMI is the idea of *decomposition of questions*. The notion of decomposition for inquiry is very intuitive and simply refers to the way the inquirer decomposes her inquiry into several steps. When we are seeing inquiry as an interrogative process, this idea can be reformulated by saying that the inquirer decomposes her *principal* or *big* question (goal of the inquiry) into several *operational* or *small* questions answerable by the oracle:

> In general, questions play two roles in interrogative inquiry. What happens is that the inquirer tries to answer a "big" (principal) question by means of a number of "small" (operational) questions. In any one inquiry, the two questions have to be distinguished from each other sharply. [22, p. 246]

However, in scientific practice, *big* and *small* are not labels that can be associated to questions in a rigid way. It might well be that, given the improvement of the experimental technology, a big question of the past becomes a small (operational) question of a later period. Even at a given moment, a small question for a given scientist can be, at the same time, a big question for another scientist. For instance, if the former is a theoretical scientist and the latter an experimental one:

> [W]hat for a higher-level inquiry is an operational ("small") question can for the purposes of a lower-level inquiry be the principal question of a complex inquiry in which it is to be answered by means of a number of lower-level operational questions. This, I find, is how we must view typical controlled experiments. For the purposes of a higher-level inquiry, the entire functional dependence (of the observed variable on the controlled variable) that is the outcome of the experiment is an answer to an operational question on the higher (theoretical) level. For the experimental scientist, in contrast, it is an answer to a principal question, and the experimentalist's operational answers are particular data brought to light during the experiment, for instance, instrument registration. [22, p. 246]

In the forth chapter of this thesis, we will investigate the notion of interrogative inquiry decomposition, through the notions of *decomposition of tasks* and *decomposition of questions*, from an epistemic and dynamic perspective.

1.1.4 The limitations of inquiry

In any model of scientific inquiry, an important issue concerns the way one represents the *limitations* of inquiry. In the interrogative model of inquiry, the limitations are represented within the theory of questions adopted. According to Hintikka's theory of questions, the two main limitations are

- the *presuppositions* of questions that the agent must establish in order to be able to address them to the oracle,
- the *availability of answers* from the oracle (the source of information).

We will define precisely the notion of presupposition of question in the next chapter. The main idea being that the inquirer must have established the presupposition of a question in order to meaningfully ask it, restricting thereby the scope of questions that the inquirer can address to the oracle. It is in this sense that presuppositions constitute a general limitation of any inquiry:

> [T]he limits of inquiry are obviously determined to a large extent by the available presuppositions of questions and answers. [...] It follows that all doctrines concerning the limitations of scientific or other kinds of knowledge-seeking will have to be discussed by reference to the presuppositions of questions and questioning. [25, p. 84]

The second limitation concerns the availability of answers from the oracle. This limitation is indeed a very familiar feature of scientific inquiry since it corresponds to the 'power' of observation of the scientist, described by Hintikka as a 'complex of matters of fact':

> Since nature's answers are often outcomes of experiments, this complex may include prominently the state of scientists' experimental technology. And this is a most familiar feature of the actual history of science. The progress of science has repeatedly been made possible by advances in our techniques of observation, measurement, and experiment. Kepler would never have been able to formulate his laws if Tycho Brahe had not improved the accuracy of astronomical measurements. [25, p. 87]

In the next chapter, we will provide a precise definition for the notion of presupposition, representing thereby the first limitation. Regarding the second limitation, we will simply consider an arbitrary set of available answers from the oracle, called the *answer set*. This is clearly not satisfying in the sense that it does not provide a specific selection of available answers. Providing a meaningful selection of available answers, i.e., a selection that would correspond to a representation of the scientist's power of observation and experimental technology, remains an open problem.

1.1.5 Historical roots of the interrogative method of reasoning

We do not intend here to provide an extensive historical account of the interrogative method of reasoning. Rather, we aim to sketch Hintikka's view on the place of the interrogative method of reasoning in ancient greek philosophy, and on the early development of deductive logic.

According to Hintikka, the first model of reasoning to emerge in the history of philosophy was the Socratic method of questioning or *elenchus*:

> The story, as I see it, begins with Socrates and his method of *elenchus*, or in other words, his questioning method. We all think we know what this method is all about. In reality, however, Socratic *elenchus* is full of logical subtleties even though on the surface it proceeds deceptively smoothly. Socrates is engaged in a question-answer dialogue with an interlocutor. He begins with an initial thesis which is often obtained as a response to Socrates' initial or, as I shall call it, principal question put to his dialogue partner. Socrates then addresses further questions to the other party, and eventually the subsequent answers lead him to a conclusion concerning the initial thesis, typically, to the rejection of this thesis. [20, p. 219]

Although a lots of things have been written on the Socratic method of questioning, no conceptual framework has been developed in order to enable a precise analysis of the logical structure of the *elenchus*. Hintikka claims that his interrogative model of inquiry provides such a framework:

> The interrogative model of inquiry which I have developed over the past several years offers to the first time satisfactory framework for understanding the nature of the Socratic *elenchus*. In fact, the interrogative model can almost be thought of as an updated and sharpened version of the Socratic method, as *elenchus* as it would be practiced by John von Neumann, as a commentator once said. Indeed, the overall similarity is obvious. In the interrogative method, too, all the new information enters into the inquirer's line of reasoning as a response to a question the inquirer has put to a given source of answers, called in my jargon an *oracle*. (This locution is to be taken merely as a *terminus technicus*.) By means of the answers, the inquirer tries to establish a given conclusion or to answer a given question. The main apparent difference between my interrogative games and Socratic *elenchus* is that at any stage of the line of argument the inquirer may, instead of putting a question to a source of answers, draw a logical inference from the results so far obtained, whereas a Socratic inquiry proceeds practically exclusively through questions and answers. [20, p. 219]

Then, the Socratic method became in Plato's academy a general method of philosophizing:

> In Plato's Academy, the technique was formalized into a method of philosophical training and philosophical inquiry by means of question-answer games. [20, p. 222]

Hintikka has a strong thesis on the beginning of the development of deductive logic. According to him: "the origin of deductive logic [is] in the dialectical games of Plato's Academy." [20, p. 228] According to Hintikka, the story began by an attempt of Aristotle to provide a systematic presentation of Academy's philosophizing method:

> Of course, we all know what happened next. An ambitious young member of the Academy called Aristotle undertook to write what Ryle has called "a training manual" for the interrogative games. This manual is of course the *Topics* together with its appendix, *De Soph. El.* It is a most practical, downtoearth handbook, full of advice as to keep your opponent in the dark but not *vice versa*. [20, p. 222]

Then, Aristotle turned his attention on specific situations in which the answers of an opponent in the interrogative game can be completely predicted. In such situations, these predictable answers are the ones that can be *logically deduced* from the previous answers of the opponent:

> [I]n looking at the possibilities of such an anticipation, he made a momentous discovery: Sometimes the answer could be predicted completely on the basis of the respondent's earlier answers or "admissions." Such answers were of course those that we would say are logically implied by the earlier replies. [...]

> Not only did Aristotle realize, however implicitly, the importance of such predetermined and hence predictable answers. He began to study them and developed a theory of them. That theory is the first deductive logic in existence, Aristotle's syllogistic logic. Of course, as we all tend to do, Aristotle ran away with the idea and began to use syllogistic logic as a paradigm of reasoning in general. In doing so, he merely anticipated the subsequent history of logic which has all too often tended to forget its own roots in the theory and practice of interrogative inquiry. [20, p. 226]

To sum up, Hintikka claims that his interrogative model can be used to formalize and investigate the Socratic method of philosophizing. He also claims that the development of deductive logic started from Aristotle's attempt to provide a systematic presentation of the Socratic method. We will not discuss further Hintikka's historical view on the interrogative method of reasoning and we invite the reader interested in these aspects to look at [20] and the volume 6 of Hintikka's selected papers [24].

1.2 Dynamic epistemic logics

Dynamic epistemic logics is a generic term to denote a family of logics dealing with *information change* and resulting from the encounter of *epistemic logic* and *dynamic modal logic*. In this section, we provide the necessary background on dynamic epistemic logics that will be assumed to be known in the remaining of the thesis. Thus, we will successively provide a succinct presentation of epistemic logic, dynamic epistemic logic, probabilistic epistemic logic and dynamic probabilistic epistemic logic.

1.2.1 Epistemic logic

The emergence of *epistemic logic* is generally traced back to the seminal work of Hintikka in his book *Knowledge and Belief: An Introduction to the Logic of the Two Notions* [15]. In this book, Hintikka, combining ideas from von Wright with the newly developed *possible world semantics*, proposes for the first time a *semantics* for the notions of *knowledge* and *belief*. The main idea is to represent knowledge as a *range of epistemically possible worlds*: an agent knows that something is the case if and only if it is the case in all the worlds epistemically possible for the agent, i.e., if it is the case in all the worlds present in the agent's *epistemic range*. This semantics allows to express information that the agents have about the world (called *'first-order information'*) but also information that the agents have about other agents' information (called *'higher-order information'*). In this section, we provide the formal bases of epistemic logic.[3]

First of all, we define the *epistemic language* $\mathcal{E}$ as follows:

Definition 1.1 (Epistemic language $\mathcal{E}$). *Let* P *be a countable set of atomic propositions and* N *be a finite set of agents. The epistemic language* $\mathcal{E}$ *is given by*

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_i\varphi$$

where $p \in \mathsf{P}$ *and* $i \in N$.

In this language, formulas of the form $K_i\varphi$ are read as *"agent i knows that φ"*. We will write $\bot$ for $p \wedge \neg p$ and $\top$ for $\neg\bot$. We now define the notion of *epistemic model*:

Definition 1.2 (Epistemic models). *Let* P *be a countable set of atomic propositions and* N *be a finite set of agents. An epistemic model is a tuple* $M = \langle W, \{\sim_i\}_{i\in N}, V\rangle$ *where:*

[3]See [32] for an up-to-date textbook on modal logic and epistemic logic.

- *W is a non-empty set of worlds,*
- *$\sim_i \subseteq W \times W$ is a binary equivalence relation representing the epistemic indistinguishability relation of agent i,*[4]
- *$V : W \to \mathcal{P}(\mathsf{P})$ is an atomic valuation function indicating the atomic propositions that are true at each world.*

We refer to pairs (M, s), where M is an epistemic model and s is a world in M, as *pointed models*. The intuitive idea behind the use of the epistemic indistinguishability relation is the following: if s denotes the actual world and t is a world such that $t \sim_i s$, then this means that, given all what the agent knows, she cannot tell between s and t which one is the actual world.

Finally, the epistemic language $\mathcal{E}$ is interpreted on epistemic models as follows:

Definition 1.3 (Semantics for $\mathcal{E}$)**.** *The semantics for the epistemic language $\mathcal{E}$ is given by*

$$\begin{array}{rll} M, s \models p & \textit{iff} & p \in V(s) \\ M, s \models \neg\varphi & \textit{iff} & \textit{not } M, s \models \varphi \\ M, s \models \varphi \wedge \psi & \textit{iff} & M, s \models \varphi \textit{ and } M, s \models \psi \\ M, s \models K_i\varphi & \textit{iff} & \textit{for all } w \textit{ such that } w \sim_i s \textit{ we have } M, w \models \varphi. \end{array}$$

The truth definition for the knowledge operator K_i is the formal counterpart of what we said in the introduction: the agent i knows that φ if and only if φ is true in all the worlds that agent i considers epistemically possible.

The set of valid formulas of $\mathcal{E}$ on the class of epistemic models can be axiomatized using the following axiom system EL:

Definition 1.4 (Logic EL)**.** *The logic EL is given by the following axiom system:*

1. *all classical propositional tautologies*
2. $K_i(\varphi \to \psi) \to (K_i\varphi \to K_i\psi)$ *(distribution of K_i over $\sim_i$)*
3. $K_i\varphi \to \varphi$ *(truth)*
4. $K_i\varphi \to K_iK_i\varphi$ *(positive introspection)*
5. $\neg K_i\varphi \to K_i\neg K_i\varphi$ *(negative introspection)*
6. *from φ and $\varphi \to \psi$, infer ψ (modus ponens)*
7. *from φ, infer $K_i\varphi$ (necessitation of K_i)*

Then, we have the following completeness result for EL with respect to the class of epistemic models:

Theorem 1.1 (Completeness for EL)**.** *EL is strongly complete with respect to the class of epistemic models.*

Proof. See Blackburn et al [5]. □

[4]In all this thesis, we make the common assumption that the indistinguishability relation is an *equivalence relation*.

1.2.2 Dynamic epistemic logic

Epistemic logic describes *static* epistemic properties of agents' informational states. The idea of *dynamic epistemic logics* is to extend the static epistemic logic by adding *dynamic operators* enabling to express and reason about *information change.*[5] Indeed, the informational states of the agents can be modified in a lots of various ways. Maybe the simplest one is the modification due to a *public announcement* represented in the so-called *public announcement logic* (*PAL* for short) developed by Plaza [29] and independently by Gerbrandy and Groenveld [11]. *PAL* acts as a canonical example of the methodology of dynamic epistemic logics. A more general approach is the one of Baltag, Moss and Solecki [2] which provides a general account of multi-agent update through *epistemic events.* In this section, we will only present the formal bases of *PAL*.

First of all, the language of *public announcement logic* $\mathcal{L}_{PAL}$ is obtained by adding a *public announcement operator* to the language of epistemic logic:

Definition 1.5 (Language $\mathcal{L}_{PAL}$). *Let* P *be a countable set of atomic propositions and* N *be a finite set of agents. The language* $\mathcal{L}_{PAL}$ *is given by*

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_i\varphi \mid [!\psi]\varphi$$

where $p \in \mathsf{P}$ *and* $i \in N$.

In this language, formulas of the form $[!\psi]\varphi$ are read as *"φ is the case after a public announcement of ψ"*. In order to provide a semantics for this language, it is necessary to describe how an epistemic model is modified after a public announcement. The following definition makes formally explicit the operation of *public announcement*, or *hard information update*, on epistemic models:

Definition 1.6 (Hard information update). *Let* $M = \langle W, \{\sim_i\}_{i\in N}, V\rangle$ *be an epistemic model and* $\varphi \in \mathcal{L}_{PAL}$. *The model* $M|\varphi = \langle W', \{\sim'_i\}_{i\in N}, V'\rangle$ *is given by*

- $W' := \{w' \in W \mid M, w' \models \varphi\}$,
- $\sim'_i := \sim \cap (W' \times W')$,
- $V' := V \restriction W'$.

Using the above definition describing the operation of hard information update, we can now provide a semantics for the language $\mathcal{L}_{PAL}$:

Definition 1.7 (Semantics for $\mathcal{L}_{PAL}$). *The semantics for* $\mathcal{L}_{PAL}$ *is given by the semantics for the epistemic language* $\mathcal{E}$ *plus the following semantic definition for the public announcement operator*

$$M, s \models [!\psi]\varphi \quad \textit{iff} \quad M, s \models \psi \textit{ implies } M|\psi, s \models \varphi.$$

The formulas of $\mathcal{L}_{PAL}$ valid on the class of epistemic models can be characterized syntactically by extending the logic EL with *reduction axioms* for the dynamic operator of public announcement. The resulting logic PAL is defined as follows:

Definition 1.8 (Logic PAL). *The logic* PAL *is obtained by adding to the axiom system* EL *the following reduction axioms:*

1. $[!\psi]p \leftrightarrow \psi \to p$
2. $[!\psi]\neg\varphi \leftrightarrow \psi \to \neg[!\psi]\varphi$

[5] For a recent textbook on dynamic epistemic logics see [7].

3. $[!\psi](\varphi_1 \wedge \varphi_2) \leftrightarrow [!\psi]\varphi_1$ *and* $[!\psi]\varphi_2$

4. $[!\psi]K_i\varphi \leftrightarrow \psi \rightarrow K_i(\psi \rightarrow [!\psi]\varphi)$.

Then, we have the following completeness result for PAL with respect to the class of epistemic models:

Theorem 1.2 (Completeness for PAL)**.** *The logic* PAL *is strongly complete with respect to the class of epistemic models.*

Proof. See Blackburn et al [5] or van Benthem [32, 33]. □

1.2.3 Probabilistic epistemic logic

Probabilistic epistemic logic (*PEL* for short) results from the encounter of *epistemic logic* and *probability theory* and originates from Fagin and Halpern [8] and Halpern and Tuttle [14]. The main idea is to provide the epistemic agent with *degrees of belief* or *subjective probabilities.* To this end, the language of epistemic logic is enriched in such a way that it can express the probabilities that the agent attributes to the different formulas of the language:

Definition 1.9 (Language of *PEL*)**.** *Let* P *be a set of atomic propositions. The language of probabilistic epistemic logic* $\mathcal{L}_{PEL}$ *is given by*

$$\varphi ::= \; p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid q_1\mathbf{P}(\varphi_1) + \ldots + q_n\mathbf{P}(\varphi_n) \geq q.$$

where $p \in \mathsf{P}$ *and* $q, q_1, \ldots, q_k \in \mathbb{Q}$.

A sentence of the form $\mathbf{P}(\varphi) \geq q$ is read as *"the probability assigned by the agent to the formula* φ *is greater than or equal to* q*".* Then, *probabilistic epistemic models* are defined in the following way:

Definition 1.10 (Probabilistic epistemic models)**.** *A probabilistic epistemic model* M *is a quadruple* $\langle W, \sim, V, P\rangle$ *such that:*

1. W *is a non-empty set of worlds,*

2. $\sim\, \subseteq W \times W$ *is the epistemic indistinguishability relation of the agent,*

3. $V : W \rightarrow \mathcal{P}(\mathsf{P})$ *is an atomic valuation function indicating the atomic propositions that are true at each world,*

4. $P : W \rightarrow (W \rightharpoonup [0, 1])$[6] *is a map such that for all* $w \in W$,

$$\sum_{v \in dom(P(w))} P(w)(v) = 1,$$

and which assigns a probability function at each world such that its domain is a non-empty subset of the set of possible worlds.

The probabilistic epistemic language $\mathcal{L}_{PEL}$ is interpreted on probabilistic epistemic models as follows:

[6] $\rightharpoonup$ means that it is a partial function: some worlds may not be in the domain of the function.

Definition 1.11 (Semantics for $\mathcal{L}_{PEL}$)**.** *Let $M = \langle W, \sim, V, P\rangle$ be a probabilistic epistemic model and w be a world in W. Then, the semantics for $\mathcal{L}_{PEL}$ is given by*

$$
\begin{array}{rcl}
M, w \models p & \textit{iff} & p \in V(w) \\
M, w \models \neg\varphi & \textit{iff} & \textit{not } M, w \models \varphi \\
M, w \models \varphi \wedge \psi & \textit{iff} & M, w \models \varphi \textit{ and } M, w \models \psi \\
M, w \models K\varphi & \textit{iff} & \textit{for all } v \in W, w \sim v \textit{ implies } M, v \models \varphi \\
M, w \models \displaystyle\sum_{i=1}^{n} q_1 \mathbf{P}(\varphi_i) \geq q & \textit{iff} & \displaystyle\sum_{i=1}^{n} q_1 P(w)(\varphi_i) \geq q
\end{array}
$$

where $P(w)(\varphi_i) = \sum_{v \in dom(P(w)) \& (M,v) \models \varphi_i} P(w)(v)$.

Finally, we have the following completeness result for PEL:

Theorem 1.3 (Completeness for PEL)**.** *Probabilistic epistemic logic is completely axiomatizable.*

Proof. See Fagin and Halpern [8]. □

In this work, we will make use of probabilistic epistemic logic in order to be able to use *Bar-Hillel and Carnap's theory of information.* To this end, we will make some assumptions on probabilistic epistemic models. More precisely, if $M = \langle W, \sim, V, P\rangle$ is a probabilistic epistemic model and s is a world in W, we will assume that:

- the domain of the probability function $P(s)$ at s is given by all the worlds epistemically indistinguishable from s, i.e., the domain of $P(s)$ is given by the $\sim$-equivalence class of s:
$$dom(P(s)) := \{w \in W \mid w \sim s\},$$
- the probability assignment is *uniform* in the sense that if $w \sim s$, then $P(w) = P(s)$,
- if at s the agent considers a world w possible, namely if w is in the $\sim$-equivalence class of s, then the agent assigns a probability strictly positive to w:
$$\forall w \in dom(P(s)),\ P(s)(w) > 0.$$

Besides, we will sometimes make use of the following abbreviations:

$$
\begin{array}{rcl}
\displaystyle\sum_{i=1}^{n} q_i \mathbf{P}(\varphi_i) \geq q & : & q_1 \mathbf{P}(\varphi_1) + \ldots + q_n \mathbf{P}(\varphi_n) \geq q \\
q_1 \mathbf{P}(\varphi) \geq q_2 \mathbf{P}(\psi) & : & q_1 \mathbf{P}(\varphi) - q_2 \mathbf{P}(\psi) \geq 0 \\
\displaystyle\sum_{i=1}^{n} q_i \mathbf{P}(\varphi_i) \leq q & : & \displaystyle\sum_{i=1}^{n} -q_i \mathbf{P}(\varphi_i) \leq -q \\
\displaystyle\sum_{i=1}^{n} q_i \mathbf{P}(\varphi_i) < q & : & \neg\left(\displaystyle\sum_{i=1}^{n} q_i \mathbf{P}(\varphi_i) \geq q\right) \\
\displaystyle\sum_{i=1}^{n} q_i \mathbf{P}(\varphi_i) > q & : & \neg\left(\displaystyle\sum_{i=1}^{n} q_i \mathbf{P}(\varphi_i) \leq q\right) \\
\displaystyle\sum_{i=1}^{n} q_i \mathbf{P}(\varphi_i) = q & : & \left(\displaystyle\sum_{i=1}^{n} q_i \mathbf{P}(\varphi_i) \leq q\right) \wedge \left(\displaystyle\sum_{i=1}^{n} q_i \mathbf{P}(\varphi_i) \geq q\right).
\end{array}
$$

We now turn to the 'dynamification' of probabilistic epistemic logic.

1.2.4 Probabilistic dynamic epistemic logic

Probabilistic dynamic epistemic logic (*PDEL* for short) has been developed by Kooi in his PhD thesis entitled *Knowledge, Chance and Change* [27], and results from the combination of probabilistic epistemic logic and dynamic epistemic logic. Thus, *PDEL* is a logic to talk and reason about *probability* and *information change.*

The language of *PDEL* is obtained by extending the previous language $\mathcal{L}_{PEL}$ with a *dynamic operator of public announcement*:

Definition 1.12 (Language of PDEL)**.** *Let* P *be a set of atomic propositions. The language of probabilistic dynamic epistemic logic* $\mathcal{L}_{PDEL}$ *is given by*

$$\varphi ::= \ p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid [!\varphi]\psi \mid q_1\mathbf{P}(\varphi_1) + \ldots + q_n\mathbf{P}(\varphi_n) \geq q.$$

where $p \in \mathsf{P}$ *and* $q, q_1, \ldots, q_k \in \mathbb{Q}$.

In this language, formulas of the form $[!\varphi]\psi$ are read as *"ψ is the case after the public announcement of φ".*

Then, Kooi proposes the following *probabilistic update operation* which defines how a probabilistic epistemic model is modified after an *incoming of hard information* or *public announcement*:

Definition 1.13 (Probabilistic update operation)**.** *Let* $M = \langle W, \sim, V, P\rangle$ *be a probabilistic epistemic model,* w *be a world in* W *and* φ *be a formula. The updated model* $M_\varphi = \langle W_\varphi, \sim_\varphi, V_\varphi, P_\varphi\rangle$ *is defined as follows:*

- $W_\varphi = \{u \in W \mid M, u \models \varphi\}$,
- $\sim_\varphi := \sim \cap (W_\varphi \times W_\varphi)$,
- $V'(p) := V(p) \cap W_\varphi$,
- $P_\varphi : W_\varphi \to (W_\varphi \rightharpoonup [0,1])$ *is obtained as follows for all* $u \in W_\varphi$*:*
  - $dom(P_\varphi(u)) = \{v \in dom(P(u)) \mid (M, v) \models \varphi\} = dom(P(u)) \cap W_\varphi$,
  - $P_\varphi(u)(v) = \frac{P(u)(v)}{P(u)(\varphi)}$ *given that* $v \in dom(P_\varphi(u))$.

The following theorem shows that probabilistic epistemic models are preserved under the probabilistic update operation:

Theorem 1.4. *If* (M, w) *is a probabilistic epistemic model such that* $M, w \models \varphi$*, then* (M_φ, w) *is still a probabilistic epistemic model.*

Proof. Let (M, w) be a probabilistic epistemic model with $M = \langle W, \sim, V, P\rangle$. The only difficulty lies in the map P. Let $u \in W_\varphi$, we want to show that

$$\sum_{v \in dom(P_\varphi(u))} P_\varphi(u)(v) = 1.$$

We have that:

$$\sum_{v \in dom(P_\varphi(u))} P_\varphi(u)(v) = \sum_{v \in dom(P(u)) \& (M,v) \models \varphi} \frac{P(u)(v)}{P(u)(\varphi)} = \frac{\sum_{v \in dom(P(u)) \& (M,v) \models \varphi} P(u)(v)}{P(u)(\varphi)} = \frac{P(u)(\varphi)}{P(u)(\varphi)} = 1.$$

□

Notice also that the three assumptions that we made are preserved after a successful update operation.

Thus, we are now in a position to define a semantics for the the language $\mathcal{L}_{PDEL}$:

Definition 1.14 (Semantics for $\mathcal{L}_{PDEL}$)**.** *Let $M = \langle W, \sim, V, P\rangle$ be a probabilistic epistemic model for $\mathcal{L}_{PEL}$ and w be a world in W. The semantics for $\mathcal{L}_{PDEL}$ is given by the semantics for $\mathcal{L}_{PEL}$ plus the following semantic definition for the public announcement operator*

$$(M, w) \models [!\varphi]\psi \quad \textit{iff} \quad (M, w) \models \varphi \ \textit{ implies } \ (M_\varphi, w) \models \psi.$$

In his PhD thesis, Kooi proved the following completeness result for *PDEL*:

Theorem 1.5 (Completeness *PDEL*)**.** *Probabilistic dynamic epistemic logic is completely axiomatizable.*

Proof. See Kooi [27]. □

We will use the tools from probabilistic dynamic epistemic logic in the third chapter. The probabilistic update operation will allow us to precisely represent the effect of getting an answer from a question in the probabilistic case. This will turn out to be very useful when we will use Bar-Hillel and Carnap's theory of information since the probabilistic update operation will provides a '*recomputation*' of the amount of information of the different formulas in the language after the obtention of an answer from a question.

Chapter 2

The IMI as a dynamic logic of questions and inferences

Introduction

According to the interrogative model of inquiry, an *inquiry* is an *information-seeking process* consisting in a sequence of *interrogative* and *deductive steps.* Thus, any formal investigation of the IMI must begin by defining the so-called *definitory rules* of inquiry, i.e., by stipulating the *admissible* interrogative and deductive steps that the inquirer can make. Since making interrogative and deductive steps amounts respectively to *asking questions* and *drawing inferences*, providing a formal framework defining the definitory rules of inquiry amounts finally to develop a *logical theory of questions and inferences.* Hintikka and colleagues have proposed such a theory through the so-called *interrogative logic* (henceforth, IL) [22, 26]. This theory is predicated on *Hintikka's theory of questions* for modeling *questions* and the *tableau method* for modeling *inferences.*

In this chapter, we propose a formalization of the IMI predicated on *dynamic epistemic logics.* The main idea is to represent *questions* and *inferences* as *actions* modifying the informational state of the inquiring agent. To this end, we will take inspiration from Hintikka's formalization of the IMI through IL, i.e., we will based our representation of questions on Hintikka's theory of questions and our representation of inferences on the tableau method. This perspective connects thereby the IMI and IL with two recent trends in dynamic epistemic logics: *dynamic logics of questions* and *dynamic logics of inferences.*[1] Thus, our project in this chapter can be stated as follows: to develop a formalization of the IMI under the form of a *dynamic logic of questions and inferences* inspired by Hintikka's interrogative logic.

To carry out this project, we will proceed as follows. In the first section, we will focus on the *interrogative steps*: we will describe how IL represents interrogative steps as *questions* and we will then take inspiration from it to develop a *dynamic logic of questions.* In the second section, we will focus on the *deductive steps*: we will present how IL uses the tableau method to represent *inferences* as *tableau construction steps* and we will then develop a *tableau-based dynamic logic of inferences.* In the third section, we will propose a combined treatment of *interrogative and deductive steps* by merging our dynamic logic of questions and dynamic logic of inferences into *one* system dealing *jointly* with questions and inferences. However, we will argue that a straightforward merge of our two previous systems does not necessarily capture the intricate relation between questions and inferences in inquiry. The main problem seems to lie in the treatment of questions for non-logically omniscient agents. In this third section, we will

[1] For recent developments on dynamic logics of questions see [37], for dynamic logics of inferences see [13, 38, 41].

then try to precisely locate this problem and to propose an alternative treatment of questions in the case of non-logical omniscience which avoids it.

2.1 Interrogative steps: modeling questions

In the framework of the interrogative model of inquiry, *interrogative steps* of reasoning are thought as *questions*. In this section, we will first present how IL formalizes interrogative steps through the so-called *definitory rule for questioning*. Then, we will propose a framework which makes explicit the dynamic and epistemic components of the definitory rule for questioning of IL. This framework will take the form of a *dynamic logic of questions* for which we will finally provide a sound and complete axiomatic system.

2.1.1 Interrogative steps in IL

In order to present the definitory rule for questioning of IL, we need to introduce its basic components. Thus, we first need to explain what it means for IL to be a *model-oriented logic*. Then, we need to present the theory of questions which lies behind the questioning rule of IL. To this end, we will introduce the concepts of *propositional question*, *presupposition* and *oracle*. We will then have all the ingredients to be able to state the definitory rule for questioning of IL.

First of all, one of the most important features of IL is the fact that IL is a *model-oriented logic*[2], i.e., IL is always defined relatively to a given model M:

> The Inquirer's aim is to prove logically either C or $\neg C$ by using as premises T plus the answers.
>
> Even though the required proof is required to be purely logical, *the process as a whole is relative to a given model M*, for Nature can answer questions only with respect to some particular model. (my emphases) [16, p. 1]

This feature simply reflects the fact that IL is designed to capture how an inquirer reasons in order to find out unknown aspects of a given model M, representing the actual world.

In this reasoning process, the inquirer can make *requests of information* about the model M. In IL, following the main idea of the IMI, these requests of information are conceived as *questions* to a particular source called the *oracle*:

> We will use as a technical term for all sources of information the word *oracle*. Since the information is new, the inquirer must somehow have received this information as a response to his or her own initiative, which is an action directed to some particular oracle. Since this source of information is therefore known, we might as well think of the new information as an answer to a question the inquirer has addressed to the oracle in question. [22, p. 47]

Thus, IL integrates in itself a theory[3] of questions. According to this theory, a question is identified by its set of possible answers. In the propositional case, a *propositional question Q* is simply identified by a set of propositions that we denote by $Q = (\gamma_1, \dots, \gamma_k)$ where $\gamma_1, \dots, \gamma_k$ are propositional formulas. Such a question $Q = (\gamma_1, \dots, \gamma_k)$ is read as *"Is it the case that γ_1, or is it the case that γ_2,..., or is it the case that γ_k?"*[4] Among propositional questions, *yes-no*

[2] In [16], Hintikka explains what he means by model-oriented logic and argues that the logic of science shall be seen as a model-oriented logic.

[3] We refer to this theory as *Hintikka's theory of questions.*

[4] Notice that, according to Hintikka's perspective, the 'or' here is *not* an exclusive or. We will argue later that it makes more sense actually to consider the 'or' in the presuppositions of propositional questions as an *exclusive* or.

questions, i.e., questions of the form $(\gamma, \neg\gamma)$, are particularly remarkable since they are often considered to be the epistemically simplest form of question.

The notion of question comes with the important notion of *presupposition*. According to Hintikka, a question can be *meaningfully* asked only if its presupposition has been established by the inquirer. In the case of propositional questions, the presupposition of a question $Q = (\gamma_1, \ldots, \gamma_k)$ is simply the disjunction of all its possible answers:

$$\mathsf{presup}(Q) := \gamma_1 \vee \ldots \vee \gamma_k.$$

It is important to notice that, from a philosophical point of view, the notion of presupposition plays a crucial role in the limitations of the inquiry process:

> [T]he limits of inquiry are obviously determined to a large extent by the available presuppositions of questions and answers. [...] It follows that all doctrines concerning the limitations of scientific or other kinds of knowledge-seeking will have to be discussed by reference to the presuppositions of questions and questioning. [25, p. 84]

The last remark that we have to make concerns the *oracle*. In IL, the oracle is formalized via an answer set Φ containing all the available answers from the oracle. Besides, the following hypotheses are made:[5]

1. There is only one oracle,
2. The set of answers the oracle will provide remains constant throughout the inquiry,
3. All of the oracle's answers are true, and known by the inquirer to be true.

We now have all the ingredients to state the definitory rule for questioning of IL:

> If the presupposition of a question occurs on the left side of a *subtableau*, the inquirer may address the corresponding question to the oracle. If the oracle answers, the answer is added to the left side of the *subtableau*. [22, p. 51]

In IL, the left side of the initial tableau contains all the initial premisses. Then, during the inquiry, the left side records all what has been established by the inquirer, either through logical inferences or by questioning. Thus, what the definitory rule for questioning says is the following: as soon as the inquirer has established the presupposition of a question Q, then she can ask the corresponding question to the oracle, the answer depending on the information available from the oracle.

The definitory rule for questioning of IL has a strong dynamic-epistemic flavor: the left side represents in some sense the epistemic situation of the inquirer, the action of questioning having for effect to modify this epistemic situation. Thus, we now propose to represent interrogative steps in a dynamic-epistemic setting.

2.1.2 A dynamic logic of questions

In this section, we propose a representation of *interrogative steps* which makes *explicit* the *dynamic* and *epistemic* components of the definitory rule for questioning of IL. To this end, we will represent the epistemic situation of the inquiring agent using the framework of epistemic logic. Then, we will represent the action of questioning dynamically as a *model operation* and we will add to the language of epistemic logic a *dynamic 'question to the oracle' operator*. Notice that

[5] The kind of interrogative inquiry governs by these assumptions is called the case of *pure discovery* by Hintikka.

we will provide a representation of interrogative steps that takes inspiration from the definitory rule for questioning of IL but which will be distinct from IL as a formal system. This section provides all the ingredients leading to the semantic definition of the dynamic 'question to the oracle' operator.

The static perspective

We have seen in the previous section that a question is identified with its set of possible answers. We shall then define what we mean by a possible answer. To this end, we define an *inquiry language* $\mathcal{I}$ which delimits the scope of the formulas that can be the answer to some questions. Since we focus on the propositional case, we will only consider *propositional questions*, i.e., questions for which a possible answer is simply a propositional formula. Thus, the inquiry language $\mathcal{I}$ will be, in our case, the propositional language:

Definition 2.1 (Inquiry language $\mathcal{I}$). *Let* P *be a countable set of atomic propositions. The inquiry language* $\mathcal{I}$ *is given by*

$$\gamma ::= p \mid \neg\gamma \mid \gamma \wedge \gamma$$

with $p \in \mathsf{P}$.

The static language that we consider is the language of *epistemic logic* to which we add an *oracle operator*:

Definition 2.2 (Epistemic language $\mathcal{E}_0$). *Let* P *be a set of atomic propositions. The epistemic inquiry language* $\mathcal{E}_0$ *is given by*

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid \mathit{\Phi}\gamma$$

where $p \in \mathsf{P}$ *and* $\gamma \in \mathcal{I}$.

In this language, formulas of the form $K\varphi$ are read as *"the agent knows that φ"* and formulas of the form $\mathit{\Phi}\gamma$ are read as *"γ is in the answer set of the oracle"*.

We have seen in the previous section that IL is a model-oriented logic. This means that IL is always defined relatively to a given model, representing the actual world, and to an oracle associated to this model, representing the source of information about the actual world. In our framework, the actual world is represented by a designated world in the given epistemic model. Since all the worlds of an epistemic model $M = \langle W, \sim, V\rangle$ can be potentially designated to be the actual world, we will associate an oracle to each world w in W. Thus, we will define the oracle as a function:

$$\Phi : w \in W \mapsto \Phi(w) \in \mathcal{P}(\mathcal{I}).$$

Following Hintikka, we will make the following assumptions on the oracle: for each world $w \in W$

- There is only one oracle associated to w (represented by the answer set $\Phi(w)$),
- The answer set $\Phi(w)$ remains constant throughout the inquiry,
- The oracle's answers are true.

We then define the notion of *epistemic inquiry model* as follows:[6]

Definition 2.3 (Epistemic inquiry model). *Let* P *be a countable set of atomic propositions. An epistemic inquiry model is a tuple* $M = \langle W, \sim, V, \Phi\rangle$ *where:*

[6] We provide here a general definition for epistemic inquiry model. We will then restrict it, when we will define our intended class of models, in order to integrate the assumptions on the oracle.

- W *is a non-empty set of worlds,*
- $\sim \subseteq W \times W$ *is the epistemic indistinguishability relation of the inquiring agent,*
- $V : W \to \mathcal{P}(\mathsf{P})$ *is an atomic valuation function indicating the atomic propositions that are true at each world,*
- $\Phi : W \to \mathcal{P}(\mathcal{I})$ *is a function representing the oracle which associates to each world* $w \in W$ *a set of formulas* $\Phi(w) \subseteq \mathcal{I}$.

In the definition of epistemic inquiry models, we already integrate the two first hypotheses on the oracle. We will integrate the hypothesis of truthfulness when we will define our intended class of models. Before that, we first define the semantics for the language $\mathcal{E}_0$:

Definition 2.4 (Semantics for $\mathcal{E}_0$)**.** *The semantics for the epistemic language* $\mathcal{E}_0$ *is given by the semantics for the epistemic language* $\mathcal{E}$ *plus the following semantic definition for the oracle operator* $\mathit{\Phi}$

$$M, s \models \mathit{\Phi}\gamma \quad \textit{iff} \quad \gamma \in \Phi(s).$$

In this work, we will impose the following restrictions on epistemic inquiry models: let $M = \langle W, \sim, V, \Phi \rangle$ be an epistemic inquiry model,

Veridicality for the oracle: we will require that the oracle is always truthful[7]: for all $w \in W$,

$$\text{if } \gamma \in \Phi(w), \text{ then } M, w \models \gamma,$$

Coherence property for the oracle: we will require a *coherence property* for the oracle: for all $w \in W$,

$$\text{if } \gamma \in \Phi(w), \text{ then } \gamma \in \Phi(u) \text{ for all } u \in W \text{ such that } u \sim w \text{ and } M, u \models \gamma.$$

Thus, our intended class of models $\mathbf{E_I}$, which is a subclass of the class of epistemic inquiry models, is defined as follows:[8]

Definition 2.5 (Class of models $\mathbf{E_I}$)**.** *Let* $M = \langle W, \sim, V, \Phi \rangle$ *be an epistemic inquiry model.*

$$M \in \mathbf{E_I} \textit{ if and only if}$$

1. *for all* $w \in W$*, if* $\gamma \in \Phi(w)$*, then* $M, w \models \gamma$*,*
2. *for all* $w \in W$*, if* $\gamma \in \Phi(w)$*, then* $\gamma \in \Phi(u)$ *for all* $u \in W$ *such that* $u \sim w$ *and* $M, u \models \gamma$*.*

We will now provide a semantic definition for the *'question to the oracle' operator.*

The dynamic perspective

In the dynamic perspective, we aim to provide a semantic definition for a *dynamic 'question to the oracle' operator* which would represent the *action* of making an *interrogative steps.* To this end, we first have to extend our previous language into an *epistemic inquiry language* $\mathcal{E}_\mathcal{I}$ by adding a *dynamic 'question to the oracle' operator* (henceforth, *question operator*):

Definition 2.6 (Epistemic inquiry language $\mathcal{E}_\mathcal{I}$)**.** *Let* P *be a set of atomic propositions. The epistemic inquiry language* $\mathcal{E}_\mathcal{I}$ *is given by*

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid \mathit{\Phi}\gamma \mid [(\gamma_1, \ldots, \gamma_k)?]\varphi$$

where $p \in \mathsf{P}$*,* $\gamma, \gamma_1, \ldots, \gamma_k \in \mathcal{I}$ *and* $k \geq 1$*.*

[7]This corresponds to our third hypothesis on the oracle.

[8]In the following, by *epistemic inquiry models* we will mean models of this class.

In this language, formulas of the form $[(\gamma_1, \ldots, \gamma_k)?]\varphi$ are read as *"φ is the case after having asked the question 'Is it the case that γ_1, or is it the case that γ_2, ... , or is it the case that γ_k?'"*.

The semantic definition for the question operator will have two main components: the first one is the definition of the *question operation* on epistemic models, the second one is the definition of the *precondition* to this operation.

In order to define the question operation, we first have to recall how an epistemic model is modified after an incoming of *hard information*:

Definition 2.7 (Hard information update). *Let $M = \langle W, \sim, V, \Phi\rangle$ be an epistemic inquiry model. Let $\gamma \in \mathcal{I}$. The epistemic inquiry model $M|\gamma = \langle W', \sim', V', \Phi'\rangle$ is given by*

- $W' := \{w' \in W \mid M, w' \models \gamma\}$,
- $\sim' := \sim \cap (W' \times W')$,
- $V' := V \restriction W'$,
- $\Phi' := \Phi \restriction W'$.

We then represent the effect of *asking* a question to the oracle under the form of a *conditional incoming of hard information*: if the answer to the question is available from the oracle, then the action of asking a question will lead to an hard information update with the answer. Formally, this *'asking a question to the oracle' operation* (henceforth, *question operation*) is defined as follows:

Definition 2.8 (Question operation). *Let (M, s) be a pointed epistemic inquiry model where $M = \langle W, \sim, V, \Phi\rangle$, let $Q = (\gamma_1, \ldots, \gamma_k)$ be a propositional question and let $A = \{\gamma_1, \ldots, \gamma_k\} \cap \Phi(s)$. The model $M_{(\gamma_1, \ldots, \gamma_k)?}(s)$ is obtained as follows*

- *if $A = \emptyset$, then $M_{(\gamma_1, \ldots, \gamma_k)?}(s) := M$,*
- *if $A \neq \emptyset$, then $M_{(\gamma_1, \ldots, \gamma_k)?}(s) := M| \bigwedge A$ where $\bigwedge A$ denotes the conjunction of all the formulas in A.*

The second component of the definition of the question operator is the *precondition* to the question operation. It is through the notion of precondition that we will introduce the last ingredient that we need to integrate in our framework: the notion of *presupposition*.

As we have seen in the questioning rule for IL, the agent *must* have established the presupposition of a question in order to be able to address it to the oracle. In our epistemic framework, this can be translated as follows: the agent must *know* the presupposition of a question in order to be able to address it to the oracle. Formally, if $M = \langle W, \sim, V, \Phi\rangle$ is an epistemic inquiry model, $s \in W$ represents the actual world and $Q = (\gamma_1, \ldots, \gamma_k)$ is a propositional question, then the following condition must be satisfied in order for the inquiring agent to ask the question:

$$M, s \models K(\gamma_1 \vee \ldots \vee \gamma_k).$$

However, Hintikka's way to treat presuppositions of propositional questions leads to two important problems. The first one concerns the situation in which more than one possible answer of a question are true: in this case, how does the oracle choose among the possible answers that it can give? The second one concerns the idea that presuppositions act as a limitation of inquiry: in the way Hintikka treats presuppositions, if $Q = (\gamma_1, \ldots, \gamma_k)$ is a propositional question for which the inquirer has not established the presupposition $(\gamma_1 \vee \ldots \vee \gamma_k)$, it suffices for her to ask the question $Q = (\gamma_1, \ldots, \gamma_k, \top)$ for which the presupposition is a tautology, then presuppositions are not a form of limitation anymore.

Our proposal to solve these problems is to use a stronger notion of presupposition for propositional questions. Let $Q = (\gamma_1, \ldots, \gamma_k)$ be a propositional question, instead of requiring that the agent knows that *at least one* of the possible answer to Q is the case, we will require that the agent knows that *one and only one* possible answer to Q is the case. This solves the first problem regarding the choice of answer from the oracle since there is now only one possible choice of answer, due to the fact that the oracle is truthful. Besides, this solves the second problem since the trick consisting in transforming the question by adding a tautology does not work anymore, the question cannot have two true answers. Our proposal can formally be stated as follows: if $M = \langle W, \sim, V, \Phi \rangle$ is an epistemic inquiry model, $s \in W$ represents the actual world and $Q = (\gamma_1, \ldots, \gamma_k)$ is a propositional question, then the following condition must be satisfied in order for the inquiring agent to address the question to the oracle:

$$M, s \models K\mathsf{presup}(Q) \quad \text{where} \quad \mathsf{presup}(Q) := (\gamma_1 \vee \ldots \vee \gamma_k) \wedge \bigwedge_{j_1 \neq j_2 \text{ and } j_1, j_2 \in [\![1,k]\!]} \neg(\gamma_{j_1} \wedge \gamma_{j_2}).$$

We thereby get the precondition to the question operation with respect to the pointed model (M, s) and the question Q. Thus, we integrate the notion of presupposition under the form of a *precondition* to the question operation on the model, the precondition being that the agent knows the presupposition to the question.

We now have all the ingredients to provide the semantic definition of the dynamic *question operator*:

Definition 2.9 (Semantics for $\mathcal{E}_{\mathcal{I}}$). *The semantics for the epistemic inquiry language $\mathcal{E}_{\mathcal{I}}$ is given by the semantics for the epistemic language $\mathcal{E}_0$ plus the following semantic definition for the question operator*

$$M, s \models [(\gamma_1, \ldots, \gamma_k)?]\varphi \quad \textit{iff} \quad M, s \models K\mathsf{presup}(\gamma_1, \ldots, \gamma_k) \ \textit{ implies } \ M_{(\gamma_1, \ldots, \gamma_k)?}(s), s \models \varphi,$$

where

$$\mathsf{presup}(\gamma_1, \ldots, \gamma_k) := (\gamma_1 \vee \ldots \vee \gamma_k) \wedge \bigwedge_{j_1 \neq j_2 \textit{ and } j_1, j_2 \in [\![1,k]\!]} \neg(\gamma_{j_1} \wedge \gamma_{j_2}).$$

If $Q = (\gamma_1, \ldots, \gamma_k)$, we will denote by $\mathsf{pre}(Q)$, or $\mathsf{pre}(\gamma_1, \ldots, \gamma_k)$, the precondition to the question operation by Q, i.e., $\mathsf{pre}(Q) := K\mathsf{presup}(\gamma_1, \ldots, \gamma_k)$.

We now provide a sound and complete axiomatic system for our dynamic logic of questions.

2.1.3 Soundness and Completeness

First of all, we define the logic $\mathsf{E_I}$ aiming to characterize syntactically the formulas of $\mathcal{E}_{\mathcal{I}}$ that are valid on the class of models $\mathbf{E_I}$:

Definition 2.10 (Logic $\mathsf{E_I}$). *The logic $\mathsf{E_I}$ is built from the static epistemic logic EL plus the following axioms*

1. $\Phi\gamma \rightarrow \gamma$
2. $\Phi\gamma \rightarrow K(\gamma \rightarrow \Phi\gamma)$

and the following reduction axioms:

1. $[(\gamma_1, \ldots, \gamma_k)?]p \ \leftrightarrow \ \mathsf{pre}(\gamma_1, \ldots, \gamma_k) \rightarrow p$

2. $[(\gamma_1, \ldots, \gamma_k)?]\neg\varphi \;\leftrightarrow\; \mathsf{pre}(\gamma_1, \ldots, \gamma_k) \rightarrow \neg[(\gamma_1, \ldots, \gamma_k)?]\varphi$
3. $[(\gamma_1, \ldots, \gamma_k)?]\varphi \wedge \psi \;\leftrightarrow\; [(\gamma_1, \ldots, \gamma_k)?]\varphi \wedge [(\gamma_1, \ldots, \gamma_k)?]\psi$
4. $[(\gamma_1, \ldots, \gamma_k)?]\Phi\gamma \;\leftrightarrow\; \mathsf{pre}(\gamma_1, \ldots, \gamma_k) \rightarrow \Phi\gamma$
5. $[(\gamma_1, \ldots, \gamma_k)?]\, K\varphi \;\leftrightarrow$

$$\mathsf{pre}(\gamma_1, \ldots, \gamma_k) \rightarrow \left((\neg\Phi\gamma_1 \wedge \ldots \wedge \neg\Phi\gamma_k \wedge K\varphi) \vee \bigvee_{1 \leq i \leq k} \Phi\gamma_i \wedge K(\gamma_i \rightarrow [(\gamma_1, \ldots, \gamma_k)?]\varphi) \right).$$

We now show that the logic $\mathsf{E_I}$ is sound and complete with respect to the class of models $\mathbf{E_I}$:

Theorem 2.1 (Soundness and Completeness of $\mathsf{E_I}$)**.** *For every formula $\varphi \in \mathcal{E_I}$:*

$$\models_{\mathbf{E_I}} \varphi \quad \textit{if and only if} \quad \vdash_{\mathsf{E_I}} \varphi.$$

Proof. See Appendix A.1. □

2.2 Deductive steps: modeling inferences

In the framework of the interrogative model of inquiry, *deductive steps* of reasoning are thought as *inferences.* In this section, we will first present how IL formalizes deductive steps using the *tableau method.* Then, we will propose a framework in which inferences, according to the tableau method, are represented in a dynamic-epistemic setting. This framework will take the form of a *tableau-based dynamic logic of inferences* for which we will provide a sound and complete axiomatic system.

2.2.1 Deductive steps in IL

Deductive steps in IL are represented as *tableau building steps* according to the usual rules of tableau construction:

> Initially, there are certain initial premisses on the left side of the *tableau* and the proposition to be interrogatively established (proved) on the right side. There are two kinds of moves, logical inference moves and interrogative moves. The logical inference moves are simply a variant of the *tableau*-building rules of the usual *tableau* method. [22, p. 48]

We will now provide some background information on the tableau method in the propositional case. The presentation that we adopt is based on Smullyan's *unsigned semantic trees* [30] (henceforth, *semantic trees*). These semantic trees are defined as follows:

Definition 2.11 (Semantic tree)**.** *A semantic tree for $\gamma \in \mathcal{I}$ is a binary tree whose the nodes are labelled with formulas of the inquiry language $\mathcal{I}$, which has for root γ and which is generated by the following tableau-construction rules:*

$$\begin{array}{c} \gamma_1 \wedge \gamma_2 \\ | \\ \gamma_1 \\ | \\ \gamma_2 \end{array} \qquad\qquad \begin{array}{c} \neg(\gamma_1 \wedge \gamma_2) \\ \diagup \quad \diagdown \\ \neg\gamma_1 \quad\; \neg\gamma_2 \end{array} \qquad\qquad \begin{array}{c} \neg\neg\gamma \\ | \\ \gamma \end{array}$$

In this work we will represent semantic trees as indexed sets of branches, where branches are sets of formulas. Thus, if $\mathcal{T}$ is a semantic tree, we identify $\mathcal{T}$ with the indexed set $\{\mathcal{B}_i\}_{i \in \mathbb{N}}$, where $\mathcal{B}_i \in \mathcal{P}(\mathcal{I})$ for all $i \in \mathbb{N}$, such that

- $\mathcal{B}_0, \dots, \mathcal{B}_n$ are the non-empty sets of formulas corresponding to the $n+1$ branches of $\mathcal{T}$,
- $\mathcal{B}_i = \emptyset$ for all $i > n$.

We will denote by $\mathsf{STrees}(\mathcal{I}) \subseteq \mathcal{P}(\mathcal{I})^{\mathbb{N}}$ the class of all semantic trees on the inquiry language $\mathcal{I}$. For convenience reasons, we sometimes abuse notation and just write $\mathcal{T} = \{\mathcal{B}_0, \dots, \mathcal{B}_n\}$.

Due to the fact that IL is a model-oriented logic, Hintikka's use of the tableau method is inscribed in a *model-checking perspective*[9]: the aim of the inquirer is to find out the truth value of certain propositions in the model representing the actual world. In the most basic case, the inquiring agent aims to establish a certain formula γ by eliminating all the possible scenarios which are both compatible with her knowledge and the negation of γ. To this end, the inquiring agent entertains a semantic tree $\mathcal{T}$ with root $\neg\gamma$ and tries to close all its branches, each branch corresponding to a possible scenario compatible with $\neg\gamma$. Thus, the knowledge of the inquiring agent must be integrated into the tableau closure rules. To do so, we represent the knowledge of the inquiring agent by a set of formulas $\mathsf{E} \subseteq \mathcal{I}$ and we state the *closure rules* for semantic trees as follows:

Definition 2.12 (Closure rules)**.** *Let $\mathsf{E} \subseteq \mathcal{I}$ be a set of formulas, $\mathcal{T} \in \mathsf{STrees}(\mathcal{I})$ be a semantic tree and $\mathcal{B}$ be a branch of $\mathcal{T}$. We say that the branch $\mathcal{B}$ is closed w.r.t. E when we are in one of the two following cases:*

- *there exists a formula $\varphi \in \mathcal{B}$ such that φ and $\neg\varphi$ are in $\mathcal{B}$,*
- *there exists a formula φ such that $\varphi \in \mathcal{B}$ and $\neg\varphi \in \mathsf{E}$, or $\neg\varphi \in \mathcal{B}$ and $\varphi \in \mathsf{E}$.*

We say that a semantic tree is closed if all its branches are closed.

We will now show that the tableau-based method of reasoning is *sound*, i.e., we will show that, if the agent has managed to close a semantic tree $\mathcal{T}$ with root $\neg\gamma$ with respect to her knowledge E, then γ is the case in the actual world. To this end, we first show the following lemma:

Lemma 2.1. *Let v be a boolean valuation, let E be a set of true formulas with respect to v and let $\gamma \in \mathcal{I}$. We have that:*

if γ is true w.r.t. v, then any semantic tree with root γ is open w.r.t. E.

Proof. Let v be a boolean valuation and let E be a set of true formulas with respect to v. We prove the lemma by induction on the formula γ:

- Let $\gamma := p$ be an atomic proposition, true with respect to v. Then, there is only one semantic tree with root p, namely $\{\{p\}\}$. Since E is a set of true formulas w.r.t. v, the semantic tree $\{\{p\}\}$ is necessarily open.
- Let $\gamma := \gamma_1 \vee \gamma_2$ be a true formula w.r.t. v. Let $\mathcal{T}$ be a semantic tree with root $\gamma_1 \vee \gamma_2$. Since $\gamma_1 \vee \gamma_2$ is true w.r.t. v, at least one of the formulas γ_1 and γ_2 is true, say γ_1. Then, by induction hypothesis, any semantic tree with root γ_1 is open, so the subtree of $\mathcal{T}$ with root γ_1 is open, and thereby $\mathcal{T}$ is open.

The other cases of the induction are done in a similar way. We finally conclude that, for any formula $\gamma \in \mathcal{I}$,

[9] One may ask if the tableau method should be seen as representing *model-checking operations* or *inferences*. It seems that Hintikka uses both terminology to describe the tableau method. This is maybe due to the fact that on may think of model-checking operations as a special kind of inference. Thus, our view with respect to this issue is the following: we think of the tableau method as a model-checking technique and thereby as modeling a special kind of inference.

if γ is true w.r.t. v, then any semantic tree with root γ is open w.r.t. E.

$\square$

Then, the following theorem shows the soundness of the tableau method:

Theorem 2.2. *Let v be a boolean valuation, let E be a set of true formulas with respect to v and let $\gamma \in \mathcal{I}$. We have that:*

if there exists a closed tree w.r.t. E with root $\neg\gamma$, then γ is true with respect to v.

Proof. The theorem follows directly from the contraposition of the previous lemma. $\square$

We will now develop a dynamic logic of inferences based on the tableau method.

2.2.2 A tableau-based dynamic logic of inferences

Acts of *inference* produce significant information by making implicit knowledge explicit. Recently, this way to produce information has been investigated within the general program of dynamic logics, leading to the development of the so-called *dynamic logics of inferences* [13, 38, 41]. As we have just seen, Hintikka and colleagues have used the *tableau method* to represent inferences in IL. Thus, we propose, in this section, to merge these two approaches by developing a *tableau-based dynamic logic of inferences.*

The static perspective

We first introduce the distinction between *explicit* and *implicit knowledge* by defining an epistemic framework in which the agent is not logically omniscient. Then, we propose a way to represent semantic trees in this framework, modeling thereby the *on-going inferential processes* that the agent is engaged in in order to make explicit some of her implicit knowledge. We focus here on the static aspects, leaving the dynamic aspects for the next section.

Explicit and implicit knowledge

The first thing that we have to do is to define the notions of *implicit* and *explicit knowledge.* To this end, we will adopt the same approach as the one of dynamic logics of inferences, i.e., we will use the *two-level semantic-syntactic format* proposed in [31]. According to this approach, *implicit knowledge* is represented in the same way knowledge is traditionally represented in epistemic logic using possible-worlds semantics, and *explicit knowledge* is represented via a set of formulas associated to each world in the model.

In this work, we also introduce a distinction between *local* and *global explicit knowledge*: *local explicit knowledge* is represented by a set of true formulas associated to each world in the model and represents the information that the agent has about each of these worlds; then a formula γ is *global explicit knowledge* if γ is local explicit knowledge in all the worlds present in the agent's epistemic range.

In order to express local and global explicit knowledge, we add to the language of epistemic logic a modal operator E:

Definition 2.13 (Explicit/implicit epistemic language)**.** *Let P be a set of atomic propositions. The explicit/implicit epistemic language is given by*

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid E\gamma$$

where $p \in \mathsf{P}$ and $\gamma \in \mathcal{I}$.

In this language, formulas of the form $K\varphi$ are read as *"the agent implicitly knows that φ"*, formulas of the form $E\gamma$ are read as *"the agent explicitly knows locally that γ"* and formulas of the form $KE\gamma$ are read as *"the agent explicitly knows globally that γ"*.

As we said, local explicit knowledge will be represented by a set of formulas associated to each world in the epistemic model. This leads to the following definition of *explicit/implicit epistemic models*:

Definition 2.14 (Explicit/implicit epistemic model)**.** *Let P be a countable set of atomic propositions. An explicit/implicit epistemic model is a tuple $M = \langle W, \sim, V, \mathsf{E}\rangle$ where:*

- *W is a non-empty set of worlds,*
- *$\sim \subseteq W \times W$ is the epistemic indistinguishability relation of the inquiring agent,*
- *$V : W \to \mathcal{P}(\mathsf{P})$ is an atomic valuation function indicating the atomic propositions that are true at each world,*
- *$\mathsf{E} : W \to \mathcal{P}(\mathcal{I})$ is a function which associates to each world $w \in W$ a set of formulas of the inquiry language $\mathcal{I}$.*

Then, the explicit/implicit epistemic language is interpreted on explicit/implicit epistemic models as follows:

Definition 2.15 (Semantics for the explicit/implicit epistemic language)**.** *Let (M, s) be a pointed explicit/implicit epistemic model where $M = \langle W, \sim, V, \mathsf{E}\rangle$. The semantics for the explicit/implicit epistemic language is given by the semantics for the epistemic language plus the following semantic definition for the modal operator E*

$$M, s \models E\gamma \quad \textit{iff} \quad \gamma \in \mathsf{E}(s).$$

In this work, we will impose the following restrictions on explicit/implicit models: let $M = \langle W, \sim, V, \mathsf{E}\rangle$ be an explicit/implicit epistemic model,

Veridicality for local explicit knowledge: we will require that local explicit knowledge is always truthful: for all $w \in W$,

$$\text{if } \gamma \in \mathsf{E}(w), \text{ then } M, w \models \gamma,$$

Coherence property for local explicit knowledge: we will require a *coherence property* on sets of local explicit knowledge, also called 'weak introspection' by van Benthem and Velázquez-Quesada in [38]: for all $w \in W$,

$$\text{if } \gamma \in \mathsf{E}(w) \text{ and } w \sim u \text{ with } u \in W, \text{ then } \gamma \in \mathsf{E}(u).$$

It is important to notice that, since we require local explicit knowledge to be true, all global explicit knowledge is also implicit knowledge, i.e., the following principle is valid on our intended class of models:

$$KE\gamma \to K\gamma.$$

Besides, due to the coherence property and the fact that we consider the epistemic indistinguishability relation to be an equivalence relation, we have that local and global epistemic knowledge coincide, i.e., the following principle is valid on our intended class of models:

$$KE\gamma \leftrightarrow E\gamma.$$

The main reason why we assume the coherence property for the sets of local explicit knowledge is to obtain a completeness result.[10] If we want to make use of the distinction between local and global explicit knowledge, we obviously need to give up the coherence property, but this yields a very different logic. In section 2.3.2., we propose a way to deal with the present framework while giving up the coherence property.

Representing semantic trees in a modal framework

As we said at the beginning, it is through acts of *inference* that the agent can acquire local and global explicit knowledge. In this work, we will commit ourself to the following perspective: global explicit knowledge is the result of acts of inference producing local explicit knowledge. This means that we consider that, in order to obtain global explicit knowledge of a certain formula γ, the agent has to obtain γ, through acts of inference, in each worlds present in her epistemic range. In other words, we consider that *global acts of inference* producing *global explicit knowledge* are the result of several *local acts of inference* producing *local explicit knowledge.*[11]

Following IL, we will model acts of inference using the *tableau method.* To this end, we will provide the inquiring agent with a set of semantic trees, i.e., we will associate a set of semantic trees to each world of a given explicit/implicit epistemic model. These semantic trees represent the different on-going local inferential processes that the agent is engaged in in order to extend her local explicit knowledge.

First of all, we extend the explicit/implicit epistemic language into a *tableau epistemic language* $\mathcal{TE}_0$:

Definition 2.16 (Tableau epistemic language $\mathcal{TE}_0$)**.** *Let* P *be a set of atomic propositions. The tableau epistemic language* $\mathcal{TE}_0$ *is given by*

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid E\gamma \mid Br_i^j\gamma$$

where $p \in \mathsf{P}$, $\gamma \in \mathcal{I}$ *and* $i, j \in \mathbb{N}$.

Formulas of the form $Br_i^j\gamma$ are read as *"the formula* γ *is present in the* i^{th} *branch of the* j^{th} *semantic tree entertained by the agent"*. Thus, the modal operators Br_i^j are used to talk about the presence, or the absence, of a formula of the inquiry language in a branch of a particular semantic tree entertained by the agent at a given world. Since a tableau construction rule is applied to a specific formula in a specific branch, the modal operators Br_i^j will allow us latter to express, in the dynamic perspective, the precondition to a tableau construction update.

We now define *tableau epistemic models* as explicit/implicit epistemic models where a finite set[12] of semantic trees is associated to each world in the model:

[10] This assumption is also made in the recent literature on dynamic logics of inferences (see [38, 41]).

[11] This perspective seems natural in a lots of real situations. Consider for instance a situation in which each world in the agent's epistemic range represents a model of a given theory. In this case, the local explicit knowledge of the agent is constituted, in each world, by the axioms and principles of each one of these theories. Then, the local acts of inference enable to obtain more information about the theorems and properties that are logically implied by the considered theories. If we consider that the true theory is among the theories whose the models are present in the epistemic range, then, by deducing from each theory a certain proposition γ, the agent can obtain explicit global knowledge about γ. In this situation, explicit global knowledge might represent the propositions obtaining consensus among the proponents of the different theories. In this example, global explicit knowledge, interpreted as the consensus propositions in a given scientific community, is obtained as the result of several local acts of inference.

[12] Strictly speaking, we associate to each world a set of semantic trees indexed by ω where only the first n semantic trees, where n is finite ($n \in \mathbb{N}$), are non-empty.

Definition 2.17 (Tableau epistemic model)**.** *Let* P *be a countable set of atomic propositions. A tableau epistemic model is a tuple* $M = \langle W, \sim, V, \mathsf{E}, \mathsf{T}\rangle$ *where:*

- W *is a non-empty set of worlds,*
- $\sim \subseteq W \times W$ *is the epistemic indistinguishability relation of the inquiring agent,*
- $V : W \to \mathcal{P}(\mathsf{P})$ *is an atomic valuation function indicating the atomic propositions that are true at each world,*
- $\mathsf{E} : W \to \mathcal{P}(\mathcal{I})$ *is a function which associates to each world* $w \in W$ *a set of formulas of the inquiry language* $\mathcal{I}$*,*
- $\mathsf{T} : W \to \mathsf{STrees}(\mathcal{I})^{\mathbb{N}}$*, is a function which associates to each world* $w \in W$ *an indexed set of semantic trees* $\mathsf{T}(w) = \{\mathcal{T}^j(w)\}_{j\in\mathbb{N}}$*, where* $\mathcal{T}^j(w) \in \mathsf{STrees}(\mathcal{I})$ *for all* $i \in \mathbb{N}$ *and has for root a formula of the form* $\neg\gamma$ *with* $\gamma \in \mathcal{I}$*, such that there exists* $p \in \mathbb{N}$ *for which*
  - $\mathcal{T}^0(w), \dots, \mathcal{T}^p(w)$ *are non-empty semantic trees,*
  - $\mathcal{T}^j(w) = \emptyset$ *for all* $j > p$*.*

We will denote by $\mathcal{T}^j(w)$ the j^{th} semantic tree of $\mathsf{T}(w)$ and we will denote by $\mathcal{B}^j_i(w)$ the i^{th} branch of $\mathcal{T}^j(w)$. For convenience reasons, we sometimes just write $\mathsf{T}(w) = \{\mathcal{T}^0(w), \dots, \mathcal{T}^p(w)\}$ and $\mathcal{T}^j(w) = \{Br^j_0(w), \dots, Br^j_n(w)\}$.

Then, the tableau epistemic language is interpreted on tableau epistemic models as follows:

Definition 2.18 (Semantics for the tableau epistemic language $\mathcal{TE}_0$)**.** *Let* (M, s) *be a pointed tableau epistemic model where* $M = \langle W, \sim, V, \mathsf{E}, \mathsf{T}\rangle$*. The semantics for the tableau epistemic language* $\mathcal{TE}_0$ *is given by the semantics for the explicit/implicit epistemic language plus the following semantic definition for the modal operators* Br^j_i

$$M, s \models Br^j_i\gamma \quad \textit{iff} \quad \mathcal{T}^j(s) \neq \emptyset \textit{ and } \mathcal{B}^j_i(s) \neq \emptyset \textit{ and } \gamma \in \mathcal{B}^j_i(s).$$

As for explicit/implicit epistemic models, we will also impose restrictions on tableau epistemic models. Thus, in addition of the restrictions that we imposed on explicit/implicit epistemic models, we will require that tableau epistemic models satisfy a coherence property with respect to the set of semantic trees entertained by the agent: let $M = \langle M, \sim, V, \mathsf{E}, \mathsf{T}\rangle$ be a tableau epistemic model,

Coherence property for semantic trees: we require that, for all $w \in W$,

$$\text{if } \gamma \in \mathcal{B}^j_i(w) \text{ and } u \sim w \text{ with } u \in W, \text{ then } \gamma \in \mathcal{B}^j_i(u).$$

Thus, our intended class of models **TE**, which is a subclass of the class of tableau epistemic model, is defined as follows:[13]

Definition 2.19 (Class of models **TE**)**.** *let* $M = \langle W, \sim, V, \mathsf{E}, \mathsf{T}\rangle$ *be a tableau epistemic model. Then:*

$$M \in \mathbf{TE} \quad \textit{if and only if}$$

1. *for all* $w \in W$*, if* $\gamma \in \mathsf{E}(w)$*, then* $M, w \models \gamma$*,*
2. *for all* $w \in W$*, if* $\gamma \in \mathsf{E}(w)$ *and* $w \sim u$ *with* $u \in W$*, then* $\gamma \in \mathsf{E}(u)$*,*

[13]In the following, by tableau epistemic models we will mean models of this class.

3. *for all $w \in W$, if $\gamma \in \mathcal{B}_i^j(w)$ and $w \sim u$ with $u \in W$, then $\gamma \in \mathcal{B}_i^j(u)$.*

As for local explicit knowledge, by assuming the coherence property for the sets of semantic trees entertained by the agent and by considering the epistemic indistinguishability relation to be an equivalence relation, we necessarily have that local and global inferential processes coincide. In section 2.3.2., we propose a way to deal with this framework while giving up the coherence property for the sets of semantic trees.

Expressing openness and closeness of semantic trees in the language $\mathcal{TE}_0$

The tableau epistemic language allows us to express a certain number of properties about semantic trees. More specifically, in addition of the presence or absence of a formula in a branch, we can easily express that a branch of a tree entertained by the agent at a certain state s is open or closed. To see this, let $M = \langle W, \sim, V, \mathsf{E}, \mathsf{T}\rangle$ be a tableau epistemic model, let $s \in W$ and let $\mathcal{T}^j(s) \in \mathsf{T}(s)$ such that $\mathcal{T}^j(s) \neq \emptyset$. According to our definition of closeness for semantic trees, the i^{th} branch $\mathcal{B}_i^j(s)$ of $\mathcal{T}^j(s)$ is closed if we are in one of the two following cases:

- there exists a formula $\varphi \in \mathcal{B}_i^j(s)$ such that φ and $\neg\varphi$ are in $\mathcal{B}_i^j(s)$,
- there exists a formula φ such that $\varphi \in \mathcal{B}_i^j(s)$ and $\neg\varphi \in \mathsf{E}(s)$, or $\neg\varphi \in \mathcal{B}_i^j(s)$ and $\varphi \in \mathsf{E}(s)$.

The notion of closed branch can then formally be expressed in the language $\mathcal{TE}_0$ as follows:

$$\text{the branch } \mathcal{B}_i^j(s) \text{ is closed } \Leftrightarrow M, s \models \mathsf{closed}(\mathcal{B}_i^j)$$

where

$$\mathsf{closed}(\mathcal{B}_i^j) := \bigvee_{\varphi \in \mathcal{B}_i^j(s)} (Br_i^j\varphi \wedge Br_i^j\neg\varphi) \;\vee \bigvee_{\varphi \in \mathcal{B}_i^j(s)} (Br_i^j\varphi \wedge E\neg\varphi) \;\vee \bigvee_{\neg\varphi \in \mathcal{B}_i^j(s)} (Br_i^j\neg\varphi \wedge E\varphi).$$

Then, the notion of closed tree can be expressed in $\mathcal{TE}_0$ as follows:

$$\text{the tree } \mathcal{T}^j(s) \text{ is closed } \Leftrightarrow M, s \models \mathsf{closed}(\mathcal{T}^j)$$

where

$$\mathsf{closed}(\mathcal{T}^j) := \bigwedge_{\mathcal{B}_i^j(s) \neq \emptyset} \mathsf{closed}(\mathcal{B}_i^j).$$

It is important to notice that the closure rules, as we defined them, depend crucially on the explicit knowledge of the inquiring agent. This is due to the fact that the inquiring agent uses her explicit knowledge in order to close the different branches of the semantic tree that she entertains, i.e., in order to eliminate all the possible scenarios which are both compatible with her knowledge and the negation of the conclusion that she wants to establish.

Finally, the following theorem says that if the agent has managed to close a tableau with root $\neg\gamma$ at a particular world s, then γ is true at s:

Theorem 2.3. *Let (M, s) be a pointed tableau epistemic model with $M = \langle W, \sim, V, \mathsf{E}, \mathsf{T}\rangle$ and let $\mathcal{T}^j(s) \in \mathsf{T}(s)$ where $\neg\gamma \in \mathcal{I}$ is the root of $\mathcal{T}^j$. We have*

$$M, s \models \mathsf{closed}(\mathcal{T}^j) \;\Rightarrow\; M, s \models \gamma.$$

Proof. This theorem follows directly from theorem 2.2. □

This theorem reflects the soundness of the tableau method and will assure latter that the conclusion obtained through a closed semantic tree is true and can safely be added to the set of local explicit knowledge.

The dynamic perspective

In the previous section, we have presented a language able to describe static properties of the semantic trees entertained by the agent. In the dynamic perspective, we want to extend this language in order to be able to represent *inferences* dynamically as model operations. To this end, we need to introduce two kinds of model operations: one dealing with *tableau construction steps*, the other dealing with the *creation* and the *elimination* of semantic trees.

Operation of tableau construction

The model operation of tableau construction consists in expanding a semantic tree, present in all the worlds of the agent's epistemic range, by applying the suitable expanding rule to a formula present in the tree. Formally, the *tableau construction operation* takes as input a semantic tree, a branch and a formula, and is defined as follows:

Definition 2.20 (Tableau construction operation)**.** *Let (M, s) be a pointed tableau epistemic model with $M = \langle W, \sim, V, \mathsf{E}, \mathsf{T}\rangle$ and let $\gamma \in \mathcal{B}_i^j(s)$. The model $M_{(\mathcal{T}^j,i,\gamma)}(s) = \langle W', \sim', V', \mathsf{E}', \mathsf{T}'\rangle$ is given by*

- $W' := W$, $\sim' := \sim$, $V' := V$, $\mathsf{E}' := \mathsf{E}$,
- *for every* $w \in W$ *such that* $w \nsim s$, $\mathsf{T}'(w) := \mathsf{T}(w)$,
- *for every* $w \in W$ *such that* $w \sim s$,
  - *if an expanding rule has already been applied to* γ *in the tree* $\mathcal{T}^j(w)$, *then* $\mathsf{T}'(w) := \mathsf{T}(w)$,
  - *if no expanding rule has already been applied to* γ *in the tree* $\mathcal{T}^j(w)$, *then*

$$\mathsf{T}'(w) \text{ is such that } \mathcal{T}^k(w)' := \mathcal{T}^k(w) \text{ for all } k \neq j,$$

and

$$\mathcal{T}^j(w)' \text{ is such that } \mathcal{B}_l^j(w)' := \mathcal{B}_l^j(w) \text{ for all } l \neq i \text{ and } l \neq n+1,$$ [14]

and $\mathcal{B}_i^j(w)'$ *and* $\mathcal{B}_i^{n+1}(w)'$ *are obtained in the following way:*

$\wedge$: *if* $\gamma := \gamma_1 \wedge \gamma_2$, *then* $\mathcal{B}_i^j(w)' := \mathcal{B}_i^j(w) \cup \{\gamma_1, \gamma_2\}$ *and* $\mathcal{B}_i^{n+1}(w)' := \mathcal{B}_i^{n+1}(w)$,

$\vee$: *if* $\gamma := \neg(\gamma_1 \wedge \gamma_2)$, *then* $\mathcal{B}_i^j(w)' := \mathcal{B}_i^j(w) \cup \{\neg\gamma_1\}$ *and* $\mathcal{B}_i^{n+1}(w)' := \mathcal{B}_i^j(w) \cup \{\neg\gamma_2\}$,

$\neg$: *if* $\gamma := \neg\neg\gamma_1$, *then* $\mathcal{B}_i^j(w)' := \mathcal{B}_i^j(w) \cup \{\gamma_1\}$ *and* $\mathcal{B}_i^{n+1}(w)' := \mathcal{B}_i^{n+1}(w)$.

Notice that the result of applying a tableau construction operation on a tableau epistemic model is still a tableau epistemic model, due to the fact that this operation is done according to the tableau construction rules. Besides, the coherence property for the set of semantic trees is preserved by a tableau construction operation due to the fact that the modifications on the sets of semantic trees are done in a uniform way on the epistemic range of the agent.

We now extend the tableau epistemic language with a *dynamic operator for tableau construction* (henceforth, *tableau construction operator*):

[14] Here $n+1$ is the index of the first empty branch of the tree $\mathcal{T}^j(w)$ ($\mathcal{T}^j(w) = \{\mathcal{B}_0^j(w), \ldots, \mathcal{B}_n^j(w)\}$).

Definition 2.21 (Tableau epistemic language $\mathcal{TE}_1$)**.** *Let* P *be a set of atomic propositions. The tableau epistemic language* $\mathcal{TE}_1$ *is given by*

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid E\gamma \mid Br_i^j\gamma \mid [\mathcal{T}^j, i, \gamma]\varphi$$

where $p \in \mathsf{P}$, $\gamma \in \mathcal{I}$, $i, j \in \mathbb{N}$ *and* $\mathcal{T}^j \in \mathsf{Strees}(\mathcal{I})$.

The precondition for a tableau construction operation is the following: the formula γ to which the operation is applied has to be present in the i^{th} branch of the tree $\mathcal{T}^j$. This leads to the following semantic definition for the *tableau construction operator*:

Definition 2.22 (Semantics for the language $\mathcal{TE}_1$)**.** *Let* (M, s) *be a pointed tableau epistemic model where* $M = \langle W, \sim, V, \mathsf{E}, \mathsf{T}\rangle$*. The semantics for the language* $\mathcal{TE}_1$ *is given by the semantics for the language* $\mathcal{TE}_0$ *plus the following semantic definition for the tableau construction operator*

$$M, s \models [\mathcal{T}^j, i, \gamma]\varphi \;\; \textit{iff} \;\; M, s \models Br_i^j\gamma \;\; \textit{implies} \;\; M_{(\mathcal{T}^j,i,\gamma)}(s), s \models \varphi.$$

Formulas of the form $[\mathcal{T}^j, i, \gamma]\varphi$ are read as *"φ is the case after the tableau construction operation on the formula γ in the i^{th} branch of $\mathcal{T}^j$"*. This language then enables us to express a certain number of properties relative to *change* in the semantic trees entertained by the agent, for instance:

$M, s \models [\mathcal{T}^j, i, \gamma]Br_i^j\varphi$	*expresses that*	'the formula φ is in the i^{th} branch of the tree $\mathcal{T}^j(s)$ after the tableau construction operation with input $(\mathcal{T}^j, i, \gamma)$'
$M, s \models [\mathcal{T}^j, i, \gamma]\mathsf{closed}(\mathcal{B}_i^j)$	*expresses that*	'the branch $\mathcal{B}_i^j(s)$ of $\mathcal{T}^j(s)$ is closed after the tableau construction operation with input $(\mathcal{T}^j, i, \gamma)$'

We now turn to the second kind of model operations, dealing with the creation and the elimination of semantic trees.

Operations of tableau creation and tableau elimination

The *tableau creation operation* consists simply in adding a semantic tree to the set of semantic trees entertained by the agent at a particular state. Thus, the tableau creation operation takes as input a formula in $\mathcal{I}$ which will be the root of the new semantic tree. This operation is formally defined as follows:

Definition 2.23 (Tableau creation operation)**.** *Let* (M, s) *be a pointed tableau epistemic model with* $M = \langle W, \sim, V, \mathsf{E}, \mathsf{T}\rangle$ *such that* $\mathsf{T}(s) = \{\mathcal{T}^1(s), \ldots, \mathcal{T}^p(s)\}$*, and let* $\gamma \in \mathcal{I}$ *be a formula of the form* $\neg\gamma_1$ *with* $\gamma_1 \in \mathcal{I}$*. The model* $M_{\mathsf{T}+\{\gamma\}}(s) = \langle W', \sim', V', \mathsf{E}', \mathsf{T}'\rangle$ *is given by*

- $W' := W$, $\sim' := \sim$, $V' := V$, $\mathsf{E}' := \mathsf{E}$,
- *for all* $w \in W$ *such that* $w \nsim s$, $\mathsf{T}'(w) := \mathsf{T}(w)$,
- *for all* $w \in W$ *such that* $w \sim s$, $\mathsf{T}'(w)$ *is such that*

$$\mathcal{T}^j(w)' := \mathcal{T}^j(w) \textit{ for } j \neq p+1 \textit{ and } \mathcal{T}^{p+1}(w)' := \{\{\gamma\}\}.$$

The *tableau elimination operation* consists in modeling the step going from a closed tableau with root $\neg\gamma$ to the conclusion that γ is true, becoming thereby explicit knowledge. Formally, the tableau elimination operation takes as input a (closed) semantic tree and is defined as follows:

Definition 2.24 (Tableau elimination operation)**.** *Let (M, s) be a pointed tableau epistemic model with $M = \langle W, \sim, V, \mathsf{E}, \mathsf{T}\rangle$ and $\mathsf{T}(s) = \{\mathcal{T}^0(s), \ldots, \mathcal{T}^p(s)\}$, and let $\mathcal{T}^j(s) \in \mathsf{T}(s)$ be a closed tableau w.r.t. $\mathsf{E}(s)$ with root $\neg\gamma$. The model $M_{\mathsf{T}-\mathcal{T}^j}(s) = \langle W', \sim', V', \mathsf{E}', \mathsf{T}'\rangle$ is given by*

- $W' := W$, $\sim' := \sim$, $V' := V$,
- *for all* $w \nsim s$, $\mathsf{E}'(w) := \mathsf{E}(w)$,
- *for all* $w \sim s$, $\mathsf{E}'(w) := \mathsf{E}(w) \cup \{\gamma\}$,
- $\mathsf{T}' := \mathsf{T}$.

Notice that applying a tableau elimination operation to a tableau epistemic model yields a tableau epistemic model. The first reason is that, due to theorem 2.3, the formula γ added to the sets of explicit knowledge is true in all the worlds of the agent's epistemic range. The second reason is that, the tableau elimination operation being done in a uniform way on the epistemic range of the agent, the coherence property for the sets of explicit knowledge is preserved by this operation.

We now extend our previous tableau epistemic language with a *dynamic operator of tableau creation* (henceforth, *tableau creation operator*) and a *dynamic operator of tableau elimination* (henceforth, *tableau elimination operator*):

Definition 2.25 (Tableau epistemic language $\mathcal{TE}$)**.** *Let P be a set of atomic propositions. The tableau epistemic language $\mathcal{TE}$ is given by*

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid E\gamma \mid Br_i^j\gamma \mid [\mathcal{T}^j, i, \gamma]\varphi \mid [\mathsf{T} + \{\gamma\}]\varphi \mid [\mathsf{T} - \mathcal{T}^j]\varphi$$

where $p \in \mathsf{P}$, $\gamma \in \mathcal{I}$, $i, j \in \mathbb{N}$ and $\mathcal{T}^j \in \mathsf{Strees}$.

In this language, formulas of the form $[\mathsf{T} + \{\gamma\}]\varphi$ are read as *"φ is the case after the creation of a new semantic tree with root γ"* and formulas of the form $[\mathsf{T} - \mathcal{T}^j]\varphi$ are read as *"φ is the case after the agent has concluded from the closeness of $\mathcal{T}^j$ that the root of $\mathcal{T}^j$ is false"*. There is no precondition to a tableau creation operation. For the tableau elimination operation with input $\mathcal{T}^j$, the precondition is the closeness of $\mathcal{T}^j$. This leads to the following semantic definitions for the tableau creation and tableau elimination operators:

Definition 2.26 (Semantics for the language $\mathcal{TE}$)**.** *Let (M, s) be a tableau epistemic model where $M = \langle W, \sim, V, \mathsf{E}, \mathsf{T}\rangle$. The semantics for the tableau epistemic language $\mathcal{TE}$ is given by the semantics for the language $\mathcal{TE}_1$ plus the following semantic definitions for the tableau creation and tableau elimination operators*

$$\begin{array}{lll} M, s \models [\mathsf{T} + \{\gamma\}]\varphi & \textit{iff} & M_{\mathsf{T}+\{\gamma\}}(s), s \models \varphi \\ M, s \models [\mathsf{T} - \mathcal{T}^j]\varphi & \textit{iff} & M, s \models \mathsf{closed}(\mathcal{T}^j) \ \textit{implies} \ M_{\mathsf{T}-\mathcal{T}^j}(s), s \models \varphi. \end{array}$$

We will now provide a sound and complete logic for our tableau-based dynamic logic of inferences.

2.2.3 Soundness and Completeness

In this section, we will show that our tableau-based dynamic logic of inferences is *completely axiomatizable.* To this end, we first need to enrich the language $\mathcal{TE}_0$ to be able to express the internal structure of semantic trees, in order to provide a sound and complete logic for the static fragment. We will then provide reduction axioms for the dynamic operators of tableau management.

In order to be able to express the internal structures of the semantic trees, we need to extend our static language $\mathcal{TE}_0$ into a language $\mathcal{TE}_0^*$ defined as follows:

Definition 2.27 (Language $\mathcal{TE}_0^*$). *The language $\mathcal{TE}_0^*$ is obtained by adding to the language $\mathcal{TE}_0$ the recursive rules*

$$R^j\gamma \mid C_i^j\gamma \mid \mathsf{empty}(\mathcal{B}_i^j) \mid \mathsf{empty}(\mathcal{T}^j)$$

where $\gamma \in \mathcal{I}$ and $i, j \in \mathbb{N}$.

In this language, formulas of the form $R^j\gamma$ are read as *"γ is the root of the tree $\mathcal{T}^j$"*, formulas of the form $C_i^j\gamma$ are read as *"γ results from the application of a tableau construction rule to a formula in the i^{th} branch of the tree $\mathcal{T}^j$"*, formulas of the form $\mathsf{empty}(\mathcal{B}_i^j)$ are read as *"the i^{th} branch of $\mathcal{T}^j$ is empty"* and formulas of the form $\mathsf{empty}(\mathcal{T}^j)$ are read as *"the tree $\mathcal{T}^j$ is empty"*.

The semantics for the extended language $\mathcal{TE}_0^*$ is defined as follows:

Definition 2.28 (Semantics for $\mathcal{TE}_0^*$). *Let (M, s) be a pointed tableau epistemic model where $M = \langle W, \sim, V, \mathsf{E}, \mathsf{T}\rangle$. The semantics for the language $\mathcal{TE}_0^*$ is given by the semantics for the language $\mathcal{TE}_0$ plus the following semantic definitions*

$$\begin{array}{rcl} M, s \models R^j\gamma & \textit{iff} & \gamma \textit{ is the root of } \mathcal{T}^j(s) \\ M, s \models C_i^j\gamma & \textit{iff} & \gamma \textit{ results from the application of a tableau construction rule to a formula in } \mathcal{B}_i^j(s) \\ M, s \models \mathsf{empty}(\mathcal{B}_i^j) & \textit{iff} & \mathcal{B}_i^j(s) = \emptyset \\ M, s \models \mathsf{empty}(\mathcal{T}^j) & \textit{iff} & \mathcal{T}^j(s) = \emptyset. \end{array}$$

We now define the static logic TE_0:

Definition 2.29 (Logic TE_0). *The logic TE_0 is built from the axioms and rules for the static epistemic logic EL plus the following axioms*

1. $E\gamma \to \gamma$
2. $p \to Ep$ *and* $\neg p \to E\neg p$
3. $E\gamma \to KE\gamma$
4. $Br_i^j\gamma \to KBr_i^j\gamma$
5. $Br_i^j\gamma \to R^j\gamma \vee (Br_i^j\gamma \wedge C_i^j\gamma)$
6. $\mathsf{empty}(\mathcal{T}^j) \to \neg Br_i^j\gamma$.

The logic TE_0 is sound and complete with respect to the class of models **TE**:

Theorem 2.4 (Soundness and Completeness of TE_0). *For every formula $\varphi \in \mathcal{TE}_0^*$:*

$$\models_{\mathbf{TE}} \varphi \quad \textit{if and only if} \quad \vdash_{\mathsf{TE}_0} \varphi.$$

Proof. See appendix A.2.1. □

We then obtain the logic TE by extending the static logic TE_0 with the reduction axioms for the dynamic operators of tableau construction, creation and elimination:

Definition 2.30 (Logic TE). *The logic TE is built from the static logic TE_0 plus the reduction axioms listed in the appendix A.2.2.*

We can now show that the logic TE is sound and complete with respect to the class of models **TE**:

Theorem 2.5 (Soundness and Completeness of TE). *For every formula $\varphi \in \mathcal{TE}^*$:*[15]

$$\models_{\mathbf{TE}} \varphi \quad \textit{if and only if} \quad \vdash_{\mathsf{TE}} \varphi.$$

Proof. See appendix A.2.3. □

[15] The language $\mathcal{TE}^*$ is the extension of the language $\mathcal{TE}_0^*$ with the dynamic operators of tableau management.

2.3 Combining interrogative and deductive steps

Asking questions and *making inferences* are two different, but complementary, ways to obtain information:

- By *making inferences*, the agent can extend her explicit knowledge by transforming any implicit knowledge into explicit one. However, the agent cannot extend her implicit knowledge by making inferences.

- By *asking questions*, the agent can extend both her explicit and implicit knowledge. However, the agent cannot, in most of the case, turn all implicit knowledge into explicit one by asking questions.

Most of the time, people use both questions and inferences when they are involved in knowledge-seeking processes. For instance, in the case of scientific practice, inferences are used to make predictions from a given theory, whereas questions, taking the form of observations or experiments, are used to obtain information in order to test these predictions, leading eventually to a corroboration or a rejection of the theory. From a logical perspective to inquiry, this speaks for a joint treatment of questions and inferences, as Hintikka puts it:

> Deduction (logic) and interrogation appear as two interacting and mutually reinforcing components of inquiry. Neither is dispensable. Questions are needed to bring in substantially new information, and deductions are needed both for the purpose of spelling out the consequences of such information and, more importantly, for the purpose of paving the way for new questions by establishing their presuppositions. [...]
>
> [T]here is no absolute sense in which one of the two intertwined components of interrogative inquiry, deductions and questioning, is more important or more difficult, absolutely speaking. Such judgments can only be made on the basis of some particular assumptions concerning the "cost" of different kinds of moves in the interrogative games of inquiry. A game theorist would codify such assumptions in the "payoffs" of the game. [22, p. 35]

In the two previous sections, we have treated questions and inferences separately by developing on one hand a dynamic logic of questions, and on the other hand a dynamic logic of inferences. In this section, we will first merge these two systems into a *dynamic logic of questions and inferences*, which will be the straightforward combination of the two systems developed in the previous sections. However, this system has something unsatisfactory in the way it deals with the relation between incoming of information through inferences and incoming of information through questions. In the second subsection, we will locate the problem and propose a way to avoid it.

2.3.1 A dynamic logic of questions and inferences

Combining our previous dynamic logic of questions and dynamic logic of inferences is, for the most part, straightforward. Two particular points deserve special attention: one is to define the question operation while working with implicit/explicit knowledge; the other is to introduce explicit knowledge into the precondition to the question operation. We will deal with these two issues when they will appear in the presentation of the system.

First of all, we define the *tableau epistemic inquiry language* $\mathcal{TE}_{\mathcal{I}}$ as the combination of the languages $\mathcal{TE}$ and $\mathcal{E}_{\mathcal{I}}$:

Definition 2.31 (Language $\mathcal{TE_I}$). *Let* P *be a set of atomic propositions. The tableau epistemic inquiry language* $\mathcal{TE_I}$ *is given by*

$$\varphi ::= p \mid \neg\varphi \mid \varphi\wedge\varphi \mid K\varphi \mid E\gamma \mid \Phi\gamma \mid Br_i^j\gamma \mid [\mathcal{T}^j, i, \gamma]\varphi \mid [\mathsf{T}+\{\gamma\}]\varphi \mid [\mathsf{T}-\mathcal{T}^j]\varphi \mid [(\gamma_1,\ldots,\gamma_k)?]\varphi$$

where $p \in \mathsf{P}$, $\gamma, \gamma_1, \ldots, \gamma_k \in \mathcal{I}$, $i, j, k \in \mathbb{N}$ *with* $k \geq 1$ *and* $\mathcal{T}^j \in \mathsf{Strees}(\mathcal{I})$.

Then, a *tableau epistemic inquiry model* is defined as a tableau epistemic model plus an oracle function:

Definition 2.32 (Tableau epistemic inquiry model). *A tableau epistemic inquiry model is a tuple* $M = \langle W, \sim, V, \mathsf{E}, \mathsf{T}, \Phi\rangle$ *where:*

- W *is a non-empty set of worlds,*
- $\sim\, \subseteq W \times W$ *is the epistemic indistinguishability relation of the inquiring agent,*
- $V : W \to \mathcal{P}(\mathsf{P})$ *is an atomic valuation function indicating the atomic propositions that are true at each world,*
- $\mathsf{E} : W \to \mathcal{P}(\mathcal{I})$ *is a function which associates to each world* $w \in W$ *a set of formulas of the inquiry language* $\mathcal{I}$,
- $\mathsf{T} : W \to \mathsf{STrees}(\mathcal{I})^{\mathbb{N}}$, *is a function which associates to each world* $w \in W$ *an indexed set of semantic trees* $\mathsf{T}(w) = \{\mathcal{T}^j(w)\}_{j\in\mathbb{N}}$, *where* $\mathcal{T}^j(w) \in \mathsf{STrees}(\mathcal{I})$ *for all* $i \in \mathbb{N}$ *and as for root a formula of the form* $\neg\gamma$ *with* $\gamma \in \mathcal{I}$, *and such that there exists* $p \in \mathbb{N}$ *for which*
  - $\mathcal{T}^0(w), \ldots, \mathcal{T}^p(w)$ *are non-empty semantic trees,*
  - $\mathcal{T}^j(w) = \emptyset$ *for all* $j > p$.
- $\Phi : W \to \mathcal{P}(\mathcal{I})$ *is a function representing the oracle which associates to each world* $w \in W$ *a set of formulas* $\Phi(w) \subseteq \mathcal{I}$.

The restrictions that we put on the class of tableau epistemic inquiry models are the same as before, yielding our intended class of models $\mathbf{TE_I}$:

Definition 2.33 (Class of models $\mathbf{TE_I}$). *Let* $M = \langle W, \sim, V, \mathsf{E}, \mathsf{T}, \Phi\rangle$ *be a tableau epistemic inquiry model.*

$$M \in \mathbf{TE_I} \quad \textit{if and only if}$$

1. *for all* $w \in W$, *if* $\gamma \in \mathsf{E}(w)$, *then* $M, s \models \gamma$,
2. *for all* $w \in W$, *if* $\gamma \in \mathsf{E}(w)$ *and* $w \sim u$ *with* $u \in W$, *then* $\gamma \in \mathsf{E}(u)$,
3. *for all* $w \in W$, *if* $\gamma \in \mathcal{B}_i^j(w)$ *and* $w \sim u$ *with* $u \in W$, *then* $\gamma \in \mathcal{B}_i^j(u)$,
4. *for all* $w \in W$, *if* $\gamma \in \Phi(w)$, *then* $M, w \models \gamma$,
5. *for all* $w \in W$, *if* $\gamma \in \Phi(w)$, *then* $\gamma \in \Phi(u)$ *for all* $u \in W$ *such that* $u \sim w$ *and* $M, u \models \gamma$.

The three model operations of tableau management, i.e., the tableau construction, creation and elimination operations, are defined in the same way as in the previous section. However, the question operation needs to be adapted to the implicit/explicit knowledge setting. Our proposal to do so is the following: when the answer to a question is available from the oracle, the model undergoes an hard information update with the answer, and the answer is added to each set of local explicit knowledge in the agent's epistemic range. Formally, this leads to the following definition:

Definition 2.34 (Question operation)**.** *Let (M, s) be a pointed tableau epistemic inquiry model where $M = \langle W, \sim, V, \mathsf{E}, \mathsf{T}, \Phi\rangle$, let $Q = (\gamma_1, \dots, \gamma_k)$ be a propositional question and let $A = \{\gamma_1, \dots, \gamma_k\} \cap \Phi(w)$. The model $M_{(\gamma_1,\dots,\gamma_k)?}(s)$ is obtained as follows*

1. *if $A = \emptyset$, then $M_{(\gamma_1,\dots,\gamma_k)?}(s) := M$,*
2. *if $A \neq \emptyset$, then $M_{(\gamma_1,\dots,\gamma_k)?}(s) := \langle W', \sim', V', \mathsf{E}', \mathsf{T}', \Phi'\rangle$ where*
   - $W' := \{w' \in W \mid M, w' \models \bigwedge A\}$,
   - $\sim' := \sim \cap\, (W' \times W')$,
   - $V' := V \restriction W'$,
   - $\mathsf{E}(w) := \mathsf{E}(w) \cup A$ *for all* $w \sim s$ *and* $\mathsf{E}(w)' := \mathsf{E}(w)$ *for all* $w \nsim s$,
   - $\mathsf{T}' := \mathsf{T} \restriction W'$,
   - $\Phi' := \Phi \restriction W'$.

The semantics for the language $\mathcal{TE_I}$ is directly obtained from the semantics for the languages $\mathcal{TE}$ and $\mathcal{E_I}$ except for the question operator since we need to adapt it to the implicit/explicit knowledge setting. To this end, it seems natural to say that, in order to ask a question, the agent needs to have *global explicit knowledge* of the presupposition to the question. Formally, this amounts to change the operator K into KE in the precondition to the question operation:

Definition 2.35 (Semantics for the language $\mathcal{TE_I}$)**.** *Let (M, s) be a pointed tableau epistemic inquiry model where $M = \langle W, \sim, V, \mathsf{E}, \mathsf{T}, \Phi\rangle$. The semantics for the language $\mathcal{TE_I}$ is given by the semantics for $\mathcal{TE}$ and the semantics for $\mathcal{E_I}$ in which the semantic definition of the question operator is replaced by the following*

$$M, s \models [(\gamma_1, \dots, \gamma_k)?]\varphi \quad \textit{iff} \quad M, s \models KE\mathsf{presup}(\gamma_1, \dots, \gamma_k) \ \textit{ implies } \ M_{(\gamma_1,\dots,\gamma_k)?}(s), s \models \varphi,$$

where

$$\mathsf{presup}(\gamma_1, \dots, \gamma_k) := (\gamma_1 \vee \dots \vee \gamma_k) \wedge \bigwedge_{j_1 \neq j_2 \textit{ and } j_1, j_2 \in [\![1,k]\!]} \neg(\gamma_{j_1} \wedge \gamma_{j_2}).$$

Finally, by extending the language $\mathcal{TE_I}$ in a suitable way, i.e., by adding the additional operators introduced in the language $\mathcal{TE}^*$, we can provide a sound and complete logic for our dynamic logic of questions and inferences:

Theorem 2.6 (Soundness and completeness)**.** *The dynamic logic of questions and inferences is completely axiomatizable for a suitable extension of the language $\mathcal{TE_I}$.*

Proof. See appendix A.3 for the details. □

The way we deal with incoming of information through inferences and questions is very similar to the way Fernando Velázquez-Quesada deals with the relation between inference and update in [41]. However, one may argue that this way to proceed is not accurate when dealing with non-logically omniscient agent. In the next section, we explain the problem and we propose a solution to avoid it.

2.3.2 Questioning and logical omniscience

The previous treatment of the relation between incoming of information through inferences and questions has something unsatisfactory while dealing with non-logically omniscient agents. The problem comes from the way incoming of information through questioning modifies the epistemic situation of the agent. Given our definition of the question operation, obtaining the answer to a question has for effect to eliminate all the states in the agent's epistemic range which were not compatible with the answer. However, this implicitly assumes that the agent has the capacity to *recognize* that some of the states in her epistemic range are *incompatible* with the obtained answer. In other words, this means that, if γ is the obtained answer, the agent knows somehow that γ is not true in some of the states that she considers epistemically possible. But if we consider a non-logically omniscient agent, it might well be that the agent does not know explicitly that γ is false in some of these states, i.e., $\neg\gamma$ might not be local explicit knowledge for the agent. Thus, it seems that, in order to eliminate a certain world after getting the answer γ, a non-logically omniscient agent should *explicitly* know that γ is not true in this world. Indeed, this way to deal with the relation between incoming of information through inferences and questions is closer to scientific practice: in order to reject a theory after getting new data, one needs to infer from this theory a statement which 'clashes' with the data.

In this section, we will propose a way to deal with the relation between inferences and questions which avoids the problem that we have just mentioned. First, we will explain informally how we can deal with incoming of information through questioning in the case of non-logical omniscience. Then, we will explain how to modify our previous dynamic logic of inferences and questions accordingly. Finally, we will end with some remarks on the possibility to always transform implicit knowledge into explicit one by making inferences.

Dealing with questioning in the case of non-logical omniscience

Our main critic of the way we treated the relation between questions and inferences can be summarized as follows: in order to eliminate a state in her epistemic range after an incoming of information, a non-logically omniscient agent *must* have explicit knowledge that this state is *incompatible* with the newly acquired information. In order to integrate this 'non-omniscient' feature of the information updating process through questioning, we need to make two main modifications to our previous framework.

The first modification consists in separating the *'questioning'* operation with the *'world elimination'* operation. To this end, we first need to represent, in two separated ways, information obtained through inferences and information obtained through questioning.

One possible way to do so is to introduce, in addition to the set of local explicit knowledge $\mathsf{E}(w)$ associated to each world w in the model, a *'data' set* of formulas $\mathsf{D}(w)$ recording the information obtained about the actual world through questioning. This distinction is indeed at the heart of scientific practice, where data obtained through observations and experiments are sharply distinguished with the predictions, or any information, logically deduced from the theory(ies) considered by the scientist.

Then, the respective roles of the 'questioning' and the 'world elimination' operations are the following:

- the role of the *'questioning' operation* is to represent the process of obtaining information through questioning, which consists in adding the obtained information to the data set,
- the role of the *'world elimination' operation* is to represent the process of eliminating worlds from the epistemic range of the agent, when it turns out that the information the

agent has about the actual world, recorded in the data set, is contradictory with her local explicit knowledge about these worlds.

Thus, formally, we will need to split our question operation into a *question operation* and a *world elimination operation*, and our question operator into a *question operator* and a *world elimination operator*.

The second modification concerns the assumptions we made in our tableau-based dynamic logic of inferences: we need to give up the two coherence properties for the sets of local explicit knowledge and the sets of semantic trees associated to each world in the model. This means that we allow the agent to have different sets of local explicit knowledge for the different worlds present in her epistemic range.[16] Besides, we allow the agent to entertain different semantic trees, i.e., different inferential processes, in the different worlds present in her epistemic range, in order to obtain (potentially different) local explicit knowledge about these worlds.[17]

We will now *sketch* how we can modify our previous formal framework along these lines. We first focus on the second modification regarding the tableau-based dynamic logic of inferences. Then, we will propose a way to deal with the first modification by splitting the previous question operation into a questioning operation and a world elimination operation.

Modifying the tableau-based dynamic logic of inferences

First of all, we need to modify the definition of tableau epistemic models by adding a representation of the *data sets*. Thus, the new models with data sets are of the form

$$M = \langle W, \sim, V, \mathsf{E}, \mathsf{D}, \mathsf{T} \rangle \text{ where } \mathsf{D} : W \to \mathcal{P}(\mathcal{I}).$$

We will make some assumptions on these models: we will give up the coherence property for the sets of local explicit knowledge and the sets of semantic trees, and we will assume veridicality for the data sets. Thus, our assumptions on these models are the following: let $M = \langle W, \sim, V, \mathsf{E}, \mathsf{D}, \mathsf{T} \rangle$,

Veridicality for local explicit knowledge: we will require that local explicit knowledge is always truthful: for all $w \in W$,

$$\text{if } \gamma \in \mathsf{E}(w), \text{ then } M, w \models \gamma,$$

Veridicality for data sets: we will require that the formulas in the data sets are always true: for all $w \in W$,

$$\text{if } \gamma \in \mathsf{D}(w), \text{ then } M, w \models \gamma.$$

By giving up the coherence property for the sets of local explicit knowledge and the sets of semantic trees, we open the possibility for the agent to make different inferential processes in the different worlds present in her epistemic range. It then seems natural to transform our dynamic operators of tableau management in such a way that we can express, at a given world, modifications of the semantic trees entertained at another world. One possible way to do so is to use the tools from *hybrid logic*[18], and to transform our dynamic operators of tableau construction,

[16] It is crucial that we give up the coherence property since it constraints, when we assume that epistemic indistinguishability relations are equivalence relations, to have the same set of local explicit knowledge in all the worlds present in the agent's epistemic range.

[17] Here again it is crucial, for the same reason as for local explicit knowledge, to give up the coherence property for the sets of semantic trees entertained by the agent.

[18] See [5] and [6] for a presentation of hybrid logic.

creation and elimination, into *hybrid* dynamic operators.

To this end, we first need to transform our static tableau epistemic language $\mathcal{TE}_0$ into an *hybrid* language by introducing *nominals*, denoted by l, along with *satisfaction operators*, denoted by $@_l$. To this end, a set of nominals N is added to the set of atoms P, the new set of atoms becoming $\mathsf{P} \cup \mathsf{N}$. Then, we also need to introduce a modality D expressing the presence of a formula in the data set at a given world. Formally, our static language is obtained by adding to the recursive rules of $\mathcal{TE}_0$ the following ones:

$$l \mid D\gamma \mid @_l\varphi \text{ with } l \in \mathsf{N} \text{ and } \gamma \in \mathcal{I}.$$

In this language, nominals are true in exactly one state, formulas of the form $D\gamma$ are read as *"γ is a datum for the agent"* and formulas of the form $@_l\varphi$ are read as *"φ is true in the world denoted by the nominal l"*. The semantics for these operators is the following: let (M, s) be a pointed model with $M = \langle W, \sim, V, \mathsf{E}, \mathsf{D}, \mathsf{T}\rangle$, then

$$\begin{array}{rcl} M, s \models l & \text{iff} & s \in V(l) \\ M, s \models D\gamma & \text{iff} & \gamma \in \mathsf{D}(s) \\ M, s \models @_l\varphi & \text{iff} & M, d \models \varphi \text{ where } d \text{ is the denotation of } l \text{ under } V. \end{array}$$

On the dynamic side, we want to define *hybrid* dynamic operators for tableau construction, creation and elimination, which are able to express *change* in semantic trees entertained by the agent at a world different from the one where the formula is evaluated. To this end, we need to modify the corresponding model operations in such a way that they operate transformations with respect to the set of semantic trees at only one particular world.[19] Then, we add to our previous static language *hybrid* dynamic operators of tableau management which have as subscript the nominal for the world on which they operate. Thus, these operators are of the form

$$[\mathcal{T}^j, i, \gamma]_l\varphi \mid [\mathsf{T} + \{\gamma\}]_l\varphi \mid [\mathsf{T} - \mathcal{T}^j]_l\varphi \text{ with } l \in \mathsf{N}.$$

Formulas of the form $[\mathcal{T}^j, i, \gamma]_l\varphi$ are read as *"φ is the case after the tableau construction operation on the formula γ in the i^{th} branch of $\mathcal{T}^j$ at the world denoted by l"*, formulas of the form $[\mathsf{T} + \{\gamma\}]_j\varphi$ are read as *"φ is the case after the creation of a new semantic tree with root γ at the world denoted by l"* and formulas of the form $[\mathsf{T} - \mathcal{T}^j]_l\varphi$ are read as *"φ is the case after the agent has concluded from the closeness of $\mathcal{T}^j$ that the root of $\mathcal{T}^j$ is false at the world denoted by l"*.

Then, the semantics for these operators are similar to our previous definitions, except that the preconditions are relativized to the worlds in which the transformations operate: let (M, s) be a pointed model with $M = \langle W, \sim, V, \mathsf{E}, \mathsf{D}, \mathsf{T}\rangle$, the semantics for the hybrid dynamic operators of tableau construction, creation and elimination are given by

$$\begin{array}{rcl} M, s \models [\mathcal{T}^j, i, \gamma]_l\varphi & \text{iff} & M, s \models @_l(Br_i^j\gamma) \text{ implies } M_{(\mathcal{T}^j, i, \gamma)}(d), s \models \varphi \\ M, s \models [\mathsf{T} + \{\gamma\}]_l\varphi & \text{iff} & M_{\mathsf{T}+\{\gamma\}}(d), s \models \varphi \\ M, s \models [\mathsf{T} - \mathcal{T}^j]_l\varphi & \text{iff} & M, s \models @_l(\mathsf{closed}(\mathcal{T}^j)) \text{ implies } M_{\mathsf{T}-\mathcal{T}^j}(d), s \models \varphi, \end{array}$$

where d is the world denoted by l under V.

We can now add to this system a treatment of question which separates the questioning and the world elimination operations.

[19]The definition of such operations can be straightforwardly obtained from the previous ones.

Adding the 'Questioning' and 'world elimination' operations

As we said in the introduction, we need to split our previous question operation into two distinct operations: a *'questioning' operation* and a *'world elimination' operation.* In this section, we will first present the 'questioning' operation along with the semantic definition for the new question operator, then we will present the 'world elimination' operation along with a semantic definition for the world elimination operator. The class of models that we work with is the one that we have just defined, to which we add for each model an oracle function.

As mentioned in the introduction, the role of the *questioning operation* will now consist in adding the answers obtained by asking questions to the data set. This operation takes as input a pointed model and a propositional question Q, and is formally defined as follows:

Question operation: let (M, s) be a pointed model where $M = \langle W, \sim, V, \mathsf{E}, \mathsf{D}, \mathsf{T}, \Phi \rangle$ and let $Q = (\gamma_1, \ldots, \gamma_k)$ be a propositional question. The model $M_Q(s) = \langle W', \sim', V', \mathsf{E}', \mathsf{D}', \mathsf{T}', \Phi' \rangle$ is given by

- $W' := W$, $\sim' := \sim$, $V' := V$, $\mathsf{E}' := \mathsf{E}$,
- if $\gamma_i \in \Phi(s)$ for some $i \in [\![1, k]\!]$, then $\mathsf{D}'(s) := \mathsf{D}(s) \cup \{\gamma_i\}$ and $\mathsf{D}'(w) := \mathsf{D}(w)$ for all $w \neq s$, otherwise $\mathsf{D}' := \mathsf{D}$,
- $\mathsf{T}' := \mathsf{T}$, $\Phi' := \Phi$.

Thus, asking a question will have for effect, when the answer is available from the oracle, to add the answer to the data set. Then, the precondition to a question operation is defined as before and the semantics for the question operator too. The only difference lies in the question operation: let $M = \langle W, \sim, V, \mathsf{E}, \mathsf{D}, \mathsf{T}, \Phi \rangle$, the semantic definition for the *question operator* is given by

$$M, s \models [Q?]\varphi \text{ iff } M, s \models \mathsf{pre}(Q) \text{ implies } M_Q(s), s \models \varphi.$$

We now define the *world elimination operation.* This operation takes as input a model M and a world u in M and simply eliminates u from the model M:

World elimination operation: let $M = \langle W, \sim, V, \mathsf{E}, \mathsf{D}, \mathsf{T}, \Phi \rangle$ and let $u \in W$. The model $M_{-u} = \langle W', \sim', V', \mathsf{E}', \mathsf{D}', \mathsf{T}', \Phi' \rangle$ is given by

- $W' := W \backslash \{u\}$,
- $\sim' := \sim \cap (W' \times W')$, $V' := V \upharpoonright W'$, $\mathsf{E}' := \mathsf{E} \upharpoonright W'$, $\mathsf{D}' := \mathsf{D} \upharpoonright W'$, $\mathsf{T}' := \mathsf{T} \upharpoonright W'$, $\Phi' := \Phi \upharpoonright W'$.

We now have to provide a semantic definition for the *elimination world operator.* The main idea is the following: the agent can eliminate a world from her epistemic range when she has explicit knowledge that this world is incompatible with the data obtained about the actual world. More precisely, we will consider that an agent can eliminate a world u from her epistemic range if there exists a formula γ such that *either* γ is in the data set and the agent has local explicit knowledge that $\neg\gamma$ is true at u, *or* $\neg\gamma$ is in the data set and the agent has local explicit knowledge that γ is true at u. Thus, our elimination world operator will have the form $[\gamma, l]\varphi$ with $\gamma \in \mathcal{I}$ and $l \in \mathsf{N}$, and will be read as *"φ is the case after the elimination of the world denoted by l from the agent's epistemic range, due to an incompatibility with respect to the formula γ".* Then, if (M, s) is a pointed model with $M = \langle W, \sim, V, \mathsf{E}, \mathsf{D}, \mathsf{T}, \Phi \rangle$, the precondition to an elimination world operation of l with respect to γ will be the following:

$$M, s \models @_l(\neg K \neg s) \wedge ((D\gamma \wedge @_l(E\neg\gamma)) \vee (D\neg\gamma \wedge @_l(E\gamma))),$$

where:

- $M, s \models @_l(\neg K \neg s)$ says that the world denoted by l is in the agent's epistemic range,
- $M, s \models ((D\gamma \wedge @_l(E\neg\gamma)) \vee (D\neg\gamma \wedge @_l(E\gamma)))$ says that *either* γ is in the data set at s and $\neg\gamma$ is local explicit knowledge about the world denoted by l, *or* $\neg\gamma$ is in the data set at s and γ is local explicit knowledge about the world denoted by l.

This leads to the following semantic definition for the *elimination world operator*: let (M, s) be a pointed model with $M = \langle W, \sim, V, \mathsf{E}, \mathsf{D}, \mathsf{T}, \Phi\rangle$, the semantics for the *elimination world operator* is given by

$$M, s \models [\gamma, l]\varphi \text{ iff } M, s \models @_l(\neg K\neg s) \wedge ((D\gamma \wedge @_l(E\neg\gamma)) \vee (D\neg\gamma \wedge @_l(E\gamma))) \text{ implies } M_{-d}, s \models \varphi,$$

where d is the world denoted by l under V.

We now make a last remark regarding the possibility to always turn implicit knowledge into explicit one by making inferences.

Making all implicit knowledge accessible through inferences

In a system dealing with implicit and explicit knowledge, and how to transform the latter into the former, it seems intuitive to have the property that all implicit knowledge can be turned explicit by making inferences. In this subsection, we propose a way to make sure that this is the case in the framework that we have just sketched.

By giving up the coherence property for the sets of local explicit knowledge and the sets of semantic trees, we have a way to make sure that the agent can turn any implicit knowledge by making inferences, by simply assuming that the agent has all the information about the atoms. Formally, this consists in adding the following restriction on our class of intended models: let $M = \langle W, \sim, V, \mathsf{E}, \mathsf{D}, \mathsf{T}, \Phi\rangle$, then

Local explicit knowledge of the atoms: we require that each set of local explicit knowledge contains all the information relative to the atoms, i.e., for all $w \in W$,

$$\{p \mid p \in V(w)\} \cup \{\neg p \mid p \notin V(w)\} \subseteq \mathsf{E}(w).$$

We can now show that, under this assumption, the agent can turn any implicit knowledge into explicit one by making inferences:

Theorem 2.7. *Let (M, s) be a pointed model with $M = \langle W, \sim, V, \mathsf{E}, \mathsf{D}, \mathsf{T}, \Phi\rangle$ and let $\gamma \in \mathcal{I}$. Then, we have*

$$M, s \models K\gamma \;\Rightarrow\; M, s \models [O_1] \ldots [O_n]KE\gamma \text{ for some } n \in \mathbb{N},$$

where for all $i \in [\![1, n]\!]$, O_i is either an instance of a tableau construction operator, a tableau creation operator or a tableau elimination operator.

Proof. Let (M, s) be a pointed model and let $\gamma \in \mathcal{I}$. Assume that $M, s \models K\gamma$. This means that $M, w \models \gamma$ for all $w \sim s$. In order to have global explicit knowledge that γ, the agent can create a semantic tree with root $\neg\gamma$ and develop it completely up to the atoms. Since the agent has local explicit knowledge about the atoms and γ is true in all the worlds $w \sim s$, the agent can close the completely developed tree with root $\neg\gamma$ in all the worlds $w \sim s$. Then, it suffices to apply the tableau elimination operation in all the worlds $w \sim s$ to finally have γ as global explicit knowledge. □

Besides, the following theorem says that, after a world elimination operation, the agent can still transform any implicit knowledge into explicit one:

Theorem 2.8. *Let (M, s) be a pointed model with $M = \langle W, \sim, V, \mathsf{E}, \mathsf{D}, \mathsf{T}, \Phi \rangle$, let $\gamma \in \mathcal{I}$ and let $d \in W$ such that $d \neq s$. Then, we have*

$$M_{-d}, s \models K\gamma \;\Rightarrow M_{-d}, s \models [O_1] \ldots [O_n] KE\gamma \text{ for some } n \in \mathbb{N},$$

where for all $i \in [\![1, n]\!]$, O_i is either an instance of a tableau construction operator, a tableau creation operator or a tableau elimination operator.

Proof. This theorem follows directly from the previous one. □

In this section, we have first presented a dynamic logic of questions and inferences based on the systems developed in the two previous sections. Then, we have argued that a straightforward merge of our two previous systems does not necessarily capture the intricate relation between questions and inferences in the inquiry process. Thus, we have proposed an alternative way to deal with questioning in the case of non-logical omniscience, using tools from hybrid logic. However, we have only sketched the bases of such a system. The next step would be to provide a detailed presentation of this system along with a sound and complete axiomatic system for the resulting dynamic logic of questions and inferences. We leave this issue to further developments.

Conclusion

In this chapter, we have proposed a formalization of the IMI under the form of a *dynamic logic of questions and inferences*. However, we have made several assumptions and restrictions, leaving important aspects of inquiry outside of the scope of our investigation. In this conclusion, we will suggest several possible ways to extend the work that has been presented in this chapter in order to capture and investigate other important aspects of inquiry.

One of the main limitations of our approach concerns the assumptions that we made on the oracle, in particular regarding our choice of inquiry language. Thus, one straightforward way to extend our system is to enrich our inquiry language, which was only the propositional language, to an *epistemic language*, expressing higher-order information, and/or to a *first-order language*, opening thereby a bunch of new issues:

Inquiry and higher-order information. Inquiry about what other people know or believe is a common activity of real life. Thus, it makes perfect sense to develop a system in which questions can be asked about other agents' epistemic state. To this end, one should extend the inquiry language into an *epistemic language*, allowing to ask questions about what the other agents know and believe. An important issue which will occur here concerns the notion of presupposition: we might face situation in which the presupposition of a question is established by the agent before asking a question but not anymore after having obtained the answer. Thus, an important issue here is to determine what is a *meaningful* question about other agents' informational state.

Inquiry in the first-order case. Changing our inquiry language into a first-order one would be a very interesting step. We can hope that our system can be extended without too much difficulties to the first-order case. Important issues will appear here, in particular due to the interaction between epistemic/dynamic operators and first-order formulas. Besides, this will allow discussion on topics addressed by Hintikka within the framework of interrogative logic such as the issue of *identifiability*[20].

In this chapter, we have always worked with only *one* inquiring agent. Thus, another straightforward extension would be to move to the *multi-agent case*:

[20] See [22, p. 64] for a discussion of the notion of *identifiability* within the framework of interrogative logic.

The social dimension of inquiry. Scientific inquiry, and inquiry in general, is often a *social activity*. In order to account for some of these aspects, one possible way is already to introduce suitable *group actions*. Then, one should develop a proper dynamic logic of questions and inferences for the multi-agent case. Taking in account the social dimension of inquiry would also require to introduce operations of *interaction* and *communication* between agents. The investigation of inquiry and its social components is thereby directly connected to important themes in dynamic logics such as *logics for interaction* and *logics of communication*.

Inquiry and other epistemic attitudes. In this chapter, we have thought of inquiry as a *knowledge*-seeking process. However, knowledge is a strong epistemic attitude, often hard to reach. Thus, it seems that we should also investigate inquiry for other epistemic attitudes. An interesting case is the probabilistic one in which the agent attributes *probabilities* or *degrees of belief* to formulas. Then, it might be interesting to develop a dynamic logic of questions and inferences in the probabilistic case, and to investigate situations in which the agent is not seeking for knowledge but only for a certain degree of certainty.

Questions and inferences. Questions and inferences are two different but intertwined ways to progress in an inquiry. In the section 3.2, we have sketched a system which aims to account for some aspects of the relation between questions and inferences. Thus, it might be interesting to go further and to try to develop logical theories which do justice to the intricate relation between questions and inferences in information-seeking processes.

Chapter 3

Entailment, informativity and relevance

Introduction

One of the most interesting issues to investigate within the interrogative model of inquiry is the so-called *strategic aspects* of inquiry. Hintikka introduces this notion by making a distinction between *definitory* and *strategic rules*, and by using the following analogy with *games*:

> In games, there are rules and there are rules. There are such rules as serve to define the game, e.g., the rules of chess. I shall call them 'definitory rules'. They tell which moves are possible, or, as it is sometimes put, which moves are admissible. The crucial fact about definitory rules is that they say absolutely nothing about which moves are good, which ones are bad, and which ones are better than others. Such questions are handled by rules of another kind. I shall call them 'strategic rules'. [22, p. 27]

In this chapter and the following, we will focus on *pure information-seeking inquiries*, i.e., inquiries only constituted of sequences of questions.[1] In this case, investigating the strategic aspects of inquiry amounts to determine the *'best'* questions to ask in a particular inquiry, given a certain epistemic situation.

One possible way to tackle this issue is to attribute *values* to the different questions that the agent can meaningfully ask. Introducing a *measure of value* for questions will enable the agent to compare different questions and to determine which one(s) has (have) the greatest value, providing thereby a way to determine the 'best' questions to ask.

It turns out that such an enterprise has already been carried out in the study of questions and answers from the point of view of the semantics and pragmatics of natural language. In particular, Groenendijk and Stokhof have proposed in [12] an analysis of questions in terms of *partitions* which allows to define a notion of *entailment* for questions. This work has then been extended by Robert van Rooij in [39] and [40] who proposes an analysis of the notions of *informativity* and *relevance* for questions and answers using tools from information and decision theory. Thus, even if these works have been carried out in order to better understand the semantics and pragmatics of questions and answers in natural language, they appear to be directly relevant, from an epistemological perspective, to the investigation of pure information-seeking inquiries.

Based on these works, we will introduce, in this chapter, three notions that yield particular value measures for both *propositions* and *questions*: the notions of *entailment*, *informativity*

[1]Thus, the background framework of this chapter is the one of the *dynamic logic of questions* presented in section 2.1.2. This means that we are working with logical omniscient agent and we only consider questions as a mean to obtain information.

and *relevance*. These three notions form a hierarchy in terms of dependence with respect to the current epistemic situation of the agent and her on-going inquiry:

- the notion of *entailment* is independent of both the current epistemic situation of the agent and the inquiry,
- the notion of *informativity* is dependent on the current epistemic situation of the agent but independent of the inquiry,
- the notion of *relevance*, or *informativity with respect to the inquiry*, is dependent on both the current epistemic situation of the agent and the inquiry.

The last notion, the notion of relevance, depends crucially on the *goal* of the inquiry. In Hintikka's writings on the interrogative model, two different kinds of inquiry goals are mentioned, leading to two different *perspectives* on inquiry:

the establishing perspective: according to this perspective, the goal of an inquiry is, starting from an initial epistemic situation (M, s), to establish a certain conclusion γ,

the determining perspective: according to this perspective, the goal of an inquiry is, starting from an initial epistemic situation (M, s), to determine whether or not γ is the case.

Thus, we will need to investigate the notion of relevance for both propositions and questions from these two different perspectives on inquiry.

In this chapter, we will successively study the notions of *entailment*, *informativity* and *relevance*. In the first section, we will present the partition-based analysis of questions and we will show that this approach can be adopted in our framework, allowing us to define a notion of *entailment* for questions *à la* Groenendijk and Stokhof. In the second section, we will define the notion of *informativity*, for both propositions and questions, adopting first a semantic approach before turning to a quantitative approach using Bar-Hillel and Carnap's theory of information. In the third section, we will investigate and propose a definition of the notion of *relevance* for propositions and questions in the *establishing perspective*. In the forth section, we will use Robert van Rooij's work in order to define a notion of *relevance* in the *determining perspective*, again for both propositions and questions. We end with some remarks on the importance of investigating the possibility to define measures of value for propositions and questions in order to progress in our understanding of the strategic aspects of interrogative inquiry.

3.1 The notion of *entailment*

For propositions, the notion of entailment is usually defined in the following way:

Definition 3.1 (Entailment for propositions)**.** *Let $\varphi, \psi \in \mathcal{I}$. The proposition φ entails the proposition ψ if and only if $\models \varphi \rightarrow \psi$.*

The intuitive idea behind the notion of entailment for propositions can be stated as follows: φ entails ψ if and only if whenever φ is the case ψ is also the case. One natural question that comes to mind then is the following: can we also define a notion of entailment for questions? Groenendijk and Stokhof answer yes to this question and propose a way to do so using a *partition-based analysis* of questions.

In this section, we will present the partition-based analysis of questions and we will see that this approach can be adopted in our framework. This will then allow us to define a notion of entailment for questions *à la* Groenendijk and Stokhof.

3.1.1 The partition-based analysis of questions

Groenendijk and Stokhof have developed in [12] an analysis of questions in terms of *partitions* of a given set of possible worlds[2] (called *indices* in the following quotation):

> [W]e will view questions as partitions of the set of indices [...]. If we view a question as a partition of the set of indices I, each element of that partition, a set of indices, represents a proposition, a possible semantic answer to that question. Consider the question *whether* ϕ. This question has two possible semantic answers: *that* ϕ, and *that not* ϕ. The two sets of indices corresponding to these two propositions divide the total set of indices in two non-overlapping parts. So, a single whether-question makes a bipartition on the set of indices [...].
>
> Constituent questions can be viewed as partitions as well. The possible semantic answers to the question *who G's*, are propositions that express that the objects $a_1, \ldots, a_n$ are the ones that G. Such propositions exhaustively and rigidly specify which objects have the property G at an index. The sets of indices that represent the possible semantic answers form a partition of I. [12, p. 214]

From an epistemic logic perspective, each cell of the partition induced by a question on the epistemic range of the agent represents an area where the agent would like to be. In other words, what the agent wants by asking a question is to know the answer, i.e., to reduce her epistemic range to a subset of one of the partition cells induced by the question.

It turns out that the partition-based analysis of questions can be adopted in our framework. To see this, we shall first recall that, in order for the inquiring agent to be able to ask a given question, she must have established the presupposition to this question. More precisely, if (M, s)[3] represents the agent's current epistemic situation and $Q = (\gamma_1, \ldots, \gamma_k)$ is a propositional question, then the following condition has to be satisfied in order for the agent to be able to (meaningfully) ask the question Q:

$$M, s \models K\left((\gamma_1 \vee \ldots \vee \gamma_k) \wedge \bigwedge_{j_1 \neq j_2 \text{ and } j_1, j_2 \in [\![1,k]\!]} \neg(\gamma_{j_1} \wedge \gamma_{j_2})\right).$$

In words, this means that the agent knows that *one* and *only one* of the possible answers to the question Q is true. If this is the case, it follows that Q induces a partition Q on the $\sim$-equivalence class of the actual world s:

$$\mathsf{Q} := \{[\![\gamma_i]\!]_s \mid i \in [\![1,k]\!]\} \text{ where } [\![\gamma_i]\!]_s := \{w \in W \mid w \sim s \text{ and } M, w \models \gamma_i\}.$$

To fix notations, we will denote by:

- Q the partition associated to Q when $M, s \models \mathsf{pre}(Q)$,
- $[\![\varphi]\!]_s$ the set $\{w \in W \mid w \sim s \text{ and } M, w \models \varphi\}$,
- $[s]$ the set $\{w \in W \mid w \sim s\}$.

Thus, the partition-based analysis of questions can be adopted in our framework. We are now in a position to define a notion of entailment for questions *à la* Groenendijk and Stokhof.

[2] A partition of a set of worlds W is a set of mutually exclusive subsets of W such that their union equals W.

[3] In all this chapter, we will always deal with *epistemic inquiry models* and *probabilistic epistemic inquiry models*. In order to simplify the presentation, we will simply talk of *models* and *probabilistic models*. We refer to the second chapter for the definition of epistemic inquiry models, and to the first chapter for the definition of probabilistic epistemic models. A *probabilistic epistemic inquiry model* is simply a probabilistic epistemic model with an *oracle function*.

3.1.2 The notion of entailment for questions

According to Groenendijk and Stokhof, a question Q_1 entails a question Q_2 if and only if every (complete) answer to Q_1 also provides a (complete) answer to Q_2. From an epistemic logic perspective, we can rephrase this by saying that a question Q_1 entails a question Q_2 if and only if whenever the agent knows the answer to Q_1 she also knows the answer to Q_2. Using the partition-based analysis of questions, this notion of entailment can formally be defined as follows:

Definition 3.2 (Entailment for questions). *Let $Q_1 := (\gamma_1, \ldots, \gamma_k)$ and $Q_2 := (\chi_1, \ldots, \chi_l)$ be propositional questions. We say that Q_1 entails Q_2 if and only if for all pointed models (M, s)*

1. *if $M, s \models \mathsf{pre}(Q_1)$, then $M, s \models \mathsf{pre}(Q_2)$,*
2. *if $M, s \models \mathsf{pre}(Q_1)$, then $\mathsf{Q}_1 \sqsubseteq \mathsf{Q}_2$,*

where

$$\mathsf{Q}_1 \sqsubseteq \mathsf{Q}_2 \;\; \textit{iff} \;\; \forall A \in \mathsf{Q}_1, \exists B \in \mathsf{Q}_2 \textit{ s.t. } A \subseteq B \;\; \textit{iff} \;\; \forall \gamma_i \in Q_1, \exists \chi_j \in Q_2 \textit{ s.t. } [\![\gamma_i]\!]_s \subseteq [\![\chi_j]\!]_s.$$

From a dynamic perspective, we would also expect some intuitive behaviors of the notion of entailment. One of the most natural dynamic properties to expect is the following: if Q_1 entails Q_2, then by obtaining the answer to Q_1 the inquiring agent will be brought into an epistemic situation in which she knows the answer to Q_2, whatever her initial epistemic situation is. It turns out that this property is true of the above definition of entailment:

Theorem 3.1. *Let $Q_1 = (\gamma_1, \ldots, \gamma_k)$ and $Q_2 = (\chi_1, \ldots, \chi_l)$ be questions. If Q_1 entails Q_2, then for all pointed models (M, s) where $M = \langle W, \sim, V, \Phi \rangle$ and such that $M, s \models \mathsf{pre}(Q_1)$ and $\gamma_i \in \Phi(s)$ for some $i \in [\![1, k]\!]$, we have*

$$M, s \models [Q_1?]K\chi_j \;\; \textit{where } \chi_j \textit{ is the answer to } Q_2.$$

Proof. Let $Q_1 = (\gamma_1, \ldots, \gamma_k)$ and $Q_2 = (\chi_1, \ldots, \chi_l)$ be questions such that Q_1 entails Q_2. Let (M, s) be a pointed model, where $M = \langle W, \sim, V, \Phi \rangle$, such that $M, s \models \mathsf{pre}(Q_1)$ and $\gamma_i \in \Phi(s)$ for some $i \in [\![1, k]\!]$. Then, we have by definition that

$$M, s \models [Q_1?]K\chi_j \;\Leftrightarrow\; M|\gamma_i, s \models K\chi_j.$$

Since Q_1 entails Q_2 and γ_i and χ_j are respectively the answers to Q_1 and Q_2, we necessarily have for (M, s) that $[\![\gamma_i]\!]_s \subseteq [\![\chi_j]\!]_s$. It follows that in $(M|\gamma_i, s)$ we have $[\![\chi_j]\!]_s = [s]$, i.e., $M|\gamma_i, s \models K\chi_j$. We conclude from the above equivalence that $M, s \models [Q_1?]K\chi_j$. □

This theorem shows that the notion of entailment, based on the analysis of questions in terms of partitions, behaves dynamically in the intended way. Indeed, some of the motivations behind such a definition appeal implicitly to the dynamic epistemic effect of receiving answers from questions. Thus, theorems of this kind simply exhibit these dynamic aspects through an explicit formulation of the dynamics of the questioning-answering process. We will see, in the next section, that a similar theorem can be proved for the notion of semantic informativity.

In this section, we have seen that the partition-based analysis of questions can be adopted in our framework, and enables us to define a notion of entailment for questions. The notion of entailment is already a way to compare questions: we can say that a question Q_1 is 'better', or has 'more value', than a question Q_2 if and only if Q_1 entails Q_2. However, this notion yields a partial ordering on the set of possible questions that the inquiring agent can ask, leading to situations in which two questions can be incomparable. Besides, as we mentioned it in the introduction, the notion of entailment is independent of both the current epistemic situation of the agent and the on-going inquiry.

In the next section, we will provide a measure value for propositions and questions which yields a complete ordering and takes in account the epistemic situation of the inquiring agent.

3.2 The notion of *informativity*

In this section, we will focus on the notion of *informativity* for both propositions and questions. We will adopt two approaches to the notion of informativity: a *semantic approach* and a *quantitative approach*. These two approaches differ with respect to the underlying notion of *information* considered. In the semantic approach, the notion of information that we adopt is the one lying at the heart of dynamic epistemic logics, identified by van Benthem and Martinez [36] as *information as range*. In the quantitative approach, the notion of information that we consider is the one defined by *Bar-Hillel and Carnap's information theory* [4].

3.2.1 The semantic approach

According to van Benthem and Martinez [36], the intuitive understanding of the notion of *information as range* can be summarized as follows: "The greater one's range of options for what the real world is like, the less information one has." [36, p. 1] Thus, when the agent gets new information, this has for consequence to eliminate options that the agent considered possible, leading thereby to a growth of the agent's own knowledge. Given this notion of information, we can then compare the informativity of two propositions with respect to their *'eliminative power'*. In the next two subsections, we define formally this notion of informativity successively for propositions and questions.

Informativity of a proposition

Let (M, s) be a pointed model where s denotes the actual world. The *epistemic range* of the agent is given by $[s]$, the $\sim$-equivalence class of s. In words, $[s]$ corresponds to all the states that the agent considers possible given her current knowledge. Then, we can define a notion of *informativity* for propositions based on their *eliminative power*: a proposition φ_1 is more informative than a proposition φ_2 with respect to (M, s) if and only if the worlds in $[s]$ that would be eliminated by an announcement of φ_2 would also be eliminated by an announcement of φ_1. This notion can be defined formally as follows:

Definition 3.3 (Semantic informativity for propositions)**.** *Let (M, s) be a pointed model and let $\varphi_1, \varphi_2 \in \mathcal{I}$. We say that φ_1 is semantically more informative than φ_2 w.r.t. (M, s) if and only if*

$$[\![\varphi_1]\!]_s \subseteq [\![\varphi_2]\!]_s.^4$$

Notice that the notion of semantic informativity does not depend on the fact that the announcement of a given proposition is possible or not, namely it does not depend on the truth value of this proposition in the actual world. The idea is rather that the agent can *simulate* the announcement of different propositions in order to evaluate their respective eliminative power on her epistemic range. Notice also that the extreme cases correspond to the situations in which either $[\![\varphi]\!]_s = [\![\bot]\!]_s$ (the agent knows that $\neg\varphi$) and $[\![\varphi]\!]_s = [\![\top]\!]_s$ (the agent knows that φ). In these two cases, we will say that the proposition is *not informative* since in the first case the agent already knows that φ, and in the second case the announcement is impossible. Finally, we shall remark that comparing propositions with respect to their semantic informativity yields

[4] We can also say that φ_1 is *strictly more informative* than φ_2 w.r.t. (M, s) if and only if $[\![\varphi_1]\!]_s \subset [\![\varphi_2]\!]_s$, and that φ_1 is *as informative as* φ_2 w.r.t. (M, s) if and only if $[\![\varphi_1]\!]_s = [\![\varphi_2]\!]_s$.

a partial ordering on $\mathcal{I}$.

Like the notion of entailment, the notion of semantic informativity also displays some intuitive dynamic behaviors: if φ_1 is semantically more informative than φ_2, then the agent knows *more* after an announcement of φ_1 than after an announcement of φ_2. This property can be proved formally in our framework as follows:

Theorem 3.2. *Let (M, s) be a pointed model and let $\varphi_1, \varphi_2 \in \mathcal{I}$. Then,*

$$\varphi_1 \textit{ is semantically more informative than } \varphi_2 \textit{ w.r.t. } (M,s) \Rightarrow \textit{ for all } \varphi \in \mathcal{E}_{\mathcal{I}}, M, s \models [!\varphi_2]K\varphi \to [!\varphi_1]K\varphi.$$

Proof. Let (M, s) be a pointed model, let $\varphi_1, \varphi_2 \in \mathcal{I}$. Assume that φ_1 is semantically more informative than φ_2 w.r.t. (M, s). This means by definition that $[\![\varphi_1]\!]_s \subseteq [\![\varphi_2]\!]_s$. Now let $\varphi \in \mathcal{E}_{\mathcal{I}}$ and assume that $M, s \models [!\varphi_2]K\varphi$. Then, either $M, s \models \neg\varphi_2$ or $M|\varphi_2, w \models K\varphi$.

In the first case, since $[\![\varphi_1]\!]_s \subseteq [\![\varphi_2]\!]_s$ we have that $M, s \models \neg\varphi_1$. Thus, the announcements of φ_1 and φ_2 are unsuccessful and we get $M, s \models [!\varphi_2]K\varphi \to [!\varphi_1]K\varphi$.

In the second case, we have that $M|\varphi_2, w \models K\varphi$ and since $[\![\varphi_1]\!]_s \subseteq [\![\varphi_2]\!]_s$, we have that $M|\varphi_1$ is a submodel of $M|\varphi_2$ and thereby $M|\varphi_1, w \models K\varphi$. We conclude that $M, s \models [!\varphi_1]K\varphi$ and thereby that $M, s \models [!\varphi_2]K\varphi \to [!\varphi_1]K\varphi$.

This shows that if φ_1 is semantically more informative than φ_2 w.r.t. (M, s), then for all $\varphi \in \mathcal{E}_{\mathcal{I}}$, $M, s \models [!\varphi_2]K\varphi \to [!\varphi_1]K\varphi$. □

Then, one natural question to ask is the following:

What is the relation between the notion of semantic informativity and the notion of entailment for propositions?

First of all, we can show that the notion of entailment is 'stronger' than the notion of semantic informativity:

Theorem 3.3. *Let (M, s) be a pointed model and let $\varphi_1, \varphi_2 \in \mathcal{I}$. We have that:*

$$\varphi_1 \textit{ entails } \varphi_2 \;\Rightarrow\; \varphi_1 \textit{ is semantically more informative than } \varphi_2 \textit{ w.r.t. } (M, s).$$

Proof. The proof is direct: if φ_1 entails φ_2, we have for all pointed models (M', s') that $[\![\varphi_1]\!]_{s'} \subseteq [\![\varphi_2]\!]_{s'}$, and so in particular for (M, s) we have that $[\![\varphi_1]\!]_s \subseteq [\![\varphi_2]\!]_s$. □

Besides, it turns out that the notion of entailment can be obtained in return through the notion of semantic informativity:

Theorem 3.4 (Reversal property)**.** *Let $\varphi_1, \varphi_2 \in \mathcal{I}$. If φ_1 is semantically more informative than φ_2 w.r.t. every pointed model (M, s), then φ_1 entails φ_2.*

Proof. Assume that φ_1 is more informative than φ_2 w.r.t. to every pointed epistemic model (M, s). Let (M, s) be a pointed model such that $M, s \models \varphi_1$. Since φ_1 is more informative than φ_2 w.r.t. (M, s) by assumption, we have that $[\![\varphi_1]\!]_s \subseteq [\![\varphi_2]\!]_s$ and thereby that $M, s \models \varphi_2$. It follows that for all pointed models (M, s), we have $M, s \models \varphi_1 \to \varphi_2$, i.e., φ_1 entails φ_2. □

Given these two results, the notion of informativity appears as a *generalization* of the notion of entailment which takes in account the epistemic situation of the agent.

We will now define the notion of semantic informativity for questions.

Informativity of a question

Using the notion of semantic informativity for propositions, we can define a notion of semantic informativity for questions by saying that a question Q_1 is more informative than another question Q_2 w.r.t. (M, s) if and only if whatever the answers to Q_1 and Q_2 are, the answer to Q_1 will be more informative than the answer to Q_2. This notion can be defined formally as follows:

Definition 3.4 (Semantic informativity for questions). *Let $Q_1 = (\gamma_1, \ldots, \gamma_k)$ and $Q_2 = (\chi_1, \ldots, \chi_l)$ be questions and (M, s) be a pointed model such that $M, s \models \mathsf{pre}(Q_1)$ and $M, s \models \mathsf{pre}(Q_2)$. We say that Q_1 is semantically more informative than Q_2 w.r.t. (M, s) if and only if for all $i \in [\![1, k]\!]$ there exists $j \in [\![1, l]\!]$ such that*

$$[\![\gamma_i]\!]_s \subseteq [\![\chi_j]\!]_s.$$

From a dynamic perspective, we would expect that, if Q_1 is semantically more informative than Q_2, then by obtaining the answer to Q_1 the agent will be brought into an epistemic situation in which she would know *more* than in the epistemic situation resulting from obtaining the answer to Q_2. This property of semantic informativity for questions is actually true and can be proved as follows:

Theorem 3.5. *Let (M, s) be a pointed model, le $Q_1 = (\gamma_1, \ldots, \gamma_k)$ and $Q_2 = (\chi_1, \ldots, \chi_l)$ be propositional questions such that $M, s \models \mathsf{pre}(Q_1)$ and $M, s \models \mathsf{pre}(Q_2)$. If*

- *Q_1 is semantically more informative than Q_2,*
- *$\gamma_i \in \Phi(s)$ for some $i \in [\![1, k]\!]$,*
- *$\chi_j \in \Phi(s)$ for some $j \in [\![1, l]\!]$,*

then we have for all $\varphi \in \mathcal{E}_{\mathcal{I}}$

$$M, s \models [Q_2?]K\varphi \to [Q_1?]K\varphi.$$

Proof. Assume that $M, s \models [Q_2?]K\varphi$. Since $\chi_j \in \Phi(s)$ for some $j \in [\![1, l]\!]$, we have that $M, s \models [Q_2?]K\varphi$ is equivalent to $M, s \models [!\chi_j]K\varphi$. Besides, since $\gamma_i \in \Phi(s)$ for some $i \in [\![1, k]\!]$, we have that $M, s \models [Q_1?]K\varphi$ is equivalent to $M, s \models [!\gamma_i]K\varphi$. Since Q_1 is semantically more informative than Q_2, we necessarily have that $[\![\gamma_i]\!]_s \subseteq [\![\chi_j]\!]_s$. It follows that $M|\gamma_i$ is a submodel of $M|\chi_j$ and we get from the hypothesis $M, s \models [Q_2?]K\varphi$ that $M, s \models [Q_1?]K\varphi$.

We conclude that $M, s \models [Q_2?]K\varphi \to [Q_1?]K\varphi$. □

We can notice that the definition of semantic informativity for questions is very similar to the definition of entailment for questions, except that it depends on the current epistemic situation of the agent. We will now make explicit the relation between the two notions.

What is the relation between the notion of semantic informativity and the notion of entailment for questions?

As for propositions, we can show that the notion of entailment is 'stronger' than the notion of semantic informativity:

Theorem 3.6. *Let $Q_1 := (\gamma_1, \ldots, \gamma_k)$ and $Q_2 := (\chi_1, \ldots, \chi_l)$ be questions and (M, s) be a pointed model such that $M, s \models \mathsf{pre}(Q_1)$. Then, we have that:*

$$Q_1 \textit{ entails } Q_2 \;\Rightarrow\; Q_1 \textit{ is semantically more informative than } Q_2 \textit{ w.r.t. } (M, s).$$

Proof. The proof of the theorem is direct: if Q_1 entails Q_2, then we have for all pointed models (M', s') that for all $i \in [\![1, k]\!]$ there exists $j \in [\![1, l]\!]$ such that $[\![\gamma_i]\!]_{s'} \subseteq [\![\chi_j]\!]_{s'}$. Thus, we have in particular for (M, s) that for all $i \in [\![1, k]\!]$ there exists $j \in [\![1, l]\!]$ such that $[\![\gamma_i]\!]_s \subseteq [\![\chi_j]\!]_s$, i.e., Q_1 is semantically more informative than Q_2. □

Notice that the converse to this theorem is not true. To see this, we can easily construct a counter-example by taking two questions $Q_1 = (p, q, r)$ and $Q_2 = (s, t)$ such that there exist two valuations for which Q_1 is more informative than Q_2 in one case but not in the other. Thus Q_1 might be more informative than Q_2 in some special cases although Q_1 does not entail Q_2.

Besides, we also have the following reversal property:

Theorem 3.7 (Reversal property)**.** *Let $Q_1 = (\gamma_1, \ldots, \gamma_k)$ and $Q_2 = (\chi_1, \ldots, \chi_l)$ be questions. If for all pointed models (M, s) such that $M, s \models \mathsf{pre}(Q_1)$ we have that $M, s \models \mathsf{pre}(Q_2)$ and that Q_1 is more informative than Q_2, then Q_1 entails Q_2.*

Proof. Assume that for all pointed models (M, s) such that $M, s \models \mathsf{pre}(Q_1)$ we have that $M, s \models \mathsf{pre}(Q_2)$ and that Q_1 is more informative than Q_2. This means that for all pointed models (M, s):

1. if $M, s \models \mathsf{pre}(Q_1)$, then $M, s \models \mathsf{pre}(Q_2)$,
2. if $M, s \models \mathsf{pre}(Q_1)$, then for all $i \in [\![1, k]\!]$ there exists $j \in [\![1, l]\!]$ such that $[\![\gamma_i]\!]_s \subseteq [\![\chi_j]\!]_s$.

We conclude by definition that Q_1 entails Q_2. □

As for propositions, the notion of semantic informativity for questions can be seen as a *generalization* of the notion of entailment which takes in account the epistemic situation of the agent.

We will now adopt a quantitative approach to the notion of informativity for propositions and questions.

3.2.2 The quantitative approach

In the quantitative approach, we adopt the theory of information developed by Bar-Hillel and Carnap [4]. The general idea behind this theory, which is sometimes called the *inverse relationship principle*, is the following: the amount of information associated with a proposition is *inversely proportional* to the *probability* associated with that proposition. Another way to put it is to say that the informativity of a proposition corresponds to its *degree of surprise* for the agent: φ_1 is more informative than φ_2 if and only if φ_1 would be more surprising for the agent than φ_2. One possible way to formalize this idea is to consider a set of possible worlds along with a probability distribution on this set. This can be done in our framework by introducing *probabilistic epistemic models*, i.e., by associating a probability distribution to the epistemic range of the agent.[5] In the two following subsections, we will define the notion of *quantitative informativity* for both propositions and questions based on Bar-Hillel and Carnap's information theory.

Informativity of a proposition

Let (M, s) be a pointed epistemic inquiry model, where s denotes the actual world, and $[s]$ denotes the epistemic range of the agent. In the probabilistic setting, we introduce a probability function P, transforming the epistemic model into a *probabilistic epistemic model*, which provides

[5] We refer to the first chapter for a detailed presentation of *probabilistic dynamic epistemic logic* and for the definition of *probabilistic epistemic models*. In this chapter, we consider *probabilistic epistemic inquiry models* which are probabilistic epistemic models plus an *oracle function*. Then, the definitions of the *questioning operation* and the *question operator* for epistemic inquiry models can be straightforwardly adapted to probabilistic epistemic inquiry models. This is due to the fact that we see questions as *conditional hard information updates*, and thereby it suffices to replace the hard information update operation with the one for probabilistic epistemic models to obtain the questioning operation on probabilistic epistemic inquiry models. Then, the semantic definition for the question operator is the same as before, except that the questioning operation is replaced by its probabilistic version.

a probability distribution over $[s]$. Then, we can compute the probability of a proposition φ at s[6] as follows:

$$P(\varphi) = \sum_{w \in [s] \text{ s.t. } M,w \models \varphi} P(s)(w).$$

Finally, a *probabilistic epistemic inquiry model* is simply a probabilistic epistemic model to which we add an *oracle function.*

Following the inverse relationship principle, we define formally the notion of *quantitative informativity* of a proposition as the logarithm in base 2 of the inverse of the probability associated to this proposition:

Definition 3.5 (Quantitative informativity for propositions)**.** *Let (M, s) be a pointed probabilistic model*[7] *and let $\varphi \in \mathcal{I}$. We define the quantitative informativity of φ w.r.t. (M, s) by*

$$inf(s)(\varphi) = \log_2 \left(\frac{1}{P(s)(\varphi)} \right) = -\log_2 P(s)(\varphi).$$

We can notice that the extreme cases in which $inf(\varphi) = 0$ and $inf(\varphi) = \infty$ correspond respectively to the cases in which the agent knows that φ and in which the agent knows that $\neg\varphi$. Now, we can easily compare the propositions with respect to their quantitative informativity:

Definition 3.6. *Let (M, s) be a pointed probabilistic model and let $\varphi_1, \varphi_2 \in \mathcal{I}$. We say that*

$$\varphi_1 \textit{ is quantitatively more informative than } \varphi_2 \textit{ w.r.t. } (M,s) \quad \textit{iff} \quad inf(s)(\varphi_1) \geq inf(s)(\varphi_2).$$

As we said in the introduction, the informativity of a proposition, according to the inverse relationship principle, can be interpreted as the *degree of surprise* of the proposition for the agent. This means that a proposition φ_1 is more informative than a proposition φ_2 if and only if φ_1 would be more surprising for the agent than φ_2. Notice that, since the use of the logarithm reverses the ordering of the propositions based on their probabilities, we actually have this property:

$$\begin{aligned} \varphi_1 \text{ is quantitatively more informative than } \varphi_2 \text{ w.r.t. } (M,s) \quad &\Leftrightarrow \quad inf(s)(\varphi_1) \geq inf(s)(\varphi_2) \\ &\Leftrightarrow \quad P(s)(\varphi_1) \leq P(s)(\varphi_2). \end{aligned}$$

We now turn to the following issue:

What is the relation between the quantitative notion of informativity and the notions of entailment and semantic informativity?

The following theorem makes explicit the hierarchy between the notions of entailment, semantic informativity and quantitative informativity for propositions:

Theorem 3.8. *Let (M, s) be a pointed probabilistic model and let $\varphi_1, \varphi_2 \in \mathcal{I}$. We have that:*

$$\begin{aligned} \varphi_1 \textit{ entails } \varphi_2 \quad &\Rightarrow \quad \varphi_1 \textit{ is semantically more informative than } \varphi_2 \textit{ w.r.t. } (M,s) \\ &\Rightarrow \quad \varphi_1 \textit{ is quantitatively more informative than } \varphi_2 \textit{ w.r.t. } (M,s). \end{aligned}$$

Proof. The proof is direct. □

[6]In our probabilistic setting (see chapter 1), the probability function associated to each $w \in [s]$ is identical to the probability function associated to s.

[7]We recall that, for convenience reasons, we refer to *epistemic inquiry models* as *models*, and to *probabilistic epistemic inquiry models* as *probabilistic models.*

We then get, through the notion of quantitative informativity, a *total* ordering on the set of propositions which extends the partial ordering obtained by the semantic definition of informativity, which itself is already an extension of the entailment ordering. Besides, we also have a reversal property showing that the notion of entailment can be obtained in return from the notion of quantitative informativity:

Theorem 3.9 (Reversal property). *Let $\varphi_1, \varphi_2 \in \mathcal{I}$. If φ_1 is quantitatively more informative than φ_2 with respect to every pointed probabilistic model (M, s), then φ_1 entails φ_2.*

Proof. Let $\varphi_1, \varphi_2 \in \mathcal{I}$. We will prove the theorem by contraposition. Assume that φ_1 does not entail φ_2. This means that there exists a valuation V_1 such that $V_1(\varphi_1) = 1$ and $V_1(\varphi_2) = 0$. Now consider the pointed probabilistic model (M, s) where $M = \langle \{s\}, \sim, V_1, P, \Phi \rangle$ and $P(\varphi_1) = 1$ and $P(\varphi_2) = 0$. In this case, $P(\varphi_2) < P(\varphi_1)$ and thereby φ_2 is quantitatively more informative than φ_1. Thus, this counter-example shows that it is not the case that φ_1 is quantitatively more informative than φ_2 with respect to every probabilistic model (M, s).

By contraposition, we conclude that if φ_1 is quantitatively more informative than φ_2 with respect to every probabilistic model (M, s), then φ_1 entails φ_2. □

Here again, the notion of quantitative informativity for propositions appears as a *generalization* of the notions of entailment and quantitative informativity.

We now turn to the notion of quantitative informativity for questions.

Informativity of a question

Robert van Rooij has developed, in a series of papers[8], a way to quantitatively measure the informativity of questions using Bar-Hillel and Carnap's theory of information along with notions borrowed from Shannon's information theory. The central idea is the following: the quantitative informativity of a question can be equated to the estimated amount of information conveyed by its answers, i.e., the average amount of information of the answers:

> To determine the informative value of a question, we will again follow the lead of Bar- Hillel & Carnap [4]. They discuss the problem how to determine the *estimated* amount of information conveyed by the outcome of an *experiment* to be made. They equate the value of an experiment with its estimated amount of information, and they assume that the possible outcomes denote propositions such that the set of outcomes are mutually exclusive and jointly exhaust the whole state space. In other words, they assume that the set of possible outcomes *partitions* the set of relevant states. This suggests, obviously, that we can also equate the informative value of a *question* with the estimated amount of information conveyed by its (complete) answers. The estimated amount of information of the answers will simply be the *average* amount of information of the answers. [40, p. 11]

Using the notion of *entropy* from Shannon's information theory, the notion of quantitive informativity for questions can be defined formally:

Definition 3.7 (Quantitative informativity for questions). *Let (M, s) be a pointed probabilistic model and $Q = (\gamma_1, \ldots, \gamma_k)$ be a question such that $M, s \models \mathsf{pre}(Q)$. We define the quantitative informativity, or entropy, of Q w.r.t. (M, s) as follows:*

$$E(s)(Q) = \sum_{1 \leq i \leq k} P(s)(\gamma_i) \times inf(s)(\gamma_i).$$

[8]See in particular [39] and [40].

Given this definition, the quantitative informativity of a question Q is:

- *minimal* when the inquiring agent already knows the answer, in this case we have that

$$E(Q) = P(answer) \times inf(answer) = 1 \times \log_2(1) = 0,$$

- *maximal* when the answers to Q have all the same probability, in this case we have that

$$E(Q) = \log_2(k)$$

  where k is the number of possible answers to Q (which are mutually exclusive when the inquiring agent has established the presupposition to Q).

These two properties show features that we would intuitively expect of a quantitative measure of informativity for questions: if the agent already knows the answer to a question, then the informativity of the question is null; if the agent has no evidence in order to differentiate the different answers to the question, then the question is maximally informative. We can then compare questions with respect to their quantitative informativity:

Definition 3.8. *Let (M, s) be a pointed probabilistic model and let Q_1, Q_2 be propositional questions. We say that*

$$Q_1 \textit{ is quantitatively more informative than } Q_2 \textit{ w.r.t. } (M, s) \quad \textit{iff} \quad E(s)(Q_1) \geq E(s)(Q_2).$$

We now turn to the following issue:

What is the relation between the notion of quantitative informativity and the notions of entailment and semantic informativity for questions?

As for propositions, we have the following hierarchy between the notions of entailment, semantic and quantitative informativity:

Theorem 3.10. *Let (M, s) be a pointed probabilistic model and let Q_1 and Q_2 be questions such that $M, s \models \mathsf{pre}(Q_1)$ and $M, s \models \mathsf{pre}(Q_2)$. We have that:*

$$\begin{aligned} Q_1 \textit{ entails } Q_2 \quad &\Rightarrow \quad Q_1 \textit{ is semantically more informative than } Q_2 \textit{ w.r.t. } (M, s) \\ &\Rightarrow \quad Q_1 \textit{ is quantitatively more informative than } Q_2 \textit{ w.r.t. } (M, s). \end{aligned}$$

Proof. This is still a conjecture for the moment since we do not have a proof of this theorem yet. □

Thus, we can see that the notion of quantitative informativity for questions provides a total ordering on the set of questions that the agent can meaningfully ask, extending the partial orderings provided by the semantic notion of informativity and the notion of entailment.

We also have the following reversal property:

Theorem 3.11 (Reversal property). *If $E(s)(Q_1) \geq E(s)(Q_2)$ with respect to every pointed probabilistic model (M, s) such that $M, s \models \mathsf{pre}(Q_1)$ and $M, s \models \mathsf{pre}(Q_2)$, then Q_1 entails Q_2.*

Proof. See van Rooij [40]. □

Given these two results, the notion of quantitative informativity for questions appears as a *generalization* of the notions of entailment and semantic informativity.

In this section, we have proposed definitions of the notion of informativity, for both propositions and questions, adopting first a semantic approach based on the notion of *information as* *range*, then a quantitative approach using *Bar-Hillel and Carnap's theory of information.*

As we have said in the introduction, the notion of informativity depends on the epistemic situation of the inquiring agent but is independent of the on-going inquiry. Thus, we will now explore the possibility to define the notion of *relevance*, i.e., a measure value for propositions and questions which depends on both the epistemic situation of the agent and the on-going inquiry. In the two following sections, we will propose definitions of the notion of *relevance*, first in the *establishing perspective* to inquiry, then in the *determining perspective.*

3.3 The notion of *relevance* in the establishing perspective

In the *establishing perspective*, the goal of an inquiry is, starting from an initial epistemic situation (M, s), to establish a certain conclusion γ. In this perspective, it seems that the agent has somehow an idea about what is the case in the actual world. More specifically, it seems natural to say that the agent *expects* γ to be true. Thus, the establishing perspective to inquiry introduces a new epistemic ingredient, the notion of *expectation*, which deserves to be investigated.

In some cases, the expectations of the agent differ from the *actual information* that the agent has about the world. For instance, in scientific practice, it is not rare to see a scientist expecting a certain hypothesis γ to be true, although, given the state of the knowledge in the field, there are more evidence supporting $\neg\gamma$ than γ. In such a situation, the scientist will direct her inquiry in the direction she expects to be fruitful, in this case by trying to find evidence supporting γ.

Thus, it seems that, in the establishing perspective to inquiry, we are in the presence of a *two-level* epistemic situation:

- one level is the level of the *actual information* that the inquiring agent has,
- the other level is the level of her *expectations.*

How can we represent expectations in our framework? One possibility is to consider that the agent expects a certain proposition γ to be true when she attributes to γ a probability greater than a certain value $\alpha > 0.5$, where α depends on the agent. Then, if we want to represent the two-level epistemic situation, we can simply consider two probability functions, one representing the actual evidence that the agent has, the other representing her expectations. In this section, we will consider only one probability function, i.e., we will consider that the expectations of the agent depend directly on her information about the actual world. Nevertheless, our work can straightforwardly be extended to the case where actual information and expectations differ, by simply introducing another probability function representing the agent's expectations.

The objective of this section is to define a notion of *relevance* in the establishing perspective to inquiry. As in the previous section, we first adopt a *semantic approach* to the definition of relevance for propositions and questions, before turning to a *quantitative approach.* We will see that the notion of expectation that we have just discussed will be particularly helpful to define a notion of relevance for questions from a given notion of relevance for propositions. The main idea being that a question is relevant to a given inquiry in the establishing perspective if the *expected answer* to the question is relevant.

3.3.1 The semantic approach

In the semantic approach, we aim to define a notion of *relevance* as *semantic informativity with respect to the goal of the inquiry.* More precisely, since in order to establish a proposition γ the agent has to eliminate all the $\neg\gamma$-worlds, our proposal is to think of the notion of relevance of a proposition in terms of its *'eliminative power'* with respect to the worlds that the inquiring agent *wants* to eliminate. We will now make this notion of relevance precise for both propositions and questions.

Relevance of a proposition

Let (M, s) be an epistemic model representing the initial epistemic situation of the agent and γ be the proposition that the agent wants to establish (in this situation, we will talk of γ-*inquiry*). What the inquiring agent has to do in this case, in order to reach her inquiry goal, is to eliminate all the $\neg\gamma$-worlds. Then, one possible way to define semantically a notion of relevance, in the establishing perspective, is the following: a proposition φ_1 is more relevant to the γ-inquiry than a proposition φ_2 w.r.t. (M, s) if and only if the $\neg\gamma$-worlds in the agent's epistemic range $[s]$ that would be eliminated by an announcement of φ_2 would also be eliminated by an announcement of φ_1. This notion of relevance can be defined formally as follows:

Definition 3.9 (Relevance of a proposition)**.** *Let (M, s) be a pointed model, let $\gamma, \varphi_1, \varphi_2 \in \mathcal{I}$. We say that φ_1 is semantically more relevant to the γ-inquiry than φ_2 w.r.t. (M, s) if and only if*

$$[\![\varphi_1 \wedge \neg\gamma]\!]_s \subseteq [\![\varphi_2 \wedge \neg\gamma]\!]_s.$$

Thus, it makes sense to think of this notion of relevance as *informativity with respect to the γ-inquiry*, since the above definition can be rewritten as follows: φ_1 is more relevant to the γ-inquiry than φ_2 w.r.t. (M, s) if and only if $(\varphi_1 \wedge \neg\gamma)$ is more informative than $(\varphi_2 \wedge \neg\gamma)$, namely φ_1 is semantically more informative than φ_2 w.r.t. the $\neg\gamma$-area[9].

We will now 'export' this definition in the case of questions.

Relevance of a question

As we suggested in the introduction of this section, one possible way to define a notion of relevance for questions in the establishing perspective is to define the notion of *expected answer* and to make use of the previous definition of relevance for propositions. More precisely, the notion of relevance for questions can be defined as follows: a question Q_1 is more relevant to the γ-inquiry than a question Q_2 w.r.t. (M, s) if and only if the expected answer to Q_1 is more relevant to the γ-inquiry than the expected answer to Q_2. In order to formally define this notion, we first need to define the notion of *expected answer.*

Let $Q = (\gamma_1, \ldots, \gamma_k)$ be a propositional question. One possible way to define the expected answer to Q is the following: γ_i is the *expected answer* to Q if and only if the agent attributes to γ a probability greater than $\alpha > 0.5$, where α depends on the agent. Formally, this leads to the following definition:

Definition 3.10 (Expected answer)**.** *Let (M, s) be a pointed probabilistic model, let $\alpha \in\,]0.5, 1]$, let $Q_1 = (\gamma_1, \ldots, \gamma_k)$ be a question such that $M, s \models \mathsf{pre}(Q)$ and let $\gamma \in \mathcal{I}$. If $M, s \models P(\gamma_i) \geq \alpha$ for some $i \in [\![1, k]\!]$, then we say that γ_i is the α-expected answer to Q.*

Notice that the expected answer to a propositional question is not always defined. Notice also that, if the expected answer to a question is defined, then it is unique due to the fact that Q induces a partition on the agent's epistemic range.

We can now define the notion of relevance for questions formally using the previous definition of relevance for propositions and the notion of expected answer:

[9] By $\neg\gamma$-area we mean the worlds in $[\![\neg\gamma]\!]_s$, i.e., the $\neg\gamma$-worlds in the agent's epistemic range.

Definition 3.11 (Relevance of a question). *Let (M, s) be a pointed probabilistic model, let $\alpha \in]0.5, 1]$, let Q_1, Q_2 be propositional questions such that (i) $M, s \models \mathsf{pre}(Q_1)$ and $M, s \models \mathsf{pre}(Q_2)$ and (ii) the α-expected answers to Q_1 and Q_2 are defined. Let $\gamma \in \mathcal{I}$. We say that Q_1 is more relevant to the γ-inquiry than Q_2 w.r.t. (M, s) if and only if the α-expected answer to Q_1 is semantically more relevant to the γ-inquiry than the α-expected answer to Q_2.*

Some remarks are in order here. First of all, we shall notice that, since the notion of expected answer is not always defined, we cannot always compare two questions with respect to their relevance. In other words, the notion of relevance leads a *partial ordering* on the questions that the agent can meaningfully ask[10]. Secondly, the notion of relevance for questions depends on the coefficient α, which itself depends on the agent, and which says when a proposition becomes *likely* or *expected* by the agent.

We will now adopt a quantitative approach to the notion of relevance for propositions and questions.

3.3.2 The quantitative approach

Our initial idea to define the notion of relevance for propositions and questions can be straightforwardly adapted to the quantitative setting based on Bar-Hillel and Carnap's theory of information. In the quantitative approach, this idea becomes the following: to define the notion of *relevance* as *quantitative informativity with respect to the goal of the inquiry.* This means that we will measure the relevance of a proposition by its quantitative informativity with respect to the area of the worlds that the agent wants to eliminate. Then, the notion of relevance for questions will be defined as before, i.e., as the relevance of the expected answer to the question.

Relevance of a proposition

Let (M, s) be a probabilistic model representing the initial epistemic situation of the agent and γ be the proposition that the agent wants to establish. Following the idea that the notion of relevance of a proposition can be defined in terms of its eliminative power with respect to the worlds that the inquiring agent wants to eliminate, which are in the case of a γ-inquiry the $\neg\gamma$-worlds, we can say that: a proposition φ_1 is quantitatively more relevant to the γ-inquiry than a proposition φ_2 w.r.t. (M, s) if and only if the quantitative informativity of φ_1 with respect to the $\neg\gamma$-area is greater than the one of φ_2. Formally, this leads to the following definition:

Definition 3.12 (Relevance of a proposition). *Let (M, s) be a pointed probabilistic model and let $\gamma, \varphi_1, \varphi_2 \in \mathcal{I}$. We say that φ_1 is quantitatively more relevant to the γ-inquiry than φ_2 if and only if*

$$inf(s)(\varphi_1 \wedge \neg\gamma) \geq inf(s)(\varphi_2 \wedge \neg\gamma).$$

Thus, the intuitive idea behind this definition is to define a notion of quantitative informativity relative to the *target area* defined by the goal of the inquiry, i.e., the $\neg\gamma$-area for a γ-inquiry. This is the reason why we talk about relevance as *quantitative informativity with respect to the goal of the inquiry.*

We will now export this definition in the case of questions.

[10]We recall here that these questions that can be meaningfully ask are the questions whose the presuppositions have been established by the agent.

Relevance of a question

As in the semantic approach, we define the notion of *relevance* for questions from the notions of *expected answer* and *relevance* for propositions. The notion of expected answer is defined as before. Then, the notion of relevance for questions is obtained as follows:

Definition 3.13 (Relevance of a question). *Let (M, s) be a pointed probabilistic model, let $\alpha \in]0.5, 1]$, let Q_1, Q_2 be propositional questions such that (i) $M, s \models \mathsf{pre}(Q_1)$ and $M, s \models \mathsf{pre}(Q_2)$ and (ii) the α-expected answers to Q_1 and Q_2 are defined. Let $\gamma \in \mathcal{I}$. We say that Q_1 is quantitatively more relevant to the γ-inquiry than Q_2 w.r.t. (M, s) if and only if the α-expected answer to Q_1 is quantitatively more relevant to the γ-inquiry than the α-expected answer to Q_2.*

As in the semantic approach, the definition of relevance for questions yields a partial ordering on the set of questions that the agent can ask. Besides, the notion of relevance also depends on the coefficient α, relative to the agent, through the notion of expected answer.

In this section, we have proposed definitions for the notion of relevance in the establishing perspective for both propositions and questions. However, other definitions might also make sense. Thus, from a methodological point of view, it would be interesting, maybe even necessary, to relate a given notion of relevance with the performance of the agent in her inquiry based on this notion: for instance, we can extract a *questioning strategy* from a given notion of relevance by simply saying that the agent chooses the *most relevant* question at each interrogative step. To this end, one would need a way to measure performance in inquiry. We will not try to do so in this thesis, leaving this issue to further investigations.

3.4 The notion of *relevance* in the determining perspective

In the *determining perspective*, the goal of the inquiring agent is, from an initial epistemic situation (M, s), to determine whether or not a certain proposition γ is the case. In this perspective, the goal of the inquiring agent amounts to answering the yes-no question $(\gamma, \neg\gamma)$. Indeed, we do not need to restrict ourselves to yes-no questions and we can generalize this kind of inquiry goal to any propositional question $Q = (\gamma_1, \ldots, \gamma_k)$ whose the presupposition has been established by the agent.

In [40], Robert van Rooij proposes a definition of the notion of relevance for both propositions and questions when the goal of the inquiry is to determine which one of the mutually exclusive hypotheses of a set $\{h_1, \ldots, h_n\}$ is the case in the actual world s, where the set $\{h_1, \ldots, h_n\}$ forms a partition on the $\sim$-equivalence class of s. This setting coincides exactly with the situation, in our framework, where the inquiring agent has established the presupposition of a question $Q = (\gamma_1, \ldots, \gamma_k)$, Q inducing in this case a partition on $[s]$. In such a situation, the inquiry goal embodied by the question Q can be seen as the decision problem: 'which one of the mutually exclusive hypotheses of the set $Q = \{\gamma_1, \ldots, \gamma_k\}$ should be chosen'. The work of van Rooij on the notion of relevance can then be directly transfered in our framework. Since in the determining perspective, the semantic approach is just a particular case of the quantitative approach in which all the worlds are given the same probabilities, we will directly adopt the quantitative approach.

3.4.1 Relevance of a proposition

Robert van Rooij's proposal to measure the relevance of a proposition φ with respect to the decision problem 'which one of the mutually exclusive hypotheses of $Q = \{\gamma_1, \ldots, \gamma_k\}$ should be chosen', is to use the notion of *information value* of an assertion φ with respect to the partition Q. This notion is defined as follows:

Definition 3.14 (Information value)**.** *Let (M, s) be a probabilistic model, let $\varphi \in \mathcal{I}$ and let $Q = (\gamma_1, \ldots, \gamma_k)$ be a question such that $M, s \models \mathsf{pre}(Q)$. We define the information value of φ w.r.t. Q and (M, s) by*

$$IV_Q(s)(\varphi) = E(s)(Q) - E_\varphi(s)(Q)$$

where $E(s)(Q) = \sum_{1 \leq i \leq k} P(s)(\gamma_i) \times inf(s)(\gamma_i)$ and $E_\varphi(s)(Q) = \sum_{1 \leq i \leq k} P(s)(\gamma_i|\varphi) \times inf(s)(\gamma_i|\varphi)$.

This notion of information value corresponds to the reduction of entropy (uncertainty) of the partition Q when the proposition φ is learned.

We can now say that an assertion φ is *relevant* with respect to Q when $IV_Q(\varphi) > 0$. Notice that it might be the case that an assertion flattens the probability distribution over Q, leading to a situation where $IV_Q(\varphi) < 0$. In this case, the assertion φ should be considered relevant even if it makes the decision problem more difficult. Thus, we will rather say that φ is *relevant* with respect to Q when $IV_Q(\varphi) \neq 0$.

We can then compare the different propositions with respect to their relevance to the decision problem Q by saying that φ_1 is more relevant than φ_2 with respect to Q if and only if the informational value of φ_1 is higher than the informational value of φ_2:

Definition 3.15 (Relevance of a proposition)**.** *Let (M, s) be a probabilistic model, let $\varphi_1, \varphi_2 \in \mathcal{I}$ and let $Q = (\gamma_1, \ldots, \gamma_k)$ be a question such that $M, s \models \mathsf{pre}(Q)$. We say that φ_1 is more relevant to Q w.r.t. (M, s) than φ_2 if and only if*

$$IV_Q(s)(\varphi_1) > IV_Q(s)(\varphi_2), \text{ or } IV_Q(s)(\varphi_1) = IV_Q(s)(\varphi_2) \text{ and } inf(s)(\varphi_1) < inf(s)(\varphi_2).$$

This definition says that φ_1 is more relevant than φ_2 with respect to Q if φ_1 reduces the entropy of Q more than φ_2 does. In the case where $IV_Q(\varphi_1) = IV_Q(\varphi_2)$, the more relevant proposition between φ_1 and φ_2 is the one which is the less informative (we privilege in this the case the more 'economical' proposition in terms of informativity).

Thus, since the situation in which the inquiring agent tries to answer the question $Q = (\gamma_1, \ldots, \gamma_k)$ is identical formally to the situation in which she tries to solve the decision problem 'which one of the mutually exclusive hypotheses of $Q = \{\gamma_1, \ldots, \gamma_k\}$ should be chosen', it makes perfect sense to define the notion of relevance for proposition using the decision-theoretic concept of informative value. We will now see that the notion of relevance can also be define for questions along the same line.

3.4.2 Relevance of a question

We now want to measure the relevance of a question Q' with respect to another question Q. To this end, Robert van Rooij proposes to use the notion of *expected informational value* of Q' with respect to Q, which is the average reduction of entropy of Q when an answer to Q' is learned:

Definition 3.16 (Expected informational value)**.** *Let (M, s) be a probabilistic model, let $Q = (\gamma_1, \ldots, k)$ and $Q' = (\chi_1, \ldots, \chi_l)$ be two questions such that $M, s \models \mathsf{pre}(Q)$ and $M, s \models \mathsf{pre}(Q')$. We define the expected informational value of Q' w.r.t. Q and (M, s) by*

$$EIV_Q(s)(Q') = \sum_{1 \leq i \leq l} P(s)(\chi_i) \times IV_Q(s)(\chi_i).$$

We can now define the notion of relevance for questions: Q' is *relevant* with respect to Q if the true answer to Q' is expected to reduce the uncertainty about which one of the answers to Q is true: $EIV_Q(Q') > 0$. Thus, we "equate the *usefulness* of question Q with respect to the *decision problem* which of the hypotheses of H should be chosen, with the reduction of uncertainty about H due to Q, i.e. $EIV_H(Q)$." [40, p. 15] Formally, this leads to the following definition:

Definition 3.17 (Relevance of a question). *Let (M, s) be a probabilistic model, let Q, Q_1, Q_2 be questions such that $M, s \models \mathsf{pre}(Q)$, $M, s \models \mathsf{pre}(Q_1)$ and $M, s \models \mathsf{pre}(Q_2)$. We say that Q_1 is more relevant to Q w.r.t. (M, s) than Q_2 if and only if*

$$EIV_Q(s)(Q_1) > EIV_Q(s)(Q_2), \text{ or } EIV_Q(s)(Q_1) = EIV_Q(s)(Q_2) \text{ and } \mathsf{Q}_2 \sqsubseteq \mathsf{Q}_1.$$

As in the case of propositions, the fact that, formally, the situation in which the agent aims to answer to the question $Q = (\gamma_1, \ldots, \gamma_k)$ and in which she aims to solve the decision problem 'which one of the mutually exclusive hypotheses of $Q = \{\gamma_1, \ldots, \gamma_k\}$ should be chosen' coincide, allow us to directly make use of the decision-theoretic notion of expected informational value to define the notion of relevance for questions.

The concluding remarks that we made for the notion of relevance in the establishing perspective apply also to the determining perspective, namely we shall relate notions of relevance with inquiry performance. Notice that we might also find notions of relevance through an investigation of *questioning strategies*, by looking at the way the agent picks the questions she will ask according to a given questioning strategy. Thus, investigating the notion(s) of relevance and investigating questioning strategies seem to be two very intertwined enterprise. This constitutes thereby a very interesting line of research for further investigations of the strategic aspects of interrogative inquiry.

Conclusion

In this chapter, we have proposed several ways to define *measures of value* for propositions and questions. More precisely, we have investigated the possibility to define the notions of *entailment*, *informativity* and *relevance* for propositions and questions, each notions having specific dependencies with respect to the agent's epistemic situation and her on-going inquiry. To this end, we have shown how to import results from studies in the semantics and pragmatics of questions and answers, in particular the works of Groenendijk and Stokhof [12] and the ones of van Rooij [40]. In this conclusion, we will propose several directions for further investigations regarding the main issues addressed in this chapter.

First of all, we shall explain why we think that developing and investigating the possibility to define measures of epistemic value for propositions and questions can help to progress in our understanding of interrogative inquiries and in the development of the IMI:

Strategic aspects of inquiry. It seems that the investigation of possible measures value for questions and the study of *questioning strategies* are intimately related. The intuitive idea behind this is the following: the best questions to ask are the ones that are the *most informative* or the *most relevant* at a given moment in a given inquiry. Thus, given a measure value for questions, one can extract a questioning strategy from it by simply choosing at each step a question that *maximizes* the given measure value. This also works in the other way around, namely from a given questioning strategy, one may extract a measure value for questions. However, in order to investigate these ideas precisely, we need a framework which enables us to measure the *performance* of the inquiring agent. Hintikka strongly suggests that *game theory* provides such a framework, but *learning theory* or framework merging temporal logic and dynamic epistemic logic such as the one developed in [34] might also be possible candidates.

Decomposition of interrogative inquiries. In the following chapter, we will investigate how one can *decompose* an interrogative inquiry into several 'small' steps, i.e., how one can establish an *inquiry plan* given an inquiry goal and certain means to potentially reach it. In the notion of decomposition for interrogative inquiries, there is the idea that we decompose a *'big'* question, goal of the inquiry, into a number of *'small'* questions. To make this idea precise, one needs a way to *'weigh'* questions, i.e., to provide a precise meaning to 'big' and 'small'. This issue coincides exactly with the principal issue of this chapter, namely determining *measures of value* for questions. For instance, one may say that a question is smaller than another if it is *entailed by* or *less informative* than the other question. In the following chapter, we will develop this idea further.

Some issues have emerged from the investigations conducted in this chapter. One is the idea that there are different *ways* to conduct an inquiry, leading to the study of different *perspectives* on inquiry:

Different perspectives on inquiry. We have seen, through Hintikka's writings, that there are already two different possible perspectives on inquiry, that we have called the *establishing* and the *determining perspectives*. We have also seen that an investigation of the notion of *relevance* strongly depends on the perspective that we take on inquiry.[11] Thus, it would be interesting to see if there are *others* meaningful perspectives on inquiry. To this end, a conceptual analysis from a philosophy of science or epistemological perspective on the very notion of *inquiry* would be necessary.

Another line emerging from what has been done in this chapter is the idea that there might also be some interests in considering the work of Groenendijk and Stokhof, and the work of van Rooij, from a *dynamic-epistemic perspective*:

Dynamic-epistemic aspects of the questioning-answering process. The questioning-answering process is clearly a dynamic and epistemic process involving *flow* of information. Indeed, the dynamics is already implicitly present in the work of van Rooij when he considers and describes the effect, on the epistemic range of the agent, of receiving a partial or complete answer to a question. The DEL-based frameworks, and in particular the dynamic logic of questions that we have developed in the previous chapter, make these operations on epistemic models *explicit* and *formally precise*. Such an enterprise is sometimes quiet difficult, witness for instance the definition of probabilistic update developed by van Benthem, Gerbrandy and Kooi in [35]. Thus, in such situations, it might be necessary to have these model operations precisely defined in order to investigate notions of *informativity* and *relevance* for questions and answers.

[11] It will also be the case for the notion of decomposition of questions that we will treat in the next chapter.

Chapter 4

Decomposing interrogative inquiry

Introduction

Developing an *inquiry plan* or a *research agenda*, given a certain inquiry goal and certain means to potentially reach it, is an important mechanism of scientific inquiry and inquiry in general. Within the framework of the interrogative model of inquiry, this issue is addressed by Hintikka under the heading of *decomposition of questions*:

> In general questions play two roles in interrogative inquiry. What happens is that the inquirer tries to answer a "big" (principal) question by means of a number of "small" (operational) questions. In any one inquiry, the two questions have to be distinguished sharply. [22, p. 246]

However, even though Hintikka claims that the interrogative model of inquiry accounts for this notion of decomposition, he has never proposed a conceptual or formal analysis of this mechanism, although he seems to suggest that the interrogative logic based on the tableau method accounts for it. Thus, we propose in this chapter to investigate conceptually and formally what it could mean to *'decompose'* an interrogative inquiry.

To this end, we will study the notion of interrogative inquiry decomposition through the two following aspects:

Partition decomposition: the decomposition of an inquiry into several steps is a *partition decomposition* if by following the steps of the decomposition the inquirer will reach her inquiry goal.

Small-pieces decomposition: the decomposition of an inquiry into several steps is a *small-pieces decomposition* if each step from the decomposition is 'small', in a sense to be defined, with respect to the inquiry goal.

It turns out that the notion of interrogative inquiry decomposition depends on the chosen perspective on inquiry. In the previous chapter, we have identified two perspectives on inquiry: the *determining perspective* and the *establishing perspective*. Thus, we will approach, in this chapter, the issue of interrogative inquiry decomposition from these two perspectives:

- In the case of the *establishing* perspective, we will think of interrogative inquiry decomposition in terms of *decomposition of tasks*:

  Partition decomposition for tasks: a set of tasks is a *partition decomposition* of a principal task if and only if after having carried out all the tasks of the decomposition the agent would have carried out the principal task.

Small-pieces decomposition for tasks: a set of tasks is a *small-pieces decomposition* of a principal task if and only if each task in the set is 'smaller', in a sense to be defined, than the principal task.

- In the case of the *determining* perspective, we will think of interrogative inquiry decomposition in terms of *decomposition of questions*:

Partition decomposition for questions: a set of questions is a *partition decomposition* of a principal question if and only if by getting the answers to all the questions of the decomposition the agent would know the answer to the principal question.

Small-pieces decomposition for questions: a set of questions is a *small-pieces decomposition* of a principal question if and only if each question in the set is 'smaller', in a sense to be defined, than the principal question.

We will structure our investigation as follows. In the first section, we will work in the *establishing* perspective to inquiry and we will investigate the notion of *decomposition of tasks*. In the second section, we will work in the *determining* perspective to inquiry and we will investigate the notion of *decomposition of questions* from a *static* perspective. In these two first sections, we will examine the claim made by Hintikka according to which the tableau method can be interpreted as an inquiry decomposition mechanism. In the third section, we will take in account the *dynamics of information* into the decomposition mechanism, leading to the notion of *dynamic decomposition of questions*. In the fourth section, we will present the concept of *erotetic search scenario* developed by Andrzej Wiśniewski which aims to capture the idea of inquiry decomposition in the framework of *erotetic logic*. In the fifth section, we compare the two notions of dynamic decomposition of questions and erotetic search scenario.

4.1 Decomposition of tasks

In the establishing perspective, the goal of the inquiring agent is, from an initial epistemic situation (M, s), to establish a certain proposition γ. In our framework[1], this goal can be restated as follows: to eliminate all the $\neg\gamma$-worlds from the epistemic range $[s]$ of the inquiring agent. We propose to think of this kind of inquiry goals in terms of *tasks* that we define in the following way:

Definition 4.1 (Tasks)**.** *Let φ be a formula of $\mathcal{I}$. We denote by $\mathbf{T}_\varphi$ the task consisting in eliminating all the φ-worlds from the agent's epistemic range*[2].

Thus, the goal of a γ-inquiry can be reformulated as follows: to carry out the task $\mathbf{T}_{\neg\gamma}$. Then, one may ask how to *decompose* the 'big' task $\mathbf{T}_{\neg\gamma}$, goal of the γ-inquiry, into a number of 'small' tasks.

In this section, we are interested in this particular issue, i.e., the notion of *decomposition of tasks*. In the first two subsections, we will define for tasks the two aspects of the notion of decomposition: *partition decomposition* and *small-pieces decomposition*. In the third subsection, we will interpret the tableau method as a mechanism of task decomposition and we will show that a decomposition obtained via the tableau method is actually a partition and small-pieces decomposition according to our definitions.

[1]As in the previous chapter, the background framework of this chapter is the dynamic logic of questions presented in section 2.1.2.

[2]Thus, the notion of task is always defined with respect to a pointed epistemic model (M, s). Thereafter, it will always be clear from the context on which pointed model the task is interpreted.

4.1.1 Partition decomposition

The first aspect of the notion of decomposition is what we have called *partition decomposition*: a set of tasks $\{\mathbf{T}_{\varphi_1}, \ldots, \mathbf{T}_{\varphi_k}\}$ is a partition decomposition of a task $\mathbf{T}_\varphi$ if and only if after having carried out all the tasks $\{\mathbf{T}_{\varphi_1}, \ldots, \mathbf{T}_{\varphi_k}\}$ the task $\mathbf{T}_\varphi$ is also carried out. In order to formally define the notion of partition decomposition, we will first define the notions of *addition* and *inclusion* of tasks:

Definition 4.2 (Task addition)**.** *Let $\varphi, \psi \in \mathcal{I}$. The addition of the tasks $\mathbf{T}_\varphi$ and $\mathbf{T}_\psi$ is defined as follows*

$$\mathbf{T}_\varphi + \mathbf{T}_\psi := \mathbf{T}_{\varphi \vee \psi}.$$

This definition captures the idea that, in order to carry out the tasks $\mathbf{T}_\varphi$ *and* $\mathbf{T}_\psi$, we need to eliminate all the φ-worlds *and* all the ψ-worlds, which is equivalent to eliminate all the $(\varphi \vee \psi)$-worlds. We now define the notion of *inclusion* of tasks:

Definition 4.3 (Task inclusion)**.** *Let (M, s) be an epistemic model and $\varphi, \psi \in \mathcal{I}$. We say that the task $\mathbf{T}_\varphi$ is included into the task $\mathbf{T}_\psi$, written $\mathbf{T}_\varphi \subseteq_s \mathbf{T}_\psi$, if and only if*

$$[\![\varphi]\!]_s \subseteq [\![\psi]\!]_s.$$

Intuitively, saying that the task $\mathbf{T}_\varphi$ is included in the task $\mathbf{T}_\psi$ means that, by carrying out the task $\mathbf{T}_\psi$, the agent will also have carried out the task $\mathbf{T}_\varphi$.

From the two notions of addition and inclusion of tasks, we can define the notion of *partition decomposition* of tasks as follows:

Definition 4.4 (Partition decomposition of tasks)**.** *Let (M, s) be an epistemic model, $\mathbf{T}_\varphi$ be a task and $\{\mathbf{T}_{\varphi_1}, \ldots, \mathbf{T}_{\varphi_k}\}$ be a set of tasks. We say that $\{\mathbf{T}_{\varphi_1}, \ldots, \mathbf{T}_{\varphi_k}\}$ is a partition decomposition of $\mathbf{T}_\varphi$ w.r.t. (M, s) if and only if*

$$\mathbf{T}_\varphi \subseteq_s \mathbf{T}_{\varphi_1} + \ldots + \mathbf{T}_{\varphi_k}.$$

This definition says that a set of tasks $\{\mathbf{T}_{\varphi_1}, \ldots, \mathbf{T}_{\varphi_k}\}$ is a partition decomposition of a task $\mathbf{T}_\varphi$ if and only if by carrying out all the tasks $\{\mathbf{T}_{\varphi_1}, \ldots, \mathbf{T}_{\varphi_k}\}$, namely by carrying out the task $\mathbf{T}_{\varphi_1} + \ldots + \mathbf{T}_{\varphi_k}$, the inquiring agent will have carried out the task $\mathbf{T}_\varphi$. This definition captures thereby our initial idea about partition decomposition for tasks.

4.1.2 Small-pieces decomposition

The second aspect of the notion of decomposition is what we have called *small-pieces decomposition*: a set of tasks $\{\mathbf{T}_{\varphi_1}, \ldots, \mathbf{T}_{\varphi_k}\}$ is a small-pieces decomposition of a task $\mathbf{T}_\varphi$ if and only if the tasks $\mathbf{T}_{\varphi_1}, \ldots, \mathbf{T}_{\varphi_k}$ are individually 'smaller' than the task $\mathbf{T}_\varphi$. One possible way to attribute a meaning to *big* and *small* for tasks is in terms of *difficulty*. In this perspective, the notion of *small-pieces decomposition* becomes the following: a set of tasks $\{\mathbf{T}_{\varphi_1}, \ldots, \mathbf{T}_{\varphi_k}\}$ is a small-pieces decomposition of a task $\mathbf{T}_\varphi$ if and only if the tasks $\mathbf{T}_{\varphi_1}, \ldots, \mathbf{T}_{\varphi_k}$ are individually *less difficult* than the task $\mathbf{T}_\varphi$. In order to define this notion of small-pieces decomposition precisely, we need to provide a way to measure the difficulty of tasks.

Our proposal is to measure the difficulty of a task in terms of the *minimal quantity of information* that the agent needs to carry out the task. The intuitive idea is that a task is more difficult than another if it requires *more* information in order to be carried out. Using the notion of quantitative informativity for propositions from the previous chapter, we can formally define this measure of task difficulty as follows:

Definition 4.5 (Task difficulty). *Let (M, s) be a pointed probabilistic epistemic model and let $\varphi, \psi \in \mathcal{I}$. We say that*

$$\mathbf{T}_\varphi \text{ is more difficult than } \mathbf{T}_\psi \text{ if and only if } inf(\neg\varphi) \geq inf(\neg\psi).^3$$

By defining the notion of task difficulty in this way, we implicitly assume that the minimal quantity of information that one needs to carry out the task $\mathbf{T}_\varphi$ is $inf(\neg\varphi)$. It is not difficult to see that this is indeed the case since the more economical way to carry out the task $\mathbf{T}_\varphi$ is to eliminate *all* and *only all* the φ-worlds, which is exactly the effect of getting the information that $\neg\varphi$.

We can now define the notion of small-pieces decomposition for tasks:

Definition 4.6 (Small-pieces decomposition). *Let (M, s) be a probabilistic epistemic model and let $\varphi, \varphi_1, \ldots, \varphi_k \in \mathcal{I}$. We say that $\{\mathbf{T}_{\varphi_1}, \ldots, \mathbf{T}_{\varphi_k}\}$ is a small-pieces decomposition of $\mathbf{T}_\varphi$ w.r.t. (M, s) if and only if*

1. *$\{\mathbf{T}_{\varphi_1}, \ldots, \mathbf{T}_{\varphi_k}\}$ is a partition decomposition of $\mathbf{T}_\varphi$,*
2. *for all $i \in [\![1, k]\!]$, $\mathbf{T}_{\varphi_i}$ is less difficult than $\mathbf{T}_\varphi$.*

According to this definition, a small-pieces decomposition can still globally be stronger than what is needed in the sense that the decomposition might require the elimination of *more worlds* than the minimal required by the 'big' task. This speaks for an analysis of *optimal* decompositions. In this work, we will only propose two possible definitions of optimal decompositions. The first one is based on the remark that we have just made, namely that an optimal decomposition does not require the elimination of more worlds than the minimal needed:

Definition 4.7 (Optimal-1 decomposition). *Let (M, s) be a probabilistic epistemic model and let $\varphi, \varphi_1, \ldots, \varphi_k \in \mathcal{I}$ such that $\{\mathbf{T}_{\varphi_1}, \ldots, \mathbf{T}_{\varphi_k}\}$ is a small-pieces decomposition of $\mathbf{T}_\varphi$ w.r.t. (M, s). We say that $\{\mathbf{T}_{\varphi_1}, \ldots, \mathbf{T}_{\varphi_k}\}$ is an optimal-1 decomposition of $\mathbf{T}_\varphi$ w.r.t. (M, s) if and only if*

$$inf(\neg\varphi) = inf(\neg\varphi_1 \wedge \ldots \wedge \neg\varphi_k).$$

This definition says that a decomposition is *optimal-1* if the agent needs the same amount of information to carry out the big task as to carry out all the tasks from the decomposition. Notice that the minimal information required to carry out all the tasks in $\{\mathbf{T}_{\varphi_1}, \ldots, \mathbf{T}_{\varphi_k}\}$ is identical to the minimal information required to carry out the task $\mathbf{T}_{\varphi_1} + \ldots + \mathbf{T}_{\varphi_k}$, which is $inf(\neg(\varphi_1 \vee \ldots \vee \varphi_k)) = inf(\neg\varphi_1 \wedge \ldots \wedge \neg\varphi_k)$.

There are other alternative definitions of the notion of optimal decomposition. Another intuitive definition is the following:

Definition 4.8 (Optimal-2 decomposition). *Let (M, s) be a probabilistic epistemic model and let $\varphi, \varphi_1, \ldots, \varphi_k \in \mathcal{I}$ such that $\{\mathbf{T}_{\varphi_1}, \ldots, \mathbf{T}_{\varphi_k}\}$ is a small-pieces decomposition of $\mathbf{T}_\varphi$ w.r.t. (M, s). We say that $\{\mathbf{T}_{\varphi_1}, \ldots, \mathbf{T}_{\varphi_k}\}$ is an optimal-2 decomposition of $\mathbf{T}_\varphi$ w.r.t. (M, s) if and only if*

$$inf(\neg\varphi) = inf(\neg\varphi_1) + \ldots + inf(\neg\varphi_k).$$

This definition says that a decomposition is *optimal-2* if the formulas $\neg\varphi_1, \ldots, \neg\varphi_k$ form a *partition* on the $\neg\varphi$-area. In other words, this means that the tasks of the decomposition *do not overlap*. Notice also that an optimal-2 decomposition is always an optimal-1 decomposition.

[3] We use a shortcut in all this chapter: instead of writing $inf(s)(\varphi)$ we write $inf(\varphi)$, the point s in the model with respect to which the quantitative informativity is computed being always clear from the context.

It is important to remark that these definitions of the notion of optimal decomposition only take one parameter in account, namely the difficulty of the tasks. It might be interesting to define a notion of optimal decomposition which also takes in account the *number*[4] of tasks. For instance, it seems intuitive to say that a decomposition into a lots of very small tasks is not optimal. This speaks for a notion of optimal decomposition which yields a good *trade-off* between *time* and *difficulty*. We will not pursue this idea here, leaving it for further investigations.

In the following section, we will investigate the possibility to interpret the tableau method as a mechanism to decompose tasks.

4.1.3 The tableau method as a mechanism of task decomposition

In his writings on the interrogative model of inquiry, Hintikka suggests that the tableau method can be seen as a way to decompose 'big' questions into 'small' ones. However, the tableau method only deals with decomposition of propositions and not of questions. Thus, due to the fact that tasks, as we defined them, are associated with propositions and not with questions, we suggest that thinking in terms of decomposition of tasks better fits the idea that the tableau method can be seen as a decomposition mechanism. In this section, we will propose a way to interpret the tableau method as a mechanism of task decomposition and we will show that decompositions obtained by this method are small-pieces decompositions according to the definition of the previous section.

Assume that the goal of the inquiring agent is to establish a certain proposition γ. Then, what the agent has to do is to carry out the task $\mathbf{T}_{\neg\gamma}$ which consists in eliminating all the $\neg\gamma$-worlds from her epistemic range. As we have already seen, given a semantic tree $\mathcal{T}$ with root $\neg\gamma$, eliminating all the $\neg\gamma$-worlds amounts to eliminate all the worlds compatible with the different branches of $\mathcal{T}$. This idea leads to the notion of *$\mathcal{T}$-decomposition*:

Definition 4.9 ($\mathcal{T}$-decomposition)**.** *Let $\mathcal{T} = \{\mathcal{B}_1, \ldots, \mathcal{B}_n\}$ be a semantic tree ($\mathcal{T} \in \mathsf{Strees}(\mathcal{I})$) with root $\neg\gamma$. Then, we say that $\{\mathbf{T}_{B_1}, \ldots, \mathbf{T}_{B_n}\}$ is the $\mathcal{T}$-decomposition of $\mathbf{T}_{\neg\gamma}$, where for all $i \in [\![1, n]\!]$ we let*

$$B_i := \bigwedge_{\varphi \in \mathcal{B}_i} \varphi.$$

We can then define the notion of *tableau decomposition* of tasks as follows:

Definition 4.10 (Tableau decomposition)**.** *Let $\varphi, \varphi_1, \ldots, \varphi_n \in \mathcal{I}$. We say that $\{\mathbf{T}_{\varphi_1}, \ldots, \mathbf{T}_{\varphi_n}\}$ is a tableau decomposition of $\mathbf{T}_\varphi$ if and only if there exists a semantic tree $\mathcal{T} \in \mathsf{Strees}(\mathcal{I})$ such that $\{\mathbf{T}_{\varphi_1}, \ldots, \mathbf{T}_{\varphi_n}\}$ is the $\mathcal{T}$-decomposition of $\mathbf{T}_\varphi$.*

We first show the following lemma which says that all $\mathcal{T}$-decompositions are small-pieces decompositions:

Lemma 4.1. *Let (M, s) be a probabilistic epistemic model, let $\gamma \in \mathcal{I}$ and let $\mathcal{T}$ be a semantic tree ($\mathcal{T} \in \mathsf{Strees}(\mathcal{I})$) with root $\neg\gamma$. Then*

the $\mathcal{T}$-decomposition of $\mathbf{T}_{\neg\gamma}$ is a small-pieces decomposition.

Proof. Let (M, s) be a probabilistic epistemic model, let $\gamma \in \mathcal{I}$ and let $\mathcal{T} = \{\mathcal{B}_1, \ldots, \mathcal{B}_n\}$ be a semantic tree with root $\neg\gamma$. Let $\{\mathbf{T}_{B_1}, \ldots, \mathbf{T}_{B_n}\}$ be the $\mathcal{T}$-decomposition of $\mathbf{T}_{\neg\gamma}$.

We first have to show that $\{\mathbf{T}_{B_1}, \ldots, \mathbf{T}_{B_n}\}$ is a partition decomposition of $\mathbf{T}_{\neg\gamma}$, namely that

$$\mathbf{T}_{\neg\gamma} \subseteq \mathbf{T}_{B_1} + \ldots + \mathbf{T}_{B_n}.$$

[4]Counting the number of tasks can also be seen as a way to take in account the *temporal* aspects of inquiry.

To show this, we have to show that $[\![\neg\gamma]\!]_s \subseteq [\![B_1 \vee \ldots \vee B_n]\!]_s$, which is equivalent to show that $[\![\neg B_1 \wedge \ldots \wedge \neg B_n]\!]_s \subseteq [\![\gamma]\!]_s$. Then, since we know that if all the branches of a semantic tree with root $\neg\gamma$ are closed at a world w, then γ is true at that world, we get that the inclusion $[\![\neg B_1 \wedge \ldots \wedge \neg B_n]\!]_s \subseteq [\![\gamma]\!]_s$ is always true. This shows that $\{\mathbf{T}_{B_1}, \ldots, \mathbf{T}_{B_n}\}$ is a partition decomposition of $\mathbf{T}_{\neg\gamma}$.

We now show that $\{\mathbf{T}_{B_1}, \ldots, \mathbf{T}_{B_n}\}$ is a small-pieces decomposition of $\mathbf{T}_{\neg\gamma}$, namely that for all $i \in [\![1, k]\!]$, $\mathbf{T}_{B_i}$ is less difficult than $\mathbf{T}_{\neg\gamma}$. We directly have this property since for all $i \in [\![1, k]\!]$ we have that $\neg\gamma$ is present in the branch $\mathcal{B}_i$ so $\neg\gamma$ is in the conjunction B_i and thereby we have that $[\![B_i]\!]_s \subseteq [\![\neg\gamma]\!]_s$. This shows that $\{\mathbf{T}_{B_1}, \ldots, \mathbf{T}_{B_n}\}$ is a small-pieces decomposition of $\mathbf{T}_{\neg\gamma}$.

We conclude that the $\mathcal{T}$-decomposition of $\mathbf{T}_{\neg\gamma}$ is a small-pieces decomposition. □

It follows directly from this lemma that any *tableau decomposition* is a *small-pieces decomposition*:

Theorem 4.1. *Let $\varphi, \varphi_1, \ldots, \varphi_n \in \mathcal{I}$. We have that*

if $\{\mathbf{T}_{\varphi_1}, \ldots, \mathbf{T}_{\varphi_n}\}$ is a tableau decomposition of $\mathbf{T}_\varphi$, then $\{\mathbf{T}_{\varphi_1}, \ldots, \mathbf{T}_{\varphi_n}\}$ is a small-pieces decomposition of $\mathbf{T}_\varphi$ w.r.t. any pointed model (M, s).

Proof. The theorem follows directly from the previous lemma. □

Indeed, it turns out that a $\mathcal{T}$-decomposition is always an optimal-1 decomposition:

Lemma 4.2. *Let $\gamma \in \mathcal{I}$ and let $\mathcal{T}$ be a semantic tree with root $\neg\gamma$. Then*

the $\mathcal{T}$-decomposition of $\mathbf{T}_{\neg\gamma}$ is an optimal-1 decomposition w.r.t. any pointed model (M, s).

Proof. Let (M, s) be a probabilistic epistemic model, let $\gamma \in \mathcal{I}$ and let $\mathcal{T} = \{\mathcal{B}_1, \ldots, \mathcal{B}_n\}$ be a semantic tree with root $\neg\gamma$. Let $\{\mathbf{T}_{B_1}, \ldots, \mathbf{T}_{B_n}\}$ be the $\mathcal{T}$-decomposition of $\mathbf{T}_{\neg\gamma}$. We already know given the previous theorem that the $\mathcal{T}$-decomposition of $\mathbf{T}_{\neg\gamma}$ is a small-pieces decomposition. What we want to show now is the following equality

$$inf(\gamma) = inf(\neg B_1 \wedge \ldots \wedge \neg B_n).$$

To this end, we will show that $[\![\gamma]\!]_s = [\![\neg B_1 \wedge \ldots \wedge \neg B_n]\!]_s$. We have that:

- $[\![\gamma]\!]_s \subseteq [\![\neg B_1 \wedge \ldots \wedge \neg B_n]\!]_s$ since γ is present in all the conjunctions B_i for $i \in [\![1, n]\!]$,
- $[\![\neg B_1 \wedge \ldots \wedge \neg B_n]\!]_s \subseteq [\![\gamma]\!]_s$ since we know that if all the branches of a tree with root $\neg\gamma$ are closed at a world w, then γ is true at that world.

We conclude that the $\mathcal{T}$-decomposition of $\mathbf{T}_{\neg\gamma}$ is an optimal-1 decomposition. □

It follows directly from this lemma that any *tableau decomposition* is also an *optimal-1 decomposition*:

Theorem 4.2. *Let $\varphi, \varphi_1, \ldots, \varphi_n \in \mathcal{I}$. We have that*

if $\{\mathbf{T}_{\varphi_1}, \ldots, \mathbf{T}_{\varphi_n}\}$ is a tableau decomposition of $\mathbf{T}_\varphi$, then $\{\mathbf{T}_{\varphi_1}, \ldots, \mathbf{T}_{\varphi_n}\}$ is an optimal-1 decomposition of $\mathbf{T}_\varphi$ w.r.t. any pointed model (M, s).

Proof. The theorem follows directly from the previous lemma. □

However, a tableau decomposition is not always an optimal-2 decomposition. To see this, takes for instance the following semantic tree

$$\begin{array}{cc} \multicolumn{2}{c}{p \vee q} \\ p & q \end{array}$$

and consider an epistemic model (M, s) such that there exists w with $w \sim s$ and $M, w \models p \wedge q$. In this case, we have that

$$inf(p \vee q) < inf(p) + inf(q).$$

To sum up, we have presented in this section a way to think of inquiry decomposition in the establishing perspective through the notion of *decomposition of tasks*. We have defined the notions of *partition* and *small-pieces decomposition of tasks*, along with two proposals for defining notions of *optimal decomposition*. Then, we have seen that the tableau method can be interpreted as a decomposition mechanism for tasks. We have finally proved that the tableau method yields small-pieces decompositions and even optimal-1 decompositions, but not optimal-2 decompositions.

We shall make two important remarks at this point. The first one is that it makes more sense to think in terms of decomposition of tasks, in the establishing perspective to inquiry, rather than in terms of decomposition of questions. The reason is the focus on the notion of *proposition* rather than the notion of *question* in this perspective: the inquirer does not try to answer a question but to establish a proposition. This leads to the second remark: it seems to make more sense to think of the *tableau method* as a decomposition mechanism for tasks rather than as a decomposition mechanism for questions. Nevertheless, we will see in the next section, where we will focus on the notion of *decomposition of questions*, that the tableau method can also be interpreted as a mechanism of question decomposition.

4.2 Static decomposition of questions

In the determining perspective, the goal of the inquiring agent is, from an initial epistemic situation (M, s), to determine whether or not a certain proposition γ is the case. In other words, in such a situation, the goal of the inquiring agent is to answer the yes-no question $(\gamma, \neg\gamma)$, and more generally to answer any propositional question $Q = (\gamma_1, \dots, \gamma_k)$. Hintikka argues that, in scientific inquiry, *big* questions are decomposed into *small* or *operational* questions that are potentially answerable by Nature, i.e., questions to which the scientific can obtain an answer given her experimental technology. However, Hintikka does not propose a precise definition of what this decomposition mechanism consists in, although he seems to suggest that the tableau method can be seen as a mechanism of question decomposition.

In this section, we aim to provide a formal definition of the notion of *decomposition of questions* from a static perspective. In the two first subsections, we will define for questions, as we did for tasks, the two aspects of the notion of decomposition, *partition decomposition* and *small-pieces decomposition*, and we will see that there are several possible ways to define the latter depending on how we 'weigh' questions. In the third subsection, we will interpret the tableau method as a mechanism of question decomposition and we will show that a decomposition obtained via the tableau method is a partition decomposition, but not generally a small-pieces decomposition for the different ways we propose to weigh questions.

4.2.1 Partition decomposition

The notion of *partition decomposition* for questions aims to capture the following informal idea: if $\{Q_1, \ldots, Q_k\}$ is a decomposition of a question Q, then after having asked the questions $Q_1, \ldots, Q_k$ the agent will be in an epistemic situation in which she knows the answer to the question Q. Using the partition-based analysis of questions, we will be able to formally capture this notion of partition decomposition. To this end, we first have to define the notion of *combination* of questions:

Definition 4.11 (Combination operation)**.** *Let (M, s) be an epistemic model and let $Q_1 = (\gamma_1, \ldots, \gamma_k)$ and $Q_2 = (\chi_1, \ldots, \chi_l)$ be questions such that $M, s \models \mathsf{pre}(Q_1)$ and $M, s \models \mathsf{pre}(Q_2)$. The combined question Q resulting from the combination of the questions Q_1 and Q_2 is given, in terms of a partition on the $\sim$-equivalence class of s, by*

$$\mathsf{Q} = \mathsf{Q}_1 \sqcap \mathsf{Q}_2 = \{[\![\gamma_i]\!]_s \cap [\![\chi_j]\!]_s \mid \gamma_i \in Q_1 \textit{ and } \chi_j \in Q_2\},$$

which corresponds to the question

$$Q := (\gamma_i \wedge \chi_j \mid i \in [\![1, k]\!] \textit{ and } j \in [\![1, l]\!]).$$

Intuitively, the idea of the combination operation is to *merge* several questions into one question in such a way that asking this combined question will have the same effect than successively asking each individual question of the combination. Thus, we might want to be sure that asking successively each question in $\{Q_1, \ldots, Q_k\}$ is indeed equivalent to asking the combined question characterized by the partition $\bigsqcap_{1 \leq i \leq k} \mathsf{Q}_i$. This can be shown formally within the framework provided by our dynamic logic of questions:

Proposition 4.1. *Let (M, s) be an epistemic inquiry model, let $Q_1, \ldots, Q_k$ be questions such that $M, s \models \mathsf{pre}(Q_1) \wedge \ldots \wedge \mathsf{pre}(Q_k)$ and let Q be the combination of $Q_1, \ldots, Q_k$. Then*

$$M, s \models [Q_1?] \ldots [Q_k?]\varphi \leftrightarrow [Q?]\varphi.$$

Proof. Let $\varphi_1, \ldots, \varphi_k$ be the answers respectively to $Q_1, \ldots, Q_k$. We then have that the answer to Q is $\varphi_1 \wedge \ldots \wedge \varphi_k$. Since we only consider questions whose answers are propositional formulas, we have that $M|\varphi_1| \ldots |\varphi_k$ is identical to $M|\varphi_1 \wedge \ldots \wedge \varphi_k$. Thus, we have that:

$$M|\varphi_1| \ldots |\varphi_k, s \models \varphi \;\Leftrightarrow M|\varphi_1 \wedge \ldots \wedge \varphi_k, s \models \varphi.$$

It then follows that

$$M, s \models [Q_1?] \ldots [Q_k?]\varphi \leftrightarrow [Q?]\varphi.$$

□

We are now in a position to define the notion of *partition decomposition* for questions:

Definition 4.12 (Partition decomposition)**.** *Let (M, s) be an epistemic inquiry model and let $Q, Q_1, \ldots, Q_k$ be questions such that $M, s \models \mathsf{pre}(Q) \wedge \mathsf{pre}(Q_1) \wedge \ldots \wedge \mathsf{pre}(Q_k)$. We say that $\{Q_1, \ldots, Q_k\}$ is a partition decomposition of the question Q if and only if*

$$\bigsqcap_{1 \leq i \leq k} \mathsf{Q}_i \sqsubseteq \mathsf{Q}.$$

According to this definition, if $\{Q_1, \ldots, Q_k\}$ is a partition decomposition of Q with respect to (M, s), then by asking the combination of the questions $Q_1, \ldots, Q_k$, which is equivalent to successively ask the questions $Q_1, \ldots, Q_k$ in an arbitrary order, the agent will automatically be brought in an epistemic state in which she knows the answer to Q since she will end up in one of the cells of the partition Q. It is interesting to notice the parallel with the definition of the notion of partition decomposition for tasks: these two notions of combination and partition inclusion of questions are respectively similar to the notions of addition and inclusion of tasks, these two pairs of notions are then used to build, following the same scheme, the notions of partition decomposition respectively for tasks and questions.

We will now propose a way to define the notion of small-pieces decomposition for questions.

4.2.2 Small-pieces decomposition

In order to define the notion of *small-pieces decomposition* for questions, we need a way to 'weigh' questions, i.e., we need to provide a precise meaning for the notions of *'big'* and *'small' questions.* In the previous chapter, we have proposed several ways to attribute values to questions, which then enabled us to compare and rank the different questions that the inquiring agent can ask at a given moment. The three main notions that we have introduced are the notions of *entailment*, *informativity* and *relevance*[5]. These notions can then be used to weigh questions: we might say, for instance, that a question Q_1 is 'smaller' than a question Q_2 if and only if Q_2 entails Q_1. Indeed, any value measure for questions leads to a specific notion of small-pieces decomposition. Thus, we will first provide the definitions of the notions of small-pieces decompositions associated respectively to the notions of entailment, informativity and relevance. Then, we will provide a general definition of *measure value* for questions, that we will use to provide a general definition of small-pieces decomposition for any measure value.

The definitions of the notions of *small-pieces decomposition* respectively associated to the notions of *entailment*, *informativity* and *relevance* are the following:

Definition 4.13 (E-small-pieces decomposition)**.** *Let (M, s) be a pointed epistemic model and let $Q, Q_1, \dots, Q_k$ be questions such that $M, s \models \mathsf{pre}(Q) \wedge \mathsf{pre}(Q_1) \wedge \dots \wedge \mathsf{pre}(Q_k)$. We say that $Q_1, \dots, Q_k$ is an E-small-pieces decomposition of Q with respect to (M, s) if and only if*

- *$Q_1, \dots, Q_k$ is a partition decomposition of Q w.r.t. (M, s),*
- *for all $i \in [\![1, k]\!]$, Q entails Q_i.*

Definition 4.14 (I-small-pieces decomposition)**.** *Let (M, s) be a pointed probabilistic epistemic model and let $Q, Q_1, \dots, Q_k$ be questions such that $M, s \models \mathsf{pre}(Q) \wedge \mathsf{pre}(Q_1) \wedge \dots \wedge \mathsf{pre}(Q_k)$. We say that $Q_1, \dots, Q_k$ is an I-small-pieces decomposition of Q with respect to (M, s) if and only if*

- *$Q_1, \dots, Q_k$ is a partition decomposition of Q w.r.t. (M, s),*
- *for all $i \in [\![1, k]\!]$, $E(Q_i) \leq E(Q)$.*

Definition 4.15 (R-small-pieces decomposition)**.** *Let (M, s) be a pointed probabilistic epistemic model and let $Q, Q_1, \dots, Q_k$ be questions such that $M, s \models \mathsf{pre}(Q) \wedge \mathsf{pre}(Q_1) \wedge \dots \wedge \mathsf{pre}(Q_k)$. We say that $Q_1, \dots, Q_k$ is an R-small-pieces decomposition of Q with respect to (M, s) if and only if*

- *$Q_1, \dots, Q_k$ is a partition decomposition of Q w.r.t. (M, s),*
- *for all $i \in [\![1, k]\!]$, Q_i is less relevant w.r.t. Q.*

Each one of these definitions are built on the same scheme: we first require that the decomposition is a partition decomposition, then we require that the value of the big question is greater than the value of each question of the decomposition taken individually for the considered measure value. We now want to provide a general definition of small-pieces decomposition

[5]Since we are working in this section in the determining perspective to inquiry, we will only consider the notion of relevance for the determining perspective.

for arbitrary measure value. To this end, we first have to provide a general definition for the notion of measure value for questions.

Intuitively, a *measure value* for questions is a partial map

- which takes as input a question and an epistemic situation and yields a value for this question with respect to this specific epistemic situation,
- which allows to compare different questions with respect to the same epistemic situation.

Thus, we formally define the notion of *measure value for questions* as follows:

Definition 4.16 (Measure value for questions)**.** *Let* $\mathbf{Q}$ *be the class of all propositional questions and* $(\mathbf{M}, \mathbf{s})$ *be the class of all epistemic pointed models. A measure value* $\mathcal{V}$ *for questions is a (partial) map*

$$\mathcal{V} : (\mathbf{M}, \mathbf{s}) \times \mathbf{Q} \longrightarrow (P, \leq) \ \ \textit{where } P \textit{ is a set and } \leq \subseteq P \times P$$

such that for every $(M, s) \in (\mathbf{M}, \mathbf{s})$*,* $\mathcal{V}[(M, s), \mathbf{Q}]$ *is partially ordered by* $\leq$*.*

Notice that the three notions of entailment, informativity and relevance yield, for a given pointed probabilistic model (M, s), a partial ordering on the set of questions, and are thereby measures value for questions according to the above definition.

Now, given an arbitrary measure value $\mathcal{V}$ for questions, we can define the associated notion of *$\mathcal{V}$-small-pieces decomposition* as follows:

Definition 4.17 ($\mathcal{V}$-small-pieces decomposition)**.** *Let* (M, s) *be a pointed epistemic model, let* $\mathcal{V}$ *be a measure value for questions and let* $Q, Q_1, \ldots, Q_k$ *be questions such that* $M, s \models \mathsf{pre}(Q) \wedge \mathsf{pre}(Q_1) \wedge \ldots \wedge \mathsf{pre}(Q_k)$*. We say that* $Q_1, \ldots, Q_k$ *is a* $\mathcal{V}$*-small-pieces decomposition of* Q *with respect to* (M, s) *if and only if*

- $Q_1, \ldots, Q_k$ *is a partition decomposition of* Q *w.r.t.* (M, s)*,*
- *for all* $i \in [\![1, k]\!]$*,* $\mathcal{V}((M, s), Q_i) \leq \mathcal{V}((M, s), Q)$*.*

Of course, only specific measures of value, and thereby specific notions of small-pieces decomposition, make sense from an epistemological perspective. One interesting issue here would be to determine further properties restricting the class of admissible measures value, and thereby the associated class of small-pieces decompositions. In this work, we will stick to our general definition of measure value for questions, leaving the issue consisting in determining criteria for admissible measures value to further investigations.

We will now investigate the possibility to think of the tableau method as a decomposition mechanism for questions.

4.2.3 The tableau method as a mechanism of question decomposition

In some of his articles on the IMI, Hintikka seems to suggest that the tableau method can be seen as a technique to decompose a 'big' question into a number of 'small' ones. However, in a semantic tree, no question appears as such, only propositions are present at the nodes of the tree. This is the reason why we suggested to interpret the semantic tree as a way to decompose a big task into smaller ones, rather than as a mechanism to decompose questions. Now, is there a possible way to interpret the tableau method as a mechanism of question decomposition? One proposal consists first to associate a yes-no question to each proposition present at the different nodes of a semantic tree, and then to interpret the tree as a way to decompose a yes-no question into several ones. This idea leads to the following definition of *$\mathcal{T}$-decomposition*:

Definition 4.18 ($\mathcal{T}$-decomposition)**.** *Let $\mathcal{T}$ be a semantic tree on $\mathcal{I}$ ($\mathcal{T} \in \mathsf{Strees}(\mathcal{I})$). Let φ be the root of $\mathcal{T}$ and $\varphi_1, \ldots, \varphi_k$ be all the extremity formulas*[6] *of $\mathcal{T}$. Let $Q, Q_1, \ldots, Q_k$ be the yes-no questions associated*[7] *to $\varphi, \varphi_1, \ldots, \varphi_k$. Then, we say that $\{Q_1, \ldots, Q_k\}$ is the $\mathcal{T}$-decomposition of Q.*

We can then define the notion of *tableau decomposition* as follows:

Definition 4.19 (Tableau decomposition)**.** *Let $Q, Q_1, \ldots, Q_k$ be yes-no questions. We say that $\{Q_1, \ldots, Q_k\}$ is a tableau decomposition of Q if and only if there exists a semantic tree $\mathcal{T}$ on $\mathcal{I}$ such that $\{Q_1, \ldots, Q_k\}$ is the $\mathcal{T}$-decomposition of Q.*

It turns out that the tableau method always yields *partition decompositions*:

Theorem 4.3. *Let (M, s) be a pointed epistemic model and let $Q, Q_1, \ldots, Q_k$ be yes-no questions. We have that if $\{Q_1, \ldots, Q_k\}$ is a tableau decomposition of Q, then $\{Q_1, \ldots, Q_k\}$ is a partition decomposition of Q w.r.t. (M, s).*

Proof. Assume that $\{Q_1, \ldots, Q_k\}$ is a tableau decomposition of Q. This means that there exists a semantic tree $\mathcal{T}$ such that $\{Q_1, \ldots, Q_k\}$ is the $\mathcal{T}$-decomposition of Q. We want to show that

$$\bigsqcap_{1 \leq i \leq k} \mathsf{Q}_i \sqsubseteq \mathsf{Q}.$$

By definition, $Q, Q_1, \ldots, Q_k$ are yes-no questions where $Q = \{\gamma, \neg\gamma\}, Q_1 = \{\gamma_1, \neg\gamma_1\}, \ldots, Q_k = \{\gamma_k, \neg\gamma_k\}$, with $\gamma, \gamma_1, \ldots, \gamma_k \in \mathcal{I}$, and such that γ is the root of $\mathcal{T}$ and $\gamma_1, \ldots, \gamma_k$ are formulas at nodes of $\mathcal{T}$ to which no expanding rule has been applied. Let $q \in \bigsqcap_{1 \leq i \leq k} \mathsf{Q}_i$. We have by definition that

$$q = [\![\chi_1 \wedge \ldots \wedge \chi_k]\!]_s \text{ with } \chi_i \in Q_i \text{ for all } i \in [\![1, k]\!].$$

We want to show that either $q \subseteq [\![\gamma]\!]_s$ or $q \subseteq [\![\neg\gamma]\!]_s$. Assume first that we have $\chi = \neg\gamma_i$ for all $i \in [\![1, k]\!]$. This means that $\neg\gamma$ is the case in all the worlds in q since all the branches of the tree $\mathcal{T}$ are closed. Assume now that one of the χ_i is equal to γ_i. Then, this means that γ is the case in all the worlds in q since one branch of the tree is open. Thus, in all cases, we either have $q \subseteq [\![\gamma]\!]_s$ or $q \subseteq [\![\neg\gamma]\!]_s$.

We conclude that $\bigsqcap_{1 \leq i \leq k} \mathsf{Q}_i \sqsubseteq \mathsf{Q}$, and thereby that $\{Q_1, \ldots, Q_k\}$ is a partition decomposition of Q w.r.t. (M, s). □

Given our definition of tableau decomposition, we can provide an answer to the following question: is a tableau decomposition also an E/I/R-decomposition?

- A tableau decomposition is not necessarily an E-decomposition: consider the following example

$$\begin{array}{ccc} p \wedge q & \qquad\qquad & ((p \wedge q) \vee \neg(p \wedge q))? \\ | & & | \\ p & & (p \vee \neg p)? \\ | & & | \\ q & & (q \vee \neg q)? \end{array}$$

  so we have that $\{Q_1, Q_2\}$, where $Q_1 = (p, \neg p)$ and $Q_2 = (q, \neg q)$, is a tableau decomposition of $Q = (p \wedge q, \neg(p \wedge q))$. However, it is neither the case that Q entails Q_1, nor the case that Q entails Q_2.

[6]By *extremity formulas* of $\mathcal{T}$ we refer to the formulas associated to the nodes of $\mathcal{T}$ to which no expanding rule has been applied.

[7]The yes-no question associated to a formula $\gamma \in \mathcal{I}$ is the question $(\gamma, \neg\gamma)$.

- A tableau decomposition is not necessarily an I-decomposition: consider the same tableau decomposition as before. Consider now the probabilistic epistemic model $\langle W, \sim, V, P \rangle$ where $W = \{w_1, w_2, w_3, w_4\}$, $\sim = W \times W$, $p, q \in V(w_1)$, $p, \neg q \in V(w_2)$, $\neg p, q \in V(w_3)$, $\neg p, \neg q \in V(w_4)$ and $P(w_1) = P(w_2) = P(w_3) = P(w_4) = \frac{1}{4}$. Then we have

$$E(Q_1) = E(Q_2) = 1 \text{ but } E(Q) = \frac{1}{4} \times log_2(4) + \frac{3}{4} \times log_2\left(\frac{4}{3}\right) = 0.811 < 1.$$

  Thus, $E(Q) < E(Q_1)$ and $E(Q) < E(Q_2)$ so $\{Q_1, Q_2\}$ is not an I-decomposition of Q.

- A tableau decomposition is always an R-decomposition: this is due to the fact that, given a yes-no question $Q = (\gamma, \neg\gamma)$, we always have for any other yes-no question $Q' = (\gamma', \neg\gamma')$ that:

$$EIV_Q(Q) \geq EIV_Q(Q').$$

  To see this, notice that if γ is the answer to Q, then

$$EIV_Q(Q) = IV_Q(\gamma) = E(Q) - E_\gamma(Q) = E(Q),$$

  and

$$EIV_Q(Q') = IV_Q(\gamma') = E(Q) - E_{\gamma'}(Q) \leq E(Q).$$

  The fact that a tableau decomposition is always an R-decomposition is simply due to the fact that *any* question Q' is less relevant to Q than Q. Consequently, the fact that any tableau decomposition is an R-decomposition is not really significant since any partition decomposition of a question Q turns out to be an R-decomposition of Q.

Thus, even though the tableau method, seen as a decomposition mechanism for questions, yields partition decompositions, this method does not provide interesting small-pieces decomposition, or at least not small-pieces decompositions associated to the notions of entailment and informativity.

To sum up, we have first defined in this section the notions of partition and small-pieces decomposition for questions. Then, we have seen that the tableau method can be interpreted as a decomposition mechanism for questions which yields partition decompositions, but which does not provide E-decompositions or I-decompositions. Thus, an interesting issue here is to find mechanisms, under the form of algorithmic methods, which enable to find E-decompositions or I-decompositions for a given question.

In the next section, we move to a dynamic perspective to the decomposition of questions.

4.3 Dynamic decomposition of questions

In the static perspective to the decomposition of questions, we have assumed that the presuppositions to all the questions of a given decomposition have been previously established by the agent. However, getting the answer to a question can open the possibility for the inquiring agent to ask *new* questions. This is due to the fact that getting new information can bring the inquiring agent into an epistemic situation in which the presuppositions of new questions are then established. Thus, this aspect should be taken into account during the process of decomposing a question.

In this section, we attempt to develop a notion of *dynamic decomposition of questions* which integrates this dynamics into the process of question decomposition. We will define, in the two first subsections, the two aspects of the notion of decomposition in the dynamic perspective, i.e., the notions of *dynamic partition decomposition* and *dynamic small-pieces decomposition.* Then, we will explore, in the third subsection, some properties of the notion of dynamic decomposition of questions.

4.3.1 Dynamic partition decomposition

The main idea behind the notion of *dynamic partition decomposition* of questions is the following: given an initial epistemic situation (M, s) and a question $Q = (\gamma_1, \dots, \gamma_k)$ such that $M, s \models \mathsf{pre}(Q)$, the inquiring agent can make a *plan* of the questions that she will ask by considering, at each step, the different possible answers that she can get for a given question, and by determining the next question to be asked in each case. Thus, a dynamic partition decomposition of a question Q takes the form of a *tree* for which:

- the nodes are epistemic situations,
- the root of the tree is the agent's initial epistemic situation,
- the only expanding rule describes the effect on a particular node, i.e., on a particular epistemic situation, of getting each one of the possible answers to a question, each branch ending with the epistemic situation resulting from the incoming of information with one of the answers.

The notion of *dynamic partition decomposition* of questions can then be formally defined as follows:

Definition 4.20 (Dynamic partition decomposition)**.** *Let (M, s) be a pointed epistemic model with $M = \langle W, \sim, V\rangle$. Let $Q = (\gamma_1, \dots, \gamma_k)$ be a question such that $M, s \models \mathsf{pre}(Q)$. A dynamic partition decomposition of Q with respect to (M, s) is a tree $\mathcal{T}_Q$ built according to the following construction rules*

- *the root of the tree is $(M_0, -)$ where $M_0 = \langle W_0, V_0\rangle$ is defined as follows:*[8]

$$W_0 = [s] \text{ and } V_0 = V \upharpoonright W_0,$$

- *the only expanding rule is the following:*

  Expanding rule: *if $(M_i, -)$ is a node of the tree and $Q_i = (\chi_1, \dots, \chi_l)$ is a question*[9] *such that $M_i \models \mathsf{presup}(Q_i)$, then the node $(M_i, -)$ can be expanded as follows*

$$\begin{array}{c} (M_i, Q_i) \\ \swarrow \quad \swarrow \quad \searrow \quad \searrow \\ (M_i|\chi_1, -) \quad (M_i|\chi_2, -) \quad \dots \quad (M_i|\chi_l, -) \end{array}$$

- *for each leaf $(M_i, -)$ of the tree, either $W_i = \emptyset$ or $M_i \models \gamma_j$ for some $\gamma_j \in Q$.*

Thus, in a dynamic partition decomposition of a question Q:

- the different paths of the tree correspond to different sequences of question-answer events, all starting from the agent's initial epistemic situation,

[8] Notice that we only keep track of the informational changes on the epistemic range of the agent, not on the whole epistemic model.

[9] Notice that, in this definition, the questions Q_i do not need to be related to the principal question Q.

- each path leads either to an epistemic situation in which the agent knows the answer to the initial question Q, or to a situation in which the agent's epistemic range is empty[10].

It turns out that any static partition decomposition of a given question can be interpreted dynamically as follows:

Theorem 4.4. *Let (M, s) be a pointed epistemic model, let Q be a question such that $M, s \models \mathsf{pre}(Q)$ and let $\{Q_1, \ldots, Q_k\}$ be a static partition decomposition of Q with respect to (M, s). Then, the tree obtained by applying to all the nodes at level i the question Q_i is a dynamic partition decomposition of Q with respect to (M, s).*[11]

Proof. First of all, since $\{Q_1, \ldots, Q_k\}$ is a static partition decomposition of Q with respect to (M, s), we have that

$$M, s \models \mathsf{pre}(Q_1) \wedge \ldots \wedge \mathsf{pre}(Q_k).$$

We know that if the inquiring agent has established the presupposition of a question at a given moment, then this presupposition is still established after any acquisition of new information. Thus, the tree built from the static decomposition satisfies the construction conditions from the definition of dynamic decomposition.

Since after having asked all the questions $\{Q_1, \ldots, Q_k\}$, the agent will know the answer to the question Q, then we can easily see that the last condition in the definition of dynamic partition decomposition is satisfied as well.

We conclude that the tree obtained by applying to all the nodes at level i the question Q_i is a dynamic partition decomposition of Q with respect to (M, s). $\square$

Thus, the notion of dynamic partition decomposition of questions can be seen as a *generalization* of our previous static notion of partition decomposition.

4.3.2 Dynamic small-pieces decomposition

The main difference between the static and the dynamic setting regarding the notion of small-pieces decomposition is the following: in the static case, all the questions can be compared with respect to the same epistemic situation whereas, in the dynamic case, we take in account the dynamics of information and we are thereby dealing with several different epistemic situations. Thus, one possible way to define a notion of small-pieces decomposition in the dynamic case consists simply in comparing a 'small' question to the initial 'big' question with respect to the epistemic situation associated to the small question in the dynamic decomposition tree.

This idea, combined with a given measure value for questions $\mathcal{V}$, yields the following formal definition of the notion of *dynamic $\mathcal{V}$-small-pieces decomposition*:

Definition 4.21 (Dynamic $\mathcal{V}$-small-pieces decomposition)**.** *Let (M, s) be a pointed epistemic model, let $\mathcal{V}$ be a measure value for questions and let Q be a question such that $M, s \models \mathsf{pre}(Q)$. The tree $\mathcal{T}_Q$ is a dynamic $\mathcal{V}$-small-pieces decomposition of Q if and only if*

1. *$\mathcal{T}_Q$ is a dynamic partition decomposition of Q,*
2. *for all nodes (M_i, Q_i) of $\mathcal{T}_Q$, $\mathcal{V}((M_i, s), Q_i) \leq \mathcal{V}((M_i, s), Q)$.*[12]

[10] According to the assumptions we made, a path ending with a situation in which the agent's epistemic range is empty does not correspond to a possible inquiry scenario. This is due to the fact that we assume that (i) the actual world is in the agent's epistemic range and (ii) the oracle is truthful.

[11] The root of the tree corresponds to the level 1.

[12] If (M_i, Q_i) is a node of $\mathcal{T}_Q$ with $M_i = \langle W_i, V_i \rangle$, then (M_i, s) denotes the pointed model (M_i', s) where $M_i' = \langle W_i, \sim_i, V_i \rangle$ and $\sim_i = W_i \times W_i$.

For instance, the measures value associated respectively to the notions of entailment, informativity and relevance, yield the following notions of *dynamic E-/I-/R-small pieces decomposition*:

Definition 4.22 (Dynamic E-/I-/R-decomposition)**.** *Let (M, s) be a pointed probabilistic model and let Q be a question such that $M, s \models \mathsf{pre}(Q)$. The tree $\mathcal{T}_Q$ is a dynamic E-/I-/R-decomposition of Q if and only if*

1. *$\mathcal{T}_Q$ is a dynamic partition decomposition of Q,*
2. *for all nodes (M_i, Q_i) of $\mathcal{T}_Q$, Q_i is entailed by/less informative than/less relevant than Q w.r.t M_i.*

As in the static case, an interesting direction to pursue here would be to try to provide notions of *optimal* dynamic decompositions of questions. In particular, it would be interesting to define notions of optimality which take in account the 'weigh' of questions but also the number of questions that the inquirer needs to ask in order to reach her inquiry goal. We will not try to do so here and leave this issue for further investigations.

We now explore some properties of the notion of dynamic decomposition of questions.

4.3.3 Some properties of the notion of dynamic decomposition of questions

In this subsection, we prove some properties of the notion of dynamic decomposition of questions. First of all, we show that, if the oracle has the answers to all the questions occurring in a dynamic decomposition of a question Q, then the dynamic decomposition of Q provides the inquiring agent with a method to find the answer to Q. Then, we introduce the notion of *embedding* which aims to model the idea that a question occurring in a dynamic decomposition can in turn be decomposed, and the resulting decomposition *embedded* into the initial one. Finally, we prove some properties regarding dynamic decompositions of questions and yes-no questions.

The following theorem shows that a dynamic decomposition of a question Q provides the inquiring agent with a successful method, under some conditions on the oracle, to carry out her inquiry about Q:

Theorem 4.5. *Let $M = \langle W, \sim, V, \Phi \rangle$ be an epistemic inquiry model and let $s \in W$ be the actual world. Let $Q = (\gamma_1, \dots, \gamma_k)$ be a question such that $M, s \models \mathsf{pre}(Q)$ and let $\mathcal{T}_Q$ be a dynamic partition decomposition of Q w.r.t. (M, s). We have that:*

if the oracle has the answers to all the questions occurring in $\mathcal{T}_Q$, then by following the path in $\mathcal{T}_Q$ corresponding to the answers of the oracle the inquiring agent will end up in an epistemic situation in which she knows the answer to Q.

Proof. By following the path corresponding to the answers from the oracle, the inquiring agent will end up in an epistemic situation in which her epistemic range is the one which is at the extremity of the followed path. Since the oracle has the answers to all the questions in $\mathcal{T}_Q$ and the oracle is truthful, we have that the node $(M_i, -)$ at the extremity of the followed path is such that $W_i \neq \emptyset$ since the actual world s is in W_i (the actual world could not be eliminated by the answers of the oracle since the oracle is truthful). Thus, we get that the node M_i is such that $M_i \models \gamma_p$ for some $\gamma_p \in Q$. This means that, after having followed the path in $\mathcal{T}_Q$ corresponding to the answers of the oracle, the inquiring agent is in an epistemic situation in which γ_p is true in all the worlds of her epistemic range, i.e., the inquiring agent finally knows the answer to the principal question. □

Notice that the assumption of this theorem is stronger than necessary: it is sufficient to assume that the oracle has the answers to all the questions present in the path followed by the agent. However, since the agent cannot predict the answers from the oracle, the only way for her to evaluate the inquiry method provided by a dynamic decomposition is to check that all the questions occurring in it can be answered by the oracle. In the case of scientific inquiry, the agent knows the possibilities of her experimental technology and thereby has a way to determine the questions for which she can potentially obtain an answer. Thus, it seems that a 'realistic' dynamic decomposition is a one which does not contain questions for which the inquirer knows that she cannot obtain an answer.

We now turn to another important mechanism of scientific inquiry described by Hintikka as follows:

> [W]hat for a higher-level inquiry is an operational ("small") question can for the purposes of a lower-level inquiry be the principal question of a complex inquiry in which it is to be answered by means of a number of lower-level operational questions. [22, p. 246]

At the heart of this mechanism, there is the notion of *embedding*. The main idea behind this notion is the following: when a question Q' occurs in a dynamic decomposition of a principal question Q, the inquiring agent expects that she will obtain an answer to Q' from the oracle. However, this might not be the case and the inquiring agent might then want to decompose the question Q' into smaller ones and *embed* the resulting decomposition of Q' into the initial decomposition of Q. We now propose a formal definition of the notion of *embedding* for dynamic partition decompositions:

Definition 4.23 (Embedding)**.** *Let (M, s) be a pointed epistemic model and let $Q = (\gamma_1, \ldots, \gamma_k)$ be a question such that $M, s \models \mathsf{pre}(Q)$. Let $\mathcal{T}_Q$ be a dynamic partition decomposition of Q w.r.t. (M, s), let (M_i, Q_i) be a node of $\mathcal{T}_Q$ ($Q_i = (\chi_1, \ldots, \chi_l)$) and let $\mathcal{T}_{Q_i}$ be a dynamic partition decomposition of Q_i w.r.t. M_i. Then, the tree $\mathcal{T}'_Q$ resulting from the embedding of $\mathcal{T}_{Q_i}$ in $\mathcal{T}_Q$ is obtained by*

1. *expanding each leaf $(M_j, -)$ of $\mathcal{T}_{Q_i}$ following the questions from the subtree of $\mathcal{T}_Q$ with root $(M_i|\chi_p, -)$ where χ_p is the only formula of Q_i such that $M_j \models \chi_p$,*
2. *replacing the subtree of $\mathcal{T}_Q$ with root (M_i, Q_i) by the expended tree obtained from $\mathcal{T}_{Q_i}$ in 1.*

We shall now show that this definition of embedding is licit and yields a tree which is indeed a dynamic partition decomposition:

Theorem 4.6. *Let (M, s) be a pointed epistemic model and let $Q = (\gamma_1, \ldots, \gamma_k)$ be a question such that $M, s \models \mathsf{pre}(Q)$. If*

- *$\mathcal{T}_Q$ is a dynamic partition decomposition of Q w.r.t. (M, s),*
- *(M_i, Q_i) is a node of $\mathcal{T}_Q$ ($Q_i = (\chi_1, \ldots, \chi_l)$),*
- *$\mathcal{T}_{Q_i}$ is a dynamic partition decomposition of Q_i w.r.t. M_i,*

then the tree $\mathcal{T}'_Q$ resulting from the embedding of $\mathcal{T}_{Q_i}$ in $\mathcal{T}_Q$ is a dynamic partition decomposition of Q.

Proof. To prove this, we have to show two things: one is to check that the tree is constructed according to the construction rules of dynamic partition decomposition trees, the other is to show that for every leaf $(M_p, -)$ of the tree $\mathcal{T}'_Q$, either $W_p = \emptyset$ or $M_p \models \gamma_r$ for some $\gamma_r \in Q$.

Since $\mathcal{T}_{Q_i}$ is a dynamic partition decomposition of Q, we have that for each leaf $(M_j, -)$ of $\mathcal{T}_{Q_i}$, either $W_j = \emptyset$ or $M_j \models \chi_r$ for some $\chi_r \in Q_i$. In the first case, expanding the tree via some questions always yields an empty model, in the second case $M_j \models \chi_r$ for some $\chi_r \in Q_i$ so M_j is a submodel of $M_i|\chi_r$ and thereby every formula that is valid in $M_i|\chi_r$ is also valid in M_j, so in particular the presupposition of the questions that are used to expand the tree $\mathcal{T}_{Q_i}$. This argument can be repeated for all the applications of the expanding rule in the construction of the expansion of the tree $\mathcal{T}_{Q_i}$.

Regarding the second point, we only have to show that every leaf $(M_p, -)$ of the tree $\mathcal{T}_{Q_i}$ are such that either $W_p = \emptyset$ or $M_p \models \gamma_r$ for some $\gamma_r \in Q$. The reason why this is the case is that every node $(M_t, -)$ of the expanded part of the tree $\mathcal{T}_{Q_i}$ are such that M_t is a submodel of the corresponding node in $\mathcal{T}_Q$. This is true in particular for the leaves, and since $\mathcal{T}_Q$ is a dynamic partition decomposition tree, the property that we want to prove for $\mathcal{T}'_Q$ is true for $\mathcal{T}_Q$ and thereby transfers to $\mathcal{T}'_Q$. □

We are touching an important issue here which concerns the modification of an inquiry plan, or dynamic decomposition, during an inquiry. This issue has been investigated by Olsson and Westlund [28] in the AGM-paradigm of belief revision theory[13]. According to their perspective, a research agenda is a set of questions entertained by the inquirer. Then, their objective is to formalize how a research agenda is modified after an incoming of information. Thus, the bigger issue here is the one of the *dynamics* of *research agendas* or *inquiry plans*. Notice that the notion of embedding is already a step toward an analysis of such a dynamics since it explains how an inquiry plan can be changed when the inquirer wants to decompose one of the questions present in a dynamic decomposition into smaller ones.

The two last properties that we will prove concern the relation between dynamic decompositions of questions and *yes-no questions*. From an epistemological perspective, yes-no questions are often considered to be the 'simplest' questions and deserve thereby special attention. The following theorem says that there always exists, for a given question Q, a dynamic partition decomposition which only contains yes-no questions about immediate subformulas of the answers to Q:

Theorem 4.7. *Let (M, s) be an epistemic model and let $Q = (\gamma_1, \ldots, \gamma_k)$ be a question such that $M, s \models \mathsf{pre}(Q)$ and each answer to Q is a compound[14] formula. Then, there exists a dynamic partition decomposition of Q w.r.t. to (M, s) in which only occur yes-no questions about immediate subformulas of the answers to Q.*

Proof. First of all, we have that the set of questions $\{Q_1, \ldots, Q_k\}$, where $Q_i = \{\gamma_i, \neg\gamma_i\}$ for all $i \in [\![1, k]\!]$, constitutes a static partition decomposition of the question $Q = (\gamma_1, \ldots, \gamma_k)$. Then, since each answer γ_i to Q is a compound formula, we have that for each answer γ_i there exists a semantic tree with root γ_i of length greater or equal than 1, i.e., which has not for only node the root. Let $\mathcal{T}_1, \ldots, \mathcal{T}_k$ be semantic trees of length ≥ 1 with root respectively $\gamma_1, \ldots, \gamma_k$.

If we now take the union of all the $\mathcal{T}_i$-decomposition of the questions Q_i for $i \in [\![1, k]\!]$, we get a static partition decomposition of Q in which only occur yes-no questions about immediate subformulas of the answers to Q.

We have seen previously how to transform a static partition decomposition into a dynamic one. By transforming the static partition decomposition we have just obtained into a dynamic one, we end up with a dynamic partition decomposition of Q w.r.t. to (M, s) in which only occur yes-no questions about immediate subformulas of the answers to Q. □

[13]See [1] and [9] for a presentation of belief revision theory in the AGM-paradigm.

[14]By a *compound formula* we mean a propositional formula which is not atomic.

The following theorem says that there always exists for a given question Q a dynamic partition decomposition which only contains yes-no questions based on atomic propositions occurring in the answers to Q:

Theorem 4.8. *Let (M, s) be an epistemic model and let $Q = (\gamma_1, \ldots, \gamma_k)$ be a question such that $M, s \models \mathsf{pre}(Q)$. Then, there exists a dynamic partition decomposition of Q in which only occur yes-no questions based on atomic propositions occurring in the answers to Q.*

Proof. The proof is identical to the one of the previous theorem, except that we consider the completely developed semantic trees $\mathcal{T}_1, \ldots, \mathcal{T}_k$ with root respectively $\gamma_1, \ldots, \gamma_k$, i.e., the semantic trees to which no more expanding rules can be applied. In such semantic trees, all the leaves are then atoms occurring respectively in the formulas $\gamma_1, \ldots, \gamma_k$. □

In this section, we have defined a notion of dynamic decomposition of questions and we have investigated some of its properties. It turns out that this notion has very closed similarities with the notion of *erotetic search scenario* developed by Andrzej Wiśniewski. In the next section, we present the main points of Wiśniewski's work on erotetic search scenarios in order to make a comparison with the notion of dynamic decomposition of questions.

4.4 Erotetic search scenarios

In his 2003 paper entitled 'Erotetic search scenarios' [44], Andrzej Wiśniewski investigates the mechanism of question decomposition, consisting in answering a principal question by asking small or operational questions, from the point of view of *erotetic logic*. To this end, he introduces, develops and formalizes the concept of *erotetic search scenario*. This concept constitutes thereby another way to precisely define the idea of decomposition of questions.

In this section, we will present the notion of *erotetic search scenario* in order to be able to compare it, in the next section, to the notion of dynamic decomposition of questions. We will proceed as follows: first we will introduce the notion of erotetic search scenario informally; then we will present the formal definition proposed by Wiśniewski within the framework of *erotetic logic*; finally we will report some properties of erotetic search scenarios.

4.4.1 Informal presentation of the concept of erotetic search scenario

In [44], Wiśniewski introduces the concept of erotetic search scenario as follows:

> In this paper, we shall introduce the concept of an *erotetic search scenario*. Roughly, a scenario of this kind shows how an initial question can be answered on the basis of a given set of initial premises and by means of asking and answering auxiliary questions. [...] [W]e use here some tools borrowed from a logic of questions which allows questions to be premises and conclusions of inferences. [44, p. 391]

The best way to understand what Wiśniewski means by erotetic search scenario is to present an example. In his paper, Wiśniewski considers the following situation: a detective is trying to keep track of a certain Andrew W. and looks for an answer to the question:

- Where did Andrew W. leave for: Paris, London, Kiev, or Moscow?

At his disposal, the detective has the following premises:

- Andrew W. left for Paris or London if and only if he departed in the morning,
- Andrew W. left for Kiev or Moscow if and only if he departed in the evening,

- If Andrew W. took a train, then he did not leave for London or Moscow,
- If Andrew W. left for Paris or Kiev, then he took a train.

In such a situation, the detective cannot deduce from the premises the answer to the initial question. What the detective has to do is to develop a plan of auxiliary questions that will lead to an answer to the initial question. Here is an example of such a plan, proposed by Wiśniewski, in which the premises are italicized and the answers to the small questions are written in boldface:

Where did Andrew W. leave for: Paris, London, Kiev, or Moscow?
Andrew W. left for Paris or London if and only if he departed in the morning.
Andrew W. left for Kiev or Moscow if and only if he departed in the evening.
If Andrew W. took a train, then he did not leave for London or Moscow.
If Andrew W. left for Paris or Kiev, then he took a train.
When did Andrew W. depart: in the morning, or in the evening?

Andrew W. departed in the morning.
Andrew W. left for Paris or London.
If Andrew W. took a train,
then he did not leave for London.
If Andrew W. did not take a train,
then he did not leave for Paris.
Did Andrew W. take a train?

Yes.
A. W. did not leave for L.
A. W. left for P.

No.
A. W. did not leave for P.
A.W. left for L.

Andrew W. departed in the evening.
Andrew W. left for Kiev or Moscow.
If Andrew W. took a train,
then he did not leave for Moscow.
If Andrew W. did not take a train,
then he did not leave for Kiev.
Did Andrew W. take a train?

Yes.
A. W. did not leave for M.
A. W. left for K.

No.
A. W. did not leave for K.
A. W. left for M.

The above tree contains four branches, corresponding to four different 'stories', which constitute together an example of an *erotetic search scenario* for the main question.

The branches of an erotetic search scenario will be called *paths* of the scenario and satisfy, according to Wiśniewski, the following conditions:

- they begin with the principal question and end with a direct answer to it,
- any inferential step involved is:
  - a standard deductive step, in which premise(s) and the conclusion are declarative sentences, or
  - an erotetic step, in which one premise and the conclusion are questions,
- any declarative sentence which occurs on the path is:
  - an initial premise, or
  - a direct answer to a question (different from the initial one) which immediately precedes this sentence on the path, or
  - is entailed by some declarative sentence(s) which occur(s) earlier on the path.

An erotetic search scenario as a whole, namely all the paths taken together, satisfies the following conditions:

- no direct answer to the principal question belongs to the set of initial premises,
- if an auxiliary question is asked and then answered in one way on a given path, then the scenario contains path(s) on which this question is answered in all the other possible ways; these paths are identical up to the point at which the auxiliary question is asked, but start to differ at the level of answers to the auxiliary question.

The particularity of the notion of path lies in the fact that it involves another kind of inferential steps, in addition to standard deductive steps, called *erotetic steps.* Erotetic steps are inferential steps in which both the premise(s) and the conclusion are questions. They have been studied by Wiśniewski in the framework of *erotetic logic* through the so-called notion of *erotetic implication* which is considered as an explication of the concept "a question Q_1 arises from a question Q on the basis of a set of declarative sentences" [43, chapter 1 and 7]. Conceptually, the notion of erotetic implication is characterized by the two following conditions (here Q_1 denotes the conclusion and Q the premise):[15]

"open minded" cognitive usefulness: each direct answer to Q_1 together with the declarative premises entail some direct or partial answer to the question Q which is the premise. So *each* direct answer to Q_1 is potentially useful.

transmission of soundness: Q_1 must have a true direct answer if the question Q, which is the premise, has a true direct answer and all the declarative premises are true.

We will now present how Wiśniewski formalizes the notions of *erotetic implication* and *erotetic search scenario* in the framework of erotetic logic.

4.4.2 Formal definition of erotetic search scenarios

In [44], Wiśniewski proposes a formal definition of the notion of erotetic search scenario. Before stating this definition, we first have to define the bases of the formal framework used by Wiśniewski along with some preliminaries notions.

In [44], Wiśniewski works with the propositional language $\mathcal{L}$. The well-formed formulas, defined as usual, are called *declarative well-formed formulas* (d-wffs for short). A *question* of $\mathcal{L}$ is an expression of the form $?\{A_1, \dots, A_n\}$ where $n > 1$ and $A_1, \dots, A_n$ are syntactically distinct d-wffs. In this case, $A_1, \dots, A_n$ are called the *direct answers* to the question $?\{A_1, \dots, A_n\}$ and such a question is read as *"Is it the case that A_1, or is it the case that $A_2, \dots,$ or is it the case that A_n?"*. The symbols $Q, Q_1, \dots$ are used for metavariables for questions, and dQ denotes the set of direct answers to a question Q. The declarative well-formed formulas and the questions of $\mathcal{L}$ constitutes the *well-formed formulas* of $\mathcal{L}$, the greek letters φ, ψ, γ are used as metavariables for them.

As we have seen in the previous section, the notion of *erotetic implication* is one of the main components of the notion of erotetic search scenario. In order to define the notion of erotetic implication, we need to define the notions of *entailment* and *multiple-conclusion entailment*:

Definition 4.24 (Entailment)**.** *A set of d-wffs X entails a d-wff A if and only if A is true under each valuation in which all the d-wffs in X are true.*

[15] See [42] for detailed on the notion of erotetic implication and [43] for a presentation of the general program of *erotetic logic.*

Definition 4.25 (Multiple-conclusion entailment)**.** *A set of d-wffs X multiple-conclusion entails (mc-entails for short) a set of d-wffs Y if and only if for each valuation v: if all the d-wffs in X are true under v, then at least one d-wff in Y is true under v.*

Using this notion, Wiśniewski proposes the following formal definition aiming to capture the conceptual features of the notion of *erotetic implication*:

Definition 4.26 (Erotetic implication)**.** *A question Q implies a question Q_1 on the basis of a set of d-wffs X (in symbols: $\mathbf{Im}(Q, X, Q_1)$) if and only if*

1. *for each $A \in dQ$: $X \cup \{A\}$ mc-entails dQ_1, and*
2. *for each $B \in dQ_1$ there exists a non-empty proper subset Y of dQ such that $X \cup \{B\}$ mc-entails Y.*

If $X = \emptyset$, then we say that Q implies Q_1 and we write $\mathbf{Im}(Q, Q_1)$.

The clause 1. assures that the condition of 'transmission of soundness' is satisfied. The clause 2. aims to capture the condition of '"open minded" cognitive usefulness', and says that "each direct answer to an implied question Q_1 *narrows down* together with X the class of possibilities offered by the (set of direct answers to) implying question Q" [44, p. 401].

At this point, Wiśniewski can define the notion of *erotetic derivation* which aims to capture the notion of path of an erotetic scenario, which in turn will be used to define the notion of erotetic search scenario:

Definition 4.27 (Erotetic derivation)**.** *A finite sequence $\mathbf{e} = \varphi_1, \ldots, \varphi_n$ of wffs is an erotetic derivation of a direct answer A to a question Q on the basis of a set of d-wffs X if and only if $\varphi_1 = Q, \varphi_n = A$ and the following conditions hold:*

1. *for each question φ_k of $\mathbf{e}$ such that $k > 1$:*
   (a) *$d\varphi_k \neq dQ$, and*
   (b) *φ_{k+1} is either a question or a direct answer to φ_k;*
2. *for each d-wff φ_j of $\mathbf{e}$:*
   (a) *$\varphi_j \in X$, or*
   (b) *φ_j is a direct answer to φ_{j-1}, where $\varphi_{j-1} \neq Q$, or*
   (c) *φ_j is entailed by a certain set of d-wffs such that each element of this set precedes φ_j in $\mathbf{e}$;*
3. *for each question φ_k of $\mathbf{e}$ such that $\varphi_k \neq Q$: φ_k is implied by a certain question φ_j which precedes φ_k in $\mathbf{e}$ on the basis of the empty set, or on the basis of a set of d-wffs such that each element of this set precedes φ_k in $\mathbf{e}$.*

Finally, the notion of *erotetic search scenario* (*e-scenario* for short) is defined as follows:

Definition 4.28 (Erotetic search scenario)**.** *A finite family Φ of sequences of wffs is an erotetic search scenario for a question Q relative to a set of d-wffs X if and only if each element of Φ is an e-derivation of a direct answer to Q on the basis of X and the following condition hold:*

1. *$dQ \cap X = \emptyset$;*
2. *Φ contains at least two elements;*
3. *for each element $\mathbf{e} = \varphi_1, \ldots, \varphi_n$ of Φ, for each index k such that $1 \leq k < n$:*

(a) if φ_k is a question and φ_{k+1} is a direct answer to φ_k, then for each direct answer B to φ_k, the family Φ contains a certain e-derivation $\mathbf{e}' = \psi_1, \psi_2, \ldots, \psi_m$ such that $\psi_j = \varphi_j$ for $j = 1, \ldots, k$, and $\psi_{k+1} = B$;

(b) if φ_k is a d-wff, or φ_k is a question and φ_{k+1} is not a direct answer to φ_k, then for each e-derivation $\mathbf{e}' = \psi_1, \psi_2, \ldots, \psi_m$ in Φ such that $\psi_j = \varphi_j$ for $j = 1, \ldots, k$ we have $\psi_{k+1} = \varphi_{k+1}$.

Wiśniewski makes use of the following terminology: elements of an erotetic search scenario Φ are called *paths* of Φ, the question Q is called the *principal question* of Φ, and the elements of the set X are called *initial premises*.

In order to illustrate the formal definition of e-scenarios, we will provide the formalization of the e-scenario associated to the detective example. To this end, we first define the following atoms:

- p denotes the proposition 'A.W. left for Paris',
- q denotes the proposition 'A.W. left for London',
- r denotes the proposition 'A.W. left for Kiev',
- s denotes the proposition 'A.W. left for Moscow',
- t denotes the proposition 'A.W. departed in the morning',
- u denotes the proposition 'A.W. departed in the evening',
- w denotes the proposition 'A.W. took the train'.

Then, the formalization of the erotetic search scenario from the detective example is the following:

$$\begin{array}{c}
?\{p, q, r, s\} \\
p \vee q \leftrightarrow t \\
r \vee s \leftrightarrow u \\
w \to \neg(q \vee s) \\
p \vee r \to w \\
?\{t, u\}
\end{array}$$

$$\begin{array}{cc}
\begin{array}{c}
t \\
p \vee q \\
w \to \neg q \\
\neg w \to \neg p \\
?w \\
\begin{array}{cc}
\begin{array}{c} w \\ \neg q \\ p \end{array} & \begin{array}{c} \neg w \\ \neg p \\ q \end{array}
\end{array}
\end{array}
&
\begin{array}{c}
u \\
r \vee s \\
w \to \neg s \\
\neg w \to \neg r \\
?w \\
\begin{array}{cc}
\begin{array}{c} w \\ \neg s \\ r \end{array} & \begin{array}{c} \neg w \\ \neg r \\ s \end{array}
\end{array}
\end{array}
\end{array}$$

One can check that this tree (i) corresponds to the e-scenario from the informal detective example and (ii) satisfies the formal definition of e-scenario.

We will now report some properties of e-scenarios proved by Wiśniewski in [44].

4.4.3 Some properties of the concept of erotetic search scenario

The following result is considered by Wiśniewski to be the basic property of erotetic search scenarios:

Theorem 4.9 (Golden path theorem)**.** *Let Φ be an e-scenario for a question Q relative to a set of d-wffs X. Let v be a valuation such that at least one direct answer to Q is true under v. Then, the scenario Φ contains at least one path $\mathbf{e}$ such that:*

1. *each d-wff of* $\mathbf{e}$ *is true under v, and*
2. *each question of* $\mathbf{e}$ *has at least one true direct answer under v, and*
3. $\mathbf{e}$ *leads to a direct answer to Q which is true under v.*

Proof. See [44]. □

In words, this theorem says that an e-scenario, under the condition that the principal question has at least one true direct answer and that all the premises are true, contains at least one *'golden path'*, i.e., a path which leads to a direct answer to the principal question and which contains only true d-wff and questions with at least one true direct answer.

Among the e-scenarios, some subfamilies have remarkable properties. The two following ones are mentioned by Wiśniewski:

Information-seeking e-scenarios: these e-scenarios are the ones in which no direct answer to Q is entailed by X,

Complete e-scenarios: these e-scenarios are the ones in which each direct answer to the principal question is the endpoint of some paths.

One characteristic of complete e-scenarios is the possibility to embed one complete e-scenario into another. This means that if one replaces a question Q, occurring in a complete e-scenario in which Q is immediately followed by an answer, by a complete e-scenario with Q for principal question, then the result of this operation is still a complete e-scenario.[16]

We now present the two last main theorems presented and proved in [44] which are existence theorems of particular e-scenarios involving *yes-no questions*. To this end, we first need to provide the formal definitions of the notions involved in these theorems.

The first one is the definition of *yes-no questions*: in Wiśniewski's framework, a *yes-no question* is a question of the form $?\{A, \neg A\}$ where the declarative well-formed formulas A and $\neg A$ are respectively called the affirmative and the negative answers. Then, a yes-no question is said to be *based on a d-wff A* when its affirmative answer is A. Finally, a *query* is a question in an erotetic search scenario which is immediately followed by an answer. We now have all the ingredients to state the following theorem:

Theorem 4.10. *Let $Q = ?\{A_1, A_2, \ldots, A_n\}$ and let $X = \{A_1 \vee A_2 \vee \ldots \vee A_n\}$. Assume that each direct answer to Q is a compound d-wff. There exists a complete e-scenario for Q relative to X such that each query of this scenario is a simple yes-no question based on an immediate subformula of a direct answer to Q.*

Proof. See [44]. □

[16]This result is formally proved by Wiśniewski in [44, p. 414].

The content of this theorem can be simply stated: it says that, under the conditions of the theorem, we can always find an e-scenario for a question Q which only contains yes-no questions about subformulas of the direct answers to Q.

Finally, the last theorem is a result regarding *atomic yes-no questions*, which are questions whose sets of direct answers are constituted of an atomic proposition and its negation:

Theorem 4.11. *If Q is not an atomic yes-no question, then there exists a complete e-scenario for Q relative to a disjunction of all the direct answers to Q such that each query of this scenario is an atomic yes-no question based*[17] *on an atomic proposition that occurs in Q.*

Proof. See [44]. □

We will now compare the notion of erotetic search scenario with the notion of dynamic decomposition of questions as possible formalizations of the idea of interrogative inquiry decomposition.

4.5 Comparing dynamic decompositions of questions and erotetic search scenarios

In this section, we aim to compare the two notions of *dynamic decomposition of questions* and *erotetic search scenario*. Since both notions are presented under the form of *trees*, one way to compare them is to look at the way these trees are constructed. Thus, we can look how the *nodes*, the *construction rules* and the *extremity conditions* are defined for the trees representing dynamic decompositions of questions and erotetic search scenarios. This leads to the following comparison table:

- *Nodes:*

  DDQ: The nodes are *epistemic ranges*,

  ESS: The nodes are *propositions* or *questions*.

- *Construction rules:*

  DDQ: The only construction rule takes as input a question and a node, and constructs as many branches as possible answers to the question, each branch ending with the epistemic range resulting from the incoming of information of one of the answers on the epistemic range associated to the node.

  ESS: There are two different construction rules:

  1. One is a non-branching rule and consists in adding in a branch either a formula or a question respectively entailed by a set of preceding formulas in the branch or by a preceding question and a set of preceding formulas.
  2. The other is a branching rule which can be applied to a question and which constructs as many branches as possible answers to the question, each branch ending with an answer to the question.

- *Extremity conditions:*

  DDQ: The extremity condition requires that each branch ends with either an empty epistemic range, or with an epistemic range in which the agent knows the answer to the 'big' question (the initial question goal of the inquiry).

[17] An atomic yes-no question is said to be *based on an atomic proposition p* if p is a direct to the question.

ESS: The extremity condition requires that each branch ends with an answer to the 'big' question (the initial question goal of the inquiry).

From this table, one can see that the notions of *dynamic decomposition of questions* and *erotetic search scenario* have very strong similarities. One may say that the main difference between the two notions concerns their representational means: DDQ adopts a *semantic* approach to represent knowledge via the notion of epistemic range, whereas EES uses a *syntactic* approach via a language formed of propositions and questions. Then, as the above comparison shows, these two notions are constructed on the same scheme: the construction rules and the extremity conditions for the two notions are similar, they are just adapted to the representational mean adopted by each notion.

Nevertheless, the notions of DDQ and EES differ with respect to several important points.

The first one is the presence of *inferential steps* in EES, whereas in DDQ we only consider *pure information-seeking inquiries*, i.e., inquiries only composed of sequences of questions. Adding a representation of inferential steps in DDQ does not seem too difficult: this would simply amounts to provide a notion of dynamic decomposition of questions for non-logically omniscient agents. In this case, we would need, in particular, to represent the inferential steps leading to the establishment of presuppositions of questions. However, developing a notion of DDQ in the non-logical omniscience case will open a lots of issues regarding the notion of small-pieces decomposition. For instance, we will have to provide a measure of epistemic value for inferential steps, and more difficult, a value measure able to attribute respective value for questions and inferences, i.e., a value measure allowing to compare *deductive* and *interrogative steps*.

The second point concerns the distinction between what we have called partition decomposition and small-pieces decomposition, and more generally between the notions of *admissible* and *optimal* inquiry plans. In the notion of dynamic decomposition of questions, we started with a general notion of *admissible* decomposition through our definition of dynamic partition decomposition. Then, we have seen how we can define a notion of dynamic small-pieces decomposition. One may go further by trying to define notions of *efficient* or *optimal* dynamic decomposition. In the case of erotetic search scenario, there is no distinction between admissible and efficient e-scenarios. Indeed, the very notion of erotetic implication at the heart of e-scenario is considered by Wiśniewski to be a way to determine the 'good' questions to ask:

> A good operative question has useful answers, and since it cannot be said in advance which of the possible answers will appear to be acceptable, *each possible answer should be potentially useful* [...]. One may argue that there are some patterns which underlie the process of asking "good" operative questions in the above sense, and that these patterns are due, at least partially, to the underlying logic of questions. Such patterns are not uncovered by IMI in its present form, however. [44, p. 391]

We think it is better to separate the two issues of admissible and efficient inquiry plans. In this way we can compare different possibilities to choose questions for an inquiry plan. Then, the notion of erotetic implication appears as just one possible way among others to guide the inquirer in the construction of her inquiry plan in order to find the 'good' questions.

The last point concerns the absence of the notion of *presupposition* in erotetic search scenarios. The reason might be connected with our previous remark since the role of the notion of *presupposition* is precisely to determine the admissible questions that can be meaningfully asked by the inquiring agent. Since Wiśniewski skips a definition of admissible e-scenario and directly works with 'efficient' e-scenarios based on the use of erotetic implication, we might have here the reason why the notion of presupposition does not appear as such in Wiśniewski's presentation.

Conclusion

In this chapter, we have investigated the possibility to capture formally the idea that an interrogative inquiry can be *decomposed* into several small steps. To this end, we have proposed several ways to capture the notion of an *inquiry plan*, given a certain epistemic goal and certain means to potentially reach it, through the notions of *decomposition of tasks*, *static decomposition of questions* and *dynamic decomposition of questions*. Then, in the two last sections, we have first introduced the notion of *erotetic search scenario* developed by Wiśniewski which aims to capture the idea of question decomposition in the framework of erotetic logic, and we have compared it to the notion of dynamic decomposition of questions. In this conclusion, we propose several directions to pursue further the investigations presented in this chapter.

Inquiry plans with questions and inferences. In this chapter, we worked in the case of *pure information-seeking inquiry*, namely inquiries constituted of sequences of questions, with a logically omniscient inquiring agent. As a consequence, inferential steps do not appear in our different definitions of decomposition, contrary to erotetic search scenarios. Thus, it might be interesting to provide a definition of inquiry plan in the case of non-logical omniscience in which both *interrogative* and *deductive steps* are represented. It should not be too difficult to do so since, for instance, we can easily extend our notion of dynamic decomposition of questions by introducing inferential steps. However, this would open new issues regarding the properties of such inquiry plans. In particular, studying notions of *optimality* in this case would be quiet subtle, since one would need to compare and to choose, in a given situation, between making *interrogative* and *deductive steps*.

Dynamics of inquiry plans. In the notion of dynamic decomposition of questions, we have integrated a certain dynamics of information. More precisely, we have taken into account the possible incoming of information, through obtention of answers to questions, in the planning of the next questions to ask. The main consequence of doing so is the following: it extends the scope of the possible questions that the agent can ask, due to the fact that getting new information leads to the establishment of new presuppositions of questions. However, in addition of this, inquiry plans have also a dynamics of their own. For instance, consider a situation where the inquirer follows an inquiry plan and asks a question for which the oracle does not have the answer. Then, the inquirer must find a way to change her plan and to construct a new one. One possibility that we have already seen is to *decompose* again the 'non-answerable' question and to *embed* the resulting decomposition into the initial inquiry plan. However, there might also be other solutions to explore, such as asking annex questions. Another dynamic aspect of inquiry plans concerns the situations in which the agent obtains some information externally and changes her inquiry plan in consequence. This is very custom in scientific practice, in particular when an important discovery by a member of a field leads to a revision of the inquiry plans or research agendas of the other members. Thus, the dynamics of inquiry plans is something very complex and important which deserves to be investigated.

Optimal inquiry plans. An important issue that emerged from our investigation is the study of notions of *optimality* for inquiry plans. Studying *optimal* inquiry plans is the straightforward continuity of the work done in this chapter: what we have done is to define *admissible* inquiry plans, i.e., inquiry plans (i) which contain only admissible questions (questions whose presuppositions have been established), (ii) which in principle enable to reach the inquiry goals and (iii) which are constituted of 'small' steps. Notice here the strong parallel between *admissible* and *optimal inquiry plans* on one hand, and *definitory* and *strategic rules* of inquiry on the other hand: an optimal inquiry plan is a one which in principle leads to 'good' performance of the inquirer. In order to investigate optimal inquiry plans, on might introduce notions of *costs* or *available resources*. For instance, one might be interested into inquiry plans with a good trade-off between *time* and *cost of questions*, i.e., the cost of acquiring information. It might also be important, as we mentioned it, to compare the costs of questions with the costs of inferences.[18] The issue of designing optimal inquiry plans is very important in practice, in particular for research policy makers in order to make optimal use of the available resources. Thus, it really deserves to be investigated, both from a conceptual and a formal perspectives.

Social inquiry plans. There is also a strong *social* dimension of inquiry plans. This is due to the fact that, in scientific inquiry and inquiry in general, there is a repartition of tasks depending on the *skills* of the agents. For instance, in physics, some people are working in *theoretical physics* and others in *experimental physics*. From the point of view of the IMI, one can roughly says that theoretical physicists are dealing with deductive steps whereas experimental physicists are dealing with interrogative steps. Thus, the physicists as a group conduct inquiry by dividing tasks depending on the skills of the members of the group. Thus, it seems that investigating *social inquiry plans* requires to introduce *skillful* agents in order to describe how an inquiry can be decomposed into tasks requiring specific skills, and thereby specific agents. Investigating the social dimension of inquiry plans constitutes thereby an interesting direction for further research.

[18]There is an interesting illustration of this aspect in current *mathematical engineering* and in particular in *numerical simulation*. Today, engineers have the choice between conducting real experiments or using numerical simulation tools, for instance in the design of airplanes. This choice, from the perspective of the IMI, is really a choice between interrogative and deductive steps.

Chapter 5

Conclusion

The aim of the present thesis was to explore possible interactions between the *interrogative model of inquiry* and *dynamic epistemic logics*. Within this general project, we have chosen to investigate three main axes:

- The first one was to develop a formalization of the IMI under the form of a *dynamic logic of questions and inferences*. The main idea was to represent questions and inferences as *actions* modifying the informational state of the agent, and to investigate the intricate relation between information coming from questions and information coming from inferences in the inquiry process.
- The second one was to argue that certain works from the formal study of the semantics and pragmatics of questions and answers are relevant for an investigation of the inquiry process within the IMI. More precisely, we have shown how to import the works of Groenendijk, Stokhof and van Rooij, in the framework constituted by our dynamic logic of questions. We have also argued that the approach consisting in providing *measures of epistemic value* for questions constitutes a first step toward an analysis of the *strategic aspects* of inquiry, even though, in order to be complete, such an analysis would require to find a way to measure performance of the inquirer in the inquiry process.
- The third one was to investigate the notion of *inquiry plan*, or *interrogative inquiry decomposition*, within our formalization of the IMI. To this end, we have proposed several definitions of the notions of *decomposition of tasks* and *decomposition of questions*. The notion of inquiry plan is a very rich notion, with multiple aspects to be investigated. Thus, we have tried in this chapter to structure our investigation by separating different issues through a hierarchy between the notions of *admissible*, *small-pieces* and *optimal inquiry decompositions*. We have also presented the work of Wiśniewski on *erotetic search scenarios* and compare his approach with the one that we have proposed.

We hope, with this thesis, to have convinced the reader that the interrogative model of inquiry and dynamic epistemic logics broadly conceived can fruitfully interact, opening thereby new interesting lines of research.

In the conclusions of each one these chapters, we have proposed possible directions to extend the works that have been presented in the thesis. We will not repeat this here. Rather, we will end with some general remarks on the logical approach to inquiry.

Inquiry, either in science or in other contexts, is often a very complex process with multiple dimensions. Thus, it seems to us that a conceptual analysis of the inquiry process, from an epistemological or philosophy of science perspective, is necessary to better understand the underlying logic and rational thinking that govern information-seeking processes. For instance, it would be very useful to adopt such a perspective in order to provide a *taxonomy* of the different kinds of inquiries and inquiry goals, to investigate the different notions of information involved in inquiry processes along with the different ways to acquire information, and to study notions of informativity and relevance from the perspective of scientific practice. Maybe one of the most important task to carry out from a philosophical point of view in order to progress in the logical modeling of the inquiry process is to try to *break off* the notion of inquiry and to identify its different components. Then, one can try to provide a conceptual framework which accounts for these different aspects and the ways they interact with each other in the inquiry process. The interrogative model of inquiry constitutes already such a framework and could be extended along these lines. Such a decomposition of the notion of inquiry into small components is necessary in order to fruitfully use the different formal and logical tools available today. Indeed, a lots of formal frameworks have already been used to investigate the inquiry process, examples are: information theory, kolmogorov complexity, learning theory, belief revision theory[1] and game theory. All these frameworks certainly capture some aspects of the inquiry process. Thus, it seems important to decompose the investigation of inquiry in order to use these different formal frameworks appropriately to investigate specific and well identified aspects of the inquiry process.

This clearly speaks for a back-and-forth investigation of inquiry between *conceptual analysis*, from the perspective of epistemology and philosophy of science, and *logical modeling*. Such an approach would maybe contribute to a revival of the relation between the fields of *logic* and *philosophy of science*.

[1]We would like to mention here that recent works have been carried out by Emmanuel Genot at the intersection between the interrogative model of inquiry and belief revision theory in the AGM-approach, see for instance [10].

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Appendices

Appendix A

Appendix to the chapter 1

A.1 Completeness proof for the dynamic logic of questions

Here is the proof of the following completeness theorem for the logic $\mathsf{E_I}$ with respect to the class of models $\mathbf{E_I}$:

Theorem A.1 (Soundness and Completeness of $\mathsf{E_I}$). *For every formula $\varphi \in \mathcal{E}_{\mathcal{I}}$:*

$$\models_{\mathbf{E_I}} \varphi \quad \textit{if and only if} \quad \vdash_{\mathsf{E_I}} \varphi.$$

Proof. The soundness and the completeness of the static part is proved by usual technique. We start by proving the soundness of the reduction axioms of $\mathsf{E_I}$.

Consider the first axiom:

$$[(\gamma_1, \ldots, \gamma_k)?]p \; \leftrightarrow \; \mathsf{pre}(\gamma_1, \ldots, \gamma_k) \rightarrow p.$$

Let (M, s) be a pointed epistemic inquiry model. Assume that $M, s \models [(\gamma_1, \ldots, \gamma_k)?]p$ and $M, s \models \mathsf{pre}(\gamma_1, \ldots, \gamma_k)$. By the semantic definition of the question operator, we have that $M_{(\gamma_1, \ldots, \gamma_k)?}(s), s \models p$. Then, we have to consider two different cases:

- For all $i \in [\![1, k]\!]$, $\gamma_i \notin \Phi(s)$: in this case $M_{(\gamma_1, \ldots, \gamma_k)?}(s) := M$ and we thereby have that $M, s \models p$.
- There exists $i \in [\![1, k]\!]$ s.t. $\gamma_i \in \Phi(s)$: in this case $M_{(\gamma_1, \ldots, \gamma_k)?}(s) := M|\gamma_i$ so we get that $M|\gamma_i, s \models p$ and thereby that $M, s \models p$.

In the other way around, assume that $M, s \models \mathsf{pre}(\gamma_1, \ldots, \gamma_k) \rightarrow p$ and assume also that $M, s \models \mathsf{pre}(\gamma_1, \ldots, \gamma_k)$. Then, we have that $M, s \models p$ and we can directly see that, in all the cases, $M_{(\gamma_1, \ldots, \gamma_k)?}(s), s \models p$.

The soundness of the reduction axioms 2., 3. and 4. can be proved in a similar way. Consider now the fifth axiom:

$$[(\gamma_1, \ldots, \gamma_k)?]\, K\varphi \; \leftrightarrow$$

$$\mathsf{pre}(\gamma_1, \ldots, \gamma_k) \rightarrow \left((\neg \Phi\gamma_1 \wedge \ldots \wedge \neg \Phi\gamma_k \wedge K\varphi) \vee \bigvee_{1 \leq i \leq k} \Phi\gamma_i \wedge K(\gamma_i \rightarrow [(\gamma_1, \ldots, \gamma_k)?]\varphi) \right).$$

Let (M, s) be a pointed epistemic inquiry model. Assume that $M, s \models [(\gamma_1, \ldots, \gamma_k)?]K\varphi$ and $M, s \models \mathsf{pre}(\gamma_1, \ldots, \gamma_k)$. Then, by the semantic definition of the question operator, we have that $M_{(\gamma_1, \ldots, \gamma_k)?}(s), s \models K\varphi$. We now have to consider two different cases:

- For all $i \in [\![1,k]\!]$, $\gamma_i \notin \Phi(s)$: in this case $M_{(\gamma_1,...,\gamma_k)?}(s) := M$ and we thereby have that $M, s \models K\varphi$. Besides, since for all $i \in [\![1,k]\!]$, $\gamma_i \notin \Phi(s)$, we have for all $i \in [\![1,k]\!]$ that $M, s \models \neg\Phi\gamma_i$. Thus, we finally get that:

$$M, s \models \neg\Phi\gamma_1 \wedge \ldots \wedge \neg\Phi\gamma_k \wedge K\varphi.$$

- There exists $i \in [\![1,k]\!]$ s.t. $\gamma_i \in \Phi(s)$: in this case $M_{(\gamma_1,...,\gamma_k)?}(s) := M|\gamma_i$ and we have $M|\gamma_i, s \models K\varphi$. We directly get that $M, s \models \Phi\gamma_i$ and we want to show that

$$M, s \models K(\gamma_i \to [(\gamma_1, \ldots, \gamma_k)?]\varphi).$$

  Let $w \sim s$ such that $M, w \models \gamma_i$. We want to show that $M, w \models [(\gamma_1, \ldots, \gamma_k)?]\varphi$. Since $M, s \models \mathsf{pre}(\gamma_1, \ldots, \gamma_k)$, we have that $M, w \models \mathsf{pre}(\gamma_1, \ldots, \gamma_k)$ so we have to show that $M_{(\gamma_1,...,\gamma_k)?}(w), w \models \varphi$. Since $\gamma_i \in \Phi(s)$, $w \sim s$ and $M, w \models \gamma_i$, we have by the coherence property of the oracle that $\gamma_i \in \Phi(w)$ and thereby that $M_{(\gamma_1,...,\gamma_k)?}(w) := M|\gamma_i$. Then, since by assumption we have that $M|\gamma_i, s \models K\varphi$, we have in particular that $M|\gamma_i, w \models \varphi$ and therefore that $M_{(\gamma_1,...,\gamma_k)?}(w), w \models \varphi$. We finally get

$$M, s \models \Phi\gamma_i \wedge K(\gamma_i \to [(\gamma_1, \ldots, \gamma_k)?]\varphi).$$

We conclude that the direction from the left to the right is valid. Now assume that

$$M, s \models \mathsf{pre}(\gamma_1, \ldots, \gamma_k) \to \left((\neg\Phi\gamma_1 \wedge \ldots \wedge \neg\Phi\gamma_k \wedge K\varphi) \vee \bigvee_{1 \le i \le k} \Phi\gamma_i \wedge K(\gamma_i \to [(\gamma_1, \ldots, \gamma_k)?]\varphi) \right).$$

We want to show that $M, s \models [(\gamma_1, \ldots, \gamma_k)?]\, K\varphi$. Assume that $M, s \models \mathsf{pre}(\gamma_1, \ldots, \gamma_k)$. We then have that

$$M, s \models \left((\neg\Phi\gamma_1 \wedge \ldots \wedge \neg\Phi\gamma_k \wedge K\varphi) \vee \bigvee_{1 \le i \le k} \Phi\gamma_i \wedge K(\gamma_i \to [(\gamma_1, \ldots, \gamma_k)?]\varphi) \right).$$

We want to show that $M_{(\gamma_1,...,\gamma_k)?}(s), s \models K\varphi$. To this end, we have to consider two different cases:

- For all $i \in [\![1,k]\!]$, $\gamma_i \notin \Phi(s)$: in this case $M_{(\gamma_1,...,\gamma_k)?}(s) := M$ and $M, s \models \neg\Phi_1 \wedge \ldots \wedge \neg\Phi_k$. Then, we necessarily get that $M, s \models K\varphi$ and thereby that $M_{(\gamma_1,...,\gamma_k)?}(s), s \models K\varphi$.

- There exists $i \in [\![1,k]\!]$ s.t. $\gamma_i \in \Phi(s)$: in this case $M_{(\gamma_1,...,\gamma_k)?}(s) := M|\gamma_i$ and $M, s \models \Phi\gamma_i$. Then, we necessarily get that $M, s \models K(\gamma_i \to [(\gamma_1, \ldots, \gamma_k)?]\varphi)$. Let $w \in W_{(\gamma_1,...,\gamma_k)?}(s)$ such that $w \sim s$. Then, we necessarily have that w in the model M is such that $M, w \models \gamma_i$. Since $M, s \models K(\gamma_i \to [(\gamma_1, \ldots, \gamma_k)?]\varphi)$ and $w \sim s$, we get that $M, w \models [(\gamma_1, \ldots, \gamma_k)?]\varphi$. Since $M, w \models \mathsf{pre}(\gamma_1, \ldots, \gamma_k)$, we have that $M_{(\gamma_1,...,\gamma_k)?}(w), w \models \varphi$ and since $\gamma_i \in \Phi(w)$ by the coherence property for the oracle, we indeed have that $M|\gamma_i, w \models \varphi$. This shows that $M\gamma_i, s \models \varphi$ and therefore that $M_{(\gamma_1,...,\gamma_k)?}(s), s \models K\varphi$.

We conclude that the direction from the right to the left is valid.

Thus, we have proved that the reduction axioms 1. to 5. of $\mathsf{E_I}$ are sound. Finally, the completeness part is proved by a standard DEL-style translation argument: by working inside out, the reduction axioms translate the dynamic formulas into corresponding static ones. Then, we appeal to completeness for the static base logic. $\square$

A.2 Completeness proof for the dynamic logic of inferences

We first prove the soundness and completeness of the logic $\mathsf{TE_0}$ on the class of models **TE**. Then, we provide reduction axioms for the dynamic operators of tableau management. Finally, we obtain completeness result for the dynamic logic of inferences.

A.2.1 Completeness for the static fragment

In order to prove completeness for the logic $\mathsf{TE_0}$, we need the Lindenbaum's lemma, the existence lemma and the truth lemma, along with the definition of canonical models for $\mathsf{TE_0}$:

Lemma A.1 (Lindenbaum's Lemma)**.** *For any $\mathsf{TE_0}$-consistent set of formulas Σ, there is a maximal $\mathsf{TE_0}$-consistent set Σ^+ such that $\Sigma \subseteq \Sigma^+$.*

Definition A.1 (Canonical model for $\mathsf{TE_0}$)**.** *The canonical model of the logic $\mathsf{TE_0}$ is the tuple $M^{\mathsf{TE_0}} = \langle W^{\mathsf{TE_0}}, \sim^{\mathsf{TE_0}}, V^{\mathsf{TE_0}}, \mathsf{E}^{\mathsf{TE_0}}, \mathsf{T}^{\mathsf{TE_0}} \rangle$, where:*

- *$W^{\mathsf{TE_0}}$ is the set of all maximal $\mathsf{TE_0}$-consistent set of formulas,*
- *$\sim^{\mathsf{TE_0}}$ is the binary relation on $W^{\mathsf{TE_0}}$ defined by $w \sim^{\mathsf{TE_0}} u$ if for all formulas φ, $\varphi \in u$ implies $\varphi \in w$,*
- *$V^{\mathsf{TE_0}}$ is the valuation defined by $V^{\mathsf{TE_0}}(w) := \{p \in \mathsf{P} \mid p \in w\}$,*
- *$\mathsf{E}^{\mathsf{TE_0}}(w) := \{\gamma \in \mathcal{I} \mid E\gamma \in w\}$,*
- *$\mathsf{T}^{\mathsf{TE_0}}(w) := \{\mathcal{T}^j \in \mathcal{P}(\mathcal{P}(\mathcal{I})) \mid j \in \mathbb{N}$ and $\mathcal{T}^j = \{\mathcal{B}^j_i\}_{i\in\mathbb{N}}$ where $\mathcal{B}^j_i = \{\gamma \in \mathcal{I} \mid Br^j_i \in w\}\}$.*

Lemma A.2 (Existence Lemma)**.** *For every world $w \in W^{\mathsf{TE_0}}$, if $\langle K\rangle\varphi \in w$, then there is a world $u \in W^{\mathsf{TE_0}}$ such that $w \sim^{\mathsf{TE_0}} u$ and $\varphi \in u$.*

Lemma A.3 (Truth Lemma)**.** *For all $w \in W^{\mathsf{TE_0}}$, we have $(M^{\mathsf{TE_0}}, w) \models \varphi$ iff $\varphi \in w$.*

We can now prove the following completeness theorem for the logic $\mathsf{TE_0}$:

Theorem A.2 (Soundness and Completeness of $\mathsf{TE_0}$)**.** *For every formula $\varphi \in \mathcal{TE}^*_0$:*

$$\models_{\mathbf{TE}} \varphi \quad \textit{if and only if} \quad \vdash_{\mathsf{TE_0}} \varphi.$$

Proof. The soundness part is obtained directly by checking that the axioms of $\mathsf{TE_0}$ are actually valid on tableau epistemic models. We now show the completeness part using the canonical model technique.

First, the proposition 4.12 of Blackburn et al [5] tells us that all what we have to show is that every $\mathsf{TE_0}$-consistent set of formulas is satisfiable. Let Σ be such a set. By the Lindenbaum's lemma, we can extend Σ to a maximal $\mathsf{TE_0}$-consistent set of formula Σ^+. Then, by the truth lemma, we have $(M^{\mathsf{TE_0}}, w) \models \Sigma$, so Σ is satisfiable in the canonical model of $\mathsf{TE_0}$ at Σ^+. What we have to show now is that $M^{\mathsf{TE_0}}$ is indeed a tableau epistemic model.

First we will show that for all $w \in W^{\mathsf{TE_0}}$, $\mathsf{E}^{\mathsf{TE_0}}(w)$ is a set of true formulas. Let $w \in W^{\mathsf{TE_0}}$ and let $\gamma \in \mathsf{E}^{\mathsf{TE_0}}(w)$. Since w is a $\mathsf{TE_0}$-consistent set, we have by definition that $E\gamma \to \gamma$ is in w. Then, since $\gamma \in \mathsf{E}^{\mathsf{TE_0}}(w)$ we have that $E\gamma \in w$. Thus, since w is a maximal $\mathsf{TE_0}$-consistent set, w is closed under modus ponens and we thereby have $\gamma \in w$. We then get by the truth lemma that $(M^{\mathsf{TE_0}}, w) \models \gamma$, namely γ is true at w in $M^{\mathsf{TE_0}}$.

The last thing that we have to show is the following: for all $w \in W^{\mathsf{TE_0}}$, all the $\mathcal{T}^j(w) \in \mathsf{T}(s)$ have indeed the structure of a semantic tree. To this end, consider $w \in W^{\mathsf{TE_0}}$ and let $\mathcal{T}^j(w) \in$

$\mathsf{T}(s)$. If $\mathcal{T}^j(w) = \emptyset$ then we are done. If $\mathcal{T}^j(w) \neq \emptyset$, then we will show that every formula γ occurring in $\mathcal{T}^j(w)$ is either the root of the tree or is the result of the application of a tableau construction rule. Let γ occurring in $\mathcal{T}^j(w)$. Then, there is a branch $\mathcal{B}^j_i(w)$ such that $\gamma \in \mathcal{B}^j_i(w)$. It follows by definition of $\mathsf{T}^{\mathsf{EI}_0}$ that $Br^j_i\gamma \in w$. Then, since w is a maximal TE_0-consistent set, $Br^j_i\gamma \to R^j\gamma \vee (Br^j_i\gamma \wedge C^j_i\gamma)$ is in w and by modus ponens closure $R^j\gamma \vee (Br^j_i\gamma \wedge C^j_i\gamma)$ is in w. By the truth lemma, we get that $(M^{\mathsf{EI}_0}, w) \models R^j\gamma \vee (Br^j_i\gamma \wedge C^j_i\gamma)$ which says exactly that either γ is the root of $\mathcal{T}^j(w)$ or γ results from the application of a tableau construction rule to a formula in $\mathcal{B}^j_i(s)$.

This shows that for all $w \in W^{\mathsf{TE}_0}$, $\mathsf{T}(s) \in \mathcal{P}(\mathsf{Strees}(\mathcal{I}))$.

We conclude that M^{TE_0} is a tableau epistemic model. We have thereby proved that every TE_0-consistent set of formulas is satisfiable on some tableau epistemic model (namely the canonical model associated to TE_0). We finally conclude that the logic TE_0 is complete with respect to the class of tableau epistemic models. $\square$

A.2.2 Reduction axioms for the dynamic operators of tableau management

We provide here the list of the reduction axioms for the operators of tableaux management, i.e., the *tableau construction*, *tableau creation* and *tableau elimination operators*.

Reduction axioms for the tableau construction operator

$$
\begin{array}{rcl}
[\mathcal{T}^j, i, \gamma]p & \leftrightarrow & Br^j_i\gamma \to p \\
[\mathcal{T}^j, i, \gamma]\neg\varphi & \leftrightarrow & Br^j_i\gamma \to \neg[\mathcal{T}^j, i, \gamma]\varphi \\
[\mathcal{T}^j, i, \gamma](\varphi \wedge \psi) & \leftrightarrow & [\mathcal{T}^j, i, \gamma]\varphi \wedge [\mathcal{T}^j, i, \gamma]\psi \\
[\mathcal{T}^j, i, \gamma]K\varphi & \leftrightarrow & Br^j_i\gamma \to K[\mathcal{T}^j, i, \gamma]\varphi \\
[\mathcal{T}^j, i, \gamma]E\gamma' & \leftrightarrow & Br^j_i\gamma \to E\gamma' \\
[\mathcal{T}^j, i, p]Br^j_i\gamma & \leftrightarrow & Br^j_i p \to Br^j_i\gamma \\
[\mathcal{T}^j, i, \gamma_1 \vee \gamma_2]Br^{j'}_{i'}\gamma & \leftrightarrow & Br^j_i(\gamma_1 \vee \gamma_2) \to Br^{j'}_{i'}\gamma \quad \text{for } \gamma \neq \gamma_1, \gamma_2 \\
[\mathcal{T}^j, i, \gamma_1 \vee \gamma_2]Br^{j'}_{i'}\gamma_1 & \leftrightarrow & Br^j_i(\gamma_1 \vee \gamma_2) \to Br^{j'}_{i'}\gamma_1 \quad \text{for } j' \neq j \text{ or } i' \neq i \\
[\mathcal{T}^j, i, \gamma_1 \vee \gamma_2]Br^j_i\gamma_1 & \leftrightarrow & \top \\
[\mathcal{T}^j, i, \gamma_1 \vee \gamma_2]Br^{j'}_{i'}\gamma_2 & \leftrightarrow & Br^j_i(\gamma_1 \vee \gamma_2) \to Br^{j'}_{i'}\gamma_2 \quad \text{for } j' \neq j \\
[\mathcal{T}^j, i, \gamma_1 \vee \gamma_2]Br^j_{i'}\gamma_2 & \leftrightarrow & Br^j_i(\gamma_1 \vee \gamma_2) \to Br^j_{i'}\gamma_2 \vee (\mathsf{empty}(\mathcal{B}^j_{i'}) \wedge \neg\mathsf{empty}(\mathcal{B}^j_{i'-1})) \quad \text{for } i' > 0 \\
[\mathcal{T}^j, i, \gamma_1 \vee \gamma_2]Br^j_0\gamma_2 & \leftrightarrow & Br^j_i(\gamma_1 \vee \gamma_2) \to Br^j_0\gamma_2 \\
[\mathcal{T}^j, i, \gamma_1 \wedge \gamma_2]Br^{j'}_{i'}\gamma & \leftrightarrow & Br^j_i(\gamma_1 \wedge \gamma_2) \to Br^{j'}_{i'}\gamma \quad \text{for } j' \neq j \text{ or } i' \neq i \\
[\mathcal{T}^j, i, \gamma_1 \wedge \gamma_2]Br^j_i\gamma & \leftrightarrow & Br^j_i(\gamma_1 \wedge \gamma_2) \to Br^j_i\gamma \quad \text{for } \gamma \neq \gamma_1, \gamma_2 \\
[\mathcal{T}^j, i, \gamma_1 \wedge \gamma_2]Br^j_i\gamma & \leftrightarrow & \top \text{ for } \gamma = \gamma_1 \text{ or } \gamma = \gamma_2 \\
[\mathcal{T}^j, i, \neg\neg\gamma]Br^{j'}_{i'}\gamma' & \leftrightarrow & Br^j_i\neg\neg\gamma \to Br^{j'}_{i'}\gamma' \quad \text{for } j' \neq j \text{ or } i' \neq i \\
[\mathcal{T}^j, i, \neg\neg\gamma]Br^j_i\gamma' & \leftrightarrow & Br^j_i\neg\neg\gamma \to Br^j_i\gamma' \quad \text{for } \gamma' \neq \gamma \\
[\mathcal{T}^j, i, \neg\neg\gamma]Br^j_i\gamma & \leftrightarrow & \top \\
[\mathcal{T}^j, i, \gamma]R^{j'}\gamma' & \leftrightarrow & Br^j_i\gamma \to R^{j'}\gamma'
\end{array}
$$

The reduction axioms for the operators C^j_i, $\mathsf{empty}(\mathcal{B}^j_i)$ and $\mathsf{empty}(\mathcal{T}^j)$ can be obtained in a similar way.

Reduction axioms for the tableau creation operator

$$
\begin{array}{rcl}
[\mathsf{T}+\{\gamma\}]p & \leftrightarrow & p \\
[\mathsf{T}+\{\gamma\}]\,\neg\varphi & \leftrightarrow & \neg[\mathsf{T}+\{\gamma\}]\varphi \\
[\mathsf{T}+\{\gamma\}]\,(\varphi\wedge\psi) & \leftrightarrow & [\mathsf{T}+\{\gamma\}]\varphi\wedge[\mathsf{T}+\{\gamma\}]\psi \\
[\mathsf{T}+\{\gamma\}]\,K\varphi & \leftrightarrow & K[\mathsf{T}+\{\gamma\}]\varphi \\
[\mathsf{T}+\{\gamma\}]\,E\gamma' & \leftrightarrow & E\gamma' \\
[\mathsf{T}+\{\gamma\}]\,Br_i^j\gamma' & \leftrightarrow & Br_i^j\gamma' \quad \text{for } \gamma'\neq\gamma \\
[\mathsf{T}+\{\gamma\}]\,Br_i^j\gamma & \leftrightarrow & Br_i^j\gamma \quad \text{for } i\neq 0 \\
[\mathsf{T}+\{\gamma\}]\,Br_0^j\gamma & \leftrightarrow & Br_0^j\gamma\vee(\mathsf{empty}(\mathcal{T}^j)\wedge\neg\mathsf{empty}(\mathcal{T}^{j-1})) \quad \text{for } j>0 \\
[\mathsf{T}+\{\gamma\}]\,Br_0^0\gamma & \leftrightarrow & Br_0^0\gamma\vee\mathsf{empty}(\mathcal{T}^0) \\
[\mathsf{T}+\{\gamma\}]\,R^j\gamma' & \leftrightarrow & R^j\gamma' \quad \text{for } \gamma'\neq\gamma \\
[\mathsf{T}+\{\gamma\}]\,R^j\gamma & \leftrightarrow & R^j\gamma\vee(\mathsf{empty}(\mathcal{T}^j)\wedge\neg\mathsf{empty}(\mathcal{T}^{j-1})) \quad \text{for } j>0 \\
[\mathsf{T}+\{\gamma\}]\,R^0\gamma & \leftrightarrow & R^0\gamma\vee\mathsf{empty}(\mathcal{T}^0) \\
[\mathsf{T}+\{\gamma\}]\,C_i^j\gamma' & \leftrightarrow & C_i^j\gamma' \\
[\mathsf{T}+\{\gamma\}]\,\mathsf{empty}(\mathcal{B}_i^j) & \leftrightarrow & \mathsf{empty}(\mathcal{B}_i^j) \quad \text{for } i>0 \\
[\mathsf{T}+\{\gamma\}]\,\mathsf{empty}(\mathcal{B}_0^j) & \leftrightarrow & \mathsf{empty}(\mathcal{B}_0^j)\wedge\mathsf{empty}(\mathcal{B}_0^{j-1}) \quad \text{for } j>0 \\
[\mathsf{T}+\{\gamma\}]\,\mathsf{empty}(\mathcal{B}_0^0) & \leftrightarrow & \bot \\
[\mathsf{T}+\{\gamma\}]\,\mathsf{empty}(\mathcal{T}^j) & \leftrightarrow & \mathsf{empty}(\mathcal{T}^j)\wedge\mathsf{empty}(\mathcal{T}^{j-1}) \quad \text{for } j>0 \\
[\mathsf{T}+\{\gamma\}]\,\mathsf{empty}(\mathcal{T}^0) & \leftrightarrow & \bot.
\end{array}
$$

Reduction axioms for the tableau elimination operator

$$
\begin{array}{rcl}
[\mathsf{T}-\mathcal{T}^j]p & \leftrightarrow & \mathsf{closed}(\mathcal{T}^j)\rightarrow p \\
\left[\mathsf{T}-\mathcal{T}^j\right]\neg\varphi & \leftrightarrow & \mathsf{closed}(\mathcal{T}^j)\rightarrow\neg[\mathsf{T}-\mathcal{T}^j]\varphi \\
\left[\mathsf{T}-\mathcal{T}^j\right](\varphi\wedge\psi) & \leftrightarrow & \mathsf{closed}(\mathcal{T}^j)\rightarrow[\mathsf{T}-\mathcal{T}^j]\varphi\wedge[\mathsf{T}-\mathcal{T}^j]\psi \\
\left[\mathsf{T}-\mathcal{T}^j\right]K\varphi & \leftrightarrow & \mathsf{closed}(\mathcal{T}^j)\rightarrow K[\mathsf{T}-\mathcal{T}^j]\varphi \\
\left[\mathsf{T}-\mathcal{T}^j\right]E\gamma & \leftrightarrow & \mathsf{closed}(\mathcal{T}^j)\rightarrow E\gamma\vee R^j\neg\gamma \\
\left[\mathsf{T}-\mathcal{T}^j\right]Br_i^j\gamma & \leftrightarrow & \mathsf{closed}(\mathcal{T}^j)\rightarrow Br_i^j\gamma \\
\left[\mathsf{T}-\mathcal{T}^j\right]R^j\gamma & \leftrightarrow & \mathsf{closed}(\mathcal{T}^j)\rightarrow R^j\gamma \\
\left[\mathsf{T}-\mathcal{T}^j\right]C_i^j\gamma & \leftrightarrow & \mathsf{closed}(\mathcal{T}^j)\rightarrow C_i^j\gamma \\
\left[\mathsf{T}-\mathcal{T}^j\right]\mathsf{empty}(\mathcal{T}^j) & \leftrightarrow & \mathsf{closed}(\mathcal{T}^j)\rightarrow\mathsf{empty}(\mathcal{T}^j) \\
\left[\mathsf{T}-\mathcal{T}^j\right]\mathsf{empty}(\mathcal{B}_i^j) & \leftrightarrow & \mathsf{closed}(\mathcal{T}^j)\rightarrow\mathsf{empty}(\mathcal{B}_i^j)
\end{array}
$$

We can then define the logic TE as the extension of the static logic TE_0 with the reduction axioms listed above:

Definition A.2 (Logic TE)**.** *The logic* TE *is built from the logic* TE_0 *plus the reduction axioms for the dynamic operators of tableau construction, creation and elimination listed above.*

A.2.3 Completeness for the logic TE

We can now show that the logic TE is sound and complete with respect to the class of models **TE**:

Theorem A.3 (Soundness and completeness of TE). *For every formula $\varphi \in \mathcal{TE}^*$:*

$$\models_{\mathbf{TE}} \varphi \quad \textit{if and only if} \quad \vdash_{\mathsf{TE}} \varphi.$$

Proof. The soundness part is proved by checking that all the reduction axioms are valid on the class of models **TE**.

The completeness part is proved by a standard DEL-style translation argument: by working inside out, the reduction axioms translate the dynamic formulas into corresponding static ones. Then, we appeal to completeness for the static base logic TE_0. □

A.3 Completeness proof for the dynamic logic of questions and inferences

In this section, we will show that our dynamic logic of questions and inferences is completely axiomatizable for a suitable extension of the language $\mathcal{TE}_{\mathcal{I}}$.

The language $\mathcal{TE}_{\mathcal{I}}$ was obtained by combining the language $\mathcal{TE}$ and $\mathcal{E}_{\mathcal{I}}$. In order to prove a completeness result for the dynamic logic of questions and inferences, we define the language $\mathcal{TE}_{\mathcal{I}}{}^*$ as the combination of the language $\mathcal{TE}^*$ and $\mathcal{E}_{\mathcal{I}}$. Thus, the language $\mathcal{TE}_{\mathcal{I}}{}^*$ is an extension of the language $\mathcal{TE}_{\mathcal{I}}$ obtained by adding the additional operators introduced in $\mathcal{TE}^*$. We denote by $\mathcal{TE}_{\mathcal{I}0}^*$ the static fragment of $\mathcal{TE}_{\mathcal{I}}{}^*$.

We first focus on the completeness proof for the static fragment. To this end, we define the static logic $\mathsf{TE}_{\mathsf{I}0}$:

Definition A.3 (Logic $\mathsf{TE}_{\mathsf{I}0}$). *The logic $\mathsf{TE}_{\mathsf{I}0}$ is built by adding the axioms and rules of inference of the static fragments of the logics TE and E_{I}.*

From our previous completeness proofs for the static fragments of the logics TE and E_{I}, we directly obtain that the logic $\mathsf{TE}_{\mathsf{I}0}$ is sound and complete with respect to the class of models $\mathbf{TE}_{\mathbf{I}}$:

Theorem A.4 (Soundness and completeness for $\mathsf{TE}_{\mathsf{I}0}$). *For every formula $\varphi \in \mathcal{TE}_{\mathcal{I}0}^*$:*

$$\models_{\mathbf{TE_I}} \varphi \quad \textit{if and only if} \quad \vdash_{\mathsf{TE}_{\mathsf{I}0}} \varphi.$$

Proof. The proof is direct from the completeness proofs for the static fragments of the logics TE and E_{I}. □

Then, we obtain the logic TE_{I} by adding to the logic $\mathsf{TE}_{\mathsf{I}0}$ reduction axioms for the dynamic operators:

Definition A.4 (Logic TE_{I}). *The logic TE_{I} is built from the static logic $\mathsf{TE}_{\mathsf{I}0}$ plus the following reduction axioms for the dynamic operators:*

Question operator: *the reduction axioms for the question operator are the ones of the logic E_{I} plus the following ones:*

1. $[(\gamma_1, \ldots, \gamma_k)?]\, E\gamma \leftrightarrow \mathsf{pre}(\gamma_1, \ldots, \gamma_k) \rightarrow E\gamma$ *where $\gamma \neq \gamma_i$ for all $i \in [\![1, k]\!]$*

2. $[(\gamma_1,\ldots,\gamma_k)?]\,E\gamma \leftrightarrow \mathsf{pre}(\gamma_1,\ldots,\gamma_k) \rightarrow (\neg\Phi\gamma_1 \wedge \ldots \wedge \neg\Phi\gamma_k \wedge E\gamma_i) \vee \Phi\gamma_i$ *where* $\gamma = \gamma_i$ *for some* $i \in [\![1,k]\!]$
3. $[(\gamma_1,\ldots,\gamma_k)?]\,Br_i^j\gamma \leftrightarrow \mathsf{pre}(\gamma_1,\ldots,\gamma_k) \rightarrow Br_i^j\gamma$
4. $[(\gamma_1,\ldots,\gamma_k)?]\,R^j\gamma \leftrightarrow \mathsf{pre}(\gamma_1,\ldots,\gamma_k) \rightarrow R^j\gamma$
5. $[(\gamma_1,\ldots,\gamma_k)?]\,C_i^j\gamma \leftrightarrow \mathsf{pre}(\gamma_1,\ldots,\gamma_k) \rightarrow C_i^j\gamma$
6. $[(\gamma_1,\ldots,\gamma_k)?]\,\mathsf{empty}(\mathcal{B}_i^j) \leftrightarrow \mathsf{pre}(\gamma_1,\ldots,\gamma_k) \rightarrow \mathsf{empty}(\mathcal{B}_i^j)$
7. $[(\gamma_1,\ldots,\gamma_k)?]\,\mathsf{empty}(\mathcal{T}^j) \leftrightarrow \mathsf{pre}(\gamma_1,\ldots,\gamma_k) \rightarrow \mathsf{empty}(\mathcal{T}^j)$

Tableau management operators: *the reduction axioms for the tableau construction, creation and elimination operators are the ones of the logic* TE *plus the following ones:*

1. $[\mathcal{T}^j, i, \gamma]\Phi\gamma \leftrightarrow Br_i^j\gamma \rightarrow \Phi\gamma$
2. $[\mathsf{T} + \{\gamma\}]\Phi\gamma \leftrightarrow \Phi\gamma$
3. $[\mathsf{T} - \mathcal{T}^j]\Phi\gamma \leftrightarrow \mathsf{closed}(\mathcal{T}^j) \rightarrow \Phi\gamma$

We can finally show that the logic $\mathsf{TE_I}$ is sound and complete with respect to the class of models $\mathbf{TE_I}$:

Theorem A.5 (Soundness and completeness of $\mathsf{TE_I}$)**.** *For every formula* $\varphi \in \mathcal{TE_I}^*$*:*

$$\models_{\mathbf{TE_I}} \varphi \quad \textit{if and only if} \quad \vdash_{\mathsf{TE_I}} \varphi.$$

Proof. The soundness part is proved by checking that all the reduction axioms are valid on the class of models $\mathbf{TE_I}$.

The completeness part is proved by a standard DEL-style translation argument: by working inside out, the reduction axioms translate the dynamic formulas into corresponding static ones. Then, we appeal to completeness for the static base logic $\mathsf{TE_{I0}}$. □