Non-Standard Models of Arithmetic: a Philosophical and Historical perspective

MSc Thesis (*Afstudeerscriptie*)

written by

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Clarity is the good faith of philosophers.
(Arthur Schopenhauer)

Bernard of Chartres used to say that we are like dwarfs on the shoulders of giants,
so that we can see more than they,and things at a greater distance,
not by virtue of any sharpness of sight on our part, or any physical distinction,
but because we are carried high and raised up by their giant size.
(John of Salisbury)

Alla mia nazione

Non popolo arabo, non popolo balcanico, non popolo antico
ma nazione vivente, ma nazione europea:
e cosa sei? Terra di infanti, affamati, corrotti,
governanti impiegati di agrari, prefetti codini,
avvocatucci unti di brillantina e i piedi sporchi,
funzionari liberali carogne come gli zii bigotti,
una caserma, un seminario, una spiaggia libera, un casino!
Milioni di piccoli borghesi come milioni di porci
pascolano sospingendosi sotto gli illesi palazzotti,
tra case coloniali scrostate ormai come chiese.
Proprio perché tu sei esistita, ora non esisti,
proprio perché fosti cosciente, sei incosciente.
E solo perché sei cattolica, non puoi pensare
che il tuo male é tutto male: colpa di ogni male.
Sprofonda in questo tuo bel mare, libera il mondo.
(Pier Paolo Pasolini)

Abstract

Over the last fifty years, the study of non-standard models of arithmetic has become a fertile and highly technical mathematical branch. Nevertheless, surprising as it might seem today, the topic of non-standard models was born as a genuine philosophical issue.

My work sets out to investigate the philosophical and historical significance of non-standard models of arithmetic. Not only has non-standard models of arithmetic encountered a scarce success in what concerns philosophy, but also a detailed presentation on the beginnings of the topic has never been written.

In particular, I will discuss the origins of non-standard models of arithmetic with respect to the pioneering work of Richard Dedekind and Thoralf Skolem, and strive to shed light on the relationship between philosophy, logic and mathematics in what concerns arithmetical theories and foundational studies in mathematics.

Keywords: Non-standard models, intended model, descriptive use of logic, Peano Arithmetic, Dedekind, Skolem, Tennenbaum's Theorem, foundations of mathematics.

Introduction

To fully grasp the significance of NON-STANDARD MODELS, we have to go back to the origins of mathematical logic. That is to say, to trace the historical developments that have resulted in what we call logic today. Dry as it may seem at a first glance, the historical approach I pursue here is likely to come out as an excellent way to cast a fresh, brand-new light on age-old issues that seemed to be settled once and for all. As we gradually move towards the role of non-standard models of arithmetic, history, philosophy, logic and mathematics come out as inseparably interwoven. On top of that, not only has the topic of NON-STANDARD MODELS encountered a scarce success in what concerns philosophy, but also a philosophical and historical presentation about the beginnings of the topic has never been written. Hopefully, this research may constitute a starting point for much needed further work on the subject.

Why Non-Standard models?

Today the issue of NON-STANDARD MODELS represents a highly technical mathematical field. However, to tell the story of these models, we have to turn to philosophy prior to mathematics. Surprising as it may seem today, I argue that the mathematical research concerning non-standard models is grounded on profound philosophical roots. This work intends to investigate these roots, which may ultimately shed new light on the recent mathematical results.

Why Non-Standard models of arithmetic?

Needless to say that the phrase "non-standard models" does not pertain to the sole arithmetical sphere, it may also refer to the *set-theoretical* case (pointed out by Skolem in 1922 as the existence of "countable" models of the first order axiomatisation of set-theory — a phenomenon known today as Skolem's Paradox), or to non-standard *analysis* (which deals with the existence of "special" numbers, called non-standard, exploited to provide a rigorous definition of elements known in analysis as "infinitesimals").

Nevertheless, in this work I will confine myself to the arithmetical case. The reasons are mainly three:

- I believe that arithmetic represents, in this respect, the first step in the investigation of non-standard models. In fact, the meagre presence, in the philosophical literature, of a uniform and all-comprehensive treatment of non-standard models makes the arithmetical account a seminal and self-contained research;

- for a historical and exegetical reasons: the large number of historical papers in arithmetic lends itself to an in-depth study that can represent a good starting point for new perspectives on old debates (e.g. whether arithmetical theories could reach some degree of completeness; which logic is the most suitable for arithmetical theories; etc);
- for the relevance with respect to the present-day debate in philosophy of mathematics: the field of non-standard models can provide some results (e.g. Tennenbaum's Theorem) for the current philosophical discussion.

To sum up, in this thesis I will emphasise the importance of non-standard models of arithmetic as a genuine philosophical issue.

The philosophical and mathematical phase in the development of non-standard models of arithmetic

Commonly, the space dedicated in logical textbooks, if any, to non-standard models is quite exiguous: Hodges discusses them in half a page; Van Dalen does not write more than a couple of pages on them; the only exception is Boolos' book where a seventeen page chapter is devoted to a rather technical presentation of the topic.[1]

On the other hand, there are only two fundamental references on the non-standard models, so-called "bibles" by the insiders, that do justice to what should be regarded, by this time, as an important and promising field of research: [kaye91] and the more recent [kossak06].[2] On top of that, we should note that these two monographs are strongly mathematically oriented.

At the end of the day, it would then seem that the concept of non-standard models has little to contribute to philosophical debates. Roughly speaking, the two mathematical studies along with the meagre logical treatment exhaust the literature on non-standard models.

An exception is [smorynski84] which was indeed a forerunner attempt to give a historical account of non-standard models. Before the publishing of [kaye91] in 1991, "the only, and not easily available, source was the excellent notes of Craig Smorynski [smorynski84] from his lectures on nonstandard models at the University of Utrecht in 1978." [3]

[1] See [hodges01, p.70], [vandalen04, p. 113, pp.121-122], [boolos07, ch.25]

[2] Strictly speaking, [kossak06] presents itself as continuation of [kaye91].

[3] [kossak06, p. viii]

Although Smorynski's lecture notes have the purpose of presenting the main mathematical results, at the beginning of his text some historical considerations are put forward. According to him, the intention of the notes is "to give a partly historical account of the development of the subject."[4] In particular, Smorynski points out that two different approaches of non-standard models can be identified.

> It is also worth mentioning that Skolem's goal[5] in constructing nonstandard models was *philosophical*: He aimed to shew that first-order logic could not characterise the number series; he did not care to start a new subject. Until the 1960s, this was generally the case - nonstandard models of arithmetic were either objects of philosophical interest or tools, not objects of mathematical interest in their own right. [smorynski84, p.3, my emphasis]

For Smorynski, Skolem considered non-standard models as nothing but limitative results, whereas from the 1960s onwards these models gained their own importance as entities.

Following Smorynski, we distinguish two historical phases: one "philosophical", which is characterised by the work of Dedekind (as will argue) and Skolem, and the other one "mathematical" which goes from the 1960s and onwards. The quotation of Smorynski appears in the first paragraph that he entitles "The Beginnings (the 1950s and Earlier)". Nevertheless, Smorynski does not elaborate further on this "philosophical phase" in the lecture notes, he rather focuses mainly on the mathematical results that have built up non-standard models as a mathematical field in its own right.

The thesis strives to fill such a historical gap and provides a better picture of the philosophical roots of non-standard models of arithmetic, and their influence on the next mathematical developments.

To do so, we will discuss in detail on the importance of the philosophical phase by presenting its features with respect to the mathematical phase. We then argue that Dedekind has to be considered, together with Skolem, as the forefather the non-standard models and thus viewed as a member of the philosophical phase.

In particular, we will consider Dedekind's letter to Keferstein and his categorical characterisation of his system. After that, we will discuss Skolem's proofs of the existence of non-standard models and their relationship with Skolem's finitist mathematical credo.

Moreover, we will also argue for a possible third "hybrid" phase which could be the attempt to recover from the mismatch between the scarce interest that philosophy has paid to non-standard models over the last 60 years, and the abundant mathematical results from the 1960's on. This phase possibly sets out from the philosophical relevance of results concerning non-standard to debates in philosophy of mathematics.

[4][smorynski84, p.1]

[5]Note that Thoralf Skolem is commonly credited with being the one who discovered non-standard models. However, we will argue that, in this respect, Dedekind should be mentioned also.

Overview of the thesis

In chapter 1 we present some introductory remarks about important distinctions that are dealt with in the text — such as analytic/synthetic reasoning, deductive/descriptive use of logic in mathematics, intended/unintended models — in order to put forth their role in the discussion of non-standard models of arithmetic (NMoA for short).

In chapter 2 we consider Dedekind's work with respect to NMoA, especially his "The Nature and the Meaning of Numbers" and one of the letter he addressed to Keferstein. We argue that in the letter to Keferstein we find the very first evidence of the existence of NMoA. Moreover, we illustrate Dedekind's attempt to banish non-standard models in his axiomatization of arithmetic via employing statements expressible in second-order logic.

In chapter 3 we explore Skolem's original proof of the existence of NMoA, known in the literature as "ultrapower-like" construction, the impact that Skolem's philosophical views had on the proof and the way non-standard models were received from following philosophers.

In chapter 4 we outline the main features of the succeeding mathematical phase. In particular we focus on one mathematical result, viz. Tennenbaum's Theorem, which could be the sparkle of the return of philosophical interest to the issue of non-standard models.

In the appendix we provide the syntax and the semantics of first and second order logic.

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Chapter 1

Descriptive use of logic and Intended models

In this first part we set the stage to allow for a better understanding of the significance and role of non-standard models. We aim at discussing some background issues that will be investigated in chapter 2 and 3 through Dedekind's and Skolem's pioneering works.

Above all, in this section we present two fundamental distinctions: the distinction between descriptive versus deductive use of logic and the distinction between intended and unintended models. These notions will be fundamental throughout the discussion that will follow.

1.1 Standard models of arithmetic

One possible way to approach non-standard models is to look at their *alter ego*: the standard models. Our main goal is to investigate the dichotomy *standard/non-standard* with respect to models of arithmetic. In this sense, as the prefix "non" may suggest, a clear knowledge of what *standard models* of arithmetic are should lead us to a much clearer understanding of non-standard models.

In first instance, if we want to find out what a model of arithmetic is, it is not hard to encounter a piece of text devoted to the topic. Mostly, the exposition proceeds as follows:

> The system of first-order Peano Arithmetic (or PA), is a theory in the language $\mathcal{L}_{PA} = (S, +, \cdot\,; 0)$, where s is an unary function, + and $\cdot$ are binary functions and 0 is a constant, and with the following axioms:
>
> 1. $\forall x \neg(S(x) = 0)$
> 2. $\forall x \forall y (S(x) = S(y) \rightarrow x = y)$
> 3. $\forall x (x + 0 = x)$

4. $\forall x \forall y(x + S(y) = S(x + y))$
5. $\forall x(x \cdot 0 = 0)$
6. $\forall x \forall y(x \cdot S(y) = (x \cdot y) + x)$
7. $\forall \vec{x}[(\varphi(0, \vec{x}) \wedge \forall y(\varphi(y, \vec{x}) \rightarrow \varphi(S(y), \vec{x}))) \rightarrow \forall y \varphi(y, \vec{x})]$

> Item 7 is meant to be an axiom for every formula $\varphi(y, \vec{x})$. These axioms are called induction axioms. Such a set of axioms, given by one or more generic symbols "φ" which range over all formulas, is called an axiom scheme; in our case we talk about the induction scheme.[6]

As we continue reading, we finally come across the definition of *standard model* for PA:

> Clearly, the set N together with the element 0 and usual successor function, addition and multiplication, is a model of PA, which we call the *standard model* and denote by $\mathcal{N}$.[7]

But once again, why should we need to specify that we are dealing with *standard* models rather than models *tout court*? In particular, what exactly is suggested with the adjective "standard"? Does this lead us to assume the existence of other kinds of models? Why is it being said that we are focusing on 'the' standard model rather than on many of them? Which is the connection, if any, between such a standard model and the language $\mathcal{L}_{PA}$ in which the seven axioms are stated?

All these questions seem quite legitimate given the goal of elaborating a philosophical account of the concept of non-standard model. In other words, our aim is to investigate the reasons why mathematicians have come up with "standard/non-standard" distinction, and thus why such a dichotomy should be fruitful to use.

To put it briefly, what the standard model refers to is the subject matter of the well-established number-theory as it is learnt at school: the series of natural numbers provided with unambiguous operations such as addition and multiplication.

Nevertheless, we note that, prior to presenting the definition of standard model of arithmetic, a set of axioms, the so-called *Peano Arithmetic* (*PA* for short), has been presented. Thus, we are likely to think that there is a certain interplay between the axiomatic theory PA and the notion of "standard" model. This phenomenon calls for a better account.

To shed some light on all these matters, we embark, in turn, on the following chain of issues: we start off by discussing the nature of logical axiomatization, we then focus

[6] [oosten10, p. 55]
[7] [oosten10, pp. 55–56]

on the relation between axiomatic presentations and models of arithmetic and finally we take up the intended/unintended dichotomy.

The first point we will turn to is the relation between logical axiomatization and models of arithmetic. In other words, we want to consider the origin of the definition of *model of arithmetic* with respect to a certain axiomatization, viz. the reason why models of arithmetic are formulated in such-and-such way. As Hao Wang puts it:

> I once asked myself the question: How were the famous axiom systems, such as [...] Peano's for arithmetic [the axiom system presented above], originally obtained? This was to me more than merely a historical question, as I wished to know how the basic concepts and axioms were to be singled out, and, once they were singled out, how one could establish their adequacy. [...] The attempt to find an answer to this question led me to some interesting fragments of history. [wang57, p. 145]

This issue constitutes the central thread of this first part: how was PA originally obtained? We claim that the issue is directly related to the issue of non-standard models of arithmetic.

1.2 Axiomatics and Formal theories

Peano Arithmetic is said to be an axiomatic theory. But in which sense can we speak of axiomatic theories? Can we be more precise about that?

Aristotle is widely considered as the first who laid down the guidelines for the axiomatic method.[8] In the *Posterior analytics*, Aristotle describes what can be seen from a modern point of view as an "effective model building technique" to carry out scientific investigation. The manuscript, which can be dated around 330 B.C, calls for "precise definitions" of the concepts used in a model and its theory, and the "rigorous proof" of theorems about the scientific paradigm.

Aristotle also refers to the need to define complicated concepts, and to prove difficult theorems from simpler ones. The idea was to specify a way to systematise and organise propositions in such a way that, on the basis of self-evident truths, all other truths would follow from the former. The goal was to increase clarity, evidence and certainty of the concepts. Thus, the idea was that the certainty of the arguments relies on the way in which those are presented.

A particularly influential application of such a model to mathematics came into being around 300 B.C.: Euclid's *Elements*. Euclid's work soon became a paradigm of an evident, convincing and well-conducted way of reasoning and arguing in mathematics.

[8] [odifreddi03, p. 34]

Euclid's paradigm has been influential for more than two millennia, until the modern breakthrough took place at the turn of the 19th and 20th century. Despite some features in common with the axiomatic set-up, present-day axiomatization is characterised as a "refinement" of the then-axiomatic paradigm.[9] By this time, the idea of an axiomatics resting on intuitive, self-evident and primitive concepts has been dismissed to make room for a new notion: formality[10]. What takes place by the time is a switch from the certitude, evidence, and clarity of concepts to their construction, structure, and form.

Whereas in Euclid's *Elements* axioms are picked out in virtue of their self-evident and consequently all-accepted truth, in a formal axiomatic theory we deal with schemes of sentences rather than sentences themselves. The crucial point is that only *via* schemes[11] we can focus merely on the form of the statement and abstract away from their subject-matter. Thus, we can catalogue sentences in accordance with the schemes they exhibit, and reason on the particular types of these patterns.

To do so and deal with schemes, axiomatic theories are expressed by a *formal* language. A formal language is a language whose syntax and semantics is explicitly presented in accordance with certain rules. Leaving aside the twists and turns that the historical development of the concept of formal has been subjected to, with "formal theory" here we mean a theory whose language is not already interpreted, and whose sentences are amenable to different interpretation. Conversely, the antonym of formal theory is "intuitive theory", viz. a theory whose language comes to be interpreted with respect to a certain domain.

Once again, if axiomatics concerns the method in virtue of which arguments are presented to obtain assent (i.e. isolating a set of basic propositions from which all the others follow necessarily), then the *formal* refinement, which still hinges on the axiomatic presentation, is characterised by the fact that its axioms are schemes formulated in a formal language rather than interpreted sentences. Thus, the axioms have the property of being amenable to different interpretations rather than conveying information on a sole domain. Schemes are constituted by place holders whose characteristic is that they be replace by elements of a domain when interpreted.[12]

In line with what we just said, the issue about the nature of the individuals involved in a structure does not pertain to formal axiomatics since it focuses on structures as such, viz. as the description of relations among elements, rather than the domain on which these structures can be interpreted.

Ultimately, the formal version of axiomatics turns out to be the study of the properties that objects have as a components of a common structure. Hence, we will use formal axiomatics as an essential device to study mathematical structures.

[9] See [awodey02, p. 5]
[10] See [dutilh-novaes10] for an historical account of the notion of formality.
[11] Note that here the concept of scheme is not related to the one of *induction scheme* presented earlier.
[12] See [corcoran06] for a historical account of the concept of "schema" and "scheme".

1.3 Hintikka and the two uses of logic in mathematics

A formal language provided with a deductive consequence relation is called "logic"[13]. Since its modern re-birth, logic has been considered a powerful and fascinating tool which can be applied to several disciplines.

In particular, mathematical logic is the branch that studies the interplay between logic and mathematics in the following terms: on the one hand, we can study mathematics by means of logical tools, on the other hand, logic can be studied by referring to mathematical techniques. Here we consider mathematical logic in the former sense, namely as a helpful device for mathematicians who want to investigate a certain mathematical theory.

Consider, for example, a mathematical structure such as the natural numbers series. To study it, mathematicians have developed a corresponding mathematical theory: number theory. Foundational research sets off when we ask ourselves which are the basic assumptions underlying the theory, i.e. the nature of the principles on which the mathematical theory rests.

To accomplish this task concerning the grounds of mathematical theories, logic seems to be a good candidate, as Vänänen puts it:

> Mathematicians argue exactly but informally. This has worked well for centuries. However, if we want to understand the way mathematicians argue, it is necessary to formalize basic concepts such as the concepts of language and criteria of truth. [vaananen01, p. 506]

Logic is certainly useful for mathematics, but in which way? How can we apply logical ideas to mathematical concepts in a fruitful way?

Jaakko Hintikka tackles this question by putting forward an insightful distinction between two uses of logic in mathematics:[14]

- If we use logical notions (such as quantifiers, connectives, etc) for the purpose of "capturing" a *class of structures* studied in a particular mathematical theory, we are pursuing the descriptive use of logic. To be more precise, we exploit logic in the sense that we formulate an axiomatization of a mathematical theory in order to describe that class of structures and no other structures, as precisely as we can. Thus, a *descriptive use* of logic consists, for example, in formulating an axiom system in order to capture the class of structures which number-theory deals with, e.g. the series of natural numbers.

- If we want to systematize and formalize mathematicians' *reasoning* about the mathematical structures they are interested in, we are interested in the *deductive use* of

[13]For a more precise definition see Appendix, sections A.1, A.2.

[14][hintikka89, p. 89]

> logic. In fact, logicians have isolated those valid inference patterns, catalogued them and elaborated an effective way so that an infinity of such inference schemes is generated. This machinery appeals to the deductive consequence relation provided by the logic, which in turn is defined by a list of inference rules. A deductive use of logic consists, for example, in checking that the reasoning exploited in a certain mathematical proof is indeed valid, and thus follows from the mathematical axioms.

Once again, whereas *deductive* use exploits logic to generate and check the validity of inference patterns, in the *descriptive* we use logic as specifically applied to mathematics to describe mathematical structures.

These two uses of logic in mathematics result in a symbiotic relationship: the deductive use must take the descriptive use as its starting point, but the deductive use allows us to test the correctness of a given logical description of a structure.

In other words, the *deductive use* applies to a logical theory obtained in accordance with the *descriptive* task, and, at the same time, we say that we have a logical description of a mathematical structure when its theorems can indeed be *deduced* logically.

To go back to axiomatics, we can say that both descriptive and deductive tasks have to do with the axiomatic framework: in the *descriptive* case, we fix up some mathematical axioms in order to show that the theorems of the mathematical theories come to be captured by the axiomatic description; in the *deductive* case, the logical axioms are some of the valid inference patterns, viz. tautologies, and the logical theorems are other valid patterns derived by the rules of inference.

1.4 Types of completeness in formal theories

A sharp separation between the two uses of logic is vital to the present endeavour, viz. the analysis of non-standard models.

According to Awodey and Reck, formal axiomatization encompasses a fundamental feature: it intends not only to investigate mathematical concepts but also "to characterise them *completely*".[15] Still, the notion of completeness can be understood in different ways, so a clear account of different meanings of "completeness" is called for.

Nonetheless, the answer that Awodey and Reck put forward seems far from being adequately precise and unambiguous:

> In general, notions of completeness arise in contexts where axiomatizations are being undertaken with *specific goals in mind*. To say that an axiomatization is *complete* is, then, to say that the *axiomatizers* have achieved their goal,

[15] [awodey02, p. 5, my emphasis]

> in particular that no further addition of 'new axioms' is called for. In its mature mathematical form, formal axiomatics involves using a *formal language*, a language that is taken to be uninterpreted and for which various different interpretations can be considered and compared. [awodey02, p. 5, my emphasis]

Some phrases such as "specific goals in mind", "in particular that no further addition of 'new axioms' is called for" appear too vague and not really helpful without a further explanation. For example, what does "specific goal" mean?

Let us try to shed some light on this passage by referring back to Hintikka's distinction presented earlier. We have distinguished a two-fold use of logic in mathematics, but still we have not mentioned when these uses are fulfilled. The notion of completeness aims at this, i.e. specifying that the target, if any, we are gearing toward might be reached at a certain point. Thus, we can then talk about completeness with respect to each use of logic:

- completeness with respect to a *descriptive* use consists in a complete characterisation of the structure into consideration: the axiom system we formulate accurately describes (up to isomorphism) one and only one mathematical structure, the one we intended to study;
- completeness with respect to a *deductive* use consists in a complete derivation of all valid inferences: what we are aiming at is not only obtaining inferences which happen to be valid, but in fact all the valid patterns, viz. there is no inference pattern which we cannot deductively obtain.

Two explanatory remarks are in order. First of all, we must note that the two types of completeness apply to different domains. Whereas completeness with respect to the descriptive use of logic apply to *formal theories* (i.e. a set of formal axioms in the underlying language), and we call *categorical* a formal theory which is complete in this sense, completeness with respect to the deductive use of logic applies to *logic* (i.e. a formal language together with a deductive consequence) or some part of it. To refer to completeness in this latter case, we may use the word *exhaustiveness*[16]. Once again, categoricity hinges on the choice of the axioms formulated in order to capture a certain mathematical structure; exhaustiveness relies on the choice of the inference rule that enables us to grasp all the valid inference patterns.

In the second place, it is interesting to take into consideration the watershed between theories or logics which are complete and those which are not. Thus, we can enhance

[16] Commonly, in the literature a logic or a part of it is called "complete" when it captures all valid inference schemes. However, to avoid confusion, we use "exhaustiveness" as the corresponding term to "completeness" in the sense of all valid inferences, just as "categoricity" is used in the sense of completeness as uniqueness of models.

our knowledge of what is sufficient to transform an incomplete theory or incomplete logic into a complete one. For example, whereas we talk about a unique model when a theory is categorical, a non-categorical theory enables us to study the interplay between two kinds of models: the intended one, the one we would like to capture completely, and some unintended models, which come up as shortcomings. For the "exhaustiveness", we can investigate the relation between first order logics, which are exhaustive, and second or higher order logics, which are not.[17]

Exhaustiveness and categoricity do not exhaust all the kinds of completeness which are worth discussing. A third one, which combines the descriptive and deductive dimensions of logic, is *deductive completeness*. The ingredients that make deductive completeness a hybrid are, respectively, the reference to a formal theory, say T, formulated in the formal language L, and the reference to a logic, i.e. the tuple formed by L and the deductive consequence '$\vdash$'. So we say that the theory T is *deductively complete* if and only if for each sentence S in the language L, either S or its negation can be deduced from T by means of '$\vdash$'.

The last type of completeness that we take into account is an analogue of the deductive completeness: *semantic completeness*. A theory T is *semantically complete* if and only if for each sentence S in the language L, S or its negation is valid in the class M(T), i.e. the class of all the models which makes T true. This condition can also expressed as follows: all the models that are satisfied by the theory T make the same formulas true. By means of this definition we can easily see that semantic completeness exhibits a weaker condition on models than categoricity does. Whereas a categorical theory is also semantically complete since isomorphic models satisfy the same formulas, the models of a semantically complete theory are not taken to be isomorphic.

Moreover, we observe that if a theory is *deductively complete* and the underlying logic is *exhaustive*, it follows at once that the theory is *semantically complete* as well.[18]

Having said this, we can look at the passage by Awodey and Reck quoted above in a new light:

> To say that an axiomatization is *complete* is, then, to say that the *axiomatizers* have achieved their goal, in particular that no further addition of 'new axioms' is called for. [awodey02, p. 5, my emphasis]

What Awodey and Reck mean by completeness here is *categoricity* as long as the goal that the "axiomatizer" has in mind is a *descriptive* one. In fact, she wants to consider a mathematical structure and thus characterises it completely so that no other structure can fall under the axiom system (up to isomorphism).

[17]Note that here we do not consider second order logic with non-standard, or Henkin, semantics which is indeed exhaustive. See [vaananen01, pp. 504–506]

[18]See also section 2.5.

In such a context, it turns out quite natural to talk about *intended models*, i.e. the structure that an "axiomatizer" intends to study. Indeed, we can take an intended model to be any mathematical structure whatsoever, e.g. the natural number series, real numbers, topological spaces, lattices, etc, according to the mathematical branch we are interested in.

1.5 A heuristic to a descriptive use of logic

Let us go back to the axiomatic method for a moment. Since formal axiomatics is a "refinement" of axiomatics *simpliciter*, both kinds of axiomatics share some features.

In this respect, René Descartes noted that in axiomatics the way arguments are presented play a crucial role, and thus suggested an illuminating account of the order of exposition in axiomatics.

The distinction is discussed in Descartes' replies to the *Second Objections* to the *Meditations on First Philosophy*. Here Descartes indicates that the method of proof is two-fold: "l'une se fait par l'*analyse* ou résolution, et l'autre par la *synthèse* ou composition"[19]:

> *Analysis* shows the true path by which the thing was methodically discovered, as if a priori, so that, if the reader is willing to follow it and to pay sufficient attention to every point, he will *understand* it and *assimilate* it as perfectly *as if he had discovered it himself*. [descartes08, p. 99, my emphasis]

As if the reader speaks for herself, the analysis reconstructs the steps and chronicles the way through which she came to that discovery. However, despite its clarity, the analytic method has its drawbacks as well:

> But it has nothing that can compel the assent of a lazy or reluctant reader. For *if the slightest element in the argument is missed, the necessity of the resultant conclusions does not appear* [...] [ibidem, my emphasis]

The *analytic* method that Descartes proposes is neither an intuitive process, which would rely on the senses, nor an abstract formal deduction, which would be unable to elucidate the way the discovery was obtained. By contrast, the value of the analytic procedure lies in the connection with the "true way" which has made the invention possible, and shows the link of causal dependence.

On the other hand, the *synthetic* method proceeds differently:

> *Synthesis* [...] works in the opposite direction, retracing the path, so to speak, a posteriori. It *clearly demonstrates whatever conclusions have been drawn*,

[19][descartes96, vol. 7, p. 155, my emphasis]

> and makes use of a long string of definitions, postulates, axioms, theorems, and problems; so that if a reader should deny any of its consequences, it immediately shows that *the consequence is contained in the antecedents*, and thus *forces him*, however reluctant or recalcitrant, *to yield his assent*. But it is not so satisfying as the other, *nor so fulfilling for those who really wish to learn, since it does not reveal the method by which the thing was discovered.* [descartes08, p. 100, my emphasis]

Of course, Descartes' description of the synthetic procedure refers back to the geometry of the ancients and, in particular, to the Euclidean model. In contrast to the analytic method, this procedure gains the reader's consent by exploiting the peculiar "coercive" power of logic.

By the analytic-synthetic dichotomy, Descartes makes clear that different purposes require different methods of exposition: when for example *teaching* is considered, the analytic method must be preferred due to its propaedeutic nature; conversely, while we desire to present the *necessary entailment* between premises and conclusions, the synthetic method must be favoured.

After that, Descartes wonders whether we can benefit from a conjoint use of synthesis and analysis:

> Now I have followed this analytic method alone, as *the true and best way of teaching*, in my Meditations; but as for synthesis [...] although it plays a most valuable role in geometry, *when placed after analysis*, [...]. [ibidem, my emphasis]

Arguably, Descartes' clarification can be applied *mutatis mutandis* to present-day logic. The analogy lies in the fact that whereas logic textbook's presentations usually rely on the synthetic exposition[20], in ordinary university logical courses the analytic method complements the synthetic one since it puts itself forward as an introductory step to the formal presentation[21].

Note that also the way our investigation on non-standard models is carried out refers to the *analytic* method: our train of reasoning is indeed tracking backwards as we set off from the nature of axiomatics in order to reach an account of the non-standard models of arithmetic.

To our purposes, we claim that the *analysis* as presented by Descartes is the heuristic to the descriptive task of logic in mathematics. In particular, we will argue that Dedekind

[20]For arithmetical theories, for example, it is customary to follow the line of reasoning: we define the formal language, syntax and semantics; we present the axioms of the arithmetical theory; we list the arithmetical theorems provable within the arithmetical theory, etc.

[21]E.g. by stressing the reason why some logical notions come out to be defined in such-and-such way; by reconstructing how important logical proofs were carried out; by tracing unsuccessful attempts in order to see why they failed and where.

and Skolem, in their descriptive application of logic to arithmetic, exploit the analytic method as a heuristic.

1.6 From Standard models to Intended models

So far, we have considered the following notions: standard models, axiomatics in general, formal axiomatics and two possible uses of logic in mathematics. Since we want to examine the notion of non-standard model applied to arithmetic, we also noted that the descriptive use of logic is best suited to focus on a mathematical structure such as the series of natural numbers.

On top of that, in the descriptive task we strive to completely describe a certain model, called intended, and thus to rule out some other models, called unintended. We claim that the *intended/unintended* distinction can shed light on the *standard/non-standard* one. Indeed, for the moment let us assume that the two dichotomies are coextensive: as many authors, especially philosophers, we do not explicitly draw any distinction between standard and intended.[22] Hence, standard and intended, and accordingly non-standard and unintended, will be use interchangeably.

In the second place, besides the *synthetic* presentation of PA provided at the beginning of the chapter, we are urged to provide an *analytic* account of PA in accordance with our foundational (and thus explicative-oriented) purposes. Analysis turns out to be useful not only for propaedeutic reasons, but also for "analysing" the core results of our investigation into non-standard models.

Along these lines, to discuss Peano Arithmetic, we first put forward the insights behind the axiomatization of arithmetic, and only later return to the formal presentation of PA. Thus, the analytic method complements the synthetic in the following respect: the understanding how the axiom came to be discovered comes out as an unavoidable step for understanding the role of the axioms in the synthetic presentation.

1.6.1 Intended models and the axiomatization of arithmetic

Let us embark on the reconstruction of the main phases of a possible axiomatization of arithmetic. In doing so, we will try to put all the notions presented so far in place and draw mutual connections.

As noted above, axiomatic theories allow us to set up a list of formulas (which constitutes what we call a "theory") in a certain formal language. A theory is axiomatic in virtue of the following features: we stipulate some basic formulas to be true and we

[22] We will return to this point in chapter 4, section 4.1.2, as we assess to what extent the notions of unintended and non-standard may differ in meaning. In a sense, also Dedekind and Skolem can be considered among such authors even though they do not adopt this wording.

claim that the other formulas follow from these basic ones. As we turn to the modern logical framework, we have at our disposal an effective and mechanical notion of "follows from": the deductive consequence relation, written "$\vdash$".

Accordingly, it is easy to provide a proof if a formula is logically derivable from the set of basic formulas. Once again, since we have a recipe for dealing with derivable formulas, all the weight is carried by the axioms. This is the reason why the definition of a theory does not consider the set of all the formulas true in a certain model, but just the sole axioms which will derive all the other propositions. Now, what is highest on our "descriptive" agenda is to assess the best way to fix the list of axioms so that the outcome will be *categorical*.

Let us revisit a sketch of how an axiomatization of arithmetic may come about.

As we decide to assess which is the best way to formalize a given structure[23] (in our specific case the series of natural numbers), we must hold that its *accessibility* represents the *conditio sine qua non* to talk about it.[24] Obvious as this may seem, without granting this condition it is not possible to state any knowledge about arithmetic and, even less, to lay down the foundations of an arithmetical theory. Nevertheless, even when we speak of accessibility, we do not want to delve into the more problematic distinction between what we have access to and what we purport to have access to. Rather, we claim that as soon as we study natural numbers, we are indeed saying that the access to this subject-matter is tacitly assumed.

We should also note that, by addressing the accessibility issue in these terms, this requirement comes at a price, as Shapiro suggests:

> The "realism" indicates that mathematical discourse is taken at face value. Contra formalism, (most) mathematical assertions are regarded as meaningful assertions about mathematical entities. Mathematical truth is determined by the subject matter of mathematics and, thus, "truth" is synonymous with neither (real/ideal) "knowledge" nor "provability". [shapiro85, p. 715]

Following Shapiro, we point out that the descriptive task in mathematics as we are considering here tacitly hinges on a *realistic position* in philosophy of mathematics.[25] For our purposes, realism is understood in a loose sense, viz. we hold that there are mathematical structures that we want to study even though we do not take a precise position about the specific nature of these mathematical entities.

Once again, to account for the series of natural numbers we use logic in a descriptive fashion. So given the mathematical structure of natural numbers (in accordance with

[23]We assume no distinction in usage between the words "structure" and "model": with them we mean a framework or a context in which we can interpret formulas and verify whether in that context what these formulas say turns out to be true. For a precise definition see A.2.0.1

[24]See [giordani02, pp. 72–73]

[25]See [shapiro00, ch. 1–2, 8–10] for a general overview on the topic

our realistic assumption), our goal is two-fold: to set on record certain facts about the structure of natural numbers, and to render it as complete as possible by means of a (possibly categorical) axiomatic theory. To do so, we state a set of axioms so that we grasp the main features of the number sequence which then becomes known as the "intended model". In the end, accessibility and realism represent our primitive assumptions.

Now, insofar as we have access to the series of natural numbers, we gain insights into it. The bunch of intuitions so gathered can be shaped in a sorted and organised way. Accordingly, we want to *thematize* the insights about the intended model by means of a sort of code. [26] That is to say, we aim at determining a "theme", namely a criterion, in virtue of which we are able to sort our knowledge. Language has a key role in this step: only by means of a certain language are we entitled to codify our insights and thoughts. As a matter of fact, by "thematizing" what we have grasped about the series of natural numbers, we can start to build up the axiomatic theory.

The next step is to provide a preliminary formulation of statements which has to stand for the axioms of the theory. Since natural language is the "natural" mediator by means of which we can express features of the intended model, the tenets can be written down through natural language. For convenience, we can also rephrase some terms in the natural language by using a handy symbolism. In this case, we talk about a semi-formal language since the sentences come out as a fusion of natural language and a piece of symbolism that we have stipulated. Note that between the natural and the semi-formal presentation there is just a difference in degree, not in kind.

The statements so formulated constitute what we call the naïve step of the axiomatization. Commonly, the naïve axiomatization of arithmetic is exposed as follows:

The three primitive ideas are: 0, number, successor. By "successor" we mean the next number in the natural order. That is to say, the successor of 0 is 1, the successor of 1 is 2, and so on. By "number" we mean, in this connection, an element of the class of natural numbers. The naïve arithmetic is based on five principles:

1. 0 is a number.
2. The successor of any number is a number.
3. No two numbers have the same successor.
4. 0 is not the successor of any number.
5. Any property which belongs to 0, and also to the successor of every number which has the property, belongs to all numbers.[27]

[26] See [giordani02, pp. 222–230]
[27] See [russell19, p. 163]

As noted above, despite being axiomatic, this presentation cannot be considered formal yet. To obtain a formal axiomatization of arithmetic, we have to replace all the intuitive notions embedded in the naïve axiomatization, e.g. the concept of “number”, “successor” and “0”, with variables and constants so that we are able to reformulate each proposition in a certain formal language in accordance with its syntactic rules.

If, for instance, we pick set theory as framework, a possible formalization of the naïve arithmetic above comes out as follows:

The three primitive terms are N, 0, and S: N is a set, 0 is an element of N, and S is a function from N to N. The axioms are

1. $0 \in N$
2. $\forall x(x \in N \to S(x) \in N)$
3. $\forall x(x \in N \to S(x) \neq 0)$
4. $\forall x \forall y(x, y \in N \to (S(x) = S(y) \to x = y)$
5. $P \subseteq N \wedge (0 \in P) \wedge [\forall x(x \in P \to S(x) \in P)] \to P = N$

Even though this is rarely stated explicitly in the axiomatizations, the naïve formulation constitutes a crucial step towards a formal system as long as it comes as conceptually prior to the formal stage. Of like mind is Hodges as he presents Peano Arithmetic:

> I begin with naive arithmetic, not formal Peano arithmetic. One needs to have at least an intuitive grasp of naive arithmetic in order to understand what a formal system is. [hodges01, p. 108]

In this spirit, as soon as the formalization is carried out, we still have to check whether the formal theory is indeed suitable for the intended model we started from. To do so, we have to interpret each symbol of the formulas of the logical theory in the intended model, so that it is possible to check whether the properties stated by the formulas hold in such a structure.

With respect to arithmetic, we interpret the formulas of a logical theory, e.g. Peano arithmetic, as arithmetical statements and then verify whether they hold or not.[28]

In particular, such verification can be carried out via effective means by defining the semantic relation between models and formulas within a logical system.

Basically, the formal language is two-fold:

[28] This claim calls for a comment. Roughly speaking, “interpretation” can be seen as a mapping from the set of symbols of formal theory into the set of individuals, functions and relations of a model. The “official” formulation of Peano arithmetic exploits the same symbols that belong to everyday arithmetic. In other words, the symbol “0”, “+” or “$\cdot$” denote, in the formal theory, a constant and two binary functions, respectively, while in a mathematical book denote what we call “zero”, “plus” and “times”. According to this, by no means must the constant “0” of PA be confused with “0” of mathematical books, but rather the constant “0” can be interpreted as “0” of ordinary mathematics. To keep the distinction clear, some authors speak of “ $\bar{0}$” instead of 0 for the formal language constant.

- on the syntactic side, we focus on the axioms of the logical theory of arithmetic given that the theorems can be effectively derived by a deductive consequence relation. That is, we have an effective machinery by means of which we can write down new formulas, according to the rules, that stem from the old ones. These formulas are said to be deducible or derivable from the former;
- on the semantic side, we consider the connection between the logical theory and the model which it is supposed to characterise. By the semantic relation defined between the model and the logical theory, we are able to decide whether the model is a model of the theory (equivalently, the theory is true in the model) or not.[29]

To go back to the formal axiomatization, once the semantics and the syntax of the logical language are spelled out, we can check whether the obtained formal theory, say T, is

⋆ *sound*, i.e. the necessary condition of the formalization that all the formulas that can be derived from T are true in the intended model of natural numbers;

⋆ *complete* in the different senses of *categorical*, *syntactically complete* or *semantically complete*: the sufficient conditions to capture in a complete way the intended model and the logical theorems of T true in the model.

In particular, with respect to the descriptive use of logic in mathematics, we can express the interplay between intended model and axiomatization as follows: the *ratio essendi* of the axiomatization is achieving categoricity, i.e. the full characterisation of the model at hand; conversely, the *ratio cognoscendi* of a complete description is the formal axiomatization.

The diagram 1.1 wraps up the main phases of formal axiomatization.

- We turn to an *accessible* mathematical structure in order to study and axiomatize it. We call this model *intended* since our purpose is to describe that very model. In our case it is represented by the series of natural numbers.
- The intended model is being *thematized* in a naïve axiomatic theory via semi-formal language, e.g. the naïve arithmetic. Some concepts are still intuitive and implicit and the axioms still rely on self-evidence.
- For this reason, we go through the formalization so that each proposition of the naïve theory is rephrased into an equivalent and well-defined formal theory based on a formal language.

[29]For a detailed presentation of syntax and semantics see the Appendix.

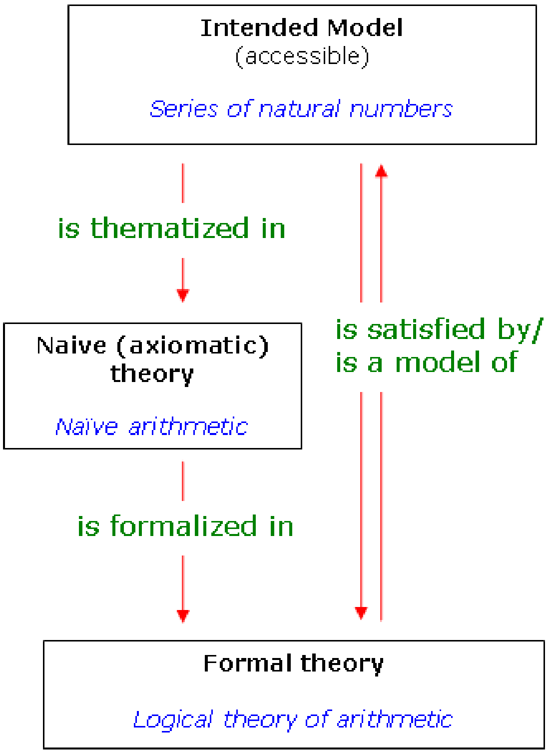

Figure 1.1: A sketch of the formal axiomatization of arithmetic

- Logic equips us with a tool to find out whether the formal theory is complete with respect to the intended model and the logical theorems of the theory true in the model.

To sum up, the adjective *intended* denotes the model which the axiomatization sets off from. Thus, only one model can be defined as intended and it corresponds to the starting point of the whole descriptive use of logic. The whole issue rests on the question: is it possible to speak of unintended models? If so, how can we characterise the correlation between unintended and non-standard models?

1.6.2 Final remarks

We conclude this section with two final considerations.

The descriptive use of logic stands or falls with the possibility to make it a meaningful question whether the theory is true only for the intended model and thus characterises the mathematical structure "completely" in the sense of "categoricity".

In the section above, we have offered a plausible reconstruction of the so to speak "way of discovery" of a formal theory with respect to an intended model. Generic as it might seem, a question arises: has this procedure ever adopted to axiomatize arithmetic?

Does this *modus operandi* have something to do with the formulation of Peano arithmetic and, accordingly, with the definition of the standard model of arithmetic?

We argue that there is evidence for a positive answer by referring to a famous unpublished letter by Dedekind. To put it briefly, Dedekind furnishes an illuminating explanation regarding his way of going about and analysing the series of natural numbers.

In the second place, we have to focus on the notion of categoricity as a possible goal, and last, but in no way least, illustrate whether such a complete axiomatization of the series of natural numbers is indeed possible. The next chapter will be devoted to logical aspects and completeness along the lines of Dedekind's work.

1.7 Non-Standard models: an attempted definition

Before turning to the historical account of Dedekind's and Skolem's perspective about non-standard models, we will commit ourselves to a first but sketchy definition of non-standard models of arithmetic.

By doing so, we want to emphasise the importance of these models as the conceptual thread of the whole text and their connection with what has been said so far. As said earlier, we stipulate that unintended models amount to non-standard models. So, if "standard model" stands for the intended model that an "axiomatizer" would like to axiomatize, "a non-standard model" will turn out to be an unwanted result of the axiomatization, namely a model which is unintended.

Along these lines, in the opening line of his "Non-Standard Models in a Broader Perspective" Haim Gaifman suggests that:

> A non-standard model is one that constitutes an interpretation of a formal system that is *admittedly different* from the intended one. [gaifman03, p. 1]

Once again, a mathematician considers a mathematical structure (namely, her standard or intended model) and uses logic to state some axioms within her favourite formal language in order to describe such a model. After that, she checks back whether the model she considered actually validates the axioms and thus the relative theorems. If this is the case, our mathematician considers whether the standard model she had in mind is indeed the only model which makes the axioms true.

However, trivial as it might seem, Gaifman's definition hides some pitfalls. We must unravel what *admittedly different* means, we must investigate the tacit assumption that the standard and non-standard are concept coextensive with intended and we must determine unintended, respectively, and how to apply Gaifman's definition to the concrete arithmetical case.

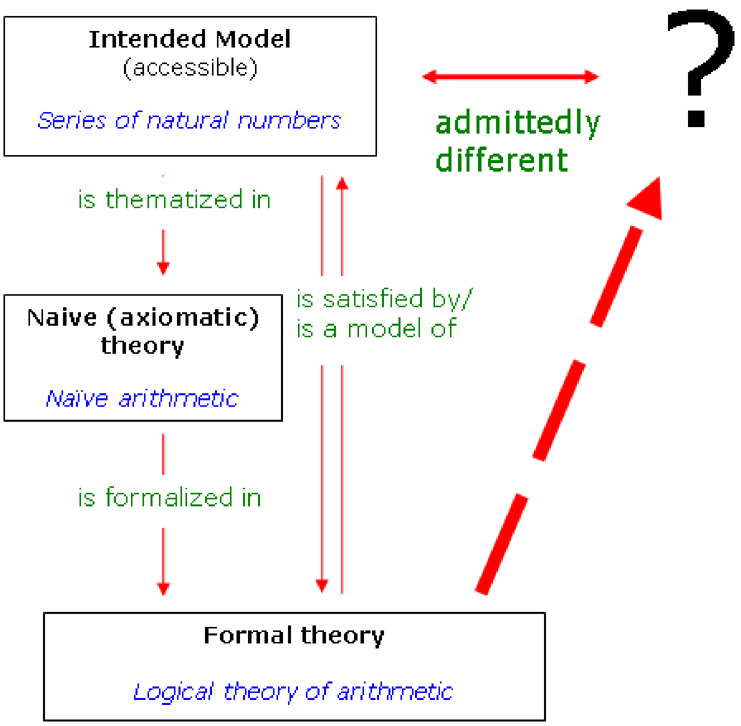

Figure 1.2: Non-standard models of arithmetic: a first attempt

Chapter 2

The philosophical phase - Dedekind and the birth of NMoA

After some preliminary considerations, we are now ready to present a historical treatment of non-standard models. In this section we claim that not only Thoralf Skolem (23 May 1887 - 23 March 1963), but also Richard Dedekind (October 6, 1831 - February 12, 1916) has to be considered as forefather of the field of non-standard models of arithmetic (NMoA for short).

First of all, we shall say a word about a plausible historical classification of NMoA over the period from their first introduction to present-days. Arguably, we can distinguish three chronological as well as conceptual phases in the development of research on NMoA.

The first period, that we call *philosophical-pathological*, goes from their "birth", that we date around 1890, to the 1950s.

The adjective "philosophical" traces back to the classification outlined by Smorynski:

> It is also worth mentioning that Skolem's goal in constructing nonstandard models was *philosophical*: he aimed to show that first-order logic could not characterise the number series; he did not care to start a new subject. Until the 1960s, this was generally the case - nonstandard models of arithmetic were either objects of philosophical interest or tools, not objects of mathematical interest in their own right. [smorynski84, p. 3, my emphasis]

Thus, we speak of philosophical phase in the following terms: Dedekind (as we will show) and Skolem did not go beyond a philosophical-oriented observation. Indeed, they assess the pros and cons of non-standard models as a facet of foundational inquiry in mathematics.

In the second place, the reason for "pathological" lies in the way NMoA were initially conceived. At their origins, non-standard models were put in a pretty negative light: as soon as they came up, they were conceived as a negative side effect of the formal axiomatization. In this respect, both Dedekind and Skolem refer to the existence of NMoA as an undesirable shortcoming: in modern terms, for Dedekind the outcome is due to an inappropriate axiomatization, for Skolem there is an intrinsic dependence between NMoA and the choice of logical framework to adopt. To sum up, both mathematicians find these models so philosophically troubling that they have to seek for a remedy to cleanse the logical framework on which the arithmetical theory is based:

- ⋆ Dedekind suggests to exploit the notion of *chain* as a possible way out. To use the modern terminology, by the notion of chain Dedekind employs an induction axiom formulated in second order language;

- ⋆ Skolem confines himself to the mathematical purely finite aspects so that we cannot reach non-standard elements: his philosophical view had a profound impact on his mathematical results.

Still, the crux of the problem brings us back to the intrinsic connection that ties up a certain formal axiomatization with the expressive power of the underlying language.
Along these lines, NMoA are pathological in the sense that, however we fix an axiomatization that is able to capture the series of natural numbers, there will be certain models that satisfy the very same formulas describing the intended model but which are also *admittedly different* from the standard one.[30]

In other words, non-standard models of arithmetic trace back to the notion of unintended model to in the following sense:

> [in the arithmetical case] *non-standard models* show an *essential shortcoming of a formalized approach*: the *failure to fully determine the intended model*. The reason for the difference is obvious: in as much as the intended model is problematic, the existence of non-standard models support one's doubts. But when the intended model is accepted as a *basic precondition of our mathematical investigations*, the existence of non-standard models points to the *inability of the formalization to characterize the intended model*. [gaifman03, p. 3, my italics]

Just as a "pathology" is the study and diagnosis of a disease through examination of the body of a human being, the analysis of non-standard models of arithmetic comes out as the study and the diagnosis of a "logical disease" through examination of the axiomatization of the series of the natural numbers.

Common view on non-standard models of arithmetic In spite of the fact that non-standard models of arithmetic do not meet with large success in logic textbooks, one

[30] As will see, this actually applies to first-order theories of arithmetic, but not to the second-order ones.

of well-established fact concerning them is that the Norwegian mathematician Thoralf Skolem has to be credited with their "discovery":

> [...] Such models of Δ are called non-standard models of arithmetic. They were first constructed by Skolem [1934], and today people hold conferences on them. [hodges01, p. 70]

> Non-standard models have been introduced by Skolem, in a series of papers from 1922 to 1934, in two cases: set theory and arithmetic. [gaifman03, p. 2]

In other introductory texts, e.g. [vandalen04] and [boolos07], inasmuch as the technical facet supersedes historical details on Skolem's endeavour, Van Dalen and Boolos simply put the existence of non-standard models as a consequence of Gödel's *Compactness* theorem or, more trivially, of the *Upward Löwenheim-Skolem* theorem:

> The proof of the existence of an (enumerable) nonstandard model of arithmetic is as an easy application of the compactness theorem (and the Löwenheim-Skolem theorem). [boolos07, p. 302][31]

We argue that both claims are not correct in the following respects:

- In contrast to Hodges and Gaifman, we claim that Dedekind has a role to play in the discovery of non-standard models of arithmetic. The whole argument hinges on a Dedekind's letter dated 1890 which provides concrete evidence of a construction of a non-standard model of arithmetic more than 40 years before Skolem's.
- Along the historical lines, the application of the *Compactness* or the *Upward Löwenheim-Skolem* theorem cannot do justice to the train of thought employed by Skolem in his papers [skolem33] and [skolem34] where NMoA were presented, nor does it take into account the fact that Skolem does not fully acknowledge the *Compactness* result and denies the *Upward Löwenheim-Skolem* due to his adamant *finitistic* position in mathematics. Van Dalen and Boolos neglect this important historical aspect in the presentations of NMoA.

While the latter point will be discussed in detail in chapter 3, the following sections will be devoted to Dedekind's work.

[31] See also [vandalen04, pp. 113,121]

2.1 Dedekind and the birth of non-standard models

Dedekind's work in arithmetic represents a milestone in foundational studies and an unavoidable step towards understanding the significance of the nature of NMoA. This section sets forth the core of our argument.

We shall argue that Dedekind's letter written in 1890 to his colleague Hans Keferstein (a mathematician then active in Hamburg) shows clear evidence of Dedekind's awareness of the existence of non-standard models of arithmetic.

After that, we will consider Hao Wang's paper of 1957 discussing this very letter. Although Hao Wang unearthed the letter and was the first who envisaged Dedekind as the forefather of the study of NMoA, in the paper he puts forward general and ill-conducted arguments in supports of this historical account. As a matter of fact, Wang's approach steered philosophers and historians away from the idea that Dedekind could have portrayed a concrete example of NMoA.

Not only will we present Dedekind's reasoning with respect to non-standard models, but we will also point out that his text provides an extraordinary example of formal axiomatization for the arithmetic. This corroborates what said in

- section 1.5, where we ventured to explain why the *analytic* approach can be useful for the mathematical case. With the letter to Keferstein, Dedekind actually offers an "analytic" account of the synthetic presentation preferred in "the Nature and Meaning of Numbers", published two years before.
- section 1.6.1, where a possible reconstruction of the way formal axiomatics could be carried through was presented. Dedekind's letter actually covers all the steps he followed in order to axiomatize arithmetic. This makes it an unavoidable landmark if we want to study the interplay between formal axiomatization and the intended model of arithmetic.

2.1.1 Dedekind's work on the series of natural numbers

In 1888 Dedekind published his well-known work on the series of the natural numbers, entitled "the Nature and Meaning of Numbers"[32] (NMN for short). The text came out as one of the masterpieces both in mathematical and philosophical respects.

- In the first place, Dedekind exploits, for the first time, revolutionary set-theoretical techniques. In this respect, we find a clearly stated train of definitions and theorems that hardly differs from modern presentation in set theory[33].

[32]We will cite the English translation [dedekind01]

[33]For details see [reck03, pp. 374–375]

- Secondly, Dedekind offers a brand-new philosophical standpoint concerning mathematics and its subject matter. Notably, Dedekind is widely recognised as a forerunner of the modern view of structuralist view in philosophy of mathematics.[34]

- In addition, Dedekind sets the agenda for what later became the modern axiomatization of arithmetic. In other words, in characterising the series of natural numbers he pioneers an extensional approach[35] (in contrast to the intensional one pursued by Frege)[36] and puts himself as one of the founders of the logical study of arithmetic. His insights paved the way for what is known today as Peano Arithmetic. Along these lines, PA can be seen as a refinement of what was already articulated and elaborated in Dedekind's NMN.[37]

NMN is an axiomatic presentation made up of 172 propositions (divided in definitions, theorems and remarks) structured in 14 main sections. Roughly speaking, we can distinguish six conceptual parts:

- §1 - §5 constitutes the preliminary definition of the main concepts that will be involved afterwards (e.g. system[38], being part of, functions, similar function, similar systems, chain, etc...)

- §6 - §8 closely considers the notion of simply infinite system[39] and traces a connection with the series of natural numbers;

- §9 - §10 concerns meta-theoretical properties of the natural numbers so defined, namely the kind of definitions allowed (definition by induction) and properties of the class of simply infinite systems (categoricity).

- Finally §11 - §15 deals with the arithmetical operations and relative theorems that are indeed definable in the framework.

Inasmuch as we consider the revolutionary impact that NMN had in mathematics at that time, it comes as no surprise that the text was not immediately appreciated and fully grasped by then-contemporary scholars. Among them, Dedekind had a significant correspondence with Keferstein about issues concerning "the Nature and Meaning of Numbers". In particular, Keferstein addresses strong criticism on Dedekind's tenets. What is at issue is the choice of Dedekind's assumptions and the way he had developed them throughout the paper.

[34]See [tait96], [reck03] and [reck00]

[35]See [wang57, pp. 156–157]

[36]See [tait96] for comparison with Frege, and [reck03] for similarities with Kronecker

[37]See the table on page 31

[38]From now on, the word *system* should be taken in Dedekind's sense, i.e. a mathematical set. See [ferreiros07]

[39]Roughly speaking, a simply infinite system represents a mathematical structure that has the same properties of the structure of the series of the natural numbers.

Roughly speaking, Keferstein disagrees with Dedekind about the starting presuppositions: he argues that some of them are superfluous and thus are to be discarded. However, Keferstein's suggestions for amending the text actually reveal "his lack of real understanding of some fundamental points"[40].

In response, Dedekind puts forth forcefully his analysis by a firm reply. He provides a point by point justification of the underlying reasons according to which he conducted his work. Thus Dedekind puts forward the thorough analysis he carried out prior to the wording of his work on the natural numbers. Dedekind is utterly convinced that Keferstein's critique is due to a lack of understanding of his arguments on which the entire NMN rests, despite being only latent in the phrasing of NMN. To sum up, in the letter to Keferstein Dedekind unwraps the whole plan he had exploited in NMN.

2.2 Letter to Keferstein

Dedekind writes his response to Keferstein in 1890, two years after the publishing of NMN. The letter consists in a short text, five pages in the unabridged English translation[41], which can be conceptually divided in three parts: the introductory remarks, the exposition of the nine key intuitions on which Dedekind based his essays NMN, and the last part dedicated to a point by point refutation of Keferstein's misconceptions. Here we will focus on the first two parts.

Hao Wang has to be credited as the first who drew attention to this letter and translated it into English. In his 1957 paper, Wang strives to account for the following foundational issue, as he puts it in the very opening line:

> I once asked myself the question: How were the famous axiom systems, such as Euclid's for geometry, Zermelo's for set theory, *Peano's for arithmetic*, originally obtained? This was to me *more than merely a historical question*, as I wished to know *how the basic concepts and axioms were to be singled out*, and, once they were singled out, *how one could establish their adequacy*. [wang57, p. 145, my emphasis]

This is nothing but the very issue which Dedekind pursues in his letter. In fact, Dedekind puts forward the logical framework of arithmetic by supplying an "analytic" account (in Descartes' sense) that sheds light on the "synthetic" presentation given in NMN, i.e. to explain how the "basic concepts and axioms were to be singled out".

In this respect, Dedekind wants to elucidate the key intuitions behind NMN by tracing the order in which they were presented. Thus, he explains the basic steps underlying

[40] [vanheijenoort67, p. 98]
[41] [dedekind90, pp. 99-103]

NMN, i.e. how he first obtained the basic tenets on which NMN rest and the way they were shaped into that essay.

One opening consideration is in order. Descartes' analytic-synthetic dichotomy can be successfully exploited in Dedekind's letter in the following terms: Dedekind considers the analytic exposition as an important step to unravel the meaning of the axiomatic presentation provided in the "the Nature and Meaning of Numbers". In confirmation of that, Dedekind employs an explicative approach that sets forth to reconstruct all the phases he followed so that he is able to present the reasons underlying the such-and-such list of axioms and principles.
Moreover, it must be said that Dedekind's choice comes by no coincidence. Not only does Dedekind's letter, in virtue of its explicative purpose, fall under what Descartes calls analytic, but also Dedekind himself points out a similar classification at the end of the second part of the letter, just after the discussion of the nine basic insights of NMN:

> Thus the analysis was completed and the synthesis could begin; but this still caused me trouble enough! Indeed the reader of my essay does not have an easy task either; apart from sound common sense, it requires very strong determination to work everything through completely. [dedekind90, p. 101]

The parallel between Dedekind and Descartes is striking: over a "coercive-synthetic" way of reasoning, peculiar to the axiomatic presentation such as NMN, the analytic presentation, which unravels the underlying tenets, is preferred by Dedekind as a first step. We should point out that the habit of leaving most of the analytic-like considerations as implicit is a widespread mathematical as well as logical.

At the time of NMN, Dedekind's work represented a revolutionary exposition executed with brand-new concepts and techniques. Thus, the appeal to the analytic method was likely to be a helpful device in order to establish more consensus among then-contemporary mathematicians since it would have been much harder to reach the same agreement with a purely axiomatic presentation.

To sum up, Dedekind's text corroborates the argument for the importance of the analytic method in foundational studies. Let us turn now to the core of the letter.

2.2.1 Dedekind's introductory remarks

In the very first line of the letter Dedekind points out the necessary conditions for a careful analysis of the series of the natural numbers. They represent the *condition sine qua non* on which the work is based:

> But I would ask you not to rush anything in this matter and to come to a decision only after you have once more carefully read and thoroughly considered the most important definitions and proofs in my essay on numbers, if

> you have time. I should like to ask you to lend your attention to the following train of thought, which constitutes the genesis of my essay. [dedekind90, p. 99]

The point that Dedekind wants to make here is that, if one carefully follows his line of reasoning, then this will lead one to full assent. Dedekind is persuaded that this suffices to chronicle the steps he followed, which were not fully brought up in NMN, and trace again his reasoning once again.

> How did my essay come to be written? Certainly not in one day; rather, it is a synthesis constructed after protracted labor, based upon a prior analysis of the sequence of natural numbers just as it presents itself, in experience, so to speak, for our consideration. [ibidem]

The preliminary step consists in looking at the series of natural numbers and investigating its properties "as it presents itself, in experience". As suggested in the previous chapter, this amounts to saying that there is a structure, viz. the sequence of natural numbers, which is accessible to us.

Before going through the second part of the letter, there are two aspects that need to be spelt out. To do this, we have to introduce some notations of NMN.

To begin with, it must be said that the ultimate purpose of Dedekind's NMN is to trace arithmetic, once characterised in a logical fashion, to our understanding. That is to say that the most primitive notions of arithmetic are assumed to fall under a few logical tenets which according to Dedekind correspond to what he calls "the primitive notions of the understanding".

In particular, Dedekind considers three primitive concepts:

- the *thing*, i.e. "every object of our thought"[42];
- the *system*, which is formed by "different things" that "for some reason can be considered from a common point of view, can be associated in the mind"[43];
- the *transformation*, i.e. "the ability of the mind to relate things to things, to let a thing correspond to a thing, or to represent a thing by a thing"[44].

According to Dedekind, things, systems and transformations constitute the necessary conditions "without which no thinking is possible"[45] since things and systems represent the primitive objects and transformations the primitive activity.[46]

[42] [dedekind01, p. 21]
[43] [ibidem]
[44] [dedekind01, p. 15]
[45] [ibidem]
[46] Note that here functions are taken as primitive rather than being reduced to sets, as customary today.

Overall, Dedekind's pursuit consists in reconstructing the whole arithmetical building by only referring to these three notions, as he points out in the letter:

> And how should we **divest** these properties of their **specifically arithmetic character** so that they are **subsumed under more general notions** and **under activities of the understanding** *without* which no thinking is possible at all but *with* which a foundation is provided for the reliability and completeness of proofs and for the construction of consistent notions and definitions? [dedekind90, p. 100, original emphasis, bold mine]

This quotation puts forward in a clearer way Dedekind's project. Dedekind is not indeed interested in providing the basic principles solely for arithmetic but rather in studying the natural numbers in terms of "general notions" and "activities of the understanding". According to Dedekind, arithmetic is indeed a part of logic, viz. "an immediate result from the laws of thought"[47], and thus if one accounts for the logical notions of *thing*, *system* and *transformation*, then one will be able to account for the arithmetical principles.

This remark becomes important inasmuch as we consider Dedekind's detractors[48] who argue that Dedekind's principles or conditions do not turn out to be genuine axioms. By contrast, Dedekind seems to favour a position according to which the "proper axioms" are the logical ones, viz. the most general notions of the understanding, and the arithmetical principles are formulated in terms of these notions.

The second observation has to do with the Dedekind's impact on the way logic has been developed over the following years.

There is no doubt that Dedekind's primitive notions do resemble current set-theoretical terminology. In confirmation of the deep influence that Dedekind's work had on the field, his constructions still look very familiar to us. On top of that, the following passages of Dedekind speak for themselves:

> 1. In what follows I understand by *thing* every object of our thought. In order to be able easily to speak of things, we designate them by symbols, e. g., by letters, and we venture to speak briefly of the thing a or of a simply, when we mean the thing denoted by a and not at all the letter a itself. A thing is completely determined by all that can be affirmed or thought concerning it. [dedekind01, p. 21, original emphasis]
>
> 2. It very frequently happens that different things, a, b, c, . . . for some reason can be considered from a common point of view, can be associated in the mind, and we say that they form a *system* S; we call the things $a, b, c, \ldots$

[47] [dedekind01, p. 14]

[48] See [awodey02, p. 7] and [reck03, p. 378]

> *elements* of the system S, they are *contained* in S; conversely, S *consists* of these elements. [ibidem, original emphasis]

> 21. Definition. By a *transformation* φ of a system S we understand a law according to which to every determinate element s of S there *belongs* a determinate thing which is called the *transform* of s and denoted by $\varphi(s)$ [...]. For convenience, we shall denote the transforms of elements s and parts T respectively by s' and T';[49] [dedekind01, p. 24, original emphasis]

It is easy to see that we can recast things, systems, and transformations in modern terminology as elements, sets and functions, respectively, of the familiar set-theoretical framework.

Dedekind considers two special cases in as far as the transformation is concerned: *similar transformation* and *transformation in itself*.

> 26. Definition. A transformation φ of a system S is said to be *similar* or *distinct* when to different elements a, b of the system S there always correspond different transforms $a' = \varphi(a)$, $b' = \varphi(b)$. [...] in this case conversely from $s' = t'$ we always have $s = t$ [...]. [dedekind01, p. 25, original emphasis]

There is no doubt that Dedekind's similar (or distinct) transformation can be rephrased as the modern injective function, i.e. a function, say f, is injective if for all $x, y \in dom(f)$, if $x \neq y$ then $f(x) \neq f(y)$.

Besides the similar transformation, Dedekind introduces the notion of transformation on itself:

> 36. Definition. If φ is a similar or dissimilar transformation of a system S, and $\varphi(S)$ part of a system Z, then φ is said to be a transformation of S *in* Z, and we say S is transformed by φ in Z. Hence we call φ a transformation of the system S *in itself*, when $\varphi(S) \subseteq$ [50]S, and we propose in this paragraph to investigate the general laws of such a transformation φ. In doing this we shall use the same notations as in II.[51] and again put $\varphi(s) = s'$, $\varphi(T) = T'$. These transforms s', T' are by (22), (7) themselves again elements or parts of S, like all things designated by italic letters. [dedekind01, p. 27, original emphasis]

This function represents what we call today "a set closed under a given function". Note that the two kind of transformation are compatible with each other: a transformation can be both *similar* and *in itself*.

Equipped with all the notions at hand, we can now move on to Dedekind's presentation of the key principles, viz. the second and most important part of the letter.

[49] This amounts to say that $s' = \varphi(s)$ and $T' = \varphi(T)$ for a transformation φ.

[50] Henceforth, we replace the symbol '3' used in the original text with '$\subseteq$'. See def. 3, [dedekind01, pp. 21–22]

[51] See footnote 49.

2.2.2 Dedekind's tenets for the series of natural numbers

Dedekind offers an illuminating explanation of the features of the series of the natural numbers in the following five points:

> (1) The number sequence N is a $system$ of individuals, or elements, called numbers. This leads to the general consideration of systems as such (§1 of my essay).
> (2) The elements of the system N stand in a certain relation to one another; a certain order obtains, which consists, to begin with, in the fact that to each definite number n there corresponds a definite number n', the succeeding, or next greater, number. This leads to the consideration of the general notion of a $mapping$[52] φ of a system (§2), and since the image $\varphi(n)$ of every number n is again a number n', and therefore $\varphi(N)$ is a part of N, we are here concerned with the mapping φ of a system N *into itself*, of which we must therefore make a general investigation (§4).
> (3) Distinct numbers a and b are succeeded by distinct numbers a' and b'; the mapping φ, therefore, has the property of distinctness, or *similarity* (§3).
> (4) Not every number is a successor n'; in other words, $\varphi(N)$ is a proper part of N. This (together with the preceding) is what makes the number sequence N infinite[53](§5).
> (5) And, in particular, the number 1 is the $only$ number that does not lie in $\varphi(N)$. [dedekind90, p. 100, original emphasis]

To put it in modern notation, Dedekind actually characterises natural numbers as a logical entity satisfying the following conditions:

1. The series of natural numbers is a set [system], say N.

2. The elements of N stand in successor relation to one another, i.e. there is a function [mapping or transformation] φ defined for every number n in N. $\varphi(n)$ represents the successor of n, in symbol n'. The system N is closed under the successor function φ [transformation of a system into itself], i.e. $\varphi(N) \subseteq N$.

3. The successor function φ is also injective [similar], i.e. for all x, y in N, $x \neq y \rightarrow \varphi(x) \neq \varphi(y)$.

4. There is a number n which is not in the range of φ, i.e. $n \notin \varphi(N)$ for some n. Hence, $\varphi(N) \subset N$. According to Dedekind's definition, this makes N infinite.

[52] Note that Dedekind adopts the term "mapping" as a synonym of "transformation".

[53] Dedekind defines a system to be "infinite" when there exists an injective function on a proper part of itself. [dedekind90, p. 31]

5. The first element of the sequence of N is indeed the only element with this property, i.e. $1 \notin \varphi(N)$ where 1 is a constant.[54]

Dedekind's exposition is accurate: this synopsis gathers together all his ingenious set-theoretical constructions that he had adopted in NMN to logically characterise the succession of natural numbers.

For the reasons exhibited above, these five points are presented as principles rather than axioms. Still, if we look at NMN, Dedekind had presented them as conditions in def.71 as follows:

α. $\varphi(N) \subseteq N$ [...]
γ. The element 1 is not contained in $\varphi(N)$
δ. The function φ is similar.[55]

To have an idea what a system satisfying these three conditions looks like, we consider the following diagram:

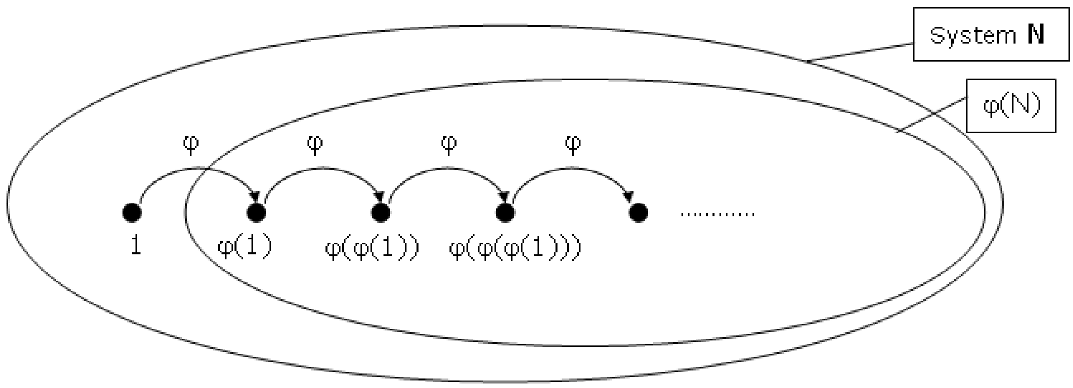

Figure 2.1: A system satisfying conditions α, γ, δ

To sum up, both Dedekind's presentations, the one in the letter and the one in NMN, result in the same naïve axiomatization of the series of the natural numbers.

On top of that, in some theorems and definitions Dedekind means with the same symbol both a variable and a constant.[56]

Disregarding the logical twists and turns, Dedekind's insights still represent the basis of modern formulation of Peano Arithmetic.

[54]Note that Dedekind argues that 1 rather than 0 is the first number of the series of natural numbers. Troubling as it might seem, we do not run into difficulty if we take 1 to represent just a symbol in Dedekind's language that stands for the first number of the series of the natural numbers. That is, the symbol "1" is interpreted as the actual first element of the series of the natural numbers "zero".

[55][dedekind01, p. 34]. Note that we replaced N' with $\varphi(N)$ in accordance with the footnote 49.

[56]See [dedekind01, Thm.71, 72] and [reck03, pp. 379–380]

Recall Peano Aritmetic, i.e. the customary way the formal axiomatization of arithmetic. PA consists of seven axioms:

1. $\forall x \neg (S(x) = 0)$
2. $\forall x \forall y (S(x) = S(y) \rightarrow x = y)$
3. $\forall x (x + 0 = x)$
4. $\forall x \forall y (x + S(y) = S(x + y))$
5. $\forall x (x \cdot 0 = 0)$
6. $\forall x \forall y (x \cdot S(y) = (x \cdot y) + x)$
7. $\forall \vec{x}[(\varphi(0, \vec{x}) \wedge \forall y(\varphi(y, \vec{x}) \rightarrow \varphi(S(y), \vec{x}))) \rightarrow \forall y \varphi(y, \vec{x})]$

The idea which lies behind the axiomatization is the following: each axiom aims at characterising, as it were, the behaviour of the symbols of $\mathcal{L}_{PA}$, namely S, +, $\cdot$, and 0. Axioms 3, 4 and 5, 6 pinpoint the meaning of the function symbols + and $\cdot$, respectively. They were all formulated as such by Herman Grassmann in 1861 and remained unchanged in this very form to this day.[57] Dedekind is aware of Grassmann's formulation and actually exploits it in section 11 of NMN. By contrast, axioms 1, 2, 7 come out as the result, in varying degree, of the logical developments throughout the last century. Putting aside the last axiom, the so-called induction axiom, axioms 1 and 2 came into being in Dedekind's NMN. To trace a brief history of the logical developed of axioms 1 and 2 from Dedekind's work to the present day, we consider the following table:

Dedekind's letter to Keferstein: principles (2), (3), (5)
1. "Since the image $\varphi(n)$ of every number n is again a number n', and therefore $\varphi(N)$ is a part of N"   2. "Distinct numbers a and b are succeeded by distinct numbers a' and b'"   3. "The number 1 is the *only* number that does not lie in $\varphi(N)$"
Dedekind's "The Nature and Meaning of Numbers": def.71, conditions α, γ, δ
1. "$N' \subseteq N$".[58]

[57] "He [Grassmann] was probably the first to introduce recursive definitions for addition and multiplication, and prove on such a basis ordinary laws of arithmetic by mathematical induction." [wang57, p. 147]

[58] Recall def. 26: "$N' = \varphi(N)$".

2. "The function φ is similar."[59] 3. "The element 1 is not contained in N'."
Peano's "Arithmetices Principia, Nova Methodo Expositia"[60]: axioms 6, 7, 8
1. $(a \in N) \Rightarrow (a + 1 \in N)$ 2. $(a, b \in N) \Rightarrow ((a = b) = (a + 1 = b + 1))$ 3. $(a \in N) \Rightarrow (a + 1 \neq 1)$[61]
Gödel's "On formally undecidable proposition of Principia Mathematica and related systems I"[62]: axioms 1, 2
1. 2. $f(x_1) = f(x_2) \rightarrow x_1 = x_2$ 3. $\neg(f(x_1) = 0)$
Modern presentation: "Peano Arithmetic": axioms 1, 2
1. 2. $\forall y(S(x) = S(y) \rightarrow x = y)$ 3. $\forall x \neg(S(x) = 0)$

Table 2.1: From Dedekind's letter to PA

In the end, Dedekind's principles represent the basis of modern Peano Arithmetic. Despite the lack of acknowledgement in the name, the discovery of the so-called "Peano axioms", i.e. the axioms of PA, should also be credited to Dedekind. In fact, Peano actually presents Dedekind's principles in a formal fashion by providing an effective symbolism that enables him to replace any intuitive and intended notion.[63] Thus, we are able

[59]Recall def. 26: "A transformation φ of a system S is said to be similar when to different elements a, b of the system S there always correspond different transforms $a' = \varphi(a)$, $b' = \varphi(b)$."

[60][peano89, p. 94]

[61] We replace Peano's "deducitur", i.e. "is a consequence of", with $\Rightarrow$. We replace Peano's parenthesis convention with modern notation. Peano also defines a, b as "indetermined objects" (i.e. elements), the sign N as number (i.e. the set of natural numbers), the sign 1 as unity (base element), a+1 as successor of a, and the sign = as "is equal to". See [peano89, pp. 86–87]

[62][godel31, p. 600]

[63]Indeed, whether Peano came to discover those axioms independently of Dedekind or not is still debated. We know that Peano acknowledged having consulted Dedekind's book while preparing his own work in

to deal with a formal theory: as soon as the principles are formulated as formal axioms, we are entitled to study the logical structure they define regardless of what the possible interpretation of the symbols might be. On the other hand, Dedekind assesses this task only partially since his arithmetical principles rely on the intuitive notions of thing, system and transformation.

This consideration goes hand in hand with the second and third steps of the formal axiomatic process brought up in section 1.6.1: the insights about the intended model are firstly stated in a naïve-interpreted language and then formalized in a logical framework.

Still in section 1.6.1, we discussed a last step to accommodate regarding completeness of the theory: when we are assured that the features of the series of natural numbers have been grasped first by a naïve formulation, and accordingly by its formal refinement, we have to check that not only does the formal theory satisfy the intended model, but also it comes out as *completely* characterised.[64]

Surprisingly, even this issue is addressed by Dedekind himself. At face value, the five tenets discussed so far suffice to characterise "completely" the series of the natural numbers. The diagram 2.1 depicts nothing more than the structure of the series of the natural numbers: we have an initial element and all the other elements are linked with this first one by a succession function. Yet, Dedekind curbs our enthusiasm:

> Thus we have listed the facts that you [Mr. Keferstein] regard as the complete characterisation of an ordered, simply infinite system N. [ibidem]

Keferstein's main criticism consisted in the fact that for him the aforementioned principles are sufficient to characterise the series of natural numbers. In arguing against Keferstein, Dedekind does not consider the above five points enough to *completely* define natural numbers. In fact, Dedekind spells out two important features of his research: he explicitly pursues completeness as a necessary constraint of foundational investigation; secondly, in doing so he provides a better understanding why the previous principled do not result in a complete axiomatization.

Along the lines of the earlier discussion in section 1.4 of the possible meaning of completeness, we shall claim that Dedekind actually holds the phrase "complete characterisation" as *categorical* characterisation.[65]

Let us now return to the letter to discuss the sixth principle of the list.

[peano91, p. 93] and [peano89, p. 86]. This suffices for Van Heijenoort to say that "Peano acknowledges that his axioms come from Dedekind"[vanheijenoort67, p. 84]. On the other hand, according to H. C. Kennedy [kennedy74, p. 389] Peano had already arrived at his axioms by then, and Dedekind's work only confirmed his results. See [ferreiros07, p. 235]

[64]Recall Awodey's quotations on page 6.

[65]See also section 2.6.2 below

2.3 Dedekind's sixth principle

In this section we provide evidence that the following sixth principle of the letter to Keferstein turns out to be fundamental for our research. In particular, we argue that we are confronted with the first clear statement of the existence of non standard models of arithmetic in the history of mathematics. We will first quote the sixth point in full and then comment on it point by point.

> (6) I have shown in my reply, however, that these facts are still far from being adequate for completely characterising the nature of the number sequence N. All these facts would hold also for every system S that, besides the number sequence N, contained a system T, of arbitrary additional elements t, to which the mapping φ could always be extended while remaining similar and satisfying $\varphi(T) = T$. But such a system S is obviously something quite different from our number sequence N, and I could so choose it that scarcely a single theorem of arithmetic would be preserved in it. What, then, must we add to the facts above in order to cleanse our system S again of such alien intruders t as disturb every vestige of order and to restrict it to N? This was one of the most difficult points of my analysis and its mastery required lengthy reflection. If one presupposes knowledge of the sequence N of natural numbers and, accordingly, allows himself the use of the language of arithmetic, then, of course, he has an easy time of it. He need only say: an element n belongs to the sequence N if and only if, starting with the element 1 and counting on and on steadfastly, that is, going through a finite number of iterations of the mapping φ, (see the end of article 131 in my essay)[66], I actually reach the element n at some time; by this procedure, however, I shall never reach an element t outside of the sequence N. But this way of characterising the distinction between those elements t that are to be ejected from S and those elements n that alone are to remain is surely quite useless for our purpose; it would, after all, contain the most pernicious and obvious kind of vicious circle. The mere words "finally get there at some time", of course, will not do either; they would be of no more use than, say, the words "karam sipo tatura", which I invent at this instant without giving them any clearly defined meaning. Thus, how can I, without presupposing any arithmetic knowledge, give an unambiguous conceptual foundation to the distinction between the elements n and the elements t? Merely through consideration of the *chains* (articles 37 and 44 of my essay), and yet, by means of these, completely! If I wanted to avoid my technical expression "chain" I would say: an element n of S belongs to the sequence N if and only if n is an element of *every* part K

[66] We discuss theorem 131 in section 2.6.2

> of S that possesses the following two properties: (i) the element 1 belongs to K and (ii) the image $\varphi(K)$ is a part of K. In my technical language: N is the intersection [Gemeinheit] 1_0, or $\varphi_0(1)$, of all those chains K (in S) to which the element 1 belongs. Only now is the sequence N characterised completely. [...] [dedekind90, pp. 100–101, original emphasis]

We can distinguish five parts in the quoted passage. Let us discuss in turn each of them.

2.3.1 The existence of non-standard models of arithmetic

As already noted, Dedekind emphasises that a further principle must be added to get the series of natural numbers "completely characterised".

Suppose that the principles (1) - (5) of the letter above did fully characterise the natural number series. We seek for a contradiction.

Let N be a set provided with a function φ defined according to the principles (1) - (5). Then, by assumption, N fully characterises the series natural numbers. Assume that by "completely characterise" we mean "categorical", i.e. (1) - (5) capture all and only the models[67] that make all arithmetical statements true. Hence N is uniquely characterised up to isomorphism, i.e. (N,φ) is the only model which satisfies (1) - (5) except for the case of models isomorphic to N.

But this is not the case. We show that there is a set, say S, provided with a function ψ and closed under it, such that S is a proper superset of N and satisfies (1) - (5) as well.

Let S be a set such that $S = N \cup T$ where the elements $\{t_1, t_2, \dots\}$ of T do not belong to N, in symbols $N \subseteq S$, $T \subseteq S$ and $T \cap N = \emptyset$. Clearly, $N \subset S$. Now, we can define a function ψ. For all $n \in N$, $\psi(n) = \varphi(n)$
For each $t_i \in T$, $\psi(t_i) \in T$ and $\psi(t_i) \notin \varphi(N)$.
For each $t_1, t_2 \in T$, if $\psi(t_1) = \psi(t_2)$ then $t_1 = t_2$.
We can do this for example by defining $\psi(t_i) = t_i$ or by defining $\psi(2i) = 2i - 1$ and $\psi(2i - 1) = 2i$, etc.

Clearly, the system S satisfies conditions (1) - (5) as well:

- $\psi(S) \subseteq S$
- ψ is injective
- There is a n such that $n \notin \psi(S)$, namely 1, because $1 \notin \varphi(N) = \psi(N)$ and $\psi(T) \subseteq T$.

But $N \subset S$ and therefore $N \neq S$. Since by assumption (N, φ) is the unique model satisfying (1) - (5) and (N, φ) $\ncong$ (S, ψ), we get a contradiction.

[67] A model represents, to use Dedekind's terminology, a system provided with some transformations. See, for example, the definition of similar system on page 52.

Neglecting the modern set-theoretical notation we exploited, Dedekind provides the very same reasoning:

> All these facts would hold also for every system S that, besides the number sequence N, contained a system T, of arbitrary additional elements t, to which the mapping φ could always be extended while remaining similar and satisfying $\varphi(T) = T$. [dedekind90, p. 100]

Dedekind introduces a system S which contains the systems N and T. First of all, T contains arbitrary additional elements with respect to N (intuitively we take it that the intersection between N and T is empty). Secondly, he extends the map defined on N to those arbitrary t elements. We have assessed this by defining a new function ψ.

Note that, despite extending the function φ, Dedekind maintains the same name for it. This is not really puzzling since the same procedure is exhibited by Dedekind in NMN as well:

> If now T is any part of S [included], then in the transformation φ of S is likewise contained a determinate transformation of T, which for the sake of simplicity may be denoted by the same symbol φ and consists in this that to every element t of the system T there corresponds the same transform $\varphi(t)$, which t possesses as element of S; [dedekind01, p. 24]

Finally, Dedekind sets φ so extended such that $\varphi(T) = T$. This calls for an important remark.

Note that the construction $\varphi(T) = T$ has never been exploited by Dedekind throughout NMN: he puts $\varphi(T) = T'$ for convenience[68], $\varphi(T) = R$ and $\varphi(R) = T$ to define similar systems[69], or $\varphi(T) \subseteq T$ to define a transformation on itself[70], but he never mentions a transformation on a system T such that $\varphi(T) = T$.

To sum up, Dedekind discusses in the letter a new construction which is not presented in NMN, viz. the construction of non-standard models for the set of the conditions α, γ and δ.[71] Once again, it is important to consider that there is no trace of a similar reasoning in NMN. Dedekind considers these non-standard models insofar as he wants to put forth a counterexample to Keferstein's argument for the sufficiency of conditions (1) - (5).

The construction that we have provided along Dedekind lines exhibits a model as in diagram 2.2.

This represents a counterexample to the fact that principles (1) - (5) describe the series of natural numbers completely. Accordingly, since (S, ψ) is admittedly different from the

[68] See def.21 on page 28 and footnote 49.

[69] See def.32 on page 52.

[70] See def.36 on page 28.

[71] For convenience, we call the set of the conditions α, γ and δ DT^-. See section 2.6 for the discussion on the difference between DT and DT^-

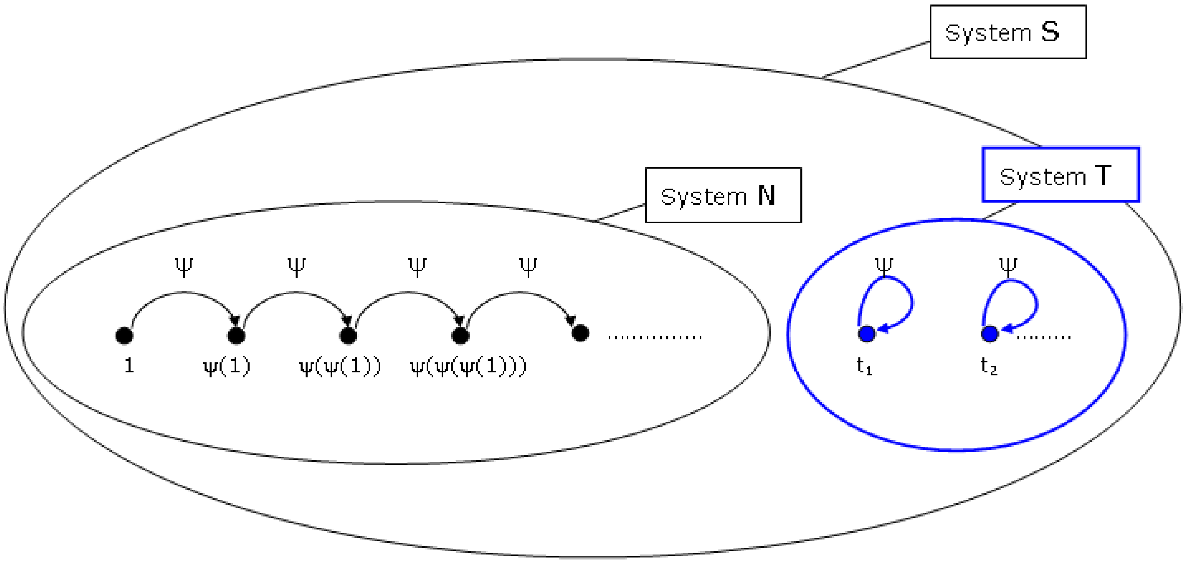

Figure 2.2: An example of Dedekind's non-standard model

series of natural numbers, the existence of a non-standard model of the principles (1) - (5) is provided.

However, Dedekind does not say much about the system T and the transformation defined on it, except for the fact that is formed by "arbitrary additional elements t" and that the mapping ψ satisfies "$\psi(T) = T$".

Indeed, other configurations for (T,ψ) are also possible. In fact, the mapping ψ such that "$\psi(T) = T$" does not necessarily have to be defined as in diagram 2.2 for allowing non-standard models of DT^-.[72]

Depending on the way we define (T, ψ), we can construct four types of models (S,ψ) similar to the one in diagram 2.2. If T is chosen to be finite, then we can define ψ as in diagram 2.3.

The two models so defined clearly satisfy the conditions α and δ so that we can use them to construct a non-standard model (S, ψ) for DT^-.

If T is chosen to be denumerably infinite, we can define ψ so that the conditions α and δ are satisfied, as diagram 2.4 shows.

Also in this case, we can use (T,ψ) to construct non-standard models (S, ψ) for DT^- as shown above.

One last technical observation is in order. From a modern point of view, three of the four cases just considered result in uninteresting non-standard models due to the cy-

[72]Recall that DT^- stands for the set of conditions α, γ, δ, and thus it also represents the set of conditions (1) - (5).

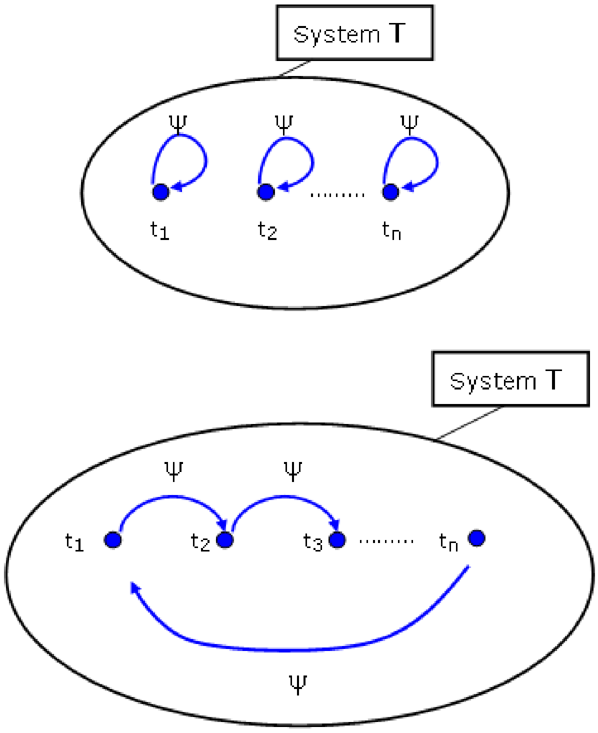

Figure 2.3: Possible models (T,ψ) to construct non-standard models of DT^-

cles.[73] We can prevent such cycles by adding the condition that ψ is nowhere the identity, and in general $\psi^n = \psi \dots \psi$ is nowhere the identity. We can rephrase it in modern terminology as $S(x) \neq x$, and in general $S^n(x) \neq x$ where $S^n(x)$ stands for $\underbrace{S \dots S}_{n\ times}(x)$.[74] Once this condition is added, the only type of (T,ψ) that can be taken into account to construct non-standard models is the fourth one.[75]

To go back to the letter, Dedekind goes on as follows:

> But such a system S is clearly *something quite different* from our number sequence N, and I could so choose it that scarcely a single theorem of arithmetic would be preserved in it. [my emphasis]

[73] Strictly speaking, all the non-standard models possibly definable are quite uninteresting since we do not have an induction axiom in DT^- and thus we cannot define most of the usual arithmetical properties and operations.

[74] Note that this axiom together with Peano Axioms 1 and 2 give a *syntactically complete* system for the language with only 0 and S. Moreover, it is an interesting point whether Dedekind could have access to something like $S^n(x) \neq x$. He might have seen it as a vicious circle due to a not entirely clear distinction between mathematics and languages that talk about mathematics.

[75] In this case the non-standard models would resemble the ones we study today. Compare with the diagram 2.4 in chapter 4.

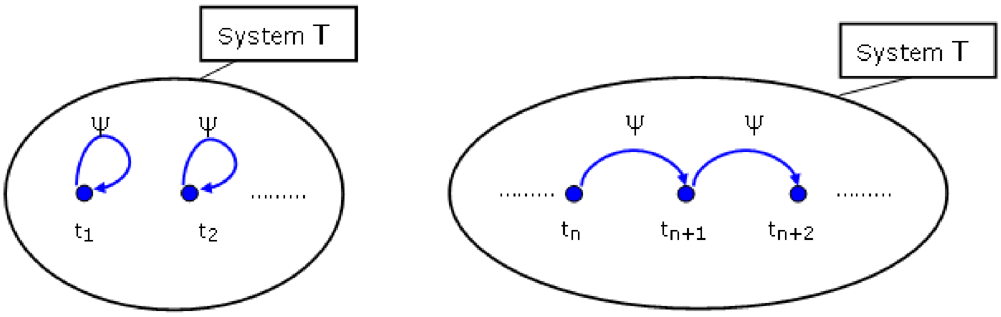

Figure 2.4: Possible models (T,ψ) to construct non-standard models of DT^{-}

These passage needs an attentive and accurate discussion. Dedekind has just constructed a non-standard model of DT^{-}, viz. a model that satisfies DT^{-} as the standard model (N, φ), but it is clearly different from it. Although standard and non-standard models do make true the same axioms, Dedekind points out that we can choose non-standard models such that they do not preserve theorems that are indeed true in the standard model.

By contrast, it is well-known today that is actually this very feature, namely the fact that the standard and non-standard models are indistinguishable with respect to the first-order properties, that make the non-standard models so interesting.

To explain this, we should recall that Dedekind, at least in writing, had not taken into consideration the possibility of NMoA until he had to respond to Keferstein's criticism. In other words, it seems that NMoA are brought up by Dedekind uniquely to argue against Keferstein's position about the way to characterise the series of natural numbers.

Dedekind certainly had a feeling for the peculiarity of those non-standard models, but still he did not pay much attention to them as long as he believed the inadequacy of those models for his foundational purposes.

At any rate, whatever way we interpret the word "different", it turns out that Dedekind has discovered the existence of non-standard models of his arithmetical axiomatization. Once again, he sets off from the series of natural numbers as intended model, and formulates some principles and verifies that they hold for such an intended model. Yet, the principles also hold for other models although different from the intended one. Those are what we call unintended or non-standard models: even though those models are admittedly different from the intended one, in the sense just shown, they actually satisfy the same principles. This is the main feature of the unintended models that Dedekind first introduced.

In the end, Dedekind's distinction between the systems N and S leads us to the following conclusions:

- As we saw, Dedekind does not provide a name for these unintended models: he simply deals with them as a counterexample to Keferstein's argument about the sufficiency of the three conditions.
- Dedekind actually draws the distinction between "the" model we want to study and "bad" and "unwanted" models that come out as an alarm bell warning us that we are off the completeness track.
- The unintended models are a concrete threat since they can be easily constructed from the intended one.

2.3.2 Alien intruders and Chains

Dedekind is aware of the danger that those "unpleasant" elements t can bring up (in virtue of their properties) to the logical foundation of arithmetic. As immediate reaction, he wants to banish them as alien intruders whose presence can only harm any foundational investigation:

> What, then, must we add to the facts above in order to cleanse our system S again of such alien intruders t as disturb every vestige of order and to restrict it to N? This was one of the most difficult points of my analysis and its mastery required lengthy reflection.

Necessity breeds invention and Dedekind's solution is quite sophisticated: the notion of *chain*. Despite being a not easy problem, Dedekind's solution in terms of Chain can be considered as an ingenious *deus ex machina*.

First of all, Dedekind is conscious that, for this particular issue, looking back to how "the sequence of natural numbers just as it presents itself, in experience, so to speak" would turn out to be quite fruitless. For example, if we try to rely on an intuitive notion such as "iterative process", we run into the following difficulties:

> If one presupposes knowledge of the sequence N of natural numbers and, accordingly, allows himself the use of the language of arithmetic, then, of course, he has an easy time of it. He need only say: an element n belongs to the sequence N if and only if, starting with the element 1 and counting on and on steadfastly, that is, going through a finite number of iterations of the mapping φ, [...], I actually reach the element n at some time; by this procedure, however, I shall never reach an element t outside of the sequence N. But this way of characterising the distinction between those elements t that are to be ejected from S and those elements n that alone are to remain is surely quite useless for our purpose; it would, after all, contain the most pernicious and

> obvious kind of vicious circle. The mere words "finally get there at some time", of course, will not do either; they would be of no more use than, say, the words "karam sipo tatura", which I invent at this instant without giving them any clearly defined meaning.

The concept "iterative" comes out as a tempting but deceiving solution. For the nth time, Dedekind points out that what he is looking for is a precise characterisation of the series of natural numbers. Arithmetic is given as a mathematical branch and it stands by itself. On the other hand, the issue which are the very basic principles of arithmetic on which the entire arithmetical building rests pertains to foundational research, viz. the research of primitive and unavoidable tenets of mathematics.

Even if Dedekind does not have the modern logical framework at his disposal, he pursues a precise and unambiguous concept of definition. We can understand in this light Dedekind's attitude to dismiss both "finite number of iteration" and "karam sipo tatura" as unclear statements. If we cannot assign to these phrases a "clear" meaning within the logical and mathematical framework, viz. we can define them by the definitions and theorems that we have already settled, we are dealing with mere strings of words which can lead us into vicious circles.

It is somehow impressing that Dedekind's goal of regimentation is clear from the very first line of NMN as "In science nothing capable of proof ought to be accepted without proof"[76]. Thus, for Dedekind proofs turn out to be the privileged way to admit new knowledge in science.

For this reason, instead of adopting an outright and intuitive solution that trace back to a former "analysis in experience", we can obtain a complete axiomatization of the natural number series by referring to the general notions and the activities of the understanding:

> Thus, how can I, without presupposing any arithmetic knowledge, give an unambiguous conceptual foundation to the distinction between the elements n and the elements t? Merely through consideration of the $chains$ (articles 37 and 44 of my essay), and yet, by means of these, completely! If I wanted to avoid my technical expression "chain" I would say: an element n of S belongs to the sequence N if and only if n is an element of every part K of S that possesses the following two properties: (i) the element 1 belongs to K and (ii) the image $\varphi(K)$ is a part of K. In my technical language: N is the intersection [Gemeinheit] 1_0, or $\varphi_0(1)$, of all those chains K (in S) to which the element 1 belongs. Only now is the sequence N characterised completely. [dedekind90, p. 101]

[76][dedekind01, p. 14]

The solution is accomplished by the notion of chain that can be rephrased in current mathematical notation as follows: the chain S under f with base element a is the smallest set which includes a and is closed under f, i.e. it is the intersection of all sets with these properties. This represents the condition β which was previously omitted in def.71 on page 30,[77] namely:

$$\beta.\ N = 1_0$$

According to def.37 in NMN, a system S is a chain when there is a transformation of the system S on itself, i.e. $\varphi(S) \subseteq S$. In particular the chain S under φ with base element a is the chain defined by φ such that a is in S. Now, the statement $N = 1_0$ says that N is the smallest chain closed under a transformation, say φ, and that contains the base element 1.

The notion of chain has a place in Peano's formalization as well. In fact, Peano equips his formal arithmetical system with an axiom that states the condition on chains just mentioned as follows:[78]

$$\{(k \in K) \wedge (1 \in k) \wedge [(x \in N) \wedge (x \in k) \Rightarrow (x + 1 \in k)]\} \Rightarrow (N \in k)$$

which can be rephrased in modern set-theoretical terms as

$$P \subseteq N \wedge (0 \in P) \wedge [\forall x(x \in P \rightarrow S(x) \in P)] \rightarrow P = N$$

where N, 0, and S are the three primitive terms: N is a set, 0 is an element of N, and S is a function from N onto N.

Unlike the other principles stated so far, this last condition on chains requires a second-order language to be formulated, i.e. it is a statement concerning not only the elements but also subsets of elements. The use of the second order quantification lies in the phrase "the smallest set": among all the sets that satisfy those properties, we have to pick the smallest one. Regardless of the expressive power of the logical language used, Dedekind's attempt to wrap up a complete axiomatization for arithmetic stands or falls with this last chain condition. We will say a last word about this in section 2.6.

To sum up, Dedekind fixes on the following four conditions:

- $\alpha.$ $\varphi(N) \subseteq N.$
- $\beta.$ $N = 1_0.$
- $\gamma.$ The element 1 is not contained in $\varphi(N)$.
- $\delta.$ The function φ is similar.

[77] Recall that whereas in NMN Dedekind exposes his argument synthetically, and exhibits immediately the four conditions that characterise completely the series of natural numbers, in the letter he wants to stress the analytical reasoning underlying his results.

[78] [peano89, p. 94]. For Peano's notation see footnote 61.

Every model (i.e. a tuple of a system, and a transformation defined on the system) which satisfies the aforementioned conditions is called by Dedekind a *simply infinite system*.

We note that the condition β is two-fold: on the one hand, it guarantees that the system N satisfies *at least* all the properties of the series of the natural numbers, viz. the conditions α, γ and δ; on the other hand, β specifies that N has *at most* these properties, otherwise it would not be the smallest set closed under φ that contains the initial element 1.

Finally, we can define the series of natural numbers in accordance with Dedekind's notions:

> 73. Definition. If in the consideration of a simply infinite system N set in order by a transformation φ we entirely neglect the special character of the elements;[79] simply retaining their distinguishability and taking into account only the relations to one another in which they are placed by the order-setting transformation φ, then are these elements called *natural numbers* or *ordinal numbers* or simply *numbers*, and the base-element 1 is called the *base-number* of the *number-series* N. [...] The relations or laws which are derived entirely from the conditions α, β, γ, δ in (71) and therefore are always the same in all ordered simply infinite systems, whatever names may happen to be given to the individual elements (compare 134), form the first object of the *science of numbers* or *arithmetic*. [dedekind01, p. 34]

The whole discussion we have put forward about the sixth point of Dedekind's letter seems to be in flat contradiction with a claim in [awodey02]:

> At the same time, Dedekind clearly intends *various different systems* to fall under the concept 'simply infinite system'. He even considers systems that *satisfy only some of the four clauses in it but not others*; see [dedekind90], pp.100-101. [awodey02, p. 7, fn. 21, emphasis mine]

Apparently, Awodey and Reck fail to grasp what Dedekind made extremely clear. Dedekind points out that the three conditions α, γ and δ are not enough for a complete axiomatization and then, as we saw, to achieve such a completeness he puts forward a further clause, viz. β. These four conditions define the "simply infinite systems".

Accordingly, it does not make much sense to talk about "various different systems" that fall under the four clauses unless we are considering a non-categorical theory. In fact, if the four conditions define a categorical theory, as Dedekind himself shows,[80] it

[79] This passage has been used by Reck and Tait to support the view that sees Dedekind as a structuralist. In fact, Dedekind's axioms can only define the natural numbers up to isomorphism. See [tait96] and [reck03]

[80] See section 2.6.2 below

is dismissed the possibility of having various different systems as long as the conditions capture the series of the natural numbers uniquely.

Moreover, the reference "see [dedekind90], pp. 100-101", that Awodey and Reck give in support of the claim that "[Dedekind] considers systems that satisfy only some of the four clauses in it but not others", is exactly Dedekind's sixth principle that we have discussed and fully quoted.

Even if we consider the sole three conditions that give non-standard models, Awodey's and Reck's is still false as they continue: "He [i.e. Dedekind] even considers systems that satisfy only some of the four clauses in it but not others". As we discussed, by no means does Dedekind consider arbitrary systems that satisfy some clauses and not others, but rather he observes that there are different systems, i.e. intended and unintended, that indeed satisfy the very same three conditions.

To conclude, even if completeness calls for a detailed account, Dedekind's construction of non-standard models as well as his perspective about the threat of those alien intruders led us to a better picture on what Dedekind means with "complete characterisation".

Thus, if the set of axioms is not adequately chosen, models *admittedly different* from the intended one can come up. Dedekind says that the axiomatization is complete if they do not. So completeness is arguably ascribable to the set of axioms.

In this section we have put forward that Dedekind is indeed the forefather of non-standard models of arithmetic. At face value, a question arises: why has the sixth principle that we have discussed never been recognised as evidence for Dedekind's discover of the existence of non-standard models of arithmetic? Indeed, it has by Hao Wang in a paper of 1957, but nonetheless the claim encountered scarce success in philosophical discussion.

2.4 Hao Wang: Dedekind and Non-standard models

We have already mentioned that Hao Wang was the first philosopher to draw attention to Dedekind's letter to Keferstein. In the very same paper, Wang notes that in Dedekind's letter there is evidence to say that Dedekind deals with non-standard models:

> The notion of non-standard models (unintended interpretations) of axioms for positive integers is, for instance, brought out quite clearly in Dedekind's letter. [wang57, p. 145]

However, Wang seems not to push this argument through. The point we want to make here is that Wang puts forward poor arguments in support of this claim as well as misconceives some of Dedekind's results.

To begin with, Hao Wang provides a different translation of the sixth point quoted above. In particular, his translation undermines the evidence for Dedekind's construction of the non-standard models.

Let us compare Bauer-Menglberg's translation, which we have quoted above, published in [vanheijenoort67]

> All these facts would hold also for every system S that, besides the number sequence N, contained a system T, of arbitrary **additional** elements t, to which the mapping φ could always be **extended** while remaining similar and satisfying $\varphi(T) = T$. But such a system S is obviously something quite different from our number sequence N, and I could so choose it that scarcely a single theorem of arithmetic would be preserved in it. (*Bauer-Menglberg's translation*, [dedekind90, p. 100, my bold])

with Hao Wang's

> Indeed, all these facts also apply to every system S which, in addition to the number-sequence N, contains also a system T of arbitrary **other** elements t. One can always define the mapping φ so as to **preserve** the character of similarity and so as to make $\varphi(T) = T$. But such a system S is obviously something quite different from our number-sequence N, and I could so choose the system that scarcely a single arithmetic theorem holds for it. (*Wang's translation*, [wang57, p. 150, my bold])

We note that Bauer-Menglberg's use of the word "additional" instead of "other" and of the crucial word "extended" rather than "preserve" puts forth in a clearer manner the construction that Dedekind is hinting at.

Thus, Bauer-Menglberg's translation stresses even more than Wang's how such NMoA[81] would look.[82]

In the second place, Wang considers Dedekind's construction as a mere "clue" for ruling out NMoA rather than the actual exhibition of them:

> His [Dedekind's] letter supplies a useful clue, when he discusses under (6)[i.e. the sixth condition presented above] the question of excluding undesirable interpretations of the set N for which some ordinary arithmetic theorems would fail to hold. [wang57, p. 154]

In the sixth tenet Dedekind characterises non-standard models as the ones that do satisfy the same conditions as the standard ones in spite of being different from them.

[81] Recall that NMoA stands for non-standard models of arithmetic.

[82] However, in fairness to Wang it must be said that Bauer-Menglberg's translation came out as refinement of Wang's as long as Wang was the first who translated the letter into English.

As we mentioned, although Dedekind deals with non-standard models, the purposes he pursues in NMN and the spirit of his project steer him away from studying them as mathematical entities, and accordingly to verify whether the ordinary theorems may hold for them or not. As we will see, they might (or can be forced to), and this caused all the philosophical trouble that was sparked by Skolem.

Nevertheless, Wang understands the importance of non-standard models and their relation to Dedekind's work

> For then we shall have two models for the axioms, one of which fits our intentions while the other contains additional alien elements which cannot be reached from the base-element by finite numbers of steps. [...] In recent years a good deal of research in mathematical logic has been devoted to the question of unintended interpretations (nonstandard models) of theories of positive integers. It is therefore of interest to find this question raised in Dedekind's letter. [wang57, p. 155]

However, Wang did not ascribe such an analysis to Dedekind, but rather he considers it as something just recently achieved:

> Dedekind's discussion in his letter, especially under (6), is very articulate and instructive except that the last step in his argument, which leads to the conclusion that N is completely determined, can be elaborated further. [ibidem]

Following Wang, non-standard models are brought up as a controversial, confusing and complex notion, something that presents itself as mysterious and obscure:

> Usually the models directly obtained from these general theorems [i.e. non standard models] are quite complex or artificial, and to find intuitively transparent unintended models is in most cases very difficult. [wang57, p. 156]

Arguably, a plausible although trivial explanation of Wang's position lies in the fact that the time was not yet ripe for looking at non-standard models as interesting mathematical structures in their own right. At that day (the end of 1950s), Wang still had a blurred view on the nature of non-standard models. The time of important mathematical results and detailed studies that legitimated NMoA as a field in its own right still had to come. Despite that, there is no doubt that the mathematical approach that came later was based on the same philosophical considerations that had been gathered by Dedekind before.

We draw one last but in no way least consideration. As noted above, Wang takes unintended models as non-standard models: he uses both expressions interchangeably:

"the notion of non-standard models (unintended interpretations)" [83] and "question of unintended interpretations (nonstandard models)"[84].

Indeed, Hao Wang has to be credited as the one who drew this connection between the adjective unintended and non-standard. What should be quite clear by now is that the standard/non-standard dichotomy is posterior to the intended/unintended one: Dedekind starts off the study of an intended model, e.g. the series of natural numbers, and tries to provide axioms which are supposed not to be satisfied by the unintended models, viz. models admittedly different from the series of the natural numbers. Unintended/intended dichotomy makes sense only in an axiomatic framework, namely we necessarily need an intended model as privileged starting point.[85] At any rate, Wang explicitly holds that standard and intended as well as non standard and unintended are taken as coextensive notions.

This fits in what we said at the beginning of the chapter, viz. the position that characterises the period from Dedekind to the end of 1950s is the idea that NMoA has to be considered as a philosophical and logical threat. This view was progressively abandoned due to a mathematical highly technical shift that took places in the 1960s.

There is no doubt that Wang foresaw that Dedekind had an important role in the birth of non-standard models. However, the way his arguments were conducted, the lack of important results in the field and the historical period when the paper was written all contributed to prevent the topic of NMoA from taking off.

Let us now make a leap and move to present-day formal logic. We will present some formal results that will turn to be useful both to better understand what Dedekind meant with "completeness" and to set the stage for the discussion on Skolem's work.

2.5 Meta-properties and Arithmetic

In the appendix we provide an explicit presentation of the syntax and the semantics for first-order and second-order logic. By doing so, we define the notions of deductive consequence, on the one hand, and we focus on semantic concepts such as model, interpretation, truth, and semantic consequence, on the other hand.

Here we deal with meta-properties of these formal languages. First of all, we have to draw a distinction between properties of a formal language with respect to the *deductive consequence relation* and properties of a formal language with respect to a certain *formal theory*.

As far as the deductive consequence is concerned, we state three important results with respect to first order logic. Let the tuple $(L, \vdash)$ define a first order logic where L

[83][wang57, p. 145]
[84][wang57, p. 155]
[85]See further discussion in section 4.1.2.

is a first-order language, and $\vdash$ the relative deductive consequence. Then the following theorems hold for it:

Theorem 2.5.0.1. (Soundness) *The deductive consequence relation* $\vdash$ *is* sound *with respect to the semantic consequence relation* $\models$*, i.e. for all L-sentences* φ *and all L-theories* Γ*: if* $\Gamma \vdash \varphi$ *then* $\Gamma \models \varphi$

Theorem 2.5.0.2. (Completeness)[86] *The deductive consequence relation* $\vdash$ *is* complete *with respect to the semantic consequence relation* $\models$*, i.e. for all L-sentences* φ *and all L-theories* Γ*:* $\Gamma \vdash \varphi$ *if and only if* $\Gamma \models \varphi$

Theorem 2.5.0.3. (Compactness) *If every finite subset of a theory* Γ *has a model, then* Γ *has a model. Or Equivalently, if* Γ *is a L-theory, and* φ *is an L-sentence such that* $\Gamma \models \varphi$*, then there is a finite* Δ*,* $\Delta \subseteq \Gamma$ *such that* $\Delta \models \varphi$*.*

These theorems are three important meta-properties of first-order logic, as we will discuss in section 3.7.1.

Another aspect we should discuss is the presence of meta-properties for formal theories formulated in a certain logic. Whereas the previous notions specify some properties of the deductive relation, we now turn to definitions that entitle us to characterise a specific theory Γ.

Definition 2.5.0.4. *Let L be a language. A* theory *in L, or an L-theory, is a set of L-sentences, i.e. formulas of L. If* Γ *is an L-theory, a model of* Γ *is an L-structure* $\mathcal{M}$ *such that* $\mathcal{M} \models \varphi$ *for all* $\varphi \in \Gamma$

An L-theory Γ *is called* satisfiable *if* Γ *has a model.*

We abbreviate $\mathcal{M} \models \Gamma$ *for* $\mathcal{M} \models \varphi$ *for all* $\varphi \in \Gamma$

We say that φ logically follows *from* Γ*, in symbols* $\Gamma \models \varphi$ *iff for every model* $\mathcal{M}$*, if* $\mathcal{M} \models \Gamma$ *then* $\mathcal{M} \models \varphi$[87]

Now, a theory can be said "semantically" and "syntactically" complete in the following cases:

Definition 2.5.0.5. *A L-theory* Γ *is called* semantically complete *with respect to a given semantics if, for all L-sentences* φ*, either* $\Gamma \models \varphi$ *or* $\Gamma \models \neg\varphi$

Definition 2.5.0.6. *A L-theory* Γ *is called* syntactically complete *with respect to a given deductive relation if, for all L-sentences* φ*, either* $\Gamma \vdash \varphi$ *or* $\Gamma \vdash \neg\varphi$

A theory which is semantically complete has the following property:

[86] Recall that here completeness stands for *exhaustiveness*. See footnote 16.

[87] Unfortunately, we are accustomed to use the same symbol '$\models$' in two completely different situations: in A.2.0.3 as relation between a model and a L-formula and now as relation between a L-theory and a L-formula.

Theorem 2.5.0.7. *The following conditions are equivalent:*

1. *For all L-formulas φ and all models $\mathcal{M}, \mathcal{N}$ of Γ, $\mathcal{M} \models \varphi$ iff $\mathcal{N} \models \varphi$*
2. *For all L-formulas φ, either $\Gamma \models \varphi$ or $\Gamma \models \neg\varphi$*
3. *For all L-formulas φ, either $\Gamma \models \varphi$ or $\Gamma \cup \{\neg\varphi\}$ is not satisfiable.*
4. *There is no L-formula such that both $\Gamma \models \varphi$ and $\Gamma \cup \{\neg\varphi\}$ are satisfiable.*

Proof. The equivalence between (2), (3) and (4) easily follow from the remark in 2.5.0.4 and the definition of truth in A.2.0.3. So we prove the equivalence between (1) and (2).
For (1) $\Rightarrow$ (2) suppose that $\mathcal{M} \models \varphi$ iff $\mathcal{N} \models \varphi$ for an arbitrary L-formula φ and for two arbitrary models $\mathcal{M}$ and $\mathcal{N}$ such that $\mathcal{M} \models \Gamma$ and $\mathcal{N} \models \Gamma$. We want to show that either $\Gamma \models \varphi$ or $\Gamma \models \neg\varphi$ holds. We assume that it is not the case and we try to get a contradiction.
Case.1 Suppose that $\Gamma \models \varphi$ and $\Gamma \models \neg\varphi$. By hypothesis, we know that $\mathcal{M}$ and $\mathcal{N}$ are models of Γ. Hence, by definition 2.5.0.4 it follows that $\mathcal{M} \models \varphi$ and $\mathcal{N} \models \neg\varphi$. Also, we know by assumption that since $\mathcal{N} \models \neg\varphi$, $\mathcal{M} \models \neg\varphi$ holds, whence it is not the case that $\mathcal{M} \models \varphi$. So, by def. A.2.0.3 we get $\mathcal{M} \not\models \varphi$ and $\mathcal{M} \models \varphi$ previously obtained. Contradiction.
Case.2 Suppose that $\Gamma \not\models \varphi$ and $\Gamma \not\models \neg\varphi$. By def. 2.5.0.4, there exist two models $\mathcal{M}'$ and $\mathcal{N}'$ s.t. they both are models of Γ and $\mathcal{M}' \not\models \varphi$ and $\mathcal{N}' \not\models \neg\varphi$. Now, by hypothesis we know that $\mathcal{M}' \models \varphi$ iff $\mathcal{N}' \models \varphi$. So, by def. A.2.0.3, $\mathcal{N}' \not\models \neg\varphi$ is equivalent to $\mathcal{N}' \models \varphi$ and therefore we get $\mathcal{M}' \models \varphi$. But again $\mathcal{M}' \not\models \varphi$ and $\mathcal{M}' \models \varphi$. Contradiction.
Now, for (2) $\Rightarrow$ (1) let us assume that either $\Gamma \models \varphi$ or $\Gamma \models \neg\varphi$ holds for an arbitrary L-formula φ. We want to show that $\mathcal{M} \models \varphi$ iff $\mathcal{N} \models \varphi$ for any $\mathcal{M}, \mathcal{N}$ which satisfies Γ. Suppose that it is not the case.
Let $\Gamma \models \varphi$. If $\mathcal{M}$ is a model of Γ, then it is also model of φ. But, by assumption it follows that $\mathcal{N}$ does not satisfy the same formula φ and therefore $\mathcal{N} \not\models \varphi$. Also, $\mathcal{N}$ is a model of Γ and therefore $\mathcal{N} \models \varphi$. Contradiction.
Now, let $\Gamma \models \neg\varphi$. If $\mathcal{M}$ is a model of Γ, then it is also model of $\neg\varphi$. But by assumption it follows that $\mathcal{N}$ does not satisfy the same formula $\neg\varphi$ and therefore $\mathcal{N} \models \varphi$. Also, $\mathcal{N}$ is a model of Γ and therefore $\mathcal{N} \models \neg\varphi$. Contradiction. $\square$

If (1) says that all the models of the theory satisfy the very same formulas, (2), (3) and (4) suggest that all the sentences in the language are semantically determined, i.e. according to an interpretation, the L-formulas either holds or not in the model. Thus, no undefined situations are possible with respect to the theory, namely there are no statements such that they come out as independent of the theory.

Once again, the idea behind (1) is that all models of the theory are 'logically equivalent' or, in the first order case, 'elementarily equivalent'.

Recall that if two L-structures are logically equivalent, then they are models of exactly the same sentences of L. In saying so, for any two L-models of a theory T which are logically equivalent, we have $\mathcal{M} \models \varphi$ iff $\mathcal{N} \models \varphi$ for any L-formula φ, that is condition (1).

Another significant property that can hold between two L-structures is the isomorphism:

Definition 2.5.0.8. *Let $\mathcal{M}$ and $\mathcal{N}$ be two L-structures. By an* isomorphism *from $\mathcal{M}$ to $\mathcal{N}$, written $f : \mathcal{M} \cong \mathcal{N}$, we mean a function f from M to N which is a bijection with the following properties:*

- *For each constant c of L, $f(c^{\mathcal{M}}) = c^{\mathcal{N}}$*
- *For each n-ary function symbol F of L and n-tuple $t_1, \ldots, t_n$ from M, $f(F^{\mathcal{M}}(t_1, \ldots, t_n)) = F^{\mathcal{N}}(f(t_1), \ldots, f(t_n))$*
- *For each n-ary relation symbol R of L and n-tuple $t_1, \ldots, t_n$ from M, $(t_1, \ldots, t_n) \in R^{\mathcal{M}}$ iff $(f(t_1), \ldots, f(t_n)) \in R^{\mathcal{N}}$*

As a matter of fact, if two structures are isomorphic, then they share all model-theoretic properties. To go back to the previous definition, we also claim that two isomorphic structures are indeed logically equivalent, i.e. they are models of exactly the very same sentences of L. We will return to this point later. Let us turn to the last definition for this section: categoricity.

Definition 2.5.0.9. *A L-theory Γ is called* categorical *with respect to a given semantics, if for all model $\mathcal{M}$ and $\mathcal{N}$ of Γ, there exists an isomorphism between $\mathcal{M}$ and $\mathcal{N}$.*

Remark 2.5.0.10. *By categoricity, we can easily refer to a circumstance when the models that satisfy some theory Γ turns out to be indistinguishable from a model-theoretical point of view; they are, as it were, essentially the same. But then, since they all share the same model-theoretic properties, it follows that they are logically equivalent and they make true the very sentences. Thus, we can draw a connection between 2.5.0.9 and 2.5.0.5: if a theory T is categorical and all its models are essentially the same model-theoretically speaking, then they are also the same logically speaking, viz. they satisfies the same formulas. Less trivial is the converse: does a semantically complete theory only have isomorphic models?*

To this respect, Dedekind seems to have provided an answer at least in his axiomatization of the natural numbers.

We have so presented in a precise way all the notions that we outlined in chapter 1.

2.6 Dedekind: completeness and categoricity

Having settled the formal notion and the logical framework, we are now able to shed some light on an aspect that we have not made explicit yet, viz. in which sense we should understand Dedekind's attempt to pursue a "complete characterisation of the nature of the number-sequence", as he refers in the letter. [88]

In this section we argue that Dedekind's notion of "completeness" can be unraveled by means of the considerations that we have amassed so far. In doing so, we show that the meta-theoretical properties, such as categoricity and semantic completeness that we have just defined, have actually to do with Dedekind's idea of "complete characterisation".

2.6.1 Complete characterisation

In which way should we understand the word "complete" to which Dedekind refers before presenting the sixth tenet of his letter?

First of all, we must note it would certainly be anachronistic to say that Dedekind exploited notions such as deductive and semantic consequence relations, which were not clearly formulated until Gödel's (some ideas were already in Frege's texts and accordingly in Russell-Whitehead's) and Tarski's works, respectively. Thus, for the lack of "deducibility relation" in his thought, it is reasonable to dismiss that with "complete characterisation" Dedekind means completeness in the sense of "exhaustiveness" in theorem 2.5.0.2, and therefore any hint at the deductive use of logic in his work.

Still, in spite of the lack of a modern framework and a clear notion of "deducibility", we advocate that interesting logical aspects can be brought up in Dedekind's work in connection with the semantic relation.

On page 7, before turning to Dedekind, we put forward the definition of categoricity as the main purpose of the descriptive task of logic in mathematics.

This way of looking at Dedekind's completeness, viz. as an attempt to use logic in a *descriptive* way, seems the most plausible. In fact, as we noted so far, Dedekind is extremely clear in which case a characterisation is *in-complete*, i.e. we acknowledge the existence of a non-standard model of arithmetic. Thus, he seems to suggest that the list of conditions (axioms) is not *complete* insofar as we are able to construct a model which embodies all the properties specified in such a list, but at the same time such model turns out as admittedly different from the intended structure we were supposed to completely characterise.

In other words, the series of natural numbers is completely characterised (in Dedekind sense) if the axioms we fix on are in such a way that unintended models do not appear. This is the reason why Dedekind comes up with the notion of chain: to rule out non-standard models and thus achieve in characterising completely the series of natural num-

[88] See page 33.

bers. Along these lines, Dedekind hints at an axiomatization that is *categorical*: the set of axioms come to be satisfied by a unique model (the intended one) up to isomorphism.

This is actually the case for the four axioms α - δ, viz. it is proved to be categorical in NMN by two important theorems.

2.6.2 Complete characterisation as Categoricity

To prove that an axiomatization is categorical, Dedekind has to show that there is indeed an isomorphism between all the models that satisfy α - δ.

In section 2.2.1, we considered Dedekind's similar transformations and we noted that can be rephrased in modern terminology as injective functions. However, by using similar transformation Dedekind seems to run into one hazard, as Reck stresses

> While Dedekind is not completely precise, or at least not completely explicit, about the notion of isomorphism involved here - he defines "similarity" in terms of the existence of a 1-1 ("similar") function, while what he really needs is a bijective function (1-1 and "onto") - his proofs of these two theorems show that he essentially understands the notion of a simple infinity, thus the Dedekind-Peano Axioms, to be categorical. [reck03, p. 377]

If Dedekind only employs injective (similar) transformation, what he obtains is a mere embedding rather than isomorphism between structures. For this reason, it is crucial that the function has to be not only an injection but also onto, and thus bijective.

However, in contrast to what Reck claims, Dedekind is aware of the necessity of bijective rather than injective functions. The controversy lies in the way Dedekind defines "similar transformations" and "similar system". Whereas a *similar transformation* represents an injective function, two systems are defined to be *similar* as follows:

> 32. Definition. The systems R, S are said to be similar when there exists such a similar transformation φ of S that $\varphi(S) = R$, and therefore $\overline{\varphi}(R) = S$.

We would expect Dedekind to define a similar system as a system provided with a similar function. But Dedekind actually considers a similar function in order to compare a pair of systems. Indeed, the definition not also says that a similar transformation holds between S and R, but that the inverse transformation covers the whole system R. This is to say that, φ is a function from S to R, such that f is injective and $f[S] = R$. In other words, f is taken as a bijection.

This position is corroborated by the way Dedekind defines the inverse transformation introduced in def.26:

> an *inverse* transformation of the system S′, to be denoted by $\overline{\varphi}(S')$, which consists in this that to every element s′ of S′ there corresponds the transform $\overline{\varphi}(s') = s$, and obviously this transformation is also similar.

Since s′ and S′ denote the range of the injective function φ, it does make sense to consider the inverse function φ^{-1} for the elements that fall under the range. Normally, it is not known whether the range of φ covers the whole codomain of φ. If this is the case the function is indeed bijective, otherwise only injective.

Mainly, Dedekind studies similar transformations and transformations on the system itself. This is the reason why he adopts the convenient way to denote with S′ or s′ the range of S or s under a transformation. For example the notion of chain: the idea is to consider those sets whose similar transformations have the set itself as a range. Still, in the definition of "similar set" what it at issue is a transformation between R and S which is not only injective, but also surjective: "φ of S that $\varphi(S) = R$, and therefore $\overline{\varphi}(R) = S$".

To sum up, in contrast to Reck's and other scholars' idea, the definition of similar system provided by Dedekind does not fall short as we try to characterise an isomorphism between simply infinite systems.

Once we have set matters straight about that, we can eventually turn to theorems 132 and 133 of NMN which the categoricity of the axioms rests on:

> 132. Theorem. All simply infinite systems are similar to the number-series N and consequently by (33) also to one another.
>
> 133. Theorem. Every system which is similar to a simply infinite system and therefore by (132), (33) to the number-series N is simply infinite.

Theorem 132 claims that any simply infinite system (i.e. a structure that makes true conditions α - δ) is isomorphic to the model of natural numbers. In short, for Dedekind the difference between them lies in the fact that in the number-series case "we entirely neglect the special character of the elements; simply retaining their distinguishability and taking into account only the relations to one another in which they are placed by the order-setting transformation φ"[89]. Although such a claim would require further discussion, for our purposes it suffices to say that, arguably, Dedekind urges for a separation between the standard model, the series of the natural numbers, i.e. what we normally mean with positive integers, and all other models that indeed satisfy the same properties. Moreover, by transitivity of similarity[90], we also get that two simply infinite systems isomorphic to N are indeed isomorphic to each other.

Theorem 133 completes the argument since it claims that every system which is isomorphic to a simply infinite system (and therefore to the number series N by thm. 132) comes out as simply infinite as well. Both results entitle Dedekind to conclude as follows:

> 134. Remark. By the two preceding theorems (132), (133) all simply infinite systems form a class in the sense of (34).

[89][dedekind01, p. 34]

[90]"33. Theorem. If R, S are similar systems, then every system Q similar to R is also similar to S." [dedekind01, p. 26]

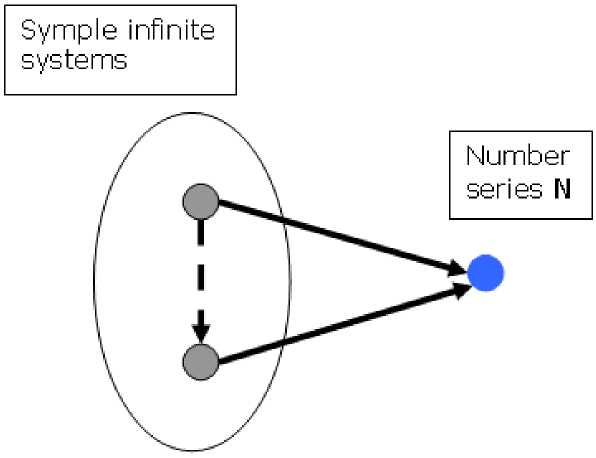

Figure 2.5: Theorem 132

Thus, we obtain a class[91] in the modern sense of the word which includes all the models isomorphic to the standard model N. Therefore, the theory defined by the four conditions α - δ comes out as categorical since all its models are isomorphic. We call this theory "Dedekind Theory", DT for short.

But Dedekind continues:

> At the same time, with reference to (71), (73) it is clear that **every theorem regarding numbers**, i.e. regarding the elements n of the simply infinite system N set in order by the transformation φ, and indeed every theorem in which we [...] discuss only such notions as arise from the arrangement φ, **possesses perfectly general validity for every other simply infinite system** [...] [dedekind01, p. 48]

Thus, even if without a formal proof as in the previous theorems, Dedekind puts forward the fact that from categoricity of the his logical account of the series of natural numbers it follows that all the theorems (viz. L-formulas) regarding numbers (viz. true in the standard model of arithmetic), once they are formulated in what we would now call a certain formal language,[92] are valid (viz. with respect to the semantic relation $\models$) in every other simply infinite system (viz. a model that satisfies DT). But this states nothing but that the *semantically completeness* of DT that Dedekind actually derives from the *categoricity* of DT.

[91] "34. Definition. We can therefore separate all systems into *classes* by putting into a determinate class all systems Q, R, S, . . . , and only those, that are similar to a determinate system R, the *representative* of the class; according to (33) the class is not changed by taking as representative any other system belonging to it." [dedekind01, p. 26]

[92] Roughly speaking, Dedekind hints at this step as long as he talks about a "translation" from the language of the number-series N to the language of any simply infinite system

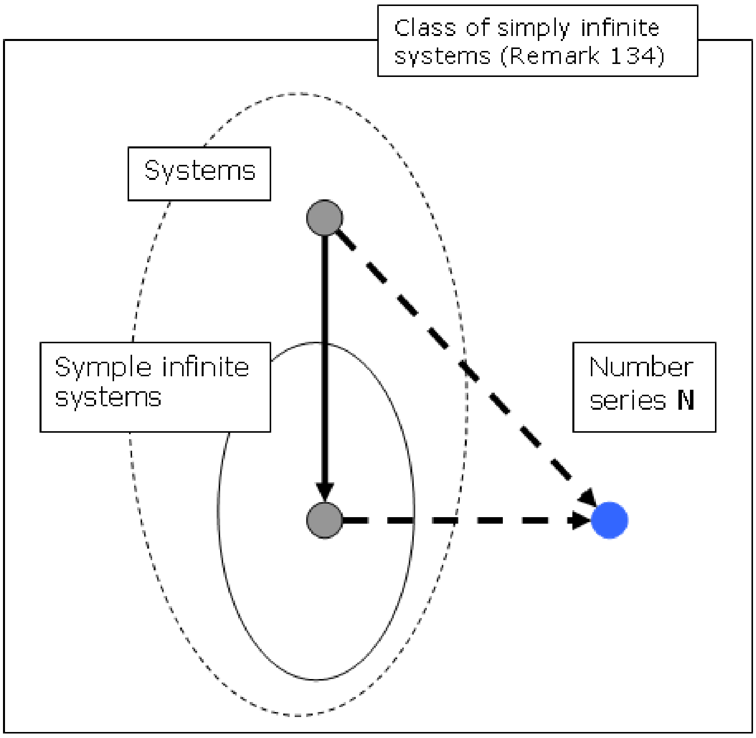

Figure 2.6: Theorem 133-134

Going back to the issue in the remark 2.5.0.10, Dedekind has also something to say about the converse of this implication. Does, for DT, semantically completeness imply categoricity?

Dedekind does not say much about that with respect to DT, but we are likely to say that Dedekind might have answered positively as regards the theory DT^- rather than DT.

In fact, Dedekind holds a stronger claim, viz. the theory DT^- is both non-categorical and semantically incomplete.[93]

Consider the theory DT^-. Now, Dedekind proves that the theory DT^-, is satisfied by models which are not isomorphic to the standard one, say N, and therefore DT^- is not categorical. In addition, Dedekind claims that we could choose one of these non-standard models such that "scarcely a single theorem of arithmetic would be preserved in it"[94]. Let M be one of these models. Now, N and M are both models of DT^- but there is some arithmetical theorem which is true in N but not in M. That is, if N $\models \varphi$ then M $\not\models \varphi$ for almost all arithmetical theorems φ. Hence, DT^- is semantically incomplete.

[93]From this easily follows that if DT^- is non-categorical then it is semantically incomplete. Hence, by contraposition the claim follows.

[94][dedekind90, p. 100]

2.7 To sum up

We return now to what we said at the beginning of the chapter 1, namely the presentation of PA:

> The system of first-order Peano Arithmetic (or PA), is a theory in the language $\mathcal{L}_{PA} = (S, +, \cdot; 0)$, where s is an unary function, + and $\cdot$ are binary functions and 0 is a constant, and with the following axioms:
>
> 1. $\forall x \neg (S(x) = 0)$
> 2. $\forall x \forall y (S(x) = S(y) \rightarrow x = y)$
> 3. $\forall x (x + 0 = x)$
> 4. $\forall x \forall y (x + S(y) = S(x + y))$
> 5. $\forall x (x \cdot 0 = 0)$
> 6. $\forall x \forall y (x \cdot S(y) = (x \cdot y) + x)$
> 7. $\forall \vec{x} [(\varphi(0, \vec{x}) \wedge \forall y (\varphi(y, \vec{x}) \rightarrow \varphi(S(y), \vec{x}))) \rightarrow \forall y \varphi(y, \vec{x})]$
>
> Item 7 is meant to be an axiom for every formula $\varphi(y, \vec{x})$. These axioms are called induction axioms. Such a set of axioms, given by one or more generic symbols "φ" which range over all formulas, is called an axiom scheme; in our case we talk about the induction scheme. [...]
>
> The set N together with the element 0 and usual successor function, addition and multiplication, is a model of PA, which we call the standard model and denote by $\mathcal{N}$. [oosten10, pp. 55–56]

According to what we have been saying, PA so defined is quite different from the way it was when it came into being. Even though this first-order formulation of PA was introduced by Gödel in 1931[95], its original formulation was actually in second-order logic as stated by Dedekind.

In fact, to render the condition β, Dedekind should have used second order logic. In other words, Dedekind puts forward his attempt to use logic *descriptively* in arithmetic. After all, he seeks for a categorical characterisation of the series of natural numbers.

We can easily show that Dedekind's condition concerning the notion of chain requires second order logic to be expressible. To be more precise, it amounts to the following second order formula:[96]

$$\forall X [(X0 \wedge \forall x (Xx \rightarrow XS(x))) \rightarrow \forall x Xx]$$

[95] See [godel31]

[96] See appendix for further details on second-order logic.

The way this whole line of research was born suggests that the most natural framework for foundational inquiry is the second (or high) order logic rather than the first order one.[97] Arguably, the preference for first order logic, which had grown in importance only later, was fuelled by the the discovery of certain beautiful meta-properties for first order languages. For this reason, it was thought that those properties were a good reason to prefer first order over the second one.

However, as he was aiming at categoricity, Dedekind came across the existence of non-standard models of arithmetic. Dedekind was aware that non-standard models such as (S, ψ) undermine the characterisation of the series of natural numbers only from a foundational point of view, not from an experimental one.

It is funny to say that at its origin the axiomatization was profoundly different, namely defined in second order arithmetic and purported completeness. Interesting not only for historical reason but also for philosophical and logical ones is the pathway to such a first-order formalization.

PA departs from the original formulation in these respects, viz. it was a semi-formal formulation and it embodied propositions expressible only in second-order language.

We can now go back to the issue about the set-up of formal axiomatics for foundational purposes. Diagram 2.7 wraps up Dedekind's work in the light of the process of axiomatization presented in chapter 1.

Three degrees of reality are interwoven in the diagram: the ontological, epistemic and linguistic dimensions.[98]

The ontological dimension is represented in the topmost part of the drawing and concerns the way things are in themselves. In our case, we consider there mathematical structures which we are able to have access and study. We first have access to the standard, or intended, model of the series of natural numbers ("the sequence of numbers as it is presented itself in experience", as Dedekind argues).

The epistemic dimension is indicated by the red arrows: it allows mutual interplay between the ontological and linguistic dimensions. By green words we denote the type of relationships held among dimensions in successive steps of the axiomatization.

Last but no way least comes the linguistic dimension which is distinguished, as it were, in two sub-domains, i.e. the semi-formalized language, and the formal language.

As we turn to the series of natural numbers for foundational purposes, we gather insights and intuitions about it. We then thematize the knowledge acquired by this accurate analysis in a list of principles. Those principles are written down by Dedekind in the semi-formal language of systems and transformations. Still, it has to be cleansed from the intended interpretation of symbols and parts which are not fully explicit. This step

[97]See section 3.7.1

[98]See [giordani02, pp. 13–24]

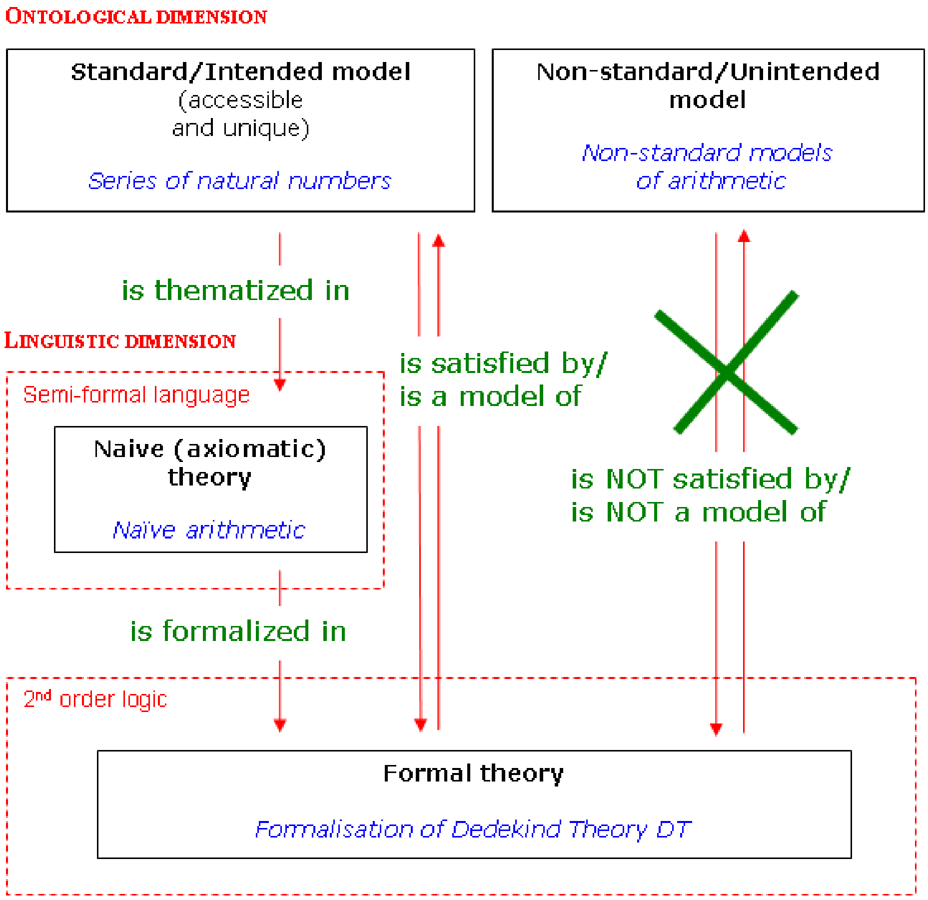

Figure 2.7: The Dedekind-Peano approach

is indeed carried out by Peano, who provides a formalization of Dedekind's axioms. In this way, it becomes clear that from a modern perspective the chain condition relies on second-order logic in a non-trivial way.

In order to provide a categorical axiomatization, the intended and standard model has to be the unique one to satisfy the axioms. However, non-standard models of arithmetic are ontological entities and do exist also. For this reason, Dedekind sees them as a threat to the formal axiomatization as long as they satisfy the axioms even though they are non-isomorphic to the intended model. Nevertheless, by making refinements to the theory, Dedekind manages to ban access to them.

By considering Dedekind's letter as an analytic line of reasoning, we are now entitled to mark off two key-features of his treatment:

- The *causa formalis*[99] of Dedekind Theory, viz. the expression of what DT is, corre-

[99]To recall Aristotle's four-fold classification: "i) The material cause: "that out of which", e.g., the bronze

sponds to a set of principles independent from each other;

- The *causa finalis* of DT, viz. the end for which DT is, amounts to the goal of a complete axiomatization

In the next chapter we will discuss an alternative way of going about the axiomatization of the series of the natural numbers, i.e. the Gödel-Skolem approach (See diagram 2.8). The Gödel-Skolem approach puts aside the quest of categoricity and favours the *deductive* use of logic since it deals with a first-order arithmetical theory such as Peano Arithmetic.[100]

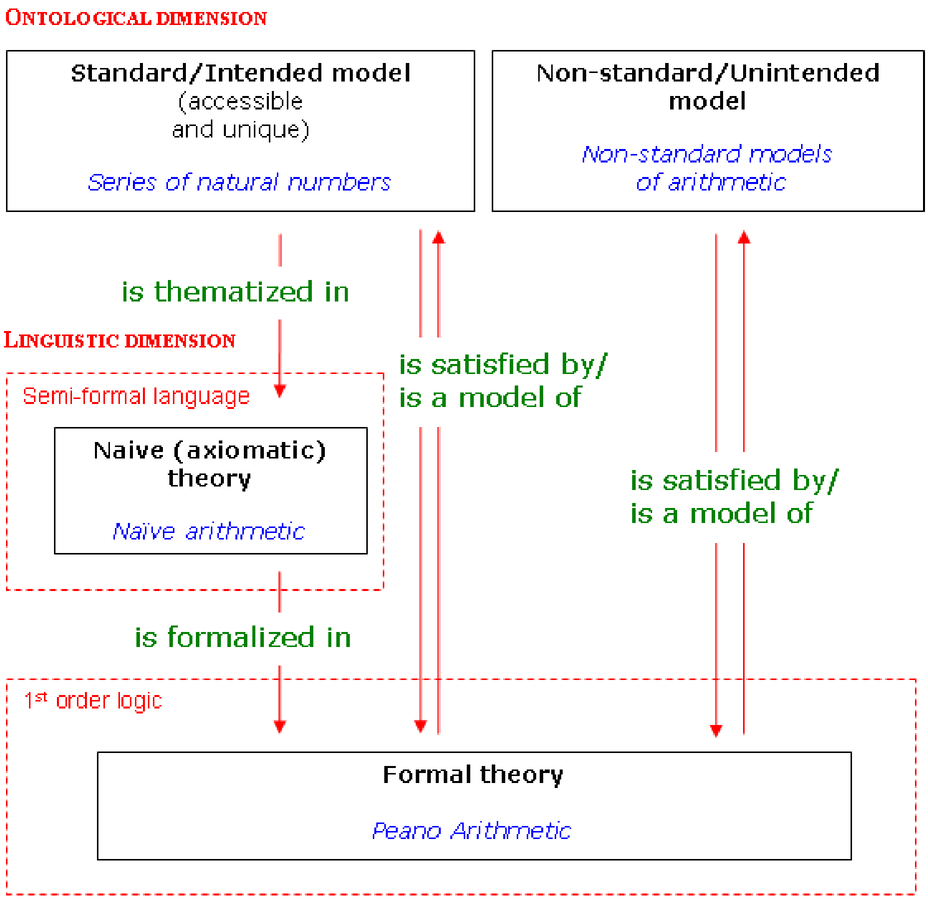

Figure 2.8: The Gödel-Skolem approach

Let us move on to Skolem's work so that we can shed some light on the reasons why we should prefer one of these approaches over the other.

of a statue. ii) The formal cause: "the form", "the account of what-it-is-to-be", e.g., the shape of a statue. iii) The efficient cause: "the primary source of the change or rest", e.g., the artisan, the art of bronze-casting the statue, the man who gives advice, the father of the child. iv) The final cause: "the end, that for the sake of which a thing is done", e.g., health is the end of walking, losing weight, purging, drugs, and surgical tools. " [falcon08]

[100]See section 3.7.1 below.

Chapter 3

The philosophical phase - Skolem and the proof of the existence of NMoA

In the previous chapter we have provided some arguments in favour of the fact that Dedekind must be acknowledged as the forefather of non-standard models. By doing so, we have put forward two principal features ascribable to NMoA:

- a non-standard model is indistinguishable from a standard one if we consider the axiom system: the same axioms are satisfied by both models;
- still, the isomorphism type of non-standard models is admittedly different from that of the standard model.

Now, Dedekind formulates the semi-formal theory DT so as to completely capture the structure of the natural numbers and thus prevent these non-standard models from satisfying the theory. DT appeals in a non-trivial way to a condition that can be only stated in a more powerful language than a first-order one.

In this respect, one of Dedekind's passages deserves our attention once again:

> But such a system S [i.e. the non-standard model constructed by Dedekind] is obviously something quite different from our number sequence N, and I could so choose it that *scarcely a single theorem of arithmetic would be preserved in it*. [dedekind90, p.100, my emphasis]

At face value, the quote seems to be in contradiction with the very basic features of non-standard models. In fact, Dedekind suggests that we could choose a non-standard model such that it does not satisfy the arithmetical theorems of the standard one.

Apparently, Dedekind runs into the following hazard: on the one hand, Dedekind gains evidence for the existence of non-standard models of DT^{-}[101] insofar as there are models according to the two criteria mentioned above, and thus no categorical axiomatization for the series of natural numbers can be achieved via DT^{-}; on the other hand, by saying that possibly "scarcely a single theorem of arithmetic would be preserved" in NMoA, Dedekind dismisses the first condition that makes that model non-standard, viz. both models are indistinguishable if we consider the axiom system.

If this were the case, the conspicuous attempt of Dedekind to flesh out these non-standard models as dangerous intruders would not make sense. In fact, if NMoA did not preserve arithmetical theorems, we would not have any difficulty to dismiss them from the study of the series of natural numbers. But this is not the case: the three conditions stated in DT^{-} are not only satisfied by the standard model of arithmetic but also by NMoA. Hence, as long as theorems are proved on the basis of axioms, it seems quite controversial to say that NMoA make true the axioms but not their theorems.

This point is quite subtle. First of all, we have to distinguish between two sets of statements: on the one hand, we can consider the *logical theorems* that follow from the axioms of a logical theory. In our case, we are dealing with sets of formal axioms for arithmetic such as PA or DT^{*}. Thus, the set of all the theorems valid in PA and DT^{*} is given as follows:

$$Th(PA) = \{\varphi \mid PA \vdash \varphi\}$$

$$Th(DT^{*}) = \{\varphi \mid DT^{*} \vdash \varphi\}$$

given a deductive consequence relation $\vdash$ in the respective logics.

On the other hand, we can focus on the theorems true in the arithmetical models. In this case, we want to refer to the arithmetical theorems that are true in the everyday number theory. Thus, we look at the theorems true in the standard model, say $\mathcal{N}$,[102] viz. :

$$Th(\mathcal{N}) = \{\varphi \mid \mathcal{N} \vDash \varphi\}$$

This set is often called True Arithmetic.

For instance, the whole issue of the *deductive use* of logic in mathematics lies in the fact that, given a statement true in the True Arithmetic, we are able to make it a meaningful question whether the arithmetical theory that we formulated does either derive the statement, or refute it, or neither.

Going back to Dedekind, his claim could be understood as follows: there are some non-standard models that do not preserve the statements true in the True Arithmetic,

[101] Recall that DT is formed by the four conditions α, β, γ, δ while DT^{-} does not contain β. We will also use DT^{*} to denote the corresponding formal version of DT.

[102] The signature of the standard model $\mathcal{N}$ is given, for example, on page 1.

and are thus not relevant to the foundational studies.

Moreover, another consideration is in order. Dedekind is aware of the importance of the "induction principle" in mathematics. In NMN, he carefully points out that the induction principle can be derived from the axioms of DT as a theorem[103] and shows step by step how to apply it to prove some basic arithmetical theorems[104].

Arguably, a model of DT^- could not preserve "a single theorem" in the sense that we are not able to prove theorems in a theory such as DT^- since no induction axiom is provided in it.[105] Let us elaborate on this point. DT^- is a theory satisfied by the standard model as well as the non-standard ones. Dedekind constructs models which are different from the series of natural numbers because he is not specific about the three conditions in DT^-. To rule out non-standard models and obtain a categorical theory, Dedekind then adds a further condition in terms of chains. As showed earlier, this condition can be rephrased in logic as the second-order induction principle. Now, we note that DT^- does not have any induction principle at all: by considering only three conditions, Dedekind does not provide any induction axiom, which is an absolutely necessary tool to prove any basic theorem on natural numbers. Thus, we could pick some non-standard model in which mathematical induction does not hold. Of course, his chains are a formulation of induction, and what he was aiming at was a categorical characterisation of the series of natural numbers. To sum up, Dedekind sets up the weaker theory DT^- as a step toward the complete characterisation of natural numbers, and not *per se* as a possible axiomatization for arithmetic. Accordingly, since DT^- does not have induction principle, it is obvious that "scarcely a single theorem of arithmetic would be preserved in it".

In the end, Dedekind has a bias about non-standard models as mathematical concepts as such. In fact, Dedekind pays little attention to NMoA not because he considers them as innocuous, but rather since they have a secondary role in the overall purpose of the letter to Keferstein. NMoA are just presented as a counterexample to Keferstein's position and not considered as a topic on its own.

To conclude, Dedekind did actually discover the existence of NMoA even if he was not fully aware of their relevance for his work. Nevertheless, Dedekind seems quite unconscious of what non-standard models actually are, what properties they could have, believing that non-standard models have little to do with research in arithmetic.

After Dedekind's work, the issue of NMoA had to wait for Skolem's work, 40 years later, to be explicitly brought up as a limitative result for foundational research in mathematics.

[103][dedekind01, pp. 42–46]
[104][dedekind01, pp. 48–54]
[105]For example $S(x) \neq x$, as considered earlier.

Skolem is commonly considered as the father of non-standard models. Yet, as we showed, he has to share this credit with Dedekind. Unlike Dedekind's, Skolem's work takes place in a period which was characterised by a change in research goals and in the status of the field of mathematical logic. This chapter will be devoted to the main differences between Skolem's and Dedekind's attempt, to Skolem's proof of the existence of NMoA for PA, to Skolem's philosophical views and its influence on the development of his work, and finally we will go back to the relation between descriptive and deductive uses of logic in mathematics in the light of the first versus second order debate over the choice of the logical framework.

3.1 Skolem and non-standard models

In the 1930s, logic had reached its own autonomy, clear boundaries, importance as a subject matter, and thus its own tools and methods. In particular, mathematical logic, the field that was developed aiming at a fruitful application of logical means to mathematics, by then had already achieved one of its most important results: Gödel's Incompleteness Theorems.

The work of Thoralf Skolem on non-standard models of arithmetic can be situated in this fertile period: whereas Dedekind can be considered as one of the pioneers of mathematical logic for his attempt to pursue a logical account of arithmetic, Skolem puts himself forward as a mature mathematical logic scholar who brought up brilliant results so as to break new ground to this field. In his introductory paper to the collection of Skolem's texts, Hao Wang puts it as follows:

> Skolem has made many fundamental contributions to modern investigations on the foundations of mathematics. [...] [Skolem's] work dealt with mathematical logic before such fragmentations became prevalent and has quite a different flavor from the more abstract current approaches: rather he participated in the founding of these subjects by introducing initially several of the basic ideas in their concrete and naked form. [wang70, p.17]

Skolem's research sets off in a period when logic had progressively acquired maturity and its own tools and methods. To this extent, we can say that Skolem had at his disposal a neat formulation of first-order logic together with the most important results pertaining to it. Before Gödel proved the completeness of first-order logic[106] in 1930, Skolem had proved an important meta-property of first-order theories, today stated as: "if a set of sentences has a model, then it has a denumerable model."[107]

[106]Recall that here completeness stands for "exhaustiveness", see footnote 16.
[107][boolos07, p. 147]

The theorem was, put forward for the first time by Löwenheim in 1915, Skolem mended one mistake in the proof and generalized it in 1920 by means of the application of the ingenious device now known as Skolem functions[108].

Skolem's approach to non-standard models differs, to some extent, from Dedekind's: the increasing foundational interest and the fundamental and revolutionary results achieved in logic characterise the status of mathematical logic at Skolem's time. The issue of "non-standard models of mathematical concepts" was discussed by Skolem in connection with the very same 1920 results concerning models of first-order theories. In fact, Skolem takes note in 1923 that, if there is a model satisfying the axioms of set theory (which are stated in first-order language), then, because of his theorem of 1920 (well-known as the Löwenheim-Skolem theorem), a countable structure will be a model of the axioms as well.

According to Skolem, a serious difficulty now lies in the fact that a set-theoretic axiomatization is provided in order to capture in a natural way the mathematical structure represented by set theory whose main feature is the "uncountability" of its domain. Thus, by his 1920 result, first order axiomatization fails to render this fundamental property due to the presence of models whose domain is not "uncountable". In this light, Skolem introduces the notion of "non-standard models"[109]: a model which satisfies the axioms of set theory but does not exhibit the natural "uncountability" that it would be supposed to have.[110] This apparent contradiction is known in the literature as Skolem's paradox [111].

From that moment on, Skolem began to question the formal framework as a reliable foundation for mathematics. For instance, the so-called Skolem paradox is used by Skolem to claim that set-theory cannot serve as a "foundation of mathematics".

However, this attitude is not motivated by a rejection of foundational research as a pseudo-issue, but rather, by means of his criticisms, Skolem hopes to find the solid and unshakable foundations on which mathematical methods can be build. In this light, he also questions infinitary methodes as not safe enough.[112]

The 1923 remark on set theory represents the starting point of Skolem's discussion on non-standard models: once again, the notion of *non-standard* embodies the idea of undesirable results suggesting a revision of the assumptions in the light of what we would like to achieve. Nonetheless, Skolem does not restrict himself only to the set-theoretical case, but he envisages that this issue also concerns other mathematical structures. As Jens Erik Fenstad reports in the biographical note "Thoralf Albert Skolem in Memoriam":

[108] For a definition of Skolem functions, see [wang70, p.19–22]

[109] Indeed, Skolem never used the term "non-standard" for these models except for the last paper on the topic, namely [skolem55]. Before than, Skolem used to adopt long periphrases instead of special terms.

[110] Today we actually know that there are other types of non-standard models of set theory other than the countable ones.

[111] See [bays09]

[112] See the discussion in section 3.6

> Toward the end of the 1929 paper[113] Skolem expressed some doubts as to the complete axiomatizability of mathematical concepts. His scepticism was based on the set-theoretic relativism which follows from the Löwenheim-Skolem theorem. In 1929 he could give only some partial results, but in a paper from 1934 (and a previous one from 1933[114]) Über die Nicht-characterisierbarkeit der Zahlenreihe[115] he could prove that there is no finite or countably infinite set of sentences in the language of Peano arithmetic which characterises the natural numbers. [fenstad70, p.14]

Before turning to the arithmetical case, we put forward another point on which Skolem and Dedekind seem to differ significantly. As a consequence of the mature status of the research in mathematical logic, we can credit Skolem as the father of non-standard models of arithmetic in a "modern sense". This claim is not in contradiction with what we have argued about Dedekind.

Dedekind brings up *in nuce* that non-standard models are different from the intended model of arithmetic, and that it is not possible to ban them as long as we only consider the theory DT^{-}. However, Dedekind mentions the construction of NMoA just in passing as he steers toward a categorical axiomatization of natural numbers and, in the second place, makes a controversial claim about the possibility of theorems of arithmetic not being preserved in these non-standard models. On the other hand, Skolem is aware that such theorems do actually hold in *properly constructed* non-standard models, and that we cannot distinguish these with respect to first-order properties (i.e. both axioms and theorems). This enables Skolem to fully recognise NMoA as a relevant issue to foundational research.

To sum up, Skolem's study of non-standard models of arithmetic differs from Dedekind's in two respects:

1. Unlike Dedekind, Skolem introduces non-standard models by considering a formal theory for arithmetic, viz. a first order theory which contains the PA axioms and thus is provided with an induction axiom, strong enough to derive at least most of the ordinary theorems of arithmetic;

2. In virtue of the then-developments in mathematical logic, Skolem was able to address his discussion on non-standard models as a limitative result to some attempted lines of research and corroborate his position by referring to analogous phenomena in other mathematical branches.

Nevertheless, both Skolem and Dedekind agree on one important point: both consider NMoA as a limitative result which comes out as a bound to the descriptive task of

[113] Ed. note: [skolem29]
[114] Ed. note: [skolem33]
[115] Ed. note: [skolem34]

pursuing completeness in the sense of categoricity.

3.2 Skolem and the existence of NMoA

In his introduction to Skolem's logical works, Hao Wang stresses the importance of Skolem's interest in non-standard models with respect to his wide range of interests:

> Skolem has made many fundamental contributions to modern investigations on the foundations of mathematics. If one has to single out one most intriguing item, it would probably be his work on non-standard models of set theory and number theory. [wang70, p.17]

As we mentioned earlier, Skolem comes up with the idea of non-standard models looking at the axiomatization of set theory. As far as non-standard models of arithmetic are concerned, Skolem foresees to apply *mutatis mutandis* his set-theoretical insights to the arithmetical case. In the last section of his paper 1929 [116], Skolem considers a fragment of number theory and puts forward the idea of "some simple nonstandard model by taking a suitable set of polynomials as the natural numbers"[117]. Elsewhere, Wang puts even more forcefully that in [skolem29] "it is for the first time suggested in print [...] that given any set of theorems on natural numbers, we can find a nonstandard model in which these theorems are true."[118].

But how does Skolem describe these non-standard models? Which are their properties? Skolem himself hints at non-standard models as a shortcoming that may affect any attempt to exhibit any categorical axiomatization of mathematical concepts:

> A very probable consequence of this relativism[119] is again that it cannot be possible to completely characterise the mathematical concepts; *this already holds for the concept of the natural number*. Thereby arises the question, *whether the unicity or categoricity of mathematics* might not be an illusion. [wang96, p.125, my emphasis]

In this way, Skolem suggests that categoricity, which was the guiding thread of Dedekind's analysis of natural numbers, may not be reached at all. In particular, the presence of non-standard models of PA is indeed evidence that categoricity cannot be obtained for that theory. Thus, it seems that Skolem's standpoint comes out in the face of Dedekind's attempt to pursue a categorical axiomatization of the natural numbers.

[116][skolem29, pp. 269–272]
[117][wang96, p.125]
[118][wang70, p.41]
[119]Skolem uses the word "relativism" in the following technical sense: the impossibility to capture completely the "absolute" features of mathematical structures. See earlier discussion on set-theoretical models on page 64

After having only sketched this possibility, in 1933 ([skolem33]) Skolem provides the actual proof of the existence of NMoA:

> In 1933 Skolem published his famous result on the concept of natural number [...] a nonstandard model which has the same true (first-order) sentences as its standard model. [wang96, pp.126]

Roughly speaking, Skolem's paper says that standard and non-standard models make the same first-order formulas true despite being different from each other (this condition is implicitly assumed in this passage). Thus, Skolem states out clearly what was proposed in Dedekind's letter. The title of the paper speaks for itself: "On the impossibility of a complete characterisation of the number sequence by means of a finite axiom system."[120] Once again, Skolem does not consider non-standard models *per se*, but he relegates them to second place. NMoA are indeed consequences of the "impossibility of complete characterisation" and they are not mentioned in the title. We also note that Skolem eventually spells out the interconnection between standard/non-standard models and axiom systems. His result relies on an arbitrary "axiom system" which is given prior to the proof.

The proof of the "On the impossibility of a complete characterisation of the number sequence" is not just presented in [skolem33]. Skolem puts forth this result in other three papers: [skolem34], [skolem41] and [skolem55]. Let us discuss them in turn. The 1933 and 1934 papers are both written in German and both reviewed by Gödel in [godel34] and [godel35] respectively. Gödel acknowledges the fact that the two papers are essentially one, as he reports in the review to [skolem34]:

> With respect to its result, this work is identical to the same author's 1933a[121] it merely gives a somewhat simplified proof and the following formulation of the result [...] [godel86, p. 385]

However, Smorynski brings up the fact that the subtle difference between the two papers may lie in the following respect:

> In Skolem 1933, he was able to prove this [the existence of non-standard elements] relative to any finite set of axioms. [...] in the following year in Skolem 1934 lie published a proof of the existence of *strong* nonstandard models of arithmetic [i.e. an elementary extension of the standard model][122] [...] [smorynski84, p.2, my emphasis]

[120]"Über die Unmöglichkeit einer Charakterisierung der Zahlenreihe mittels eines endlichen Axiomensystems". Title translated in [godel86, p.379]

[121]Ed. note: [skolem33]

[122]"A nonstandard model was called "strong" if it elementarily extended the standard model [...]" [smorynski84, p.1]. However, it is plausible that the concept of elementary extension was not known at the time of Skolem, but only the one of elementary equivalence.

In fact, it suffices to compare the titles of the two papers to note this mathematical nuance: [skolem33] is entitled "On the impossibility of a complete characterisation of the number sequence by means of a finite axiom system" whereas [skolem34] is entitled "On the non-characterisability of the number sequence by means of finitely or *denumerably* many propositions containing variables only for numbers"[123]. By considering elementary extensions of the standard model, Skolem generalises from a conclusion of the finite case to the denumerable one. At any rate, we can summarize that in both papers Skolem proves "a general theorem on all formulas of elementary number theory"[124].

[skolem41] is the French report of Skolem's 1938 lecture in Zürich. What makes this paper interesting is its philosophical character. In fact, it must be kept in mind that Skolem wrote only few philosophically oriented essays, and most of them were on the occasion of giving a lecture.[125]

In 1955 Skolem provided an English version of the previous papers in [skolem55] in the proceedings of a conference. This version does not present relevant differences in content with respect to the previous papers, but presents the most common reference to Skolem's proof of the existence of NMoA, as Smorynski notes:

> Although this was merely a repetition of his earlier proof, it is this later paper to which most people refer. Perhaps this is because this paper is in English, a much easier language than German for many of us; perhaps it is simply that this paper, appearing in a slim little volume that has been reprinted, is in more private libraries and hence more accessible; [...] [smorynski84, p.2]

We shall now elaborate more on Skolem's proof, viz. the proof usually presented in present day text books.

3.3 Compactness and the modern proof of the existence of NMoA

Besides the widespread (but erroneous) belief that Skolem must be considered the only father of non-standard models of arithmetic, in several textbooks Skolem's name is oftentimes mentioned in connection with the application of Completeness, Compactness or Upwards Löwenheim-Skolem theorems in the proof of the existence of NMoA for PA.[126] As we will try to show, this view is profoundly mistaken for two reasons. To begin with, Skolem's proof comes out as an independent and valuable attempt to settle the limits and weaknesses of first-order axiomatization of number-theory. Secondly, Skolem is abundantly clear on his firm position against the use of infinitary reasoning such as in the

[123] "Über de Nichtcharakterisierbarkeit der Zahlenreihe mittels endlich oder abzählbar unendlich vieler Aussagen mit ausschliesslich Zahlenvariablen". Title translated in [godel86, p.385]

[124] [wang70, p.41]

[125] See [wang96, p.127]

[126] See on page 21.

Gödel Completeness theorem and in the Upwards Löwenheim-Skolem theorem. In this respect, to associate Skolem's name to the commonly presented proof of the existence of NMoA turns to be a historical as well as conceptual mistake. Let us discuss this point in more detail.

In 1930, Gödel proves the completeness theorem for first order logic[127]. The following year, Gödel obtains what is arguably the most important result for mathematical logic[128], known as the first incompleteness theorem, which can be informally phrased as follows.

Let T be a sufficiently strong formal theory for arithmetic. Recall that T is called *consistent* if for no statement A in the language of T, $T \vdash A$ and $T \vdash \neg A$ and T is called *syntactically incomplete* if for some statement A in the language of T, neither $T \vdash A$ nor $T \vdash \neg A$.

Theorem. (Gödel [1931] as improved by Rosser [1938]): If T is consistent, then it is syntactically incomplete.[129]

This theorem represents a dramatic consequence for any attempt to axiomatize the series of natural numbers: for any axiomatization (under certain conditions) that we fix to characterise the series of natural numbers, viz. capable to derive true theorems of number-theory, there is some arithmetical sentence that is neither provable nor refutable in that axiomatization.

The first and foremost consequence of this theorem regards the limits of the use of logic in mathematics: even for an elementary mathematical structure such as the natural numbers, we will never succeed in acquiring a syntactically complete theory.

Going back to NMoA, Gödel's Completeness and the First Incompleteness Theorem play a crucial role in our purposes. In fact, the existence of non-standard models of PA could have been established by a simply appeal to these theorems. In fact, today the proof of the existence of NMoA follows as a rather simple consequence of the Compactness Theorem (which is in turn a consequence of the Completeness theorem). Yet, it must not be overlooked that at that time Gödel did not come up with the idea of applying compactness theorem to obtain non-standard models of PA:

> This simple fact about models of Peano Arithmetic [i.e. the proof of existence of NMoA by the completeness theorem] was not pointed out by Gödel in any of the publications connected with the Completeness Theorem from that time, and it seems not to have been noticed by the general logic community until much later. [...] Gödel in his review (1934c)[130] of Skolem's paper

[127] See theorem 2.5.0.2 on page 48. Once again, according to the distinction drew in section 1.3, here completeness refers to a meta-property pertaining to logic and we call it "exhaustiveness".

[128] See [hintikka89]

[129] See [barendregt10]

[130] Ed. note: [godel34]

> also does not mention this fact, rather observing that the failure of categoricity for arithmetic follows from the incompleteness theorem. [kennedy08]

On the other hand, Skolem did not exploit in his proof completeness and compactness either:

> But Skolem never mentions the fact that the existence of such models follows from the completeness and compactness theorems. [kennedy08]

The methods Skolem used in his proof have surely been influenced by his philosophical views: he favoured non-constructive methods which are weaker than the infinitary reasoning exploited in the proof of Completeness. In addition, we have to bear in mind that Skolem had already hinted at the notion of non-standard models for arithmetic in 1929, and thus before Gödel's results.

As just said, the currently most frequent proof of the existence of NMoA for PA rests on the use of compactness. This proof can be traced back to Henkin's paper: "Completeness in the Theory of Types". In 1950, Leon Henkin first proves the existence of non-standard models by an appeal to Compactness[131]

> [...] it is not immediately clear that any non-standard models exist. However, they do exist [...] and *we shall give a method of constructing every general model* without resorting to impredicative processes. [henkin50, pp.84–85]

> By appealing to Theorem 3 [i.e. Compactness theorem], however, it becomes a simple matter to construct a model containing a non-standard number system which will satisfy all of the Peano postulates as well as any preassigned set of further axioms. We have only to adjoin a new primitive constant u_ι, and add to the given set of axioms the infinite list of formulas $u_\iota \neq 0$, $u_\iota \neq S_{\iota\iota}0$, $u_\iota \neq S_{\iota\iota}(S_{\iota\iota}0)$, ... Since any finite subset of the enlarged system of formulas is clearly satisfiable, it follows from theorem [i.e. Compactness theorem] that some denumerable model satisfies the full set of formulas, and such a model has the properties sought. By adding a non-denumerable number of primitive constants v_ι^ξ, together with all formulas $v_\iota^{\xi_1} \neq v_\iota^{\xi_2}$ for $\xi_1 \neq \xi_2$, we may even build models for which the Peano axioms are valid and which contain a number system having any given cardinal. [henkin50, p.90]

An explicit version of Henkin's proof can be exhibited in the following way:

[131] In a footnote, Henkin acknowledges that Malcev had carried out a similar proof in algebra: "A similar result for formulations of arithmetic within the first order functional calculus was established by A. Malcev, Untersuchungen aus dem Gebiete der mathematischen Logik, Recueil mathematique, n.s. vol. 1 (1936), pp. 323-336. Malcev's method of proof bears a certain resemblance to the method used above. I am indebted to Professor Church for bringing this paper to my attention. (Added February 14, 1950.)" [henkin50, p.90, fn.12]. See in this respect [hodges09]

Theorem 3.4. *PA has non-standard models*

Proof. Let $\mathcal{N}$ be the standard model of PA, i.e. $\mathcal{N} = (\mathbb{N}, 0, S, +, \cdot)$ where $\mathbb{N}$ is the set of natural numbers, 0 is the element zero, S, +, $\cdot$ are the usual successor function, addition and multiplication. We want to show that there is a model which also satisfies the theory PA but it is not isomorphic to $\mathcal{N}$.

First, we define, for every $n \in \mathbb{N}$, a term $\overline{n}$ of $\mathcal{L}_{PA}$ by recursion: $\overline{0} \equiv 0$ and $\overline{n+1} \equiv S(\overline{n})$. Terms of the form $\overline{n}$ are called numerals.

Now let c be a new constant. Consider in the language $\mathcal{L}_{PA} \cup \{c\}$ the following set of axioms:

$$T_m := \{axioms\ of\ PA\} \cup \{\neg(c = \overline{n}) |\ n \in \mathbb{N} \wedge n < m\}$$

Clearly, for some $m \in \mathbb{N}$, T_m represents the set which contains all the axioms of PA along with an axiom $\neg(c = \overline{n})$ for each number n less than m. Suppose that PA is consistent, in symbols $PA \nvdash \perp$. Clearly, for each $m \in \mathbb{N}$, $T_m \nvdash \perp$ as well. For given a T_m we put $c^{\mathcal{N}} = \overline{m}^{\mathcal{N}}$ so that we get that $\mathcal{N}$ is a model of T_m when we assign to c an element which is greater than all the numerals that occur in the theory T_m. And this element is indeed m.

Now, since $T_0 \subseteq T_1 \subseteq \ldots$ the set $T_\omega = \bigcup_{n \in \mathbb{N}} T_n$ is also consistent. Then, by completeness $T_\omega \nvDash \perp$, and hence there is a model $\mathcal{M} = (M, 0, S', +', \cdot')$ such that $\mathcal{M} \vDash \varphi$ for all $\varphi \in T_\omega$. Clearly, $\mathcal{M}$ satisfies the PA-axioms with respect to the formulas in the language $\mathcal{L}_{PA}$.

Now, we show that it is not the case that $\mathcal{M} \cong \mathcal{N}$. In particular, we show that there is no bijection between M and N. For this, if c is an element of M defined as above, it is easy to see by the theorem $\forall x(x = 0 \vee \exists y(x = S(y)))$ that in M there is a downward sequence $c, c-1, c-2, \ldots$, and this cannot exist in N. $\square$

A similar proof can be exploited to show that the True Arithmetic TA has non-standard models as well.

Moreover, we note that this proof is highly *non-constructive* in the sense that the Completeness Theorem (and thus Compactness) can be applied to every first-order system, even some non-axiomatizable one (i.e. a theory whose set of axioms is not recursive enumerable). In other words, we are thus allowed to claim that a certain formal theory has indeed a model although we are not even able to exhibit a recipe to list its formulas. In this sense we say that this particular approach to non-standard models would not very likely have been to Skolem's taste. In fact, Skolem dislikes such reasoning, and calls it non-constructive reasoning and thus aims at establishing the definition of non-standard models in a *perfectly constructive way*, as Skolem himself states:

> [...] I intend to show how models of a similar kind [non-standard model of arithmetic] can be set up in a *perfectly constructive way* when we consider

some very restricted arithmetical theories. [skolem55, p. 1]

Before investigating the philosophical motivations behind Skolem's mathematical *modus operandi*, we shall present a sketch of Skolem's proof.

3.5 Skolem's proof of the existence of NMoA

One preliminary observation is needed. Skolem's writing style is not really simple in his first 1933 and 1934 papers. Robert Vaught argues, in fact, that Gödel's reviews of Skolem's proof are likely to be found much clearer than the original text. For example, Vaught notes that in [skolem33] and [skolem34] "Skolem does not use the word 'definable' and replaces it in each paper with a different discussion over a page long!"[132]. Thus, for a clearer and more lucid presentation, we will not only refer to Skolem's five papers that contain the proof but also to the explicative remarks offered by Gödel in his reviews, by Wang[133], by Vaught[134] and by Smorynski[135].

To begin with, Skolem specifies the formal language, called P, in which the arithmetical formulas can be formulated. Wang, following Gödel[136], summarises P in five points:

> Skolem proves a general theorem on all formulas of elementary number theory P. i.e. those obtained in the usual manner from: (1) variables ranging over natural numbers; (2) + (addition) and $\cdot$ (multiplication); (3) $<$and = ; (4) operations of the propositional calculus; (5) quantifiers with number variables. (More complicated recursive and indeed arithmetic functions can be defined in P; alternatively, they can be added at the beginning without affecting Skolem's proof). [skolem70, p.41]

Skolem's theorem can be formulated as follows:

> There exists a system $\mathcal{N}^*$ of things, for which two operations + and $\cdot$, and two relations = and $<$are defined, such that N^* is not isomorphic to the system N of natural numbers, but, nevertheless, all sentences of P which are true of N are true of N^*.[137]

Since the original [skolem33] is in German, this is the English translation presented by Wang which hardly differs from the English translation of Gödel's review.[138] Skolem

[132] [godel86, p. 376]
[133] [wang70]
[134] [godel86, pp. 376–379]
[135] [smorynski84]
[136] See [godel34, p.379]
[137] Wang's translation on [wang70, p. 41]
[138] [godel34, p. 379]

defines $\mathcal{N}$ to be the structure $(N, +, \cdot, <)$, where $N = \{1, 2, 3, \dots\}$ and thus constructs a model $\mathcal{N}^*$ that has the same true first-order sentences as N. The proof is articulated in three steps.

P represents the underlying language in which the concepts of the elementary number theory can be formulated. As arithmetical theory, Skolem provides a list of ten axioms[139], formulated in P, which is slightly different from the usual PA presentation in the sense that, along with the seven PA axioms, he also considers three axioms for defining '=' logically.

However, Skolem does not want to deal with generic P-statements as axioms or theorems of the "theory of natural numbers"[140], say T, but rather looks at them as equations between arithmetical functions. In this way, instead of considering the language of the arithmetical theory as a list of formulas which, by construction, are built by at least one relation, he considers arithmetical functions as primitive elements, and thus proves the theorem only by dealing with such functions and the relations occurring among them. To do so, Skolem has to demonstrate that, given the set F of all arithmetical functions definable in P, "every statement can be replaced by an equivalent equation between two elements of F containing only free variables" [141] and therefore "it is evident that all true [P-]formulas may be listed as an enumerated set S. To each of them we may find an equivalent equation whose both sides are functions belonging to F."[142].

This is done in the "Preliminary Remarks", the first section of [skolem55], just after he has set up the formal theory for arithmetic.

Once he has shown how to rephrase each axiom of T as a free-variable equation between arithmetical functions,[143] Skolem has simply to prove the existence of a model $\mathcal{N}^*$ different from $\mathcal{N}$ for the set S of statements, as follows:

> Let S be a set of equations whose both sides are elements of a denumerable set of functions closed with regard to nesting. Assuming the equations belonging to S all valid for the natural number series $\mathcal{N}$ we may define a greater series $\mathcal{N}^*$ such that by suitable extension of all notions concerning $\mathcal{N}$ to corresponding ones in $\mathcal{N}^*$ all equations in S are also valid for $\mathcal{N}^*$.

Since S represents the set of first-order sentences definable in P, the claim amounts to saying that there is a structure $\mathcal{N}^*$ not isomorphic to $\mathcal{N}$ which verifies the same first-order sentences as $\mathcal{N}$. In short, $\mathcal{N}^*$ embodies the two features (first-order indistinguishability

[139] [skolem55, p. 2]
[140] [ibid.]
[141] [skolem55, p. 4]
[142] [ibid.]
[143] Skolem replaces existential quantifiers with Skolem functions, offers a way to restate formulas where negation, conjunction and disjunction occur by the sole means of arithmetical functions. See [skolem55, pp. 2–4]

and non-isomorphism) with respect to $\mathcal{N}$ to be a called a non-standard model of arithmetic. In fact, if $\mathcal{N}^*$ is a greater series than $\mathcal{N}$, than they are non-isomorphic and the equivalences valid in $\mathcal{N}$ will remain valid in $\mathcal{N}^*$. This completes the first step which enables Skolem to focus on the set F made up of all P-definable functions. Before providing the proof of the theorem, which represents the third and final step, Skolem elaborates an arithmetical lemma which will turn out to be fundamental later on.

Lemma 3.5.1. *If F represents the enumerated sequence of arithmetical functions, i.e.* $\{f_i | i \in \mathrm{N}\}$ *where* $f_i : N \to N$*, an arithmetical function* $g : \mathrm{N} \to G \subseteq \mathrm{N}$ *exists such that for any pair i, j the same relation* $<$*, = or* $>$ *takes place between* $f_i(g(t))$ *and* $f_j(g(t))$ *for all t* $>$ *max(i, j). Moreover g is onto G and G is infinite.*

Proof. Consider the following sequence $f_1, f_2, f_3, \ldots$ where for any $m \in N$, $f_m : N \to N$.

We can pick one by one these functions and compare them as follows:

Consider f_1 and f_2. We define the following sets: $N^{(1)} = \{x | f_1(x) < f_2(x)\}$, $N^{(2)} = \{x | f_1(x) = f_2(x)\}$ and $N^{(3)} = \{x | f_1(x) > f_2(x)\}$. Let N_1 be $N^{(s)}$ where s is the least index for which $N^{(s)}$ is infinite.

Now consider f_3. If N_1 is $N^{(2)}$, then we have to consider three cases: $N_1^{(1)} = \{x | f_1(x) < f_3(x)\}$, $N_1^{(2)} = \{x | f_1(x) = f_3(x)\}$ and $N_1^{(3)} = \{x | f_1(x) > f_3(x)\}$. If N_1 is $N^{(1)}$ or $N^{(3)}$, there are five cases to consider. Suppose that $N_1 = N^{(1)}$, then we define $N_1^{(1)} = \{x | f_1(x) < f_3(x) < f_2(x)\}$, $N_1^{(2)} = \{x | f_3(x) < f_1(x)\}$, $N_1^{(3)} = \{x | f_1(x) = f_3(x)\}$, $N_1^{(4)} = \{x | f_2(x) = f_3(x)\}$ and $N_1^{(5)} = \{x | f_2(x) > f_3(x)\}$.

As before, we put $N_2 = N_1^{(s)}$ where s is the least index for which $N_1^{(s)}$ is infinite.

By construction, N_2 is a subset of N_1. Thus, by continuing in the same way for the other f_m, we obtain an infinite sequence of infinite subsets of N (which represents the set of natural numbers and thus the domain of the standard model)

$$N = N_0 \supset N_1 \supset N_2 \ldots$$

Now we can define a linear order on F as follows. By definition, for all $t \in N_{n-1}$ for $n \in N$ the relations $<$, $>$or = take place between $f_1(t), \ldots f_n(t)$. Let $g : N \to N_{m \in N}$ such that $g(n)$ is the least number in N_{n-1}. Clearly it exists for all N_n. It follows that for any f_i, f_j, eventually

$$f_i(g(n)) \gtreqless f_j(g(n))$$

where $n = max(i, j)$, then we have accordingly

$$f_i(g(t)) \gtreqless f_j(g(t))$$

for all $t \geq n$. This g is the function we were looking for. Note that g is monotone increasing and does not have an upper bound.[144]. □

[144]Skolem also claims that the intersection of all N_s is the empty set. It is not entirely clear how this follows.

Now, by means of this lemma, we can consider the range of the function g, say G, that is an infinite subset of N in order to construct the model $\mathcal{N}^*$ according to the theorem:

Theorem 3.5.2. Skolem 1933. *There exists a system $\mathcal{N}^*$ of things, for which two operations + and $\cdot$, and two relations = and $<$ are defined, such that $\mathcal{N}^*$ is not isomorphic to the system $\mathcal{N}$ of natural numbers, but, nevertheless, all sentences of P which are true of $\mathcal{N}$ are true of $\mathcal{N}^*$.*

Proof. Let F be an enumerated sequence of arithmetical functions definable in P. By Lemma 3.5.1, there exists a function $g : N \to G$ such that for any pair $i, j \in N$ the same relation $<$, = or $>$ takes place between $f_i(n)$ and $f_j(n)$ for almost all (i.e. all but finitely many) $n \in G$.

Now, we define $f_i \equiv f_j$ iff $f_i(n) = f_j(n)$ for almost all $n \in G$, and we put the domain of $\mathcal{N}^*$, say N*, such that is the quotient set $F/\equiv$. It is easy to show that $F/\equiv$ is an equivalence class on F. Moreover, the operations +, $\cdot$ can be defined on N* pointwise such that the P-statements valid on $\mathcal{N}$ remains also valid on $\mathcal{N}^*$.

Again, by Lemma 3.5.1, we define a linear order $<$ on the domain $F/\equiv$ for almost all $n \in G$.

We have so obtained the model $\mathcal{N}^* = (N^* = F/\equiv, +/\equiv, \cdot/\equiv, </\equiv)$ which makes valid all the sentences valid in $\mathcal{N}$ by construction, i.e. $\mathcal{N}$ and $\mathcal{N}^*$ are elementary equivalent.

To complete the proof we have to show that $\mathcal{N}$ and $\mathcal{N}^*$ are not isomorphic. To do so, one can form statements that hold for $\mathcal{N}^*$ but not for $\mathcal{N}$.

Skolem is not very explicit on this point in [skolem55]. However, in [skolem41] he informally reasons that:

> Consequently, the whole recursive theory is valid in $\mathcal{N}^*$. On the other hand, the models $\mathcal{N}$ and $\mathcal{N}^*$ cannot be isomorphic because a bi-univocal correspondence that would keep the order, between $\mathcal{N}$ and $\mathcal{N}^*$, should then exist. But in any correspondence of this type, $\mathcal{N}$ elements are represented on themselves, and the element $f(x) = x$ of $\mathcal{N}^*$ is $>$ than any constant, viz. than any element of $\mathcal{N}$. In the language on the set theory, $\mathcal{N}^*$ represents a type of much higher order than $\mathcal{N}$.[145]

By using the diagonalizing "g", Skolem forms a statement that holds in $\mathcal{N}^*$ but not in $\mathcal{N}$.

For example, we could each time leave out the smallest element out of N_s. Then it does surely follow.

[145]"Par conséquent, la théorie récursive tout entiére est valable dans $\mathcal{N}^*$. D'autre part, les modéles $\mathcal{N}$ et $\mathcal{N}^*$, ne peuvent être isomorphes ; car alors il devrait exister une correspondance biunivoque conservant l'ordre, entre $\mathcal{N}$ et $\mathcal{N}^*$. Mais dans toute correspondance de ce genre les éléments $\mathcal{N}$ se représentent sur eux-mêmes, et l'élément $f(x) = x$ de $\mathcal{N}^*$ est $>$ que toute constante, c'est-á-dire que tout élément de $\mathcal{N}$. Dans le langage de la théorie des ensembles, $\mathcal{N}^*$ représente un type d'ordre beaucoup plus haut que $\mathcal{N}$." [skolem41, p. 475, my translation]

$f(x) = x$ is valid for almost all $x \in G$ and thus $f(g(y)) = g(y)$ for $y \in N$. Now, if N elements are represented on themselves, $f(g(y)) = g(y)$ is necessarily greater than any constant and therefore than any $n \in N$, i.e. $g(y) > n$.

This shows that $\mathcal{N}$ and $\mathcal{N}^*$ are non-isomorphic and indeed $\mathcal{N}^*$ is a proper elementary extension of $\mathcal{N}$. □

An important observation is in order. Note that in the proof Skolem tacitly switches from the formal framework of the P-theory T to the consideration of True Arithmetic $Th(\mathcal{N})$. In fact, he starts off presenting the formal language P. Thus, Skolem wants to show that all the P-sentences are also true in the non-standard model $\mathcal{N}^*$. But he does not actually work with P since he provides right away an effective way to translate P-sentences into free-variable equation between arithmetical functions. At this point, Skolem's proof relies in a non-trivial way on considerations about arithmetical functions, and thus on theorems true in the True Arithmetic. Skolem is not fully explicit how to return to the formal theory T and state the formula that holds in $\mathcal{N}^*$ but not in $\mathcal{N}$.

In his 1950 paper, Henkin summarises the result as follows:

> Skolem makes ingenious use of a theorem on sequences of functions (which he had previously proved) to construct, for each set of axioms for the number sequence [...] a set of numerical functions which satisfy the axioms, but have a different order type than the natural numbers. [henkin50, p. 90]

Leaving aside the observation on the different order-type, it is clear from what we have argued so far that the result holds for *each set of* P-axioms: Skolem has provided a general method to restate any P-statement in terms of arithmetical functions. Thus, Skolem notes that the following corollary follows from theorem 3.5.2:

Corollary 3.5.3. *No recursively enumerable axiom system using only the notation of P (i.e. using only concepts of elementary number theory) can determine uniquely the structure of the sequence of natural numbers.*[146]

Theorem 3.5.2 is stronger than corollary 3.5.3 since it applies to any given set of true sentences, and thus even if it is not axiomatizable (i.e. not a recursively enumerable set).

Corollary 3.5.3 can be also proved by Gödel's First Incompleteness Theorem, as Gödel himself notes in the review to [skolem33]:

> From this [theorem 3.5.2] it follows that there is no axiom system employing only the notions mentioned at the outset (and therefore none at all employing only number-theoretic notions) that uniquely determines the structure of the sequence of natural numbers, a result that also follows without difficulty from the investigations of the reviewer in his 1931. [godel34, p. 379]

[146] [wang70, p. 41]

Along this line, we can prove a weaker claim than 3.5.3 but still relevant for our purposes. Instead of looking at it as a consequence of theorem 3.5.2, we prove it by applying the First Incompleteness Theorem:

Corollary 3.5.4. *PA is not categorical*

Proof. The First Incompleteness Theorem is applied as follows:
by Gödel's incompleteness theorems, there is a formula in the language of PA, say $\overline{\varphi}$ such that is independent of PA, i.e. neither $PA \vdash \overline{\varphi}$ nor $PA \vdash \neg\overline{\varphi}$ holds. By completeness of first order logic, we get that neither the $\overline{\varphi}$ nor its negation is a semantic consequence of PA, i.e. neither $PA \vDash \overline{\varphi}$ nor $PA \vDash \neg\overline{\varphi}$ holds. This is enough to say that PA is not semantically complete, and therefore PA is not categorical. □

Instead of saying that PA cannot determine uniquely the structure of the sequence of natural numbers, we adopt the concept of categoricity to restate this result in a shorter way. Indeed, this will become even clearer when we look at Skolem's 1938 lecture in which he himself talks about the notion of categoricity and characterises it in the very same way we use it today:

> We say that a field [champ] is categorically defined by certain axioms if two models of this domain, M and M' namely any two exemplifications of this domain, are isomorphic with respect to any properties and relations occurring in the axioms, viz. M and M' can be univocally represented one on the other in such a way that all the properties and relations are conserved in the representation. [147]

Among other things, Skolem considers the non-categoricity of first order arithmetic in connection with Gödel's Incompleteness Theorems in terms of existence of non-decidable theorems in PA which makes it a non-categorical theory:

> [...] a result by K. Gödel, according to which in any formal system or in any set theory that embraces ordinary arithmetic, non-decidable theorems can be formulated. If Σ is a non-decidable theorem, a model M for which Σ is true must exist as well as a model M' for which Σ is false, and then M and M' are certainly non-isomorphic, viz. no categoricity holds. [148]

[147] "On dit parfois qu'un champ est catégoriquement défini par certains axiomes, si deux modéles de ce champ, M et M' c'est-á-dire deux réalisations quelconques de ce champ, sont isomorphes par rapport á toutes les propriétés et relations dont il est question dans les axiomes, c'est-á-dire si M et M' peuvent être représentés univoquement l'un sur l'autre de telle façon que toutes ces propriétés et relations soient conservées dans la représentation." [skolem41, p.471, my translation]

[148] "[...] un résultat connu de K. Gödel, d'aprés lequel dans tout systéme formel ou dans toute théorie des ensembles qui embrasse l'arithmétique ordinaire, des théorémes peuvent être formulées qui ne sont pas décidables.
Si Σ est un théoréme non décidable, il doit alors exister un modéle M pour lequel Σ est vrai ainsi qu'un modéle M' pour lequel Σ est faux, et alors M et M' sont sûrement non isomorphes c'est-á-dire qu'il n'existe pas de catégoricité." [skolem41, pp. 471–472, my translation]

To sum up, Skolem shows that a first-order theory such as PA turns to be satisfied by at least one non-standard model other than the standard and intended one, and comes out as a non-categorical theory.

Three final considerations are in order. To begin with, we recall that Skolem claims that his proof is "purely constructive". But is that indeed true? The method Skolem uses to construct the function g and $\mathcal{N}^*$ is arguably constructive[149]. However, the whole proof heavily relies on an important assumption, viz. the fact that the arithmetical language is taken as denumerable. This implies that F, the set of arithmetical functions definable in the language, is also denumerable. If we dismiss this assumption, the proof is not able to construct a non-standard model anymore, as Smorynski notes:

> [Skolem's] method does not yield the existence of nonstandard models when, say, a continuum of predicates naming all sets of natural numbers is added to the language. For this latter, one must use one of the standard abstract existence theorems of logic. [smorynski84, p. 3]

Ultimately, Skolem's proof does not apply to the general case, but only to arithmetical theories whose language is fixed as denumerable. If it is necessary to appeal to the "abstract existence theorems of logic" such as Löwenheim-Skolem, then Skolem's requirement to seek for an all-constructive proof will fail.

After that, it must be observed that in the literature, Skolem's proof is well-known as to resemble the model-theoretic construction of "ultrapowers"[150]. In fact, as Skolem tries to give a direct proof of a non-standard model of arithmetic, he comes up with an ultrapower-like construction, as Fenstad claims: "The existence of non-standard models of arithmetic was discovered by Thoralf Skolem (1934). Skolem's method foreshadows the ultrapower construction [...]"[151].

Nevertheless, Skolem's proof can be seen only as an embryonic "ultrapower construction" that came to inspire the successive mathematical work.

The main divergence between Skolem's construction and the modern ultrapower lies in the following fact: an ultrapower can be defined once an ultrafilter is also given. To define in P the same non-standard model $\mathcal{N}^*$ via ultraproducts, we take C to be a non-principal ultrafilter which contains all the cofinal subsets N_i of N such that G, i.e. the range of the function g, is almost included in each N_i. In other words, N is the index set, C is the non-principal ultrafilter on N The equivalence relation $\equiv$ is defined as follows:

[149] It is certainly not constructive in the sense of intuitionistic.
[150] See [hodges93, p.450] for a definition of "ultrapower".
[151] [fenstad70, p. 14]. See also Kennedy in [kennedy08], Wang on [wang70, p. 42], Kossak on [kossak06, p. v]

$f_i \equiv f_j$ iff $\{n \in G | f_i(n) = f_j(n)\} \in C$. The quotient set $\mathcal{N}^N/C$ forms the ultrapower.[152] So we take for $\mathcal{N}^*$ the elementary substructure of the ultrapower consisting of all f/C such that f is definable in $\mathcal{N}$.

In the second place, Skolem's is reasoning has a quite different shape than modern ultrapower construction. As pointed out by Smorynski:

> It should be noted that, although Skolem's construction resembles the ultrapower construction so much that one feels like calling it such, an important element is missing. Skolem's use of the diagonalising g in place of the now-usual ultrafilter relies heavily on the countability of the arithmetic language; [smorynski84, p.3]

The use of g rather than an ultrafilter such as C represents the crucial difference with respect to the ultrapower construction.

From the historical side, Smorynski puts forth that the misguided reference to ultrapower is probably motivated by the presence, in the same proceedings of a conference, of the most common English version of Skolem's proof, [skolem55], together with the important Łos's paper on ultraproducts:

> [Skolem] republished the proof in English in Skolem 1955 in the proceedings of a conference – the same proceedings in which Łos published his theorem on ultraproducts. [...] it is this later paper to which most people refer. [...] perhaps it is the juxtaposition with Łos' paper that strikes one's fancy – for, Skolem's construction of a nonstandard model is something of an ultrapower construction. [smorynski84, p.3]

In that volume Łos introduced the notion of ultraproduct and implicitly gave the basic result about when a sentence holds in an ultraproduct. However, there is no reference in Łos' paper to Skolem's work. It is more likely that the presence of Skolem's and Łos's paper in the very same book fuelled the idea of the existence of a conceptual link between Skolem's reasoning and ultrapower construction. Still, this does not exclude that Skolem's work, which was published 20 years before Łos's paper, was indeed a *source of inspiration* for the model-theorists who actually developed the notion of ultrapower.

Finally, the last remark concerns a second impact that Skolem's proof had on the later developments on non-standard models of arithmetic. In 1961[153], MacDowell and Specker showed an important result about NMoA: any model $\mathcal{M}$ of the usual first-order Peano axioms PA has a *proper elementary extension* $\mathcal{M}^*$ in which "all new elements follow all old". In this case, Skolem's notion of "definability" presented earlier comes out as essential for

[152] $\mathcal{N}^N$ represents the Cartesian product $\prod_{i \in N} \mathcal{N}_i$.

[153] See [macdowell-specker61]

MacDowell's and Specker's argument. In fact their proof is related to Skolem's insofar as MacDowell and Specker construct $\mathcal{M}^*$ by extending $\mathcal{M}$: the desired model $\mathcal{M}^*$ is taken to be as the set of all elements definable using the elements of M (i.e. the domain of $\mathcal{M}$) as parameters, plus a certain single element of $M^* - M$. And to carry out the proof, MacDowell and Specker use a result which is more or less the formalization of Skolem's arithmetical lemma discussed above.[154]

3.6 Skolem's philosophy and the proof of the existence of NMoA

As we noted so far, the proof of the existence of non-standard models of arithmetic can be obtained in different ways. We have presented, on the one hand, the now-common one based on the compactness theorem and presented by Henkin in 1950, and on the other hand, the original proof exhibited by Skolem via definable arithmetical functions.

The investigation of the two proofs turns out not only historically but also philosophically relevant. In this respect, we claim that Skolem' attempt is encouraged by a well-established philosophical perspective. To understand the reason why his proof on NMoA was carried out in such "constructive" way, we shall turn to Skolem's philosophical views.

Roughly speaking, Skolem has a strong inclination towards *finitism* and *relativism*. Nevertheless, it would be quite difficult to illustrate his view by referring to some "-ism" or even tracing his view back to other philosophical schools. By contrast, for Skolem it would be more appropriate to speak of his finitism and relativism as *sui generis* if we compare them to the main streams in philosophy of mathematics.

To give an idea of Skolem's peculiarity, we will sketch an account of what "finitistic" and "relative" might have meant for Skolem.

First of all, it is particular noteworthy that Hao Wang describes Skolem as a "free spirit"[155]. Wang, in fact, authored the most comprehensive survey on Skolem's work in logic, which appears as the introduction to the most important collection of Skolem's logical papers, namely [skolem70]. Here Wang makes the acute observation that:

> [Skolem] did not belong to any school, he did not found any school of his own, he did not usually make heavy use of known results in more specialized developments, rather he was very much an innovator and most of his papers can be read and understood by people without much specialized knowledge. It seems quite likely that if he were young today, logic, in its more developed stage, would not have appealed to him. [wang70, pp.17–18]

[154] See [godel86, pp. 378–379]

[155] [wang70, p.17]

To get a real bite of Skolem's work it is necessary to consider it as independent of any logical school.[156] In particular, if we look closely at Skolem's methodology to approach problems, his peculiar way to treat and solve mathematical problems stands out, as Wang reports:

> Skolem has a tendency of treating general problems by concrete examples. Often *proofs seem to be presented in the same order as he came to discover them*. This results in a fresh informality as well as a certain inconclusiveness. Many of his papers give the impression of reports on work in progress. Yet his ideas are often pregnant and potentially capable of wide applications. [wang70, p.17, my emphasis]

As repeatedly noted, this way of presenting mathematical results by favouring the way one discovers results rather than the way arguments gain the most assent reminds us of Descartes' analytic method. Once again, we can draw a connection between this way of formulating arguments and foundational research in mathematics, as Skolem implicitly believed. So, Skolem seemed to develop his results and methodology quite independently from other research lines of his time.

In addition, Skolem's account comes out as a valuable one insofar as it constitutes a "coherent and fruitful philosophical viewpoint about the nature of mathematics and mathematical activity"[157]. Despite that, Skolem rarely attempted to provide an articulate presentation of his philosophical perspective. Testimonials for this view are the records of his lectures (one of the few occasions in which Skolem was most explicit on his ideas), scattered remarks in his mathematical essays, along with an important letter that Gödel addresses to Wang. Let us consider them in turn.

As seen earlier, Skolem's proof of the existence of NMoA does not rely on the Completeness result of first-order logic. Yet, the following fact seemed quite controversial for many historians: although by 1929 Skolem had amassed some important results that would have allowed him to prove the Completeness theorem for first-order logic before Gödel, Skolem never drew this conclusion. Wang, who was one of these historians, puts his puzzlement as follows:

> Since about 1950 I had been struck by the fact that all the pieces in Gödel's proof of the completeness of predicate logic had been available by 1929 in the work of Skolem [...], supplemented by a simple observation of Herbrand's [...]. In my draft I explained this fact and said that Gödel had discovered the theorem independently and given it an attractive treatment. [wang96, p.122]

[156]See the influence of the algebraic tradition on Skolem's formation in [burris09] and his isolated condition from the research community for a long period of his life in [fenstad70, pp. 9–16]

[157][wang96, p.119]

However, Gödel himself casts some doubts on Wang's remark. In a letter on 7 December 1967 to Wang, Gödel clarifies both the relation between Skolem's and his own work on the completeness of predicate logic and his own views on the relation between philosophy and the study of logic:

> [...] It seems to me that, in some points, you don't represent matters quite correctly. [...] You say, in effect, that the completeness theorem is attributed to me only because of my attractive treatment. Perhaps it looks this way, if the situation is viewed from the present state of logic by a superficial observer. The completeness theorem, mathematically, is indeed an almost trivial consequence of Skolem 1922. However, the fact is that, at that time, nobody (including Skolem himself) drew this conclusion (neither from Skolem 1922 nor, as I did, from similar considerations of his own). This blindness (or prejudice, or whatever you may call it) of logicians is indeed surprising. But I think the explanation is not hard to find. *It lies in a widespread lack, at that time, of the required epistemological attitude toward metamathematics and toward nonfinitary reasoning*. [wang96, p.122, my emphasis]

Gödel points out that what made his treatment revolutionary for that time was his appeal to metamathematics and nonfinitary reasoning. By contrast, Skolem does not accept nonfinitary reasoning. The reason is unlikely to be Skolem's lack of familiarity with these methods. In fact, Skolem sticks to a radical finitary position also after Gödel's result. But what do Skolem's finitary methods exactly mean? To answer this question, Fenstad for example brings into consideration one of Skolem's papers in [fenstad70]:

> His [Skolem's] own preferences are better represented by the important 1923 paper. Here Skolem tries to build up elementary arithmetic without applying the unrestricted quantifiers "for all" and "there exists" to the infinite completed totality of natural numbers. Historically this is perhaps "a first" paper in the theory of recursive arithmetic. [fenstad70, p.12]

Furthermore, during one of his lectures, Skolem explicitly refers to his finitism, as Fenstad continues:

> But, whereas recursion theory has witnessed a rich development, the *strictly finitistic construction of mathematics* advocated by Skolem has aroused less interest. In a lecture to the 1950 International Congress of Mathematicians he said that he had hoped that "the very natural feature of my considerations would convince people that *this finitistic treatment of mathematics was not only a possible one but the true or correct one*". For an ordinary mathematician this is a rather extreme position, so Skolem hastens to add: "Now I will not be

> misunderstood, I am no fanatic". To this, everyone who knew Skolem would surely agree. [ibid., my emphasis]

Skolem believed that the only safe way to talk about mathematics is to stick to what we can construct in finite steps. Unrestricted quantifiers, for example, go beyond what we can actually enumerate. By maintaining a finitistic position, we can safely make mathematical claims and dismiss unrestricted considerations which could turn out to be only pseudo-mathematical claims.

Despite this radical view on finitistic treatment of mathematics, Skolem was likely to take into account logical completeness, although informally:

> Skolem did not think of the theorems of elementary logic as given in a formal system and, therefore, that the question of full completeness had no meaning for Skolem. [...] In any case, it is clear that Skolem had little interest in the formalization of logic, so that, as Gödel suggested, Skolem implicitly proved an informal version of the completeness theorem in Skolem 1923. [wang96, p.124]

Among other things, we should not neglect the fact that Skolem did actually employ non-finitary reasoning in his papers, for example in his early proof of Löwenheim's theorem.[158]. Still, in these cases Skolem strove to offer other proofs, in accordance with his finitary methods, as a refinement and improvement of the previous ones.

Despite being extreme and oftentimes disputable, Skolem's position has remained a coherent view with respect to his mathematical practice, as Gödel acknowledges:

> Skolem's epistemological views were, in some sense, diametrically opposed to my own. [...], evidently because of the transfinite character of the completeness question, he tried to *eliminate* it, instead of answering it, using to this end a new definition of logical consequence, whose idea exactly was to *avoid* the concept of mathematical truth. Moreover, he was a firm believer in set theoretical relativism and in the sterility of transfinite reasoning for finitary questions [...]. [wang96, p.124, original emphasis]

It comes out at once that Skolem's view focuses more on the limits and boundaries in mathematics than on thrusting the subjects into non-constructive waters. Skolem proved the so-called Löwenheim-Skolem theorem in 1920. According to his view, this theorem has a deep impact on foundational research in set theory. Even if we naturally have in mind an uncountable model, when we work with the first-order set-theoretical axiomatization, everything that we prove for this uncountable model is also true for a countable model.

[158]See [wang96, p.124]

So "relativity of set theoretic notions" has to be understood as the phenomenon of the impossibility for the first order axioms of set theory to uniquely capture the intended uncountable model. First-order axioms cannot say which of the two kinds of models (countable or uncountable) we are dealing with. Statements assume the same meaning independently of whether the domain on which are interpreted is countable or not.

One last consideration is in order. Notably, besides the Löwenheim-Skolem theorem, now-called Downward Löwenheim-Skolem theorem, there is an analogous Upward version: "any theory formulated in a first-order language of cardinality α which has an infinite model has a model of every cardinality greater than or equal to α." [159]

Despite the name, the Upward Löwenheim-Skolem has nothing to do with either Löwenheim or Skolem since the complete form of the theorem is only due to Tarski. However, the name is motivated by content-based reasons rather than historical ones: both theorems exhibit some properties of first-order theories with respect to the cardinality of their models.

Moreover, it is not difficult to see that if we apply the Upward Löwenheim-Skolem to Peano Arithmetic, we obtain immediate evidence of the existence of NMoA. In fact, the theorem guarantees that some models of greater cardinality than (and thus non-isomorphic to) the standard model of arithmetic exist for PA.

However, even more than the Compactness theorem, Skolem takes a firm position against the Upward Löwenheim-Skolem theorem, as Hodges points out

> I follow custom in calling Corollary 6.1.4 the upward Löwenheim-Skolem theorem. But in fact Skolem didn't even believe it, because he didn't believe in the existence of uncountable sets. [hodges93, p.267].
>
> The upward Löwenheim-Skolem theorem [...] is from Tarski & Vaught [1957][160]. [...] Skolem [...] rejected the result as meaningless; Tarski [...] very reasonably responded that Skolem's formalist viewpoint ought to reckon the downward Löwenheim-Skolem theorem meaningless just like the upward. [hodges93, pp. 318–319]

In the end, Skolem's rejection of Tarski's result corroborates what we have been arguing so far. It comes as no surprise that Skolem, who has maintained a coherent and constant position for all his life about *infinitary methods*, was likely to be annoyed by the fact that the Upward version given his name:

> Legend has it that Thoralf Skolem, up until the end of his life, was scandalized by the association of his name to a result of this type, which he con-

[159] [read97, p.83]
[160] Ed. note: [tarski57]

sidered an absurdity, nondenumerable sets being, for him, fictions without real existence. [poizat00, pp. 53–54].

3.7 Descriptive vs deductive use: 1st vs 2nd order logic

So far, we have discussed Dedekind's and Skolem's account on non-standard models of arithmetic. Dedekind has been presented as the pioneer of a foundational study in arithmetic which pursues categoricity and thus aims at ruling out non-standard models. On the other hand, we considered Skolem's mature result which acknowledges the existence for non-standard models of a first-order theory such as Peano Arithmetic.

We should note that the two positions differ with respect to for the logical framework adopted as well as their foundational purposes. In fact, whereas Skolem' result applies to Peano Arithmetic which is stated in first-order logic, Dedekind's theory DT can be formulated only in a more powerful language: second-order logic. The choice of logic for the arithmetical theories is far from being innocuous: the two logics embody different meta-properties that make them suitable for alternative foundational purposes. In fact, first order arithmetic allows for NMoA, but second-order arithmetic does not. Let us elaborate on these two points.

3.7.1 Two logical frameworks

A detailed discussion on which logic should be preferred for foundational purposes in mathematics goes beyond the purposes of this work. It suffices to say that this debate has become an important stream of research only recently.[161] This issue is not a new one, though. In the history of logic, mathematicians have favoured different positions regarding this point: fathers of modern logic such as Frege and Dedekind dealt with a second order language while from Gödel's first important results onward, first order languages took progressively the place that second order logic had in foundational studies. Indeed, Gödel was one of the "strongest (and probably the most influential) proponents of first-order languages"[162]. The issue, which has forcefully returned to the ongoing philosophical debate, is crucial for our purposes. In fact, we have stressed that the expressive power of the formal language does have consequences for the occurrence of non-standard models.

What we want to claim here is that the choice of which logic to employ should not be based on the different meta-properties that each logic has *per se*, but rather on the foundational purpose in mathematics that we want to pursue.

[161] Commonly, [shapiro91] represents the landmark in this debate.
[162] [shapiro85, p.715]

Meta-properties of first and second order logic are recapitulated in the table 3.1.

	1st order logic	2nd order logic
Sound	x	x
Complete (or Exhaustive)	x	–
Compact	x	–
Downward Löwenheim-Skolem	x	–
Upward Löwenheim-Skolem	x	–

Table 3.1: Meta-properties: 1st order versus 2nd order logic

Except for soundness (i.e. the basic requirement that inference rules are chosen to preserve truth), second-order logic does not present any of the beautiful meta-properties that first order-logic does. Nonetheless, what we are interested in here is not the logic as such, but rather the formal theories, formulated in a certain logic that are capable of capturing the features of a given mathematical structure.

As soon as we compare the first-order theory Peano Arithmetic with the formal version of Dedekind Theory, say DT*, formulated in second-order language, we move on to an account between arithmetical theories:

	PA	DT*
Syntactically complete	–	–
Semantically complete	–	x
Categorical	–	x
Decidable	–	–

Table 3.2: Meta-properties: PA versus DT*

By recalling what we said in section 3.3, by Gödel's First Incompleteness Theorem we know that PA cannot be decidable and thus syntactically complete. By completeness of first-order logic, we also know that PA is not semantically complete and by corollary 2.5.0.10 it is not categorical either. Conversely, DT* has been proved categorical by Dedekind and therefore is also semantically complete. Still, since the incompleteness theorem applies to PA and its extensions, DT* as well as PA are neither decidable nor syntactically complete.

Now, if we recall Hintikka's distinction between uses of logic, we can easily see that the choice of arithmetical theory (and accordingly of the underlying logic) may be based on the aim we want to achieve, namely either deductive or descriptive:

In short, the Deductive task pursues the syntactic completeness of a theory, i.e. to state

Task of logic	Descriptive	Deductive
Logical system	2nd order logic	1st order logic
Aim	Categoricity	Syntactic completeness

Table 3.3: Logical tasks in mathematics

a theory which is capable, for every formula in its language, to say whether the formula is provable or refutable (according to a given notion of deductive consequence). Although first-order logic is semi-decidable, i.e. if a statement is true, then the logic will eventually provide a proof for it, neither PA nor DT* are syntactically complete as a consequence of the First Incompleteness Theorem.

On the other hand, the Descriptive task of logic aims at categoricity, viz. to state a theory which fully captures the mathematical structure at hand. This is accomplished by ruling out the possibility for that theory to be satisfied by any non-standard model. To do so, the logic underlying the theory should not be chosen to be the first-order one. Let us discuss this point in detail.

In contrast to the deductive use, the descriptive goal can indeed be established notwithstanding the First Incompleteness Theorem. In fact, Gödel's Incompleteness result shows the impossibility, under certain conditions, to formulate a syntactically complete theory for arithmetic. Even though the theorem applies to both PA and DT*, it does not follow that it is not possible to achieve any completeness whatsoever.

The syntactic completeness of a theory rests on two basic assumptions: the "exhaustiveness" of the fragment of the underlying logic, and the specification of the model of the theory for which we effectively check the logical theorems. Now, if we put aside the first ingredient and we adopt a second-order logic, the *syntactic* incompleteness of DT* does not imply the *semantic* incompleteness of DT*. Indeed, DT* is categorical as Dedekind showed, and therefore it is *semantically complete*.

To sum up, Gödel's result did imply that the deductive task of logic in mathematics as defined earlier cannot be carried out. However, the same does not hold for the descriptive task, but in the end we can reach descriptive completeness, namely categoricity, only by sacrificing deductive completeness. That is, we can have either kind of completeness, but not both.

3.7.2 Dedekind, Skolem and their foundational goals

Diagrams 3.1 and 3.2 illustrate what we have been discussing so far, that is the interplay between the choice of the logic and the existence of non-standard models according to Dedekind's and Skolem's results.

Dedekind favours a theory for arithmetic that can be formulated in second order logic,

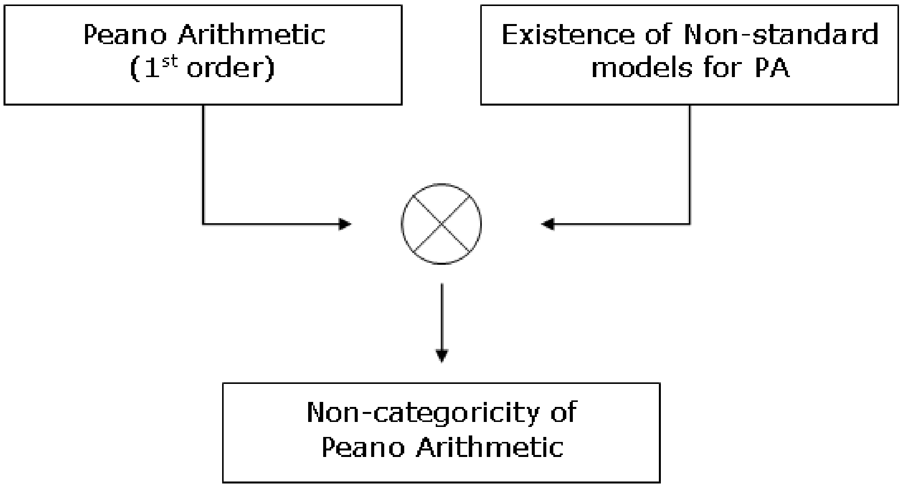

Figure 3.1: Peano Arithmetic and non-standard models

while Skolem studies the limits of first order arithmetical theories.

However, we should probably not conclude that from Dedekind's NMN and letter to Keferstein to Skolem's proof of the existence of NMoA a switch in logical framework has occurred. Indeed, we claim that Dedekind and Skolem agree on many points, including the foundational aim. As noted earlier, if both want to tackle the descriptive task, then they will have the same opinion about the underlying logic, viz. that first-order logic is not suitable for the task.

In this respect, it is particularly noteworthy that Skolem himself hints at a distinction, fairly similar to Hintikka's deductive and descriptive distinction, between "completely formalized mathematics" and "ordinary mathematical practice":

> If one works within a *completely formalized mathematics*, based on a finite number of precisely stated axioms, there is nothing to discuss but questions of consistency[163] and the ease of manipulation. But in *ordinary mathematical practice* [...] which are never given by a set of specified rules [...]. [skolem70, p. 14, my emphasis]

Whereas completely formalized mathematics is based on the study of structures satisfying the axioms, ordinary mathematical practice corresponds to the study of one particular model, namely the intended one. Along these lines, the distinction brings us back to Hintikka's two-fold use of logic.

The descriptive use takes into consideration the ordinary mathematical practice: as Dedekind showed, if we want to provide an axiomatization for natural numbers, we

[163]Note that consistency for Skolem is a semantic notion, not syntactic.

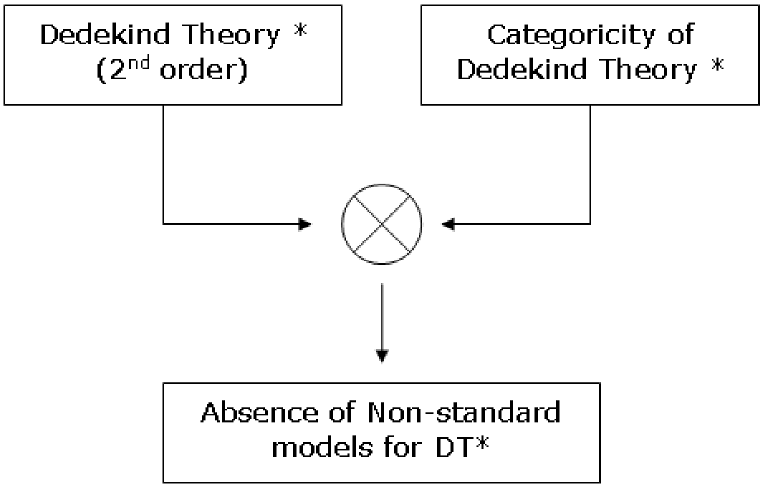

Figure 3.2: Dedekind Theory and non-standard models

have to study their particular structure denoted as the "intended model". Thus, this model constitutes the subject matter of the descriptive inquiry in the sense that we aim at capturing this model by means of the descriptive use of logic.

On the other hand, if we consider some axioms rather than the intended model, our research focuses on the class of models, if any, that come to be true for those axioms. So the deductive use of logic enables us to study whether, for derived theorems, there is a model satisfying them (viz. the notion of consistency in Skolem's sense).

Now, Skolem has ideas as clear as Hintikka does on the nature of foundational research in mathematics. In particular, he puts forward that the existence of NMoA represents a limitative result for what concerns "completely formalized mathematics". The theory PA, as well as set theory in its first-order axiomatization, has a severe weakness: it only captures, to use Skolem's terminology, in a *relative* rather than absolute way the structure of the natural numbers. In other words, we cannot be aware that we are dealing with the standard models of arithmetic, i.e. the intended model of the ordinary practice in number theory, unless we stipulate it. As a consequence, Skolem is unlikely to stick to first-order logic insofar as we deal with ordinary mathematical practice. Can we infer that Skolem is indeed suggesting that ordinary mathematical practice calls for a more powerful logic insofar as logic is seen as a helpmeet to attempted solutions? As we argued in chapter 2, Dedekind favours the descriptive task for arithmetic. His overall aim is to render the structure of natural numbers completely up to isomorphism. Is it plausible to ascribe the same view to Skolem?

In the record of the lecture that Skolem gave in Zürich in 1938[164], we can find evidence that Skolem favoured the descriptive task. At the end, a discussion between Bernays and Skolem is reported. Bernays observes that results like the "Skolem paradox" were to be understood as a limitation of the use of formal systems to capture our intuitive mathematical concepts of set and number. Skolem's reply is reported as follows:

> Mr. Skolem thinks that one does not have to view the situation from such an angle. In his view, the best way is to refer in each domain of research to an appropriate formalism. This manner of proceeding does not imply any restriction on the possibilities of reasoning, for one has always the liberty of passing to a more extended formalism [wang96, pp. 127–128][165]

By suggesting that for each domain of research we should refer to an appropriate formalism, Skolem makes clear his preference to a more extended formalism if some restrictions on the reasoning occur. His claim makes perfect sense with the "ordinary mathematical practice" which is always seeking for appropriate formalisms.

For this reason, speaking of a Skolem approach in contrast to a Dedekind one as we did at the end of chapter 2 may be quite misleading. It would be rather more appropriate to talk about complementary results: whereas Dedekind discusses the case in which the arithmetical theory turns out categorical, Skolem deals with the weaknesses of first-order theories of arithmetic.

To conclude this chapter, we sum up briefly the similarities between Skolem's and Dedekind's position that we have encountered throughout our analysis:

- the existence of NMoA constitutes a limitative result: Dedekind comes up with them as an evidence of the inadequacy of the non-categorical axiomatization; Skolem adopts them to emphasise the "relativity" (i.e. the impossibility to capture uniquely some features) of PA in characterising arithmetical concepts;

- the existence of NMoA is conceived as a philosophical result: neither Dedekind nor Skolem are interested in studying non-standard models for themselves. These models are rather viewed as a boundary to foundational research. As soon as the obstacle is removed and NMoA are weeded out, foundational research can continue;

- both Dedekind and Skolem pursue a descriptive task of logic in mathematics, rather than a deductive one. One of the first items of their philosophical agenda is the attempt to characterise mathematical structures by means of logical tools.

[164][skolem41]
[165]This is Wang's translation of [skolem41, p.480]

Chapter 4

The mathematical phase and the attempt of a philosophical revival

In this last chapter we draw some final considerations about the status of the field of non-standard models after Skolem's result. Above all, we will settle some exegetical and historical issues related with the terminology ″standard/non-standard″; we will sketch out the main features of what we call the *mathematical phase* of non-standard models, and briefly discuss two important mathematical results concerning NMoA; finally, we will present how one of these mathematical results, viz. Tennenbaum's Theorem, turns out to be relevant to the present-day debates in philosophy of mathematics.

4.1 An exegetical and historical clarification

In chapter 1 we brought up the issue of how non-standard models of arithmetic can be characterised. The way we approached it led us to investigate the ″descriptive use″ of logic in mathematics, and thus to focus on the relevance of an ″intended /unintended″ distinction for models. After that, we stipulated that, for what we are concerned with, the notion of non-standard model amounts to the one of unintended model.

Here we suggest that, as we understand and use it today, the two wordings ″unintended″ and ″non-standard″ do not quite have the same meaning.

Let us discuss the etymology of the phrase ″non-standard″, and then attempt to figure out to what extent ″non-standard″ differs from ″unintended″.

4.1.1 The origin of the phrase ″non-standard model″

The dichotomy *standard/non-standard* came to be applied to models as late as 1950. Before then, non-standard models remained unnamed entities. This comes as no surprise if we recall the ″pathological″ and ″limitative″ role that they have in Dedekind's and Skolem's work.

As we argued earlier, Dedekind considered non-standard models insofar as he had to provide a counterexample to Keferstein's position. Brief as the treatment was, Dedekind did not have any particular reason to give to non-standard models a specific name.[166]

Despite offering a more attentive and detailed study, Skolem was not driven by the need to name non-standard models either. This was perhaps due to his "analytic style"[167], in fact Skolem has rarely been fond of creating versatile names for the new concepts he was progressively developing in his papers.

On top of that, Skolem seemed quite reluctant to use words whose meaning had not fully been established previously, especially as far as the then-new logical terminology was concerned. We note for example that expressions such as "non-standard model" or "definable" do not occur neither in [skolem33], nor [skolem34]. In this respect, Robert Vaught points out: "in these papers[168] Skolem does not use the word 'definable' and replaces it in each paper with a different discussion over a page long!"[169]. Conversely, Skolem favoured long periphrasis and *ab ovo* presentations instead of defining a new terminology and settling the new concepts once and for all.

To go back to the phrase "non-standard model", Skolem simply designed NMoA as $\mathcal{N}^*$, viz. the proper extension of the standard model $\mathcal{N}$, rather than denoting them as an independent class of models.[170]

The *standard/non-standard* terminology was adopted for the first time by Henkin in his Ph.D. dissertation [henkin47]. An extract of his dissertation was published in 1950 in The Journal of Symbolic Logic with the title "Completeness in the Theory of Types"[171]. This paper was a landmark in the history of logic for its influence in later developments.[172]

In the paper Henkin exploits the distinction standard and non-standard in a two-fold way: on the one hand, he applies it to *models* to denote standard and non-standard arithmetical models. In particular, he proves the existence of a non-standard model of arithmetic by using the Compactness Theorem.[173] On the other hand, the paper is famous for providing a different *interpretation* of second and higher-order quantifiers. Commonly, second or higher-order quantifiers range over arbitrary subsets of the domain. For this case Henkin speaks of interpretations with the "standard meaning", or validity in a "standard sense". Conversely, the "non-standard" case occurs when quantifiers are chosen to range over a fixed set of subsets of the domain. Henkin discusses this "non-standard" interpretation in detail and puts forward that second-order logic provided with this "non-

[166]See page 40.
[167]E.g., see Wang's passage on page 81.
[168]That is [skolem33] and [skolem34]
[169][godel86, p. 376]
[170]E.g., see the claim of the theorem 3.5.2.
[171]Ed. note: [henkin50]
[172]E.g. see [vaananen01] and [odifreddi03, p. 274]
[173]See section 3.3 above.

standard semantics" has the same meta-properties as first-order logic. [174]

The bivalent use of the adjectives "standard" and "non-standard" has originated confusion. In fact, Henkin did not fully establish a neat and clear formulation. For example, we can observe that the definition of *standard model*, that Henkin puts as follows:

> By a *standard model*, we mean a family of domains, one for each type-symbol, as follows: D_ι is an arbitrary set of elements called *individuals*, D_o is the set consisting of two truth values, T and F, and $D_{\alpha\beta}$ is the set of all functions defined over D_β with values in D_α. [henkin50, p. 83]

should rather represent the definition of "standard interpretation" according to what we said earlier. Conversely, in the following passage "non-standard" refers correctly to non-standard models of arithmetic:

> By appealing to Theorem 3, however, it becomes a simple matter to construct a model containing a *non-standard number system* which will satisfy all of the Peano postulates as well as any preassigned set of further axioms [henkin50, p. 90, my emphasis]

Ultimately, depending on which domain it comes to be applied, the "standard/non-standard" dichotomy assumes two different meanings.

In 1989, Jakko Hintikka returned to the issue of Henkin's terminology in the following terms:

> [...] it is not clear what, if anything, connects Henkin's notion of standard interpretation of higher-order logics with the idea that certain first-order mathematical theories have designated or "standard" models. In is not clear what two notions of standardness are supposed to have in common, except the term. [hintikka89, p. 78]

Once we have well-distinguished between the two uses, viz. we have acknowledged that "standard/non-standard" distinction may concern either models or quantifier semantics, it is reasonable to address the issue of what link, if any, lies indeed between the non-standard interpretation of second-order quantifiers and non-standard models of arithmetic.[175]

Still, Hintikka's passage seems to fuel a philosophical controversy. In fact, there is no reason to assume that the two uses of the "standard/non-standard" terminology ought to have something in common. Once again, the former use concerns the model in itself, i.e. the mathematical structure independently of formal systems; the latter regards

[174] See footnote 17.

[175] Today it is difficult to find the expression "non-standard semantics. Commonly, mathematicians talk about Henkin semantics instead of non-standard semantics.

the semantic relation defined in the logic that rules the interplay between logical statements and a given model. Thus, Hintikka's question can be reasonably seen as a pseudo-issue.[176]

To go back to the etymological issue, we have introduced Leon Henkin as the first who adopted the standard/ non-standard terminology. However, only in the 1950 paper "Non-Standard Models for Formal Logics"[177] Wang and Rosser dwell at length on the notion of *non-standard model*.

Indeed, at the 8th Congress of Polish mathematicians in September 1953 Andrzej Mostowski did credit Wang and Rosser (rather than Henkin) with the terminology non-standard model of arithmetic, as Richard Kaye reports:

> What kind of structure have the models of Peano's arithmetic differing from a model composed of natural numbers; in particular, what is their ordinal [i.e., order] type like? *After Rosser and Wang [23]*[178] *we term such models non-standard*. [kaye06, p. 2, my emphasis]

Unlike Henkin, Wang and Rosser focus specifically on the study of non-standard models, so that the scope of the dichotomy "standard/non-standard" is presented clearly enough.

Still, the opening lines of Wang-Rosser's paper may sound quite ambiguous insofar as they seem to lack the rigour that a mathematical presentation usually requires:

> For the purposes of the present paper, *we do not need a precise definition of what is meant by a standard model of a formal logic*. The non-standard models which we shall discuss will be *flagrantly non-standard*, as for instance a model of the sort whose existence is proved by Henkin. [wang-rosser50, p. 113, my emphasis]

The passage that should represent a conceptual clarification seems to leave the question open rather than attempting to delve into it. When the following year Skolem himself reviewed Wang-Rosser's paper, he openly pointed out how vague and approximate the standard/non-standard definition was:

> The non-standard models for formal logics which are considered here are non-standard in the sense that either the equality relation in the model is not

[176]Arguably, we can speak of connection in the following terms: if you have PA (or a weaker theory) + second order quantifiers + induction, then if you allow only standard interpretation of quantifiers you get only the standard model. Conversely, if you allow non-standard interpretation of quantifiers, then you get non-standard models.

[177]Ed. note: [wang-rosser50]

[178]Ed. note: [wang-rosser50]

> the equality relation in the logic or the part of the model which represents the positive integers is not well-ordered or the part of the model representing the ordinals of the logic is not well-ordered. *The authors state that a precise definition of the notion "standard model" is not necessary for the understanding of this paper. The reviewer would nevertheless have appreciated such a definition.* Indeed there are many statements about standard models. [skolem51, p. 145, my emphasis]

By simply reporting on Wang-Rosser's definition, Skolem has the strong feeling that a more accurate definition ought to be established. Once we want to use a handy dichotomy as the standard/non-standard one, the scope of the two terms must be sufficiently neat in order to avoid ambiguities.

To sum up, in the beginning of 1950s a rigorous definition of "non-standard model" is still debated. Yet, Wang-Rosser's paper results in a further step toward the disclosure of the extent of non-standard models.

In fact, despite his prima facie sceptical attitude, Skolem comes to be convinced in the course of time by the "standard/non-standard" terminology. In his 1955 paper [skolem55], Skolem uses for the first time the expression "non-standard models" as he discusses the proof of the existence of NMoA:

> I would like to add some remarks on the setting up of models of certain fragments of number theory, [...]. In these simple cases *the definition of non-standard models* can often be established in a perfectly constructive way. [skolem55, p. 9]

This is the only passage where we find the phrase "non-standard models" in the whole Skolem's written work devoted to the topic of the existence of NMoA. However, even in this particular case we should bear in mind that Skolem is still uneasy about the standard/non-standard dichotomy. In effect, Skolem speaks of "non-standard models" only once in the whole paper, namely in the fourth and last section of the text entitled "Some more Special Results", in confirmation of the minor role that the term has in the presentation of the proof.

To sum up, Henkin's and Wang-Rosser's paper played a crucial role for the favourable reception of "standard/non-standard" terminology among the mathematical community, including Skolem.

4.1.2 Does non-standard amount to unintended?

As discussed above, the standard/non-standard distinction came to be applied both to models and to the interpretation of second-order quantifiers. Still, we have not mentioned which features lie behind a standard and non-standard model or interpretation.

In other words, we want to elaborate more on the reason that would lead us to call a model or an interpretation ”non-standard” rather than ”standard”.

For instance, the standard interpretation of second-order quantifiers does not seem to be historically motivated, as Hintikka suggests

> A terminological point is in order here. By calling one of the two contrasting *interpretations* ”standard”, we are not passing any judgement as to which view ought to be adopted or which one is historically the usual one. [hintikka92, p. 147, my emphasis]

Hintikka points out that the standard semantics of second-order logic relies on the notion of arbitrary set and arbitrary function that in the past was far from being the standard approach in mathematics.

> The idea of standard interpretation is virtually identical with the idea of a completely arbitrary function. Hence the gradual development of what in effect was the notion of standard interpretation can be partially followed by tracing the history of the notion of a (completely) arbitrary function. [hintikka95, p. 110]

Conversely, as far as standard and non-standard models are concerned, Hintikka does not go further than saying that with respect to arithmetic the ”standard model” is nothing but the intended model:

> [...] it is far from clear how Henkin’s notion of standard interpretation or standard model is related to logician’s idea of the standard model of such first-order theories as elementary arithmetic, in which usage ”standard model” means simply ”intended model”. [hintikka95, p. 106]

That, Along Hintikka’s lines, NMoA amount to the unintended models of arithmetic follows at once. Although in the previous chapters we agreed with Hintikka and we adopted interchangeably intended and standard as well as unintended and non-standard, here we want to say something more about the interplay between these two dichotomies.

In this respect, Wang has once again a leading role in the way non-standard models came to be developed in philosophy. In his 1957 paper, Wang drew for the first time the important connection between non-standard and unintended models.[179]

To some extent, Wang puts himself as a spokesperson of the *philosophical phase* of non-standard models as we described previously: he presents the existence of non-standard models as a consequence of the failed attempt to capture completely the intended model.

[179]See page 46.

Along these lines, to talk about intended and unintended models is natural insofar as we aim at describing a certain mathematical structure. To put this in another way, when a theory is formulated in order to capture such a structure, the model (or models) exemplifying this structure is called the intended or standard model (or intended or standard models). The others are called unintended or non-standard models. The descriptive incompleteness of a theory amounts to the fact that the theory does not rule out all non-standard models.

However, if the descriptive task of logic is put aside, the expressions intended and unintended will come out as vacuous.

In fact, as soon as we decide on the structure to describe, we are able to speak of the intended model (viz. the model that we stipulated as the one our study is all about), and the unintended models (viz. models that represent the outcome of a description "not yet complete"). The basis of the intended-unintended terminology is the ability to single out a model as privileged with respect to the others. Outside the descriptive task of logic, we have no need in picking out a certain model rather than another.

On the other hand, the "standard / non-standard" terminology can be seen as mathematically nuanced. Along these lines, non-standard models represent entities independently of the descriptive task, so that we can decide to study non-standard models as models in themselves. In accordance with today's mathematical practice, we say that the standard model of arithmetic has order type ω while a denumerable non-standard model has a completely different order type than the standard one.[180] In this case we have no privileged model, but we catalogue and name the different classes of models in virtue of their features.

In the previous chapters we have mainly dealt with an exclusively descriptive framework. Thus, we assumed for convenience that non-standard (or standard) models amount to unintended (or intended) ones.

Here, according to what we have just suggested, we can conclude that the two dichotomies can also be viewed as pertaining to two distinct dimensions: one ontological and one epistemic. On the one hand, the terms "standard" and "non-standard" can be taken to range over classes of arithmetical models as considered in virtue of their properties. In this sense this distinction is viewed as ontological since we denote the models *per se* in order to study how they actually are. In this case the name "standard" is due to historical reasons, namely the class that came to be studied prior to the non-standard ones.

On the other hand, the "intended-unintended" distinction is epistemic as long as we consider the way we come to know the arithmetical models. In other words, we assume that we have access to a privileged class of models, called intended, and insofar as we describe it as completely as possible, we may come to know some other kind of models,

[180] See section 4.2 below.

called unintended, that undermine our knowledge of the intended or privileged structure.

4.2 The mathematical phase

Both Dedekind's and Skolem's accounts of non-standard models were characterised by the following: the attitude to non-standard models of arithmetic as shortcomings of the first-order axiomatization. Since their attempts were philosophically oriented, we called that period the "philosophical phase" of non-standard models. Although the existence of non-standard models was considered as an embarrassing result from a philosophical point of view, in 1950 Leon Henkin showed his interest toward these models in their own by studying their order type.

Henkin's paper paved the way for the mathematical results on non-standard models culminating in the 1960s in the first important results : the Tennenbaum and MacDowell-Specker Theorems. With Henkin we witness a radical change with respect to the consideration of non-standard models of arithmetic. So we call this period "the mathematical phase of NMoA".

4.2.1 From the philosophical phase to the mathematical one

In the previous chapters, we tried to give an account of non-standard models of arithmetic through Dedekind's and Skolem's work. Although they did not have a great deal to say about non-standard models themselves, we noted that both Dedekind and Skolem rendered them as an end extension of the standard model.[181]

The 1950s have witnessed a shift in the way of conceiving non-standard models of arithmetic. Not only does Henkin provide a new terminology as we discussed, but his paper can also be considered as the landmark for the study of non-standard models, as Abraham Robinson and Craig Smorynski put forth forcefully:

> Skolem was interested only in showing that no axiomatic system specified in a formal language [...] can characterize the natural numbers categorically; and he did not concern himself further with the properties of the structures whose existence he had established. In due course these and similar structures became known as *nonstandard models of arithmetic* [...] [robinson67, p.818]

> Skolem's goal in constructing nonstandard models was philosophical: He aimed to shew that first-order logic could not characterise the number series; he did not care to start a new subject. Until the 1960s, this was generally

[181] Yet, we have to be careful to use the word *extension* in this context: while Dedekind described non-standard models as simply supersets, Skolem actually shows that there exists a non-standard model which is a proper extension of the standard one.

> the case– nonstandard models of arithmetic were either objects of philosophical interest or tools, not objects of mathematical interest in their own right. [smorynski84, p. 3]

While Dedekind and Skolem consider non-standard models as undesired shortcomings, Henkin paves the way for acknowledging full dignity and interest to these entities.

Once we know that non-standard models exist, it is natural to ask how different they are from the standard one. With Skolem's result, we can only say that non-standard models of arithmetic are proper extensions of the standard one, as depicted in diagram 4.1 [182].

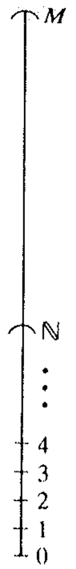

Figure 4.1: A non-standard model of arithmetic

Nevertheless, the mathematical phase is indeed characterised by the interest in investigating such non-standard models into detail. In particular, the study of the order type of the models tries to give an answer to the question to which extent NMoA are different from the standard ones.

4.2.2 Henkin and the order type of NMoA

Henkin is interested in providing the "size" of non-standard models in terms of "order types", as Smorynski continues:

> The major counterexample to this [i.e. non-standard models as either objects of philosophical interest or tools] was an observation made by Leon

[182] The diagram is taken form [kaye91, p. 12]

> Henkin in his paper Henkin 1950 on the Completeness Theorem for Type Theory. He announced the order type of a nonstandard model of arithmetic to be $\omega + (\omega * + \omega)\eta$, where η is a dense linear order. [smorynski84, p. 3]

Indeed, in this respect Henkin considers only the class of denumerable non-standard models:

> It, therefore, becomes of practical interest to number-theorists to study the structure of such models [i.e. non-standard models]. A detailed investigation of these numerical structures is beyond the scope of the present paper. As an example, however, we quote one simple result: Every non-standard *denumerable* model for the Peano axioms has the order type $\omega + (\omega^* + \omega)\eta$ where η is the type of the rationals. [henkin50, p. 91][183]

Henkin restricts himself to the class of denumerable non-standard models. This is due to the complex picture obtained by a closer scrutiny of the class of non-standard models. In fact, even a quick glance at a non-standard model reveals a very rich structure.

Even though the diversity among non-standard models is so vast that no coherent picture in terms of a relative classification can be painted, we can consider some well defined as well as more tameable subclasses of non-standard models.

Going back to the order type, Henkin claims that any countable non-standard model of arithmetic has order type $\omega + (\omega^* + \omega) \cdot \eta$. Recall that ω is the order type of the standard natural numbers, ω^* is the dual order (an infinite decreasing sequence) and η is the order type of the rational numbers. In diagram 4.2[184] we have a picture of how these denumerable non-standard models look.

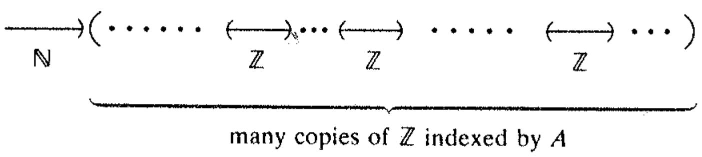

Figure 4.2: The order type $\mathbb{N} + \mathbb{Z} \cdot A$

To sum up, [henkin50] can be considered as a crucial paper to the radical change of the role of non-standard models. Henkin was indeed interested in studying those models by providing, for example, their "size" in terms of "order types". Thus, he paved the way for the study of non-standard models as a mathematical interest in its own right.

[183] Also Robinson recognises Henkin as the one who provided the order type of the non-standard models. See [robinson96, p. 88]

[184] The diagram is taken form [kaye91, p. 74]. Note that in the diagram $\omega^* + \omega$ is replaced for convenience with the order type of the integers, denoted by $\mathbb{Z}$.

4.2.3 Two mathematical results

As repeatedly mentioned, the second half of the XXth century abounded with fascinating and intriguing results concerning non-standard models. For reasons of space, we will discuss only two of the first important mathematical results that were proved between the end of 1950s and the beginning of 1960s: the MacDowell-Specker Theorem and Tennenbaum's Theorem. In particular, we will take into consideration Tennenbaum's Theorem since it has aroused the interest of philosophers with respect to the discussion about structuralism as a position in philosophy of mathematics.

In their paper [macdowell-specker61] Robert MacDowell and Ernst Specker generalized Skolem's result. As we saw, Skolem proved that the standard model $\mathcal{N}$ has a proper elementary extension $\mathcal{N}^*$. The MacDowell-Specker Theorem consists in a refinement of Skolem's method in order to show that *every model* of Peano Arithmetic (not just the standard model) has a proper elementary extension, and elementary extensions have to be end extensions.

In the second of these, Stanley Tennenbaum proved that in no non-standard model can either addition or multiplication have a recursive presentation.

Theorem 4.1. (Tennenbaum, 1959). *Let* $\mathcal{M} = (M, +, \cdot, 0, 1, <)$ *be a countable model of PA, and not isomorphic to the standard model* $\mathcal{N} = (\mathbb{N}, +, \cdot, 0, 1, <)$. *Then* $\mathcal{M}$ *is not recursive.*

One first consideration concerning Skolem's proof of the existence of NMoA is in order. By Tennenbaum's theorem we now know that no construction of non-standard model can be finitarily given. Tennenbaum's result points to an essential difficulty in constructing non-standard models, viz. there is no non-standard recursive model of Peano Arithmetic. Hence, Skolem's attempt to build up a non-standard model of arithmetic in a finitary way was doomed to fail. Ultimately, Tennenbaum's Theorem says that non-standard models do not belong to the world of constructive mathematics.

To use Kossak's slogan, the two theorems can be seen as complementary results: whereas the MacDowell-Specker Theorem assures us that "some interesting set-theoretic constructions are easily available", Tennenbaum says "there are no effective constructions".[185]

4.3 A philosophical revival?

As mathematicians continue to respond to the more and more appealing call of NMoA and their beautiful properties, non-standard models made a comeback in the philosophy debate as well.

In particular, philosophers are likely to be fascinated by *limitative results*: as the philosophical phase of non-standard models represented the borderline between a good and

[185][kossak06, p. vi]

bad description of the arithmetical structure, Tennenbaum's Theorem concerns the distinction between models whose operations are recursive and those whose operations are not.

Tennenbaum's Theorem gives rise to a new philosophical discussion about non-standard models of arithmetic. What we want to consider here is the possibility of recovering from the mismatch between the scarce interest that philosophy has paid towards non-standard models over the last 60 years and the abundant mathematical results from the 1960's on. On top of that, what we will take into account is the impact that Tennenbaum's Theorem along the lines proposed in [halbach-horsten05].

4.3.1 Tennenbaum's Theorem and Structuralism

The structuralist account in philosophy of mathematics was sketched in [benacerraf65]. Structuralism comes out as an attempt to bring philosophical explanations more in line with actual mathematical practice. In particular, Benacerraf's structuralism is inspired by the consideration that certain questions about the way the numbers "really are" seem to be somewhat meaningless insofar as answers to such questions are unlikely to have an effect on mathematical practice. On top of that, right after the discussion on the possibility of identifying the numbers with sets, Benacerraf comes to the following conclusion:

> Therefore, numbers are not objects at all, because in giving the properties (that is, necessary and sufficient) of numbers you merely characterize an *abstract structure* — and the distinction lies in the fact that the "elements" of the structure have no properties other than those relating them to other "elements" of the same structure. [benacerraf65, p. 291]

In the passage Benacerraf articulates the main feature of structuralism: the subject matter of the study of mathematics is abstract structure, and the elements of that structure have properties only in virtue of being members of the domain of that structure.

To sum up, the structuralist point of view claims that the actual subject-matter of mathematics is not made of numbers, shapes, etc, but rather mathematical structures. Thus, arithmetic is about a certain structure independently of the nature of the natural numbers that form the domain of the structure. We then conclude that, according to structuralism, arithmetic is about a *single* structure (up to isomorphism).

Immediately, the phrase "single model" brings us back to no other issue than the one of non-standard models of arithmetic. As we know, a formal problem threatens the structuralist view: first-order Peano Arithmetic has both standard models and non-standard models. Then, the whole issue rests on the possibility to actually rule out non-standard models of arithmetic.

To do so, Structuralists have tried several strategies. Specifically, we can distinguish three important methods.[186]

To begin with, in [benacerraf65] two conditions are established: on the one hand the structure is an ω-sequence, and on the other hand the ordering of the elements is recursive. In [benacerraf96], Benacerraf drops the requirement of recursiveness and leaves only the condition concerning the order type. After all, to assume that each model has to be an ω-sequence is surely sufficient to rule out non-standard models of arithmetic whose order type is different than ω.

However, by considering only ω-sequences, we run into one crucial hazard: Benacerraf's condition simply begs the question. In a sense, if we require all the models to have the order type ω, it amounts to say that all the models have to be isomorphic to the standard model.

Recall, in fact, that for the order type of non-standard models discussed earlier, we have first encountered non-standard models and then studied their features among which the order type. What Benacerraf proposes instead is to fix the order type and thus study the models with that particular order type.

A second method for ruling out non-standard models relies on naming every element in the model by some numeral of the language of arithmetic.[187] The purpose is to flesh out all the models whose elements are not named by numerals. Since non-standard numbers are not named by standard numerals, non-standard models cannot meet this requirement. First of all, we can prove in Peano Arithmetic that any object is named by its numeral. Hence, for any object (standard or not) there is a numeral for it. Thus, we are able to rule out non-standard models by necessitating that each element of the model be named by some standard numeral. But also in this case we end up to a *petitio principii*: since a non-standard element will be coded by a non-standard numeral, the point is indeed to distinguish between standard and non-standard numerals, and therefore between standard and non-standard models.

The third and most popular approach for ruling out non-standard models hinges on second-order logic. As we discussed, Dedekind was the forefather of this approach. More recently, Stewart Shapiro has advocated forcefully this approach in several works.[188] As suggested in the previous chapters, by a second-order axiomatization we are able to formulate a categorical theory such as it describes one single structure up to isomorphism: the standard model of arithmetic. This third method does not beg the question as the two previous approaches do, and thus comes out as a viable solution.

Still, the use of a more powerful logic than the first-order one comes at a price. Crucially, the understanding of second-order consequence takes for granted an a priori un-

[186] See [halbach-horsten05, pp. 175–176]
[187] See [halbach-horsten05, p. 176]
[188] E.g., [shapiro85] and [shapiro91]

derstanding of the notion of power set. In fact, the power set is a more problematic than the notion of natural number itself.[189]

Despite the feasibility of the third approach, what Halbach and Horsten propose in [halbach-horsten05] is a forth and alternative way-out, called *computational structuralism*. The idea is to return to Benacerraf's original attempt so as it was stated in [benacerraf65], and review it in the light of Tennenbaum's Theorem.[190]

Tennenbaum's Theorem provides us with a good recipe for ruling out non-standard models by means of a condition on operations of the model. By putting the requirement on the recursiveness of the ordering, Benacerraf has argued that it is indeed possible to determine *effectively*, given two elements of a model, which one is greater.

Now, we can easily extend Benacerraf's requirement to the operations of addition and multiplication by saying that there is an algorithm or a procedure that enables us to compute the sum and the product of any two given numbers.

The condition of sum and product thus becomes the main restriction on models, as Halbach and Horsten put forward:

> **REC1**: In a intended [or standard] model the relation $>$ and the operations of addition and multiplication are recursive. [halbach-horsten05, p. 177]

Now, by Tennenbaum's Theorem 4.1 we know that the standard model of Peano arithmetic is the only one to satisfy REC1, i.e. the operations of addition and multiplication are just recursive in the standard model. And therefore REC1 suffices for ruling out non-standard models of arithmetic.

As a result, Benacerraf's original restriction to the order type of the models comes to be superfluous once we have assumed REC1 as a requirement. Moreover, Tennenbaum's result plays a crucial role in the way the argument is conducted.

However, the solution is less straightforward than it might seem at a first glance. The ubiquitous threat is begging the question yet again. Viewing the matter in more detail, Halbach and Horsten note that the notion of recursiveness hinges on functions and predicates defined *on natural numbers*. Thus, condition REC1 applies only to models whose domains are subsets of the set of natural numbers. Conversely, what we would like to have is a general notion of recursiveness that actually applies to arbitrary relations and functions on arbitrary objects.

Halbach and Horsten put forward two different options:

- they consider the notion of coding, viz. a one-one mapping of the class of the objects in question to the set of the natural numbers, and rephrase REC1 as follows:

  > **REC2**: For every intended model there is a coding of the set of its elements such that the relation $>$ and the operations of addition and mul-

[189] Note that in this respect, the notion of power set would range over at least an infinite set.

[190] [halbach-horsten05, p. 176]

> tiplication on the codes, as they are induced by the relations on the intended model, are recursive. [halbach-horsten05, p. 178]

Still, the reference to the notion natural numbers is likely to bring us back to the previous difficulty.

- they turn to an informal notion of *effective procedure* and thus of computability in the informal sense in order not to presuppose number theory. Whereas the *theoretical* notion of computability is a purely mathematical notion that applies to numbers, the *practical* notion does not completely belong to theoretical mathematics as long as it relies on the informal notion of algorithm and procedure. By appealing to Church's thesis, the result is that the theoretical and practical notions of computability are coextensive. Now we can replace the theoretical notion of recursiveness used in Tennenbaum's theorem by its practical version so that all the weight is carried by the notion of algorithm and the symbols (or notations) that algorithms manipulate. Since a notation system is indeed a structure, Halbach and Horsten offer a third and last version of REC1 as follows:

  > **REC3**: Intended [or standard] models are notation systems with recursive operations on them satisfying the Peano axioms. [halbach-horsten05, p. 178]

  This last proposal claims that arithmetic is exclusively about notations.

To conclude, Tennenbaum's Theorem can be seen as a relevant result for philosophical discussion. Moreover, we can also look at Tennenbaum's result in connection with mathematical practice: pure mathematicians are opposed to mathematical logicians in the sense that pure mathematicians do not seem to be interested in studying these non-standard models as long as they want something more concrete. If one looks at weaker systems than PA, mathematicians' interest is likely to start when systems are weak enough to disallow Tennenbaum's theorem, which is considerably weaker than PA.

At any rate, the theorem constitutes a starting point for much needed further work on the notion of recursiveness and the notion of series of natural numbers with respect to the structuralist account.

Conclusion

We attempted to characterise the *philosophical phase* of non-standard models by discussing Dedekind's and Skolem's work. In particular, we put forward a new role of Dedekind in the history of non-standard models, and discussed the philosophical credo that lay behind the original proof of Skolem. So, we strove to emphasis the features that they had in common and how they contributed to the development of non-standard models of arithmetic.

This allowed us to show that the distinction between the two historical phases of NMoA did not make sense as long as we were focusing on the main differences that distinguish them. Furthermore, we argued that the philosophical phase actually gains its importance in connection with the later mathematical developments, and at the same time we stressed the importance of some mathematical results for philosophical discussions.

On the background, we brought up some crucial foundational issues concerning arithmetic such as the use of logic in mathematics, the notion of intended model, first order logic versus second order logic, strategies for ruling out non-standard models, some features of non-standard models, mathematical structuralism, etc.

Ultimately, our main purpose was to show how the topic of non-standard models of arithmetic relies on the phenomenon of the interaction between formal languages and mathematical structures. This issue seems to compete mathematical logician rather than pure mathematicians. In fact, in contrast to pure mathematicians, mathematical logicians not only do consider mathematical structures, but also the language that talk about these structures.

So, one of next items on the philosophical agenda is certainly the attempt to fill the hiatus between the mathematically oriented research and the philosophical discussion with respect to non-standard models. That is to say that non-standard models seem to require both mathematical and philosophical point of view to be properly investigated.

Still, there is much work left to do even with respect to mathematics. I would like to conclude with a quote of Laurence Kirby who wrote a review of Kaye's book [kaye91]. This remark was written in 1992, but seems still appropriate today as she renders what seems to be the actual status of the study of non-standard models:

> Our vocabulary lacks a term to denote a person whose calling is the study of models of arithmetic. Model theorists, topologists, even functional analysts can identify themselves succinctly, but we have to resort to such locutions as ”I’m in models of arithmetic.” And the name of the field itself—”models of arithmetic”—also seems to bespeak an insecurity about whether it is a field at all: the objects of study are baldly named without any pretensions to a grand theory or -ology. Models of arithmetic certainly is a bona fide field. It has its own meetings, folklore, and stars. It has built up a coherent body of knowledge relevant to some of the central problems of modern logic. But it has never sat comfortably within the traditional fourfold division of logic, it is sparsely populated and has been known to lie dormant for decades, and it has never had a ”bible.” Access to this difficult terrain has been daunting to outsiders. [kirby92, p.461]

Appendices

Appendix A

First order and Second order languages

This appendix intends to provide a precise framework which enables us to gather up all the technical notions we have discussed so far and exhibit them explicitly. In other words, we introduce the formal language of first-order and second-order logic in order to define key concepts and shed some light on their mutual relationship. In particular, we will discuss deductive consequence, semantic consequence, two different kinds of completeness, and categoricity.[191]

A.1 Syntax

To begin with, we illustrate what a logic is, viz. a formal language (a set of formulas built out of symbols according to a set of formation rules) provided with a deductive consequence relation. We in turn present the syntax of the language: the symbols, the terms, the formulas and deductive consequence.

Definition A.1.0.1. *A* language of similarity type ST, *in symbols ST(L), is given by three sets of symbols: constants, function symbols and relation symbols.*

$$ST(L) = (con(L); fun(L); rel(L))$$

The set ST(L) is not fixed in advance, but as soon as it is fixed, the definition of ST(L) becomes completely precise. When the similarity type is clear form the context we simply refer to the language as L.

Note that each function symbol f and each relation symbol R, which belong respectively to the sets $fun(L)$ and $rel(L)$, are assumed to be labelled somehow to indicate

[191] The following exposition is a blend of [oosten10], [hodges01], [vandalen04] and [awodey02]

that they are an n-place function constant and an m-place predicate constant (for some positive integer n and m), and we call it the $arity$ of f or of R.

Definition A.1.0.2. *Given such a language L, we can build terms (which denote elements) and formulas (which state properties) by using the following symbols:*

- *An infinite set of* variables *for individuals, say V_I, and its elements are denoted by $x, y, z, \dots$ or $x_0, x_1, \dots$;*[192]
- *A set of the following* auxiliary symbols*:*
  - *The equality symbol =*
  - *The symbol $\perp$ ('falsum')*
  - *Connectives: the symbols $\wedge$ ('and') for conjunction, $\vee$ ('or') for disjunction, $\rightarrow$ ('if ... then') for implication and $\neg$ ('not') for negation*
  - *Quantifiers: the universal quantifier $\forall$ ('for all') and the existential quantifier $\exists$ ('there exists')*
  - *Some readability symbols such as commas and brackets.*

The formation rules for terms and formulas are the following:

Definition A.1.0.3. *The set of* terms *of a language* L *is inductively defined as follows:*

- *any constant c of L is a term of L.*
- *any variable x in V_I is a term of L.*
- *if $t_1, \dots, t_n$ is an n-tuple of terms of L and f is an n-place function symbol of L, then $f(t_1, \dots, t_n)$ is a term of L.*

A term which does not contain variables (and hence is built up only from constants and function symbols) is called *closed*.

Definition A.1.0.4. *The set of* formulas *of a given language L is inductively defined as follows:*

- *If t and s are terms of L, then (t = s) is a formula of L.*
- *If $t_1, \dots, t_n$ is an n-tuple of terms of L and R is an n-place relation symbol of L, then $R(t_1, \dots, t_n)$ is a formula of L.*

[192]Note that we might have defined the set of variables as part of the language ST(L). In that case we would be committed to defining a function, called suitable assignment, for free variables. For the pros and cons of this alternative approach, see [hodges01, §14]. As here we only deal with arithmetical theories, this presentation turns out to be more convenient.

- $\perp$ *is a formula of L.*
- *If* φ *and* ψ *are formulas of L, then so are* $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \to \psi$ *and* $\neg\varphi$.
- *If* φ *is a formula of L, and x is a variable of* V_I*, then also* $\forall x \varphi$ *and* $\exists x \varphi$ *are formulas of L.*
- *Nothing is formula of L except as required by the previous conditions.*

We also provide the following syntactic tool:

Definition A.1.0.5. *Suppose* φ *is a formula of L, and t a term of L. By* substitution $\varphi[t/x]$ *we mean the formula which results by replacing each occurrence of the variable x by the term t, provided x is a free variable in* φ*, and no variable in the term t becomes bound in* φ*.* [193]

Remark A.1.0.6. First order and other kinds of languages *Within first-order language we can only talk about some or all elements of a structure, but by no means is the language powerful enough to talk about subsets of elements, unless we are considering relation constants.*

To talk about 'some' or 'all' subsets of elements, we are required to quantify over properties. This is allowed in more powerful languages such as second order language.[194]*.*

To put it briefly, a higher-order language (in this place we just consider second-order logic) is an extension of the first-order language. Formally, this means that to define a second-order language we have to add the following conditions:

- *in A.1.0.2 we include: an infinite set of* variables for relations*, say* V_R*, and its elements are denoted by capital letters* $X, Y, Z, \ldots$ *or* $X_0, X_1, \ldots$*;*
- *in A.1.0.3 we make the following additions:*
  - *any n-ary relation variable X in* V_R *is a term of L.*
  - *if* $t_1, \ldots, t_n$ *is an n-tuple of terms of L and X is an n-ary relation variable of L, then* $X(t_1, \ldots, t_n)$ *is a term of L.*
- *in A.1.0.4 we include: If* φ *is a formula of L, and X is a variable of* V_R*, then also* $\forall X \varphi$ *and* $\exists X \varphi$ *are formulas of L.*

To avoid complication, we consider just the case of unary second-order variables because in PA pairing function exists so that we can simulate n-ary variables easily.

[193] Although fundamental, the definition of when a variable is either free or bounded in a formula will be presented here for reasons of space. For an explicit account, see e.g. [vandalen04, pp. 63–65]

[194] The issue of assessing the pros and cons of first-order language over second-order one in formal axiomatics is discussed in section 3.7.1

Finally, we can say what a *first order language* is:
by *first-order language L of similarity type ST* we mean an ordered triple $(ST, T(ST), F(ST))$ where ST is the similarity type of L, and $T(ST)$ and $F(ST)$ are respectively the set of all terms and formulas of similarity type ST defined according to A.1.0.3 and A.1.0.4.
Conversely, a second order language L_2 of a given similarity type is an extension of a first order language. In particular it consists in an ordered triple $(ST, T(ST), F(ST))$ where ST is the similarity type of L_2, and $T(ST)$, $F(ST)$ are respectively the set of all terms and formulas of similarity type ST defined according to A.1.0.3, A.1.0.4 and A.1.0.6.

The second indispensable component to obtain a *logic* is the deductive consequence relation:[195]

Definition A.1.0.7. *A* formal proof calculus, *call it Σ, is a device for proving statements in the form 'if-then' in a language L. First of all, we have to make explicit a* set of rules *for writing down arrays of symbols on a page. An array which is written down according to the rules is called a* formal proof in Σ. *The rules must be such that we can effectively check by calculation whether or not an array is a formal proof. After that, we say that the calculus contains a rule to tell us how we can mechanically work out what are the premises and the conclusion of each formal proof.*

Having said this, we are now able to define the deductive relation:

Definition A.1.0.8. *Let a formal proof calculus Σ be a set of rules which allows to write a proofs, φ be L-formula and Γ be a set of L-formulas. We define the relation $\Gamma \vdash_\Sigma \varphi$ if there is a proof with conclusion φ by application some Σ-rule on Γ-formulas.*
We abbreviate: $\Gamma \vdash_\Sigma \varphi$ as $\Gamma \vdash \varphi$ when the set of rules is clear from the context; $\{\varphi\} \vdash \psi$ as $\varphi \vdash \psi$; $\emptyset \vdash \psi$ as $\vdash \psi$; $\Gamma \cup \{\varphi\} \vdash \psi$ as $\Gamma, \varphi \vdash \psi$.

A.2 Semantics

Having considered the syntax of the language L, we turn to the semantic side. Whereas syntax concerns the properties of the formal language in itself, semantics stresses the relationship between the formal language and the structures on which the language is interpreted.

At this point, we can say in formal terms what a structure is:

Definition A.2.0.1. *An L-structure $\mathcal{M}$ is an ordered tuple $(M, (\cdot)^{\mathcal{M}})$ where:*

- *M is a nonempty set which is called the* domain *of $\mathcal{M}$. The elements of M are called the $elements$ of $\mathcal{M}$, and the cardinality of M is called the $cardinality$ of $\mathcal{M}$.*

[195] For our purposes, deductive consequence does not play a role as important as the semantic counterpart. So we offer just a general-informal definition of it.

- $(.)$ *is a function from* L *to* $\mathcal{M}$ *which assigns:*
  - *to each constant* c *of* ST *an element* $(c)^{\mathcal{M}}$ *of* $\mathcal{M}$
  - *to each n-place function symbol* f *of* ST, *a function* $(f)^{\mathcal{M}} : \mathcal{M}^n \to \mathcal{M}$
  - *to each n-place relation symbol* R *of* ST, *a subset* $(R)^{\mathcal{M}} \subseteq \mathcal{M}^n$

Henceforth, by structure or model $\mathcal{M}$ we mean the L-structure $\mathcal{M}$ for some language L. We say that the element $c^{\mathcal{M}}$ the interpretation of c in $\mathcal{M}$, and similarly, $f^{\mathcal{M}}$ and $R^{\mathcal{M}}$ are called the interpretations of f and R, respectively.
Given an L-structure $\mathcal{M}$, we can also consider the language $L_{\mathcal{M}}$, i.e. the language of the structure $\mathcal{M}$: it consists of L together with, for each element m of $\mathcal{M}$, an extra constant (also denoted m). Here it is assumed that $con(L) \cap \mathcal{M} = \emptyset$.
Note that if we stipulate that the interpretation in $\mathcal{M}$ of each new constant m is the element m, then $\mathcal{M}$ is also an $L_{\mathcal{M}}$-structure. Once again, we are saying that $L_{\mathcal{M}}$ is strong enough to talk about the entire domain of $\mathcal{M}$, i.e. we have enough names in the formal language $L_{\mathcal{M}}$ to name and refer to all elements in $\mathcal{M}$.

Since we understand the natural numbers as a structure in which every number n is named by a numeral constant n of the language, we have an L-structure whose elements are all named by individual constants of L.

Definition A.2.0.2. *For each closed term t of the language* $L_{\mathcal{M}}$, *we define its* interpretation $t^{\mathcal{M}}$ *as element of* $\mathcal{M}$, *by induction on t, as follows.*

- *If t is a constant, then its interpretation is already defined since* $\mathcal{M}$ *is an* L*-structure.*
- *If t is of the form* $f(m_1, \ldots, m_n)$ *then also* $m_1, \ldots, m_n$ *are closed terms of* $L_{\mathcal{M}}$, *so by induction hypothesis their interpretations* $m_1^{\mathcal{M}}, \ldots, m_n^{\mathcal{M}}$ *have already been defined. So we simply put* $t^{\mathcal{M}} = f^{\mathcal{M}}(m_1^{\mathcal{M}}, \ldots, m_n^{\mathcal{M}})$

That is to say that for the function $(\cdot)^{\mathcal{M}}$ the principle of compositionality holds.

Now we can define for a closed formula[196] φ of the language $L_{\mathcal{M}}$ to be *true in* $\mathcal{M}$, written $\mathcal{M} \models \varphi$. Equivalently, we say that $\mathcal{M}$ *satisfies* φ or that $\mathcal{M}$ *is a model* of φ.

Definition A.2.0.3. *For a closed formula* φ *of the language* $L_{\mathcal{M}}$, *the relation* $\mathcal{M} \models \varphi$ *is defined by induction on* φ*:*

- *If* φ *is an atomic formula, it is either equal to* $\perp$, *or of the form* $(t_1 = t_2)$, *or of the form* $R(t_1, \ldots t_n)$ *with* $t_1, t_2, \ldots t_n$ *closed terms. Define:*

  $\mathcal{M} \models \perp$ *never holds*

  $\mathcal{M} \models (t_1 = t_2)$ *iff* $t_1^{\mathcal{M}} = t_2^{\mathcal{M}}$

  $\mathcal{M} \models R(t_1, \ldots t_n)$ *iff* $t_1, \ldots t_n \in R^{\mathcal{M}}$, *for some* $n \in \mathbb{N}$

[196] Note that to consider only closed formulas is enough for our arithmetical purpose.

where $t_i^{\mathcal{M}}$ for $i \leq n$ are the interpretations of the terms (according to definition A.2.0.2), and $R^{\mathcal{M}}$ is the interpretation of R in the structure $\mathcal{M}$.

- *If φ is on the form $\varphi_1 \wedge \varphi_2$ then $\mathcal{M} \models \varphi$ iff $\mathcal{M} \models \varphi_1$ and $\mathcal{M} \models \varphi_2$.*
- *If φ is on the form $\varphi_1 \vee \varphi_2$ then $\mathcal{M} \models \varphi$ iff $\mathcal{M} \models \varphi_1$ or $\mathcal{M} \models \varphi_2$.*
- *If φ is on the form $\varphi_1 \rightarrow \varphi_2$ then $\mathcal{M} \models \varphi$ iff it is not the case $\mathcal{M} \models \varphi_1$ or $\mathcal{M} \models \varphi_2$.*
- *If φ is on the form $\neg\psi$ then $\mathcal{M} \models \varphi$ iff not $\mathcal{M} \models \psi$*
- *If φ is on the form $\forall x\psi$ then $\mathcal{M} \models \varphi$ iff $\mathcal{M} \models \psi[m/x]$ for all $m \in M$*
- *If φ is on the form $\exists x\psi$ then $\mathcal{M} \models \varphi$ iff $\mathcal{M} \models \psi[m/x]$ for some $m \in M$*

Note that in the last two clauses $\psi[m/x]$ is the result of the substitution of the new constant m for x in ψ. To be a bit fussy, note that m in $[m/x]$ is not the same m of $m \in M$: recall that for each m in M, we defined a constant with the same name m, but in the language $L_{\mathcal{M}}$, and the substitution involves such a constant m.

This came out as a fundamental result in the development of logic, as Corcoran puts it:

> Today we *can* speak of mathematical systems without reference to particular formal languages interpreted in them, but we often do not. For example, when we speak of the system of natural numbers we often mean the intended interpretation of one of the formal languages commonly used for number theory. And we have to be reminded of the fact that we can refer to the system of natural numbers in itself, so to speak. We also have to be reminded of the fact that an interpretation of a formal language is not merely mathematical system but it also involves (among other things) a precise specification of which formal symbols get assigned to which distinguished relations, functions, and elements. In general, a set of formal axioms can be interpreted in a given system in more than one way. [corcoran80, p. 189]

We obtain a sharp distinction between a formal language and what it interprets, freeing it from the intended interpretation that prevents the language from having just one particular assignment. As already mentioned, this is one of the advantages of a formal theory over a naïve theory: it is amenable to multiple domains and allows various interpretations.

To go back to def. A.2.0.3, we move to more general definitions and results applicable to a second order language.

For the second order case, we have to account for quantifiers applied to relation variables, namely we want to express phrases such as "for every property of objects (in the domain under consideration)" and "for some property of objects (in the domain under consideration)". It suffices to add an additional case to def. A.2.0.3:

- If φ is of the form $\forall X \psi$ (where X is a unary relation variable) then $\mathcal{M} \models \varphi$ iff for all unary relations P subset of M we have $\mathcal{M} \models \psi[P/X]$
- If φ is of the form $\exists X \psi$ (where X is a unary relation variable) then $\mathcal{M} \models \varphi$ iff for some unary relation P subset of M we have $\mathcal{M} \models \psi[P/X]$

Note that in these two clauses, $\psi[P/X]$ is the result of the substitution of the new set constant P for X and no clash of variable whatsoever occurs. Of course, we assume that we have names for all subsets.

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