Buridan's Theory of Logical Consequence

MSc Thesis (*Afstudeerscriptie*)

written by

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under the supervision of **Dr Catarina Dutilh-Novaes**, and submitted to the Board of Examiners in partial fulfillment of the requirements for the degree of

MSc in Logic

at the *Universiteit van Amsterdam*.

Date of the public defense:
May 30 2011

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ABSTRACT

In this thesis we develop a proof-theoretic formalization for John Buridan's theory of modal propositions. This thesis provides a complete exegesis of the second chapter of *De Consequentia* and develops the system $G3_{syl}$ which we show is an adequate formalization of Buridan's theory.

Coming to the end of writing this thesis leaves me with far too many people I would like to thank. To my fellow students who helped deepen my understanding of logic and philosophy as well as keeping me sane throughout this process, you have my deepest thanks. I would like to especially thank, Hanne Berg, Lorenz Demey, David Fiske, Kian Mintz-Woo and Matthew Wampler-Doty. Their patience, willingness to help me and kindness cannot be repaid. Likewise, to the church community that essentially adopted me and helped me through my two years of living in the Netherlands, I want to offer my deepest thanks. I would like to especially thank: Jolanda Dane, Alastair and Sophie MacDonald, Matthijs van Engelen,Bert de Jong, Marco Klaue, and Thomas and Miriam Jones. Your friendship was a source of great encouragement to me.

Special thanks go to my supervisor Catarina Dutilh-Novaes, who was a wonderful and patient guide through this entire writing process. Her constant wisdom and guidance was tremendous through this entire process, as was her patience for working through half a dozen different versions of the $G3_{syl}$ rule systems, often where the only difference was a change in the world label. I would like to thank Sara Uckelman for sparking my interest in medieval logic and supervising two of the January projects on various historical topics in logic. I owe a profound debt of gratitude to Stephen Read, my current PhD supervisor in St. Andrews, for providing me with a draft of his excellent translation of Buridan's *De Consequentia*. I would like to thank Dick de Jongh for his mentoring of me for my time with the ILLC. In addition I would like to thank him and Benedikt Lwe for sitting on my thesis committee.

I am very grateful and pleased to thank the University of Amsterdam for the financial support they provided for me during my two years in Amsterdam.

I would never have become interested in Logic if it had not been for the encouragement of my philosophy professors during my undergraduate. I would like to particularly thank Dr. Davis for introducing me to symbolic logic, encouraging my study of logic and introducing me to modal logic. I would like to thank Dr. Weed for encouraging me to study at the ILLC. I would also like to thank Dr. Urqhart for introducing me to meta-logic, and allowing me to attempt the graduate level logic course at the University of Toronto.

Finally, I would like to thank my parents, Gordon and Charlene Johnston, whose constant love, encouragement and support for me has been amazing and a source of great strength to me. It is to them I dedicate this thesis.

Fides quaerens intellectum.

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1. INTRODUCTION

> The idea of logical consequence has been with us since the time of Plato and Aristotle and since then has been central for our understanding of philosophy and science... somehow the idea occurred that the new practice [of Greek mathematics] was to be seen as involving the drawing of logically valid inferences.[13, Pg 671]

These opening lines from Prawitz's discussion of logical consequence bring two important observations together. The first is that the 'problem' of logical consequence has an impressive historical pedigree. The second is that from antiquity their existed a close connection between logical reasoning, and mathematics.

This raises the question, what is a logical consequence? What does it mean to say that one proposition, expression or whatever we take the relata of consequences to be formed out of, is a consequence or entails another?

Within the modern logical literature, there are two main approaches to the notion of consequence that have taken central stage: the model theoretic and proof theoretic notions of (logical) consequence . We will explore both of these in just a moment. However, it is important to observe that both of these ideas are *very* modern developments, the former having its historic roots in the writings of Bolzano, but being given its mathematical form by Tarski in the 1930's and 40's. The proof-theoretic account finds its modern inception with Hilbert and his finitist project in Mathematics. These are both distinctly 20th century developments, which raises the question, what about the rest of the literature on consequence before the 20th century?

The goal of this work is to explore one such historical figure's work on logical consequence: John Buridan. The aim of this work is threefold. First, we want to unpack and analyze Buridan's treatment of logical consequences. Specifically we will focus on his treatment of modal consequences.[1] The goal here will be primarily textual in nature. We want to try and understand how Buridan understood logical consequence when applied to modal propositions. Here we will argue that 1) Buridan's treatment of modal propositions is different from what is normally found in first order modal logic. Within this discussion some attention will be paid to how Buridan's understanding of supposition leads to his preferred analysis of modal propositions and their consequences. 2) Buridan's account displays a strong rule-like characteristic to it. 3) In virtue of 1) Buridan's account of modal consequence represents an interesting alternative approach to modal logic.

The second goal of this thesis will be to take Buridan's theory of logical consequence and reconstruct it from the point of view of one modern logical system. Due to the strong rule-like nature that the theory contains, we will formulate Buridan's logic using a modification of the sequent Calculi developed by Negri and Von Plato.[10][8]. We will proceed as follows: First,

[1] A proposition A is said to be a modal consequence of B, if A is a consequence of B, and one of A,B is a proposition in which a modal expression occurs. Modal expressions are things like 'possible' , 'necessary', 'contingent', 'believes' etc.

we will develop a calculus that can capture the medieval term-based relationships, as well as Buridan's unique understanding of $\Box$ and $\Diamond$. Second, we will show that the system is sound with respect to Buridan's claims, and that if cut is admissible/not present in the system, it is also 'complete' in a particular sense to specified.

Finally, we will provide some brief philosophical reflections on the role that formalization plays in the history of logic. Specifically we will look at how we approach the formalization of historic figures. Here we will distinguish two functions formalization can play in historical analysis. We will then argue that a formalization needs to be more then formally adequate. It has to meet two criteria. First, It must be faithful to the underlying spirit of the author that it is trying to model. Second, it needs to elucidate Buridan's theory in such a way that we learn new things about how Buridan understands consequence.

1.1 Logical Consequence: The Modern Debate

Modern theories of logical consequence can be classified into two broad categories. The first is the model theoretic approach, which traces its origins back to Tarski. The second are proof theoretic accounts of consequence. Within this discussion we will focus on the tradition started by Gentenz and given fresh expression by Prawtiz and many others.

The model theoretic approach understands a consequence as preservation of truth in all possible situations. The intuition behind the idea is as follows:

> Let us consider an arbitrary class of sentences K and an arbitrary sentence X which follows from the sentences of this class. [2.3.3] From the point of view of everyday intuitions it is clear that it cannot happen that all the sentences of the class K would be true but at the same time the sentence X would be false.[16, Pg.183]

Setting aside the comment about everyday intuitions[2] what the semantic account wants is an analysis of consequence where truth is preserved. I.e. A sentence X follows from a class of sentences K just in case there is no situation where K is true and X is false. Tarski expresses this idea formally in the following definition:

> If in the sentences of the class K and in the sentence X we replace the constant terms which are not general-logical terms correspondingly by arbitrary other constant terms (where we replace equiform constants everywhere by equiform constants) and in this way we obtain a new class of sentences K' and a new sentence X', then the sentence X' must be true if only all sentences of the class K' are true.[16, Pg.183-184]

The idea here is that we take the class K and the sentence X. We fix the terms in K and X that are logical [3]. We substitute the non-logical terms in K and X with other terms, to obtain K' and X'. The idea is that X is a consequent of K if and only if for there is no K', X' such that K' is true and X' is false. The way this notion of permutation is made precise is via the concept of a model.[4]

[2] which is another complex and philosophically interesting problem

[3] for example terms like 'and', 'or', 'all', some, not, etc

[4] [12, Pg 155].

Another way of analyzing consequence is done via proof-theory. Here consequence is understood in terms of the results of provability relations from a given set of rules. Prawitz, citing Lorenzen draws the analogy between logical consequence and chess,

> the possibility of moving the horse four squares in any direction by two consecutive moves is a consequence of the rules of the game in the same way as the implication $A \to C$, which can be understood as a rule, is a Consequence of $A \to B$ and $B \to C$. [12, Pg. 155]

The idea here is that we first fix the pieces of our game, together with a collection of rules that tell us what can and cannot be done. A is then a consequence of B, if given B, we can apply the rules in such a way that we arrive at A.

Both of these accounts have a number of very nice properties. First, both accounts are reductive. In the case of the model theoretic account, the notion of consequence can be reduced to truth in a class of models. In the proof-theoretic analysis, consequence is reduced to the presence of derivation of the conclusion from the premises.[5] Reduction is a useful property to have since, it ideally should enable a deeper understanding of the consequence relationship.[6] Second, both of these accounts can be made mathematically precise and can be formally represented.

However, both accounts of consequence are open to a number of objections and misgivings. Etchemendy has argued that the Tarskian model-theoretic account of logical consequence both over and under-generates with respect to our intuitive understanding of logical consequence. The model theoretic account also does not give a definite answer to what counts as a logical constant and why. For example, why treat a term like 'and' as a logical connective, and not other relationships, like brother.

The proof theoretic account has other problems. The first is making sure that it is actually closed under the appropriate notions of entailment. Second, since proof theory consists of a string of symbols with various rules, it is not clear what the resulting interpretation of these rules should be. Connected with this is the issue of introducing rules that lead systems having undesirable properties.[7]

All of these questions have been addressed and debated in various places. However, our concern in this work is to explore how one historical theory of consequence relates to this modern debate. Our goal for this work is to unpack Buridan's theory of logical consequence and see how it can be related to modern notions of logical consequence. As we do this, it will become clear that Buridan's theory has a number of interesting logical features, but also that it is in some sense quite different from how modern thinkers have conceived of consequence. However, before we can begin our discussion of Buridan we will need to briefly discuss a few features of medieval logic that are foreign to modern logicians and relevant to understanding Buridan's theory of logical consequence.

---

[5] a derivation is a sequence of applications of inference rules to the premises.

[6] It should be pointed out that this claim is contentious. Some, e.g. Etchemendy, contend that such a reduction do not work.

[7] for example, the rule 'tonk' which has the introduction rule for $\vee$ and the elimination rule for $\wedge$.See[14, Pg. 39].

1.2 Logical Consequence: Middle Ages

As Prawtiz quote reminded us, the problem of logical consequence is not unique to the modern period. The problem goes all the way back to Aristotle and his treatment of syllogisms in the *Prior Analytics*. The goal of this section will be to sketch how logic was understood at the time of Buridan, and explain a number of concepts that are relevant to our understanding of his *De Consequentia*. Here we will sketch the development of theories of consequence as they emerged in the late middle ages and discuss an important part of medieval semantic theory, the theory of supposition.

1.2.1 Origins of 14th Century views on consequence

Discussions of logical consequence have historical roots going back to the ancient Greeks, at least in the western tradition. However, at the turn of the 14th century there began to be an emergence of treatises that focused specifically on logical consequence. It is still a topic of debate among scholars as to what exactly caused such treatises to be written. One hypothesis is that these treatises developed out of the tradition of commentary on Aristotle's Topics. For further discussion of this hypothesis see [5, Pg468-469] and the references given there.

However, broadly speaking, the picture that emerges is that:

> At least four traditions - topics, syncategoremata, hypothetical syllogisms and *Prior Analytics* must be taken in account to explain the raise of theories of consequence in the 14th century. [5, Pg 470]

Within the 14th century we can group the discussion of consequence into four groups:[5, IBID]

> Early 14th century treats that provide guidelines for dealing with various sophismata that emerge from the treatment of some syncategorematic terms.
>
> Works like Burley's *De Puritate* and the chapters on consequence in Ockham's Summa where a deeper and more systematic study of the theory of consequence is made. Here general definitions and criteria are developed to account for various kinds of consequences.
>
> Buridan's *De consequentia* and various treaties that were inspired by it. Here a number of ideas developed in the previous two groups are absent (e.g. discussion of intrinsic and extrinsic middles) Formal consequence is formulated based on a substitutivity criteria.
>
> The final group is composed of treaties written by predominantly British authors. This group is characterized by the ideas that formal consequence can be understood in terms of "containment of the consequent in the antecedent." This theory of consequence is distinctly epistemic in form, in contrast with group three.

As this tradition developed the Medieval logicians began to recognize a number of different kinds of consequence, all satisfying a modal definition of consequence. The modal definition of consequence is the following:

The consequent cannot be false while the antecedent is true.

This was generally held to be a necessary condition for consequence.[5, Pg 472] However, there was quite a bit of debate as to what other conditions might also be required for a consequence to be a good one. It is into this collection of writing that Buridan's *de consequentia* is situated. Throughout the 14th century the various medieval authors realized that there were different kinds of consequences, i.e. there were various pairs/triples that satisfied that modal definition but could be grouped into different, possibly overlapping sub-classes. Three main distinctions were drawn:[8]

> Natural vs. accidental consequences[9]
>
> Absolute vs. as-of-now consequences
>
> Formal vs. material consequences

In Buridan's witting, it is the last distinction that is of importance to us. For Buridan, a consequence is formal:

> If it is valid in all terms retaining a similar form. Or if you want to put it explicitly, a formal consequence is one where every proposition similar in form which was formed would be a good consequence.[10]

The definition of formal consequence provides an illuminating formulation of the substitutivity criteria that Buridan employs. In essence Buridan is saying the following: a consequence is formal if it satisfies the modal definition and when the common logical features of the sentence are held constant, the argument remains valid. By the common logical features, he means that the logical constants are not changed, but the terms are substituted with other other categorical terms resulting in a good argument. As an example of this Buridan cites the conversion from B is A to A is B. In this case, regardless of what A and B stand for, the argument will remain valid.[11] A consequence is called a material consequence if it satisfies the modal definition, but does not satisfy the substituivity condition. When Buridan treats modal consequences in book two, his focus will only be on formal consequences.

Supposition and Ampliation

Within the 14th century, logic together with grammar and rhetoric, formed the first courses a medieval student took.[5, Pg.433] In a sense, logic was foundational to the study of any other academic discipline. At this time, treatises on logic covered a number of diverse topics such as:

> insolubilia (paradoxical propositions); modal propositions; supposition; the analysis ('proof') of propositions; obligations; and consequence[5, Pg.434]

One of the key components that linked the analysis of all of these issues was supposition. Indeed, supposition plays an important role in Buridan's treatment of modal consequences. Another and equally important idea was that of ampliation. Ampliation plays a critical role

[8] [5, Pg 473]

[9] This distinction quickly fell out of use in the 14th century.

[10] Quae in omnibus terminis ualet retenta forma consimili. Vel si uis expresse loqui de ui sermonis, consequentia formalis est cui omnis propositio similis in forma quae formaretur esset bona consequentia.[15, 1.4.2]

[11] Here, as in other places in this thesis, we will use capital letters as schematic letters for terms

in Buridan's definition of the 'possibility' modality and since it is connected with supposition, will also be treated here. Our goal in this section is to sketch the basic idea of supposition and ampliation so that we can understand Buridan's application of it in *De Consequentia*

Supposition can be defined as follows:

> Supposition is the property of terms (occurring in propositions) of standing for things, so that these things can be talked about by means of propositions, and supposition theory (in its different versions) is a theory codifying the different uses of terms in propositions, based on the idea that one and the same term can stand for different things when occurring in different propositional contexts. [6, Pg 1]

Of key importance to our discussion about Buridan is that supposition is a theory that differentiated the different uses of terms within a proposition. Supposition was intended to capture the various ways terms can be used in expressions, and focused on four main kinds of semantic variation:

> - Whether a term is taken literally or metaphorically.
> - The different ontological kinds of the things that one and the same term can stand for - the supposita: (extra-mental) individual entities, universal entities, mental entities, or linguistic entities.
> - The different temporal and modal statuses of the supposita: present, past, future, actual, or possible.
> - The different quantities of entities required to verify or falsify a proposition.
>
> [6, Pg 2]

In terms of how supposition is used in *De Consequentia*, our main focus will be on the second way supposition can function. When we come across supposition in chapter two, for the most part, it will be used to differentiate the different kinds of entities things stand for in the various propositions Buridan considers.

This kind of supposition was traditionally divided into three main kinds:

1. The word itself
2. It's related form or concept
3. The individual that falls under the form or concept

These three kinds of suppositions were referred to as material, simple and personal supposition respectively. Consider the following three propositions:

1. Donkey is a six letter word
2. Humans are rational animals
3. Ralph is a donkey

In the first case, the word does not supposit for anything in the mind or in the outside word. The word supposits for the linguistic entity named by it. In this case, it supposits for the written term ‘donkey’. Another example of material supposition would be ’material is polysyllabic’. Here it supposits for the uttered word ‘material’ In both cases the word is said to supposit for itself. It is important to note that the medievals never developed a kind of symbolization to distinguish when the term in question is referring to itself, or when it is referring to other things. In modern witting it is customary to differentiate donkey from ‘donkey’. Throughout the thesis we will follow the modern convention on this point, but it should always be remembered that supposition was used to draw this distinction.

The second case is an example of simple supposition. In this kind of supposition, the term supposits for the form or essence of the term. Buridan does not make much of this notion of supposition. Like William of Ockham, Buridan is committed to nominalism. The result of this is that ”Buridan rejects the existence of extra-mental universals, so for him simple supposition becomes a superfluous concept.[6, pg 6]”

The third case is an example of personal supposition. Here, the term in question supposits for ”the individuals instantiating the universal nature.[6, pg 4]” In other words, when the term ‘donkey’ has personal supposition, it supposits for the individuals that fall under the term. In the case of the example, we are told that Ralph is one of the individuals that falls under the term donkey.

Ampliation can be seen as a natural extension of supposition in the following way:

> In a non-modal present tense categorical proposition, common terms typically do not supposit for everything that they indirectly (or secondarily) signify. Instead they supposit only for the presently existing things that they indirectly signify. [11, Pg. 202]

The medievals realized that there were a number of propositions that required the supposition of terms to extend beyond those things that exist at the present time and are actual. Expressions like believes, want, was, can be, will be, etc. all ampliate the terms for which they supposit. Hence in the proposition ‘every human is mortal’ human is restricted to the present time, actually existing humans. In essence, what ampliation does is changes the range of objects that supposit for a given term. Consider the following two sentences:

> Every horse was running

> A person can be running

In both cases ampliation changes the scope of the objects that supposition is applied to. In the first example, we do not only consider the horses that currently exist, but we also consider all of the horses alive at the time the proposition implicitly references. This is required to ensure that we get the correct truth conditions for these expressions. In the case of past tense expressions, the first sentence is true if everything that is or was a horse was running. Analogous formulations can be obtained for future tense propositions and modal propositions.

The importance of supposition and ampliation to Buridan’s theory of consequence emerges very quickly from the text. To be able to formalize Buridan’s logic we will need to understand the relevant truth conditions for modal propositions, and hence how he understands the supposition of modal terms. As Parsons points out, ”Supposition is important because it, together with the structure of the proposition, determines the truth conditions of the proposition.[11, Pg.

188]” Since we are working with modal propositions, ampliation is also an important part of how we determine the truth-conditions for Buridan’s logic. Hence, we will need to ensure that our logic can faithfully reflect how supposition works in the various propositions that Buridan treats.

As a precursor to how Buridan uses supposition consider the following example:

> [A] divided proposition of possibility has a subject ampliated by the mode following it to supposit not only for things that exist but also for what can exist even if they do not.[12] [15, 2.4.1]

Here it is helpful to understand what Buridan means by supposition, and to be aware that the kind of supposition being used here is simple supposition. Likewise, when we encounter Buridan’s discussion of composite modal claims, Buridan uses material supposition to analyze these propositions. All of this will be dealt with in due time, but it will be important to keep this background information in mind as we turn to the second book of *De Consequentia*.

Conclusion

The goal of this chapter was to provide a brief sketch of some questions about logical consequence that are relevant in both modern philosophical discussions and were important at the time Buridan. On that point it is time that we look at Buridan’s treatment of modal consequences.

[12] propositio diuisa de possibili habet subiectum ampliatum per modum sequentem ipsum ad supponendum non solum pro his quae sunt sed etiam pro his quae possum esse quamuis non sint

2. DE CONSEQUENTIA

The manuscripts of *De Consequentia* were compiled into a critical edition by Hubert Hubien, whose Latin text we follow throughout this work. All English translations unless otherwise stated, are due to a currently unpublished translation of this edition by Stephan Read. Page references are made relative to the word documents provided.

2.0.2 Syntax

The goal of this chapter is twofold. First, we will provide commentary on Buridan's overall argument and his views on modal logic. Second, we will provide formalizations of the divided modal propositions and arguments that occur within chapter 2. Throughout this chapter we will provide formalizations of various expressions to clarify what Buridan is doing. To that end we will introduce our full language, and the relevant portions of the syntax. The syntax of our logic is as follows:[1] Our language $\mathfrak{L}_S = \{P, \wedge, \vee, \rightarrow, \neg, \Diamond, \Box, \forall, \exists, \circ,\}$
where P=a,b,c...;A,B,C...;$\bar{A}, \bar{B}, \bar{C} \ldots$
Our terms are formed as follows:
Terms:

> lower case letters a,b,c...denote individuals, except for the symbols, x,y,z, x',y',z'..., which are reserved as labels for possible worlds.
>
> Uppercase letters A,B,C...denote terms and $\bar{A}, \bar{B}, \bar{C} \ldots$ denote negative terms.
>
> $\forall$ $\exists$ followed by a term e.g. $\forall A$ is a quantified term.
>
> $\Diamond, \Box$ are modes.[2]
>
> $\circ$ is used to denote the basic present tense copula between expressions.
>
> $\neg\circ$ denote the negative copula.
>
> $\Box\circ, \Diamond\circ$ are modalized copula.
>
> $\neg\Box\circ, \neg\Diamond\circ$ are negative, modalized copula.

Our base level formulae are of the form
'$\alpha' * \circ\beta$
where

> α is either a term, a quantified term or a mode,

---

[1] this is a modification of the notation used in[4]

[2] Throughout this thesis we will also use the meta-variable ∇ to range over both of these modes, either on their own, or when they occur with other symbols, like $\circ$

$*\circ$ denotes any of the copulas

β is either a term, a negative term, or a mode.

For example, the formula $\forall A \Diamond \circ \bar{B}$ is a basic formula, and means, every is A is possibly non-B.[3] In addition, composite modals (which we will come to in due time) are constructed as follows
$\phi \circ \psi$ where

One of ϕ, ψ is a mode, and the other is a basic formulae.

2.0.3 Chapter 1: Introduction

In the second book of*De Consequentia* Buridan deals with particular class of modal consequences, namely, those that:

> Aristotle and other masters have treated... modals of possibility and impossibility, of necessity and contingency, and of truth and falsity.[4] [15, Pg. 1]

The consequences that Buridan focuses on are those that have been addressed by many of the major thinkers of Western Philosophy. This is his explicit reason for restricting his attention to these 6 modal expressions.[5] It should also be remembered that these six modalities are traditionally the ones most explored and are most relevant to many areas of philosophy and theology. Book two only looks at consequences that hold between single premise propositions. Buridan's full treatment of syllogistics is developed in the fourth book of this work.

Book two proceeds as follows:

In the first three chapters Buridan lays down a number of definitions about modal terms, and draws a number of distinctions that will serve as the basis for his analysis of modal consequences. In chapter four he outlines the relevant information on supposition required throughout this book. Chapter five introduces a number of important equivalences while Chapters six and seven deal with divided and composite modal claims respectively. After covering this material, Buridan argues for 19 conclusions based on the assumptions and definitions he has made. Eight of these conclusions are about divdied propositions, the rest are about composite modal propositions, and the way the two kinds of propositions interact. By the end of this chapter Buridan will have outlined the inferences for modal propositions he takes to be valid for both divided and composite modal claims.

2.0.4 Chapter 2: Divided and Composite Modal Propositions

Buridan starts his treatment of modal consequence by drawing the following distinction:

---

[3] Throughout this thesis we will use the English expression non-B to denote the negative term $\bar{B}$ while using the expression 'not B', to refer to $\neg \circ B$

[4] Aristoteles et alii magistri tractauerunt, sunt modales de possibili et impossibili, de necessario et contingenti, de uero et falso.

[5] it should be noted that Buridan holds that there are other modals, which he refers to from time to time. e.g. See Bk 4 Conclusion three.'

> It should be noted that propositions are not said to be of necessity or of possibility in that they are possible or necessary, rather, from the fact that the modes 'possible' or 'necessary' occur in them, and the same for other modes. [6] [15, Pg 2]

For Buridan, for a proposition to be modal, it must explicitly contain a modal term within it. To illustrate this, he cites Aristotle, and the proposition: 'every man is an animal.' For Aristotle, this claim is not only true but necessarily so. However, the sentence only expresses an asserotric proposition. For Buridan, It is not enough that the proposition is in fact true/false necessarily/possibly etc. In the case of 'every man is an animal' the proposition is assertoric and true[7].

From this, it is clear what counts as a modal proposition. A proposition is modal just in case a modal term occurs in it. The next distinction that Buridan draws tells us what kinds of modal propositions there are. According to Buridan there are two types of modal propositions:

> They[the modal claims] are called composite when a mode is the subject and a dictum is the predicate, or vice versa. I call the terms 'possible', 'necessary', 'contingent' and suchlike, 'modes'. I call a 'dictum' that whole occurring in the proposition in addition to the mode and copula and negations and signs or other determinations of the mode or the copula.[8] [15, Pg 2]

A composite modal proposition is one where the modal term occurs as either the predicate or the subject of an expression and the copula is not modalized. The following are all examples of composite modal proposition, 'That a man runs is possible,' 'It is necessary that a man is an animal.' Formally, we define a composite modal claim as a formulae of the form $\phi \circ \psi$ where one of ϕ, ψ is a mode, and the other is a basic formula.[9] In contrast to this, a divided modal claim is one where:

> Part of the dictum is the subject and the other part the predicate. The mode attaches to the copula as a determination of it.[10] [15, Pg 2]

In a divided modal claim neither of the terms are modal expressions. Instead it is the copula that is modalized. Formally, we represent this as $A\nabla \circ B$.

At the heart of this distinction is that the two different kinds of modal claims have different truth conditions, and are *not* generally speaking, equivalent to each other. Composite modal propositions are assertoric in nature, i.e. they contain a modal term, but the copula is not modalized and so the modal term does not range over the entire proposition that occurs as the second term in the composite proposition. In the case of an expression like "that a man runs' is

[6] Sed notandum est quod propositiones non dicuntur 'de necessario' aut 'de possibili' ex eo quod sunt possibiles aut necessariae, immo ex eo quod in eis ponuntur isti modi 'possibile' aut 'necessarium', et sic de alius modis.

[7] assuming that some men exist

[8] 'Compositae' uocantur in quibus modus subicitur et dictum praedicatur uel econuerso. Et uoco 'modos' istos terminos 'possibile', 'necessarium', 'contingens' et huiusmodi. Et uoco 'dictum' illud totum quod in propositione ponitur praeter modum et copulam et negationes et signa aut alias determinationes modi uel copulae.

[9] It should be noted at this point that if we were to use the falsum modality, we obtain a form of negation that is very close to what modern logicians would call (external) propositional negation. In modern logic we often read $\neg\phi$ as asserting that ϕ is not the case. In Buridan's term logic he can do something very similar. He can construct the composite modal claim, $\perp \circ \phi$ where $\perp$ is taken to be falsum.

[10] Sed 'diuisae' uocantur in quibus pars dicti subicitur et alia pars praedicatur. Modus autem se tenet ex parte copulae, tamquam eius quaedam determinatio.

possible' the possibility is modifying the proposition 'that a man runs.' In contrast to this. It is asserting that the proposition given by 'that a man runs' is possible.

While this distinction may at first glance seem somewhat foreign to the modern reader, it has an interesting parallel with the de re/de dicto distinction. While the two distinctions are not the same it can be helpful to keep this distinction in mind. In this case, divided modal propositions can be understood as modal claim de re, while composite modal expressions can be understood as de dicto modal claims. It should be stressed that the distinction between composite and divided propositions is a much broader kind of scope distinction then de re/de dicto, but the distinction is huristically helpful.

But the question still remains, when are divided and composite modal claims true? And besides the linguistic difference, what is the logical difference between them? To answer this question, Buridan needs to first distinguish postiive from negative propositions, since the truth conditions for these are different. Buridan defines when propositions are negative or positive in the follow way:

> [the first is where] negation occurs in the mode... [the second is where] the negation does not occur in the mode but follows it.[11] [15, Pg 3]

As an example of the first consider the proposition 'a man is not possibly an ass' while for the latter, consider the expression 'A man is possibly not an ass.' The difference has to do with where negation occurs in the proposition. This is further complicated, since grammatically, there are more places for the expression 'non' to occur in Latin then there are in English. English grammar makes it difficult to capture the three places in Latin that negation can occur. The negation can occur in front of the modal term which is modifying the verb, it can occur in front of the verb, or it can occur in front of the term. For Buridan a proposition is negative just in case either the modalized copula or the verb is negated. If the predicate is negated the term is affirmative.[12] In the following sections, we will refer to negative propositions using the standard, A is not B, or A is possibly not B etc, and when we intend the affirmative reading, we will use the construction A is non-B.

To complete his treatment of positive and negative propositions, Buridan defines the following equivalences:

> negation occurs twice, once [attached] to the mode, the other to the predicate... are really affirmatives, since they are equivalent to ones which are clearly affirmative.[13] [15, Pg 3]

Formally, we can state these equivalences as follows:

$$A\neg\Diamond\neg \circ B \leftrightarrow A\Box \circ B$$

$$A\neg\Box\neg \circ B \leftrightarrow A\Diamond \circ B.$$

[11] Aliae sunt negatiuae, et illae sunt duplices. Quaedam sunt in quibus negatio fertur in modum... Aliae sunt in quibus negatio non fertur in modum sed sequitur ipsum...

[12] The reason for this goes back to Aristotle. See chapter 9 of the Categories, especially 13b 26-35

[13] Aliae sunt in quibus ponitur duplex negatio, una ad modum, alia ad praedicatum... Et credo quod istae secundum ueritatem sunt affirmatiuae, quia aequipollent aliquibus manifeste affirmatiuis.

These equivalences show how modalized copula interact with terms. Informally, the first tells us that from the proposition A is not possibly not B, we may conclude A is necessarily B and vice versa. Likewise, A is not necessarily not B, is equivalent to A is possibly B. To the modern reader, there is little surprising here. These principles simply reflect the standard interdefinability of $\Box$ and $\Diamond$, only in a divided modal context.

2.0.5 Chapter 4: Ampliation and Supposition

From these distinctions and definitions Buridan moves on to discuss how ampliation works in divided modal expressions. Ampliation, as we discussed in our introduction, was a feature of the theory of supposition. Buridan's discussion of ampliation starts with possibility, he observes that

> A divided proposition of possibility has a subject ampliated by the mode following it to supposit not only for things that exist but also for what can exist even if they do not.[14] [15, Pg 3]

A few observations should be made at this point. First, the class of objects that possibility ranges over. For Buridan possibility includes more than the things that currently are. It also includes things that are not, but at some point can come into existence. This can be clearly seen in Buridan's remarks that

> it is true that air can be made from water, although this may not be true of any air which exists.[15] [15, Pg 4]

The point here is simply that something can be possible even if there are currently no objects that would instantiate both of the terms. Second, it follows from this that,

> The proposition 'B can be A' is equivalent to 'That which is or can be B can be A'.[16] [15, Pg 4]

The idea here is that when we evaluate the proposition 'B can be A' we do not only look at the objects currently are B. Instead we need to consider all the objects that can be B as well. When we say 'B can be A' we look at everything that is or can be B, and see if it can be A as well. From this, Buridan goes on to consider if the proposition

> 'That which is B can be A or that which can be B can be A'.[17] [15, Pg 4]

is equivalent to B can be A. Buridan rejects this equivalence, and to do so he offers the following counter-example: 'A creating God can fail to be God.' If we read this expression as saying, 'That which is a creating God or can be a creating God can fail to God', then this expression is false, since, as Buridan points out, its contradictory, namely, 'nothing which is or can be a creating God can fail to be God', is true. This is true because only God can be a creating God. However, if we follow the latter reading of B can be A, we get the following disjunction:

[14] propositio diuisa de possibili habet subiectum ampliatum per modum sequentem ipsum ad supponendum non solum pro his quae sunt sed etiam pro his quae possum esse quamuis non sint.

[15] sic est uerum quod aer potest fieri ex aqua, licet hoc non sit uerum de aliquo aere qui est.

[16] haec propositio: B potest esse A aequiualet isti: Quod est uel potest esse B potest esse A

[17] Quod est B erit A uel quod erit B erit A

That which is a creating God can fail to be God or that which is a creating God can fail to be God'
In this case, the first disjunct is true, given the assumption that God is not now creating. The reasoning is as follows: Assume that God is not creating. God is a term that only has one object in it. Since God is not creating the term 'that which is a creating God' is not satisfied by any object, since only God can be God, and per assumption, He is not creating. The expression 'that which is a creating God' is empty. Buridan treats this as being synonyms with 'nothing' and points out that nothing can fail to be God. Hence, the two propositions are not equivalent.

The use of amplition can naturally extended to temporal clauses. For example, the claim A will be B, is to be analyzed as ' that which is or will be A will be B.' Likewise for past-tense claims. It should be noted that the connection between temporal and modal expressions is not explicitly dealt with in this book. None of the consequences explicitly deal with the relationship between temporal and other types of modal claims.

2.0.6 Chapter 5: Equivalences

After Buridan's discussion of ampliation, he moves on to discuss equivalences between modal propositions. Here Buridan follows Aristotle as well as most medieval logicians by stating that

> 'necessarily' and 'impossibly not' are equivalent, and 'necessarily not' and 'impossibly' are also equivalent.[18] [15, Pg 5]

Buridan treats these as fairly obvious claims. To the modern reader this equivalence should be equally straight forward, noting that the standard formalization of 'impossible' is $\Box\neg\phi$. Then, given double negation elimination, it is fairly straight forward to show that $\Box\neg\neg\phi \leftrightarrow \Box\phi$ holds in any normal, classical modal logic. In addition Buridan also takes

> 'impossibly' and 'not possibly' to be equivalent, because a negation is implicit in the term 'impossibly'. So 'B is not possibly A' and 'B is impossibly A' are equivalent, and similarly, 'B cannot be A' and 'B is not possibly A', because 'can be' and 'is possibly' mean the same [19] [15, Pg 5]

This is equally straight forward and reflects the inter-definability of possibility and necessity. As well, Buridan treats the terms "can be" and "is possibly" as equivalent in the divided context.[20] The final set of 'equivalences' that Buridan lays down are the contradictories. Here

> I take it that a universal affirmative contradicts a particular negative, and a universal negative a particular affirmative in the same way, so that in the negative the negation governs the mode.[21] [15, Pg 6]

For example,

[18] quod aequipollent 'necesse esse' et 'impossibile non esse', et etiam aequipollent 'necesse non esse' et 'impossibile esse'

[19] 'impossibile' et 'non possibile', quondam in hoc nomine "impossibile" implicatur negatio. Et ideo istae aequipollent: 'B non possibile est esse A' et: 'B impossibile est esse A' et similiter: 'B non potest esse A' et: 'B non possibile est esse A' quia idem significat 'potest esse' et 'possibile est esse'.

[20] It should be observed that 'is possible' can also be used for composite modal claims where the mode is in the second term.

[21] Suppono etiam quod uniuersalis affirmatiua contradicit particulari negatiuae et uniuersalis negatiua particulari affirmatiuae de eodem modo, ita quod in negatiua negatio cadet super modum.

> Every B is possibly A' contradicts 'Some B is not possibly A' and similarly 'No B can be A' and 'Some B can be A.'[22] [15, Pg 6]

In the case of both examples, these are the usual contradictions within the Aristotelian square of oppositions augmented with the modal terms already defined. In the first example, the expressions can be formalized as saying: $\forall A \Diamond \circ B$ and $\exists A \neg \Diamond \circ B$ are contradictory.[23]

These are all of the assumptions and definitions that Buridan lays down concerning equivalences of divided modal propositions. Before moving on to state a number of conclusion that follow from these definitions, he makes one important final assumption:

> Further, in the sixth chapter, it will be assumed that modal propositions can be expressed without any restriction of the subject by 'that which is' or 'that which was', and suchlike, e.g., 'Every B can be A', 'Every C is necessarily D', and so on. But sometimes they can also be expressed by adding 'that which is' or 'that which was' or 'that which can be' and suchlike to the subject. For example, 'That which is B can be A'[24] [15, Pg 7]

As we discussed within the context of ampliation and supposition there are certain expressions that can restrict the type of objects over which the first term of a given proposition supposit. In this case, expressions of the form 'that which is', 'that which was' etc do exactly this. An expression of the form 'every B can be A', says of every B, regardless of when or if it exists in the present, it is possibly[25] A. In other words, the proposition covers those objects both are, and can be B. With the addition of the expression 'that which is', 'that which was' etc, we restrict the range over which the term is ampliated. So, in the case of 'That which is B can be A' the ampliation ranges over the things that are B now, not over all B's at all times or places. In this case, it's as though the ampliation of the expression 'possible' is ignored, and we only look at the present when considering the first term. The modern analogy would be seeing this as a kind of restriction on the domain and worlds over which a given modal operator ranges. In the case of 'that which is B can be A', we restrict our evaluation of the claim to range over only those objects which currently exist and are B. We then evaluate from that world, if they can be A. Likewise, if we were to evaluate 'that which was B is necessarily A' we would go back in time to a specific point (normally given by the context of the proposition) and from that point, see if in every accessible world, those which are (at that point in time) B's are also A's.

It should also be noted that the expression 'that which' need not only range over temporally dependent clauses.

> Similarly, we can say 'That which can be B is necessarily A', or 'That which is necessarily B is contingently A', and so on.[26] [15, Pg 6]

[22] istae contradicunt: Omne B possibile est esse A et: Quoddam B non possibile est esse A similiter istae duae: Nullum B potest esse A et: Quoddam B potest esse A.

[23] justification for this will be provided when we prove the contradictories in chapter 3

[24] Ulterius, in sexto capitulo, supponendum est quod propositiones modales possum proponi sine aliqua restrictione subiecti per 'quod est' uel 'quod fuit', et huiusmodi, ut: 'Omne B potest esse A', 'Omne C necesse est esse D' et sic de aliis. Aliquando etiam possum proponi addendo ad subiectum 'quod est' uel 'quod fuit' uel 'quod potest esse', et huiusmodi. Verbi gratia: Quod est B potest esse A

[25] recall Buridans equivalence above

[26] Similiter potest dici: 'Quod potest esse B necesse est esse A' uel: 'Quod necesse est esse B contingit esse A' et sic de aliis.

When we make use of the operation ‘that which’ it’s really a way of making explicit the scope of the modal operation in question. When the expression ’that which’ is absent, Buridan in essence tells us to interpret it as broadly as possible, covering the objects that are B and can be B. When we use the expression ’that which’ we restrict the scope over which object we consider.

2.1 Divided Modal Propositions

Having drawn this distinction, Buridan goes on to prove a number of conclusions about divided modal propositions that are not restricted.

First Conclusion

> From any proposition of possibility there follows as an equivalent another of necessity and from any of necessity another of possibility, such that if a negation was attached either to the mode or to the dictum or to both in the one it is not attached to it in the other and if it was not attached in the one it is attached in the other, other things remaining the same.[27] [15, Pg 7]

This follows from his definition of when a divided modal term is negative, along with his equivalences between modal propositions. For example, if we have an proposition of the form, B is necessarily A, we can form another claim of possibility, by observing that necessarily is equivalent to impossibly not, and that impossibly is equivalent to not possible, we obtain the expression, A is not possibly not B, which is clearly a claim of possibility. Recall that this is a claim of possibility because the mode ‘possible’ occurs in the proposition.

After giving a number of examples of this form, Buridan goes on to note another important property of the definitions that he has given.

> If something was established for [propositions] of possibility having an affirmed mode it will be established for those of necessity having a negated mode since these are equivalent other things remaining the same, as was shown. Similarly, if something was established of those of necessity having an affirmed mode it will be established of those of possibility having a negated mode, since these are also equivalent, as was said. [28] [15, Pg 7]

In essence, he is saying that if we have proven a result for a given proposition either of possibility or necessity, then the result will also hold for all expression that are equivalent to the original expression. This is easily seen. Assume that we have proven such a proposition, then by the argument given above, we can transform that proposition into its equivalent form. Pasting these two proofs together yields the desired conclusion. Because of this:

[27] Ad omnem propositionem de possibili sequi per aequipollentiam aliam de necessario et ad omnem de necessario aliam de possibili, sic se habentes quod si fuerit apposite negatio uel ad modum uel ad dictum uel ad utrumque in una non apponatur ad illud in alia et si non fuerit apposite in una apponatur in alia, aliis manentibus eisdem.

[28] Ex dictis inferendum est correlarie quod si determinatum fuerit de illis de possibili habentibus modum affirmatum sufficienter erit determinatum de illis de necessario habentibus modum negatum, quia illae aequipollent caeteris manentibus eisdem, ut apparuit. Et consimiliter si fuerit determinatum de illis de necessario habentibus modum affirmatum erit determinatum de illis de possibili habentibus modum negatum, quia illae etiam aequipollent, ut dictum est.

> We will only establish [conclusions] for those of possibility and of necessity which have an affirmed mode; when we speak of [propositions] of possibility or of necessity we will always understand [those] having an affirmed mode.[29] [15, Pg 7]

Second Conclusion

> In every divided proposition of necessity the subject is ampliated to supposit for those which can be.[30] [15, Pg 7]

To see this remember the equivalence that Buridan laid down concerning ampliation and supposition. He did not provide a treatment of necessity, only possibility. This conclusion is the reason why he did not. It follows from what he has already laid down. Buridan says that the conclusion is clear, and offers a justification by reductio. He points out that if this were not so then the equivalences laid down in chapter 5 would not hold.

> For otherwise those of necessity would not be equivalent to those of possibility having a negated mode, since in those of possibility the subject is clearly granted to be so ampliated[31] [15, Pg 8]

As an illustration of this, consider the expression 'every A is necessarily B'. By the equivalences laid down in chapter 5 this is equivalent to 'every A is not possibly not B.' In this case A is ampliated for all things that are B and can be B, per Buridan's definition of 'possible' given in chapter 4. Now, suppose that the second conclusion is false, and instead it only supposits over those things which are A. Then we can construct a counter-example to the proposition. Consider the following situation: Every turkey that currently exists is necessarily green but there could be a turkey that is not green. Then the proposition 'every turkey is necessarily green' is true, since every turkey that currently exist is necessarily green, and necessity does not ampliate for that which can be a turkey. However, the proposition some turkey is possibly not green is also true, since here turkey ampliates for all those things which are and can be turkeys, and per our assumption, there could be some turkey that is not green. This means that every turkey is not possibly not green, is false. And this contradicts the equivalence between necessity and possibility that Buridan laid down. Given this analysis the following claims are true,

> 'Something creating is of necessity God' and 'Something creating is necessarily God' are true, even on the hypothesis that God is not now creating.[32] [15, Pg 8]

To see this notice that these statements are equivalent to 'Something creating is or can of necessity be God' and 'something creating is or can necessarily be God'. In both cases, since God can be creating, and God is of necessity God the claims follow as desired. The claims will also follow if the term is a universal, since it is possible that God is the only thing creating, and he is

[29] Ideo solum determinabimus de illis de possibili et de necessario quae habent modum affirmatum; et quando de caetero loquemur de illis de possibili uel de necessario semper intelligemus de habentibus modum affirmatum

[30] In omni propositione de necessario diuisa subiectum ampliatur ad supponendum pro his quae possum esse.

[31] Quia aliter illae de necessario non aequipollerent illis de possibili habentibus modum negatum, cum in illis de possibili subiectum manifeste concedatur sic ampliari

[32] 'Creans de necessitate est deus' uel: 'Creantem necesse est esse deum' licet ponatur casus quod deus non sit modo creans.

necessarily God. So, the universal, everything creating is of necessity God, is true. As Buridan points out, it is VERY important to keep this understanding in mind when discussing modal claims, since one may be tempted to analysis the previous claim as a conjunction of individuals i.e. take every member in the class of creating things, and say for each of them, x is of necessity God, and y is of necessity God. . . . As Buridan points out this is clearly false whenever x,y etc are not God.

Third Conclusion

Buridan's third conclusion is that

> from no proposition of necessity does there follow an assertoric or vice versa, except that from a universal negative of necessity a universal negative assertoric follows.[33] [15, Pg 8]

To see that an assertoric does not entail a proposition of necessity, Buridan reminds us that

> an assertoric can describe a contingency of which that of necessity is not true. For example, 'Every, or some, man necessarily runs' does not follow [p.65] from 'Every man runs'[34] [15, Pg 8]

The other direction however, is more interesting. First, it is important to keep in mind that Buridan is dealing with *unrestricted* divided modal propositions.[35] As a counter-example to the second, he offers the following theological example

> Everything creating is necessarily God, so the one creating is God.'[36] [15, Pg 8]

The first expression is true regardless of whether God is creating right now or not. Again, this is because of ampliation since 'Everything creating is necessarily God' is equivalent to 'everything that is or can be creating is necessarily God.'[37]. This is clearly true since God can be creating. However, it does not follow from this that the one currently creating is God, since for this assertoric to be true, God needs to be creating at this moment. Given that there are times when God is not creating, the assertoric is false at those times, and so does not follow. While Buridan uses a theological counter-example for this, it is important to realize that there are also non-theological ones that are equally illuminating. In fact, Buridan gives the following example from medieval astronomy

> 'Some planet shining on our hemisphere is necessarily not the Sun, so some planet shining on our hemisphere is not the Sun' [38] [15, Pg 8]

[33] Ad nullam propositionem de necessario sequi aliquam de inesse uel econverso, praeter quod ad uniuersalem negatiuam de necessario sequitur uniuersalis negatiua de inesse.

[34] quia illa de inesse potest esse in materia contingenti, ubi non est uera illa de necessario. Verbi gratia, non sequitur: Omnis homo currit; ergo omnem (uel 'aliquem') hominem necesse est currere

[35] He points out that if the expression is properly restricted then the move from necessity to assertoric is sound.

[36] Omne creans necesse est esse deum; ergo creans est deus'

[37] This follows from conclusion two

[38] Quendam planetam lucentem super nostrum hemisphaerium necesse est non esse solem; ergo quidam planeta lucens super nostrum hemisphaerium non est sol

In both cases the lack of equivalence is due to the way terms supposit in divided modal expressions. In the case of the first expression

> the reason is that the subject in the [proposition] of necessity supposits for other things besides the Sun, because of the ampliation.[39] [15]

In other words, in the first expression, the clause 'planet shining on our hemisphere' ranges over many objects that are not the sun and depending on the point in time, the object that causes this to be true changes. However, in the case of the second expression, that shift in object being refered to cannot happen. There must be some planet, now, which is shining on our hemisphere, but is not the sun. But this is not true (at least according to medieval astronomy) all of the time. However, there is one case where the conclusion is a good one

> It is clear that 'Every B is necessarily not A, so no B is A' is a good consequence.[40] [15, Pg 9]

The reason this is a good conclusion has to do with how terms supposit in universal negative propositions. In such a proposition, the B clause is distributed over everything that falls under B. Hence:

> In the first B is distributed for everything for which it is distributed in the second, and perhaps also for more besides. [41] [15, Pg 9-10]

The idea here is that in the first proposition B may have more elements fall under it then it does in the second clause, but it will include all of the B's that are treated in the second clause. So it still follows that:

> If the first is true of everything, the second will also be true of everything.[42] [15, Pg 10]

And this covers the case where B is non-empty. If B does not supposit for anything in either

> then the cause of truth will be the same for both; so neither will be true without the other. But if B supposits for something in the first but for nothing in the second, then the second will be true, whether the first is true or false. So no situation can be given in which the first is true while the second is false.[43] [15, Pg 10]

In the case where B does not supposit for anything, the proposition will be true but vacuously. In the case where B does supposits for something in the first proposition i.e. 'Every B is necessarily not A' but does not suposit in the second, then the second proposition again is vacuously true, and so no counter-example can be constructed. From the absence of such a counter-example, Buridan concludes that the claim is proven. In essence what Buridan has done is proven his claim by cases. He has shown that, no matter how one would proceed, the claim will always be true and hence the conclusion is proven.

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[39] Et causa est quia subiectum in illa de necessario supponit pro aliis a sole, propter ampliationem, et in illa de inesse non supponit pro aliis.

[40] Omne B necesse est non esse A; ergo nullum B est A manifestum est

[41] in prima distribuitur B pro omnibus pro quibus in secunda distribuitur, et forte etiam cum hoc pro pluribus.

[42] ideo si pro omni est uera prima, pro omni etiam erit uera secunda

[43] tunc eadem erit causa ueritatis utriusque; ideo neutra erit uera sine alia. Si autem B supponit pro aliquo in prima et pro nullo in secunda, tunc secunda erit uera, siue prima sit uera siue falsa. Ideo nullus casus potest dari in quo prima sit uera secunda existente falsa

Fourth Conclusion

> From no proposition of possibility does there follow an assertoric or vice versa, except that from every affirmative assertoric proposition there follows an affirmative particular of possibility.[44] [15, Pg 10]

The claim here is twofold. First Buridan claims that there are no propositions of possibility that entail an assertoric proposition, and second, that only affirmative assertoric propositions entail particular affirmatives of possibility. Buridan proves this conclusion in two ways. The first proof is by using the contradictory properties of the square of opposition.

> The conclusion is proved by the fact that affirmative particulars of possibility contradict universal negatives of necessity, and universal affirmatives particular negatives, and similarly universal negatives [p.66] of possibility affirmative particulars of necessity and particular negatives universal affirmatives. Between assertorics and those of necessity, except universal negatives of necessity, there are no consequences so neither will there be between their contradictories, which are all the assertorics and all those of possibility, except a particular affirmative of possibility.[45] [15, Pg 10]

The proof rests on the results of the previous conclusion. Buridan points out that particular propositions of possibility contradict universal negatives of necessity, and universal affirmatives contradict particular negatives etc. These contraditions all follow from the definitions given in chapter five of this book. If we consider the contradictories, we know from the previous conclusion that from a conclusion of necessity, we cannot conclude that the assertoric is true, except in the case when we are dealing with a universal negative proposition. Taking this, togeather with the Buridan's observation in conclusion one, that

> If something was established for [propositions] of possibility having an affirmed mode it will be established for those of necessity having a negated mode [15]

The conclusion follows.

Buridan's second way of proving the claim proceeds as follows,

> It clearly follows if some B is A that it can be A. It is also clear that 'Every white thing can be black, so a white thing is black' does not follow, nor does 'Everything running can fail to be running, so everything running is not running' follow. Nor does 'Something creating is not God, so something creating can fail to be God' follow, because if God is not now creating, the first is true and the second false.

[44] Ad nullam propositionem de possibili sequi aliquam de inesse uel econtra, praeter quod ad omnem propositionem affirmatiuam de inesse sequitur particularis affirmatiua de possibili.

[45] Conclusio probatur per hoc quod particulares affirmatiuae de possibili contradicunt uniuersalibus negatiuis de necessario et uninersales affirmatiuae particularibus negatiuis, et similiter uniuersales negatiuae de possibili particularibus affirmatiuis de necessario et particulares negatiuae uniuersalibus affirmatiuis. Sed inter illas de inesse et illas de necessario, excepta uniuersali negatiua de necessario, nulla est consequentia; ergo nec erit inter contradictorias earum, quae sunt omnes de inesse et omnes de possibili, praeter particularem affirmatiuam de possibili.

> What went before can be confirmed by this conclusion since they mutually convert by the third conclusion of Book I.[46] [15, Pg 10]

The inference pattern here is the following. Take an object that is both B and A. Then clearly, it is or can be B and it can be A, because it already is B and A. Hence, it follows from $B \circ A$ we can conclude that $B\Diamond \circ A$. Buridan's examples all rest on the idea that if something is currently the case, then it is also possible. In modern propositional logic this is schematically represented as $\phi \rightarrow \Diamond\phi$, which is easily recognizable as one form of T-axiom of modal logic.

To the see the second part of the proposition holds, Buridan invokes counter-examples. He shows that there are situations where the first clause is true and the second false. As an upshot of this discussion Buridan also notices that

> It should also be realized that from every proposition of necessity there follows a proposition of possibility...But since this is self-evident, I have not called it a Conclusion. [15, Pg 10]

The inference itself is fairly straight forward. If something is necessary, then it could also be the case. Buridan treats this claim as self-evident and clear to his readers. A modern reader might think that this is equally obvious; assuming that what Buridan is developing here is a kind of modern propositional or quantified modal logic. We already noted that to the modern logician, Buridan is appealing to something that looks like the T axiom. It is easily shown in modern propositional logic that T entails the validity of D axiom, which states that $\Box\phi \rightarrow \Diamond\phi$. However, it is important to realize that Buridan's modal logic is *not* a propositional nor a quantified modal logic. It is a term based modal logic and because of this, things become much more complicated. In part, this is because there are more ways for negation and modalities to interact. It is also due to the disjunctive reading of the modal operation that Buridan uses.

This points to an important dissimilarity between Buridan's treatment of divided modal propositions and the two schemas of modal logic we have just referenced. In propositional modal logic the formula, $\phi \rightarrow \Diamond\phi$ is equivalent to the claim $\Box\phi \rightarrow \phi$ Translating this into the term logic that Buridan is using, we obtain the following equivalence,
$A \circ B \rightarrow A\Diamond \circ B$ is equivalent to the claim $A\Box \circ B \rightarrow A \circ B$[47] However, Buridan argues that while the inference from $A \circ B$ to $A\Diamond \circ B$ holds, the inference $A\Box \circ B \rightarrow A \circ B$ only holds when it is of the form $\forall A\Box\neg \circ B$[48] This raises two important questions, first, if Buridan is invoking something like the T-axiom, has he simply made a logical mistake here? The second, and related question is how does Buridan's analysis of when an expression is true fit into this question? These two questions are important to ask for a number of reasons that will come into sharper focus in chapters four and three of this thesis respectively. Traditionally, when modern logicians and historians of logic have noticed 'errors' of this kind, they state that the author has simply made a logical mistake and gone on to correct the system so as to be consistent with modern formal logic. This is perhaps best seen in Łukasiewiczs treatment of Aristotle's modal syllogistic[7], but has occurred elsewhere in the history of logic. This is not the approach that

[46] quia manifeste sequitur si quoddam B est A quod ipsum potest esse A. Manifestum est etiam quod non sequitur: 'Omne album potest esse nigrum; ergo album est nigrum' nec sequitur:'Omne currens potest non esse currens; ergo currens non est currens' Nec etiam sequitur:'Quoddam creans non est deus; ergo quoddam creans potest non esse deus' quia si deus non creat modo prima est uera et secunda falsa. Et per hanc conclusionem potest confirmari praecedens, quia conuertuntur ad inuicem per tertiam conclusionem primi libri.

[47] We also have the corresponding instances for the universal and negative propositions.

[48] Every A is necessarily not B, or No A is necessarily B.

we will adopt in this work. The problem with this approach is that, a) it assumes the author we are interpreting has the same idea of what modal logic is, and how the various operations are to be understood. b) it does not offer a very good explanation of why the author believed these two propositions to be dissimilar. c) in simply writing this off as a mistake, we miss an opportunity to better understand the differences between medieval and modern approaches to modal logic. d) It makes the logical model that we develop substantially less interesting then it could have been. Points a through c will be taken up in our philosophical reflection on what makes a model of a given historical figure's logic a good one. Point d will be addressed when we present our formalization of Buridan's logic. What we will see is that Buridan's term-based logic is quite different from modern propositional modal logic, and that this non-equivalence can be easily preserved.

Fifth Conclusion

> From every affirmative of possibility there follows by conversion of the terms a particular affirmative of possibility, but not a universal, and from no negative of possibility does there follow by conversion of the terms another of possibility. [49] [15, Pg 11]

Buridan is laying out the conversion rules for modal terms. He is exploring when an expression of the form $A\nabla \circ B$ can be converted to $B\nabla \circ A$ for each of the four possible propositions. The first consequence he proves is that

> From every affirmative of possibility there follows by conversion of the terms a particular affirmative of possibility.[50] [15, Pg 11]

He proves this claim using exthesis.[51] The proof is as follows: Assume that B can be A. Now, pick some C such that this is or can be B and that it can be A. Clearly, C can be B, since if C is B, then it follows by the fourth conclusion that C can be B. From this it follows that, 'that which can be A, (namely C) can be B', as we have just shown. Now, it remains to show 'that which can be A can be B' entails 'A can be B.' To prove this, Buridan makes implicit use of the fact that if the scope of supposition is the same for A, then it covers the same objects, and hence, the two propositions are equivalent. To see that the two terms have the same scope of supposition, Buridan notes that A supposits at least as broadly in 'that which can be A can be B' as it does in 'A can be B'. This makes sense, since 'that which can be A can be B' ranges over all the objects that can be A. This includes those objects which are now A, as we have already seen in Conclusion four.[52] When the 'that which' clause is dropped we have the normal ampliation of the expression 'possible.' i.e. it ranges over that which is or can be A.

[49] Ad omnem affirmatiuam de possibili sequi per conuersionem in terminis particularem affirmatiuam de possibili, sed non uniuersalem, et ad nullam negatiuam de possibili sequi per conuersionem in terminis aliam de possibili.

[50] Ad omnem affirmatiuam de possibili sequi per conuersionem in terminis particularem affirmatiuam de possibili

[51] For those not familiar with this proof technique, Ekthesis is an idea taken from Aristotle's Prior Analytics. It involves making a term based assumption and 'picking' one such instance of the assumption, and then proving the claim.

[52] Alternatively, we could look at this as saying that the clause 'that which could be' is redundant in a proposition.

Given this, it is quite clear to see that from a universal of possibility, a particular of possibility follows by conversion. I.e. From $\forall B \Diamond \circ A$, it follows that $\exists A \Diamond \circ B$. To prove this Buridan argues as follows. By the subaltern we have that $\exists B \Diamond \circ A$, and from what Buridan has shown in the previous paragraph, it follows that $\exists A \Diamond \circ B$. It is also easy to see that a universal of possibility does not convert. Buridan leaves the exercise to his reader, observing that it is exactly the same as in the case of assertorics. To prove the second part of this conclusion Buridan offers the following counter-example:

> Every God can fail to be creating, so something creating can fail to be God [53] [15, Pg 11]

The second clause clearly does not follow from the first, since Every God can fail to be creating is true. Just think of a time when God is the only thing that exists, and he is not creating. However, at that time, God cannot fail to be God. Since He is the only thing that exists, the second proposition is false

Sixth Conclusion

> From no proposition of necessity does there follow by conversion of the terms another of necessity, except that from a universal negative there follows a universal negative.[54] [15, Pg 11]

The first part of the claim is illustrated by the following counter-example:

> Everything creating is necessarily God, so God is necessarily creating.[55] [15, Pg 11]

In this case, the first premise is true because of the same reasoning that Buridan used to prove the claim 'something creating can fail to be God' in conclusion five. However, the second proposition is clearly false, since there are times at which God is not creating. For the particular negative Buridan uses the example

> Some animal necessarily fails to be an ass, so some ass necessarily fails to be an animal.'[56] [15, Pg 11]

The first again, is clearly true, since a cow is not a donkey. However, the second proposition is false, and according to many medievals, an impossible conclusion, since every ass is necessarily an animal. The second part of Buridan's conclusion, that:

> From a universal negative there follows a universal negative. [57] [15]

This follows by transposition of the claim and the previous conclusion. This follows because the contrapositive of a universal negative is a particular affirmative. Hence the proposition becomes:

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[53] Omnis deus potest non esse creans; ergo creans potest non esse deus

[54] Ad nullam propositionem de necessario sequi per conuersionem in terminis aliam de necessario, praeter quod ad uniuersalem negatiuam sequitur uniuersalis negatiua.

[55] Omnem creantem necesse est esse deum; ergo deum necesse est esse creantem

[56] Quoddam animal necesse est non esse asinum; ergo quendam asinum necesse est non esse animal

[57] praeter quod ad uniuersalem negatiuam sequitur uniuersalis negatiua

> Some A can be B, so some B can be A'[58] [15, Pg 11-12]

Which follows from what was shown in conclusion 5. As far as this conclusion is concerned, the addition of the expression 'that which' will create additional valid arguments. In the first case, the claim

> Every affirmative of necessity can be converted by use of 'that which' in the result of the conversion. For example, 'Something creating is necessarily God, so that which is necessarily God is or can be creating' does follow. [59] [15, Pg 12]

The conversion hold because of the presence of the 'that which' clause. These clauses change the ampliation of terms in the expression to range over fewer objects. Using Buridan's example, the inference from something creating is necessarily God, to God is necessarily creating, is invalid, since God could not be creating. However, the inference from something creating is necessarily God, to that which is necessarily God is or could be creating, is good, since there are times at which God was creating. The second case that Buridan points out is that,

> Affirmatives of necessity can be converted into an affirmative of possibility.[60] [15, Pg 12]

As an example of this Buridan argues that from B is necessarily A, we may conclude that B can be A, on the basis that

> From the [proposition] of necessity that of possibility follows.[61] [15, Pg 12]

To see this Buridan offers the following argument:

> For example, 'B is necessarily A, so B can be A' follows; from there, by conversion, 'A can be B'; so, from first to last, etc.[62] [15, Pg 12]

The proof goes as follows: Assume that B is necessarily A, then clearly, B can be A. While Buridan does not spell out the exact justificaition for this, it is easily seen to follow from what he laid down in terms of ampliaiton of modal terms. Assume that B is necessarily A, then that which is or can be B is necessarily A, hence there is at least one possible way for B to be A. Hence B is possibly A. Given this, it follows by conclusion five that A can be B. The case for the universal is essentially the same.

To quickly highly another connection with modern logic, a modern logician could see this as saying that positive propositions are closed under the D axiom of modal logic.

[58] Quoddam A potest esse B; ergo quoddam B potest esse A

[59] Tamen sciendum est quod omnis affirmatiua de necessario potest conuerti secundum resolutionem conuertentis per 'quod'. Verbi gratia, sequitur: 'Creantem necesse est esse deum; ergo quod necesse est esse deum est uel potest esse creans'

[60] quod affirmatiuae de necessario possum conuerti in affirmatiuam de possibili.

[61] B necesse est esse A; ergo B potest esse A

[62] Verbi gratia, sequitur: B necesse est esse A; ergo B potest esse A deinde per conuersionem: ergo A potest esse B ergo, de primo ad ultimum etc.

Seventh Conclusion

> Every proposition of each way-contingency having an affirmed mode is converted into [one of] the opposite quality with an affirmed mode, but none is converted if the result of conversion or what was converted had a negated mode.[63] [15, Pg 12]

The expression 'way-contingency'[64] is synonymous with what is sometimes referred to as contingency. A proposition is contigent just it can is it possible that it be true, and possible that it be false. Formally, we write this as $\phi \diamond \circ\neg\psi \wedge \phi \diamond \circ\psi$

In this case Buridan first argues that every affirmed contingent mode can be converted. This is immediately clear, since if something is contingent, it can be true and it can be false, hence 'Every B is contingently A' is equivalent to 'Every B contingently fails to be A'. I.e. If every B can be A contigently, i.e. every B is possibly A and every B is possibly not A, then this is the same as saying that B can fail to be A and B cannot fail to be A.[65]. Buridan's second point is that

> It is contingently, so it is not contingently does not follow, indeed, they are opposed.[66] [15, Pg 12]

The distinction is important, and has to do with where the negation occurs in the expression. The expression 'not contingently' negates the contingency of the expression. For example, A is not contingently B means that either 'B is necessarily A or B is necessarily not A.' This is clearly contradictory to the claim that it is possible that B be A and it is possible that B not be A. From this Buridan points out that the expressions

> For 'Every B is contingently A' and 'No B is contingently A' are contraries.[67] [15, Pg 12]

These two propositions cannot both be true at the same time. If every B is contingently A, then every B is possibly A and every B is possibly not A. However, if no B is contigently A, then it follows that No B is possibly A, and no B is possibly not A. To see that these are contraries, notice that there is no situation where the can both be true. If no B is contingently A, then every B is either necessarily not A or every B is necessarily A.[68] In either case, we have a contradiciton with the assumption that every B is possibly A. Given this, it is clear why:

> So it is well said that contingency excludes necessity and impossibility.[69] [15, Pg 11]

This follows from the definition of contingency, and when modal propositions are contradictory. It is also immediately clear from this that

> Whence from every proposition of necessity having an affirmed mode there follows a proposition of contingency having a negated mode.[70] [15, Pg 11]

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[63] Omnem propositionem de contingenti ad utrumlibet habentem modum affirmatum conuerti in oppositam qualitatem de modo affirmato, sed nullam sic conuerti si conuertens uel conuersa fuerit de modo negato.

[64] de contingenti

[65] not fail here would be rendered as a double negative reducing to B can be A

[66] Quoddam B contingit esse A; ergo quoddam B contingit non esse A

[67] Istae enim sunt contrariae: Omne B contingit esse A et: Nullum B contingit esse A

[68] This will be made clear in Chapter three. The term 'no' introduces a negation in front of the modalized copula.

[69] bene dicitur quod contingens excludit necessarium et impossibile

[70] Unde ad omnem propositionem de necessario habentem modum affirmatum sequitur propositio de contingenti habens modum negatum.

Eigith Conclusion

Buridan's eighth and final conclusion dealing with divided modal propositions highlights conversion relationships between contingent propositions. He states

> No proposition of contingency can be converted in terms into another of contingency, but any having an affirmed mode can be converted into another of possibility.[71] [15, Pg 12]

He shows the first claim is false by counterexample. Consider

> 'God is contingently creating, so something creating is contingently God.' [72] [15, Pg 12]

The first proposition is seen by Buridan as true. The idea is that there are times when God could not be creating, and there are times when he is creating. The second expression is false, since nothing can contingently be God. This example also works in the negative case.

> 'Nothing creating is contingently God, so no God is contingently creating.'[73] [15, Pg 12]

The reasoning again is the same.

In the case of the second part of this conclusion, Buridan only sketches the argument. He states that

> But the second part of the conclusion is clear from the fact that from any [proposition] of contingency having an affirmed mode there follows an affirmative of possibility, which is converted into another of possibility. So from first to last...[74] [15, Pg 13]

The expression 'from first to last' can be thought of as a kind of transitivity of hypotheticals. The idea is that you lay down a chain of hypotheticals, where the consequent of the first is the antecedent, and repeat this as necessary. Then, 'from first to last', you conclude that the antecedent of the first hypothetical entails the consequent of the last hypothetical. The proof of Buridan's propositions starts from the proposition A is contingently B. Then, by what was said in conclusion seven, it follows from this that A is possibly B. Hence, by conclusion four it follows that B is possibly A.[75]

At this point Buridan ends his conclusions about divided modal expression. Now one may very well ask, why does Buridan end with this claim? Buridan is building up his arguments and system so that he can offer a full treatment of modal syllogisms.[76]To this end, Buridan's conclusions can be seen as a collection of lemmas that will be applied in the proofs of various syllogisms. Also, the rules and principles presented here are useful for dealing with modal propositions as they occure in other medieval sciences. As is clear from the prolific use of theological examples, Buridan's reasoning has applications to theology.

[71] Nullam propositionem de contingenti posse conuerti in terminis in aliam de contingenti, sed omnem habentem modum affirmatum posse conuerti in aliam de possibili.

[72] Deum contingit esse creantem; ergo creantem contingit esse deum

[73] Nullum creantem contingit esse deum; ergo nullum deum contingit esse creantem

[74] Sed secunda pars conclusionis ex hoc patet quia ad quamlibet de contingenti habentem modum affirmatum sequitur affirmatiua de possibili, quae conuertitur in aliam de possibili. Ideo, de primo ad ultimum etc.

[75] In this case the two conclusions are the intermediary hypotheticals

[76] A full treatment of Buridan's modal syllogisitic is outside the scope of the present work.

2.2 Chapter 7: Composite Modal Propositions

Buridan now moves on to treat composite modal propositions. It should be recalled that a composite modal proposition is an expression where the modal term occurs as one of the terms in the statement, and not with the copula. While the proposition 'A can be B' is a divided modal proposition, a composite modal proposition is something of the form "A is B' is possible', or 'it is necessary that 'A is B"'. Again, universal and indefinite expressions are formed in the same way. For example:

> 'Every possibility is that 'B is A" is universal and 'It is a possibility that 'B is A" is indefinite, and similarly one can say 'some possibility' or 'this possibility'.[77] [15]

It is important to realize that possibility will function differently in these types of modal propositions.

> Here 'possibility' is taken not for what can be but for a possible proposition, which is said to be possible in so far as things can be altogether as it signifies. In the examples above, saying 'Every possibility is that B is A' is the same as to say 'Every possible proposition is that B is A'. If this noun 'possibility' were used in any other way, there would be an equivocation...The same must be said of the other modes.[78] [15, Pg 13]

The reason for this difference is because of what a composite modal proposition is. In the case of divided modal propositions, the modal operations ampliated over terms. In composite propositions, the modal operator is a term, and it is interacting with a proposition. A proposition is considered possible just in case things can happen in the way the proposition says it can. For example the expression 'it is possible that 'some man is running" is true just in case there could be some man running. The expression 'ever possibility/necessity' here is synonymous with every possible/necessary proposition. Again, this is different not only beacuse the modality ranges over propositions, but also because the modality is not bound to the copula. In the divided modal proposition $A\Diamond \circ B$, the copula is modalized. In the composite modal proposition $\Diamond \circ (A \circ B)$, the possibility occurs as a subject and the copula remains unmodalized. In composite modal propositions the supposition is material and the expression 'A is B' supposits for the proposition A is B. The reason it's material, is because the term 'A is B' is an utterance or written expression, and those are not propositions. This is slightly clearer in the Latin, where Buridan uses an accusative construction to expression the proposition. To take Buridan's example 'Omne possibile est B esse A,' here B and A will be in the accusative case, so that the expression 'A esse B' can be the grammatical object of the verb to be. The second instance of the verb 'to be' needs to be infinitive for exactly the same reasons. This kind of modality should be more familiar to our modern reader, since here, the idea is that the modality ranges over an entire proposition.

Buridan goes on to consider the various ways these expressions can be formed. In the first case,

[77] haec est uniuersalis: 'Omne possibile est B esse A' et haec indefinita: Possibile est B esse A et ita possum dicere 'quoddam possibile' uel 'hoc possibile'.

[78] Et capitur hic 'possibile' non quia possit esse sed pro propositione possibili, quae ex eo dicitur 'possibilis' quia qualitercumque significat ita potest esse. Unde in proposito idem ualet dicere: 'Omne possibile est B esse A' sicut dicere: 'Omnis propositio possibilis est B esse A' Et si aliter caperetur hoc nomen 'possibile', esset aequiuocatio et non uocaretur talis propositio 'de possibili'. Et consimiliter est dicendum de aliis modis.

> Now if the mode is the predicate and the dictum is the subject, the proposition can still be universal, particular and so on.[79] [15, Pg13]

In this case, expressions of the form 'everything is 'B can be A' is possible, is an example of a universal proposition, while something of the form 'every B can be A' is possible, is indefinite. The difference in these two propositions has to do with where the quantifiers occur in the term. This is easier to see if we make use of our formal language. Buridan's first example is of the form $\forall`\forall A\Diamond \circ B' \circ \diamond$ This is why in our formation sequence we make use of the ' ' over the first term. An example of the second proposition is $`\forall B\Diamond \circ A' \circ \Diamond$. What we see is that the first proposition is of the form $\forall B \circ A$, while the second is of the form $B \circ A$. When this is done in natural language, it is very important to sharply separate these two kinds of propositions. The ambiguity between these two expressions has to do with the scope of the expression 'every.' In the first case it is ranging over propositions, i.e. $\forall A\Diamond \circ B$. In the second case, it is part of one of the basic formulae. Because modern English makes use of quotation marks '', we can unambiguously deal with the scope of these propositions. However, in the Lain, the expressions look much more alike, and so confusion could have easily abounded.

Ninth Conclusion

Having noted this ambiguity, Buridan proceeds to lay down a number of conclusions about these modal propositions.

> In all composite modals in which the dictum is subject, from a particular there follows a universal, the rest being unchanged.[80] [15, P 14]

From the proposition, 'some proposition 'A is B'' is possible it follows that, 'every proposition 'A is B' is possible'' To see this, Buridan points out that

> The reason is that among all the propositions 'B is A', each signifies whatever the others signify and altogether as the others signify. [81] [15, Pg 14]

The idea here is that among all the expressions of the form 'B is A' each one of these supposits for the same proposition, namely B is A. Since they all supposit for the same proposition, they all refer to the same object, and will have the same truth-value. If the proposition is false, they will all be false. Likewise if the proposition is true, they will all be true. Thus we can see why the universal follows from the particular in this case. The inference from some 'A is B' is possible to all 'A is B' is possible is good, since if ''Some B is A' is possible', then the proposition some B is A, is in fact possible. But by what we just observed above, everything that materially supposits for B is A, will also be possible. Hence the universal claim is also true. In the case of negative propositions, this rule also hold. For example from 'some 'A is B' is not true/possible etc...' we can conclude 'no proposition 'A is B' is true/possible etc...' The reasoning is basically the same. The expression 'A is B' supposits for the proposition A is B. Now, according to Buridan, there are two ways for the proposition, A is B to be false.

[79] Si autem modus praedicetur et dictum subiciatur, adhuc potest propositio esse uniuersalis, particularis, etc.

[80] In omnibus modalibus compositis in quibus dictum subicitur ad particularem sequi uniuersalem caeteris non mutatis.

[81] Causa est quia omnium talium propositionum 'B est A' una significat quid quid alia significat et qualitercumque alia significat

First, it can simply be an empty expression. For example it could be a claim like ‘Rebecca is a chimera’, or ‘Pegasus is a horse’. According to Buridan, both of these are false since in the case of the first, Chimeras are impossible objects[82] and in the case of the second, the name Pegasus does not refer to anything existing. In both of these cases the expression supposits for nothing, and so vacuously makes the universal negative claim true. From some proposition ‘Rebbecca is a Chimera’ is not true, we may conclude no proposition ‘Rebbecca is a Chimera’ is true.

The second way for a proposition to be false if both terms are non-empty but that the proposition is simply false. For example, in the expression, ’all dogs are two legged animals’ both dog, and two legged animals are non-empty, but the expression does not supposit for a true proposition because some dogs are not two-legged animals. But again, by the same reasoning, we have that from some proposition ‘all dogs are two legged animals’ is false it follows that ‘no proposition ’all dogs are two legged animals’ is true‘ since the proposition, all dogs are two legged animals, is false.

Tenth Conclusion

> Any composite modal in which the dictum is subject is simply converted in [its] terms except a universal affirmative, which is only converted accidentally. [83] [15, Pg 15]

The distinction between simple and accidental conversion is an important one in term based logics. Simple conversion is results when move from $A \circ B$ to $B \circ A$. For example the inference from, ’some A is a B’ to ’some B is an A’ is an instance of simple conversion since whatever objects make the first proposition true will also make the second true. In contrast, when we convert accidentally, we take one of the objects that is true because of the proposition, and show that it converts. This kind of conversion is of the form ‘All A is a B’ to ‘Some B is an A.’[84] What Buridan is saying is that when we convert composite modal propositions where the dictum is the subject, all of the conversions are simple, except in the case of the universal affirmative.

Buridan gives an outline of the proof of accidental conversion, noting that

> This is proved just as in the assertoric case, and can be proved by an expository syllogism. [85] [15, Pg 15]

. Assume that ‘all ‘B is A’ is possible’. We need to show that there is some possible proposition such that ‘B is A’. Take one particular instance of the universal.[86] Then we have an instance C, which is of the form ‘B is A’ is possible. But because of supposition we know that the proposition B is A, is possible. Hence, by conversion, A is B is also possible.

In the same manner, Buridan only sketches the proofs of the simple conversion of negative universal propositions, again noting that the proof is just like in the assertoric case. The only

[82] In traditional Greek Mythology a chimera had the head of a goat, the torso of a female human, body of a lioness and a tail which was the head of a snake. For Aristotle and many of the medievals, such an object was impossible because it had conflated essences.

[83] Quamlibet modalem compositam in qua dictum est subiectum conuerti simpliciter in terminis praeter uniuersalem affirmatiuam, quae conuertitur solum per accidens.

[84] Note that in Buridan’s system, the terms A and B can both be empty, but that, if they were empty, the positive propositions would be false.

[85] hoc probatur sicut in illis de inesse, et posset probari, per syllogismum expositorium.

[86] This is how a proof by expository syllogism proceeds.

case that requires some attention is when we are dealing with o propositions. Here, it is normally invalid to move from some A is not B to some B is not A. However, as Buridan noted before[87], from some 'A is B' is not possible we can show that no 'A is B' is possible. For this to hold, they both must convert in the same manner, otherwise we could invalidate the ninth conclusion. A negative particular must convert, and it must be an instance of simple conversion. This is highly unusual. Normally, in the square of oppositions, only the positive particular and the negative universal convert simply. Particular negative propositions do not convert at all, and universal affirmative propositions can only be converted accidentally into particular propositions. The reason why composite modal propositions are so different is because of conclusion nine. The supposition of these terms allows universal and particulars to be equivalent. It should also be clear that conversion is not simple in the case of universal propositions. From the expression 'all 'B is A' is possible' it may not follow that it is possible that every proposition 'B is A'. The latter may be false in cases were more then one proposition is formed.

Eleventh Conclusion

> Every composite modal in which the mode is subject is converted simply, except a particular negative, which is not converted. [88] [15, Pg15]

Here Buridan leaves the proofs to his readers, simply noting that the conclusion is clear, because it is analogous to the proof of assertoric expressions. To see that the negative particulars do not convert, consider the proposition, 'B is A' is not necessary. Conversion would yield the proposition, it is necessary that 'B is A' is false.[89] These are clearly not equivalent. To see this observe that the proposition "an animal is a man' is not necessary', is true, since it is possible that no men exist, but other animals do. However, the conversion of this, 'it is necessary that the proposition 'an animal is a man' is false' is not true, since if a man exists, he is necessarily an animal, and so in this case, it is possible that an animal is a man, making the second proposition false.

Twelfth Conclusion

Having considered conversion as it applies to composite modal propositions as a whole, Buridan goes on to consider 'nested' conversion, i.e. conversions where the dictum, is converted, while the rest of the composite modal proposition remains the same. Buridan states:

> Every affirmative composite modal of truth, of possibility and of necessity is converted as regards the dictum however the dictum would be converted in itself.[90] [15, Pg 16]

What Buridan means is that the 'A is B' expressions convert in the same way as they do in the various conclusions for composite modal cases. For example:

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[87] see conclusion nine

[88] Omnem modalem compositam in qua modus subicitur conuerti simpliciter, praeter particularem negatiuam, quae non conuertitur.

[89] Formally: '$B \circ A' \neg \circ \Box$ and $\Box \neg \circ B \circ A$

[90] Omnem modalem affirmatiuam compositam de uero, de possibili et de necessario conuerti quantum ad dictum sicut dictum per se conuerteretur.

> 'It is possible that some man is running, so it is possible that something running is a man'; similarly, 'It is necessary that every God is just, so it is necessary that some just person is God'; similarly, 'That no man is an ass is true, so that no ass is a man is true'[91] [15, Pg 16]

. What Buridan illustrates in each of these propositions is the way conversion works. The general idea is that we can substitute valid conversion of the dictum in composite modal propositions. Because of Buridan's commitment to nominalism, both of these propositions must be formed. Hence he adds the following caveat:

> Then it should be noted that these conversions are absolutely good consequences only if an existence postulate about the terms is added [15, Pg 16]

After adding this caveat, he goes on to treat two objections. First, he argues against the idea:

> That [since] a consequence cannot be formed unless each [of its parts] is formed, so if the consequence were formed the existence of the terms [would follow] without being added as a postulate.[92] [15, Pg 16]

Take a particular consequence, and then according to this objection, for that consequence to be formed, each part of it is formed in the process of stating the consequence. We do not need the formation clause. This objection is not a good one in Buridan's mind, because it conflates sentences with the propositions they supposit for. In the expression 'it is possible that 'B is A" 'B is A' is formed and it supposits for a proposition, but is not itself a proposition. To suggest that it is would be to conflate a sentence with its proposition, which Buridan thinks would be bad. As long as this distinction is maintained, it is possible for the proposition to be unformed, while the sentence is formed. Hence, the formation requirement is required. The second objection Buridan treats is the following:

> For an absolutely good consequence it is impossible for things to be as the antecedent signifies unless they are as is signified by the consequent when the consequence is formed.[93] [15, Pg 16]

Buridan objects to this definition of consequence. He thinks that it is too strong. Instead Buridan embraces the following definition:

> It is impossible for things to be as is signified by the antecedent unless they are as is signified by the consequent[94] [15, 16]

The reason Buridan uses this defintion is to block inference like the following:

[91] 'Possibile est quendam hominem currere; ergo possibile est' quendam currentem esse hominem similiter: 'Necesse est omnem deum esse iustum; ergo necesse est quendam' iustum esse deum similiter : 'Nullum hominem esse asinum est uerum; ergo nullum asinum esse hominem est uerum.'

[92] scilicet quod non potest formari consequentia quin utraque formetur, ideo sine suppositione posita est constantia terminorum si consequentia formetur.

[93] scilicet quod non potest formari consequentia quin utraque formetur, ideo sine suppositione posita est constantia terminorum si consequentia formetur.

[94] Immo simpliciter requiritur quod impossibile sit esse ita sicut per antecedens significatur quin sit ita sicut per consequens significatur.

'Some proposition is affirmative, so some proposition is particular [95] [15, Pg 16]

The converse of this, 'no proposition is particular, so no proposition is affirmative' is clearly seen to be invalid, since a universal positive is affirmative, but not particular, making the antecedent true and the consequent false. Buridan's definition of consequence blocks this inference, even when the signification conditions are met.

Thirteenth Conclusion

Buridan's thirteenth consequence deals with the relationship between particular and universal dictums.

> Every particular dictum is converted into a universal in an affirmative composite modal proposition of falsity or of impossibility, but a universal is not converted into a particular; and a universal negative dictum and a particular affirmative are converted simply, but a universal affirmative is not converted.[96] [15, Pg 17]

Buridan proves the first part this conclusion by observing that:

> If the consequent is false or impossible it is necessary that the antecedent is false or impossible [97] [15, Pg 17]

This is the principle of modus tollens, applied to the cases of negation and impossibility. To see that the principle follows consider the example of every impossibility is that 'some B is A'. Buridan claims that this can be converted to every proposition 'every A is B'is impossible. Assume that the consequent is false. Then some proposition 'every A is B' is not impossible. This is equivalent to some proposition 'every A is B' is possible. Observing that 'every A is B' supposits for every A is B, we have by conversion that some B is A, and so we can form the expression 'some B is A' is possible, which is a contradiction. The remaining cases are proven in a similar way.

To see the second part, consider the proposition: 'every proposition 'every animal is a donkey' is impossible.' This is true on the hypothesis that there are animals other then donkeys that are possible. However, if we convert this, we obtain the proposition every proposition 'some donkey is an animal' is impossible, which is clearly false. Hence the second part of this conclusion follows.

Fourteenth Conclusion

Buridan's fourteenth consequence deals with how conversion works when contingency is introduced into composite modal propositions.

> Every dictum in an affirmative composite proposition of contingency is converted according to the opposite quality into the contradictory dictum, not into the contrary.[98] [15, Pg 17]

[95] Quaedam propositio est affirmatiua; ergo quaedam propositio est particularis

[96] Omne dictum particulare conuerti in uniuersale in propositione modali composite affirmatiua de falso uel de impossibili, sed non conuerti uniuersale in particulare; item, dictum uniuersale negatiuum et particulare affirmatium conuerti simpliciter, sed uniuersale affirmatiuum non conuerti.

[97] si consequens est falsum uel impossibile oportet antecedens esse falsum uel impossibile

[98] Omne dictum in propositione de contingenti composite et affirmatiua conuerti secundum oppositam qualitatem in dictum contradictorium, non in contrarium.

Here Bruidan gives us the rules for converting the dictum when it occurs in a contingent composite modal proposition. As we saw in the previous section, a proposition is contingent, just in case it can be true, and it can be false. Because of this, when we convert the dictum of a proposition, we must make sure that this division is respected. When we convert the proposition, we need to ensure that preserves the contradictory relationship. To do this, when we convert contingent modal propositions, we substitute the dictum with its contradictory proposition. For example, if we have the proposition:

'it is contingent that 'every intelligent being is God"
The converse of this proposition is that 'it is contingent that' some intelligent being is not God"

Fifteenth Conclusion

> from every affirmative composite proposition of truth its dictum follows and conversely; and from every such [proposition] of necessity its dictum follows and propositions of truth and of possibility follow, but not conversely; and from every such [proposition] of truth that of possibility follows but not conversely; and from every proposition a proposition of possibility of which it is the dictum follows, but not conversely; and from every such [proposition] of contingency one of possibility follows but not conversely.[99] [15, Pg 17]

The first principle is straight forward. For example, from some proposition: "A is B' is true" it follows that A is B. The converse is equally straightforward. In both cases these follow because of the definition of truth that Buridan is using. Since our thesis is not looking at theories of truth, we will not treat this in detail. The second principle, that:

> and from every such [proposition] of necessity its dictum follows and propositions of truth and of possibility follow, but not conversely [15]

Is equally clear since necessity implies truth and possibility, and truth implies possibility. The reasoning here is the same as it was in the case of divided propositions. Buridan points out that the positive part of this conclusion is simply obvious. He remarks that:

> I have put all of these forward together because they are obvious, but perhaps they should really be put forward as principles rather than conclusions.[100] [15, Pg17-18]

If worse comes to worse, these can be added to his system as rules, instead of as conclusions. If we want to see the negative part of this conclusion, consider the following:

'that a man runs' is possible, therefore 'that a man runs', is true. Consider a situation where no men are currently running. Then the first premise is true, and the second false. For the case of necessity, consider the example: it is possible that a donkey is white therefore it is

[99] Ad omnem propositionem de uero compositam et affirmatiuam sequi dictum suum et econuerso; et ad omnem talem de necessario sequi dictum suum et sequi propositionem de uero et de possibili, et non econuerso; et ad omnem talem de uero sequi illam de possibili et non econuerso; et ad omnem propositionem sequi propositionem de possibili cuius ipsa erit dictum et non econuerso; et, ad omnem talem de contingenti sequi illam de possibili et non econuerso.
[100] Haec omnia posui simul quia sunt manifesta, et forte potius debent reputari principia quam conclusiones.

necessary that donkey is white. Imagine that in the world there are some white donkeys, and some donkeys are of another colour. In this case, the antecedent is true, and the conclusion is false.

Sixteenth Conclusion

From these principles, Buridan discusses how necessity relates to the dictum of composite modal propositions.

> If an affirmative composite proposition of necessity is true, whatever follows from its dictum is necessary and similarly whatever follows from [the dictum of one] of possibility is possible, and whatever from one of truth is true; but this is not the case for divided [propositions] except those of truth.[101] [15, Pg 18]

The first part of this follows by the principle:

> 'If the antecedent is true, the consequent is true, if possible then possible, and if necessary then necessary' [102] [15, Pg 18]

. The rule here is a sort of K-axiom. The idea is that if we have a consequence of the form 'A therefore B' and we have A is necessary/possible/true then we may conclude that B is necessary/possible/true as the case may be However, in this case A and B have more structure to them, since they are of the form '$\phi' \circ \psi$ where '$\phi' \circ \psi$ is a composite modal proposition of either necessity, possibility or truth. The reason this is K-like, stems from nature of consequence Buridan uses:

> one proposition is antecedent to another which is such that it is impossible for things to be altogether as it signifies unless they are altogether as the other signifies when they are proposed together.[103][104] [15, Pg 18]

When we have a consequence relationship, we have necessarily (A entails B). This is why this kind of principle is K-like in nature. Given a necessary hypothetical, a necessary anticedent and modus mondens, we may conclude the consequent as well.

However, Buridan points out that:

> but this is not the case for divided [propositions] except those of truth. [105] [15, Pg 18]

As per the usual Buridan proves this with a counterexample:

[101] Si propositio de necessario composite et affirmatiua sit uera, quidquid sequitur ad eius dictum est necessarium et ita de possibili quidquid sequitur est possible, et de uero quidquid sequitur est uerum; sed non est ita de diuisis praeterquam de uero.

[102] Si antecedens est uerum, consequens est uerum, si possibile possibile et si necessarium necessarium

[103] Illa propositio est antecedens ad aliam quae sic se habet ad illam quod impossibile est qualitercumque ipsa significat sic esse quin qualitercumque illa alia significat sic sit ipsis simul propositis.

[104] While Buridan does have some minor reservations about this definition as formulated here, it is satisfactory, as long as a couple of points are kept in mind. See Book 1 Chapter 4

[105] sed non est ita de diuisis praeterquam de uero.

> 'Everything creating is necessarily God' is true, nonetheless, it need not be necessary that it follows from 'Something creating is God', for [the former] might be false. Similarly, although 'A white thing can be black' is true, nonetheless, from 'A white thing is black' there follows an impossibility.[106] [15, Pg 18]

Given what we have already seen, this counterexample is fairly clear. 'Everything creating is necessarily God' is true, because of how ampliation applies to necessity. However, the composite modal claim:, "something creating is God' is necessary' does not follow. That proposition is true, only if 'something creating is God' is true at all points, and per theology, this is not the case.[107] What this highlights is a very important fact about these two kinds of propositions. In general, they are not equivalent to each other.

After addressing this issue, Buridan deals with the principle that:

> Supposing an assertoric to be possible nothing impossible follows [108] [15]

The principle is an obvious upshot to the definition of possibility. Assume that we have an assertoric proposition that is possible, say it is of the form A is B. Then the composite modal claim, "A is B'is possible' is also true. And, from what we have seen above only the only things that follow from this are other claims that are either possible, or necessary.[109]. These are not impossible propositions. Hences, nothing impossible follows.[110]

However, Buridan is quick to remind his reader that this principle only applies to composite modal claims. When we start to mix the two kinds of propositions, things can rapidly change.

> Now there is a doubt whether it is permissible that every divided proposition of possibility warrants an assertoric, that is to say that the doubt is whether and in what way there corresponds to every true divided proposition of possibility a possible assertoric proposition.[111]
>
> [15, Pg 18]

Buridan's response is that this rule is strictly speaking false when the two kinds of modal claims interact. He uses the following example:

> For whereas every star existing in the zodiac is possibly shining on our hemisphere, nonetheless, 'Every star existing in the zodiac is shining on our hemisphere' is by the laws of nature impossible.[112] [15, Pg 18]

The difference is that, in the proposition, every star existing in the zodiac is possibly shining on our hemisphere, the exact stars that make this proposition that makes this true can change.

[106] Secunda pars patet. Quia licet haec sit uera: 'Omnem creantem necesse est esse deum' tamen non oportet esse necessarium quod sequitur ad istam:'Creans est deus' quia forte falsa est. Similiter, licet haec sit uera: 'Album potest esse nigrum' tamen ad istam: 'Album est nigrum' sequitur impossibile.

[107] i.e. it is possible that God not be creating.

[108] possibili posito inesse nihil sequitur impossibile

[109] The necessary ones follow in cases where the consequent is necessarily true

[110] This could also be proven by reductio or counterexample.

[111] Utrum autem liceat omnem propositionem de possibili diuisam ponere inesse dubitatio est, hoc est dictum quod dubium est utrum et quomodo omni propositioni de possibili diuisae uerae correspondeat propositio de inesse possibilis.

[112] Quamuis enim omnem stellam existentem in zodiaco possibile sit lucere super nostrum hemisphaerium, tamen haec est impossibilis naturaliter:Omnis stella existens in zodiaco lucet super nostrum hemisphaerium.

The consequent will require more objects to fall under the terms for the proposition, if the proposition is to be true. In "Every star existing in the zodiac is shining on our hemisphere' is possible', it must be the case that every star in the zodiac is shining on our hemisphere at the same time. But as Buridan points out, this is not possible, because for it to be true, every star in the Zodiac would need to be shining on our hemisphere, *at the same time*. Buridan points out, this is physically impossible. While that conclusion does not follow, Buridan proposes a slight modification to the principle. He suggests that it be treated as an assertoric disjunction of the singular terms in the class. In other words, if we compose the corresponding composite claim as a disjunction of the elements that fall under the first term i.e. one that says 'This star existing in the zodiac is shining on our hemisphere or that star existing in the zodiac is shining on our hemisphere or…'. This will result in a valid consequence, since the disjunction requires only one element to be true for the whole to be true. This is a weakening of the universal claim but it does preserve the validity of the rule.

There is a second problem with this principle that Buridan points out:

> It is not necessary that if a divided particular or indefinite [proposition] of possibility is true, an assertoric is possible retaining the same terms.[113] [15, Pg 18]

Consider the following divided modal claim: A white thing is possibly black. From this it does not follow that "a white thing is black' is possible' since, in the case of the first expression, the possibility ampliates over possibilities, and as such is true just in case there is something that is white now, but can be black. In contrast to this, the expression "a white thing is black' is possible' is true just in case it is possible that a white thing is black, and this is impossible, since the object would have to be black and white at the same time.[114] To deal with this counterexample, Buridan proposes that we add a pronoun to the composite modal claim. From the claim a white thing is possibly black, we convert it to, "this is black' is possible' where the expression 'this' is taken to refer to the object that is a white thing. Basically, Buridan is using supposition to pick out a given object that is white, and then uses the demonstrative to pick out the object, without it retaining the property of being white.

Hence, while the principle does have a counterexample, Buridan provides some ways to construct near-similar propositions that will be valid.

Seventeenth Conclusion

Buridan then goes on to explore how composite modal propositions relate to divided modal propositions. His seventeenth conclusion states:

> From no affirmative composite of possibility does there follow a divided one of possibility with the mode affirmed, or conversely, except that from an affirmative composite with an affirmed dictum there follows a divided particular affirmative.[115] [15, Pg 19]

[113] Non oportet si particularis uel indefinita de possibili diuisa est uera quod illa de inesse sit possibilis retentis eisdem terminis.

[114] It is implicit in this case that the objects in question are white/black with respect to the area on the object/creature etc

[115] Ad nullam compositam affirmatiuam de possibili sequi aliquam diuisam de possibili de modo affirmato nec econuerso, praeterquam ad compositam affirmatiuam de dicto affirmato sequitur particularis affirmatiua diuisa.

The exemption is to allow the move from ''some B is A' is possible' to 'some B can be A.' This is fairly easy to see. The expression ''some B is A' is possible' is true just in case 'some B is A' is a possible proposition, and this is true, just in case some B is A is possible. From which it is easily seen to follow: that which is or can be A can be B. This does not hold for any propositions For example, from ''Everything running is a horse' is possible' it does not follow that everything running can be a horse, since at that time a donkey or a human could be running. To see this, remember that the second proposition is equivalent to everything that is or can be running can be a horse. However, there are things that can never be horses, even though they run. Humans and donkeys are both examples of this. However, the singular case does follow.

The converse are disproven using counterexamples.

> Now it is clear that the converse never follows. For although everyone asleep can be awake, nonetheless 'Someone asleep is awake' is not possible; similarly, in the negative, although every star in the zodiac shining on our hemisphere can fail to shine on our hemisphere, nonetheless, 'A star in the zodiac shining on our hemisphere does not shine on our hemisphere' is not possible.[116] [15, Pg 19]

In the case of the sleeping and star counter-examples, this is clear. The expression 'someone asleep is awake' is a contradiction, and so never possible. Likewise the expression 'A star in the zodiac shining on our hemisphere does not shine on our hemisphere' is also a contradiction in terms. Since a contradiction is never possible, the consequent is false, while the antecedents true; Hence they are not good consequences.

Eighteenth Conclusion

Having outlined the relationship between possible composite modal propositions, Buridan goes on to treat necessarily composite propositions and their relationships to divided ones.

> From no composite affirmative of necessity does there follow a divided one of necessity with an affirmed mode, nor conversely, except that from a divided universal negative there does follow a composite universal with a negated dictum.[117] [15, Pg 19]

Buridan proves the exception first. From what we saw in the previous conclusion, we know that, from 'B is A' is possible it follows that B can be A. Take the contraposition of this proposition. It becomes, No B is necessarily A therefore "B is A'is not possible.'[118] Which is exactly what Buridan says follows.

The main claim is proven by counter-examples. In the case of universals, Buridan points out that:

> 'Every horse is an animal' is necessary [119] [15, Pg 19]

[116] Econuerso autem manifestum est quod nihil sequitur. Quia licet omne dormiens possit esse uigilans, tamen haec non est possibilis 'Dormiens est uigilans'; similiter, negatiue, quia licet omnis stella zodiaci lucens super nostrum hemisphaerium possit non lucere super nostrum hemisphaerium, tamen haec non est possibilis: Stella zodiaci lucens super nostrum hemisphaerium non lucet super nostrum hemiphaerium.

[117] Ad nullam compositam de necessario affirmatiuam sequi aliquam diuisam de necessario de modo affirmato, nec econuerso, praeterquam ad uniuersalem negatiuam diuisam sequitur composite de dicto negato etiam uniuersalis.

[118] Observe here that the negation is applied to the coupla, not to either of the terms.

[119] Omnis equus est animal

is true. However, from this it does not follow that 'no horse is necessarily an animal', since it is possible that horses exist. A similar counter-example can be generated for the negative universal claim using the sleeping/awake argument from conclusion eighteen.

Nineteenth Conclusion

Buridan's nineteenth and final conclusion is that

> from no proposition, [whether] assertoric, of possibility or of necessity does there follow one of contingency with both modes affirmed; similarly, from none of contingency does there follow an assertoric or one of necessity, but there does follow one of possibility.[120] [15, Pg 20]

The first part of the conclusion can be seen from the fact that from an affirmative proposition, regardless of modality, does not imply a negative proposition with the terms in the same position. Since a positive expression does not imply a negative one, it would not be possible for it to imply a contingent expression, since for a contingent expression to hold, one needs to show that a given claim is both possibly true and possibly false. The second and third part of the conclusion also follows from the definition of contingency. First, it is clear that if a given proposition is contingent, it cannot be necessary. A contingent proposition is possibly true and possibly false, while a necessary one is one that is always true, and hence never false. In the same way, since a possibly does not entail an assertoric, from a contingent proposition, an assertoric one does not follow. However, a contingent proposition does entail a possibility. It should also be noted that this conclusion holds in both divided and composite modal propositions. Buridan concludes this book with the following exhortation to his reader

> Let everyone take care that they do not take divided modals for composites or vice versa, and do not take divided [modals] with a negated mode for divided [modals] with a negated dictum, since they are very different, as has been shown.[121] [15, Pg 20]

Having explained and walked through book two of Buridan's *De Consequentia*, the challenge for formalization becomes clear. As we stated in the introduction, we intend to develop a logic for divided modal claims. First, we need to develop a formalism that is robust enough to capture the distinction between composite and divided modal propositions, prove all of the conclusions and equivalences presented within this work. The goal of our next chapter will to show that we can modify existing structural proof theoretic analysis of modal logic to capture the system that Buridan has developed here.

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[120] Ad nullam propositionem de inesse, de possibili uel de necessario sequi aliquam de contingenti modis utrobique existentibus affirmatis; similiter, ad nullam de contingenti sequi aliquam de inesse uel de necessario sed sequi similem de possibili.

[121] Ergo caueant sibi omnes ne modales diuisas sumant pro compositis uel econtra et ne diuisas de modo negato sumant pro diuisis de dicto negato, quondam illae differunt ualde, sicut apparuit.

3. FORMALIZATION

3.1 Structural Proof Theory: Introduction

As we discussed at the close of the previous chapter, Buridan's approach to logical consequence has a rule-like characteristic to it. The goal of this chapter is to present a modern proof theoretic formalization of analysis of the modal consequences that Buridan developed in book 2 of his *De Consequentia*. This will be done by modifying a system of structural proof theory to capture Buridan's term-based modal logic. We will start with a brief introduction to structural proof theory. We will then introduce the systems G3K*, and its various extensions as developed by Sara Negri in[8][9]. Using Negri's analysis as our foundation, we will develop the system $G3_{syl}$ which is our fromaliziation of Buridan's logic. This system will be a modification of the system G3K*q, where quantification is term based, and allows for empty terms. Using this system we will prove the 8 conclusions that Buridan developed for divided consequences. We will conclude this discussion with suggestions for further work that can be done on Buridan's theory of logical consequence and the system $G3_{syl}$.

In essence the goal of this chapter is to prove the following meta-logical property about $G3_{syl}$. Let $\vdash_B \phi \rightarrow \psi$ mean that Buridan asserts the consequent 'ϕ therefore ψ' is a good consequence. Let $\vdash_{G3_{syl}} \phi \rightarrow \psi$ mean that given assumptions ϕ, ψ is derivable in $G3_{syl}$

The goal of this chapter is to prove that $\vdash_B \phi \rightarrow \psi$then $\vdash_{G3_{syl}} \phi \Rightarrow \psi$. I.e. that the system $G3_{syl}$ is sound with respect to everything that Buridan claims. We will also sketch how, given a number of meta-logical properties of $G3_{syl}$, we can show that $\vdash_B \phi \nrightarrow \psi$ then $\nvdash_{G3_{syl}} \phi \Rightarrow \psi$. In other words, if Buridan says that '$\psi \rightarrow \phi$' is not a good consequence, then it is not derivable in $G3_{syl}$ either. In section 3.1 we will introduce our syntax, the system G3, and our extension $G3_{syl}$, based on the system G3K*. In section 3.2 we will prove soundness and outline how to show a proof does not follow, using root-first decomposition.

*3.2 G3K*s*

Historically, the development of a proof theory for modal logic has lagged behind semantic based approaches. The rather lengthy quote from Blackburn et al. illustrates the difficulty inherent in modal proof theory:

> As is often the case in modal logic, the proof systems discussed are basically Hilbert-style axiomatic systems. There is no discussion of natural deduction, sequent calculi, labeled deductive systems, resolution, or display calculi.... Why is this? Essentially because modal proof theory and automated reasoning are still relatively youthful enterprises; they are exciting and active fields, but as yet there is little consensus about methods and few general results. Moreover, these fields

are moving fast; much that is currently regarded as state of the art is likely to go rapidly out of date. [3, Pg. xii-xiii]

Indeed, four years after the initial publication of this work, a general method was developed for creating contraction and cut-free sequent calculi for modal logic. In [8] a general method was developed that drew on previous attempts to internalize the frame properties of modal logic. These properties can be represented within a sequent calculus that preserves many of the desirable structural properties of the proof theory. Modal logic, on this perspective, becomes an extension of classical proof theory. In such systems, the various properties of modal operations are given by axioms, together with appropriate introduction and elimination rules.

This is done by making use of labels within the sequent calculi. These labels are then prefixed to formulae. These labels function as an 'internalization' of possible worlds. In addition, a two place predicate relation Rxy is added to the language, where x and y are world labels. This is the internalization of the accessibility relationship. To this system a collection of rules are added both for the modal operators and accessibility relationship. These rules are the introduction and elimination rules for the modal operators, together with proof-theoretic analogues of the various frame properties[1]. The resulting systems are shown to be invertible, the rules for contraction and cut are admissible and there is a close connection between such sequent calculi and tabluex semantics for modal logic.cf[8][9]

The underlying classical logic that we will modify in this thesis is the system G3c. For a full discussion of G3c see[10, Pg. 61-89] The rules of G3c are as follows:
Axiom: $P, \Gamma \Rightarrow, \Delta, P$

Where P is an atomic formula, and Γ, Δ are sets of well formed formulae.

The rules of the system are as follows:

Symbol	Left Rule	Right Rule
$\perp$	$\dfrac{}{\Gamma, \perp \Rightarrow \Delta}$	No rule
$\wedge$	$\dfrac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma, \Rightarrow \Delta}$	$\dfrac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma, \Rightarrow \Delta, A \wedge B}$
$\vee$	$\dfrac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma, \Rightarrow \Delta}$	$\dfrac{\Gamma \Rightarrow \Delta, A, B}{\Gamma, \Rightarrow \Delta, A \vee B}$
$\rightarrow$	$\dfrac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma, \Rightarrow \Delta}$	$\dfrac{A, \Gamma \Rightarrow \Delta B}{\Gamma, \Rightarrow \Delta, A \rightarrow B}$
$\forall$	$\dfrac{A(t/x), \forall x A x, \Gamma \Rightarrow \Delta}{\forall x A x, \Gamma \Rightarrow \Delta}$	$\dfrac{\Gamma \Rightarrow \Delta, A(y/x)}{\Gamma \Rightarrow \Delta, \forall x A x}$
$\exists$	$\dfrac{A(y/x)\Gamma \Rightarrow \Delta}{\exists x A x, \Gamma \Rightarrow \Delta}$	$\dfrac{\Gamma \Rightarrow \Delta, A(t/x), \exists x A x}{\Gamma \Rightarrow \Delta, \exists x A x}$

The restrictions on this system are as follow:

y does not occur free in A, Γ, Δ

There are a number of important observations we should make about how formula are expressed in this system. First, proofs are easiest to understand if read from the bottom to the top. Reading the proofs in this manner makes it clear how the sequents were decomposed. Second, be aware that the 'meaning' of the comma changes depending on which side of the sequent the formula occur on. For example on the right hand side of the sequent we have: $\dfrac{A, B \Rightarrow \Delta}{A \wedge B \Rightarrow \Delta}$ While on the left hand side, we have $\dfrac{\Gamma \Rightarrow A, B}{\Gamma \Rightarrow A \vee B}$. It's important to be very clear on this point, since

[1] eg reflexivity, symmetric, seriality etc.

in a number of the rules we introduce for $G3_{syl}$ we implicitly simplify $\wedge$ and $\vee$ to their comma form.[2]

Third, the restrictions on the terms and variables need to be understood clearly. Informally, the idea is that rules applied on the left would correspond to elimination rules in a natural deduction system[3] while the rules on the right correspond to introduction rules. With this in mind, the reason for the restrictions becomes clear. If we want to eliminate (i.e. left rule) an $\exists$ quantifier, we need to make sure that we pick a specific object that does not occur in the proof already. The same holds true if we want to introduce a $\forall$ quantifier. However, unlike the comma which changes it's interpretation depending on which side of the sequent, the restrictions on the formula will only depend on where they occur when they are decomposed. For example consider the following non-proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{B(z/x), Az\Gamma \Rightarrow B(z/x)}{B(y/x), Az\Gamma \Rightarrow Bz}\,??}{B(y/x), \Gamma \Rightarrow (A(x/z) \rightarrow B(x/z))}\,R\rightarrow}{B(y/x)\Gamma \Rightarrow \forall x(Ax \rightarrow Bx)}\,R\forall}{\exists x Bx, \Gamma \Rightarrow \forall x(Ax \rightarrow Bx)}\,L\exists$$

Here, once B(y/x) occurs on the left hand side we *cannot* re-label it without keeping the restrictions previously invoked in mind. Otherwise we will end up with problematic proofs like the above one.

Formally, G3c has a number of very desirable structural properties. All of the rules are invertible,[4] weakening and contraction are height preserving admissible, and cut is also admissible. From this it is possible to give an algorithm for decomposing any formula, so that it will either yield a proof of the formula, or it will generate an infinite branching tree. From this it is possible to prove completeness in the usual manner.

Another desirable property of this system is that it admits proof search by root-first decomposition. The basic idea here is that given an arbitrary sequent $\Gamma \Rightarrow \Delta$ there is an algorithm that decomposes the tree in such a way that if a sequent is provable, the search terminates in a finite number of steps, and gives a valid tree. However, if the sequent is invalid, it will not terminate in a finite number of steps.[5] This process will be employed when we motivate the negative components of Buridan's system.

The technique for extending G3c to a modal logic is done via the more general method of adding mathematical axioms to the system. In the case of modal logic, the language of the system is extended to include 'labels' for each formula. In the propositional case, axioms are of the form $w : A, \Gamma, \Rightarrow \Delta, w : A$ where w is a label and A is an atomic formula. The various rules of G3 are all modified to incorporate these new labels. The rules for $\Box$ and $\Diamond$ are essentially the semantic truth conditions for these modal logic, only converted into normal form and added to the logic. This can be done since G3c preserves a number of its structural properties when axioms and general rules of inference are added to the system. Details for adding mathematical axioms to G3c can be found in chapter 5 of [10]. Results for the resulting class of modal logics G3K* can be found in[8].

[2] There is nothing essential in our doing this, it is only done to make proofs shorter, and make the rules easier to read.

[3] say a Fitch style natural deduction

[4] i.e. whatever follows from the direct grounds for the proposition must follow from the proposition as well. See cite[Pg6-7]Negri1

[5] if we remove the quantifiers from G3c, the resulting system is decidable and it will terminate in a finite number of steps.

The system that we will develop in this thesis will be an extension of the system G3K*c. The system we will developed, called $G3_{syl}$, will need to incorporate a number of features that are not present in G3K*c. First, the system will need to be able to express the various term-based relationships that are the hallmark of medieval logic. Rules will be needed for introducing expressions like 'some A is a B', 'all B could not be A' etc. These rules must capture the medieval understanding of supposition as it pertains to terms.[6] Second, the system needs to be able to express the various ways negation can be applied within term-based propositions. In Buridan's logic, there are four possible places for negation to go:

1. It can be in one of the terms of a composite modal claim.[7]

2. The negation can go in front of the copula, as in 'A is not B.'[8]

3. It can go in front of the modal operator, as in the proposition 'A is not necessarily B',

4. It can be applied to the term, as in the proposition, some A is non-B.

It's important that our logic be able to capture these different senses of negation correctly, since a number of Buridan's distinctions rest on the difference when dealing with supposition in negative and affirmative propositions.[9] Recalling Buridan's own example, 'Chimaeram necesse est non esse asinum[10]' this is true, even though there is nothing that falls under the term chimera. In this sense, the logic that we develop will also need to be a free logic, at least with respect to the terms. Finally, the logic will need to have the syntactic resources to capture the divided and composite senses of modal claims. As we have already seen, these propositions differed quite a bit from each other in terms of their truth conditions, and as will be clear, divided modal claims are quite different from modern analysis of modal expressions.

3.2.1 Syntax

Our language consists of the following symbols: $\mathfrak{L}_S = \{P, \bot, \wedge, \vee, \rightarrow, \neg, \Diamond, \Box, \forall, \exists, \circ, \sqsubset, :, (,)\}$ where P=a,b,c...;A,B,C...;$\bar{A}, \bar{B}, \bar{C}\ldots$ We define our notation as follows:

- lower case letters a,b,c... are labels. The symbols at the beginning of the alphabet are used for names of objects, and the ones near the end are for possible worlds, with the exception of e which will be reserved for another purpose.

- Uppercase letters A,B,C... denote terms,$\bar{A}, \bar{B}, \bar{C}\ldots$ denote negative terms, $A^e, B^e\ldots$ denote empty terms, except for the letter R which is reserved for the accessibility relationship.

  $\forall$ or $\exists$ followed by a term is a quantified term.

[6] e.g. We must be able to show that all of the principles of the traditional square of opposition are derivable

[7] For example, 'it is not the case that every A is B.' It should be noted that this is negating the entire expression and points to an interesting connection between composite modal claims of falsity and propositional negation.

[8] it should be noted that in the Latin the expression would be non est.

[9] in practice we will only need to remember points 2 and 3. Propositions with the fourth kind of negation are treated as positive claims in this language, and we will not be treating composite modal claims in this thesis.

[10] a chimera is necessarily not a donkey

$\Diamond, \Box$ are modes. Throughout this thesis we will also use the meta-variable ∇ to range over both of these modes, either on their own, or when they occur with other symbols, like $\circ$

$\circ$ is used to denote the basic (present tense) copula between expressions.[11]

$\neg\circ$ denotes the negative, present tense copula.

$\Box\circ, \Diamond\circ$ are modalized copula.

$\neg\Box\circ, \neg\Diamond\circ$ are negative, modalized copula.

Our base level formulae are of the form:

$w : a \sqsubset A$ where w is a world label, a is an individual and A is a term.

wRv where w,v are world labels, and R is an accessibility relationship.

$w : A^e$ where A is a term.

$w : \alpha * \circ\beta$
where

α is either a term, a restricted term, a quantified term or a mode,

$*\circ$ denotes any of the copulas

β is either a term, a negative term, or a mode.

w is a world-label.

Informally, $w : a \sqsubset A$ can understood as saying 'that 'a' is an A at world w', 'wRv' means 'world v is accessible from w', and $w : a \not\sqsubset A$ means 'that a is not an A at w' and '$w : A^e$' stands for 'the term A is empty at w.'

For example, the formula $w : \forall A \Diamond \circ \bar{B}$ is a basic formula, and means, at world w, everything that is A is possibly non-B. It should be noted that quantifiers in this language *only* range over terms. Quantification over labels or names is not grammatical.

We define composite and divided modal propositions as follows:

If exactly one of ϕ, ψ is a mode, and the other is a basic or quantified formulae then $\phi \circ \psi$ is a composite modal proposition.

If both ϕ, ψ are terms, quantified terms or negated terms, and the copula is modalized, then $\phi * \circ\psi$ is a divided modal proposition.

If both ϕ, ψ are terms, quantified terms or negated terms, and the copula is not modalized, then $\phi * \circ\psi$ is an assertoric proposition.

To all of this we add the 'normal' connectives. i.e. If ϕ, ψ are basic formulae then

[11] Note that $\circ$ does not distinguish grammatical form. In composite modal claims, both the dictum and the main copula will be formalized with $\circ$

$\phi \vee \psi$, $\phi \wedge \psi$ and $\phi \rightarrow \psi$ are formulae.

Throughout this thesis we use the following short hands:

$\Gamma \vdash \Delta$ for sequents with assumptions Γ and conclusions Δ.[12]

$\neg\phi$ to denote the sequent, $\phi \rightarrow \bot$ where ϕ is any well formed formulae.

$w : a \not\sqsubset A$ to denote the sequent $w : a \sqsubset A \rightarrow \bot$

$A\triangle \circ B$ to denote the sequent $w : A\Diamond \circ B \wedge w : A\Diamond\neg \circ B$

$\phi \Leftrightarrow \psi$ as a metalogical expression stating $(\phi \vdash \psi)$ and $(\psi \vdash \phi)$

ϕ^n to denote the sequent ϕ repeated n times.[13]

Conceptually, the idea is to obtain rules that capture the main aspects of medieval term-based logic. The rules are taken from the following formalization of the term operations:

$w : \forall A \circ B$	if and only if	$(w : b \sqsubset A \rightarrow w : b \sqsubset B) \wedge w : a \sqsubset A$
$w : \forall A \neg \circ B$	if and only if	$(w : b \sqsubset A \rightarrow w : b \not\sqsubset B) \vee A^e$
$w : A \circ B$	if and only if	$(w : a \sqsubset A \wedge w : a \sqsubset B)$
$w : A \neg \circ B$	if and only if	$(w : a \sqsubset A \wedge w : a \not\sqsubset B) \vee A^e$

Where b is arbitrarily chosen. To obtain the system $G3_{syl}$ we remove the rules for $\forall$ and $\exists$ and add the following rules to the system G3*K

The left rules are as follows:

A^e	$\dfrac{w : b \not\sqsubset A}{w : A^e, \Gamma \vdash \Delta}$
$\circ$	$\dfrac{w : a \sqsubset A, w : a \sqsubset B, \Gamma \vdash \Delta}{w : A \circ B, \Gamma \vdash \Delta}$
$\neg\circ$	$\dfrac{w : A^e, \Gamma \vdash \Delta \quad w : a \sqsubset A, w : a \not\sqsubset B, \Gamma \vdash \Delta}{w : A \neg \circ B, \Gamma \vdash \Delta}$
$\forall$	$\dfrac{w : a \sqsubset A, \Gamma \vdash \ w : b \sqsubset A, \Delta \quad w : a \sqsubset A, w : b \sqsubset B, \Gamma \vdash \Delta}{w : \forall A \circ B, \Gamma \vdash \Delta}$
$\forall \ldots \neg$	$\dfrac{w : A^e, \Gamma \vdash \Delta \quad \Gamma \vdash \Delta, w : b \sqsubset A \quad w : a \sqsubset A, w : b \not\sqsubset B, \Gamma \vdash \Delta}{w : \forall A \neg \circ B, \Gamma, \vdash \Delta}$

The restrictions are as follows:

1. A and B cannot be modal operations

2. a does not occur in Γ, Δ

The right rules are as follows:

[12] To be more accurate we should write$\vdash G3_{s}yl$ However, context will make it clear what system we are refering to.

[13] This is introduced only to save space in proofs and make them easier to read. We do NOT intend any explicit connection to be drawn between this shorthand, and the way this expression functions in the system

A^e	$\dfrac{\Gamma \vdash \Delta, w : b \not\sqsubset A}{\Gamma \vdash \Delta, w : A^e}$
$\circ$	$\dfrac{\Gamma, \vdash \Delta, w : a \sqsubset A \quad \Gamma, \vdash \Delta, w : a \sqsubset B}{\Gamma, \vdash \Delta, w : A \circ B}$
$\neg\circ$	$\dfrac{\Gamma \vdash \Delta, w : A^e, w : a \sqsubset A \wedge w : a \not\sqsubset B}{\Gamma, \vdash \Delta, w : A \neg \circ B}$
$\forall$	$\dfrac{\Gamma \vdash \Delta, w : a \sqsubset A \quad \Gamma, w : b \sqsubset A \vdash \Delta, w : b \sqsubset B}{\Gamma, \vdash \Delta, w : \forall A \circ B}$
$\forall\neg$	$\dfrac{\Gamma, w : b \sqsubset A \vdash \Delta, w : b \not\sqsubset B)}{\Gamma, \vdash \Delta, w : \forall A \neg \circ B}$

The restrictions are as follows:

A and B cannot be modal operations

b does not occur in Γ, Δ

Now, For the modal rules, the left rules for affirmative divided modal claims are as follows:

$\Box\circ$	$\dfrac{w : a \sqsubset A, v : a \sqsubset B, \Gamma \vdash \Delta \quad w' : a \sqsubset A, v : a \sqsubset B, \Gamma \vdash \Delta}{w : A\Box \circ B, \Gamma, \vdash \Delta}$
$\forall\Box$	$\dfrac{v : a \sqsubset A, \Gamma \vdash \ w : b \sqsubset A, w' : b \sqsubset A, \Delta \quad v : a \sqsubset A, v : b \sqsubset B, \Gamma \vdash \Delta}{w : \forall A\Box \circ B, \Gamma \vdash \Delta}$
$\Diamond\circ$	$\dfrac{w : a \sqsubset A, w'' : a \sqsubset B, \Gamma \vdash \Delta \quad w' : a \sqsubset A, w'' : a \sqsubset B, \Gamma \vdash \Delta}{w : A\Diamond \circ B, \Gamma \vdash \Delta}$
$\forall\Diamond$	$\dfrac{w'' : a \sqsubset A, \Gamma \vdash \Delta, w : b \sqsubset A, w' : b \sqsubset A \quad w'' : a \sqsubset A, w'' : b \sqsubset B, \Gamma \vdash \Delta}{w : \forall A\Diamond \circ B, \Gamma, \vdash \Delta}$

The restrictions are as follows:

The restrictions on a/b are the same as in the non-modal case.

w and w' are distinct.

w" does not occur in Γ, Δ

The right rules are as follows

Symbol	Right Rules
$\Box\circ$	$\dfrac{\Gamma \vdash \Delta, w : a \sqsubset A, w' : a \sqsubset A \quad \Gamma \vdash \Delta, v : a \sqsubset B}{\Gamma, \vdash \Delta, w : A\Box \circ B}$
$\forall\Box$	$\dfrac{\Gamma \vdash v : a \sqsubset A \quad w : b \sqsubset A, \Gamma \vdash \Delta v : b \sqsubset B \quad w' : b \sqsubset A, \Gamma \vdash \Delta, v : b \sqsubset B}{\Gamma, \vdash \Delta, w : \forall A\Box \circ B}$
$\Diamond\circ$	$\dfrac{\Gamma \vdash \Delta, w : a \sqsubset A, w' : a \sqsubset A \quad \Gamma \vdash \Delta, w'' : a \sqsubset B}{\Gamma \vdash \Delta, w : A\Diamond \circ B}$
$\forall\Diamond$	$\dfrac{\Gamma \vdash \Delta, w'' : a \sqsubset A \quad w : b \sqsubset A, \Gamma \vdash \Delta, w'' : b \sqsubset B \quad w' : b \sqsubset A, \Gamma \vdash \Delta, w'' : b \sqsubset B}{\Gamma \vdash \Delta, w : \forall A\Diamond \circ B}$

The restrictions are as follows:

the restrictions on a and b are the same as in the non-modal clauses.

w and w' are distinct.

v does not occur in Γ, Δ

We state the negative modal rules for our system separately, because they are slightly more confusing and require further explanation:

Symbol	left rule
$\Diamond\neg\circ$	$\dfrac{\Gamma, w'' : A^e \vdash \Delta \quad \Gamma, w : a \sqsubset A, w'' : a \not\sqsubset B \vdash \Delta \quad \Gamma, w' : a \sqsubset A, w'' : a \not\sqsubset B \vdash \Delta}{\Gamma, w : A\Diamond\neg \circ B, \Gamma \vdash \Delta}$
$\forall\Diamond\neg$	$\dfrac{\Gamma, w''A^e \vdash \Delta \quad \Gamma, w : b \sqsubset A, w'' : b \not\sqsubset B \vdash \Delta \quad \Gamma, w' : b \sqsubset A, w'' : b \not\sqsubset B \vdash \Delta}{\Gamma, w : \forall A\Diamond\neg \circ B, \Gamma \vdash \Delta}$
$\Box\neg\circ$	$\dfrac{\Gamma, v : A^e \vdash \Delta \quad \Gamma, w : a \sqsubset A, v : a \not\sqsubset B \vdash \Delta \quad \Gamma, w' : a \sqsubset A, v : a \not\sqsubset B \vdash \Delta}{\Gamma, w : A\Diamond\neg \circ B, \Gamma \vdash \Delta}$
$\forall\Box\neg$	$\dfrac{\Gamma, v : A^e \vdash \Delta \quad \Gamma, w : b \sqsubset A, v : b \not\sqsubset B \vdash \Delta \quad \Gamma, w' : b \sqsubset A, v : b \not\sqsubset B \vdash \Delta}{w : \forall A\Box\neg \circ B, \Gamma \vdash \Delta}$

the restrictions on a and b are the same as in the non-modal clauses.

w and w' are distinct.

w" does not occur in Γ, Δ

While these rules look quite frightening, they are actually just a transposition of the negative rules into a modal setting. As we have already seen, 'A is possibly not B' is true, if and only if whatever is or can be A can not be B, or A *can* be empty. This is captured in the rules for $\Diamond\neg\circ$. There are three cases to consider if we want to eliminate that connective. We need to look at the case where A can be empty[14], the case where that which is A can fail to fall under B, and the case where what can be A can fail to fall under B. This is likewise reflected in the universal case, where the only difference is the restrictions on the formulae.

The right rules also reflect this:

Symbol	Right rule
$\Diamond\neg\circ$	$\dfrac{\Gamma \vdash \Delta, w'' : A^e, (w : a \sqsubset A \vee w' : a \sqsubset A) \wedge w'' : a \not\sqsubset B}{\Gamma \vdash \Delta, w : A\Diamond\neg \circ B}$
$\forall\Diamond\neg$	$\dfrac{w : a \sqsubset A, \Gamma \vdash \Delta, w'' : A^e, w'' : a \not\sqsubset B \qquad w' : a \sqsubset A, \Gamma \vdash \Delta, w'' : A^e, w'' : a \not\sqsubset B}{\Gamma \vdash \Delta, w : \forall A\Diamond\neg \circ B}$
$\Box\neg\circ$	$\dfrac{\Gamma \vdash \Delta, v : b \not\sqsubset A, (w : a \sqsubset A \vee w' : a \sqsubset A) \wedge v : b \not\sqsubset A, v : a \not\sqsubset B}{\Gamma \vdash \Delta, w : A\Box\neg \circ B}$
$\forall\Box\neg$	$\dfrac{w : a \sqsubset A, \Gamma \vdash \Delta, v : A^e, v : a \not\sqsubset B \qquad w' : a \sqsubset A, \Gamma \vdash \Delta, v : A^e, v : a \not\sqsubset B}{\Gamma \vdash \Delta, w : \forall A\Box\neg \circ B}$

The restrictions on these rules are as follows:

the restrictions on a and b are the same as in the non-modal clauses.

w and w' are distinct.

v does not occur in Γ, Δ

Following Buridan we define $A\neg\Box \circ B$ to be $A\Diamond\neg \circ B$. We make this assumption to reduce the number of rules in the system and because it is closer to what Buridan did. In principle we could gives rules for both of these connectives and then prove the standard equivalences.

[14] It is important to notice that in the $\diamond$ case, the empty term does not have to be necessarily empty, but only have the possibility of being empty.

The absence of the relational predicate R in our rules is due to the following rule we add to the system:

$$\frac{wRv, \Gamma, \vdash \Delta}{\Gamma \vdash \Delta}\text{ Univ}$$

This has been done as a way of simplifying the system. In essence, what this rule says is that any two worlds are accessible from each other. Because this is a rule of our system we do not state the additional relational restrictions required for $\Box$ and $\Diamond$. It should also be noted that $G3_{syl}$ does *NOT* have the rules for cut and contraction.[15]

We do not add the rule for cut for three reasons. First, none of Buridan's proofs require explicit use of it.[16] Second, as the rules are currently presented, the proof rules are not invertible. To overcome this we can rewrite the principles so that the cases of $\forall$ and $\Box$ reduplicate their conclusion in their premise. This is not done for issues of readiblity, and because they will take us to far afield from formalizing Buridan's work. Third, and most importantly, to prove that a given formulae cannot be proven, we will prove this by root-first decomposition. To make this possible cut cannot be a rule in our system.[17]

A few observations and explanations about the rules are in order. First, it should be noted that there are two ways to introduce a negative claim. If the first term is empty, then the universal or particular negative can be introduced. As Buridan points out in a few places, propositions like 'some chimera is not an ass' are true, but it is true because 'chimera' is an empty term. The second case is that $w : a \sqsubset A$ but $w : a \not\sqsubset B$. I.e. there is something that falls under A, but it does not fall under B. In this sense there are two 'causes of truth' for the claims 'no A is B' and 'some A is not B'. It can either be that one of the terms is empty, or that while objects fall under both terms, they are disjoint/not subsets of each other. In the case of our rules, this is captured by using the A^e to denote empty terms. When we are dealing with e-terms, it entirely possible that the term be empty, and so either the predicate or the subject stands in for nothing.

Second, our rules for introducing the logical connectives $\forall$ and $\exists$ in divided modal claims are based on the 'normal' analysis of the claims. For example, the right rule for $\forall$ says that to prove $\forall A \circ B$ it suffices to show that $w : b \sqsubset A, \Gamma \vdash \ w : b \sqsubset B \wedge w : a \sqsubset A$ where $b \notin \Gamma, \Delta$. This can be seen as expressing the idea that A is a non empty sub-set of B. The non-emptiness follows because of the assumption that $w : a \sqsubset A$ picks out an actual a. Notice that is different from the normal modern paraphrase of 'Every A is B' in which it does not follow that something is A and B. This is because in medieval logic, a universal affirmative claim has two components to it. First, it says that A is non-empty, i.e. that there is at least one thing that is in A, and that anything which falls under A also falls under B. This is expressed in it's corresponding rule, however it has been simplified to the given form to reduce the length of the derivations.

Thus, the connection between the traditional terms and our logic are expressed in the following table.

[15] It is also possible to remove the assumption that axioms are of the form $A, \Gamma \vdash \Delta, A$, have them only be of the form $A \vdash A$, and then show that weakening is admissible.

[16] this is conditional on the assumption that we introduce $\neg\Box$ and $\neg\Diamond$ as a short hand for $\Diamond\neg$ and $\Box\neg$.

[17] Briefly, if cut is a rule of the system, but it is not admissible, then at each step in the decomposition we would need to consider an infinite number of formulae that result from cut. This would make any root-first decomposition produce an infinite tree.

Informal claim	formal representation
All A are B	$\forall A \circ B$
No A is B	$\forall A \neg \circ B$
some A are B	$\exists A \circ B$
some A are not B	$\exists A \neg \circ B$

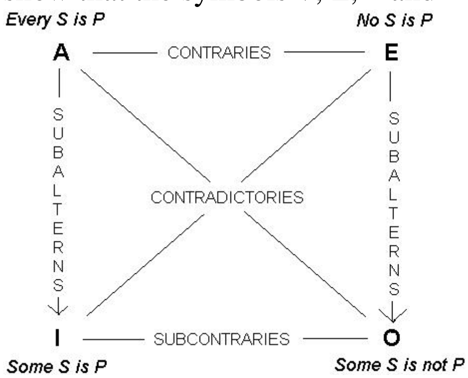
Third, for the modal rules, we observed that in Buridan's treatment of possible model claims there were ways that $\alpha \Diamond \circ \beta$ could be true:

1. There is some (collection of) objects that in the actual world fall under α and fall under B in some other world.
2. There is some (collection of) non-actual objects that satisfy A and B at the same world.
3. There is some (collection of) non-actual objects that in some other world fall under A, and in another fall under B.

This is captured with the left and right rules for our divided modal propositions. It's important to notice that the worlds w and v need to be distinct to ensure this definition works. If this restriction is removed a number of Buridan's conclusions will be lost[18] Philosophically this is interesting since while Buridan is working with metaphysical possibility, a number of inferences considered obvious to modern philosophers are not admissible in his system.

3.2.2 Proofs of Buridan's claims

In this section we will show the soundness direction of our claim. I.e. if $\vdash_B \phi$ then $\vdash_{G3syl} \phi$. To prove this we will show that the positive parts of conclusions one through eight follow from the formalization that we have developed. Before we do this however, we will prove a number of properties about formulae that will make these proofs much shorter. In essence we will show that the symbols $\forall$, $\exists$, $\circ$ and $\neg$ interact as they are defined to in the square of opposition.



[19]

Meta-rules

Throughout this this we will implicitly appeal to the following meta-logical results:

if $\vdash A$ is derivable then so is $\Gamma \vdash \Delta, A$

[18] specifically, all of the negative results of the third and fourth conclusions will be admissible, since in each case, we only need to consider the world w.

[19] image taken from the Standford Enclypedia of philosophy, http://www.seop.leeds.ac.uk/archives/spr2001/entries/square/

if $A \vdash B$ is derivable then so is $A, \Gamma \vdash B, \Delta$

Where A is any formulae. The proof of both of these claims follows immediately from weakening. In addition we show that $\bot$ is derivable from any contradictory assertion, and that the law of excluded middle is derivable for terms.

Contradiction:

$$\dfrac{\dfrac{w : a \sqsubset A \vdash w : a \sqsubset A, \bot \qquad w : a \sqsubset A, \bot \vdash \bot}{w : a \sqsubset A, w : a \sqsubset A \vdash \bot \vdash \bot} L\vdash}{w : a \sqsubset A, w : a \not\sqsubset A \vdash \bot} \text{definition of } \not\sqsubset$$

Excluded middle:

$$\dfrac{\dfrac{w : a \sqsubset A \vdash w : a \sqsubset A, \bot}{\vdash w : a \sqsubset A, w : a \sqsubset \vdash \bot} R\vdash}{\vdash w : a \sqsubset A, w : a \not\sqsubset A} \text{definition of } \not\sqsubset$$

properties of the square of opposition

We now show that all of the properties of the square of opposition hold in our system.

Contradictories:

We need to show the following:

1. $w : \forall A \circ B, w : A\neg \circ B \vdash \bot$
2. $w : \forall A\neg \circ B, w : A \circ B \vdash \bot$

For the first item, consider the following tree:

$$\dfrac{w : \forall A \circ B, w : A^e \vdash \bot \qquad w : \forall A \circ B, w : a \sqsubset A, w : a \not\sqsubset B \vdash \bot}{w : \forall A \circ B, w : A\neg \circ B \vdash \bot} L\neg\circ$$

For the first branch consider the following sub-tree:

$$\dfrac{\dfrac{w : a \sqsubset A, w : a \not\sqsubset A, w : a \sqsubset A, \bot}{w : a \sqsubset A, w : A^e \vdash w : a \sqsubset A, \bot} \qquad \dfrac{w : a \sqsubset A, w : a \sqsubset B, w : a \not\sqsubset B \vdash \bot}{w : a \sqsubset A, w : a \sqsubset B, w : A^e \vdash \bot} LB^e}{w : \forall A \circ B, w : A^e \vdash \bot} L\forall, b = a$$

Where the top branch follows by the definition of $\not\sqsubset$ and $\vdash$.

For the second branch, consider the following subtree

$$\dfrac{w : a \sqsubset A, w : a \sqsubset B, w : a \not\sqsubset B \vdash \bot \qquad w : a \sqsubset A^2, w : a \not\sqsubset B \vdash w : a \sqsubset A, \bot}{w : \forall A \circ B, w : a \sqsubset A, w : a \not\sqsubset B \vdash \bot} L\forall$$

Where the first branch closes by contradiction and the second branch is an axiom. For the second item, consider the following tree:

$$
\dfrac{
\dfrac{w : a \not\sqsubset A, w : a \sqsubset A, w : a \sqsubset B \vdash \bot}{w : A^e, w : a \sqsubset A, w : a \sqsubset B \vdash \bot}\,LA^e
\qquad
w : a \sqsubset A, , w : a \sqsubset B \vdash w : a \sqsubset A, \bot
\qquad
w : a \sqsubset A, w : a \not\sqsubset B, w : a \sqsubset A, w : a \sqsubset B \vdash \bot
}{
\dfrac{w : \forall A \neg \circ B, w : a \sqsubset A, w : a \sqsubset B \vdash \bot}{w : \forall A \neg \circ B, w : A \circ B \vdash \bot}\,L\circ
}\,LA^e
$$

Where the top branches both follow by applying the definition of $\sqsubset$ as in the previous trees.

Subalterns

We need to show the following:

$\vdash w : \forall A \circ B \vdash w : A \circ B$

$\vdash w : \forall A\neg \circ B \vdash w : A\neg \circ B$

For the first item, consider the following tree and sub-tree:

$$\dfrac{\dfrac{w : a \sqsubset A \vdash w : a \sqsubset B \vdash w : A \circ B \qquad w : a \sqsubset A, w : a \sqsubset B \vdash w : A \circ B}{w : \forall A \circ B \vdash w : A \circ B}\, L\forall,\text{ where b=a}}{\vdash w : \forall A \circ B \to w : A \circ B}\, R \to$$

Both branches close by applying $R\circ$.

For the second item, consider the following tree and sub-trees:

$$\dfrac{\dfrac{w : A^e \vdash \Delta \qquad w : a \sqsubset A \vdash w : a \sqsubset A, \Delta \qquad w : a \sqsubset A, w : a \not\sqsubset B \vdash \Delta}{w : \forall A\neg \circ B \vdash \Delta}\, L\forall\neg}{\vdash w : \forall A\neg \circ B \to \Delta}\, R \to$$

where $\Delta = \{w : A\neg \circ B\}$

For the first branch, consider the following subtree $\dfrac{w : A^e \vdash w : A^e, w : a \sqsubset A \wedge w : a \not\sqsubset B}{w : A^e \vdash w : A\neg \circ B}\, R\neg\circ$

There is nothing further to show for the second branch.

For the third branch, consider the following subtree:

$$\dfrac{\dfrac{w : a \sqsubset A, w : a \not\sqsubset B \vdash w : a \sqsubset A \qquad w : a \sqsubset A, w : a \not\sqsubset B \vdash w : a \not\sqsubset B}{w : a \sqsubset A, w : a \not\sqsubset B \vdash w : a \sqsubset A \wedge w : a \not\sqsubset B}\, R\wedge}{w : a \sqsubset A, w : a \not\sqsubset B \vdash w : A\neg \circ B}\, R\neg\circ$$

This completes the proof of the subalterns.

Conversion: We need to show the following:

$\forall A \circ B \to B \circ A$

$A \circ B \leftrightarrow B \circ A$

$\forall A\neg \circ B \leftrightarrow \forall B\neg \circ A$

The first item follows from the second item together with sub-alternation.

For the second item, consider the following tree:

$\vdash$

$$\dfrac{\dfrac{\dfrac{w : a \sqsubset A, w : a \sqsubset B \vdash w : a \sqsubset B \qquad w : a \sqsubset A, w : a \sqsubset B \vdash w : a \sqsubset A}{w : a \sqsubset A, w : a \sqsubset B \vdash B \circ A}\, R\circ}{A \circ B \vdash B \circ A}\, L\circ}{\vdash A \circ B \vdash B \circ A}\, R \vdash$$

$\dashv$ Analogous. $\vdash$

For the third item, apply $R\forall\neg$ followed by $L\forall\neg$, and observe that the following sequent are all axioms:

$$\dfrac{\dfrac{w : b \not\sqsubset A, w : b \sqsubset B \vdash w : b \not\sqsubset A}{w : A^e, w : b \sqsubset B \vdash w : b \not\sqsubset A}\, A^e a = b \qquad w : b \sqsubset B \vdash w : b \not\sqsubset A, w : b \sqsubset A \qquad w : a \sqsubset B, w : b \not\sqsubset A w : b \sqsubset B \vdash w : b \not\sqsubset A}{\dfrac{w : \forall A \neg \circ B, w : b \sqsubset B \vdash w : b \not\sqsubset A}{w : \forall A \neg \circ B \vdash w : \forall B \neg \circ A}\, R\forall\neg}\, L\forall\neg$$

Where the second branch closes because of excluded middle.

$\Leftarrow$

Anaologus.

3.2.3 Modal Properties

Modal Equivalences:
We need to show the following sequents:

$$w : \forall A \Box \circ B, w : A \Diamond \neg \circ B \vdash \bot$$

$$w : \forall A \Box \neg \circ B, w : A \Diamond \circ B \vdash \bot$$

For the first item consider the following tree:

$$\frac{\Gamma, w'' : A^e \vdash \bot \qquad \Gamma, w : a \sqsubset A, w'' : a \not\sqsubset B \vdash \bot \qquad \Gamma, w' : a \sqsubset A, w'' : a \not\sqsubset B \vdash \bot}{w : \forall A \Box \circ B, w : A \Diamond \neg \circ B \vdash \bot} L\Diamond\neg\circ$$

Where $\Gamma = w : \forall A \Box \circ B$ For the first branch consider the following sub trees:

$$\frac{\dfrac{w'' : a \sqsubset A, w'' : a \sqsubset B, w'' : a \not\sqsubset A \vdash \bot}{w'' : a \sqsubset A, w'' : a \sqsubset B, w'' : A^e \vdash \bot} \qquad \dfrac{w'' : a \sqsubset A, w'' : a \not\sqsubset A \vdash \Xi}{w'' : a \sqsubset A, w'' : A^e \vdash \Xi} LA^e}{w : \forall A \Box \circ B, w'' : A^e \vdash \bot} LA^e$$

where $\Xi = \{w : a \sqsubset A, w'' : a \sqsubset A, \bot\}$
Where both branches follow by contradiction.

For the second branch we have the following:

$$\dfrac{w'' : a \sqsubset A, w : a \sqsubset A, w'' : a \not\sqsubset B \vdash w : a \sqsubset A, w' : a \sqsubset A, \bot \qquad w'' : a \sqsubset A, w'' : a \not\sqsubset B, w : a \sqsubset A, w'' : a \sqsubset B \vdash \bot}{w : \forall A\Box \circ B, w : a \sqsubset A, w'' : a \not\sqsubset B \vdash \bot}\, L\forall\Box$$

The first is an axiom, the second follows by contradiction. The third branch is analogous to the second.

For the second item, consider the following tree:

$$\dfrac{w : a \sqsubset A, w'' : a \sqsubset B, w : \forall A\Box\neg \circ B \vdash \bot \qquad w' : a \sqsubset A, w'' : a \sqsubset B, w : \forall A\Box\neg \circ B \vdash \bot}{w : \forall A\Box\neg \circ B, w : A\Diamond \circ B \vdash \bot}\, L\forall\Box$$

We show the proof tree for the first branch, and note the second branch is analogous.

$$\dfrac{w : a \sqsubset A, w'' : a \sqsubset B, w'' : A^e \vdash \bot \qquad w : a \sqsubset A, w'' : a \sqsubset B, w : a \sqsubset A, w'' : a \not\sqsubset B \vdash \bot \qquad w : a \sqsubset A, w'' : a \sqsubset B, w' : a \sqsubset A, w'' : a \not\sqsubset B \vdash \bot}{w : a \sqsubset A, w'' : a \sqsubset B, w : \forall A\Box\neg \circ B \vdash \bot}\, L\forall\Box\neg$$

All three branches follow by contradiction.

3.2.4 Proof of Conclusions 1-8

Conclusion 1

Recall Buridan's first conclusion.

> From any proposition of possibility there follows as an equivalent another of necessity and from any of necessity another of possibility [15]

Formally, we need to show that for any divided modal claim $\alpha * \Box \circ \beta$ there is an equivalent modal claim, $\gamma * \Diamond * \circ\delta$

This conclusion follows immediately from the modal equivalences we proved to hold.

Conclusion 2

Buridan's second conclusion is that

> In every divided proposition of necessity the subject is ampliated to supposit for those which can be. [15]

For this conclusion there is nothing to be formally proven. From our perspective, we have formally built this conclusion into the proof rules themselves. Buridan's concern here is to show that necessity does not only range over those things that are now A, but also those things which could be A at some point in time. In each of the rules for $\Box$ we have given, this is ensured by the requirement that w and w' be distinct, and by the rules that govern left and right $\Box$ rules.

Conclusion 3

Buridan's third conclusion:

> from no proposition of necessity does there follow an assertoric or vice versa, except that from a universal negative of necessity a universal negative assertoric follows. I speak always of those with an affirmed mode. Also I mean if the subject is not restricted by 'that which is'; and I say this because 'That which is B is necessarily A, so B is A' does follow. [15]

To prove the positive part of the claim it suffices to show that
$\forall A\Box\neg \circ B \vdash \forall A\neg \circ B$

$$\dfrac{\dfrac{w : b \sqsubset A, A^e \vdash w : b \not\sqsubset B \qquad w : b \sqsubset A, w : b \sqsubset A, w : b \not\sqsubset B \vdash w : b \not\sqsubset B \qquad w : b \sqsubset A, w' : b \sqsubset A, w : b \not\sqsubset B \vdash w : b \not\sqsubset B}{\forall A\Box\neg \circ B, w : b \sqsubset A \vdash w : b \not\sqsubset B}\, L\forall\Box\neg}{\forall A\Box\neg \circ B \vdash \forall A\neg \circ B}\, R\forall\neg$$

The first follows by LA^e and contradiction. The second and third branches are both axioms.
Now, to see that the negative part of this follows, we have to show that the following sequents are not derivable:

$\forall A\Box \circ B \vdash \forall A \circ B$

$A\Box \circ B \vdash A \circ B$

$A\Box\neg \circ B \vdash A\neg \circ B$

We will do this by root-first decomposition. I.e. we will look at each possible rule that can be applied at a given step in a tree and then show that either the process cannot be derived from any combination of axioms or $L\bot$.
Root: $\forall A\Box \circ B \vdash \forall A \circ B$
Depth 1: we have two cases to consider:

$$\dfrac{w : a \sqsubset A \vdash w : a \sqsubset A, w' : a \sqsubset A, \Delta \qquad w : a \sqsubset A, w : a \sqsubset B \vdash \Delta \qquad w' : a \sqsubset A, w : a \sqsubset B \vdash \Delta}{\forall A\Box \circ B \vdash \Delta}\, L\forall\Box$$

Where $\Delta = \{\forall A \circ B\}$
and

$$\frac{\forall A\Box \circ B, w : b \sqsubset A \vdash w : b \sqsubset B \quad \forall A\Box \circ B \vdash w : a \sqsubset A}{\forall A\Box \circ B \vdash \forall A \circ B} R\forall$$

Call these trees 1 and 2 respectively.

Depth 2: Consider the top sequents of tree 1, we show that the third branch cannot be proven.

$$\frac{w' : a \sqsubset A, w : a \sqsubset B \vdash w : b \sqsubset A \quad w' : a \sqsubset A, w : a \sqsubset B, w : b \sqsubset A \vdash w : b \sqsubset B}{w' : a \sqsubset A, w : a \sqsubset B \vdash \forall A \circ B} R\forall$$

Neither of these are in the form of an axiom, or $L\bot$.

For tree 2 consider the second top-sequent:

$$\frac{w : a \sqsubset A \vdash w : a \sqsubset A, w' : a \sqsubset A, w : a \sqsubset A \quad w : a \sqsubset A, \Gamma \vdash \Delta \quad w' : a \sqsubset A, \vdash w : a \sqsubset A}{\forall A\Box \circ B, w : b \sqsubset A \vdash w : b \sqsubset B} L\Box\forall \text{ v=w}$$

Where$\Gamma = \{w : a \sqsubset B\}$ Notice that the third top-sequent is not in the form of an axiom. Changing the world that v stand in for will not help the situation, since if we let v=w', then the second top sequent will not be in the form of an axiom, and if we let $v \neq w', v' \neq w$ then nether the second or the third branch are of the form of an axiom.

Hence, in both cases, at least branch of the tree does does not close.

The proofs for the particular cases are similar.

The negative portion of conclusion three is extremely interesting. As we point out in our discussion of Buridan's text, his analysis of modal claims as a disjunction is different from modern accounts of modal logic. These proofs/non-proofs make formal exactly how interesting this is. As we stated in the introduction to this chapter, there are *no* restrictions on the accessibility relationship R. More specifically, it can be easily shown that the proof theoretic version of the T-axiom:

$$\frac{wRw, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}$$

is admissible in $G3_{syl}$.[20]

With this, one might think that the other three inference patterns, $w : \forall A\Box \circ B \vdash w : \forall A \circ B$ $w : A\Box \circ B \vdash w : A \circ B$ and $w : A\Box\neg \circ B \vdash w : A\neg \circ B$ should be valid. To motivate this, one might point out that all four of these rules 'look like' the normal T-axiom, just changed into a term-based context. However, as we have seen, Buridan reads the modal terms as disjunctions, and it is exactly this fact that causes the inference from $A\Box \circ B \vdash A \circ B$ to fail in the other three cases. Buridan has a different reading of the terms necessary and possible then the one that is given in contemporary modal logic. For Buridan, $A\Box \circ B$ can be true in two situations: some a falls under A in world w, and for every world w' accessible from w, a falls under B at w' There is some accessible world w', where a falls under A and from every world accessible from w', a falls under B.

This disjunctive reading of the modal operators is different from the modern propositional understanding of necessary and possible, where the operations are defined in the usual manner, i.e.

$\Box\phi$ is true, if and only if if v is accessible from w then v makes ϕ true $\Diamond\phi$ is true, if and only if there is some v is accessible from w and v makes ϕ true

Notice that in the modern case the truth conditionals are not disjunctive, and only consider the worlds accessible from the world where $\Box\phi$ is asserted. In Buridan's case, it's as though we are looking at two worlds instead of one.

Conclusion 4

Buridan's fourth conclusion is that

> from no proposition of possibility does there follow an assertoric or vice versa, except that from every affirmative assertoric proposition there follows an affirmative particular of possibility. I speak always of affirmed modes. [15]

Formally, the positive part of this proposition states that these two sequent are derivable

$A \circ B \vdash A\Diamond \circ B$

$\forall A \circ B \vdash A\Diamond \circ B$

We prove this for the sequent $A \circ B \vdash A\Diamond \circ B$ and note that the second sequent follows from the first by sub-altern.

[20] just apply the univ rule, where v=w.

$$\dfrac{\dfrac{w : a \sqsubset A, w : a \sqsubset B \vdash w : a \sqsubset A, w' : a \sqsubset A \qquad w : a \sqsubset A, w : a \sqsubset B \vdash w : a \sqsubset B}{w : a \sqsubset A, w : a \sqsubset B \vdash w : A\Diamond \circ B}\, R\Diamond \text{ w=w''}}{w : A \circ B \vdash w : A\Diamond \circ B}\, L\circ$$

To show that the negative part of the proposition holds, we need to show that

$\forall A \neg \circ B \vdash \forall A \diamond \neg \circ B$

$A \neg \circ B \vdash A \diamond \neg \circ B$

are not provable

Again, we procced by root-first decomposition Root: $A\neg \circ B \vdash A \diamond \neg \circ B$ Stage 1: We have two possible rules to consider

$$\dfrac{A^e \vdash A \diamond \neg \circ B \qquad w : a \sqsubset A, w : a \not\sqsubset B \vdash A \diamond \neg \circ B}{A\neg \circ B \vdash A \diamond \neg \circ B}\, R\Diamond\neg\circ$$

Call this tree 1

$$\dfrac{A\neg \circ B \vdash w : A^e \qquad A\neg \circ B \vdash w : a \sqsubset A, w : a \not\sqsubset B \qquad A\neg \circ B \vdash w' : a \sqsubset A, w : a \not\sqsubset B}{A\neg \circ B \vdash A \diamond \neg \circ B}\, R \diamond \neg\circ$$

Call this tree 2

Stage 2: For tree 1, we apply the only remaining rule and observe that the first branch is not an axiom.

$$\dfrac{w : a \sqsubset A, \Gamma \vdash w : A^e \qquad w : a \sqsubset A, \Gamma \vdash w : a \sqsubset A, w : a \not\sqsubset B \qquad w : a \sqsubset A, \Gamma \vdash w' : a \sqsubset A, w : a \not\sqsubset B}{w : a \sqsubset A, w : a \not\sqsubset B \vdash A \diamond \neg \circ B}\, R \diamond \neg$$

Where $\Gamma = w : a \not\sqsubset B$ For tree 2, we simply observe that the argument is analogous to the argument for tree 2 in conclusion three. Here the sequent will not be derivable because of the restrictions on $L\neg\circ$

Stage 3:

For tree 1, we apply the only remaining rule, LA^E, and then the definition of $\not\sqsubset$.

$$\dfrac{\dfrac{w : a \sqsubset A, w : a \not\sqsubset B, w : b \sqsubset A \vdash \bot}{w : a \sqsubset A, w : a \not\sqsubset B \vdash w : b \not\sqsubset A}}{w : a \sqsubset A, w : a \not\sqsubset B \vdash w : A^e}$$

And observer that there is no rule that allows us to derive $\bot$ on the right.

Conclusion 5

Buridan's fifth conclusion was that

> From every affirmative of possibility there follows by conversion of the terms a particular affirmative of possibility, but not a universal, and from no negative of possibility does there follow by conversion of the terms another of possibility. [15]

For the positive claim, we need to show that:

$\forall A\Diamond \circ B \vdash B\Diamond \circ A$

$A\Diamond \circ B \Leftrightarrow B\Diamond \circ A$

For the first proposition, it suffices to show the second item and that $\forall A\Diamond \circ B \vdash A\Diamond \circ B$ We prove the second claim, and that modal sub-alternation holds in this system.

For the second item consider the following tree:

$$\dfrac{\dfrac{w' : a \sqsubset A, w'' : a \sqsubset B \vdash w : a \sqsubset B, w''A \sqsubset B \quad w' : a \sqsubset A, w'' : a \sqsubset B \vdash w'' : a \sqsubset A}{w' : a \sqsubset A, w'' : a \sqsubset B \vdash w : B\Diamond \circ A}\; R\Diamond,\ \text{w'=w''} \qquad w : a \sqsubset A, w'' : a \sqsubset B \vdash w : B\Diamond \circ A}{w : A\Diamond \circ B \vdash w : B\Diamond \circ A}$$

Observe that the second branch closes by application of $R\Diamond$ with $w' = w''$ For modal sub-alternation, consider the following tree:

$$\dfrac{\dfrac{w' : a \sqsubset A, w'a \sqsubset B \vdash w : a \sqsubset A, w' : a \sqsubset A \quad w' : a \sqsubset A, w'a \sqsubset B \vdash w' : a \sqsubset B}{w' : a \sqsubset A, w'a \sqsubset B \vdash A\Diamond \circ B}\; R\diamond \qquad w' : a \sqsubset A \vdash A\Diamond \circ B, w : a \sqsubset A, w' : a \sqsubset A}{\forall A \diamond \circ B \vdash A \diamond \circ B}\; L\forall\diamond$$

We prove the negative universal case does not follow. In particular we will show that $\forall A \diamond \neg \circ B \vdash \forall B \diamond \neg \circ A$ and note that the particular case is similar.

Root: $w : \forall A \diamond \neg \circ B \vdash w : \forall B \diamond \neg \circ A$

Stage 1: We have two possible rules we can apply:

$$\frac{w'' : A^e \vdash \Delta \quad w : b \sqsubset A, w : b \not\sqsubset B \vdash \Delta \quad w' : b \sqsubset A, w : b \not\sqsubset B \vdash \Delta}{w : \forall A \diamond \neg \circ B \vdash \Delta} L\forall \diamond \neg$$

Where$\Delta = \{\forall B \diamond \neg \circ A\}$ Call this tree 1

$$\frac{w : \forall A \diamond \neg \circ B \vdash w : A^e, w : a \sqsubset A, w' : a \sqsubset A \quad w : \forall A \diamond \neg \circ B \vdash w : A^e w : a \not\sqsubset B}{w : \forall A \diamond \neg \circ B \vdash w : \forall B \diamond \neg \circ A} L\forall \diamond \neg$$

Call this tree 2

Stage 2: For the first tree we simply observe that the second and third top-sequents will not yeild proofs by a similar argument to the other instances of tree two in previous conclusions.

For the second tree, consider the second top-sequent. We have two possible rules to consider.

$$\frac{w : \forall A \diamond \neg \circ B \vdash w : b \not\sqsubset A, w : a \not\sqsubset B}{w : \forall A \diamond \neg \circ B \vdash w : A^e w : a \not\sqsubset B}$$

Call this tree 1.1

$$\frac{w'' : A^e \vdash w : A^e w : a \not\sqsubset B \quad w : a \sqsubset A, w'' : a \not\sqsubset B \vdash \Delta \quad w' : a \sqsubset A, w'' : a \not\sqsubset B \vdash \Delta}{w : \forall A \diamond \neg \circ B \vdash \Delta} \forall \diamond \neg$$

Where $\Delta = \{w : A^e w : a \not\sqsubset B\}$ Call this tree 1.2

The top sequents of tree 1.2 are clearly not instances of either the axiom rule or $L\bot$. After applying $L\forall \diamond \neg$ tree 1.2 will reduce to a similar tree where the world labels will not match.

Conclusion 6

Buridan's sixth conclusion was that

> from no proposition of necessity does there follow by conversion of the terms another of necessity, except that from a universal negative there follows a universal negative. [15]

For the positive part of the conclusion we need to show that $w : \forall A \Box \neg \circ B \vdash w : \forall B \Box \neg \circ A$

$$\frac{\Gamma, w : b \sqsubset A \vdash x : a \not\sqsubset A, x : a \not\sqsubset B \quad \Gamma, w' : b \sqsubset A \vdash x : a \not\sqsubset A, x : a \not\sqsubset B}{w : \forall A \Box \neg \circ B \vdash w : \forall B \Box \neg \circ A}$$

Where $\Gamma = \{w : \forall A \Box \neg \circ B\}$ For the first branch, we apply the only possible rule and need to show the following are derivable:
$w : b \sqsubset A^2, x : b \not\sqsubset B \vdash w : a \sqsubset A, x : a \not\sqsubset A, x : a \not\sqsubset B$
$w' : b \sqsubset A, x : b \not\sqsubset B, w : b \sqsubset A \vdash w : a \sqsubset A, x : a \not\sqsubset A, x : a \not\sqsubset B$
$x : b \not\sqsubset A, w : b \sqsubset A \vdash w : a \sqsubset A, x : a \not\sqsubset A, x : a \not\sqsubset B$
Which they clearly are. .

For the second branch, apply the only remaining rule, and we obtain the following three sequents:
$w : b \sqsubset A^2, x : b \not\sqsubset B \vdash x : a \not\sqsubset A, x : a \not\sqsubset B$
$w' : b \sqsubset A, x : b \not\sqsubset B, w : b \sqsubset x : a \not\sqsubset A, x : a \not\sqsubset B$
$x : b \not\sqsubset A, w : b \sqsubset A \vdash x : a \not\sqsubset A, x : a \not\sqsubset B$
These are all axioms.

For the universal positive, and the particular negative, notice that the proof of non-provability is similar to the non-modal case. I.e. the name labels will miss-match.

For the Particular affirmative, we disprove the right to left direction. The left to right direction is analogous. Consider the following decomposition.

Root: $A \Box \circ B \vdash B \Box \circ A$

Stage 1:

We have two possible rules we can apply

$$\frac{w : a \sqsubset A, w : a \sqsubset B \vdash w : B \Box \circ A \qquad w' : a \sqsubset A, w : a \sqsubset B \vdash w : B \Box \circ A}{w : A \Box \circ B \vdash w : B \Box \circ A}$$

Call this tree 1

$$\frac{w : A \Box \circ B \vdash w : a \sqsubset A, w' : a \sqsubset A \qquad w : A \Box \circ B \vdash w' : a \sqsubset B}{w : A \Box \circ B \vdash w : B \Box \circ A}$$

Call this tree 2.

Stage 2:

For both trees we only have one rule we can apply. For tree one, consider the second top-sequent.

$$\frac{w' : a \sqsubset A, w : a \sqsubset B \vdash w' : a \sqsubset A, w : a \sqsubset B \qquad w' : a \sqsubset A, w : a \sqsubset B \vdash v : a \sqsubset B}{w' : a \sqsubset A, w : a \sqsubset B \vdash w : B \Box \circ A}$$

The second top-seqent is not an axiom or an instance of $L\bot$.

For tree two consider the second top-sequent:

$$\frac{w : b \sqsubset A, w : b \sqsubset B \vdash w' : a \sqsubset B \qquad w' : b \sqsubset A, w : b \sqsubset B \vdash w' : a \sqsubset B}{w : A \Box \circ B \vdash w' : a \sqsubset B}$$

Neither of the top sequents are instances of an axiom or $L\bot$.

Conclusion 7

Buridan's seventh conclusion was

> every proposition of each way-contingency having an affirmed mode is converted into [one of] the opposite quality with an affirmed mode, but none is converted if the result of conversion or what was converted had a negated mode. [15]

Recall that we use the shorthand $A \triangle \circ B$ to refer to $A \Diamond \circ B \wedge A \Diamond \neg \circ B$ Formally, what Buridan is saying is the following:

$$\forall A \triangle \circ B \Leftrightarrow \forall A \triangle \neg \circ B$$

$$A \triangle \circ B \Leftrightarrow B \triangle \neg \circ A$$

For the first item consider the following tree:
$\vdash$

$$\dfrac{\dfrac{\forall A \diamond \circ B, \forall A \diamond \neg \circ B \vdash \forall A \diamond \neg \circ B \qquad \dfrac{\dfrac{\forall A \diamond \circ B, \forall A \diamond \neg \circ B \vdash \forall A \Diamond \circ B}{\forall A \diamond \circ B, \forall A \diamond \neg \circ B \vdash \forall A \neg \Box \neg \circ B}\text{modal Equivelence}}{\forall A \diamond \circ B, \forall A \diamond \neg \circ B \vdash \forall A \diamond \neg\neg \circ B}\text{Modal equivelence}}{\forall A \diamond \circ B, \forall A \diamond \neg \circ B \vdash \forall A \triangle \neg \circ B}\text{definition of } \triangle}{\forall A \triangle \circ B \vdash \forall A \triangle \neg \circ B}\text{definition of } \triangle$$

$\Leftarrow$

The other direction is analogous. The proof of the second item follows immediately from the definition of $\triangle$ and the 6th conclusion. For the negative portion of the conclusion, Buridan is claiming that $\triangle$ and $\neg\triangle$ are not interdefinable. I.e.

$$\forall A \diamond \circ B, \forall A \diamond \neg \circ B \not\Leftrightarrow w : \forall A \neg \triangle \circ B$$

$$A \triangle \circ B \not\Leftrightarrow B \neg \triangle \circ A$$

$$\forall A \triangle \neg \circ B \not\Leftrightarrow \forall A \neg \triangle \neg \circ B$$

$$A \triangle \neg \circ B \not\Leftrightarrow B \neg \triangle \neg \circ A$$

To prove this, Buridan points out that $\forall A \triangle \circ B$ and $\forall A \neg \triangle \circ B$ are contraries. This can be easily seen by showing that $\forall A \triangle \circ B, \forall A \neg \triangle \circ B \vdash \bot$ is derivable but that $\vdash \forall A \triangle \circ B \wedge \forall A \neg \triangle \circ B$ is not derivable.

$$\dfrac{\dfrac{\dfrac{\forall A \diamond \circ B, \forall A \diamond \neg \circ B, \forall A \Box \circ B, \forall A \Box \neg \circ B \vdash \bot}{\forall A \diamond \circ B, \forall A \diamond \neg \circ B, \forall A \neg \diamond \circ B, \forall A \neg \diamond \circ B \vdash \bot}}{\forall A \diamond \circ B, \forall A \diamond \neg \circ B, \forall A \neg \triangle \circ B \vdash \bot}}{\forall A \triangle \circ B, \forall A \neg \triangle \circ B \vdash \bot}$$

Where the initial sequent follows by modal contradictory.

To see that $\vdash \forall A \triangle \circ B \wedge \forall A \neg \triangle \circ B$ is not derivable, it suffices to show that $\vdash \forall A \diamond \circ B$ is not derivable. The proof is by root-first decomposition, and is left to the reader.

Now, Buridan does not fully spell out why the modal contrary clause is sufficient to show that the negative part of the conclusion is true. However, the upshot of Buridan's point is that, if $\triangle$ and $\neg\triangle$ were inter-changeable, then our logic would be unsound. To see this consider the following tree:

$$\dfrac{\dfrac{\forall A \triangle B, \forall A \neg \triangle B \vdash \bot}{\forall A \triangle B, \forall A \triangle B \vdash \bot}\text{Bad definition of } \neg\triangle}{\forall A \triangle B \vdash \bot}\text{R weakening}$$

Where the inital sequent follows by the modal contrary clause.

Conclusion 8

For Buridan's eight and the last conclusion we will look at here, he writes

> No proposition of contingency can be converted in terms into another of contingency, but any having an affirmed mode can be converted into another of possibility.

[15]

We need to show that:

$$w : \forall A \triangle \circ B \vdash w : \forall B \diamond \circ A$$

$$w : A \triangle \circ B \vdash w : B \Diamond \circ A$$

These both follow immetatly from the defintion of $\triangle$ and conclusion six.

For the negative portion, we need to show that

$$w : \forall A \triangle \circ B \vdash w : \forall B \triangle \circ A$$

$$w : \forall A \triangle \neg \circ B \vdash w : \forall B \triangle \neg \circ A$$

$$w : A \triangle \circ B \vdash w : B \triangle \circ A$$

$$w : A \triangle \neg \circ B \vdash w : B \triangle \neg \circ A$$

For this, it suffices to show the following do not follow:

$$w : \forall A \diamond \circ B, w : \forall A \diamond \neg \circ B \vdash w : \forall B \diamond \circ A$$

$$w : A \diamond \circ B, w : A \diamond \neg \circ B \vdash w : B \diamond \circ A$$

The proof of both of these is by root-first decomposition and are very similar to the proofs in conclusion five.

3.3 Future work

The goal of this chapter was to develop a logical formalism that is adequate to represent Buridan's notion of logical consequence. As we have seen, the system $G3_{syl}$ is sound with respect to Buridan's own claims, assuming cut to be admissible. Before we move on to consider some methodological and philosophical questions that the process of formalizing historical thinkers raises, it should be noted that there is still quite a bit of work to be done on Buridan's account of consequence.

3.3.1 Extensions of $G3_s$

As far as this system's treatment of consequence is concerned, there is still quite a bit of work still to be done. The system developed here only captures one part of Buridan's treatment of consequence. While our language is rich enough to distinguish composite modal propositions, we have not had the space to develop rules and prove Buridan's conclusions for such propositions. This is a very natural extension of the system $G3_{syl}$ and should be doable with minimum difficulty. As we noted in chapter 2, composite modal claims have a number of interesting connections with modern propositional modal logic.[21] As such, the conclusions which show how these two kinds of composite modal claims interact should prove interesting. There are three further aspects of Buridan's logic that could be used as opportunities to extend $G3_syl$. The first has to do with the addition of temporal operations to the systems. While Buridan does not provide an extensive treatment of temporal claims, Buridan does intend his theory of consequence to be able to deal with temporal claims. He mentions briefly how ampliation and supposition apply to temporal claims. And while none of the conclusions in this books deal with them explicitly it would be a natural extension of such a system. For additional information on Buridan's approach to temporality, we would need to look at some of his other writings, however the resulting system could be a very interesting exploration of multi-modal term based logics. Another part of Buridan's theory that was not formalized within $G3_{syl}$ relates to the supposition of restrictions. As we saw in Chapter 2, expressions like 'that which is' can be added to the language and have a marked impact on what a modal term supposits for. This seems to be a way of making the ampliation of the term explicit and it would be helpful to express this part of Buridan's theory within the formal system itself. For example, the proposition 'that which is A can be B', is taken to make explicit the fact that A only supposits for things that are now A. Again, this connects with the temporal material, since the restrictions can be over time as well as possibility. Formally, this corresponds to a kind of restriction on the way the rules are applied in our system. Our modal rules are of the form ($w : a \sqsubset A \vee w' : a \sqsubset A \vdash v : B$). The expression 'that which' can be viewed as an operation that modifies the world labels. For example, 'that which is' would remove the $w' : a \sqsubset A$ sequent from the disjunction. Likewise, the expression 'that which was' would shift any temporal label to a point in the past etc.

Third, Buridan's treatment of modal consequences has a particular goal in mind: the treatment of modal syllogistics. In principle the system can represent syllogisms as the following sequent: $M, m \vdash C$, where M is the major premise, m is the minor premise and C is the conclusion. A very natural and straight forward application of $G3_syl$ would be to see if it successfully captures Buridan's conclusions about modal syllogistics as developed in book four of this work.

3.3.2 Meta-theory

As we have seen the system $G3_{syl}$ is an interesting proof theoretic modal logic. It is a free logic, and it uses a non-standard definition of the modal operators. However, there has been no attempt in this thesis to look at the meta-theory of $G3_{syl}$. Can the system $G3_{syl}$ be modified in non-essential ways so that cut elimination[22] None of the syllogistic rules are invertible. However, it is in principal possible to reformulate the rules so that they are invertible.However can it be

[21] As we saw before, something like propositional negation can be obtained by using the composite modal sentence 'it is false that 'A is B'', this is also similar for composite modal claims.

[22] With the rules phrases the way they are now, the answer is no.

shown that cut is admissible? We think it should be, however the proof hinges on being able to translate the disjunctive character of the modal operations into a normal form.

3.3.3 Semantics

Another important issue is to develop a semantics that is sound and complete with respect to $G3_{syl}$. This chapter developed Buridan's theory from a purely proof-theoretic perspective. However, it is clear that Buridan has some semantics in mind when he developed this system. This can be seen by his constant appeal to counter-examples. The question then becomes, can we give a corresponding formal semantic to this system? In answer to this, the question is yes. As the reader probably already observed, there is a close connection between these label-based tree system and semantic tableaux for modal logic.[23] In her paper[9], Negri explains how to define a tableaux process that makes essential use of the proof theoretic rules. We use the proof theoretic rules to induce the semantic rules for each of our connectives in the natural way. Then, we call a tree 'open' if one of its branches ends with a formulae of the form $\Gamma \vdash \Delta$ where $\Gamma \cap \Delta = \emptyset$. We then read off a counter-model in the natural way. We let every element of Γ be true, and every member of Δ false. Assuming that the definitions of the various connectives have been imported into the semantics correctly, this should yield appropriate semantics for $G3_{syl}$ However, at this stage, this is still conjecture.

What we have successfully done here is developed a formalization of Buridan's modal logic that is able to capture the various distinctions, theories and inferences that Buridan holds to define modal logic. But, this does raise a number of interesting philosophical questions. Of interest to us, is why we think such a modern formalization of Buridan's theory is faithful to Buridan at all. After all, the techniques of structural proof theory (and mathematical logic in general) did not exist when Buridan was writing. In what way can we then claim to have captured Buridan's notion of logic consequence. We will turn to this question in the next chapter.

[23] For example, our proofs of the contradictory clauses very closely resemble semantic tableaux.

4. CONCLUSION

In our previous chapter we developed the system $G3_{syl}$ and we were able to show that it was sound with respect to our formalization of Buridan's conclusions about divided modal propositions. In concluding this thesis, I want to briefly reflect on how successful this formalization was by raising some philosophical questions that underpin the formalizations of historical theories.

4.1 Why Formalize historical theories?

Why would we want to formalize the logical writings of various historical figures? What kinds of benefits does formalization offer to our study of such theories? To put it another way, what is the point of using modern logician techniques to formalize historical systems of logic? And, assuming that we do formalize such systems of logic, if and when different systems emerge that are all 'sound' with respect to the historical system, when should we prefer one system to another? Our answer to the first part of this question will be twofold. First we will argue that logical formalization is itself a kind of interpretive tool that the historian of logic (or historian of ideas more broadly) can use to attempt to understand and clarify a given historical figure's ideas. In this sense, formalization can be seen as an interpretive technique. Like any other exegetical technique, it can illuminate what a given historical figure is doing but also needs to be sensitive to the figure's own assumptions, concepts etc. Second, we will argue that formalizing a historical system is a way of making non-formalized ideas about logic accessible and intelligible to modern logicians. As we have been at pains to show, Buridan's approach and understanding of logic is in many ways different from the research questions that drive modern logic. However, these systems present interesting, though by modern standards, informally presented ideas. Formalizing these ideas may not only lead to the development of systems that are logically interesting, but also allow modern logicians to better understand those who have gone before them, and how they understood and applied logic.[1]

In the previous two chapters of this thesis we developed the sequent calculus $G3_{syl}$ as a way of formalizing the theory of logical consequence that Buridan develops in *De Consequentia*. We have been able to show that the system is correct with respect to his analysis of divided modal propositions, and have sketched the necessary additions to prove soundness for composite modal claims. However, if we take a step back from this program, two questions naturally and quickly emerge. First, is $G3_{syl}$ a faithful formalization of Buridan's system? Second, why do we care, either as logicians, or as historians of logic, that Buridan's system can be given a formal representation?

Why should we think that $G3_{syl}$ is a faithful representation of Buridan's reasoning? $G3_{syl}$ is a mathematical language based on structural proof theory. Buridan's wittings are in Latin.

[1] The classical example of this is the logical of Arthur Prior. Prior drew inspiration from his study of various historical figures.[1, Pg 322-340]

The Latin of the late middle ages was, for the most part, a very regimented kind of Latin. But the important difference is that, at least at face value, these two kinds of languages are very different. One should wonder just how faithful any kind of modern mathematical representation of Buridan's reasoning is. This gives rise to our first problem; how do we ensure that any modern attempt (including the one presented in this work) to develop a logical formalism for Buridan's logic is a faithful one?

Broadly, faithfulness can be understood as having two components, namely intrinsic faithfulness and extrinsic faithfulness. Intrinsic faithfulness is best thought of as faithfulness to the core or essential components of the phenomena in question. In the case of something like Buridan's logic, internal faithfulness is best thought of as a faithful formalization of the logical consequences Buridan takes to be valid. Extrinsic faithfulness is faithfulness to aspects of the phenomena that are not essential to the overall theory, but are still elements of the theory that would be good or helpful to have. In the case of Buridan, we would want to capture things like his use of semantic and syntactic reasoning, try to be faithful to the general proof techniques he makes use of etc.

How do we determine when a formalization is intrinsically or extrinsically faithful? This problem is not unique to the formalizer of historical phenomena. In fact, it is one faced by anyone who wishes to apply formal logic to external phenomena, be they philosophical, linguistic, or even mathematical. What justifies a formalism? Often,

> This is seen as a matter of linguistic intuition or taken to be sufficiently legitimized by common practice.[2, Pg.94]

As Baumgartner and Lampert go on to discuss, the fact that we have a formalization for a given linguistic (or in our case, historic) phenomenon dose not tell us anything about the quality of our formalization. For example, consider the following formalization of the consequent 'Some human is white therefore some white thing is a human.' Say we formalize the expression in G3cp[2] and represent it as saying that the sequent $\Rightarrow A \rightarrow B$is derivable. Clearly, this is a very poor formalization, assuming that our goal is to show a given sequent is valid.[3] The formal argument is invalid according to G3cp,[4] as can be seen either by letting v(A)=1 and v(B)=0 or doing root-first decomposition on the sequent.

This example illustrates one of two necessary features of intrinsic faithfulness. For a given phenomenon to be correctly modelled, the formal language that we use needs to be able to abstract the right amount of information away from the phenomenon in question. Call this the granularity criteria.

This can go wrong in two ways. First, the formalization can be too coarse with respect to a given phenomenon. In the above example, the coarseness is due to how the various propositions are represented. It does not capture the relevant logical details of the term-based expressions, and so is unable to faithfully 'translate' the phenomenon into the formal language. What happens in this case is that the internal complexities of various propositions are not correctly represented. Then 'different' propositions end up being treated as the same. Again, consider the example where we represent 'Some human is white therefore some white thing is a human' as $\Rightarrow A \rightarrow B$. Here we are treating 'some human is white' and 'some white thing is human' as the propositional variables A and B respectively. The language is correct in so far as it observes

[2] That is, the classical propositional calculus, with the rule basis G3 see[10]

[3] There could be other goals one has in using logic for which this kind of formalization is adequate.

[4] see[10, Pg. 58-60] for the proof of completeness of G3cp

that these two propositions are different. However, it does this in the wrong way. It completely ignores the common elements between them, namely the copula 'is' , the presence of the word 'some' and the fact that the same terms occur in both propositions. It also fails to express that the subject and predicate switched positions. In this sense, to treat the term-based propositions[5] as mere propositional variables of the form A, B etc is too coarse. It abstracts away too much information from the phenomenon and in doing so we lose some very important internal structure. Thus, the language of formalization is too coarse in so far as it abstracts away too many of the features that are relevant to the particular set of questions we want to investigate about a given phenomenon. In our case, we are interested in Buridan's treatment of modal propositions, and so our principle concerns will relate to issues around validity.

A formalization can also be too fine-grained. A translation can be so fine-grained that two propositions which should be seen to be the 'same' have different logical forms when translated into the formalization. There are a number of factors that need to be considered when trying to figure out if two propositions should have the same form. In some cases their may be external features or data we get from the phenomenon itself that is suggestive. However, we also need to consider what exactly our goals are in developing our formalization and what we intend to apply the formalization to. In that sense, the 'sameness' criteria is also determined by the goals of the formalizer.

As an example of a language being too fine-grained, consider the following example:

> 'Some star is shining on our hemisphere therefore something shining on our hemisphere is a star.'
>
> 'Some child is sitting on a log therefore someone sitting on a log is a child.'

Now, say our formal language is a variation on G3cp, with the copula $\circ$ for 'is' and the following terms added
A;p, B;i, which respectively mean, A is a person and B is an inanimate object.

Formally, the above sentences would be
$\Rightarrow S; i \circ T \rightarrow T \circ S; i$ And
$\Rightarrow C; p \circ S; i \rightarrow S; i \circ C; p$

As can be seen, these two propositions come out with different logical forms. In the first sentence, the expression 'shining on our hemisphere' is a predicate modifying some star, and so is neither a person nor an inanimate object. But in the second case, the predicate has an inanimate term attached to it, because of the presence of the word 'log' in the example. Hence, the two terms have different (if somewhat contrived) logical forms. Now, if our goal is to provide a faithful formalization of the term-based arguments used by Buridan, as far as *De Consequentia* is concerned, the language above is too fine-grained. It does not matter, from Buridan's point of view, if the predicate contains a person or inanimate object in it.[6] In this case, we want a formalization that has the right level of granularity. If it is too fine-grained then the translation has not correctly 'abstracted' away unnecessary informal content. Instead, the formal language differentiates two propositions that the author in question has not.

To summarize this point, selecting the language that we carry out our formalization in, must have the right level of granularity. The language must also be expressive enough to capture all

[5] I.e. propositions of the form All A is possibly B, Some A is not necessarily B etc.

[6] There may be some situations where this distinction matters. It's conceivable that some logics dealing with ethics or law, for example, would want a distinction between persons and inanimate objects built into the language itself.

of the interactions and structures that are present in the various propositions. However, it must not be too fine-grained so that it cannot differentiate the various groups of propositions that the phenomenon in question take as having the same logical form.[7]

Assume that we have a language that has the right granularity to analyze the phenomenon we are interested in. At this point, a system needs to be developed that can treat the various classes of inferences that a given historical figure addresses. At this point the formalizer can encounter another set of problems. The formalism can either under-generate or over-generate[8] with respect to the phenomenon in question. In the case of under-generation, this means that the phenomenon has declared something to be the case, when in the formalism, it does not. Over-generation is the converse, the formalism says something is the case when the phenomenon says that it is not the case.

The notion of under and over-generation can be applied to the granularity criteria. A language under-generates with respect to a given phenomenon if it is too coarse grained. Likewise, a logic is said to over-generate if it is too fine-grained. In that sense, we do not need to assume that we have a sufficiently expressive language before defining over and under-generation qua the formal system. We would simply require that both the language and the consequence relationship do not over or under-generate qua the phenomenon.

This can often be seen in many formalizations of Aristotle's modal syllogistic used in introductory logic texts. Here, formulae are treated as propositional, with the modal operators understood in the modern sense. While this presentation may be useful heuristically, these presentations do *not* capture Aristotle's syllogistic correctly. There is also an important caveat that emerges when we are discussing historical theories. Often the author does not discuss all of the possible ways the various operations can interact within the language. To this end, we need to distinguish the following:

What an author says follows.

What an author says does not follow.

What they remain silent on.

From this there seems to be at least two features of an intrinsically faithful formalization. First, we need to justify the translation of the phenomena into the formalization which has the right level of granularity. Second, we need to be able to justify the rules/structure of the formalization. How might we do this?

With respect to medieval logic, the first task is, conceptually, quite straight forward. Since the medievals, like modern logicians, often use schematic letters like A,B to stand for arbitrary expressions, we can normally get a very clear idea of what aspects of the system they considered important. The texts themselves provide us with numerous examples of the various parts of a sentence that the medievals thought were important to determining the logical form of a

[7] We have deliberately avoided the philosophical questions about how exactly one distinguishes logical from non-logical constants. However, it should be stated that the formalizer must make this choice, and should do so on non ad-hoc grounds. In the case of the history of logic, it should be based on a careful reading of the author in question and the terms they treat as 'logical'.

[8] a formalism is said to under-generate with respect to some phenomenon if the phenomena declares something to be true/valid, while the formalized version of that claim comes out false/invalid. Likewise, a system is said to over-generate if the formalism declares something to be true/valid, when the phenomenon declares that it is false/invalid. The terms are taken from Etchermendy and will be useful for the discussion that follows.

proposition. This then becomes one of the places where having a good interpretation of the text is critical. If we miss-read the text, we run the risk of misunderstanding how the author understood the various logical features of propositions and what features of a given proposition allow it to interact in such a way as to cause various consequences to follow. Any formalization based on a poor reading of the text will struggle to faithfully represent the structure of the various inferences. Buridan's theory provides us with a good example of this. If the reader misunderstands the various places where modal terms can occur in a proposition, they will not be able to capture the distinction between composite and divided sense of a proposition. Any formalization that cannot capture this distinction in the language will be missing a key part of Buridan's theory. Thus the key to getting the granularity of the language correct, is understanding what the author is doing with their logic, and being able to successfully capture any syntactic[9] notions the author makes use of.

Now, turning to how we might justify the rules/structure of the formalization, the challenge before us here is one of showing that the formalization is faithful to the phenomenon. How might we understand faithfulness here? Perhaps the most natural way would be as we briefly outlined in Chapter three. A formalism is said to be correct with respect to a given phenomenon if it is sound with respect to what is the case according to the phenomenon. In other words, it is able to show everything that the author claims follows, and if the author explicitly says something does not follow, it does not follow in the case of the formalization.[10] This fact is critical to a good formalization. If a formalization lacks this, it means there is still more work to be done. There is at least one extrinsic criterion that should also be considered at this point. Ideally, the formalism should be reflective of the way in which the figure in question reasons. The formalization should, in some sense, capture not only the results, but also show some aspect of how the author is reasoning. For example, if the reasoning in question is done entirely based on the meaning of various terms in a proposition, then the formalization should account for this. It should be highly semantic in nature, since the author is also doing something that (we moderns) would understand to be semantic.[11] Likewise, if the author lays down a number of assumptions and proceeds to explore what does and does not follow from them, then a more proof-theoretic account would preserve faithfulness. While these kinds of extrinsic factors should not be decisive, it is at least one factor that is considered.

4.2 How our formalization fares

Our formalization of Buridan's logic is an adequate one, in the sense defined above. It is intrinsically faithful. In terms of the granularity, our language is able to capture everything Buridan wants to say about terms[12]. We are able to represent all of the different terms, together

[9] in the sense of which operations can go where distinctions

[10] It should be noted that this is not to say the formalization is sound and complete with respect to what the given author claims. To be sound and complete means that everything the formalism says follows, the author also says follows, which might be the case, but is a *much* stronger result.

[11] At this point we simply assume that the person carrying out the formalization is acute enough in their reading and understanding of the phenomenon to avoid unnecessary anachronisms.

[12] Or it can be easily modified to do so. The only case where this does not hold is if we want to look at the interactions between tensed and modal claims. This is a topic that is only alluded to in passing in the text, and was not considered in the formalization. To do this would require some very interesting modifications of the sequent calculus

with the ways in which Buridan uses them within his argumentation. We take this to show that the granularity is fine enough to capture the ideas that Buridan is expressing.

As far as the granularity of the language is concerned, $G3_{syl}$ has the right level of granularity. As we see throughout book two, the various expressions that Buridan does not abstract away from are left in $G3_{syl}$ as logical constants.[13] The various term-based expressions are preserved, as is the copula in the case of divided modal propositions. Further, $G3_{syl}$ can differentiate divided from composite modal expressions in the same way that Buridan does. Composite modal claims are of the form $A \circ B$ where one of the terms is a modal term, and the other is a nominalized proposition. This is how Buridan understands composite claims, and like in Buridan's discussion, it is clear from the logical form that the copula is *not* modalized. The language is clearly not too coarse. From the point of view of book two, it is also not too fine-grained. While the formalization can describe a number of propositions that Buridan does not treat in Book two[14] as far as the various term-based consequences are concerned, the language has no superfluous elements to it that could yield a too highly grained analysis. The only possible place that this could emerge is with the use of the world indexes to the various propositions. Further, for the analysis to work we do not need to restrict the class of propositions that we treat. We look at the same kind of term-based consequences that Buridan does.[15]. From this, together with the observations about the language made in chapter three, it should be clear that $G3_{syl}$ has an appropriate granularity.

In terms of the faithfulness criteria, the results are more mixed. The formalism is correct as far as Buridan's conclusions go. We are able to show that everything Buridan thinks should follow, follows, and that given a few assumptions, everything that Buridan says does not follow, does not follow in $G3_syl$. However, when we turn to the second component, difficulties emerge. One of the reasons a proof-theoretic account of Buridan's modal logic was developed has to do with the faithfulness criteria. Buridan's system, like other ancient and medieval accounts of logic (eg Aristotle) possess a strongly rule-like character. They lay down a number of assumptions (axioms) and from those axioms derive the various conclusions they wish to prove. It should also be observed that in most cases, there is an implicit appeal to what we as moderns understand as completeness. Buridan normally proves his claims by deductions from rules, however, he disproves claims by appeals to counter-examples. When Buridan invokes a counter-example, it is closer to what a modern reader would understand as involving a semantic notion. In Buridan's case, to show that a consequence does not follow, he describes a situation where the antecedent is true and the consequent is false. This is done with reference to various facts, and involves notions of truth. From this we observe that Buridan was not only concerned with pointing out what good consequences were and proving them, he was also concerned with what does not follow, and why it does not follow. In doing this, a modern logician would think that what Buridan is working with is a notion of completeness. I.e. there is a semantic notion that matches up with the notion of proof.

It becomes clear, that in terms of formalizing Buridan's modal logic, there is still more work to be done with $G3_{syl}$. As we have seen in our presentation of the negative results our way of disproving formulae was via root-first decomposition. This is an interesting modern technique, but is alien to Buridan. To be a more faithful representation of Buridan's treatment of modal consequences, we would want to explore the possibility of developing semantics for Buridan's

[13] I.e. the quantifiers, negation, the copula and modalities.

[14] For example, syllogisms which can be represented in $G3_{syl}$ as $\Rightarrow A \wedge B \rightarrow C$

[15] The only exception to this is the assumption that R is universal

logic. Ideally this semantics should allow us to show that the counter-examples that Buridan uses are in fact good counter-examples. Note that this would be a higher standard to set for a formal semantics. A weaker claim would be that the semantics is complete with respect to $G3_{syl}$. The only difficultly in going for the weak criteria is that there could be multiple semantics that are adequate for this task. This raises many interesting philosophical and methodology questions, that we will not be able to look at here. However, as formalization of Buridan's logic, it is successful in the main goals we set out for it; we have provided a faithful formalization of Buridan's *De Consequentia*.

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