Degrees of non-determinacy and game logics on cardinals under the axiom of determinacy

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written by

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Abstract

Blass showed that on each infinite cardinal, there is an algebra structure of games on it. Blass defined a reducibility relation on games via which he classified games into degrees of non-determinacy and proved nice properties of the degree structures on certain cardinals using the axiom of choice. Later Blass gave a game semantics to affine logic, an extension of linear logic, using his game algebra. He proved this game semantics is consistent (sound) but not complete. But he proved two nice completeness theorem for fragments of affine logic using the axiom of choice.

This thesis gives a detailed exposition of Blass's work on degree of non-determinacy and game semantics of linear logic, with an emphasis on the roles of cardinals and the usage of the axiom of choice, and contains our studies of degrees of non-determinacy and game logics on infinite cardinals in a set theory system without using the axiom of choice, namely in **ZF** with the axiom of determinacy.

Acknowledgements

I thank Benedikt Löwe for being supportive and encouraging to me from the very beginning of my interests in games. In May of 2010, Amsterdam hosted a set theory conference. As a first-year student who was just learning basis set theory, I sat in the lecture hall of the conference for two days. Most talks were absolutely alien to me, which made me wondering whether set theory was for me. But there was one talk which I really enjoyed and from which I learned something. It was given by Andreas Blass on games and I was fascinated by his ideas. I happened to sit very close to Benedikt, so he talked to me after that talk and I told him my intention to learn Blass's work. As a result of our informal talk, I did a self-reading project under his supervision in the summer. In January 2011 I asked Benedikt whether there were problems about games manageable for a master thesis, and he suggested to reexamine Blass's work on game semantics without using the axiom of choice, which I found quiet interesting. To my surprise, he immediately arranged me a trip to Tübingen Germany to meet Blass in person at a workshop. It turned out to be quite an experience for me. During the research and writing of my thesis, I learned a lot from discussing and corresponding with Benedikt. That he always expressed his mind clearly made our discussions enjoyable and informative.

I also want to thank Frank Veltman, Alessandra Palmigiano and George Barmpalias for their generous help during my first year in Amsterdam. They provided me valuable academic guidance and supported my application for the HSP scholarship. If I had not got the scholarship, I world have gone back to China.

Last but not least, I want to thank my parents. They spent a huge amount of their savings to finance my "escape" from China and my first year's studying in Amsterdam, the best place for me to learn logic. I deeply appreciate their wholehearted support to my pursuit of such a useless subject.

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Contents

Introduction

Infinite games without draws played by two players (player 0 and player 1) with perfect information (infinite games in short) have been studied by mathematicians for quite a long time, especially in set theory. In the larger part of the literature, infinite games are played in an alternate pattern starting with player 0. However, we will not follow this tradition but instead study general infinite games. We will call alternate games starting with player 0 *strict games* in our settings. In 1972, Blass [5] introduced a very natural algebra on the universe of all general infinite games.[1] Also in [5], Blass introduced a preordering and the associated equivalence relation which reflect the intuitive idea of classifying games according to their difficulty for a certain player, and called each equivalence class a degree of non-determinacy. He showed the degrees of non-determinacy of all infinite games formed a complete lattice (class). Using the Axiom of Choice, he showed several nice properties of this lattice restricted to each cardinal κ such that $2^\kappa = \kappa^\omega$. (We call this lattice restricted to cardinal κ the *degree structure on* κ.)

In 1992, Blass showed that the sequent calculus for affine logic (an extension of linear logic) is consistent for his semantics based on his game algebra from [5]. The information relevant for us in the context of this thesis is that linear logic and affine logic, which were studied for many other reasons, could be used in a game-theoretic setting. Using the Axiom of Choice, Blass showed that his game sequent calculus was complete for all valid additive formulas. Blass also gave a family of valid multiplicative formulas which had a very nice syntactic characterization. But the game sequent calculus is not complete for this family. Again, using the Axiom of Choice, Blass proved the family he gave was the entire family of all valid multiplicative formulas.

Following Blass's work in 1992, Abramsky and Jagadeesan [2] modified the game semantics and showed that an extended sequent calculus for linear logic was complete for the valid multiplicative fragment of linear logic. Hyland and Ong modied [15] the semantics further to get exactly multiplicative linear logic. Related work can also be found in [9], [4], [1], [16], and [3]. This work had significance in theoretical computer science. However they were in the opposite direction to our interests in this thesis, namely to study the game algebra and game logic on each infinite cardinal, rather than looking for a game semantics that is complete for a certain formal system such as linear logic.

In this thesis, we will see how the choice of axiom systems of set theory could affect

[1] In contrast to this, it is interesting that Conway [10] introduced an algebra on (finite) combinatorial games.

the structure of degrees of non-determinacy and the game semantics. Almost all lattice properties shown in [5] and all completeness results in [6] depended on diagonalization against all possible strategies using the Axiom of Choice (**AC**). Without using the Axiom of Choice, one cannot even show (in **ZF**) that there are more than two degrees of non-determinacy on ω. We are interested in studying the same subject matter without using the Axiom of Choice. Unfortunately, we can say very little in **ZF** about the degree structure on any infinite cardinal. First, **ZF** cannot prove or disprove that there is a non-determined game on κ. Also, while **ZF** proves that there are non-determined games on ω_1, it is not possible to display a concrete non-determined game in **ZF**.[1]

As we will explain in Chapter 3, **ZF** + **AD** is a natural choice for our purpose of performing concrete analysis of degree structures and game logics without using **AC**. The axiom of determinacy (**AD**) says every strict infinite game is determined. We will show in Chapter 1 that this is equivalent to say that every (general) infinite game is determined.

It is well-known that **AD** implies the existence of non-determined games on larger infinite cardinals. So there is a big difference between the degree structure and semantics on ω and those on larger cardinals. The degree structure on ω under **AD** is rather simple since it contains only two degrees, and we will show in this thesis that the game algebra on this degree structure provides a semantics for classical propositional logic. But what does the degree structure on $\kappa > \omega$ look like? What logic do we have for the semantics that the game algebra on $\kappa > \omega$ provides? Are the answers all the same for different κ's? We will try to answer these questions in this thesis and present our findings.

[1]Detailed discussion can be found in Subsection 3.1.1.

Chapter 1

Basics

1.1 Basic set-theoretic notation

Since our main aim in this thesis is to study degrees of non-determinacy and game logics on cardinals without the use of the Axiom of Choice, we will be working in the axiomatic setting of **ZF**, that is Zermelo-Fraenkel set theory without **AC**. This requires us to be careful with a number of classical results that rely on the Axiom of Choice (**AC**). This thesis will assume **ZF**. When extra axioms are used, we will explicitly say so. We aim to use set-theoretic standard notation, as can be found in [18]. In this section, we introduce some specific notations that will be used throughout the thesis.

1.1.1 Cardinals and cardinalities

We should be careful talking about cardinalities since **AC** is not included in our basic set theory. We follow [18, Chapter 3] on notations about cardinalities. In particular,

$$|X| = |Y|$$

if there is a 1-to-1 function from X onto Y;

$$|X| \leq |Y|$$

if there is a 1-to-1 function, i.e., an injection, from X to Y;

$$|X| < |Y|$$

if $|X| \leq |Y|$ and $|X| \neq |Y|$. By the famous Cantor-Bernstein theorem [18, Theorem 3.2], $|X| \leq |Y|$ and $|Y| \leq |X|$ implies $|X| = |Y|$. When $|X| = |Y|$, we say X and Y have the same cardinality, or X has cardinality $|Y|$. In particular, if $|X| = |{}^{\omega}2| = |{}^{\omega}\omega| = |\mathbb{R}|$, we say X has cardinality continuum.

Note that it is important to emphasize that $|X| \leq |Y|$ is not in general equivalent to "there is a function from Y onto X. This equivalence holds in **ZFC**, but not in **ZF** alone. However, if Y is wellorderable, then the equivalence holds in **ZF**, as the following lemma shows.

Lemma 1.1.1. *Let α be an ordinal and S a set. If there is an onto function from α to S, then $|S| \leq |\alpha|$.*

Proof. Let $f : \alpha \to S$ be onto.

$$\text{Define } g(x) \text{ to be the least } \gamma \text{ such that } \gamma \in \alpha \text{ and } f(\gamma) = x.$$

It is easy to see g is well-defined and 1-to-1, and clearly $\operatorname{ran}(g) \subset \alpha$. □

We overload the notation $|\;|$ to define a class function from the class of well-orderable sets to the class of ordinals (Ord):

$$|S| \text{ is the least ordinal } \alpha \text{ such that } |S| = |\alpha|.$$

When $|S|$ is well-defined, we say $|S|$ is the cardinal number of S. If S is not well-orderable, then $|S|$ is not defined. In this case, we say $|S|$ is not a cardinal number. If **AC** holds, then every S will be well-orderabe and hence each $|S|$ is a cardinal number.

We will use the class function ot that maps each wellorder to its order type, i.e., if (S, R) is a wellorder, $\operatorname{ot}(S, R)$ is the unique ordinal α such that $(\alpha, \in)$ is isomorphic to (S, R).

Furthermore, we'll use the class functions left and right picking the left or right side of an ordered pair, i.e.,

$$\operatorname{left}(a, b) = a \quad \text{and} \quad \operatorname{right}(a, b) = b.$$

On the class Ord × Ord of pairs of ordinals, we can define a wellordering corresponding to the so-called Gödel beta function as follows

$$\begin{aligned}(\alpha,\beta) < (\gamma,\delta) \quad \leftrightarrow \quad & \text{either } \max\{\alpha,\beta\} < \max\{\gamma,\delta\}, \\ & \text{or } \max\{\alpha,\beta\} = \max\{\gamma,\delta\} \text{ and } \alpha < \gamma, \\ & \text{or } \max\{\alpha,\beta\} = \max\{\gamma,\delta\},\ \alpha = \gamma \text{ and } \beta < \delta\end{aligned}$$

It is well-known that this is a well ordering of Ord × Ord [18, p. 30]. If α and β are fixed, consider the initial segment $X_{\alpha,\beta} := \{(\gamma, \delta) : (\gamma, \delta) < (\alpha, \beta)\}$ and let $\Gamma(\alpha, \beta) := \operatorname{ot}(X_{\alpha,\beta}, <)$. Hessenberg's theorem says that the infinite cardinals are fixed points of the Gamma function in the following sense:

Lemma 1.1.2. *For any ordinal α,*

$$\Gamma[\omega_\alpha \times \omega_\alpha] = \omega_\alpha,$$

i.e., $\Gamma \restriction \omega_\alpha \times \omega_\alpha$ is a 1-to-1 and onto function: $\omega_\alpha \times \omega_\alpha \to \omega_\alpha$

Proof. See [18, p. 30–31]. □

Theorem 1.1.3. **(a)** $\aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta = \max\{\aleph_\alpha, \aleph_\beta\}$.

(b) *If $|X| = \omega_\alpha$ and $|Y| = \omega_\beta$, and $f : X \times Y \to Z$ is onto, then $|Z| \le \omega_\alpha$ and $|Z| \le \omega_\beta$.*

Proof. Part **(a)** follows from Lemma 1.1.2. **(b)** follows from **(a)** and Lemma 1.1.1. □

1.1.2 Partitions and projections

A disjoint family of nonempty sets $\{X_y : y \in Y\}$ is a **partition** of set X iff

$$X = \bigcup\{X_y : y \in Y\}.$$

The **canonical partition** of ω_α, $\{\omega_\alpha|\gamma : \gamma < \omega_\alpha\}$ is a family of ω_α many pairwise disjoint subsets of ω_α of size ω_α, defined by

$$\omega_\alpha|\gamma = \Gamma[\{\gamma\} \times \omega_\alpha]$$

The function $\pi_\alpha : P(\omega_\alpha \times \omega_\alpha) \to \omega_{\alpha+1}$ is defined by

$$\pi_\alpha(R) = \begin{cases} \mathrm{ot}(\bigcup\bigcup R, R) & \text{if } R \text{ is a well-ordering} \\ & \text{ on } \bigcup\bigcup R \\ 0 & \text{if } R \text{ is not a well-ordering} \end{cases}$$

Theorem 1.1.4. *For each $\alpha \in \mathrm{Ord}$, $\pi_\alpha : P(\omega_\alpha \times \omega_\alpha) \to \omega_{\alpha+1}$ is well-defined and onto.*

Proof. It is easy to see that π_α is a function, since no ordinal is isomorphic to any of its initial segments and hence the order-type of any well-ordering is unique. To see that π_α has range $\omega_{\alpha+1}$, first notice that the range of π_α is a set, by Axiom of Replacement and that the domain of π_α is a set. Now we show $\mathrm{ran}(\pi_\alpha) = \omega_{\alpha+1}$.

"$\omega_{\alpha+1} \subset \mathrm{ran}(\pi_\alpha)$": Take an arbitrary $\beta \in \omega_{\alpha+1}$. If $\beta \leq \omega_\alpha$, then

$$R^{\in}_\beta = \{(\delta, \gamma) : \delta \in \gamma \in \beta\} \in P(\omega_\alpha \times \omega_\alpha)$$

and hence

$$\pi_\alpha(R^{\in}_\beta) = \mathrm{ot}(\bigcup\bigcup R^{\in}_\beta, R^{\in}_\beta) = \beta$$

by construction; otherwise $\omega_\alpha < \beta < \omega_{\alpha+1}$ and there is a 1-to-1 and onto function $g : \omega_\alpha \to \beta$, then $R^g = \{(\delta, \gamma) : \delta, \gamma \in \omega_\alpha \text{ and } g(\delta) \in g(\gamma) \in \beta\}$ satisfies that

$$\pi_\alpha(R^{\in}_\beta) = \mathrm{ot}(\bigcup\bigcup R^g, R^g) = \beta$$

since $(\bigcup\bigcup R^g, R^g)$ is isomorphic to $(\beta, \in)$ via g.

"$\mathrm{ran}(\pi_\alpha) \subset \omega_{\alpha+1}$": since there is no onto function from ω_α to $\omega_{\alpha+1}$, it is impossible for a well-ordering R on some $S \subset \omega_\alpha$ to have order-type $\geq \omega_{\alpha+1}$. □

1.1.3 Sequences

The concept of sequence will play an important role in the study of games. After all, we play a game by making a sequence of moves. Since we are primarily interested in games of length ω, we restrict ourselves to sequences of length at most countable. We should mention that all formal definitions we will give can be extended to include ordinals larger than ω.

An **infinite sequence** on set S is a function $a : \omega \to S$. A **finite sequence** on set S is a function $s : n \to S$ for some $n < \omega$. A **sequence** is either an infinite sequence or a finite sequence. The set of all finite sequences on S is denoted by $\mathrm{Fin}(S)$ and the set of all sequences on S is denoted by $\mathrm{Seq}(S)$. The **length** of a sequence s is $\mathrm{length}(s) = \mathrm{dom}(s)$. The sequence $\{(0, x)\}$ has length 1, we often write x instead of $\{(0, x)\}$ when the meaning is clear.

Theorem 1.1.5. *Let $|S| = \omega_\alpha$ for some ordinal α, $|\mathrm{Fin}(S)| = \omega_\alpha$.*

Proof. See [18, Exercise 3.6]. □

Let $s \in \mathrm{Seq}(S)$, define

$$t \prec s \ \text{ iff } \ t = s \restriction n \text{ for some } n < \mathrm{length}(s),$$

$$t \preceq s \ \text{ iff } \ t \prec s \text{ or } t = s.$$

We say t is an **initial segment** of s if $t \preceq s$, and t is a **proper initial segment** of s if $t \prec s$.

We need some useful operations on sequences. Let $s \in \mathrm{Fin}(S)$ and $t \in \mathrm{Seq}(T)$. The **concatenation** of s and t is the function $s * t : \mathrm{length}(s) + \mathrm{length}(t) \to S \cup T$ such that

$$s * t(n) = s(n) \text{ for } n < \mathrm{length}(s)$$

and

$$s * t(\mathrm{length}(s) + n) = t(n) \text{ for } n < \mathrm{length}(t).$$

We define $p - s$ to be the unique t such that $s * t = p$. Note that $p - s$ is defined only if $s \prec p$.

Let $s \in \mathrm{Seq}(S)$ and T be a set. The T-section of s, denoted by $s \searrow T$ is obtained by ignoring all elements of s that are not from T. More formally,

$$s \searrow T(n) = \begin{cases} s(m) & \text{if } m \text{ is the least such that } |s[m+1] \cap T| = n + 1, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

It is not hard to see that the domain of $s \searrow T$ is always an ordinal, so $\mathrm{dom}(s \searrow T) = \omega$ or k for some $k < \omega$.

1.1.4 The Axiom of Choice

The Axiom of Choice (**AC**) states that every family of nonempty sets has a choice function.

If S is a family of sets and $\emptyset \notin S$, then a choice funtion for S is a function f on S such that

$$f(X) \in X$$

for every $X \in S$.

The Axiom of Choice is very strong in the sense that it postulates the existence of a choice function for every family of nonempty sets. A lot of mathematical proofs do not need the full strength of **AC**. So it is useful to define weaker versions of **AC**.

Definition 1.1.6. *For non-empty sets S and T, $\mathbf{AC}_S(T)$ holds if and only if for any function $f : S \to P(T)$ such that $f(x) \neq \emptyset$ for every $x \in S$, there exists a function $g : S \to T$ such that $g(x) \in f(x)$ for every $x \in S$.*

Lemma 1.1.7. *For any S_1, S_2, T_1 and T_2 such that $0 < |S_2| \leq |S_1|$ and $0 < |T_2| \leq |T_1|$, $\mathbf{AC}_{S_1}(T_1)$ implies $\mathbf{AC}_{S_2}(T_2)$.*

Proof. Suppose $s : S_2 \to S_1$ is injective and $t : T_2 \to T_1$ is injective. Let $f : S_2 \to P(T_2)$ such that $f(u) \neq \emptyset$ for every $u \in S_2$. We want to show there exists a function $g : S_2 \to T_2$ such that $g(u) \in f(u)$ for every $u \in S_2$. Since T_1 is nonempty, we fix a $y_0 \in T_1$ and define $f' : S_1 \to P(T_1)$ as follows,

$$f'(x) = \begin{cases} t(f(u)) & \text{if } u \in S_2 \text{ and } s(u) = x \\ \{y_0\} & \text{otherwise} \end{cases}$$

It is easy to see f' is a well-defined function and $f'(x) \neq \emptyset$ for all $x \in S_1$. Then by the assumption $\mathbf{AC}_{S_1}(T_1)$, we have a function $h : S_1 \to T_1$ such that $h(x) \in f'(x)$ for every $x \in S_1$.

Now define $g : S_2 \to T_2$ as

$$g(u) = t^{-1}(h(s(u))).$$

This is well-defined, because $f'(s(u)) = t(f(u)) \subset t(S_2)$ and hence $h(s(u)) \in t(S_2)$ and $t^{-1}(h(s(u)))$ has a unique value in S_2. To see $g(u) \in f(u)$, notice that

$$h(s(u)) \in f'(s(u))$$

implies

$$g(u) \in t^{-1}(f'(s(u))) = t^{-1}(t(f(u))) = f(u).$$

Thus we have shown g has the desired property and hence $\mathbf{AC}_{S_2}(T_2)$. □

Now it is clear that **AC** is equivalent to $(\forall S \neq \emptyset)(\forall T \neq \emptyset)\mathbf{AC}_{\mathbf{S}}(T)$.

It is well-known that under **AC** all successor cardinals are regular cardinals. Surprisingly, we cannot decide for any successor cardinal whether it is regular or singular in **ZF**. In fact, Gitik [12] proved that it is consistent with **ZF** that all uncountable cardinals are singular.

1.2 Infinite games

1.2.1 The game universe $\mathcal{G}_{\omega_\alpha}$

In this subsection, we translate our intuitions about infinite games into precise mathematics. We are primally interested in the class of two-player win-lose (without draw) games with perfect information of countably infinite length, and by game we always mean game in this class.

We fix two players for all games and give several names to each of them. One is player 0, or the opponent, or player O, or "she". The other is player 1, or the proponent, or player P, or "he". We prefer to use 0 and 1 in mathematical discussions, and O and P in discussions of logics.

If we think harder about the question "what is a game", we will realize that the essence of a game A is nothing but its rules: at the beginning and after each move in the game, A should determine which player to move next; when infinitely many moves have been made, A should determine which player has lost. A should also provide a set X of possible moves to the players. Each player makes his/her move by picking a possible move out of X. So the moves in the game A occur in an ω-sequence of elements of X. That is, when A is finished after ω steps, an $s \in {}^{\omega}X$ is produced.

Formally, a **game** A on a set X is a function $A : \mathrm{Seq}(X) \to 2$.

We call the set of all games on X the **game universe** on X, denoted by $\mathcal{G}_X$. Formally,

$$\mathcal{G}_X = {}^{\mathrm{Seq}(X)}2.$$

Since we are primarily interested in games on cardinal numbers, we can always assume X is an infinite cardinal.

After identifying the game as its rules, it is not a surprise that the function A serves a dual purpose. On finite sequences, it indicates who is to move next; on infinite sequences, it indicates who has lost the play. At move n, the sequence $s{\restriction}n$ has already been produced and is known to both players; $s(n)$ is chosen by player $A(s{\restriction}n)$. When the play s is finished, i.e., $\mathrm{length}(s) = \omega$, player $A(s)$ is the loser.

In most literature of mathematics, these games are considered to be played by two players alternately. Nevertheless we will consider the concept of game in full generality as we want to systematically study all two-player infinite games. We will call games of the standard alternate form "strict" games.

Intuitively, a game should be distinguished from particular plays of it. A play of game is a possible resulting sequence of moves when the game is played by players 0 and 1. Each $a \in {}^{\omega}X$ is a **finished play** of A. Each $s \in \mathrm{Fin}(X)$ is a **partial play** or **finite play** of A.

Given a play s, $s|_0$ is the 0**-part** of s is the moves of player 0 in s, and $s|_1$ is the 1**-part** of s is the moves of player 1 in s. Formally,

$$\begin{aligned} s|_0(n) = s(m) &\text{ where } m \text{ is the least such that} \\ &|\{k : A(s{\restriction}k) = 0 \text{ and } k \leq m\}| = n+1; \end{aligned}$$

$$\begin{aligned} s|_1(n) = s(m) &\text{ where } m \text{ is the least such that} \\ &|\{k : A(s{\restriction}k) = 1 \text{ and } k \leq m\}| = n+1. \end{aligned}$$

Given $s, t \in \mathrm{Seq}(X)$, $s \star_i t$ is the play in which the sequence of moves of player i is s and that of player $1-i$ is t. Formally,

$$s \star_i t = \text{the maximal } p \in \mathrm{Seq}(X) \text{ such that } p|_i \preceq s \text{ and } p|_{(1-i)} \preceq t.$$

Very often, a finished play a of a game A is not really infinite in the sense that the outcome of the game can be predicted at some finite stage of the game.

Formally, given A and $a \in {}^{\omega}X$, a is **finitely decidable** if

$$(\exists s \prec a)(\forall b \in {}^{\omega}X)\ s \prec b \to A(b) = A(a). \tag{1.1}$$

Otherwise a is **finitely undecidable**.

A is **non-finite** if and only if every $a \in {}^{\omega}X$ is finitely undecidable. In this case A can be seen as a pure infinite game. Otherwise A is **finite**.

Intuitively, when a game has been invented, it can have many forms. First, a game A on X can be seen as a game on Y if $|X| = |Y|$. Moreover, a game can be padded with trivial moves and remain unchanged intuitively.

Definition 1.2.1. *A game A on X is a* **skeleton** *of a game B on Y, or B is a body of A, denoted by*

$$A \lhd B$$

and

$$B \rhd A$$

respectively, if and only if there is a 1-to-1 function $f : \mathrm{Fin}(X) \to \mathrm{Fin}(Y)$ such that

1. *For any $s, t \in \mathrm{Fin}(X)$, $s \prec t \to f(s) \prec f(t)$.*
2. *For any $s \in \mathrm{Fin}(X)$ and any $x \in X$,*
$$(\forall q \in \mathrm{Seq}(Y))(\forall p \not\preceq (f(s * x) - f(s))|_{1-A(s)})$$
$$[q \succeq ((f(s)|_{1-A(s)}) * p) \star_{1-A(s)} (f(s * x)|_{A(s)}) \to B(q) = 1 - A(s)].$$
3. *For any minimal s such that $s \notin \mathrm{ran}(f)$,*
$$(\forall t \succeq s) B(t) = B(s \restriction (\mathrm{length}(s) - 1))$$
4. *For any $a \in {}^{\omega}X$*
$$A(a) = B(\bigcup_{n<\omega} f(a \restriction n)).$$

The function f in the above definition translates a sequence of moves in game A into a sequence of blocks of moves in game B. The first clause says the translation is always extending. The second clause of the above definition is to say, if it is not player i's move in A, then she/he does not dominate the last block of moves in B, which means he has no control over his own moves, or his moves are determined by moves of the dominating player (i.e., the one who has made the last step in A), in the sense that any other move will cause player i to lose immediately. Clause 3 says, the first player who has made the play of B not a proper translation of any play of A loses B immediately. Clause 4 says the translation preserves the result of finished plays in A.

Lemma 1.2.2. *$\lhd$ is reflexive and transitive.*

Proof. Trivial by definition. □

Lemma 1.2.3. *Let X be an infinite set and A, B, C be games on X. If $A \lhd C$ and $B \lhd C$ then there is a game D on X such that $D \lhd A$ and $D \lhd B$.*

A and B are **similar**, denoted by $A \sim B$, if there is a C such that

$$C \lhd A \text{ and } C \lhd B.$$

Theorem 1.2.4. *For any game A on $X \subseteq Y$, there is a game B on Y such that $A \lhd B$.*

Proof. We define the **canonical extension** of A to Y, denoted by $\mathrm{Ext}(A, Y)$ as the following game B on Y:

$$B(s) = \begin{cases} A(s) & \text{if } s \in \mathrm{Seq}(X), \\ A(s \restriction k) & k \text{ is the least such that } s(k) \notin X. \end{cases}$$

Let

$$f = \mathrm{id}_{\mathrm{Fin}(X)}.$$

It is easy to see A, B and f satisfy the conditions of $A \lhd B$. □

The game $\mathrm{Ext}(A, Y)$ can be viewed the same as A in the following sense: if one of the players selects something outside X, then he is the only one allowed to make any move for the rest of the game, and he loses when the play finally becomes infinite.

We say a game is strict if it is played alternatively starting with player 0.

So a game A is **strict**, or in the strict form, if and only if

$$A(s) = \mathrm{length}(s) \mod 2$$

for all $s \in \mathrm{Fin}(X)$. Similarly, we say a game is inversely strict if it is played alternatively starting with player 1.

The following theorem states that every game on a set consisting of at least two elements has a strict form.

Theorem 1.2.5. *If $|X| > 1$, then for any game A, there is a strict game B such that $A \lhd B$.*

Proof. We will define the **strict form** of A, denoted by $S(A)$ as the game B below.

Fix two elements of X and call them 0 and 1. (In our applications, X will be an ordinal, and we can just choose the ordinals 0 and 1.) The idea is to code one move in A by a block of four successive moves in B, the first move in each block must be either 0 or 1. The second move in each block must be 0. If the first move of the block is 0, the fourth move must be 0; if the first move is 1, the third move must 0. So a legal block is either $00x0$ for some $x \in X$ which represents a move x in A made by player 0, or $100x$ for some $x \in X$ which represents a move x in A made by player 1.

Define

$$B(s) = \begin{cases} A(a) \quad \text{if } s \in {}^{\omega}X \text{ and } a = \{s(4n+2+(s(4n))) : n < \omega\}, \\ \qquad A(t \restriction n) = s(4n) \text{ for all } n < \omega \text{ and} \\ \qquad s(4n+1) = 0 \text{ and } s(4n+3-s(4n)) = 0 \text{ for all } n < \omega. \\ 0 \quad k \text{ is the largest such that} \\ \qquad t = \{s(4n+2+(s(4n)))\}_{n<k} \text{ for any } t, \\ \qquad A(t \restriction n) = s(4n) \text{ for all } n < k, \text{ and} \\ \qquad s(4n+1) = 0 \text{ and } s(4n+3-s(4n)) = 0 \text{ for all } n < k, \\ \qquad \text{and } s(4k) \neq A(t \restriction k) \\ \qquad \vee (s(4k) = A(t \restriction k \wedge s(4k+1=0) \wedge s(4k+3) \neq 0)) \\ 1 \quad k \text{ is the largest such that} \\ \qquad t = \{s(4n+2+(s(4n)))\}_{n<k} \text{ for any } t, \\ \qquad A(t \restriction n) = s(4n) \text{ for all } n < k, \text{ and} \\ \qquad s(4n+1) = 0 \text{ and } s(4n+3-s(4n)) = 0 \text{ for all } n < k, \\ \qquad \text{and } s(4k) = A(t \restriction k) \wedge s(4k+1) \neq 0. \\ \text{length}(s) \mod 2 \qquad \text{otherwise} \end{cases},$$

and

$$f(s) = t$$

where $\text{length}(t) = 4\text{length}(s)$, $t(4n) = A(s \restriction n)$, $t(4n+2+t(4n)) = s(n)$ for all $n < \text{length}(s)$, and $s(4n+1) = 0$ and $s(4n+3-s(4n)) = 0$ for all $n < \omega$.

It is easy to see these functions are well-defined and A, B and f satisfy the conditions of $A \lhd B$. □

We say a game A is **trivial** for a player i if she or he can play in such a way that the other player does not have any chance to move, and finally i wins the game when it is finished.

Formally, a game A is trivial for player i if

$$(\exists a \in {}^{\omega}X)[A(a) = 1 - i \ \wedge \ (\forall n < \omega)(A(a \restriction n) = i].$$

Corollary 1.2.6. *For any trivial game A on X, there is a non-trivial game B such that $A \lhd B$.*

Proof. Directly follows from Theorem 1.2.5. □

In fact we can generalize the concept of strict game and prove a center padding theorem for games.

Let $p : \omega \to 2$. We say p is **not eventually constant** iff

$$(\forall n < \omega)[(\exists m > n)(p(m) = 0) \wedge (\exists k > n)(p(k) = 1)].$$

A game A has **form** p iff

$$(\forall s \in \text{Fin}(X))A(s) = p(\text{length}(s)).$$

That is to say at any finite step who to make to the next step only depends on the length of the current play and is determined by the function p applied to the length of current play.

Theorem 1.2.7. *For any A where $|X| > 1$ and any not eventually constant $p : \omega \to 2$, there is a B such that B has form p and $A \lhd B$.*

Proof. By Theorem 1.2.5, there is a strict A' such that $A \lhd A'$. By Lemma 1.2.2, it is sufficient to prove the theorem for strict games. So we can safely assume A is strict.

Fix a $x_0 \in X$.

Define $p' : \omega \to \omega$ by

$$p'(0) = \text{ the least } k \text{ such that } p(k) = 0 \text{ and } p(k+1) = 1$$

and

$$p'(n+1) = \text{the least } k > p'(n) \text{ such that}$$
$$p(k) = 1 - p(p'(n)) \text{ and } p(k+1) = p(p'(n)).$$

$p'(n)$ can be seen as the n'th turning point of control described by the form p.

For any $a \in {}^{\omega}X$, define

$$B(a) = \begin{cases} p(k) & k \text{ is the such that } k \notin \text{ran}(p') \wedge s(k) \neq x_0 \\ A(a \circ p') & k \text{ otherwise.} \end{cases}$$

Define $f : \text{Fin}(X) \to \text{Fin}(X)$

$$f(s) = t$$

where

$$\text{length}(t) = p'(\text{length}(s))$$

and

$$(\forall n < \text{length}(t))[t \notin \text{ran}(p') \to t(n) = x_0]$$

and

$$s = t \circ (p' \restriction \text{length}(s)).$$

It is easy to check that A, B and f satisfy the conditions of $A \lhd B$. □

The following theorem shows that two different objects are sufficient to describe all games on natural numbers.

Theorem 1.2.8. *For any game on ω, there is a game B on 2 such that $A \lhd B$.*

Proof. We use the fact that any choice of an $n \in \omega$ can be coded as n many successive 1 followed by a 0. Let $C = \{w * 0 |\ w \in \text{Fin}(\{1\})\}$. It is easy to see there is a 1-to-1 and onto function $h : \text{Seq}(\omega) \to \text{Seq}(C)$ such that for any $s \in \text{Seq}(\omega)$, $h(s)$ is the binary code for s. There is also an almost-identity function $j : \text{Seq}(C) \to \text{Seq}(2)$ that spells each $a \in \text{Seq}(C)$ out in its 0-1 codes. Define

$$B(s) = \begin{cases} A(h^{-1}(j^{-1}s)) & \text{if } s \in \text{ran}(j), \\ B(t) & s = t * u \text{ where } t \in j(\text{Fin}(C)) \\ & \text{and } u \in \text{Seq}(\{1\}) \end{cases}$$

The above clauses cover all possible cases. It is easy to see that a player can only end his turn by playing a 0 after finite number of 1's and the next move is determined according to the definition of A and the decoding of the current play of B into A. It is also easy to see that whoever does not end his turn by playing a 0 loses.

Define $f : \mathrm{Fin}(\omega) \to \mathrm{Fin}(2)$ as

$$f(s) = j \circ h(s).$$

It is easy to show A, B and f satisfy the conditions for $A \lhd B$. $\square$

Corollary 1.2.9. *For any game on ω, there is a strict game B on 2 such that $A \lhd B$.*

Proof. By Theorem 1.2.5 and 1.2.8.

$\square$

1.2.2 Weak isomorphism

In proofs of theorems in the last section, we have seen that even if $A \lhd B$, the two games can behave differently in the sense that it is possible that A is non-finite while B is finite. Note that in $\mathrm{Ext}(A, Y)$ and $S(A)$, each player can choose to violate the rules and thereby make that play finitely decidable, with himself or herself losing.

It is natural to consider two games similar if they behave "the same" on finitely-undecidable plays. We reserve the name "isomorphism" and use **weak isomorphism** for this relation.

Definition 1.2.10. *A game A on a set X is* **weakly isomorphic** *to a game B on Y, denoted by $A \simeq B$, iff there is a function f from $\mathrm{Fin}(X)$ to $\mathrm{Fin}(Y)$ and a function g from $\mathrm{Fin}(Y)$ to $\mathrm{Fin}(X)$ such that*

1. *$s \preceq t \to f(s) \preceq f(s)$ for any $s \preceq t \in \mathrm{Fin}(X)$, and $s \preceq t \to g(s) \preceq g(s)$ for any $s \preceq t \in \mathrm{Fin}(Y)$,*
2. *$g(f(s)) \preceq s$ for any $s \in \mathrm{Fin}(X)$, and $f(g(s)) \preceq s$ for any $s \in \mathrm{Fin}(Y)$,*
3. *For any $a \in {}^{\omega}X$, if $A(a)$ is not decided at any finite stage, i.e., $(\forall s \prec a)(\exists b \in {}^{\omega}X) s \prec b \wedge A(b) \neq A(a)$, then $A(a) = B(\bigcup_{n<\omega} f(a\restriction n))$;*

   For any $a \in {}^{\omega}Y$, if $B(a)$ is not decided at any finite stage, i.e., $(\forall s \prec a)(\exists b \in {}^{\omega}Y) s \prec b \wedge B(b) \neq B(a)$, then $B(a) = A(\bigcup_{n<\omega} f(a\restriction n))$.

Theorem 1.2.11. *$\simeq$ is an equivalence relation.*

Proof. $\simeq$ is reflexive: Let $A = B$, then the identity function may serve the role of both f and g.

$\simeq$ is symmetric: Trivial from the definition.

$\simeq$ is transitive: Suppose $A \simeq B$ and $B \simeq C$. Then there are $f : \mathrm{Fin}(X) \to \mathrm{Fin}(Y)$ and $g : \mathrm{Fin}(Y) \to \mathrm{Fin}(X)$ satisfy the property of f, g in the above definition; there are $h : \mathrm{Fin}(Y) \to \mathrm{Fin}(Z)$ and $j : \mathrm{Fin}(Z) \to \mathrm{Fin}(Y)$ satisfy the property of f, g in the above definition. We want to show $h \circ f$ and $g \circ j$ have the desired property.

Conditions 1 and 2 are trivial to check. Now we left with condition 3.

Take any $a \in {}^\omega X$. We get $A(a) = B(\bigcup_{n<\omega} f(a\restriction n))$. If $\bigcup_{n<\omega} f(a\restriction n)$ is finite, then

$$B(\bigcup_{n<\omega} f(a\restriction n)) = C(h(\bigcup_{n<\omega} f(a\restriction n)))$$

by the property of h. Notice that

$$h(\bigcup_{n<\omega} f(a\restriction n)) = \bigcup_{n<\omega} (h(f(a\restriction n)))$$

by the property of h. Thus we get

$$A(a) = C(\bigcup_{n<\omega} h \circ f(a\restriction n)).$$

If $\bigcup_{n<\omega} f(a\restriction n)$ is infinite, then we have

$$B(\bigcup_{n<\omega} f(a\restriction n)) = C(\bigcup_{m<\omega} h(\bigcup_{n<\omega} f(a\restriction n))\restriction m)$$

But

$$\bigcup_{m<\omega} h(\bigcup_{n<\omega} f(a\restriction n))\restriction m) = \bigcup_{n<\omega} (h(f(a\restriction n)))$$

by the property of f and h, and the fact $\bigcup_{n<\omega} f(a\restriction n)$ is infinite. Again, we get

$$A(a) = C(\bigcup_{n<\omega} h \circ f(a\restriction n)).$$

The property of $g \circ j$ can be proved similarly. □

1.2.3 The game algebra $\mathcal{A}_{\omega_\alpha}$ on $\mathcal{G}_{\omega_\alpha}$

Intuitively, we can compose games in certain ways to get new ones. We will define several operations on games in this subsection in such a way that $\mathcal{G}_X$ will be closed under these operations for all infinite X's. Without loss of generality, we can always assume X to be an infinite cardinal ω_α for some ordinal α.

Definition 1.2.12. *For any game A, the dual of A is game $\overline{A}$ uniquely defined by*

$$\overline{A}(s) = 1 - A(s).$$

Clearly, $\overline{\overline{A}} = A$.

Our second operation acts on an indexed family $\{A_i | i \in I\}$ of games and yields a new game $\bigwedge_{i \in I} A_i$ played as follows. Player 0 begins by choosing an $i \in I$, and, from then on, the players play A_i. We choose the symbol $\bigwedge$ for this operator because it will give the greatest lower bound in the preordering that we will define in the next chapter. Another reason for this choice is historical. Lorenzen [20] proposed a semantics based upon dialogue games. The game for $\phi \wedge \psi$ is played like this: the opponent choose either ϕ or ψ and they continue to play to game for the formula just chosen and the opponent is not allowed to re-attack $\phi \wedge \psi$ again. We use $\bigwedge$ in our setting to indicate that once player 0 has made a choice, she cannot go back and choose again.

Formally we have the following definition.

Definition 1.2.13. *Given a family of games on X, $\{A_i |\ i \in I\}$ indexed by $I \subset X$, $\bigwedge_{i\in I} A_i$ is a game on X such that*

$$\bigwedge_{i\in I} A_i(s) = \begin{cases} 0 & \text{if } s = \langle\ \rangle, \\ 0 & \text{if } s(0) \notin I, \\ A_i(t) & \text{if } i \in I \text{ and } s = \langle i \rangle * t \end{cases}$$

In this definition, the first clause says 0 moves first, and the second says, in effect, that he must choose an $i \in I$, for otherwise he loses. According to the last clause, the rest of the game proceeds just like A_i.

There is an operation $\bigvee$, dual to $\bigwedge$, which will give the least upper bound in our orderings. $\bigvee_{i\in I} A_i$ is just like $\bigwedge_{i\in I} A_i$ except that the first move, the choice of which A_i to play, belongs to 1 rather than 0.

Definition 1.2.14. *The dual operator of $\bigwedge$ is*

$$\bigvee_{i\in I} A_i = \overline{\bigwedge_{i\in I} \overline{A}_i}.$$

Theorem 1.2.15. *Let A and B be two games, $\{A_i :\ i \in I\}$ and $\{B_j :\ j \in J\}$ be two finitely indexed families of games and $f : I \to J$ be 1-to-1 and onto.*

(a) *If $A \lhd B$ then $\overline{A} \lhd \overline{B}$; if $A \sim B$ then $\overline{A} \sim \overline{B}$.*

(b) *If $A_i \lhd B_i$ for each $i \in I$, then*

$$(\bigwedge_{i\in I} A_i \lhd \bigwedge_{j\in J} B_j) \quad \text{and} \quad (\bigvee_{i\in I} A_i \lhd \bigvee_{j\in J} B_j).$$

(c) *If I is finite or* **AC** *holds, then $A_i \sim B_{f(i)}$ for each $i \in I$ implies*

$$\bigwedge_{i\in I} A_i \sim \bigwedge_{j\in J} B_j \quad \text{and} \quad \bigvee_{i\in I} A_i \sim \bigvee_{j\in J} B_j.$$

Proof. Part **(a)** and **(b)** are easy by definition. For part **(c)**, we can fix an indexed family of common skeletons $\{C_i\}_{i\in I}$ since I is finite or **AC** holds, then use part **(b)**. □

In general, if I is infinite we need certain axiom of choice to prove part **(c)** of Theorem 1.2.15. But when the index set I is finite no choice is needed. We will also see later that the index set I is always finite when we study the logic of games.

We will define one more operation, the tensor product. It will be used in the definition of the orderings of games. The tensor product $\bigotimes_{i\in I} A_i$ of the indexed family $\{A_i|\ i \in I\}$ is played as follows.

The game starts with player 1 if he can make any move in any A_i. In this case he continues to play until it is player 0's move in each A_i. Whenever it is 0's move, she can choose in which A_i to make a move. After 0's move, 1 has to respond in that same A_i in which 0 made her last move. Player 0 wins if at least one of the A_i is finished (i.e., infinitely many moves are made in A_i) and won by 0; otherwise 1 wins.

Turning to a formal definition of $\bigotimes_{i\in I} A_i$, we can assume all A_i are on some infinite cardinal number without loss of generality. To define the operation $\bigotimes$, we need some coding technique. Notice that if κ is an infinite cardinal, we can code information about κ many copies of κ in κ using the canonical partition of κ, $\{\kappa|\gamma : \gamma < \kappa\}$. So given $s \in \mathrm{Seq}(\kappa)$, we can extract from it the subsequence $s \downarrow \gamma$ which can be seen as a sequence in the γ'th copy of κ.

Let $s \in \mathrm{Seq}(\omega_\alpha)$. For any $\gamma < \omega_\alpha$, define

$$s \downarrow \gamma = \mathrm{right} \circ \Gamma^{-1} \circ (s \searrow \omega_\alpha|\gamma).$$

So $s \downarrow \gamma$ extracts form s the γ'th of ω_α many sequences coded into s.

Definition 1.2.16. *Let κ be some ω_α and $I \subseteq \kappa$, and $\{A_i^\kappa : i \in I\}$ is a family of games on κ indexed by I. $\bigotimes_{i\in I} A_i$ is a game on κ defined by*

$$\bigotimes_{i\in I} A_i(s) = \begin{cases} \bigotimes_{i\in I} A_i(s\restriction k) & \text{if } k \text{ is the least such that } s(k) \notin \bigcup_{i\in I} \kappa|i \text{ or} \\ & \quad (\exists \beta \in I) A_\beta((s\restriction k) \downarrow \beta) = 1 \\ & \quad \wedge \quad \beta_0 \text{ is the least such } \beta \ \wedge \ s(k) \notin \kappa|\beta_0 \\ 1 & \text{if } s \text{ is finite and } (\exists \beta \in I) A_\beta(s \downarrow \beta) = 1), \\ 1 & \text{if the above cases do not apply and} \\ & (\exists \beta \in I)(s \downarrow \beta \in {}^\omega\kappa \wedge A_\beta(s \downarrow \beta) = 1) \\ 0 & \text{otherwise} \end{cases}$$

The first two clauses in effect say: it is player 1 to move whenever there is a sub-game A_β with $\beta \in I$ in which it is his turn to move, and the only legal move for player 1 is to make a move in A_β where β is the least one.

If s is finite and it is player 1's move in no sub-game, then it is player 0's move by the last clause, and she can choose to move in any sub-game, as long as the sub-game she has chosen has index among I by the first clause. In other words, if it is 0's turn to move, then it is her move in all sub-games. Also by the first clause, the first player who has made a illegal move continues to play the rest of the game and loses when the game is finished.

If s is infinite, i.e., the game has finished, the third clause says player 0 has won the game if at least one sub-game having index in I has been finished and won by 0. Otherwise player 1 has won by the last clause.

Each A_i in $\bigotimes_{i\in I} A_i$ is called a sub-game. If s is a play of $\bigotimes_{i\in I} A_i$, we call $s \downarrow i$ the sub-play of A_i in s, or the A_i part of s. If $A_i = G$, we often write $(s)_G$ instead of $s \downarrow i$.

Definition 1.2.17. *The dual operator of* $\bigotimes$ *is*

$$\overline{\bigotimes}_{i\in I} A_i = \overline{\bigotimes_{i\in I} \overline{A}_i}.$$

Theorem 1.2.18. *Let* $\{A_i : i \in I\}$ *and* $\{B_j : j \in J\}$ *be two finitely indexed families of games and* $f : I \to J$ *be 1-to-1 and onto.*

(a) *If* $A_i \lhd B_{f(i)}$ *for each* $i \in I$ *then*

$$\bigotimes_{i\in I} A_i \lhd \bigotimes_{j\in J} B_j \quad \textit{and} \quad \overline{\bigotimes}_{i\in I} A_i \lhd \overline{\bigotimes}_{j\in J} B_j.$$

(b) *If* I *is finite or* **AC** *holds, then* $A_i \sim B_{f(i)}$ *for each* $i \in I$ *implies*

$$\bigotimes_{i\in I} A_i \sim \bigotimes_{j\in J} B_j \quad \textit{and} \quad \overline{\bigotimes}_{i\in I} A_i \sim \overline{\bigotimes}_{j\in J} B_j$$

Proof. Similar to the proof of Theorem 1.2.15. □

Let $\bigcirc$ be any family operator, i.e., $\bigwedge$, $\bigvee$, $\bigotimes$, or $\overline{\bigotimes}$. By Theorem 1.2.15 and 1.2.18, if I is finite, we can assume A_i has a certain not eventually constant form for each $i \in A$. The resulting $\bigcirc_{i\in I} A_i$ is similar to the original one.

Theorem 1.2.19. *If* $\{I_j : i \in J\}$ *is a partition of* I*, then*

$$\bigcirc_{i\in I} A_i \lhd \bigcirc_{j\in J}(\bigcirc_{k\in I_j} A_k)$$

Proof. Follows easily from the definition. □

We follow some conventions to simplify notations. Given a family $\{A_i : i \in I\}$ where each A_i is on X_i, $\bigcirc_{i\in I} A_i = \bigcirc_{i\in I} \mathrm{Ext}(A_i, \bigcup_{i\in I} X_i)$. If $I = 2$, we often write $A_0 \circ A_1$ instead of $\bigcirc_{i\in\{0,1\}} A_i$, where $\circ$ is $\wedge$, $\vee$, $\otimes$, $\overline{\otimes}$, if $\bigcirc$ was $\bigwedge$, $\bigvee$, $\bigotimes$, $\overline{\bigotimes}$, respectively.

For family operators, we can bound the cardinalities of indexed families of games on which family operators can be applied. Let λ be a cardinal number, $\bigcirc^{<\lambda}$ means the operator $\bigcirc$ can be used on families of games that are indexed by sets of cardinality $< \lambda$, and similar for $\bigcirc^{\leq\lambda}$.

Another useful operation on games is R, the repetition. In $\mathsf{R}(A)$, the two players play ω many copies of game A. Player 0 can switch between these A's at her moves. To ensure player 0 has the control of the game, we require all the copies of game A have strict form so that it is 0's move at the beginning. Player 1 must play consistently in different copies of A, in the sense that as long as 0 makes the same moves in different copies, so must 1. We say 1 follows the consistency rule if he plays in such a way. When the game is finished, 0 has won iff at least one copy of A has been finished and won by 0. So $\mathsf{R}(A)$ can be seen as $\bigotimes_{n<\omega} A$ with the extra consistent rule put on player 1. We give the formal definition below.

Definition 1.2.20. *Let κ be some ω_α. $\mathsf{R}(A)$ is a game on κ defined by*

$$\mathsf{R}(A) = \begin{cases} \mathsf{R}(A)(s\restriction k) & \textit{if } k \textit{ is the least such that } s(k) \notin \bigcup_{i<\omega} \kappa|i \\ & \textit{or } \ (\exists m < \omega) S(A)((s\restriction k) \downarrow m) = 1 \\ & \quad \wedge \ [s(k) \notin \kappa|m \ \vee \\ & \qquad ((\exists n < \omega)(s\restriction k) \downarrow m \prec (s\restriction k) \downarrow n \\ & \qquad \wedge\, s(k) \neq ((s\restriction k) \downarrow n)(\mathrm{length}((s\restriction k) \downarrow m)))] \\ 1 & \textit{if } s \textit{ is finite and } (\exists m < \omega) S(A) = (s \downarrow m) = 1), \\ 1 & \textit{if the above cases do not apply and} \\ & (\exists m < \omega)(s \downarrow m \in {}^{\omega}\kappa \wedge S(A)(s \downarrow m) = 1) \\ 0 & \textit{otherwise} \end{cases}$$

Clause 1 in the definition says whoever makes an illegal move loses. The first line requires the two players to move only in the first ω copies of $S(A)$, the strict form of A. The lines 2 – 5 say that it is player 1's move when (exactly) one copy of $S(A)$ has 1 to move and 1 must move in that copy, following the consistency rule (line 4 and 5).

Since all copies of A have the strict form, it is easy to see that the two players move alternately. After a move by 1, it is 0's move in all copies of A; after a move by 0, it is 1's move in exactly one copy of A, so 1 never has a choice about which copy of A to move in.

The dual operator of R is $\overline{\mathsf{R}}$.

Definition 1.2.21.

$$\overline{\mathsf{R}}(A) = \overline{\mathsf{R}(\overline{A})}.$$

Let κ be an infinite cardinal. $(\mathcal{G}_\kappa, \ \overline{}, \ \wedge, \ \vee, \ \otimes, \ \overline{\otimes}, \ R, \ \overline{\mathsf{R}})$ is a well-defined algebra structure. We call this algebra structure the **game algebra** on κ, denoted by $\mathcal{A}_\kappa$.

1.2.4 Winning strategy and $\mathbf{AD}_{\omega_\alpha}$

In this subsection we will precisely define the concept of winning strategy and the general form of the Axiom of Determinacy.

Definition 1.2.22. *Let A be a game on X. A* **strategy** *for player i in A is a function σ from $\mathrm{Fin}(X) \cap A^{-1}(i)$ into X.*

Let σ be a strategy for player 0. For each $s \in \mathrm{Seq}(X)$,

$$\sigma \star s = \textit{the maximal } a \textit{ such that } (\forall n \in \omega)[A(a\restriction n) = 0 \to a(n) = \sigma(a\restriction n)] \textit{ and } a|_1 \preceq s.$$

Let σ be a strategy for player 1. For each $s \in \mathrm{Seq}(X)$,

$$s \star \sigma = \textit{the maximal } a \textit{ such that } (\forall n \in \omega)[A(a\restriction n) = 1 \to a(n) = \sigma(a\restriction n)] \textit{ and } a|_0 \preceq s.$$

A strategy σ is a **winning strategy** *for player i if*

$$(\forall a \in {}^{\omega}X)((\forall n \in \omega)(A(a\restriction n) = i \to a(n) = \sigma(a\restriction n)) \to A(a) \neq i). \tag{1.2}$$

If either 0 or 1 has a winning strategy in A, then A is **determined** *and to be a win for 0 or 1 respectively. Otherwise A is* **non-determined**.

Without loss of generality, we often assume the X in the above definition to be an infinite cardinal. Clearly at most one player can have a winning strategy in a game. A strategy σ for i should be thought of as a set of instructions for player i, telling him, if the sequence t has already been played and $A(t) = i$, to play $\sigma(t)$ next. The condition (1.2) says that if i follows these instructions then he wins. We will give another proof that non-determined games exist, using tensor product to be defined.

Lemma 1.2.23. *σ is a winning strategy for player 0 in A if and only if*

$$(\forall a \in {}^{\omega}X) A(\sigma \star a) = 1;$$

σ is a winning strategy for player 1 in A if and only if

$$(\forall a \in {}^{\omega}X) A(a \star \sigma) = 0.$$

Proof. Easy from Definition 1.2.22. □

Given any Set X, we can define two games $\mathbb{O}$ and $\mathbb{P}$ such that $\mathbb{O}$ is a win for 0 and $\mathbb{P}$ is a win for 1. $\mathbb{O}$ is defined by

$$\mathbb{O}(s) = 1$$

for all $s \in \mathrm{Seq}(X)$; $\mathbb{P}$ is defined by

$$\mathbb{P}(s) = 0$$

for all $s \in \mathrm{Seq}(X)$.

The following results will be useful later.

Lemma 1.2.24. *Let $|X| = \omega_\alpha$, and A be a game on X. There are at most $2^{|X|}$ strategies for each player in A.*

Proof. Easy. Use that $|\mathrm{Fin}(X)| = |X| = \omega_\alpha$. □

Theorem 1.2.25. *If $A \lhd B$, then player 0/1 has a winning strategy in A if and only if she/he has a winning strategy in B.*

Proof. Easy from definition of $\lhd$. □

Corollary 1.2.26. *Let $|X| > 1$. If every strict game on X is determined, then all games on X are determined.*

The next theorem will be crucial for later discussions.

Theorem 1.2.27. **(a)** *Game $\overline{A}$ is a win for player 0/1 iff A is a win for player 1/0.*

(b) *Game $\bigwedge_{i \in I} A_i$ is a win for player 0/1 iff one/each of the A_i's is; $\bigvee_{i \in I} A_i$ is a win for 0/1 iff each/one of A_i's is.*

(c) *Game $\mathsf{R}(A)$ is a win for 0/1 if A is; $\overline{\mathsf{R}}(A)$ is a win for 0/1 if A is.*

(d) *There exists* $\{A_i\}_{i\in I}$ *such that* $\bigotimes_{i\in I} A_i$ *is a win for* 0 *but none of the* A_i *are (or equivalently there exists* $\{A_i\}_{i\in I}$ *such that* $\overline{\bigotimes}_{i\in I} A_i$ *is a win for* 1 *but none of the* A_i *are); there exists* A *such that* $\mathsf{R}(A)$ *is win for* 0 *but* A *is not (or equivalently there exists* A *such that* $\overline{\mathsf{R}}(A)$ *is win for* 1 *but* A *is not).*

(e) *Suppose* I *is finite and* A_i *is nontrivial for* 1 *for every* $i \in I$. $\bigotimes_{i\in I} A_i$ *is a win for* 0 *if one of the* A_i*'s is;* $\bigotimes_{i\in I} A_i$ *is a win for* 1 *iff all the* A_i*'s are.*

(f) *Suppose* I *is finite and* A_i *is nontrivial for* 0 *for every* $i \in I$. $\overline{\bigotimes}_{i\in I} A_i$ *is a win for* 1 *if one of the* A_i*'s is;* $\overline{\bigotimes}_{i\in I} A_i$ *is a win for* 0 *iff all the* A_i*'s are.*

(g) *Suppose* $A_i(\emptyset) = 0$ *for every* $i \in I$. $\bigotimes_{i\in I} A_i$ *is a win for* 0 *if one of the* A_i*'s is;* $\bigotimes_{i\in I} A_i$ *is a win for* 1 *iff all the* A_i*'s are.*

(h) *Suppose* $A_i(\emptyset) = 1$ *for every* $i \in I$. $\overline{\bigotimes}_{i\in I} A_i$ *is a win for* 1 *if one of the* A_i*'s is;* $\overline{\bigotimes}_{i\in I} A_i$ *is a win for* 0 *iff all the* A_i*'s are.*

Proof. Parts **(a)**, **(b)** and **(c)** are easy. Part **(d)** follows from Corollary 1.2.31.

For the first part of **(e)**, notice that there is a simple winning strategy for 0 in $\bigotimes_{i\in I} A_i$, namely to make moves only in the least A_i in which she has a winning strategy and follow that strategy. The second part is similar.

The second part of **(e)** follows from the definition of $\bigotimes$. Since no A_i is trivial for 1, if he has a winning strategy σ in $\bigotimes_{i\in I} A_i$ and follows it, player 0 must have to move at some finite step. Once 0 can make a move, he can choose some A_k to continue and never to switch to another game after that move. Since σ is a winning strategy for 1, 1 can make sure he win the sub-game A_k. σ can be easily translated into a winning strategy σ' for 1 in the game A_k. The other direction is trivial. The proof for f is similar.

The proofs for **(g)** and **(h)** are similar to the proofs of **(e)** and **(f)**. □

Corollary 1.2.28. *If* $\bigotimes_{i\in I} A$ *is non-determined then* A *is non-determined; if* $\mathsf{R}(A)$ *is non-determined then* A *is non-determined.*

Given any set (often am infinite cardinal), it is natural to ask whether each game on X is determined. We say the **Axiom of Determinacy** holds for X, denoted by $\mathbf{AD}_X$, if every infinite game on X is determined. We use **AD** to mean $\mathbf{AD}_\omega$. In the literature, **AD** often means that every strict game on ω is determined. Corollary 1.2.26 tells us the two statements are equivalent. We will see in Chapter 3 that **ZF** proves that all $\mathbf{AD}_{\omega_\alpha}$'s are false for $\alpha > 0$.

If we are to focus on the true infinite part of games, we can define the concept of weak determinacy.

Definition 1.2.29. *Let* A *be a game on* X. *A strategy* σ *for player* i *in* A *is a* **weak winning strategy** *if*

$$
\begin{aligned}
(\forall a \in {}^{\omega}X) \quad (\quad & (\forall s \prec a)(\exists b \in {}^{\omega}X) s \prec b \wedge A(b) \neq A(a) \\
& \wedge\ (\forall n \in \omega)(A(a \restriction n) = i \rightarrow a(n) = \sigma(a \restriction n)) \\
\rightarrow \quad & A(a) \neq i \ )
\end{aligned} \tag{1.3}
$$

If player i has a weak winning strategy in A, then A is **weakly determined** *and to be a win for i. If neither of the players has a weak winning strategy, then A is* **weakly non-determined**.

Note that both players can have weak winning strategies in one game. But at most one play can have a weak winning strategy in a non-finite game. The condition (1.3) says that if i follows σ in a play a then he wins a if a can not be decided at any finite stage.

1.2.5 Non-determined games in ZFC

Gale and Stewart [11] showed that there is a game on natural numbers which is not determined (this was known to the Polish in the 1920). As a consequence, **AC** implies $\mathbf{AD}_\kappa$ is false for each infinite cardinal κ. In particular, **AD** is false in **ZFC**.

Theorem 1.2.30 (**AC**)**.** *There exists a non-determined game on ω.*

A diagonalization proof using **AC** can be found in [19, p. 377]. We give another proof using the game operator $\bigotimes$, which will be useful for a later theorem, following the discussion in [5, p. 155].

Proof. **AC** implies the existence of a non-principal ultrafilter on ω; see [18, p. 73-75]. Let D be a non-principal ultrafilter on ω. For any strictly increasing sequence $s \in {}^{\omega}\omega$, let

$$E(s) = \{x \in \omega |\ (\exists n \in \omega)\ s(2n-1) \leq x < s(2n)\}$$

(where $s(-1)$ means 0). If we think of ω as partitioned into the segments with endpoints $s(n)$, then $E(s)$ is the union of the even-numbered segments.

Define a game A on ω by

$$
A(s) = \begin{cases}
A(s \restriction k) & \text{if } k \text{ is the least number with } s(k) \leq s(k-1), \\
\text{length}(s) \mod 2 & \text{if there is no such } k \text{ and } s \text{ is finite}, \\
1 & \text{if there is no such } k,\ s \text{ is infinite}, \\
 & \text{and } E(s) \in D, \\
0 & \text{otherwise.}
\end{cases}
$$

The first clause forces the players to produce a strictly increasing sequence s; let us restrict our attention to such plays. By the second clause, the players move alternately, with 0 moving first. If we imagine that, by playing $s(n)$, a player "takes" the integers from $s(n-1)$ to $s(n)-1$ (inclusive), then the last two clause say that whoever takes almost all (with respect to D) integers wins.

Clearly A is not trivial for any of two players.

Let A_0 and A_1 both be A. Then $\bigotimes_{i\in\{0,1\}} A_i$ ($= A_0 \otimes A_1 = A \otimes A$) is a win for 0 by means of the following strategy. Begin by choosing i and choosing 1 in A_i. From now on, whatever your opponent makes in A_0 or A_1, immediately switch to the other game and make the same move there. If 0 follows this strategy, then $E(s \downarrow 0)$ and $E(s \downarrow 1)$ are complements of each other (except that 0 is in both), so one of them is in D, and 0 wins.

An analogous strategy shows that $\overline{A} \otimes \overline{A}$ is also a win for 0. If either A or $\overline{A}$ were a win for 0, the other would be a win for 1, and its tensor product with itself would also be a win for 1, contrary to what we have shown. So A is non-determined, even though $A \otimes A$ is a win for 0. □

Corollary 1.2.31. *There is a game A on ω such that A is non-determined but $A \otimes A$ and $\mathsf{R}(A)$ are wins for player 0.*

Proof. By the proof of Theorem 1.2.30, remembering that $\mathsf{R}(A)$ is just $\bigotimes_{n<\omega} A$ with consistent rule on player 1. □

Notice that not all non-determined games on ω satisfy that $A \times A$ or $\mathsf{R}(A)$ is a win for 0.

Theorem 1.2.32 (AC). *Let κ be an infinite cardinal. There is a game A on κ such that A and $A \otimes A$ are non-determined. There is a game B on κ such that B and $\mathsf{R}(B)$ are non-determined.*

Proof. By Corollary 1.2.28, we only need to find a A such that $A \otimes A$ is non-determined and a B such that $\mathsf{R}(B)$ is non-determind. We can easily modeify the diagonalization argument in [19, p. 377] to find such games. □

Chapter 2

From degrees of non-determinacy to game logics

2.1 Degrees of non-determinacy

In this section, we'll give an exposition of the general theory of degrees of non-determinacy as it was developed by Blass in [5]. Most of the definitions and results are from that paper.

2.1.1 Comparing the difficulty of games

Blass defined a notion of reducibility between games as follows:

$$A \leq B \text{ iff } A \otimes \overline{B} \text{ is a win for 0, or equivalently } \overline{A}\,\overline{\otimes}\, B \text{ is a win for 1.}$$

This notion captures the intuition of being **at least as difficult to win** for P since you can think of a winning strategy for P in game $\overline{A}\,\overline{\otimes}\, B$ as the game in which P continuously reads off a winning strategy in B from a winning strategy in A.

We use $A \| B$ to mean $A \nleq B$ and $B \nleq A$, and $A < B$ to mean $A \leq B$ and $B \nleq A$.

Lemma 2.1.1. *Suppose $A \leq B$. If A is a win for player 1, so is B; if B is a win for player 0, so is A.*

Proof. Trivial by definition. □

Since we want to get a equivalence relation that represents the concept of difficulty, we must first show that $\leq$ is a pre-ordering.

Theorem 2.1.2. *The relation $\leq$ is a pre-ordering (i.e., reflexive and transitive).*

Proof. Reflexivity: The game $A \otimes \overline{A}$ consists of two games of A. Player O (i.e., player 0) acts as 0 in the first but as 1 in the second, and he is free to change games at any of his moves. Suppose he uses the the following copycat strategy. Start by choosing the game where player 1 moves first. Whenever it's 0's move thereafter, switch games, and copy in this game what 1 just did in the other one. The result of this strategy is that the two plays of A are identical. As 0 plays opposite roles in the two games, she wins one (and loses the other), thereby winning $A \otimes \overline{A}$.

Transitivity: Let σ and τ be winning strategies for 0 in $A \otimes \overline{B}$ and $B \otimes \overline{C}$, respectively, and let 0 play $A \otimes \overline{C}$ according to the following strategy. Imagine, in addition to the games of A and C actually being played, a fictitious game of B. Begin by playing σ as long as it dictates moves (of 0) in A; P, i.e., player 1 must reply in A. If σ ever dictates a switch to $\overline{B}$, make any move dictated by σ (for 1) in the fictitious B, and begin playing τ as long as it dictates moves (of 1) in C. When τ dictates a switch to B, make whatever moves τ dictates (for 0) in the fictitious B. Continue playing B, using σ or τ to determine 1's or 0's moves respectively until σ or τ dictates a switch to A or $\overline{C}$ (respectively); then make the indicated move in the actual game. Thus, O use σ and τ against P in A and C, and against P in A and C, and against each other in B.

If O does not finish and win the play of A (as 0), then, since σ wins $A \otimes \overline{B}$, the fictitious play of B is finished and won by P. But then, since τ wins $B \otimes \overline{C}$, O must have finished and won the play of C (as P). Therefore, this strategy is a winning strategy for O in $A \otimes \overline{C}$. □

Once we have the means to compare the difficulty of games, we can classify games according to their difficulties. We associate an equivalence relation $\equiv$ with the pre-ordering $\leq$ in the usual way and define $A \equiv B$ iff $A \leq B$ and $B \leq A$. We have see $\equiv$ is an equivalence relation. Now games can be divided into equivalence classes and call each equivalence class a **degree of non-determinacy** or degree in short. Formally, let κ be some ω_α. (We assume all games are on some cardinal numbers without loss of generality.) For each A on κ, let

$$[A]_\kappa = \{B \text{ on } \kappa : A \equiv B\}$$

be the degree of non-determinacy of A on κ. A is called a **representative** of degree $[A]_\kappa$. When κ is clear from the context, we will write $[A]$ for $[A]_\kappa$.

Define the **degree universe** on κ, denoted by $\mathcal{S}_\kappa$, to be the set of degrees of non-determinacy on κ together with the pre-ordering $\leq$. Formally,

$$\mathcal{S}_\kappa = (\mathcal{G}_\kappa/\equiv, \leq) = (\{[A] : A \in \mathcal{G}_\kappa\}, \leq).$$

Clearly $[A] \leq [B]$ iff $A \leq B$ and hence the degrees are partially ordered by $\leq$. By Lemma 2.1.1, $A \equiv B$ implies that A and B are both wins for 0, or wins for 1, or non-determined. Thus, we can say the degree $[A]$ is a win for 0/1 if A is, or equivalently player 0/1 has a winning strategy in $[A]$ if she/he has one in A. Likewise we are also justified to say $[A]$ is non-determined if A is.

2.1.2 Algebraic structure of degree universe $\mathcal{S}_{\omega_\alpha}$

In this subsection we will show that the degree universe $\mathcal{S}_\kappa = (\mathcal{G}_\kappa/\equiv, \leq)$ is a κ-complete lattice for each infinite cardinal κ. We will also see the quotient algebra of $\mathcal{A}_\kappa^{<\omega}$ on $\mathcal{S}_\kappa$, namely $\mathcal{A}_\kappa^{<\omega}/\equiv$, is a well-defined, and if $\mathbf{AC}_\lambda(\kappa^\kappa)$ holds, $\mathcal{A}_\kappa^{\leq\lambda}/\equiv$ is well-defined.

Theorem 2.1.3. *$\bigwedge_{i\in I} A_i$ is the greatest lower bound, and $\bigvee_{i\in I} A_i$ is the least upper bound, of $\{A_i |\ i \in I\}$.*

Proof. To see $\bigwedge_{i\in I} A_i$ is a lower bound of $\{A_i |\ i \in I\}$, we show that $\bigwedge_{i\in I} A_i \otimes \overline{A}_j$ is a win for 0 for each $j \in I$. The strategy is as follows. 0 starts with playing $\bigwedge_{i\in I} A_i$ and picking A_j to continue. After that the players are in fact playing $A_j \otimes \overline{A}_j$, which is a win for 0 by mimicking.

To see $\bigwedge_{i\in I} A_i$ is the greatest lower bound, we show $C \otimes \overline{\bigwedge_{i\in I} A_i}$ is a win for 0 for any C such that $C \otimes \overline{A}_j$ is a win for 0 for each $j \in I$. 0 starts the game by choosing to play $\overline{\bigwedge_{i\in I} A_i}$. Then 1 must decide which A_j to play. Once 1 has made such a choice, the two player will be in fact playing $C \otimes \overline{A}_j$, for which 0 has a winning strategy.

That $\bigvee_{i\in I} A_i$ is the least upper bound of $\{A_i |\ i \in I\}$ can be proved similarly. □

Corollary 2.1.4. *The structure $(\mathcal{G}_{\omega_\alpha}/{\equiv}, \leq)$ is a ω_α-complete lattice.*

We use $\mathcal{S}_{\omega_\alpha}$ to denote the lattice $(\mathcal{G}_{\omega_\alpha}/{\equiv}, \leq)$.

We collect other results regarding $\leq$ and $\equiv$ in the following theorem, which will give us well-defined quotient algebras on $\mathcal{S}_{\omega_\alpha}$ and some properties of these quotient algebras.

Theorem 2.1.5. *Let κ be an infinite cardinal and A, B, A_i's and B_i's are on κ.*

(a) *$A \leq B$ iff $\overline{B} \leq \overline{A}$.*

(b) *Let $\bigcirc$ be an operator among $\bigwedge$, $\bigvee$, $\bigotimes$ or $\overline{\bigotimes}$, and $\{A_i :\ i \in I\}$ and $\{B_j :\ j \in J\}$ be two finitely indexed families of games on κ and $f : I \to J$ be 1-to-1 and onto. If I is finite or $\mathbf{AC}_{|I|}(\kappa^\kappa)$ holds, then*

$$(\forall i \in I) A_i \leq B_{f(i)} \ \rightarrow\ \bigcirc_{i\in I} A_i \leq \bigcirc_{j\in J} B_j.$$

(c) *Let $\bigcirc$ be any operator among $\bigwedge$, $\bigvee$, $\bigotimes$ or $\overline{\bigotimes}$. If $I \subset J$, then*

$$\bigcirc_{j\in J} A_j \leq \bigcirc_{i\in I} A_i.$$

(d) *$\bigotimes_{i\in I} A_i \leq \bigwedge_{i\in I} A_i$.*

(e) *$\mathbb{O} \leq A \leq \mathbb{P}$ for all A.*

(f) *If $A \lhd B$ then $A \equiv B$.*

(g) *If $A \sim B$ then $A \equiv B$.*

(h) *Let $\bigcirc$ be any operator among $\bigwedge$, $\bigvee$, $\bigotimes$ or $\overline{\bigotimes}$. If $\{I_j : i \in J\}$ is a partition of I, then*

$$\bigcirc_{i\in I} A_i \equiv \bigcirc_{j\in J} (\bigcirc_{k\in I_j} A_k).$$

(i) *$\forall A\ (\mathbb{O} \leq A \leq \mathbb{P})$.*

(j) *$A \equiv \mathbb{O}$ iff A is a win for 0; $A \equiv \mathbb{P}$ iff A is a win for 1.*

(k) *If $I = 1 = \{0\}$, then $\bigwedge_{i \in I} A_i \equiv \bigvee_{i \in I} A_i \equiv \bigotimes_{i \in I} A_i \equiv \overline{\bigotimes}_{i \in I} A_i \equiv A_0$.*

(l) *Let $\bigcirc$ be an operator among $\bigwedge$, $\bigvee$, $\bigotimes$ or $\overline{\bigotimes}$, and $\{A_i : i \in I\}$ and $\{B_j : j \in J\}$ be two finitely indexed families of games on κ and $f : I \to J$ be 1-to-1 and onto. If I is finite or $\mathbf{AC}_{|I|}(\kappa^\kappa)$ holds, then*

$$(\forall i \in I) A_i \equiv B_{f(i)} \ \rightarrow \ \bigcirc_{i \in I} A_i \equiv \bigcirc_{j \in J} B_j.$$

(m) $\mathsf{R}(A) \leq \bigotimes_{i<\omega} A \leq A$.

(n) $\mathsf{R}(A) \equiv \mathsf{R}(\mathsf{R}(A))$.

(o) *If $A \leq B$, then $\mathsf{R}(A) \leq \mathsf{R}(B)$.*

(p) $\mathsf{R}(\bigvee_{i \in I} A_i) \equiv \bigvee_{i \in I} \mathsf{R}(A_i)$.

(q) $\mathsf{R}(A) \otimes \mathsf{R}(B) \equiv \mathsf{R}(A \otimes B) \equiv \mathsf{R}(A \wedge B)$.

Proof. **(a)**, **(i)**, **(j)** and **(k)** are trivial. **(b)**-**(f)**, and **(m)** can be proved by straitforward copycat strategies. **(g)** follows from **(f)**. **(h)** follows from **(f)** and Theorem 1.2.19. **(l)** follows from **(b)**. Note that for **(b)** and **(l)**, we need that I is finite or $\mathbf{AC}_{|I|}(\kappa^\kappa)$ holds in order to fix a winning strategy in each game $A_i \otimes \overline{B_{f(i)}}$.

(n): By **(m)**, $\mathsf{R}(\mathsf{R}(A)) \leq \mathsf{R}(A)$. Now we need to show that $\mathsf{R}(A) \otimes \overline{\mathsf{R}(\mathsf{R}(A))}$ is a win for 0. Remember that $\Gamma[\omega \times \omega] = \omega$. 0 can win by a copycat strategy that makes sure that the play of the (i, j)th copy of A in $\mathsf{R}(\mathsf{R}(A))$ is the same as the play of the $\Gamma(i, j)$th copy of A in $\mathsf{R}(A)$.

(o): Suppose σ is a winning strategy for 0 in $A \otimes \overline{B}$. Then 0 wins $\mathsf{R}(A) \otimes \overline{\mathsf{R}(B)}$ by the following strategy. When player 1 chooses the ith copy of B in $\mathsf{R}(B)$, reply, using σ, in that copy of B and the ith copy of A in $\mathsf{R}(A)$. For each i, player 0 wins the ith copy of A or of B. Either he wins all the B's or at least one of the A's, so he wins $\mathsf{R}(A) \otimes \overline{\mathsf{R}(B)}$.

(p): By (o), $\mathsf{R}(A_i) \leq \mathsf{R}(\bigvee_{i \in I} A_i)$ for each $i \in I$. By Theorem 2.1.3, $\bigvee_{i \in I} \mathsf{R}(A_i) \leq \mathsf{R}(\bigvee_{i \in I} A_i)$. For the other direction, 0 wins

$$\mathsf{R}(\bigvee_{i \in I} A_i) \otimes \overline{\bigvee_{i \in I} \mathsf{R}(A_i)}$$

by the following strategy. Begin by choosing $\mathsf{R}(\bigvee_{i \in I} A_i)$ and index 0. Then 1 must reply, in the 0th copy of $\bigvee_{i \in I} A_i$, by choosing an index $j \in I$. By definition of R, he must choose the same index in all other copies that are ever played in. At your (0's) next move, choose $\overline{\bigvee_{i \in I} \mathsf{R}(A_i)}$, and there choose the same index j. From this point on, you are, in effect, playing $\mathsf{R}(A_j) \otimes \overline{\mathsf{R}(A_j)}$ which you win by mimicking.

(q): 0 wins $\mathsf{R}(A) \otimes \mathsf{R}(B) \otimes \overline{\mathsf{R}}(\overline{A \otimes B})$ by the following strategy. Copy player 1's moves in the ith copy of $\overline{A \otimes B}$ into the ith copy of A in $\mathsf{R}(A)$ or the ith copy of B in $\mathsf{R}(B)$ accordingly and force 1 to respond in one of two copies, and copy 1's respond back into the ith copy of $\overline{A \otimes B}$ as your respond. When the game is finished, if 1 has won the jth copy of $\overline{A \otimes B}$, you must have won the jth copy of A in $\mathsf{R}(A)$ or the jth copy of B in $\mathsf{R}(B)$ and hence won $\mathsf{R}(A)$ or $\mathsf{R}(B)$ by definition of R. Note that you don't have to worry about the consistency rule in $\overline{\mathsf{R}}(\overline{A \otimes B})$ because you simply copy 1's moves and if you break the consistency rule, player 1 must have broken the consistency rule in either $\mathsf{R}(A)$ or $\mathsf{R}(B)$, which makes you win the whole game by the definition of $\bigotimes$. Other directions are similar and omitted. □

Corollary 2.1.6. *For each infinite cardinal κ, the quotient algebra $\mathcal{A}_\kappa/\equiv$ on $\mathcal{S}_\kappa$ is well-defined by $\overline{[A]} = [\overline{A}]$, $R([A]) = [R(A)]$, $\overline{R}([A]) = [\overline{R}(A)]$, and $[A] \circ [B] = [A \circ B]$ where $\circ$ is $\wedge$, or $\vee$, or $\otimes$, or $\overline{\otimes}$.*

Proof. By **(a)**, **(i)** and **(o)** of Theorem 2.1.5. □

Part **(j)** of Theorem 2.1.5 says in each $\mathcal{S}_{\omega_\alpha}$ there is a unique degree in which 0/1 has a winning strategy. We denote the degree in which 0 has a winning strategy by $\mathbf{0}$ and the degree in which 1 has a winning strategy by $\mathbf{1}$. It is easy to see that $\overline{\mathbf{0}} = \mathbf{1}$ and $\overline{\mathbf{1}} = \mathbf{0}$.

With a fixed algebra on $\mathcal{S}_\kappa$, it is natural to study those algebra equations such that only variables occur in the term on the left of $=$ and the right side term is $\mathbf{1}$, and and can be satisfied by any substitution. The question is whether these equations can be characterized syntactically. This question is equivalent to the question of characterizing solvable inequations with $< \mathbf{1}$ on the right. We will discuss in the next section the connection between this question and logic. Before moving to that part, we give more lattice properties of $\mathcal{S}_{\omega_\alpha}$ in **ZFC** in the next subsection.

2.1.3 Lattice properties of $\mathcal{S}_{\omega_\alpha}$ in ZFC

In this subsection we investigate the properties of $\mathcal{S}_{\omega_\alpha}$ in **ZFC**. All results were proved by Blass [5]. Our first theorem will imply that there are at least four different degrees in $\mathcal{S}_\kappa$, if κ is an infinite cardinal such that $\kappa^\omega = 2^\kappa$.

Theorem 2.1.7 (**AC**)**.** *For every non-determined game A on κ where κ is an infinite cardinal such that $\kappa^\omega = 2^\kappa$, there is a game B on κ incomparable with A.*

Proof. Let A be given, and B be a strict game that has not been defined on ${}^\omega\kappa$

Thus, $(A \otimes \overline{B})(s)$ and $(B \otimes \overline{A})(s)$ are already defined for each finite s. Hence Σ, the set of all strategies for 0 in $A \otimes \overline{B}$, and Δ the set of all strategies for 0 $(B \otimes \overline{A})(s)$ are defined. Clearly $|\Sigma| \leq 2^\kappa$ and $\Delta \leq 2^\kappa$ by our assumption.

By **AC**, we can enumerate strategies in Σ and Δ in a sequence $\{\sigma_\alpha |\ \alpha < 2^\kappa\}$. To complete the definition of B, we will define an increasing sequence $\{B_\alpha |\ \alpha < 2^\kappa\}$ of partial functions from ${}^\omega\kappa$ into 2, and let B be any total extension of $\bigcup_{\alpha<2^\kappa} B_\alpha$.

Begin by setting $B_0 = 0$. For limit ordinals α, set $B_\alpha = \bigcup_{\beta<\alpha} B_\beta$. Suppose $\alpha = \beta{+}1$ and B_β has been defined. Suppose also that σ_β is a strategy for 0 in $B \otimes \overline{A}$. (The case that σ_β is a strategy in $A \otimes \overline{B}$ is handled similarly.) We will define B_α so as to guarantee that σ_β is not winning.

For each $s \in {}^{\omega}\kappa$, let $s'(n) = s(2n{+}1)$, so s' is the sequence of moves of 1 in the play s of B. As B_β has cardinality $\leq \alpha < 2^\kappa$ and ${}^{\omega}\kappa$ has cardinality 2^κ, we can find an $f \in {}^{\omega}\kappa$ such that $f \neq s'$ for all $s \in \text{Domain}(B_\beta)$. Consider the plays t of $B \otimes \overline{A}$ that result when 0 uses strategy σ_β, 1 plays $f(n)$ at his nth move in B, but 1 plays arbitrarily in A. For any such t, recall that $(t)_B$ and $(t)_{\overline{A}}$ be the subsequences of t consisting of the moves in B and in $\overline{A}$, respectively. (Formally, $(t)_B = t \downarrow 0$ and $(t)_{\overline{A}} = t \downarrow 1$.)

Case 1. For every such t, the play $(t)_{\overline{A}}$ of $\overline{A}$ is finished and won by 0. Then 0 wins $\overline{A}$ by means of of the following strategy. Imagine, in addition to the actual game of $\overline{A}$, a fictitious game of B; use strategy σ_β, and imagine 1 is playing f in B. Thus, $\overline{A}$ is a win for 0, contrary to the hypothesis that A is non-determined.

Case 2. For one such t, the play $(t)_{\overline{A}}$ is unfinished or won by 1. Notice that $(t)_B$ is not in the domain of B_β, for $t'_B = f \neq s'$for all s in that domain. Therefore, we may define B_α to be the extension of B_β which also maps t_B to 0. Then the play t of $B \otimes \overline{A}$ is won by 1 even though 0 used σ_β, so σ_β is not winning.

Since every possible strategy for 0 in $A \otimes \overline{B}$ and $B \otimes \overline{A}$ is σ_β for some β, we conclude that B is incomparable with A. □

Corollary 2.1.8 (AC). *For every game A on κ where κ is an infinite cardinal such that $\kappa^\omega = 2^\kappa$, there is a game B not similar to A.*

Proof. By **(g)** of Theorem 2.1.5. □

Corollary 2.1.9 (AC). *Let κ be an infinite cardinal such that $\kappa^\omega = 2^\kappa$ Every non-determined game A on κ is less than another non-determined game on κ. Every non-determined game is the first element of κ^+-long increasing well-ordered sequences in $\mathcal{G}_\kappa$.*

Proof. For the first claim, take the least upper bound of the given game and one incomparable with it. (By **(b)** of Theorem 1.2.27, $\bigvee_{i \in I} A_i$ is non-determined if all the A_i are.) The second claim follows by transfinite iteration of the first, using the completeness of $\mathcal{S}_\kappa$ given by Corollary 2.1.4. □

Corollary 2.1.10 (AC). *Let κ be an infinite cardinal such that $\kappa^\omega = 2^\kappa$. Given 2^κ or fewer non-determined games on κ, there is a game incomparable with them all. In particular, for every set of non-determined games, there is a game incomparable with them all.*

Proof. The proof is a trivial modification of the proof of Theorem 2.1.7. □

Corollary 2.1.11 (AC). *Let κ be an infinite cardinal such that $\kappa^\omega = 2^\kappa$. Then $\mathcal{S}_\kappa$ includes chains of order type κ^+ and antichains of cardinality $(2^\kappa)^+$.*

Let $(P, \leq)$ be any partially ordered set. A **term** is an expression built up from elements of P by means of the formal operations $\wedge$ and $\vee$. Whenever P is mapped into a lattice L, every term denotes an element of L, and every inequality, $S \leq T$, between such terms becomes either true or false. We define recursively the notion of a **necessary inequality** between terms.

(a) If $p \leq q$ in P, then $p \leq q$ is necessary.

(b) If $S \leq T$ and $S \leq U$ are necessary, so is $S \leq T \wedge U$.

(c) If $S \leq U$ and $T \leq U$ are necessary, so is $S \vee T \leq U$.

(d) If $S \leq U$ is necessary, so are $S \leq T \vee U$ and $S \leq U \vee T$.

(e) If $S \leq U$ is necessary, so are $S \wedge T \leq U$ and $T \wedge S \leq U$.

An inequality is necessary only if its being so follows from **(a)**–**(e)**.

If P is mapped into a lattice in an order-preserving way, then all necessary inequalities clearly become true.

Let $(L, \leq)$ and be a lattice and $P \subset L$. The sublattice generated by $\overline{P}$ consists of elements of L that can be obtained by operators $\bigwedge$ and $\bigvee$ from P. $\overline{P}$ is **free** on P if $S = T$ implies $S \leq T$ and $T \leq S$ are necessary. See [23, p. 325].

If P can be order-isomorphically embedded in a lattice L in such a way that no unnecessary inequalities become true, then the sublattice $\overline{P}$ of L generated by P is free on P. This means that any order-preserving map of P to any lattice extends uniquely to $\overline{P}$. If we take P to be an antichain, then $\overline{P}$ is a free lattice.

Our next theorem will imply that if $2^\kappa = \kappa^\omega$ then every partially ordered subset of size $\leq 2^\kappa$ can be order-isomorphically embedded in $\mathcal{S}_\kappa$ and every free lattice of size $\leq 2^\kappa$ can be lattice-isomorphically embedded in $\mathcal{S}_\kappa$.

Theorem 2.1.12 (AC). *Assume P has cardinality $\leq 2^\kappa = \kappa^\omega$. Then P can be order-isomorphically embedded in $\mathcal{S}_\kappa$ in such a way that no unnecessary inequalities become true.*

Proof. We will assign, for each $p \in P$, a strict game also called p to be defined on ${}^\omega\kappa$.

As in the proof of Theorem 2.1.7, p will be defined by an induction of length $\leq 2^\kappa$. At each step, we consider a strategy that threatens to make an unnecessary inequality true, and we make sure that it doesn't work. Also, if $p \leq q$ in P, then whenever we define $p(s) = 0$ for some s, we also define $q(s) = 0$ for the same s (at the same stage of the induction), and whenever we define $q(s) = 1$, we also define $p(s) = 1$. This ensures that $p \otimes \overline{q}$ is a win for 0 (by means of a copycat strategy) so $[p] \leq [q]$ in $\mathcal{S}_\kappa$. At each stage of the induction, only one or two new sequences s will be added to $\bigcup_{p \in P} \mathrm{dom}(p)$.

Clearly, there are only 2^κ terms and therefore only 2^κ unnecessary inequalities $S \leq T$. For each of these, there are only 2^κ strategies for 0 in the corresponding game $S \otimes \overline{T}$, so there are enough steps in the induction to make sure that each of these strategies can be defeated.

Suppose we are at a particular stage of the induction, at which the strategy σ for 0 in $S \otimes \overline{T}$ is under consideration. ($S \leq T$ is unnecessary.) There are fewer than 2^κ sequences $s \in \bigcup_{p \in P} \mathrm{dom}(p)$, so we can choose an $f \in {}^\omega\kappa$ which doesn't occur as the sequence of moves of either 0 or 1 in any such s.

A play of $S \otimes \overline{T}$ consists of two phases. In phase 1, the players are deciding which of the subterms of S and T to play. For example, if S is $X \wedge Y$, then 0's choice of X or Y belongs to phase 1. These moves continue until S and T have been reduced to atomic terms $p, q \in P$. Then the players are essentially playing $p \otimes \overline{q}$; this is phase 2.

(In fact the phases may overlap. Once S is reduced to p, the players may start to play p before finishing the reduction of T.)

Consider the following play of $S \otimes \overline{T}$. Player 0 uses strategy σ. At a phase 1 move of player 1, if S and T have been reduced to X and Y with $X \leq Y$ unnecessary, he moves so that the resulting reduction still corresponds to an unnecessary inequality. He can do this because of clause **(b)** and **(c)** in the definition of necessary. Furthermore, 0's phase 1 moves cannot produce necessary inequalities from unnecessary ones, by clause **(d)** and **(e)**. Thus, the reduced games $X \otimes \overline{Y}$ correspond to unnecessary inequalities $X \leq Y$. At his phase 2 moves, player 1 play the sequence f in each of the components p, q of $p \otimes \overline{q}$.

Case 1. In this play s, T is not finished. Then S is finished. It is ultimately reduced to some $p \in P$, and the play t of p (subsequence of s) is not in $\mathrm{dom}(r)$ for any $r \in P$, by choice of f. For $p' \geq p$, extend p' by setting $p'(t) = 0$; leave the other games unchanged. Then 0 loses the play s of $S \otimes \overline{T}$ although he used σ, so σ is not winning.

Case 2. S is not finished. This is entirely analogous to Case 1. We take the play t of the $q \in P$ to which T eventually reduced, and set $q'(t) = 1$ for all $q' \leq q$.

Case 3. Both S and T are finished in s. They are reduced to games p and q such that $p \leq q$ is unnecessary, which implies $p \nleq q$ in P by clause (a). The sub-plays s_p (moves in p) and s_q (moves in q) are not in $\mathrm{dom}(r)$ for any $r \in P$, by choice of f. If $r \geq p$, extend it by setting $r(s_p) = 0$; if $r \leq q$, set $r(s_q) = 1$; otherwise, do nothing to r. Even if $s_p = s_q$, these these definitions do not conflict with each other because $p \nleq q$.

This completes the inductive definition of the games p. The construction assures that P is mapped into $\mathcal{S}_\kappa$ in an order-preserving way, and no unnecessary inequalities hold. In particular, if $p \nleq q$ in P, then $p \leq q$ is unnecessary (by inspection of the definition of necessary), so $p \nleq q$ in $\mathcal{S}_\kappa$. The map of P into $\mathcal{S}_\kappa$ is therefore order-isomorphic. □

A lattice L is **modular** if $x \leq b$ implies $(x \vee a) \wedge b = x \vee (a \wedge b)$ for any $a \in L$.

Corollary 2.1.13 (**AC**). *Let κ be an infinite cardinal such that $\kappa^\omega = 2^\kappa$. $\mathcal{S}_\kappa$ is not modular.*

Proof. Let P be $\{0, 1, 2\}$ ordered so that $0 < 2$ and 1 is incomparable with 0 and 2. Then the modular inequality $(0 \vee 1) \wedge 2 \leq 0 \vee (1 \wedge 2)$ is unnecessary, hence false for some embedding of P into $\mathcal{S}_\kappa$. □

With P and κ in Theorem 2.1.12, let us extend the notion of term by allowing formal greatest lower bounds and least upper bounds of κ or fewer, rather than only two, terms at a time. We extend the definition of necessary to include the new infinitary terms. The clause corresponding to **(b)**, **(c)**, **(d)** and **(e)** are

(b') If $S \leq T_i$ is necessary for all $i \in I$, so is $S \leq \bigwedge_{i \in I} T_i$.

(c') If $S_i \leq T$ is necessary for all $i \in I$, so is $\bigvee_{i \in I} S \leq T$.

(d') If $S \leq T_i$ is necessary for some $i \in I$, so are $S \leq \bigvee_{i \in I} T$.

(e') If $S_i \leq T$ is necessary for some $i \in I$, then so $\bigwedge_{i\in I} S_i \leq T$.

The index set I is assumed to have cardinality $\leq \kappa$. Theorem 2.1.12 remains true with these extended definitions of term and necessary. It follows that $\mathcal{S}$ contains an isomorphic copy of the complete free κ-complete lattice on P, where P is any partially ordered set of size $\leq 2^\kappa$. It also follows that complete free κ-complete lattices satisfy only necessary inequalities.

It should be pointed out that not every lattice can be embedded in $\mathcal{S}_\kappa$. The following theorem provides us with a special property of $\mathcal{S}_\kappa$ that cannot be shared by all lattices.

Theorem 2.1.14. *If $\bigwedge_{i\in I} A_i \leq \bigvee_{j\in J} B_j$, then either $A_i \leq \bigvee_{j\in J} B_j$ for some $i \in I$, or $\bigwedge_{i\in I} A_i \leq B_j$ for some $j \in J$.*

Proof. Let σ be a winning strategy for 0 in

$$\bigwedge_{i\in I} A_i \otimes \overline{\bigvee_{j\in J} B_j} = \bigwedge_{i\in I} A_i \otimes \bigwedge_{j\in J} \overline{B}_j.$$

Since 0 moves first in a tensor-product game, σ must begin by specifying a choice of $\bigwedge_{i\in I} A_i$ or $\bigwedge_{j\in J} \overline{B}_j$; suppose it chooses the former. (The argument in the other case is analogous.) By definition of $\bigwedge$, it is still 0's move, and σ must choose an $i \in I$. From here on, the player are, in effect, playing $A_i \otimes \overline{\bigvee_{j\in J} B_j}$ and σ provides a winning strategy for 0 in this game. Hence, $A_i \leq \bigvee_{j\in J} B_j$. $\square$

It is easy to give examples of lattices where $A_1 \wedge A_2 \leq B_1 \vee B_2$ does not imply $A_i \leq B_1 \vee B_2$ or $A_1 \wedge A_2 \leq B_i$ for either i. According to Theorem 2.1.14, such a lattice cannot be lattice-isomorphically embedded in $\mathcal{S}_\kappa$.

2.2 Game logics on cardinals

2.2.1 Patterns for winning: a motivation for game logics

Let κ be an infinite cardinal. Consider the game universe $\mathcal{G}_\kappa$ and the algebra $\mathcal{A}_\kappa$. We call terms (usually with variables) of $\mathcal{A}_\kappa$ "formulas". We will give the word "formulas" a precise definition later. But let us keep informal for now. Given a formula ϕ, we call each function $\mathcal{I}$ assigning games to variables in ϕ an interpretation for ϕ. So under each interpretation $\mathcal{I}$ for ϕ, ϕ is a well-defined game. We denote this game by $\bar{\mathcal{I}}(\phi)$.

Since we are primarily concerned of which games can be won by player 1, under each interpretation $\mathcal{I}$, we can associate to the formula ϕ a winning value, our game version of truth values indicating to what extent the sentence "player 1 can win it" is true. We give value 0 to a formula ϕ, if player 0 has a winning strategy in $\bar{\mathcal{I}}(\phi)$, since there is no way I can win if my opponent want to win. Similarly, we give value 1 if player 1 has a winning strategy in $\bar{\mathcal{I}}(\phi)$, which means I can win if I want to. If ϕ is non-determined, we give it some value indicating its difficulty for me to win. A natural choice is to use $[\bar{\mathcal{I}}(\phi)]$, the degree of non-determinacy of $\bar{\mathcal{I}}(\phi)$, as the winning value of $\bar{\mathcal{I}}(\phi)$. Notice that the winning values we gave to $\bar{\mathcal{I}}(\phi)$ in the first two cases are also $[\bar{\mathcal{I}}(\phi)]$. We say a formula ϕ is valid if it has winning value 1 under all interpretations. Notice that if ϕ is valid, all its substitutes are also valid.

Now we have a language and a semantics. A a natural question is to find a consistent and complete proof system for this semantics. It is not yet clear whether such a system exists for the given cardinal κ. We will investigate this question and provide partial answers to it later in this thesis.

We can repeat the above discussion with $\mathcal{S}_\kappa$ and $\mathcal{A}_\kappa/\equiv$ instead of $\mathcal{G}_\kappa$ and $\mathcal{A}_\kappa$ respectively. Thereby we get the notion of degree language and that of degree semantics. Nevertheless, we prefer to use the names "game language" and "κ-game semantics" to mean degree language and degree semantics. There are two immediate benefits: 1. According to Theorem 1.2.7 and **(f)** of Theorem 2.1.5, for each representative of a degree, we can safely assume it has any not eventually constant form. In particular, we can assume each game is strict or reversely strict. 2. The winning value for formula ϕ under interpretation $\mathcal{I}$ is $\bar{\mathcal{I}}(\phi)$ itself. Thus interpretations also function as winning value assignment functions, making our game semantics less complicated, and more close to our algebraic problem stated near the end of subsection . Now the left terms of those algebraic equations in that problem are the same as valid formulas.

2.2.2 Game languages and game semantics on cardinals

In the last subsection we introduced the notions of game language and game semantics. We will develop these notions precisely in this subsection.

The **language of game logic**, or **game language** is $\mathcal{L}$. Variables for $\mathcal{L}$ are lower case English letters a, b,...,z, a_0,...,a_α,...b_0,...,b_α,... and so on. The game language $\mathcal{L}$ has two constant symbols 0 and 1, three unary operator $\overline{}$, R and $\overline{\mathsf{R}}$, and four binary operators $\wedge$, $\vee$, $\otimes$ and $\overline{\otimes}$.

Well-formed formulas, or **formulas** in $\mathcal{L}$ are recursively defined as follows.

- Constant symbols and variables in $\mathcal{L}$ are well-formed formulas.
- If ϕ is a well-formed formula, $\overline{\phi}$, $\mathsf{R}(\phi)$ and $\overline{\mathsf{R}}(\phi)$ are well-formed formulas.
- If ϕ and ψ are well-formed formulas, $\phi\wedge\psi$, $\phi\vee\psi$, $\phi\otimes\psi$, and $\phi \mathbin{\overline{\otimes}} \psi$ are well-formed formulas.

Formulas whose operators are among $\overline{}$, $\wedge$ and $\vee$ are called **additive** formulas. Those whose operators are among $\overline{}$, $\otimes$ and $\overline{\otimes}$ are called **multiplicative** formulas.

The degree universe $\mathcal{S}_\kappa$ is called the **(winning) value universe** of the κ-game semantics. Let $\mathcal{I}$ be a function whose domain is a subset of variables of $\mathcal{L}$ and whose range is a subset of $\mathcal{S}_\kappa$. $\mathcal{I}$ is an **variable κ-interpretation** for formula ϕ if variables occurring in ϕ are in the domain of $\mathcal{I}$. Define $\bar{\mathcal{I}}$, the **(formula) κ-interpretation** under $\mathcal{I}$, recursively as follows.

- $\bar{\mathcal{I}}(0) = \mathbf{0} \in \mathcal{S}_\kappa$; $\bar{\mathcal{I}}(1) = \mathbf{1} \in \mathcal{S}_\kappa$.
- $\bar{\mathcal{I}}(G) = \mathcal{I}(G)$ for each variable $G \in \mathrm{dom}(\mathcal{I})$.
- $\bar{\mathcal{I}}(\overline{\phi}) = \overline{\bar{\mathcal{I}}(\phi)}$, $\bar{\mathcal{I}}(\mathsf{R}(\phi)) = \mathsf{R}(\bar{\mathcal{I}}(\phi))$ and $\bar{\mathcal{I}}(\overline{\mathsf{R}}(\phi)) = \overline{\mathsf{R}}(\bar{\mathcal{I}}(\phi))$, if $\bar{\mathcal{I}}(\phi)$ has been defined.

- $\bar{\mathcal{I}}(\phi \wedge \psi) = \bar{\mathcal{I}}(\phi) \wedge \bar{\mathcal{I}}(\psi)$, $\bar{\mathcal{I}}(\phi \vee \psi) = \bar{\mathcal{I}}(\phi) \vee \bar{\mathcal{I}}(\psi)$,
  $\bar{\mathcal{I}}(\phi \otimes \psi) = \bar{\mathcal{I}}(\phi) \otimes \bar{\mathcal{I}}(\psi)$, $\bar{\mathcal{I}}(\phi \mathbin{\overline{\otimes}} \psi) = \bar{\mathcal{I}}(\phi) \mathbin{\overline{\otimes}} \bar{\mathcal{I}}(\psi)$,
  if $\bar{\mathcal{I}}(\phi)$ and $\bar{\mathcal{I}}(\psi)$ have been defined.

It is easy to see if $\mathcal{I}$ is a variable κ-interpretation for ϕ, $\bar{\mathcal{I}}(\phi)$ is well-defined. It is also easy to see the following.

Lemma 2.2.1. *Let ϕ and ψ be arbitrary formulas in $\mathcal{L}$.*

- $\bar{\mathcal{I}}(\overline{0}) = \bar{\mathcal{I}}(1)$, $\bar{\mathcal{I}}(\overline{1}) = \bar{\mathcal{I}}(0)$.
- $\bar{\mathcal{I}}(\overline{\overline{\phi}}) = \bar{\mathcal{I}}(\phi)$.
- $\bar{\mathcal{I}}(\overline{\mathsf{R}(\phi)}) = \bar{\mathcal{I}}(\overline{\mathsf{R}}(\overline{\phi}))$, $\bar{\mathcal{I}}(\overline{\overline{\mathsf{R}}(\phi)}) = \bar{\mathcal{I}}(\mathsf{R}(\overline{\phi}))$.
- $\bar{\mathcal{I}}(\overline{\phi \wedge \psi}) = \bar{\mathcal{I}}(\overline{\phi} \vee \overline{\psi})$, $\bar{\mathcal{I}}(\overline{\phi \vee \psi}) = \bar{\mathcal{I}}(\overline{\phi} \wedge \overline{\psi})$.
- $\bar{\mathcal{I}}(\overline{\phi \otimes \psi}) = \bar{\mathcal{I}}(\overline{\phi}\overline{\otimes}\overline{\psi})$, $\bar{\mathcal{I}}(\overline{\phi\overline{\otimes}\psi}) = \bar{\mathcal{I}}(\overline{\phi} \otimes \overline{\psi})$.

For any formula ϕ in $\mathcal{L}$, NF(ϕ)

Proof. By definition. □

We have a normal form theorem for our game logic.
Define the function NF (for normal form) on formulas recursively as follows.

- $\mathrm{NF}(\overline{0}) = 1$, $\mathrm{NF}(\overline{1}) = 0$.
- $\mathrm{NF}(\phi) = \phi$ if ϕ is a variable, or a constant, or the negation of a variable.
- $\mathrm{NF}(\phi \wedge \psi) = \mathrm{NF}(\phi) \wedge \mathrm{NF}(\psi)$, $\mathrm{NF}(\phi \vee \psi) = \mathrm{NF}(\phi) \vee \mathrm{NF}(\psi)$.
- $\mathrm{NF}(\phi \otimes \psi) = \mathrm{NF}(\phi) \otimes NF(\psi)$, $\mathrm{NF}(\phi \mathbin{\overline{\otimes}} \psi) = \mathrm{NF}(\phi)\overline{\otimes}\mathrm{NF}(\psi)$.
- $\mathrm{NF}(\mathsf{R}(\phi)) = \mathsf{R}(\mathrm{NF}(\phi))$, $\mathrm{NF}(\overline{\mathsf{R}}(\phi)) = \overline{\mathsf{R}}(\mathrm{NF}(\phi))$.
- $\mathrm{NF}(\overline{\overline{\phi}}) = \mathrm{NF}(\phi)$.
- $\mathrm{NF}(\overline{\mathsf{R}(\phi)}) = \mathrm{NF}(\overline{\mathsf{R}}(\overline{\phi}))$, $\mathrm{NF}(\overline{\overline{\mathsf{R}}(\phi)}) = \mathrm{NF}(\mathsf{R}(\overline{\phi}))$.
- $\mathrm{NF}(\overline{\phi \wedge \psi}) = \mathrm{NF}(\overline{\phi} \vee \overline{\psi})$, $\mathrm{NF}(\overline{\phi \vee \psi}) = \mathrm{NF}(\overline{\phi} \wedge \overline{\psi})$.
- $\mathrm{NF}(\overline{\phi \otimes \psi}) = \mathrm{NF}(\overline{\phi}\overline{\otimes}\overline{\psi})$, $\mathrm{NF}(\overline{\phi\overline{\otimes}\psi}) = \mathrm{NF}(\overline{\phi} \otimes \overline{\psi})$.

Theorem 2.2.2. *For each formula ϕ in $\mathcal{L}$,* NF(ϕ) *is a formula built from variables, their negations and constants via operators except for the dual operator.* NF(ϕ) *has the same set of variables as ϕ, and $\bar{\mathcal{I}}(\mathrm{NF}(\phi)) = \bar{\mathcal{I}}(\phi)$ for any variable κ-interpretation for ϕ.*

Proof. Easy induction on the complexity of ϕ. □

A formula ϕ in language $\mathcal{L}$ is **κ-valid** if $\bar{\mathcal{I}}(\phi) = \mathbf{1}$ for any variable κ-interpretation $\mathcal{I}$ for ϕ. In particular, ϕ is κ-valid if and only if NF(ϕ) is.

2.2.3 A syntactic characterization of valid multiplicative formulas

All results in this subsection were proved by Blass [6]. We will sometimes omit long and tedious recursive definition and inductive proofs and replace them with informal discussion which can be easily translated into precise mathematics by a careful reader.

By Theorem 2.2.2 if we have a syntactic characterization of normal forms of valid multiplicative formulas, we thereby also have a syntactic characterization of valid multiplicative formulas themselves. So in the rest of this section by multiplicative formulas, we mean norm forms of them.

Multiplicative formulas can be read as formulas in classical propositional logic, with $\otimes$ and $\overline{\otimes}$ read as conjunction and disjunction, respectively (and $\overline{\ \ }$, 1 and 0 read as a negation, truth and falsity, respectively). We do not speak of translating multiplicative formulas into the standard symbolism of classical logic (replacing $\otimes$ with $\wedge$, etc.), as this might lead to confusion with the additive connectives. Instead, we pretend that classical logic is formulated with $\overline{\ \ }$, $\otimes$ and $\overline{\otimes}$ as its connectives, so that multiplicative formulas of game logic are also formulas of classical logic. So it makes sense to speak of a multiplicative formula being a tautology, or of a positive or negative occurrence of a variable in a multiplicative formula, or of any other concept familiar from classical propositional logic. In particular, by an instance or (substitution instance) of a multiplicative formula A, we mean a formula obtained by replacing the variables in A uniformly by some multiplicative formulas. By the literals in a multiplicative formula C, we will mean the occurrences of variables and negated variables from which C is built by $\otimes$ and $\overline{\otimes}$. We call a multiplicative formula binary if each variable has at most one positive and one negative occurrence.

Note that being a tautology is a syntactic feature of a multiplicative formula, since there are consistent and complete proof systems for classical propositional logic.

Theorem 2.2.3. *A multiplicative formula (normal form) in $\mathcal{L}$ is κ-valid if it is an instance of a binary tautology.*

Proof. Since any instance of a valid formula is clearly also valid, it suffices to prove that all binary tautologies are valid. In fact, it will suffice to consider binary tautologies in which every variable occurs exactly twice (once positively and once negatively) and 1 and 0 do not occur. For brevity, we call such tautologies **special**.

To see that we may confine attention to special tautologies, suppose these were known to be valid, and consider an arbitrary binary tautology C. Starting with C, we repeatedly replace subformulas according to the following rules as long as any of the rules apply.

(1) Any literal whose variable having only one occurrence is replaced by 0.

(2) A subformula of the form $1 \otimes A$ or $0 \mathbin{\overline{\otimes}} A$ is replaced by A.

(3) A subformula of the form $1 \mathbin{\overline{\otimes}} A$ (respectively, $0 \otimes A$) is replaced by 1 (respectively 0).

It is clear that all the formulas produced are binary tautologies, that the process terminates (because each replacement reduces the total number of occurrences of propositional variables, $\otimes$, and $\overline{\otimes}$), and that the final result C' is either a special tautology or simply 1. So by our assumption (and the obvious validity of 1), C' is valid. We intend to infer from this that C is valid, as desired. But this is easy: if a valid formula B' is obtained from a formula B by a single replacement of the form (1), (2) or (3), then B is also valid. Thus, we have shown that we can safely confine our attention to special tautologies.

Before continuing our proof, we introduce a notational simplification for the ease of reading. We do not distinguish games and degrees in the following sense. We often use games rather than degrees to interpret variables, and interpret formulas by games built up from games interpreting variables and game operators the way stated by the formulas literally and we use the formulas themselves to name games interpreting them, and then show P has a (no) winning strategy in the game interpreting a given formula ϕ. But by doing this, we have shown that $\bar{\mathcal{I}}(\phi) = 1$ ($\bar{\mathcal{I}}(\phi) \neq 1$) where $\mathcal{I}$ is such that $\mathcal{I}(g) = [g]$ for each variable g. (Note that the g on the right of $=$ is the game interpreting variable g, according to our naming method.)

Let us show that each special tautology is κ-valid. Fix a special tautology C and fix a game interpretation. We assume that the games assigned to variables are strict games. We will complete the proof of the 'if' half of the theorem by describing a winning strategy for P in the game C.

Recall that the literals in C are the occurrences of variables and negated variables from which C is built by $\otimes$ and $\overline{\otimes}$. Thus, a negative occurrence of a variable p does not count as a literal; rather its context $\overline{p}$ is a literal. As C is special, the literals come in pairs, each containing p and $\overline{p}$ for some variable p. The game C consists of subgames, one for each literal, and who is to move in a given position (finite play) of C and who has lost a given play of C can be obtained by a truth-table method from the same information about those literal games. More precisely, we think in terms of the parse tree of C, with C at the root, literals at the leaves, and $\otimes$ or $\overline{\otimes}$ labeling the internal nodes. A position or play gives a labeling of the leaves as we give truth values to formulas in classical logic (True if P has won or O is to move, False if O has won or P is to move), and the truth values propagate from the leaves to the other nodes according to the truth tables for conjunction ($\otimes$) and disjunction ($\overline{\otimes}$). This can be easily justified from the definitions of $\otimes$ and $\overline{\otimes}$ and we omit the details here.

Since C is a tautology, the root will have label True provided each pair of literals, p and $\overline{p}$, have opposite truth values, so that the labeling is really a truth assignment (in the sense of classical logic). In fact, since only positive connectives are used in the tree, the root will also have label True if some p and $\overline{p}$ are both labeled True. But it is, of course, entirely possible that p and $\overline{p}$ are both labeled False in a particular position or play (e.g., if P is to move in both of these subgames), and then the root C may well be labeled False.

Whenever P is to move at a certain position in game C, i.e., when C is labeled False, his move consists of choosing a path through the parse tree from the root to a leaf l, such that all the nodes along the path are labeled False, and then making a move in l. (The choice of path will involve a real choice at $\overline{\otimes}$ nodes, where a False label means that both successors are also labeled False. At $\otimes$ nodes, usually (i.e., expect at the first visit to this node) only one successor will be labeled False; see the discussion following the definition of $\otimes$.)

The essential idea for P's winning strategy is to make sure that the plays in paired subgames p and $\overline{p}$ are identical. Since he plays opposite roles in these two subgames, he will (if infinitely many moves are made in each of them) win one and lose the other, so the final labeling of the tree will be a real truth assignment, C will be labeled True, and so P will win C. We must still show that P can carry out the proposed strategy and that he will win even if some of the subgames are unfinished (i.e., have only finitely many moves made in them). For this purpose, we must describe the strategy in somewhat more detail.

P is to ensure that, at each moment during the play of the game, for each pair p, $\overline{p}$ of literals, either the positions (=sequences of moves already played) in p and $\overline{p}$ are identical or else one of them equals the other plus one subsequent move made by O. This condition is certainly satisfied initially, as all positions are initially the empty sequence.

Furthermore, this condition cannot be destroyed by a move of O. To see this, suppose the condition is satisfied at a certain moment, which we call before, and that O then moves, say in subgame p . If the positions in p and $\overline{p}$ were identical before, then afterward the position in p is that in $\overline{p}$ plus the single move just made by O, so the condition remains satisfied. If the position in p before were that in $\overline{p}$ plus a move by O, then O could not have moved in p as it would be P's turn there.

Finally, if the position in $\overline{p}$ before were that in p plus a move of O, then the presence of that move of O in $\overline{p}$ means that the position without the move, the before position in p, is a position with O to move in $\overline{p}$, hence is a position with P to move in p; so again O could not move in p. This shows that a move of O in p (or, for symmetrical reasons, in $\overline{p}$) cannot destroy the condition that P is trying to maintain.

The preceding discussion shows furthermore that, as long as the condition is satisfied, whenever the positions in p and $\overline{p}$ are different, it is P's turn to move in both of them, whereas of course if the positions in p and $\overline{p}$ are equal, then P is to move in one and O in the other.

We show next that if the condition holds at a certain position and if P is to move, then he can move so as to maintain the condition. More precisely, we show that there is a literal l such that (1) the path from l to the root in the parse tree of C is labeled entirely with False, so that P can legally move in the subgame l, and (2) the position in $\overline{l}$ is one move longer than that in l. Here (2) means that O has made a move in $\overline{l}$ which P can simply copy in l, thereby maintaining the desired condition. We call a literal l **good** at a given position if (1) and (2) hold.

Lemma 2.2.4. *Let the parse tree of C be labeled, using the appropriate truth tables at the interior nodes but arbitrary labels at t he leaves. If C is labeled False, then there is a pair p, $\overline{p}$ of literals such that the paths joining them to the root are both labeled entirely with False.*

Proof of Lemma 2.2.4. Suppose we had a labeling that is a counterexample to the lemma. We saw earlier that, because C is a tautology yet labeled False and because the connectives at interior nodes are monotone, there must be a pair of literals p, $\overline{p}$ both labeled False. As the labeling is a counter-example to the lemma, the path from one of $p, \overline{p}$, say p, to the root contains a label True. Alter the labeling of the leaves by changing p from False to True, and consider the resulting new labeling of the parse tree (in accordance with the connectives, as always). The change at the leaf p can affect only the labels along the path from p to the root and indeed can only increase these labels (i.e., change False to True) because the connectives are monotone. There was already a True label somewhere on this path. That label will therefore be unchanged. But then all labels between that True and the root are also unaffected by our change at p. In particular, the root C retains its previous label, False. This fact, and the fact that no True has been changed to False, means that our modified labeling is still a counterexample to the lemma. It has strictly fewer leaves labeled False than the original counterexample. So, by repeating the process, we have a contradiction. □

Lemma 2.2.4 shows that whenever P is to move, there is a pair of subgames p, $\overline{p}$ such that P can legally move in either of them. In particular, the positions in p and $\overline{p}$ cannot be identical, for then it would be P's move in only one of them. If the condition that P wants to maintain holds, then the position in one of p, $\overline{p}$ is one move longer than in the other. But then the latter is a good subgame.

We have seen that if P is to move and the condition holds, then there is a good subgame and P can move in any good subgame so as to maintain the condition. Once the good subgame is chosen, the appropriate move for P is unique; it consists of copying O's last move in the paired subgame. We specify P's strategy more completely by requiring that if there are several good subgames, then he should move in one where the sequence of previous moves is as short as possible. If several are equally short, choose the leftmost one in the parse tree.

Having described P's strategy and verified its feasibility, we show that it is a winning strategy. Suppose x were a play in which P used this strategy but lost. For each literal l, we let $(x)_l$ be the subsequence of moves in x in subgame l. We indicated a proof earlier that P wins if each $(x)_l$ is infinite; the possibility of some $(x)_l$'s being finite necessitates a subtler argument.

Label the nodes of the parse tree of C in the usual way for the play x. As P lost, C is labeled False. Apply Lemma 2.2.4 to obtain (the leftmost) p and $\overline{p}$ such that the paths joining these literals to the root are both labeled entirely with False. Then $(x)_p$, and $(x)_{\overline{p}}$, cannot both be infinite, for then they would be identical, thanks to P's strategy, and would have opposite labels (by definition of $\overline{}$). Nor can one be finite and the other infinite, for P's strategy ensures that their lengths never differ by more than one. So both are finite, and one, say $(x)_p$, without loss of generality, is one move shorter than the other. Fix such a p.

While playing the game C, leading to the play x, the players arrive after finitely many moves at a position with the following properties for every literal l.

(1) If $(x)_l$ is finite, then all moves that will ever be made in subgame l have already been made. (Subgames that will remain unfinished have been permanently abandoned.)

(2) If $(x)_l$ is infinite, then the number of moves already made in subgame l exceeds the number that have been (or ever will be) made in any finite $(x)_{l'}$.

Of course, once (1) and (2) hold, they continue to hold at all later positions. Call a position 1,2-late if conditions (1) and (2) hold.

Consider any 1,2-late position with P to move. By (1), P's move is in some l such that $(x)_l$ is infinite. But, by his strategy, P moves in a good literal where the current move sequence is as short as possible. By (2), the current move sequence in p is shorter than in I, since $(x)_p$ is finite and $(x)_l$ infinite. So if p were good, P would not have moved in l. Therefore, p is not good. But the position in $\overline{p}$, i.e., $(x)_{\overline{p}}$, is one move longer than the position in p which is $(x)_p$, because of our choice of p. So the only way for p not to be good is that, on the path from p to the root, there is a node labeled True.

We claim that, from some moment on, the positions leading to the play x satisfy

(3) Some node between the root and p is labeled True.

We have just shown this for positions where P is to move. When P moves, however, labels only increase. (One leaf goes from False (with P to move) to True (with O to move), the other leaves are unchanged, and internal nodes are given by monotone connectives.) So P's move cannot destroy (3). Thus, all 1,2-late positions, except possibly the first, are in fact 1,2,3-late (in the obvious sense).

At any 1,2,3-late position, consider the location of the True label nearest p on the path from p to the root. Consider how this location changes as the play proceeds. A move of P is always at a good literal, is therefore never at a leaf beyond this (or any) True label, and therefore never affects either this True label or the False labels between it and p. So the location is unchanged when P moves. A move of O can only decrease labels (from True to False), by the dual of the argument in the preceding paragraph. So the False labels between p and the location being studied are not changed; the True label at this location may change to False, and in this case the new location of the True nearest p (which still exists by (3)) is nearer the root. In summary, the location of the True nearest p moves, at 1,2,3-late stages of the play, only toward the root. As the path on which it moves is finite, it must eventually stop moving. Let X be its final location. Thus, at all sufficiently late stages of the play, we have

(4) X is labeled True.

At such stages, P will never move in literals that are beyond X in the parse tree (i.e., are subformulas of X), because he only moves in good literals and these have only False labels between them and the root. Therefore, at 1,2,3,4-late stages, O moves at most once in any literal beyond X, for once he moves in such a literal, it is P's turn there, and it remains P's turn there forever since P does not move there any more. Therefore, from some stage on, we have

(5) No moves are made in literals beyond X.

But this means that the labeling of leaves beyond X does not change any more. This labeling is therefore the same for any 1,2,3,4,5-late stage as for the final (infinite) play x. The same therefore holds for the label of X. But X is labeled True at a 1,2,3,4,5-late stage, by (4), and is labeled False for x, by our choice of p. This contradiction shows that, when he uses the strategy we described, P cannot lose C. This completes the proof. □

With the help of the Axiom of Choice, we can show that the syntactic characterization we just gave for multiplicative formulas is complete.

Theorem 2.2.5 (**AC**). *If a multiplicative formula (normal form) in $\mathcal{L}$ is κ-valid, it is an instance of a binary tautology.*

Proof. Let C be a multiplicative formula that is not an instance of a binary tautology. We construct (the degrees of) reversely strict games on 2 to interpret the game variables in C so that P has no winning strategy in the interpretation of C, or just game C for easier reading. The game G_p associated to a game variable p will be reversely strict and on 2 so that G_p has been defined on finite sequences. We have yet to specify the G_p on infinite sequences, but G_p being reversely strict is enough to determine the set of positions in the game C and the set of strategies for P in C. As there are only countably many positions (finite plays), the number of strategies is the cardinality $|{}^{\omega}2| = 2^{\aleph_0}$, or the cardinality $\mathfrak{c}$ of the continuum. By **AC**, fix a well-ordering of the set of all strategies for P, having order-type $2^{\aleph_0}$: thus, each strategy σ has fewer than $2^{\aleph_0}$ predecessors in this well-ordering.

We will define G_p on infinite sequences by transfinite recursion over this well-ordering. At the recursion step associated to a strategy σ, we will decide, for finitely many $x \in {}^{\omega}2$, the value of $G_p(x)$. We say that these x's are decided at stage σ. These decisions will be made in a way that ensures that σ is not a winning strategy for P in C. As every possible strategy for P in C occurs in our well-ordering, the whole construction will ensure that P has no winning strategy for C. The rest of the proof consists of showing how to carry out one step in the induction, say the step associated to σ.

There have been fewer than $2^{\aleph_0}$ previous steps, each deciding only finitely many $x \in {}^{\omega}2$. We split each of these xs into the two subsequences of moves attributable to the two players, i.e., $x|_0$ and $x|_1$. Thus, for each decided x, one subsequence consists of the even-numbered moves in x, the other of the odd-numbered moves, because we are dealing with strict games. There are fewer than $2^{\aleph_0}$ subsequences so obtained – two from each of fewer than $2^{\aleph_0}$ decided x's – so we can do the following.

For each occurrence l of a literal (i.e., a positive occurrence of a variable or negated variable) in C, choose a different sequence $z_l \in {}^{\omega}2$ that does not occur as the sequence of moves of either player in any previously decided x. Note that the same variable or negated variable may have several occurrences, corresponding to several subgames of C; these count as different literals l and have different z_l's assigned to them.

Construct a play of C as follows. P uses σ. O chooses, at each of his moves, a subgame (=occurrence of literal) l in which (1) he can legally move (i.e., the current labels between l and the root of the parse tree are all True), and (2) the current position in l contains as few moves as possible, subject to (1). In l, O uses z_l as his sequence of moves.

If l and l' are two occurrences of the same literal, and if the play x that we have just produced has infinite subsequences $(x)_l$, and $(x)_{l'}$ of moves in these two subgames, then $(x)_l \neq (x)_{l'}$, because O's moves in these two plays are $z_l \neq z_{l'}$.

On the other hand, it is possible that l and l' are occurrences of p and $\overline{p}$, respectively, and that $(x)_l = (x)_{l'}$ and these subsequences are infinite. Indeed, O's moves z_l in $(x)_l$ might match P's moves in $(x)_{l'}$, (since l' is the negation of l, the players have reversed roles) and vice versa; for example, P's strategy σ might involve copying O's moves between l and l'. If this occurs, we say that l and l' are matched. Notice that, by the preceding paragraph, any l is matched with at most one l'.

Consider the formula C^* obtained from C by changing all occurrences of variables to distinct variables except that matched occurrences of literals p and $\overline{p}$ retain the same variable. Clearly, C is an instance of C^* and C^* is binary. But we assumed that C is not an instance of a binary tautology. So C^* is not a tautology. Fix a truth assignment making C^* false.

We regard this truth assignment as assigning truth values as labels to the leaves of the parse tree of C. This labeling, which we extend in the usual way to the whole parse tree and call the **preferred** labeling, need not be a real truth assignment for C, since different occurrences of the same variable in C became different variables in C^*, and may thus have received different truth values. However, if l and l' are matched literals in C, then one remained the negation of the other in C^*, so they received opposite truth values. Summarizing the properties of the preferred labeling that we will need later, we have

(1) matched literals have opposite truth values, and

(2) the root is labeled False.

For each literal occurrence l such that $(x)_l$ is infinite in the play x described above, we note that $(x)_l$ is a member of ${}^{\omega}2$ that was not decided at any previous stage of the definition of the G_p's. Indeed, z_l, the subsequence of Os moves in $(x)_l$, was chosen to differ from the subsequence of either players moves in any previously decided sequence. We can therefore freely define $G_p((x)_l)$ for p's. We use this freedom to try to make the labeling of the parse tree associated to x match the preferred labeling. Thus, if l is an occurrence of p (respectively, $\overline{p}$) and is labeled True (respectively, False) in the preferred labeling, then we define $G_p((x)_l) = 0$. On the other hand, if l is an occurrence of p (respectively, $\overline{p}$) and is labeled False (respectively, True) in the preferred labeling, then we define $G_p((x)_l) = 1$. (Other decisions, about $G_p((x)_l)$ when l is neither p nor $\overline{p}$, can be made arbitrarily.) The decisions just described do not conflict with one another. Indeed, the only possibility for conflict would be if $(x)_l = (x)_{l'}$ for two distinct occurrences of literals, l and l'. But then l and l' are matched and therefore get opposite truth values in the preferred labeling. Since one of l and l' is an occurrence of some p and the other of $\overline{p}$, opposite labels ensure that $G_p((x)_l) = G_p((x)_{l'})$.

This completes the description of stage σ of the construction of the G_p's. It remains to verify that this stage ensures that σ is not a winning strategy for P in the game C. For this purpose, we consider the play x used for stage σ. It was defined as a play where P uses strategy σ, so we need only check that O wins this play.

If the sequence (x), of moves in (the game corresponding to) l were infinite for every occurrence l of a literal in C, then our task would be trivial. The decisions made at stage σ would ensure that the labeling of the parse tree of C associated to the play x agrees, at all leaves and therefore at all other nodes as well, with the preferred labeling. Since the latter makes the root false, it follows (by the truth-table descriptions of $\otimes$ and $\overline{\otimes}$ games) that O wins the play x. Unfortunately, there is no reason to expect each $(x)_l$ to be infinite, and a finite $(x)_l$ may give l a label (as always, True if O is to move, False if P is to move) different from the preferred label. So a subtler argument is needed. This argument is quite similar to one already used in the proof of Theorem 2.2.3, so we omit some details.

In the play of the game C, at all sufficiently late stages, we have, for each literal occurrence l,

(1) if $(x)_l$ is finite, then all moves that will ever be made in subgame l have already been made, and

(2) if $(x)_l$ is infinite, then the number of moves already made in subgame l exceeds the length of every finite $(x)_{l'}$.

At moves of O this late in the game, she does not move in any subgame l for which $(x)_l$ is finite (by (l)), but he would move in such a subgame if he legally could (by the second clause in the description of how O chooses her moves in x, and by (2)). So, when O is to move this late in the game, the path from each such l to the root must contain a label False. A move of O only decreases labels, so such a False is still present afterward, when P is to move next. So, at all sufficiently late stages

(3) if $(x)_l$ is finite, then the path from l to the root contains at least one label False.

If we temporarily fix an I such that $(x)_l$ is finite and if we consider, on the path from l to the root, the False nearest l, we see that its location is unaffected by moves of O and can move only toward the root at moves of P. So this False is always at the same location X from some stage on. At such late stages, O will never move in subgames l beyond X in the parse tree (i.e., occurrences of literals within the subformula X), and therefore P will move there only finitely often. Waiting until all these moves have been made, we see that, at all sufficiently late stages in the play, nothing happens beyond X, so the labeling of the subtree with root X remains unchanged. In particular, as X was chosen to have label False at all sufficiently late stages, it also has label False in the final labeling associated to the play x.

We have shown that every l for which $(x)_l$, is finite is within a subformula $X(I)$ (meaning a subformula X containing l) whose final label is False. We complete the proof by considering the following three labelings of the parse tree of C.

(a) The preferred labeling.

(b) The final labeling associated to the play x.

(c) The labeling that agrees with **(a)** and **(b)** at all l for which $(x)_l$ is infinite but assigns False to all l for which $(x)_l$ is finite.

Notice that (c) makes sense, because we already know that **(a)** and **(b)** agree at l when $(x)_l$ is infinite. We also know that (a) labels the root C with False; by monotonicity of $\otimes$ and $\overline{\otimes}$, (c) also labels the root with False. Now consider what happens if we change labeling (b) to **(c)**. The only changes at leaves of the parse tree are decreases (from True to False) at some l's for which $(x)_l$ is finite. The only changes at interior nodes are decreases (by monotonicity again) along the paths from such l's to the root. But every such path contains a node $X(I)$ that was already labeled False in **(b)** and that is therefore unaffected by the decreases in going from **(b)** to **(c)**. But if the change at l does not affect the label at $X(I)$, it cannot affect labels nearer the root. In particular, C has the same label in (b) as in **(c)**, and we already know that the latter is False. So C is False in the labeling associated to x, i.e., O wins the play x.

This shows the stage σ of our construction prevents σ from being a winning strategy for P in C. The whole construction therefore ensures that P has no winning strategy in (this interpretation of) C, and so C is not valid. □

2.2.4 The ω_α-game logic and $\mathbf{AD}_{\omega_\alpha}$

In this subsection, we will discuss the relationship between the Axiom of Determinacy for infinite cardinal κ, and the game logic on κ.

Theorem 2.2.6. *If $\mathbf{AD}_\kappa$ holds, then the game logic on κ becomes a redundant version of classical propositional logic, and there is a consistent and complete proof system for it.*

Proof. First recall that if $\mathbf{AD}_\kappa$ is true, there are only 2 values in the value universe of the game semantics on κ. That is $\mathcal{S}_\kappa = \{\mathbf{0}, \mathbf{1}\}$. Theorem 1.2.27 tells us that on nontrivial determined games, binary game operators and the dual operator also behave like logic operators. We will make this argument precise.

According to **(c)** and **(e)** of Theorem 1.2.27, $\otimes / \overline{\otimes}$ functions exactly the same way as $\wedge/\vee$. So there are only two different binary operators and we can identify $\otimes$ as $\wedge$ and $\overline{\otimes}$ with $\vee$. By Theorem 1.2.7, **(f)** of Theorem 2.1.5, and of Theorem 1.2.27, we know that $\overline{}$, $\wedge$ and $\vee$ function on degrees the same way as $\neg$, $\wedge$ and $\vee$ in classical propositional logic on truth values. Thus, any consistent and complete proof system for classical propositional logic can be easily converted to such a system for the game logic on κ. Just replace $\top$ by 1, $\bot$ by 0 and $\neg$ by $\overline{}$, and for each axiom or inference rule containing $\wedge$ or $\vee$, add a copy of it in which $\wedge$ is replaced by $\otimes$ and $\vee$ by $\overline{\otimes}$. □

Thus $\mathbf{AD}_\kappa$ is really bad in the sense that it renders the game logic on κ very simple. In the rest of this thesis, our study will be mostly concerned with game logics for infinite cardinals κ's in settings where $\mathbf{AD}_\kappa$'s are false.

To get an initial idea of what game logic on κ will become if $\mathbf{AD}_\kappa$ is false, let us first consider the game logic on ω in **ZFC**. We have seen in Theorem 1.2.30 **ZFC** proves **AD** ($\mathbf{AD}_\omega$) is false by showing there is $\mathbf{a} \in \mathcal{S}_\omega$ such that $\mathbf{0} < \mathbf{a} < \mathbf{1}$. Moreover, by Corollary 1.2.31, $\mathbf{a} \,\overline{\otimes}\, \mathbf{a} = \mathbf{1}$ and $\overline{\mathsf{R}}(\mathbf{a}) = \mathbf{1}$. But theorem 1.2.32 tells us there is $\mathbf{b} \in \mathcal{S}_\omega$ such that $\mathbf{0} < \mathbf{b} < \mathbf{1}$ and $\mathbf{b} \,\overline{\otimes}\, \mathbf{b} < \mathbf{1}$, and there is $\mathbf{c} \in \mathcal{S}_\omega$ such that $\mathbf{0} < \mathbf{c} < \mathbf{1}$ and $\overline{\mathsf{R}}(\mathbf{c}) < \mathbf{1}$. So the behaviors of $\otimes$ and $\overline{\otimes}$ are not uniform on non-determined degrees. As a result $G \,\overline{\otimes}\, G$ is not a valid formula. But $G \,\overline{\otimes}\, \overline{G}$ is valid by Theorem 2.1.2. In this aspect, $\overline{\otimes}$ function like $\vee$ in classical 2-valued logic, while $\vee$ in game logic does not.

Notice that $G \vee \overline{G}$ is not valid because of non-determined degrees and **(b)** of Theorem 1.2.27. Similarly, $\otimes$ funcions like $\wedge$ in classical 2-valued logic. We may think that this game semantics is weird and wonder what proof system could work for it.

Surprisingly, Blass [6] proved in **ZFC** that the sequent calculus for linear logic and affine logic which had been invented for other reasons is consistent with his game semantics. Moreover by proving two completeness theorems for two nicely defined families of formulas, he showed that the sequent calculus for affine logic can derive a considerably large fragment of valid formulas of game logic. Blass used $\mathcal{S}_{\mathrm{Ord}}$ rather than some $\mathcal{S}_\kappa$ as the value universe for his semantics. In the next section we will investigate Blass's work in [6] in our settings.

2.3 Game sequent calculus

In this section, we will reproduce Blass's work in [6] in our settings.

2.3.1 Preliminaries

Let κ be an infinite cardinal. A **sequent** in $\mathcal{L}$ is an expression $\vdash \Gamma$, where Γ is a finite list of formulas in $\mathcal{L}$. Formulas in Γ are not necessarily distinct. In particular, $\vdash A, A$ is different from $\vdash A$.

Sequents whose formulas are additive are called **additive** sequents. Those whose formulas are multiplicative are called **multiplicative** sequents.

We extend the function NF so that it is defined on sequents. Let $\Gamma = \phi_1, ..., \phi_n$, then $\mathrm{NF}(\vdash \Gamma) = \vdash \mathrm{NF}(\phi_1), ..., \mathrm{NF}(\phi_n)$.

A sequent is interpreted by applying $\overline{\otimes}$ to the interpretations of the formulas in it. Formally, $\bar{\mathcal{I}}(\vdash \phi_1, ..., \phi_n) = \bar{\mathcal{I}}(\phi_1 \overline{\otimes} ... \overline{\otimes} \phi_n) = \bar{\mathcal{I}}(\phi_1) \overline{\otimes} ... \overline{\otimes} \bar{\mathcal{I}}(\phi_n)$. It is easy to see that $\bar{\mathcal{I}}(\vdash \Gamma) = \bar{\mathcal{I}}(\mathrm{NF}(\vdash \Gamma))$ for any variable κ-interpretation for Γ.

A sequent $\vdash \Gamma$ is κ-valid if $\bar{\mathcal{I}}(\vdash \Gamma) = \mathbf{1}$ for all variable κ-interpretation $\mathcal{I}$ for Γ. This follows from Theorem 2.2.2 that $\vdash \Gamma$ is valid iff $\mathrm{NF}(\vdash \Gamma)$ is valid.

The axioms and inference rules of **game sequent calculus** are the following, in which A and B represent arbitrary formulas in $\mathcal{L}$ and Γ and Δ represent arbitrary lists of formulas in $\mathcal{L}$. (We defined Γ as an order-type function in chaptor 2. This causes no trouble because the meaning of Γ is always clear.) So precisely speaking, the following are axiom schemas and rule schemas.

Logical axioms: $\vdash A, \overline{A}$

Structure rules: (Exchange) $\dfrac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, B, A, \Delta}$

(Cut) $\dfrac{\vdash \Gamma, A \qquad \vdash \Delta, \overline{A}}{\vdash \Gamma, \Delta}$

(Weakening) $\dfrac{\vdash \Gamma}{\vdash \Gamma, A}$

Additive rules: $(\wedge)$ $\dfrac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}$

(1) $\vdash 1$

$(\vee)$ $\dfrac{\vdash \Gamma, A}{\vdash \Gamma, A \vee B} \qquad \dfrac{\vdash \Gamma, B}{\vdash \Gamma, A \vee B}$

Multiplicative rules: $(\otimes)$ $\dfrac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}$

$(\overline{\otimes})$ $\dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A \mathbin{\overline{\otimes}} B}$

Exponential rules: **(Dereliction** $\overline{\mathsf{R}}$**)** $\dfrac{\vdash \Gamma, A}{\vdash \Gamma, \overline{\mathsf{R}}(A)}$

(Contraction $\overline{\mathsf{R}}$**)** $\dfrac{\vdash \Gamma, \overline{\mathsf{R}}(A), \overline{\mathsf{R}}(A)}{\vdash \Gamma, \overline{\mathsf{R}}(A)}$

(R) $\dfrac{\vdash \overline{\mathsf{R}}(\Gamma), A}{\vdash \overline{\mathsf{R}}(\Gamma), R(A)}$

In the (R) rule, $\overline{\mathsf{R}}(\Gamma)$ means the result of applying $\overline{\mathsf{R}}$ to all members of Γ.

2.3.2 Consistency of game sequent calculus

In this subsection, we prove that for the game sequent calculus for a infinite cardinal κ is consistent. That is, all its axioms are κ-valid, and all its inference rules preserve κ-validity. The following theorem was proved by Blass [6].

Theorem 2.3.1 (ZF). *Let κ be an infinite cardinal. All sequents in $\mathcal{L}$ provable in game sequent calculus are κ-valid.*

Proof. Before proving the theorem, we need some notational simplification.

1. We do not distinguish games and degrees. By a game A, we always mean its degree $[A]$.

2. We need to show that the κ-interpretation for each instance of each schema under each variable κ-interpretation gives value **1**. Note that there two levels of abstraction in each schema: A and B stand for arbitrary formulas, Γ and Δ stand for arbitrary sequences $C_1, ..., C_n$ and $D_1, ..., D_m$ where each C_i and D_i stand for arbitrary formulas. For each schema, we will 1. fix an arbitrary instant of Γ and Δ, then 2. fix arbitrary formula ϕ_A, ϕ_B, ϕ_{C_i} and ϕ_{D_i} for A, B , C_i and D_i, and then 3. fix an arbitrary variable κ-interpretation $\mathcal{I}$ for all these fixed formulas. For easier reading, we also use A, B, C_i and D_i to mean $\bar{\mathcal{I}}(\phi_A)$, $\bar{\mathcal{I}}(\phi_B)$, $\bar{\mathcal{I}}(\phi_{C_i})$ and $\bar{\mathcal{I}}(\phi_{D_i})$, and Γ and Δ to mean $\bar{\mathcal{I}}(\phi_{C_1} \mathbin{\overline{\otimes}} ... \mathbin{\overline{\otimes}} \phi_{C_n})$ and $\bar{\mathcal{I}}(\phi_{D_1} \mathbin{\overline{\otimes}} ... \mathbin{\overline{\otimes}} \phi_{D_m})$.

Recall that sequents are interpreted by combining the interpretations of the formulas with $\overline{\otimes}$; this makes soundness of the $\overline{\otimes}$ rule trivial. Recall also that, when games are combined by $\overline{\otimes}$, P has the option of switching from one component game to another at any of his moves, and P wins the compound game if he wins at least one component.

Soundness of the exchange rule is trivial, as $\overline{\otimes}$ is a commutative operation on degrees by **(b)** of Theorem 2.1.5.

Weakening rule and **1** rule are sound by **(c)** and **(k)** of Theorem 2.1.5.

Logical axioms are valid by Theorem 2.1.2.

Cut. Suppose P have winning strategies in $\Gamma \,\overline{\otimes}\, A$ and $\Delta \,\overline{\otimes}\, \overline{A}$. By definition we have $\overline{A} \leq \Gamma$ and $\overline{\Delta} \leq \overline{A}$. By transitivity of $\leq$, we get $\overline{\Delta} \leq \Gamma$, which means P has a winning strategy in $\Delta \,\overline{\otimes}\, \Gamma$.

$(\wedge)$. Suppose P has winning strategies σ for $\Gamma \,\overline{\otimes}\, A$ and τ for $\Gamma \,\overline{\otimes}\, B$. Here are instructions whereby P can win $\Gamma \,\overline{\otimes}\, (A \wedge B)$. As O moves first in $A \wedge B$ (to pick a component), by the time it is P's move in $\Gamma \,\overline{\otimes}\, (A \wedge B)$, 0 will have chosen A or B, so the game being played is effectively $\Gamma \,\overline{\otimes}\, A$ or $\Gamma \,\overline{\otimes}\, B$, and P uses σ or τ accordingly.

$(\vee)$. Suppose P has a winning strategy σ in $\Gamma \,\overline{\otimes}\, A$. Then P wins $\Gamma \,\overline{\otimes}\, (A \vee B)$ by making his first move in the component $A \vee B$, choosing A there, and then following σ.

$(\otimes)$. Suppose P has winning strategies σ in $\Gamma \,\overline{\otimes}\, A$ and τ in $\Delta \,\overline{\otimes}\, B$. The following is a strategy whereby P can win $\Gamma \,\overline{\otimes}\, (A \otimes B)$. Make sure that all your (P's) moves in Γ and A (respectively, in Δ and B) are played in accordance with σ (respectively, τ). Whenever it is P's move in $\Gamma \,\overline{\otimes}\, \Delta \,\overline{\otimes}\, (A \otimes B)$, it is (by definitions of $\overline{\otimes}$ and $\otimes$) P's move in Γ, in Δ, and in one of A and B. If it is P's move in A, then σ provides a move in Γ or in A; otherwise, τ provides a move in Δ or in B. In either case, P can move in accordance with σ and τ, so we have a well-defined strategy. To see that it is a winning strategy, consider any play using it. If P wins either the Γ or the Δ component, then he wins $\Gamma \,\overline{\otimes}\, \Delta \,\overline{\otimes}\, (A \otimes B)$. If not, then he wins A (because σ must win at least one of Γ and A) and B (similarly) and therefore $A \otimes B$ and therefore $\Gamma \,\overline{\otimes}\, \Delta \,\overline{\otimes}\, (A \otimes B)$.

Dereliction $\overline{\mathsf{R}}$ is sound by **(b)** and **(m)** of Theorem 2.1.5.

Contraction $\overline{\mathsf{R}}$. Suppose P has a winning strategy σ in $\Gamma \,\overline{\otimes}\, \overline{\mathsf{R}}(A) \,\overline{\otimes}\, \overline{\mathsf{R}}(A)$. To win $\Gamma \,\overline{\otimes}\, \overline{\mathsf{R}}(A)$, he should pretend that he is playing $\Gamma \,\overline{\otimes}\, \overline{\mathsf{R}}(A) \,\overline{\otimes}\, \overline{\mathsf{R}}(A)$, the even (respectively, odd) numbered constituents of the one actual $\overline{\mathsf{R}}(A)$ being identified with all the constituents of the first (respectively, second) imaginary $\overline{\mathsf{R}}(A)$. Using σ in the imaginary game gives, via this identification, a win for P in the real game. (In the imaginary game, the consistency rule constrains O only for pairs of A's within the same $\overline{\mathsf{R}}(A)$; in the real game, all pairs of A's are constrained. So the real game is actually a bit easier for P. In other words, σ contains information that P will never need for the real game.)

(R). Suppose P has a winning strategy σ in $\overline{\mathsf{R}}(\Gamma) \,\overline{\otimes}\, A$, i.e., $\overline{\mathsf{R}}(C_1) \,\overline{\otimes}\, ... \,\overline{\otimes}\, \overline{\mathsf{R}}(C_r) \,\overline{\otimes}\, A$. To win $\overline{\mathsf{R}}(C_1) \,\overline{\otimes}\, ... \,\overline{\otimes}\, \overline{\mathsf{R}}(C_r) \,\overline{\otimes}\, \mathsf{R}(A)$, he should proceed as follows. Within each $\overline{\mathsf{R}}(C_i)$ there are countably infinitely many copies of C_i, indexed (according to the definition of R) by ω. Use a pairing function to re-index them by $\omega \times \omega$ (for example Γ^{-1} in chapter 2); the copies indexed by (k, I) for a fixed k and varying l will be called the kth block of copies of C_i. The idea is that the kth blocks of $C_1, ..., C_r$ and the kth copy of A in $\mathsf{R}(A)$ will be treated as a copy of $\overline{\mathsf{R}}(C_1) \mathbin{\overline{\otimes}} ... \mathbin{\overline{\otimes}} \overline{\mathsf{R}}(C_r) \mathbin{\overline{\otimes}} A$, and σ will be applied to it. More precisely, when P is to move in $\overline{\mathsf{R}}(C_1) \mathbin{\overline{\otimes}} ... \mathbin{\overline{\otimes}} \overline{\mathsf{R}}(C_r) \mathbin{\overline{\otimes}} A$, it is his move in all copies of all the C_i's and in some copy, say the kth, of A. Then P should make the move prescribed by σ for the current position in this kth copy of A and the kth blocks of all the C_i's; this makes sense, as this is a position with P to move in $\overline{\mathsf{R}}(C_1) \mathbin{\overline{\otimes}} ... \mathbin{\overline{\otimes}} \overline{\mathsf{R}}(C_r) \mathbin{\overline{\otimes}} A$. To see that this strategy is a winning one, consider any play where P uses it. If P wins any component of any $\overline{\mathsf{R}}(C_i)$, then he wins $\overline{\mathsf{R}}(C_1) \mathbin{\overline{\otimes}} ... \mathbin{\overline{\otimes}} \overline{\mathsf{R}}(C_r) \mathbin{\overline{\otimes}} A$, as desired. If not, then, for each k, as P has not won any copy of any C_i in the kth block, he must have won the kth copy of A, because σ is a winning strategy. But then P has won $\overline{\mathsf{R}}(C_1) \mathbin{\overline{\otimes}} ... \mathbin{\overline{\otimes}} \overline{\mathsf{R}}(C_r) \mathbin{\overline{\otimes}} A$, as desired. □

2.3.3 Additive completeness of game sequent calculus in ZFC

The following theorem was proved by Blass [6].

Theorem 2.3.2 (AC). *Let κ be an infinite cardinal. An additive sequent $\vdash \Gamma$ is κ-valid if and only if* $\mathrm{NF}(\vdash \Gamma)$ *is provable in game sequent calculus.*

Proof. Suppose $\mathrm{NF}(\vdash \Gamma)$ is provable in game sequent calculus. By Theorem 2.3.1, $\mathrm{NF}(\vdash \Gamma)$ is valid and so is $\vdash \Gamma$. So 'if' has been proved. We will prove 'only if' in the rest of the proof.

Suppose $\mathrm{NF}(\vdash \Gamma)$ is unprovable in game sequent calculus. We want to show $\mathrm{NF}(\vdash \Gamma)$ is not κ-valid. For easier reading, we use Γ to mean $\mathrm{NF}(\vdash \Gamma)$ without causing any trouble.

First let G_0/G_1 be the only strict game on 1 that is a win for 0/1. We construct (the degrees of) reversely strict games on 2 to interpret the game variables in Γ so that P has no winning strategy in the interpretation of Γ, or just game Γ for easier reading. The game G_p associated to a game variable p will be reversely strict and on 2 so that G_p has been defined on finite sequences. We have yet to specify the G_p on infinite sequences, but G_p being reversely strict is enough to determine the set of positions in the game Γ and the set of strategies for P in Γ. As there are only countably many positions, the number of strategies is the cardinality $|^{\omega}2|$, or the cardinality $\mathfrak{c}$ of the continuum. By **AC**, fix a well-ordering of the set of all strategies for P, having order-type $2^{\aleph_0}$: thus, each strategy σ has fewer than $2^{\aleph_0}$ predecessors in this well-ordering.

We will define G_p on infinite sequences by transfinite recursion over this well-ordering. At the recursion step associated to a strategy σ, we will decide, for finitely many $x \in {}^{\omega}2$, the value of $G_p(x)$. We say that these x's are decided at stage σ. These decisions will be made in a way that ensures that σ is not a winning strategy for P in C. As every possible strategy for P in C occurs in our well-ordering, the whole construction will ensure that P has no winning strategy for C. The rest of the proof consists of showing how to carry out one step in the induction, say the step associated to σ.

There have been fewer than $2^{\aleph_0}$ previous steps, each deciding only finitely many $x \in {}^{\omega}2$. We split each of these xs into the two subsequences of moves attributable to the two players, i.e., $x|_0$ and $x|_1$. Thus, for each decided x, one subsequence consists of the even-numbered moves in x, the other of the odd-numbered moves, because we are dealing with strict games. There are fewer than $2^{\aleph_0}$ subsequences so obtained – two from each of fewer than $2^{\aleph_0}$ decided x's – so we can fix a $z \in {}^{\omega}2$ that is not such a subsequence.

Let $\Gamma = C_1, C_2, ..., C_n$ where C_i are additive formulas. Consider an arbitrary play of the game Γ. Note the game Γ is just the game $C_1 \overline{\otimes} ... \overline{\otimes} C_n$. The moves in any component game C_i come in two phases. In phase 1, the players are choosing conjuncts or disjuncts in sub-formulas for game C_i. For example, if C_i is $(p \wedge \overline{q}) \vee r$, where p, q, r are variables, then phase 1 contains P's opening move, choosing $p \wedge \overline{q}$ or r, and, if he chooses the former, then phase 1 also contains O's reply, choosing p or $\overline{q}$. Each phase 1 move replaces the ith component of Γ by one of its conjuncts or disjuncts, and phase 1 continues in the kth component until it is reduced to a literal, i.e., to a variable or the negation of one or 0 or 1. Then comes phase 2, in which the players play (the game associated to) that literal. In any component, the phase 1 moves precede the phase 2 moves, but it is possible for phase 2 to begin in one component before phase 1 is finished in another component. It is also possible for a play of Γ to have only finitely many moves in some component, and then phase 1 may not be finished there.

At any stage of the play, we write Γ' for the current list of component games. Initially, Γ' is Γ, but every phase 1 move replaces some formula in Γ' with one of its conjuncts or disjuncts.

The proceeding discussion concerned an arbitrary play of Γ. We now focus our attention on particular plays of Γ in which P follows a strategy σ while O

(1) plays phase 1 so that Γ' is never provable;

(2) plays phase 2 moves in G_0 or G_1 in the only legal way; and

(3) plays phase 2 moves in literals of the form p or $\overline{p}$ by making the fixed sequence z of moves in each such literal.

Recall also that z was chosen to be distinct from the subsequence of either players moves in every decided x. Thus, (3) ensures that the plays x in literals of the forms p and $\overline{p}$ are not yet decided.

We still need to show O can play as required by (1). Initially, $\vdash \Gamma'$ is $\vdash \Gamma$, which is unprovable, by assumption. If $\vdash \Gamma'$ is unprovable at some point during the play, and if P then makes a phase 1 move, then $\vdash \Gamma'$ will still be unprovable after this move. Indeed a phase 1 move of P replaces a component of the form $A \vee B$ with A or with B, so, up to order of components, Γ' before the move was $\Delta, A \vee B$ and Γ' after the move is either $\vdash \Delta, A$ or $\vdash \Delta, B$. But if either $\vdash \Delta, A$ or $\vdash \Delta, B$, then, by rule $(\vee)$, we have $\vdash \Delta, A \vee B$, a contradiction. Thus, phase 1 moves of P cannot make Γ' provable. A phase 1 move of O changes Γ' form $\vdash \Delta, A \wedge B$, to $\vdash \Delta, A$ or $\vdash \Delta, B$. By rule $(\wedge)$, if $\Delta, A \wedge B$ is unprovable, then so is at least one of $\vdash \Delta, A$ or $\vdash \Delta, B$. So O can make his phase 1 moves in accordance with instruction (1).

Consider a particular play of Γ where P follows strategy σ while O obeys instructions (1)-(3) above. By the preceding discussion of (1), $\vdash \Gamma'$ never becomes provable. In particular, by rule (1), the literal 1 never occurs in $\vdash \Gamma'$. Also, by the logical axioms and weakening, the literals in Γ' never include both a variable p and its negation $\overline{p}$.

For each occurrence of a literal p (respectively, $\overline{p}$) that eventually appears in Γ', if infinitely many moves are made in that component of Γ', let $x \in {}^{\omega}2$ be the sequence of phase 2 moves in that component, and define $G_p(x) = 1$. As we noticed earlier, instruction (3) ensures that the xs involved here are different from the previously decided xs, so the decisions just made (for stage σ) do not conflict with earlier decisions. Nor do they conflict with each other, even if the same x arises in several components, for p and $\overline{p}$ cannot both occur in Γ'. If (as is likely) $G_p(x)$ has just been defined for some G_p but remains undecided for other G_q's, then make these other decisions arbitrarily. For example, define $G_q(x) = 1$.

This completes stage σ in the construction of each G_p. Notice that the particular play of Γ used in the construction, with P following σ while O follows (1)-(3), is won by O. Indeed, O won components of the form p (respectively, $\overline{p}$) where infinitely many moves were made, because we defined $G_p(x) = 1$ for the corresponding xs; O wins components of the form 0 automatically (if infinitely many moves are made there); and there are no components of the form 1. So every component where infinitely many moves are made is won by O. Thus, σ is not a winning strategy for P in Γ.

After the inductive construction of the G_p's in complete (with arbitrary conventions for any xs not decided at any stage), we have a game (degree) interpretation for the variables in Γ such that no strategy for P wins Γ. Thus Γ is not κ-valid. □

It is not hard to see that the diagonalization technique used in the above proof is essentially the same as the one used in the proof of Theorem 2.1.12.

The game sequent calculus is not complete for κ-valid multiplicative formulas. The formula $[(\overline{A}\ \overline{\otimes}\ \overline{A}) \otimes (\overline{A}\ \overline{\otimes}\ \overline{A})]\ \overline{\otimes}\ [(A\ \overline{\otimes}\ A) \otimes (A\ \overline{\otimes}\ A)]$ is an instance of a binary tautology introduced in subsection 2.2.3, but unprovable in game sequent calculus (for details see [6, p. 210]). It is still an open question to find a proof system that is complete for κ-valid multiplicative formulas.

Chapter 3

Degrees of non-determinacy and game logics in $\mathbf{ZF} + \mathbf{AD}$

3.1 Preliminaries

3.1.1 Introduction to AD

The main purpose of this chapter is to study degrees of non-determinacy and game logics without using the the Axiom of Choice, and reexamine Blass's results reproduced in Chapter 2 which use the Axiom of Choice in settings without it. Unfortunately, **ZF** does not determine much regarding the game-theoretic structure of the universe. In fact, there are models of $\mathbf{ZF} + \mathbf{non\text{-}AC}$ in which there are non-determined Π^1_1 games on ω.[1] Curiously, while **ZF** proves that there are non-determined games (on ω_1, or, as we will see in Section 3.2, on every uncountable ordinal), it is not possible to display a concrete non-determined game in **ZF** (see [14, p.133–134]).

As a consequence, our analysis of game semantics without the axiom of choice cannot be done in **ZF** alone, but will need some concrete background theory as the setting. We choose $\mathbf{ZF} + \mathbf{AD}$ as our background theory from now on. There are good reasons for this choice.

Even among researchers who believe that the Axiom of Choice is the right and only sensible setting to do set theory in, there is one particular statement violating the Axiom of Choice that they cannot avoid: the Axiom of Determinacy. Introduced by Mycielski and Steinhaus in [22], the Axiom of Determinacy (**AD**) states that every strict game on ω is determined. We have seen in Chapter 1 that this is equivalent to the statement "every game on ω is determined".

Of course, we have already seen that **ZFC** proves the negation of **AD** (Theorem 1.2.30). In the last decades, it turned out, however, that even when you are studying models of **ZFC** set theory, a thorough understanding of models of $\mathbf{ZF} + \mathbf{AD}$ is

[1] This follows immediately for consistency strength reasons: Harrington [13] has proved that the determinacy of Π^1_1 games implied the existence of $0^\#$, so any $\mathbf{ZF} + \mathbf{non\text{-}AC}$ model constructed via forcing and the method of inner models from L will be such a model since $0^\#$ cannot exist in forcing extensions of L; see [13].

necessary. In this thesis, we cannot go into a lot of detail and refer the reader to the excellent exposition in [19].

In the following, we will provide all of the background we need without proofs.

3.1.2 Some facts in $\mathbf{ZF} + \mathbf{AD}$

Cantor formulated the concept of perfect set in his topological investigations of $\mathbb{R}$ in [8] and [7]. Let $s \in \mathrm{Fin}(\omega)$. Then define $O(s) := \{a \in {}^{\omega}\omega : s \prec a\}$. For $x \in {}^{\omega}\omega$ and $A \subset {}^{\omega}\omega$, x is a **limit point** of A iff for any $O(s)$ with $x \in O(s)$, $(A \cap O(s)) - \{x\} \neq \emptyset$. x is an isolated point of A iff $x \in A$ and is not a limit point of A, i.e., there is an $O(s)$ such that $A \cap O(s) = \{x\}$. For $A \subset {}^{\omega}\omega$, A is **perfect** iff it is nonempty, closed, and has no isolated points; A has the **perfect set property** iff A is countable or else has a perfect subset.

Theorem 3.1.1. *If $\omega_1 \leq 2^{\omega}$, then there is a set of reals without the perfect set property.*

Proof. See [19, p. 134]. □

Theorem 3.1.2. *Assume* **AD**.

(a) *Every set of reals has the perfect set property.*

(b) *There is no uncountable well-orderable set of reals.*

(c) $\mathbf{AC}_{\omega}({}^{\omega}\omega)$.

(d) ω_1 *is measurable.*

(e) ω_1 *is weakly compact, i.e.,* $\omega_1 \to (\omega_1)^2_2$.

Proof. For **(a)**, see [19, p. 377]. For **(b)**, suppose there is an uncountable well-orderable set of reals. Fix a well-ordering of this set. Then it must have order-type $\geq \omega_1$ since the set is uncountable. Take the initial segment of order-type ω_1 of this well-ordering and we get an injection from ω_1 to the reals. By Theorem 3.1.1, there is a set of reals without the perfect set property. But this contradicts **(a)**. For **(c)**, see [19, p. 378]. For **(d)**, see [19, p. 384–386]. **(e)** follows directly from **(d)**. □

In the world without **AC**, the cardinality of $\mathbb{R}$ is not an ordinal. Instead, we can define

$$\Theta = \sup(\{\alpha : \text{ there is a surjection: } {}^{\omega}\omega \to \alpha\})$$

and consider it the representative of the real numbers in the ordinals. Clearly, if **AC** holds, then $\Theta = (2^{\aleph_0})^+$. Without the Axiom of Choice, it is not immediately clear what Θ is. Of course Θ must be $> \omega_1$ by Theorem 1.1.4. In our setting, i.e., $\mathbf{ZF} + \mathbf{AD}$, Θ is a relatively big cardinal. In fact, Solovay proved that Θ is an $\aleph$-fixed point [19, Exercise 28.17].

Remember that without the Axiom of Choice, successor cardinals can be singular. So, it is not clear how many of the cardinals below Θ are regular. In the **AD**-situation, quite a lot is known about this. Let us start with $\aleph_1$:

Theorem 3.1.3. *Assume* **AD**. *Then* ω_1 *is regular.*

Proof. This follows from **(d)** of Theorem 3.1.2. But we give a more direct proof here.

Suppose ω_1 is singular and $lim_{n\in\omega}\alpha_n = \omega_1$. Without loss of generality, we assume $\omega < \alpha_n$ for each n.

Since $|\omega \times \omega| = \omega$, we have

$$|P(\omega \times \omega)| = |P(\omega)| = |{}^{\omega}\omega|. \tag{3.1}$$

By Theorem 1.1.4, $\pi_1 : P(\omega \times \omega) \to \omega_1$ is onto.

So $\pi_1^{-1}(\alpha) \neq \emptyset$ for any $\alpha \in \omega_1$. By Lemma 3.1.2, Lemma 1.1.7 and fact (3.1), we fix for each α_n a R_{α_n} such that $\pi_1(R_{\alpha_n}) = \alpha_n$. Once R_{α_n} is fixed, there is a unique $g_{\alpha_n} : \bigcup\bigcup R_{\alpha_n} \to \alpha_n$ such that g_{α_n} is 1-to-1 and onto and $(k, l) \in R_{\alpha_n}$ if and only if $g_{\alpha_n}(k) \in g_{\alpha_n}(l) \in \alpha_n$. This is easy to check.

Consider the function $h : \omega \times \omega \to \omega_1$ defined by

$$h(m, n) = \pi_1(\{(k, l) : k, l \in \omega \text{ and } (k, l), (l, m) \in R_{\alpha_n}\})$$

which maps each $(m, n) \in \omega \times \omega$ to the order type the initial segment of R_{α_n} below m. It is easy to see h is well-defined. Remember that any initial segment of any well-ordering is a well-ordering.

We show that $\mathrm{ran}(h) = \omega_1$. $\mathrm{ran}(h) \subset \omega_1$ is given by definition. We only need to check $\omega_1 \subset \mathrm{ran}(h)$. Take any $\alpha \in \omega_1$. Then there is a α_j for some $j \in \omega$ such that $\alpha \in \alpha_j$ by our assumption that $lim_{n\in\omega}\alpha_n = \omega_1$. Then $\{(k, l) : k, l \in \omega \text{ and } (k, l), (l, g_{\alpha_j}^{-1}(\alpha)) \in R_{\alpha_j}\}$ is isomorphic to α via g_{α_j} because $(m, n) \in R_{\alpha_j}$ if and only if $g_{\alpha_j}(m) \in g_{\alpha_j}(n) \in \alpha_j$, and hence

$$h(g_{\alpha_j}^{-1}(\alpha), j) = \alpha. \tag{3.2}$$

Thus we have shown $\mathrm{ran}(h) = \omega_1$. By Lemma 1.1.1 and the fact that $\omega \times \omega$ is countable, we get that ω_1 is countable. Contradiction. □

Steve Jackson gave a beautiful analysis of the regular cardinals below $\aleph_{\epsilon_0}$ in his PhD thesis [17]. We will not go into detail here, as for our present purposes, it only matters that there are uncountably many regular cardinals below Θ.

Theorem 3.1.4. *Assume* **AD**. *Then there are uncountably many regular cardinals below* Θ. *In particular,* ω_2 *is a regular cardinal.*

Proof. [17]; [19, p. 388]. □

Among regular cardinals below Θ, some are measurable. In particular, ω_1 and ω_2 are measurable [19, Theorem 28.2, Theorem 28.6].

3.2 Structure of $\mathcal{S}_{\omega_\alpha}$ in ZF + AD

3.2.1 Real-coding games

In this section we will see the interesting fact that our determinacy assumptions for games on ω imply certain games on higher cardinals are non-determined. Those non-determined games have similar style and we call them real-coding games. This type of games is the only non-determined one we know so far, and we will study the degree structures and game logics on uncountable cardinals through these games.

If $S \subset \kappa < \Theta$, π be a surjection from ${}^\omega\omega$ onto κ, we consider π as a **coding function** coding elements of κ by reals, amd define the set of π-codes for β to be

$$C^\pi_\beta = \{a \in \omega^\omega : \pi(a) = \beta\}.$$

Each $a \in C^\pi_\beta$ is called a π-code of β. For $S \subset \kappa$, we define the real-coding game for S and π as follows.

Definition 3.2.1. *Let $S \subset \kappa < \Theta$, $\pi : {}^\omega\omega \to \kappa$ be onto.*

The game G^π_S is the strict game such that

$$G^\pi_S(a) = \begin{cases} 0 & \text{if } a(0) \notin S \text{ or } \pi(a|_1) = a(0) \\ 1 & \text{otherwise} \end{cases}$$

for each $a \in {}^\omega\kappa$.

It is easy to see that the only move of O that matters is her first one and her first move α should be an element of S if she does not want to lose immediately. It is also easy to see P has to play some $b \in {}^\omega\omega$ if he wants to win at all. When the game is finished, P wins if and only $\alpha \notin S$ or $b \in C^\pi_\alpha$.

We give some interesting facts about real-coding games in the following.

Let $\omega < \kappa < \Theta$ and $\pi : {}^\omega\omega \to \kappa$ be onto. We define a function

$$\mathrm{FC}_\pi : \kappa \to P(\mathrm{Fin}(\omega))$$

that will give finite initial segments of π codes of β for each $\beta < \kappa$ as

$$\mathrm{FC}_\pi(\beta) = \{a{\restriction}n : n < \omega \ \wedge \ a \in C^\pi_\beta\}$$

for each $\beta < \kappa$.

It is easy to see $\mathrm{FC}_\pi(\beta)$ is the set of finite initial segments of π-codes of β, or finite codes of β in short.

If $R \subset \mathrm{Fin}(\omega)$ is such that for κ many α's, we have $\mathrm{FC}_\pi(\alpha) = R$, we say that R is FC_π-maximal. The following lemma states that FC_π-maximal sets exist.

Lemma 3.2.2. *There is $S \subset \kappa$ such that $|S| = \kappa$ and*

$$(\forall \alpha, \beta \in S)\ \mathrm{FC}_\pi(\alpha) = \mathrm{FC}_\pi(\beta).$$

Proof. By Theorem 1.1.5, $P(\mathrm{Fin}(\omega))$ has cardinality continuum. We already know that there are no uncountable well-orderable subsets of ${}^\omega\omega$, so the range of FC_π must be countable. By regularity of κ, there is a some $R \in \mathrm{ran}(\mathrm{FC}_\pi)$ such that $\{\beta : \mathrm{FC}_\pi(\beta) = R\}$ has cardinality κ. □

The following lemma says each FC_π-maximal set has size of the continuum.

Lemma 3.2.3. *If R is FC_π-maximal, then R has size continuum.*

Proof. Let $S \subset \kappa$ has cardinality κ and such that $\mathrm{FC}_\pi[S] = \{R\}$. Suppose R does not have size continuum. By Theorem 3.1.2, R is countable. $P = \{a : (\forall n < \omega)\ a{\restriction}n \in R\}$ is also countable. But $\pi[P] \supset S$, which has cardinality κ. Contradiction. □

The next lemma shows that for each $\beta \in S$, there are as many π codes of β as there are reals.

Lemma 3.2.4. *Let $S \subset \kappa$ has cardinality κ and such that $\mathrm{FC}_\pi[S] = \{R\}$. For each $\beta \in S$, C^π_β has cardinality continuum.*

Proof. Consider the function

$$(a, n) \mapsto a{\restriction}n.$$

This function is a surjection from $C^\pi_\beta \times \omega$ to R. If C^π_β is countable, $|R| = \omega$ by **(b)** of Theorem 1.1.3, contradicting our assumption. So C^π_β must have cardinality continuum by **(a)** of Theorem 3.1.2. □

The reason that we are interested in these real-coding games is that they are non-determined. Let o be the infinite sequence of 0's, i.e., $o : \omega \to 1$. We will need this o several times in the rest of this thesis.

Theorem 3.2.5. *Assume* **AD**. *If $\omega < |\kappa| < \Theta$, π is a surjection from ${}^\omega\omega$ onto κ, $S \subset \kappa$ and $|S| > \omega$, then G^π_S is non-determined.*

Proof. Clearly O does not have a winning strategy, because for each $\alpha \in S$, there is some $a \in {}^\omega\omega$ such that $\pi(a) = \alpha$.

Suppose P has a winning strategy σ.

Define the function

$$f_\sigma(\alpha) = (\alpha * o) \star \sigma|_1.$$

Note that $(\alpha * o) \star \sigma|_1$ is the unique t such that P follows σ in the play $(\alpha * o) \star_0 t$. This is a choice function for the family $\{C^\pi_\beta : \beta \in S\}$ because σ is winning strategy for P. Since S is uncountable, this contradicts Corollary 3.1.2. □

Corollary 3.2.6 (Mycielski [21])**. AD** *implies there is a non-determined game on ω_1.*

Proof. Fix a 1-to-1 and onto function $f : {}^\omega\omega \to P(\omega \times \omega)$. Then $\pi_0 \circ f$ is a surjection from ${}^\omega\omega$ onto ω_1. □

Corollary 3.2.7. *Assume* **AD**. *If $\omega < |\lambda| < \Theta$ and $\omega < |\kappa| < \Theta$, π is a surjection from ${}^\omega\omega$ onto λ and ρ is a surjection from ${}^\omega\omega$ onto κ, $S \subset \lambda$, $|S| > \omega$, $S' \subset \kappa$ and $|S'| > \omega$, then player O does not have a winning strategy in the game $G^\pi_S \mathbin{\overline{\otimes}} \overline{G^\rho_{S'}}$.*

Proof. Consider the plays in which player P never makes switches. □

Another corollary is that **ZF** proves that all $\mathbf{AD}_{\omega_\alpha}$'s are false for $\alpha > 0$, since $\mathbf{AD} \to \neg\mathbf{AD}_{\omega_\alpha}$ and $\neg\mathbf{AD} \to \neg\mathbf{AD}_{\omega_\alpha}$.

3.2.2 Partial Incomparability

Corollary 3.2.7 tells us player O has no winning strategy in the game $G^{\pi}_{S} \overline{\otimes} \overline{G^{\rho}_{S'}}$. What about player P? The following theorem says for some game of the form $G^{\pi}_{S} \overline{\otimes} \overline{G^{\rho}_{S'}}$, P does not have a winning strategy.

Remember that if $\kappa > \lambda$, $G^{\rho}_{\kappa} \overline{\otimes} \overline{G^{\pi}_{\lambda}}$ is $G^{\rho}_{\kappa} \overline{\otimes} \operatorname{Ext}(\overline{G^{\pi}_{\lambda}}, \kappa)$.

Theorem 3.2.8. *Assume* **AD**. *Given a cardinal number $\lambda > \omega$ and a cardinal number $\kappa > \lambda$, a surjection $\pi : {}^{\omega}\omega \to \lambda$ and a surjection $\rho : {}^{\omega}\omega \to \kappa$, the game $G^{\rho}_{S} \overline{\otimes} \overline{G^{\pi}_{S'}}$ is non-determined if $|S|$ is regular and $|S| > |S'|$. In particular, $G^{\pi}_{S'} \not\leq G^{\rho}_{S}$.*

Proof. Without loss of generality, we prove the case in which $S = \kappa$ and $S' = \lambda$.

By Corollary 3.2.7 O does not have a winning strategy. Now let us prove P does not have a winning strategy. Suppose, towards a contradiction, that σ is a winning strategy for P.

Notice that by definition of $\overline{\otimes}$, the first move in the game $G^{\rho}_{\kappa} \overline{\otimes} \overline{G^{\pi}_{\lambda}}$ belongs to O and she has to play an ordinal in $\overline{G^{\pi}_{\lambda}}$. Now consider the set of all finished plays of $G^{\rho}_{\kappa} \overline{\otimes} \overline{G^{\pi}_{\lambda}}$ in which P follows σ and O plays 0's in G^{ρ}_{κ} after her first move, and in which the the sub-game $\overline{G^{\pi}_{\lambda}}$ is unfinished. Let $\mathcal{P}_{\sigma}$ be the set of such plays. Formally

$$\mathcal{P}_{\sigma} = \{x \in {}^{\omega}\kappa : x = b \star \sigma \text{ for some } b \in {}^{\omega}\kappa \ \wedge \ (x)_{\overline{G^{\pi}_{\lambda}}} \in \operatorname{Fin}(\kappa)$$

$$\wedge \ (x)_{G^{\rho}_{\kappa}}|0 - (x)_{G^{\rho}_{\kappa}}(0) = o\}.$$

Clearly $|\mathcal{P}| = \kappa$. Define

$$\begin{aligned} S_{\sigma} &= \{\beta \in \kappa : O \text{ plays } \beta \text{ on her first move in } G^{\rho}_{\kappa} \text{ in some } x \in \mathcal{P}\} \\ &= \{\beta \in \kappa : (x)_{G^{\rho}_{\kappa}}(0) = \beta \text{ for some } x \in \mathcal{P}_{\sigma}\}. \end{aligned}$$

Lemma 3.2.9. *S_{σ} is at most countable.*

Proof of Lemma 3.2.16. Suppose not.

There are at most $\kappa \times \omega = \kappa$ many finite plays w's of $\overline{G^{\pi}_{\lambda}}$ (meaning $\operatorname{Ext}(\overline{G^{\pi}_{\lambda}}, \kappa)$). For each such w and $\beta \in S_{\sigma}$, there is at most 1 play x such that $x \in \mathcal{P}_{\sigma}$ and w is the $\overline{G^{\pi}_{\lambda}}$ part of x and O plays β in the sub-game G^{ρ}_{κ}. Formally,

$$g : (w \in \operatorname{Fin}(\lambda), \beta \in S_{\sigma}) \mapsto x \in \mathcal{P}_{\sigma} \text{ such that and } (x)_{\overline{G^{\pi}_{\lambda}}} = w \text{ and } x_{G^{\rho}_{\kappa}}(0) = \beta.$$

is a partial 1-to-1 function from $\operatorname{Fin}(\kappa) \times S_{\sigma}$ to $\mathcal{P}_{\sigma}$. (Take two such plays x_1 and x_2. The moves of player O are the same in both x_1 and x_2 and so must be the moves of P since P follows a strategy, which implies $x_1 = x_2$.)

For each $\beta \in S_{\sigma}$, define $T_{\beta} = \{(x)_{G^{\rho}_{\kappa}}|_1 : x \in \mathcal{P}_{\sigma} \ \wedge \ x_{G^{\rho}_{\kappa}}(0) = \beta\}$. T_{β} is a set of reals that code β, each of which is played by P in G^{ρ}_{κ} when $\overline{G^{\pi}_{\lambda}}$ is unfinished. Since g is 1-to-1, we know $1 \leq |T_{\beta}| \leq \kappa$.

By our assumption that S is uncountable, we get a well-orderable set of reals $\bigcup_{\beta \in S_{\sigma}} T_{\beta}$. Contradiction. □

Now consider all the plays of $G^\rho_\kappa \mathbin{\overline{\otimes}} \overline{G^\pi_\lambda}$ in which O plays some $\beta \in \kappa - S_\sigma$ and 0's in G^ρ_κ. In each such play, $\overline{G^\pi_\lambda}$ is finished. And it is not hard to see that the infinite sequence played by O in $\overline{G^\pi_\lambda}$ could be anything. Since σ is a winning strategy for P, whenever O has played a proper code in $\overline{G^\pi_\lambda}$, P must have played a proper code in G^ρ_κ.

We need to define the following auxiliary objects from the winning strategy σ: Given $\beta \in \kappa - S_\sigma$, let $r_\sigma(\beta)$ be the move in $\overline{G^\pi_\lambda}$ that player P makes according to σ in the game G^π_λ after player O played β in G^ρ_κ.

After β and $r_\sigma(\beta)$ have been played, the winning strategy sigma gives a definition of a continuous function reducing a code for $r_\sigma(\beta)$ into a code for β, a continuous function that maps each $a \in C^\pi_{r_\sigma(\beta)}$ to some $b \in C^\rho_\beta$.

To make it precise, let $F^\beta_\sigma = \{s : s$ is a finite play of $G^\rho_\kappa \mathbin{\overline{\otimes}} \overline{G^\pi_\lambda}$, P follows σ in s, $(s)_{G^\rho_\kappa}|_0 \prec \beta * o$ and $(s)_{\overline{G^\pi_\lambda}}|_0 \in \mathrm{Fin}(\omega)\}$ and $f^\beta_\sigma = \{(t,p) : (\exists s \in F^\beta_\sigma)\ t = (s)_{G^\pi_\lambda}|_0 \ \wedge\ p = (s)_{\overline{G^\rho_\kappa}}|_1\}$. Clearly $f^\beta_\sigma \subset \mathrm{Fin}(\omega) \times \mathrm{Fin}(\omega)$ and f^β_σ is a function since σ is a strategy for P. Let $\mathcal{F}_\sigma = \{f^\beta_\sigma : \beta \in \kappa - S_\sigma\}$. By Theorem 1.1.5,

$$|\mathcal{F}_\sigma| \leq |P(\omega \times \omega)| = |\omega^\omega|. \tag{3.3}$$

From f^β_σ we get very naturally a continuous function $\hat{f}^\beta_\sigma : C^\pi_{r_\sigma(\beta)} \to C^\rho_\beta$ defined by

$$\hat{f}^\beta_\sigma(a) = \bigcup_{n<\omega} f^\beta_\sigma(a{\restriction}n). \tag{3.4}$$

Because by the definition of f^β_σ and the fact that σ is a winning strategy for P in $G^\rho_\kappa \mathbin{\overline{\otimes}} \overline{G^\pi_\lambda}$, for any given $a \in C^\pi_{r_\sigma(\beta)}$, i.e., $\pi(a) = r_\sigma(\beta)$,

$$\rho(\bigcup_{n<\omega} f^\beta_\sigma(a{\restriction}n)) = \beta$$

which is

$$\hat{f}^\beta_\sigma(a) = \bigcup_{n<\omega} f^\beta_\sigma(a{\restriction}n) \in C^\rho_\beta. \tag{3.5}$$

Consider the function $r_\sigma : \kappa - S_\sigma \to \lambda$. Because $|\kappa - S_\sigma| = \kappa$ and κ is regular, there must be some α' in $r_\sigma(\kappa - S_\sigma)$ such that $X := \{\beta : r_\sigma(\beta) = \alpha'\}$ has cardinality κ. Take $\beta_0 \neq \beta_1$ in X and consider $f^{\beta_0}_\sigma$ and $f^{\beta_1}_\sigma$. Suppose $\pi(x) = r_\sigma(\beta_0) = r_\sigma(\beta_1) = \alpha'$. Then $\rho(\hat{f}^{\beta_0}_\sigma(x)) = \beta_0$ and $\rho(\hat{f}^{\beta_1}_\sigma(x)) = \beta_1$, which implies $\hat{f}^{\beta_0}_\sigma \neq \hat{f}^{\beta_1}_\sigma$ and hence $f^{\beta_0}_\sigma \neq f^{\beta_1}_\sigma$. Thus, the set $\mathcal{F}^X_\sigma = \{f^\beta_\sigma : \beta \in X\}$ has cardinality κ. By $\mathcal{F}^X_\sigma \subset \mathcal{F}_\sigma$ and 3.3, we get a subset of ω^ω of size κ which contradicts AD. □

Corollary 3.2.10. *Assume* **AD**. *Given a surjection π from ${}^\omega\omega$ onto ω_1 and a surjection ρ from ${}^\omega\omega$ onto ω_2, $\overline{G^\pi_{\omega_1}} \mathbin{\overline{\otimes}} G^\rho_{\omega_2}$ is non-determined. In particular, $G^\pi_{\omega_1} \not\leq G^\rho_{\omega_2}$.*

Proof. By Lemma 3.1.2, 3.1.3 and 3.1.4. □

Corollary 3.2.11. *Assume* **AD**. *Given an infinite cardinal number $\kappa > \omega$ and, a surjection $\pi : {}^\omega\omega \to \kappa$, if $|S|$ is regular and $|S| > |S'|$, the game $G^\pi_S \mathbin{\overline{\otimes}} \overline{G^\pi_{S'}}$ is non-determined. In particular, $G^\pi_{S'} \not\leq G^\pi_S$.*

Proof. Trivial modification of the proof of Theorem 3.2.8. □

Corollary 3.2.12. *If $\kappa > \omega_1$ is regular under* **AD**, *$\mathcal{S}_\kappa$ has at least four different degrees.*

It would be nice if we have also shown in Theorem 3.2.8 that $G^\rho_S \not\leq G^\pi_{S'}$, because that would imply $\mathcal{S}_\kappa$ has more than one degrees. However, we can not show that in general. Under **AD**, we can construct games of the form $G^\pi_\lambda \overline{\otimes} \overline{G^\rho_\kappa}$ with $\lambda < \kappa$ in which P has a winning strategy.

Given a cardinal number $\lambda > \omega$ and a cardinal number $\kappa > \lambda$, a surjection π from ${}^\omega\omega$ onto λ and a surjection ρ from ${}^\omega\omega$ onto κ. Define $T^\kappa_\lambda = \{\alpha \in \kappa : (\forall\beta < \alpha)(\forall\gamma < \lambda)\ \beta + \gamma \neq \alpha\}$. $|T^\kappa_\lambda| = \kappa$ since $\lambda^\gamma \in T^\kappa_\lambda$ for all $\gamma < \kappa$. From ρ and the order-type function from T^κ_λ to κ, we can get a surjection $\psi : {}^\omega\omega \to T^\kappa_\lambda$. Define $\chi : {}^\omega\omega \to \kappa$ by

$$\chi(a) = \psi(a_0) + \pi(a_1)$$

where a_0 is the even part of a and a_1 is the odd part of a. It is easy to see that the range of χ is κ.

P has the following winning strategy in $G^\pi_\lambda \overline{\otimes} \overline{G^\chi_\kappa}$: After O plays α in G^π_λ, switch to $\overline{G^\chi_\kappa}$ and play $\lambda + \alpha$. Keep switching between two games while copying the odd part of O's play in $\overline{G^\chi_\kappa}$ into your (P's) own code played in G^π_λ.

We apply the partition property (or the weak compactness, **(e)** of Theorem 3.1.2) of ω_1 to certain family of ω_1 many games to get results about comparability among these games.

Lemma 3.2.13. *Let $\pi : {}^\omega\omega \to \omega_1$ be onto. Let $\{S_\alpha\}_{\alpha<\omega_1}$ be a partition of ω_1 such that $|S_\alpha| = \omega_1$ for all $\alpha < \omega_1$. There is a set $I \subset \omega_1$ such that $|I| = \omega_1$ and either 1. $G^\pi_{S_\alpha} \| G^\pi_{S_\beta}$ for all $\alpha \in \beta \in I$ or 2. $G^\pi_{S_\alpha} \leq G^\pi_{S_\beta}$ or $G^\pi_{S_\beta} \leq G^\pi_{S_\alpha}$ for all $\alpha \in \beta \in I$.*

Proof. Easy application of $\omega_1 \to (\omega_1)^2_2$. □

The same argument can be applied to the first ω_1 regular cardinals. Let κ_α be the α-th uncountable regular cardinal under **AD**. By Theorem 3.1.4 these are all less than Θ, so there is a surjections ρ_α from ${}^\omega\omega$ onto each κ_α. Moreover, these ρ_α's can be precisely defined without using any choice; see [19, p. 397–398]. Let H_α be the game $G^{\rho_\alpha}_{\kappa_\alpha}$. Let $\mathcal{H}$ be the set of all these H_α's.

Theorem 3.2.14. *There is a set $I \subset \omega_1$ such that $|I| = \omega_1$ and either 1. $H_\alpha \| H_\beta$ for all $\alpha \in \beta \in I$ or 2. $H_\alpha > H_\beta$ for all $\alpha \in \beta \in I$. That is, there is an uncountable strictly increasing sequence of games in $\mathcal{H}$, or there is an uncountable set of incomparable games in $\mathcal{H}$.*

Proof. For any $\alpha < \beta$, $H_\alpha \not\leq H_\beta$ by Theorem 3.2.8 and hence either $H_\alpha > H_\beta$ or $H_\alpha \| H_\beta$.

Define the following colouring function:

$$c(\{\alpha, \beta\}) = \begin{cases} 1 & \text{if } \alpha < \beta \to H_\alpha > H_\beta, \\ 0 & \text{otherwise.} \end{cases}$$

The partition property gives us a homogeneous set T for c of cardinality ω_1, and this is either an uncountable strictly increasing sequence of games (if it is homogeneous for 1) or an uncountable set of incomparable games (if it is homogeneous for 0). $\square$

Corollary 3.2.15. *The degree universe $\mathcal{S}_\Theta$ has chains of order-type ω_1 or antichains of cardinality ω_1.*

3.2.3 Structure of $\mathcal{S}_{\omega_\alpha}$ under the assumption NEUCL

Consider the following question: given a surjection $\pi : {}^\omega\omega \to \kappa$ where κ is an infinite regular cardinal $> \omega$, what S and S' satisfy that $|S'| = |S| = \kappa$ and P has no winning strategy in $G^\pi_S \overline{\otimes} \overline{G^\pi_{S'}}$?

Let $\kappa > \omega$ be a regular cardinal. Consider the game $G^\pi_{S_0} \overline{\otimes} \overline{G^\pi_{S_1}}$ where $S_0, S_1 \subset \kappa$ and $|S_0| = |S_1| = \kappa$. By Corollary 3.2.7 O does not have a winning strategy. Now let us suppose that σ is a winning strategy for P. We do the same analysis as in the proof of Theorem 3.2.8.

Notice that by definition of $\overline{\otimes}$, the first move in the game $G^\pi_{S_0} \overline{\otimes} \overline{G^\pi_{S_1}}$ belongs to O and she has to play an ordinal in $\overline{G^\pi_{S_1}}$. Now consider the set of all finished plays of $G^\pi_{S_0} \overline{\otimes} \overline{G^\pi_{S_1}}$ in which P follows σ and O plays 0's in $G^\pi_{S_0}$ after her first move, and in which the the sub-game $\overline{G^\pi_{S_1}}$ is unfinished. Let $\mathcal{P}_\sigma$ be the set of such plays. Formally

$$\mathcal{P}_\sigma = \{x \in {}^\omega\kappa : x = b \star \sigma \text{ for some } b \in {}^\omega\kappa \ \wedge \ (x)_{\overline{G^\pi_{S_1}}} \in \mathrm{Fin}(\kappa)$$

$$\wedge \ (x)_{G^\pi_{S_0}}|0 - (x)_{G^\pi_{S_0}}(0) = o\}.$$

Clearly $|\mathcal{P}| = \kappa$. Define

$$\begin{aligned} S_\sigma &= \{\beta \in S_0 : O \text{ plays } \beta \text{ on her first move in } G^\pi_{S_0} \text{ in some } x \in \mathcal{P}\} \\ &= \{\beta \in S_0 : (x)_{G^\pi_{S_0}}(0) = \beta \text{ for some } x \in \mathcal{P}_\sigma\}. \end{aligned}$$

Lemma 3.2.16. *The set S_σ is at most countable.*

Proof of Lemma 3.2.16. Suppose not.

There are at most $\kappa \times \omega = \kappa$ many finite plays w's of $\overline{G^\pi_{S_1}}$ (meaning $\mathrm{Ext}(\overline{G^\pi_{S_1}}, \kappa)$). For each such w and $\beta \in S_\sigma$, there is at most 1 play x such that $x \in \mathcal{P}_\sigma$, w is the $\overline{G^\pi_{S_1}}$ part of x and O plays β in the sub-game $G^\pi_{S_0}$. Formally,

$$g : (w \in \mathrm{Fin}(\lambda), \beta \in S_\sigma) \mapsto x \in \mathcal{P}_\sigma \text{ such that and } (x)_{\overline{G^\pi_{S_1}}} = w \text{ and } x_{G^\pi_{S_0}}(0) = \beta.$$

is a partial 1-to-1 function from $\mathrm{Fin}(\kappa) \times S_\sigma$ to $\mathcal{P}_\sigma$. (Take two such plays x_1 and x_2. The moves of player O are the same in both x_1 and x_2 and so must be the moves of P since P follows a strategy, which implies $x_1 = x_2$.)

For each $\beta \in S_\sigma$, define $T_\beta = \{(x)_{G^\pi_{S_0}}|_1 : x \in \mathcal{P}_\sigma \ \wedge \ x_{G^\pi_{S_0}}(0) = \beta\}$. T_β is a set of reals that code β, each of which is played by P in $G^\pi_{S_0}$ when $\overline{G^\pi_{S_1}}$ is unfinished. Since g is 1-to-1, we know $1 \leq |T_\beta| \leq \kappa$.

By our assumption that S is uncountable, we get a well-orderable set of reals $\bigcup_{\beta \in S_\sigma} T_\beta$. Contradiction. $\square$

Now consider all the plays of $G^{\pi}_{S_0} \overline{\otimes} \overline{G^{\pi}_{S_1}}$ in which O plays some $\beta \in S_0 - S_\sigma$ and 0's in $G^{\pi}_{S_0}$. In each such play, $\overline{G^{\pi}_{S_1}}$ is finished. And it is not hard to see that the infinite sequence played by O in $\overline{G^{\pi}_{S_1}}$ could be anything. Since σ is a winning strategy for P, whenever O has played a proper code in $\overline{G^{\pi}_{S_1}}$, P must have played a proper code in $G^{\pi}_{S_0}$.

We need to define the following auxiliary objects from the winning strategy σ: Given $\beta \in S_0 - S_\sigma$, let $r_\sigma(\beta)$ be the move in $\overline{G^{\pi}_{S_1}}$ that player P makes according to σ in the game $G^{\pi}_{S_1}$ after player O played β in $G^{\pi}_{S_0}$.

After β and $r_\sigma(\beta)$ have been played, the winning strategy sigma gives a definition of a continuous function reducing a code for $r_\sigma(\beta)$ into a code for β, a continuous function that maps each $a \in C^{\pi}_{r_\sigma(\beta)}$ to some $b \in C^{\pi}_{\beta}$.

To make it precise, let $F^{\beta}_{\sigma} = \{s : s$ is a finite play of $G^{\pi}_{S_0} \otimes \overline{G^{\pi}_{S_1}}$, P follows σ in s, $(s)_{G^{\pi}_{S_0}}|_0 \prec \beta * o$ and $(s)_{\overline{G^{\pi}_{S_1}}}|_0 \in \mathrm{Fin}(\omega)\}$ and $f^{\beta}_{\sigma} = \{(t,p) : (\exists s \in F^{\beta}_{\sigma})\ t = (s)_{G^{\pi}_{S_1}}|_0 \wedge p = (s)_{\overline{G^{\pi}_{S_0}}}|_1\}$. Clearly $f^{\beta}_{\sigma} \subset \mathrm{Fin}(\omega) \times \mathrm{Fin}(\omega)$ and f^{β}_{σ} is a function since σ is a strategy for P. Let $\mathcal{F}_\sigma = \{f^{\beta}_{\sigma} : \beta \in S_0 - S_\sigma\}$. By Theorem 1.1.5,

$$|\mathcal{F}_\sigma| \le |P(\omega \times \omega)| = |\omega^\omega|. \tag{3.6}$$

From f^{β}_{σ} we get very naturally a continuous function $\hat{f}^{\beta}_{\sigma} : C^{\pi}_{r_\sigma(\beta)} \to C^{\pi}_{\beta}$ defined by

$$\hat{f}^{\beta}_{\sigma}(a) = \bigcup_{n<\omega} f^{\beta}_{\sigma}(a \restriction n). \tag{3.7}$$

Because by the definition of f^{β}_{σ} and the fact that σ is a winning strategy for P in $G^{\pi}_{S_0} \overline{\otimes} \overline{G^{\pi}_{S_1}}$, for any given $a \in C^{\pi}_{r_\sigma(\beta)}$, i.e., $\pi(a) = r_\sigma(\beta)$,

$$\pi(\bigcup_{n<\omega} f^{\beta}_{\sigma}(a \restriction n)) = \beta$$

which is

$$\hat{f}^{\beta}_{\sigma}(a) = \bigcup_{n<\omega} f^{\beta}_{\sigma}(a \restriction n) \in C^{\pi}_{\beta}. \tag{3.8}$$

Consider the function $r_\sigma : S_0 - S_\sigma \to S_1$.

Lemma 3.2.17. $|\mathrm{ran}(r_\sigma)| = \kappa$.

Proof. Because $|S_0 - S_\sigma| = \kappa$ and κ is regular, there must be some $\alpha' \in \mathrm{ran}(r_\sigma)$ such that $X := \{\beta : r_\sigma(\beta) = \alpha'\}$ has cardinality κ. Take $\beta_0 \neq \beta_1$ in X and consider $f^{\beta_0}_{\sigma}$ and $f^{\beta_1}_{\sigma}$. Suppose $\pi(x) = r_\sigma(\beta_0) = r_\sigma(\beta_1) = \alpha'$. Then $\pi(\hat{f}^{\beta_0}_{\sigma}(x)) = \beta_0$ and $\pi(\hat{f}^{\beta_1}_{\sigma}(x)) = \beta_1$, which implies $\hat{f}^{\beta_0}_{\sigma} \neq \hat{f}^{\beta_1}_{\sigma}$ and hence $f^{\beta_0}_{\sigma} \neq f^{\beta_1}_{\sigma}$. Thus, the set $\mathcal{F}_\sigma = \{f^{\beta}_{\sigma} : \beta \in X\}$ has cardinality κ. By $\mathcal{F}_\sigma \subset \mathcal{F}_\sigma$ and 3.6, we get a subset of ω^ω of size κ which contradicts AD. ☐

Define $g_\sigma : \mathrm{ran}(r_\sigma) \to S_0 - S_\sigma$ by

$$g_\sigma(\alpha) = \text{ the least } \beta \in S_0 - S \text{ such that } r_\sigma(\beta) = \alpha.$$

Clearly g_σ is 1-to-1 and has range of cardinality κ. It is easy to see r_σ is 1-to-1 on $\mathrm{ran}(g_\sigma)$. Since $\mathcal{F}_\sigma$ is countable and κ is regular, we know there is $S_0^* \subset \mathrm{ran}(g_\sigma)$ and $f \in F_\sigma$ such that $|S_0^*| = \kappa$ and $(\forall \beta, \beta' \in S_0^*)\ f_\sigma^\beta = f_\sigma^{\beta'} = f$.

Thus we get a continuous function $\hat{f} : \bigcup_{\beta \in S_0^*} C^\pi_{r_\sigma(\beta)} \to \bigcup_{\beta \in S_0^*} C^\pi_\beta$. $\hat{f}$ is a universal reduction function in the sense that $\hat{f} \restriction C^\pi_{r_\sigma(\beta)} = \hat{f}^\beta_\sigma$, or equivalently

$$(\forall \beta \in S_0^*)\ \hat{f}[C^\pi_{r_\sigma(\beta)}] \subset C^\pi_\beta.$$

Definition 3.2.18. *Let $\pi : {}^\omega\omega \to \kappa$ be a surjection, $S \subset \kappa$ and $g : S \to \kappa$ be 1-to-1. A continuous function f that is (partially) defined on ${}^\omega\omega$ is a π-**universal continuous lifting** for S and g iff*

$$(\forall \beta \in S) f[C^\pi_\beta] \subset C^\pi_{g(\beta)}.$$

It is easy to see the above $\hat{f}$ is π-universal for $r_\sigma[S_0^*]$ and g_σ.

From the above discussion, we know that if such a universal continuous lifting cannot exist, then we have two games that are incomparable. We now prove this more precisely.

Let $\kappa > \omega$ be a regular cardinal and $\pi : {}^\omega\omega \to \kappa$ be a surjection. We say the **NEUCL Assumption** (NEUCL for "non-existence of universal continuous lifting") holds for the pair (κ, π) if and only if there is no π-universal continuous lifting f for any $S \subset \kappa$ with $|S| = \kappa$ and any 1-to-1 $g : S \to \kappa$ such that $g \cap \mathrm{id} = \emptyset$.

Lemma 3.2.19. *Assume* **AD**. *Let $\kappa > \omega$ be a regular cardinal and $\pi : {}^\omega\omega \to \kappa$ be a surjection. If the NEUCL Assumption holds for (κ, π), then for any S_0 and S_1 such that $|S_0| = |S_1| = \kappa$ and $|S_0 \cap S_1| < \kappa$, it is true that $G^\pi_{S_0} \| G^\pi_{S_1}$.*

Proof. By the above analysis, P does not have a winning strategy in either $G^\pi_{S_0} \mathbin{\overline{\otimes}} \overline{G^\pi_{S_1}}$ or $G^\pi_{S_1} \mathbin{\overline{\otimes}} \overline{G^\pi_{S_0}}$; by Corollary 3.2.7, O does not have a winning strategy in either $G^\pi_{S_0} \mathbin{\overline{\otimes}} \overline{G^\pi_{S_1}}$ or $G^\pi_{S_1} \mathbin{\overline{\otimes}} \overline{G^\pi_{S_0}}$. □

It is not hard to see that if the premiss of Lemma 3.2.19 holds, we get a large family of pairwise incomparable games. The following theorem says that if the NEUCL Assumption holds for some (κ, π), then $\mathcal{S}_\kappa$ has an antichain of size κ.

Theorem 3.2.20. *Assume* **AD**. *Let $\kappa > \omega$ be a regular cardinal and $\pi : {}^\omega\omega \to \kappa$ be a surjection. If the NEUCL Assumption holds for (κ, π), then there is a family of games on κ $\{G_\alpha : \alpha \in \kappa\}$ such that $G_\alpha \| G_\beta$ for any $\alpha \neq \beta$.*

Proof. Use the canonical partition of κ and get $S_\alpha = \kappa|\alpha$ for each $\alpha \in \kappa$. Apply Lemma 3.2.19 to S_α and S_β for any $\alpha \neq \beta$. □

Corollary 3.2.21. *Let $\kappa > \omega$ be a regular cardinal under* **AD**. *If there is a onto function $\pi : {}^\omega\omega \to \kappa$ such that the NEUCL Assumption holds for (κ, π), then $\mathcal{S}_\kappa$ has antichains of size κ.*

The NEUCL Assumption cannot hold for each pair (κ, π). The following is a counterexample.

Recall that $\Gamma\lceil\omega \times \omega : \omega \times \omega \to \omega$ is 1-to-1 and onto. Define $\mathcal{R} : {}^\omega 2 \to P(\omega \times \omega)$ by

$$(m, n) \in \mathsf{R}(a) \text{ iff } f(\Gamma(m, n)) = 1$$

for each $a \in {}^\omega 2$. Clearly $\mathcal{R}$ is 1-to-1 and onto. A nice property of $\mathcal{R}$ that will be useful is that $\mathsf{R}(a)\lceil n \times n$ can be read from the first $\Gamma(0, n)$ bits of a. It is so by our choice of Γ, which was defined earlier.

Define $\pi : {}^\omega\omega \to \omega_1$ by

$$\pi(a) = \begin{cases} \pi_0 \circ \mathcal{R}(a) & \text{if } a \in {}^\omega 2, \\ 0 & \text{otherwise.} \end{cases}$$

Recall that π_0 is the canonical projection : $P(\omega \times \omega) \to \omega_1$. So π is a well-defined projection : ${}^\omega\omega \to \omega_1$.

Let $S_1 = \{\omega^\alpha : \alpha < \omega_1\}$ and $S_0 = \{\omega^\alpha \cdot 2 : \alpha < \omega_1\}$. Clearly $|S_0| = |S_1| = \kappa$ and $S_0 \cap S_1 = \emptyset$. We show that P has the following winning strategy σ in $G^\pi_{S_0} \overline{\otimes} \overline{G^\pi_{S_1}}$.

By definition of $\overline{\otimes}$, the game starts with O play a $\alpha \in S_0$ in the sub-game $G^\pi_{S_0}$. After O's starting move, P play the unique β such that $\beta \cdot 2 = \alpha$ in the sub-game $\overline{G^\pi_{S_1}}$. Then O must begin to play a code of β, if she wants to win at all. Whenever O has played $\Gamma(0, n+1)$ bits of her code b, interrupt her and extend your (P's) code a in $G^\pi_{S_0}$ to the first $\Gamma(0, 2n)$ bits so that

1. for all $m, k < n$, if $b(\Gamma(m, k)) = 1$ then $a(\Gamma(2m, 2k)) = 1$ and $a(\Gamma(2m+1, 2k+1)) = 1$,
2. for all $m, m', k, k' < n$, if $b(\Gamma(m, k)) = 1$ and $b(\Gamma(m', k')) = 1$, then $(\forall x, y \in \{m, m', k, k'\})\ a(\Gamma(2x, 2y+1)) = 1$,
3. $a(k) = 0$ for all $k < 2n$ not covered by the above clauses.

After extending a to the first $2n$ bits, switch to $\overline{G^\pi_{S_1}}$ and let O continue to play her code b.

It is easy to see this is a well-defined strategy. Note that if O starts to play anything other than 0-1 bits at any point, you just need to sit and watch and you win the game when it is finished.

It is easy to see for each $n < \omega$, if $b\lceil n$ codes a well-ordering of order-type m, $a\lceil 2n$ codes a well-ordering of order-type $m \cdot 2$, and when the game is finished, if b codes a well-ordering of order-type $\alpha \in S_0$, a codes $\alpha \cdot 2$. So the strategy is a winning one for P.

From the definition of P's strategy σ, $\mathcal{F}^\beta_\sigma = \mathcal{F}^{\beta'}_\sigma$ and hence $f^\beta_\sigma = f^{\beta'}_\sigma = f_\sigma$ for each $\beta, \beta' \in S_0$. $\hat{f}_\sigma$ is π-universal for S_0 and $g : \beta \mapsto \beta \cdot 2$, i.e.,

$$(\forall \beta \in S_0)\ \hat{f}_\sigma[C^\pi_\beta] \subset C^\pi_{\beta \cdot 2}.$$

We have seen that the NEUCL Assumption cannot be a general theorem for all pairs (κ, π) where κ is regular under **AD** and $\pi : {}^\omega\omega \to \kappa$ is onto. It is not easy to see what (κ, π) can validate the NEUCL Assumption. Now let us consider given (κ, π) what $S \subset \kappa$ and $g : S \to \kappa$ cannot falsify the NEUCL Assumption.

First, we consider the case in which $S \subset \kappa = \omega_1$ and $g : S \to S$ is 1-to-1 and onto, and

$$(\forall \alpha \in S)\ g(\alpha) \neq \alpha.$$

Lemma 3.2.22. *There is a set $T \subset S$ of cardinality κ such that $g[T] \cap T = \emptyset$*

Proof. By **(e)** of Theorem 3.1.2, $\omega_1 \to (\omega_1)^2_2$.

Define a colouring function $c : [\omega]^2 \to 2$ by

$$c(\{\alpha, \beta\}) = \begin{cases} 1 & \text{if } g(\alpha) = \beta \text{ or } g(\beta) = \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

By weakly compactness, there is a homogenous set $T \subset S$ of cardinality κ. We claim $c[T] = \{0\}$.

Suppose otherwise, $c[T] = \{1\}$. Let α, β, γ and δ are four different elements of T. Without loss of generality, we can assume that $g(\alpha) = \beta$, which implies $g(\alpha) \neq \gamma, \delta$. But since $c(\{\alpha, \gamma\}) = c(\{\alpha, \delta\}) = 1$, it has to be the case $g(\gamma) = g(\delta) = \alpha$. But that means that $g(\gamma) \neq \delta$ and $g(\delta) \neq \gamma$ and hence $c(\{\gamma, \delta\}) = 0$. Contradiction.

By the definition of c, $g(\alpha) \neq \beta$ for all $\alpha, \beta \in T$ and hence $g[T] \cap T = \emptyset$. □

Clearly if f is a π-universal continuous lifting for S and g, it is also a π-universal continuous lifting for T and g. So if we can show that for any $S \subset \omega_1$ and any g such that $|S| = \omega_1$ and $g[S] \cap S = \emptyset$, there is no π-universal continuous lifting, then we also have that for any $S \subset \omega_1$ and any 1-to-1 and onto $g : S \to S$ such that $(\forall \alpha \in S)\ g(\alpha) \neq \alpha$, there is no π-universal continuous lifting.

3.3 Game logics in ZF + AD

In this section we reexamine Blass's completeness results in Subsection 2.3.3, which used **AC**, in our new setting **ZF + AD**.

Theorem 3.3.1. *If there is family of games $\{G_n : n < \omega\} \subset \mathcal{G}_\kappa$ such that for each disjoint finite $I \subset \omega$ and finite $J \subset \omega$, $(\bigotimes_{i \in I} G_i)\ \overline{\otimes}\ (\bigotimes_{j \in J} \overline{G}_j)$ is not a win for P, then there is a variable κ-interpretation $\mathcal{I}$ such that, for any additive sequent $\vdash \Gamma$ such that* NF$(\vdash \Gamma)$ *unprovable in game sequent calculus, player P has no winning strategy in $\overline{\overline{\mathcal{I}}}(\mathrm{NF}(\vdash \Gamma))$.*

Proof. We modify the proof of Theorem and omit some details to avoid repitition.

Let $\Gamma = C_1, C_2, ..., C_m$ where C_k are additive formulas. Consider an arbitrary game interpretation and the corresponding game Γ which is the game $C_1 \overline{\otimes} ... \overline{\otimes} C_m$. The moves in any component game C_k come in two phases. In phase 1, the players are choosing conjuncts or disjuncts in sub-formulas for game C_k. For example, if C_k is $(p \wedge \overline{q}) \vee r$, where p, q, r are propositional variables, then phase 1 contains P's opening move, choosing $p \wedge \overline{q}$ or r, and, if he chooses the former, then phase 1 also contains

O's reply, choosing p or $\overline{q}$. Each phase 1 move replaces the kth component of Γ by one of its conjuncts or disjuncts, and phase 1 continues in the kth component until it is reduced to a literal, i.e., to a propositional variable or the negation of one one or $\top$ or $\bot$. Then comes phase 2, in which the players play (the game associated to) that literal. In any component, the phase 1 moves precede the phase 2 moves, but it is possible for phase 2 to begin in one component before phase 1 is finished in another component. It is also possible for a play of Γ to have only finitely many moves in some component, and then phase 1 may not be finished there.

At any stage of the play, we write Γ' for the current list of component games. Initially, Γ' is Γ, but every phase 1 move replaces some formula in Γ' with one of its conjuncts or disjuncts.

The proceeding discussion concerned an arbitrary play of Γ. We now focus our attention on particular plays of Γ in which P follows a strategy σ while O plays phase 1 so that Γ' is never provable. O can do this, as shown below. Initially, Γ' is Γ, which is unprovable, by assumption. If Γ' is unprovable at some point during the play, and if P then makes a phase 1 move, then Γ' will still be unprovable after this move. Indeed a phase 1 move of P replaces a component of the form $A \vee B$ with A or with B, so, up to order of components, Γ' before the move was $\Delta, A \vee B$ and Γ' after the move is either Δ, A or Δ, B. But if either $\vdash \Delta, A$ or $\vdash \Delta, B$, then, by rule ($\vee$), we have $\vdash \Delta, A \vee B$, a contradiction. Thus, phase 1 moves of P cannot make Γ' provable. A phase 1 move of O changes Γ' form $\Delta, A \wedge B$, to Δ, A or Δ, B. By rule ($\wedge$), if $\Delta, A \wedge B$ is unprovable, then so is at least one of Δ, A or Δ, B. So O can make his phase 1 moves in accordance with instruction we just gave.

Now consider a interpretation $\mathcal{I}$ that maps each propositional variable v_i to $[G_i]$. We show that no strategy σ of P can be a winning strategy. Consider those plays of Γ satisfying the description in the last paragraph. By the preceding discussion, Γ' never contains 1, otherwise Γ' would be provable by rule (1). Similarly, Γ' can never contain both p and $\overline{p}$ for any propositional variable p, by the logical axiom and the weakening rule. So eventually Γ' is $(\overline{\bigotimes}_{i\in I} G_i) \mathbin{\overline{\otimes}} (\overline{\bigotimes}_{j\in J} \overline{G}_j)$ where I and J are finite and $I \cap J = \emptyset$. By our assumption, σ cannot be a winning strategy for P. $\square$

Corollary 3.3.2 (Additive Completeness Theorem)**.** *Assume that there is family of games* $\{G_n : n < \omega\} \subset \mathcal{G}_\kappa$ *such that for each disjoint finite* $I \subset \omega$ *and finite* $J \subset \omega$, $(\overline{\bigotimes}_{i\in I} G_i) \mathbin{\overline{\otimes}} (\overline{\bigotimes}_{j\in J} \overline{G}_j)$ *is not a win for* P*. An additive sequent* $\vdash \Gamma$ *is* κ*-valid if and only if* $\mathrm{NF}(\vdash \Gamma)$ *is provable in game sequent calculus.*

Note that the premiss of Corollary 3.3.2 is not the same as the conclusion of Theorem 3.2.20. Having a family of pairwise incomparable games of size κ is not enough.

Let $\{G^{\pi}_{S_\alpha} : \alpha < \kappa\}$ be the same as in the proof of Theorem 3.2.20. An immediate question is whether P has a winning strategy in the game $(\overline{\bigotimes}_{i\in I} G_i) \mathbin{\overline{\otimes}} (\overline{\bigotimes}_{j\in J} \overline{G}_j)$ for some $I \subset \kappa$ and $J \subset \kappa$ such that $|I|, |J| < \omega$ and $I \cap J = \emptyset$. The answer is not clear to us.

Conclusions and future work

We have investigated game universes and game logics on infinite cardinals in $\mathbf{ZF} + \mathbf{AD}$ and proved following main results.

Theorem 2.2.6. *The game logic on ω is just classical propositional logic.*

Corollary 3.2.12. *If $\kappa > \omega_1$ is regular under* $\mathbf{AD}$*, the degree structure $\mathcal{S}_\kappa$ has at least four different degrees.*

Corollary 3.2.15. *The degree structure $\mathcal{S}_\Theta$ has chains of order-type ω_1 or antichains of cardinality ω_1.*

Corollary 3.2.21. *If $\kappa > \omega$ is a regular cardinal under* $\mathbf{AD}$ *and there is an onto function $\pi : {}^{\omega}\omega \to \kappa$ such that the NEUCL Assumption*[2] *holds for (κ, π), then $\mathcal{S}_\kappa$ has antichains of size κ.*

But a number of open technical questions remain:

- Is there a regular $\kappa > \omega$ and a onto function $\pi : {}^{\omega}\omega \to \kappa$ such that the NEUCL Assumption holds for (κ, π)? Can κ be ω_1?
- Is there a regular $\kappa > \omega$ and family of games $\{G_n : n < \omega\} \subset \mathcal{G}_\kappa$ such that for each disjoint finite $I \subset \omega$ and finite $J \subset \omega$, $(\overline{\bigotimes}_{i \in I} G_i) \ \overline{\otimes} \ (\overline{\bigotimes}_{j \in J} \overline{G}_j)$ is not a win for P?[3] Can we prove this assuming NEUCL for some (κ, π)?

There are several lines of research we can follow in the future. We could generalize infinite games to games of length α for each ordinal α and study them using the methodology provided in this thesis. We could also introduce new reducibility relations and hence new degree structures. Blass gave another reducibility in [5] which he called the weak order. This reducibility is also interesting but not studied in this thesis. We can also define different reducibility relations by putting different upper bounds on how many copies of a game can be played and consulted when playing the other. Weather we get the same degree structure for all finite upper bounds is not immediately clear.

[2]NEUCL stands for "non-existence of universal continuous lifting"; for a definition, cf. p. 61.

[3]If the answer is yes, Blass's additive completeness theorem holds for the game logic on κ; see Section 3.3.

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