COUNTERPART SEMANTICS
A FOUNDATIONAL STUDY ON QUANTIFIED MODAL LOGICS

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Abstract. Counterpart semantics is proposed as the appropriate semantical framework for a foundational investigation of quantified modal logics. It turns out to be a limit case of the categorical semantics of relational universes introduced by Ghilardi and Meloni in 1988. The main result is a deeper understanding of the interplay between substitution, quantification and identity wherever modalities are present. Languages with types and explicit substitutions are the tools used to clarify such an interplay and to disintangle classical problems related to modalities in first-order languages. It is shown that controversial modal principles are neither valid nor provable. Quine's worries are dispelled.

§1. Introduction. Early modal logic

The year 1946 is worth remembering in the history of first-order modal logic because two important papers appeared in succession in the *Journal of Symbolic Logic: 'A Functional Calculus of First order Based on Strict Implication'* by Ruth C. Barcan, and then *'Modalities and Quantification'* by Rudolf Carnap. For the first time formal systems of quantified modal logic were introduced and analysed: "So far, no forms of MFC - modal functional calculus - have been constructed, and the construction of such a system is our chief aim." Thus Carnap in 1946.[1] The logical and philosophical issues raised by Carnap and Barcan's enterprise do not depend in an essential way on the nature of the intensional operators they deal with, i.e. the alethic operators, 'it is necessary that' and its dual 'it is possible that', and can be applied to other operators as well, e.g. the temporal ones: 'always in the future' and 'always in the past'. Since foundational worries are at the core of the present work, there is no loss in limiting

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[1]See Carnap, [9], p.33.

our investigation to the 'simplest' modal language, the one Barcan and Carnap dealt with.

The problem for them was, as Willard Van Orman Quine had already argued in *'Notes on existence and necessity'*, 1943, that there were considerable difficulties in marrying either substitution or quantifiers with modalities.

Resistance to substitution[2] Quine considers intensional languages which allow composition of statements by means of 'necessarily', 'possibly', and 'necessarily if-then', and, quite rightly, wants to preserve the fundamental principle of *substitutivity*: "given a true statement of identity, one of its two terms may be substituted for the other in any true statement and the result will be true"[3] But here is Quine's example of substitution, which has become omnipresent since its first appearance:

1. The number of planets = 9.
2. 9 is necessarily greater than 7.
3. The number of planets is necessarily greater than 7.

" a substitution on the basis of the true identity (1.) transforms the truth (2.) into a falsehood (3.)". Intensional contexts "are, in fact, subject to the same defects as the contexts of quotes." " ... a name within a context of single quotes does not occur designatively ... " in the very same way as "the occurrence of '9' in (2.) is not purely designative"[4].

Resistance to quantification. " ... a statement like 'There is something which is necessarily greater than 7' is meaningless. For, would 9, that is the number of planets, be one of the numbers necessarily greater than 7 ? But such affirmation would be at once true in the form of (2.) and false in the form of (3.)."[5]. Therefore the quantifier 'there is' misses its primary function of designating an individual independently of the mode of presentation.

Does the formula $\Box(x > 7)$ express a genuine predicate? Is the meaning of $\Box(x > 7)$ a set of objects, or otherwise said, does x have a referential function? If it were so, then

1. $\Box(x > 7)$ applied to a singular term t, $\Box(t > 7)$, would express a property of the object $[t]$ which is the referent of t.
2. t could be substituted in $\Box(t > 7)$ by any co-referential term s, *salva veritate*.
3. the variable x in $\Box(x > 7)$ could be quantified and the sentence $\exists x\Box(x > 7)$ be either true or false.

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[2]In analogy to Quine's expression 'resistance to quantification', [62], p.124.
[3]See Quine, [62], p.113.
[4]See Quine, [62], p.123.
[5]See Quine, [62], p.124.

Since (1.)-(3.) seem rather disputable on the basis of the examples produced by Quine, it has been argued that in the presence of intensional operators one should abandon any 'objectual interpretation' such as Tarski's and indeed Quine's.[6]

These objections of principle to the feasibility of any 'sound' formal system of intensional logic did not stop Carnap from elaborating such systems. Carnap is willing to study intensional calculi from inside, by looking at what their proof theory and semantics might be like once we have them. In *Meaning and Necessity*, 1947, Carnap presents the well-known system S_2 whose language is very rich: it is a first-order language with identity and individual constants; moreover it contains both the iota-operator for individual descriptions and the lambda-operator for abstraction. "If a sentence consists of an abstraction expression followed by an individual constant, it says that the individual has the property in question. Therefore, '$\lambda x(\ldots x \ldots)a$' means the same as '$\ldots a \ldots$', that is, the sentence formed from '$\ldots x \ldots$' by substituting 'a' for 'x'. The rules of our system will permit the transformation of '$\lambda x(\ldots x \ldots)a$' into '$\ldots a \ldots$' and viceversa; these transformations are called conversions."[7] The admission of the λ-conversion rule without restrictions is the key feature of Carnap's system that we want to address. Already in 1964, Feys raises his finger against this rule: "It is well known that $Ax \to (x = y \to Ay)$ holds. Let us take for A the predicate $\lambda y \Box (x = y)$. Then we have

$$\Box(x = x) \to (x = y \to \Box(x = y)).$$

Since $\Box(x = x)$ holds, it follows that

$$x = y \to \Box(x = y),$$

hence we should come to this paradoxical result that whenever factual identity exists, necessary identity must exist. ... But this argumentation takes for granted that both sentences '$\Box Ay$' and y has the predicate 'to be an x, which is necessarily an A', i.e. '$(\lambda x \Box Ax)y$' are logically equivalent. But this not the case. '$\Box Ay$' is a sentence expressing a necessary proposition, whereas '$(\lambda x \Box Ax)y$' is a modally ambiguous statement attributing a necessary predicate to y."[8]

Carnap does not accept Feys' analysis and replies: "The problems which Feys discusses in the last section of his essay are indeed serious, and I would agree they must be solved if a satisfactory system of modal logic is to be constructed. ... Feys analyses the sentences '$\Box Ay$' and '$(\lambda x \Box Ax)y$'. ... In my languages these two sentences are L-equivalent by virtue of the

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[6]Ruth Barcan Marcus in [6], 1961, proposes the so-called 'substitutional interpretation' according to which $\exists x A(x)$ means $A(t)$ for some *expression* t.

[7]See, Carnap, [9], p.3.

[8]See Feys, [17], p.297. Ruth Barcan Marcus in [4], 1947, had already shown that $x = y \to \Box(x = y)$ obtains immediately from the Leibniz' principle of substitutivity of identity.

customary rule of λ-conversion."[9]
Rule of λ-conversion:

$$(\lambda x A x)t \leftrightarrow A(t).$$

We have dwelt upon those early papers by Carnap and Quine and the observation of Feys to stress that the core problems of quantified modal logics were singled out from the very beginning, and we can summarize them in the following questions: What is a suitable language to deal with modalities? Should it contain the λ operator? Is the λ-conversion rule acceptable in the presence of modalities?

The rule of λ-conversion is nothing but a rule of substitution for terms and it implies that substitution commutes with respect to the truth-functional connectives and quantifiers, i.e.:

$$(\lambda x.\neg Bx)t \leftrightarrow \neg((\lambda x Bx)t)$$

$$(\lambda x.Bx \wedge Cx)t \leftrightarrow (\lambda x.Bx)t \wedge (\lambda x.Cx)t$$

$$(\lambda x.\forall y B(y,x))t \leftrightarrow \forall y((\lambda x.B(y,x))t)$$

where y does not occur in t. In analogy with the equivalences above one is tempted to assume with Carnap that the following equivalence is also true

$$(\lambda x.\Box Bx)t \leftrightarrow \Box((\lambda x.Bx)t).$$

But, as we shall see, this is the source of the difficulties. Let us observe first that both particular instantiation and substitutivity of identity, as usually formulated, involve commutativity of substitutions. For

$$A(t) \rightarrow \exists x A(x)$$

is the fusion of two principles, 'proper' particular instantiation:

$$(\lambda x.Ax)t \rightarrow \exists x A(x)$$

and λ-conversion:

$$(\lambda x.Ax)t \leftrightarrow A(t).$$

Analogously, substitutivity of identity in its usual form:

$$s = t \rightarrow (A(s) \rightarrow A(s/t))$$

is the fusion of 'proper' substitutivity of identity:

$$s = t \rightarrow (\lambda x.A(x)s \rightarrow \lambda x A(x)t)$$

and λ-conversion:

$$\lambda x.A(x)s \leftrightarrow A(s) \text{ and } \lambda x.A(x)t \leftrightarrow A(t).$$

Of course we are ready to accept as valid both $(\lambda x.Ax)t \rightarrow \exists x Ax$ and $s = t \rightarrow (\lambda x.A(x))s \rightarrow \lambda x.A(x)t)$, but we are in doubt as to the validity of the following formulas:

(a) $\exists x \Box(x = t)$

(b) $t = s \rightarrow \Box(t = s)$.

[9] See Carnap, [11], pp.907-908.

Now let us examine the 'usual' proofs of (a) and (b) paying attention to what forms of λ-conversion are used. To this end we consider separately the two arrows of the biconditional $(\lambda x.\Box Bx)t \leftrightarrow \Box((\lambda x.Bx)t)$:

(R)[10] $\qquad (\lambda x.\Box Ax)t \to \Box((\lambda x.Ax)t)$

and

(CR)[11] $\qquad \Box((\lambda x.Ax)t) \to (\lambda x.\Box Ax)t$

$(\lambda x\Box(x=t))t \to \exists x\Box(x=t)$	particular instantiation
$\Box(t=t) \to \exists x\Box(x=t)$	by CR
$t=t$	reflexivity of identity
$\Box(t=t)$	by the rule of necessitation
$\exists x\Box(x=t)$	by the rule of modus ponens

$(\lambda x\Box(x=s))s \to (t=s \to$ $\qquad \to (\lambda x\Box(x=s))t)$	'proper' substitutivity of identity
$(\lambda x\Box(x=s))s \to (t=s \to \Box(t=s))$	by R
$\Box(s=s) \to (t=s \to \Box(t=s))$	by CR
$s=s$	reflexivity of identity
$\Box(s=s)$	by the rule of necessitation
$t=s \to \Box(t=s)$	by the rule of modus ponens

A natural and obvious solution to the problems presented so far, would be to retain a modal language with a λ-operator and to restrict λ-conversion in order to stop the unwarranted derivations. But consider the formula $\exists x\Box P(x) \to \Box\exists xP(x)$. Its validity is rather questionable from an intuitive point of view, nevertheless it can be proved without making use, on the surface at least, of any form of λ-conversion:

$P(x) \to \exists xP(x)$	'proper' particular instantiation
$\Box P(x) \to \Box\exists xP(x)$	by the distributivity of $\Box$
$\exists x\Box P(x) \to \Box\exists xP(x)$	by the rule of $\exists$-introduction

Modal languages with a λ-operator have been considered by Stalnaker and Thomason, 1968, and more recently by Fitting, 1991, and Fitting and Mendelsohn, 1998.[12] A different point of view has been put forward by Ghilardi and Meloni in 1988[13] who recognize that the principle of λ-conversion with respect to formulas containing modal operators needs careful scrutiny, and at the same time suggest that the language itself

[10]The letter 'R' is due to the fact that this principle is characteristic of those terms which are 'rigid' designators, see section 2.

[11]Converse of R.

[12]See Stalnaker and Thomason, [76], Fitting, [20] and Fitting and Mendelsohn, [22].

[13]See Ghilardi and Meloni, [30].

needs to be deeply revised in such a way to bring to the surface distinctions that do not matter if there are no intensional operators, but that are crucial if they are present. Strengthening the language of classical logic by adding the abstraction operator does not lead to any better understanding of the subtleties and/or anomalies that take place in the presence of the modal operators. On the contrary we need to refine the syntax of modal languages so as to control the free variables occurring in the formulas and to discover those features of the operation of substitution that are innocuous in the absence of modalities, but that become troublesome otherwise. With this aim Ghilardi and Meloni introduce modal languages in which both terms and formulas have *types*. On the semantical side, *counterpart semantics* is the appropriate tool if we are to investigate and clarify the main concepts and problems connected with modalities. The present work is devoted to this approach.

Worlds to be considered and transworld identification

Since Kripke's paper, 1963, possible world sematics has been 'the' semantics for modal languages.[14] We here assume familiarity with Kripke semantics, and we limit ourselves to observing that in Kripke semantics we cannot render formally the idea that an individual satisfies 'now' the open formula 'x will always be good' iff that individual is good in all future worlds in which it exists. For in order to evaluate a formula like $\Box G(x)$ at w under an assignment σ, we need to check if $\sigma(x)$ belongs to the interpretation of G in every world v accessibile to w, whether $\sigma(x) \in D_v$ or not. Moreover the very same individual $\sigma(x)$ may exist in different worlds, for the codomain of any assignment function is the entire universe V. An opposite view has been defended, according to which individuals are worldbound so that any individual can belong to the domain of just one world. The assumption that individuals are worldbound poses the problem of transworld identification. Whom is a sentence like 'Peter would have been happy, if only he had married Mary' talking about ? The actual Peter did not marry Mary, so it can be argued that he is not the one the sentence is talking about. On the other hand, it is the actual Peter that regrets not having married Mary, so it must be he that would have been happy, had things gone otherwise. But how and where is the world in which 'things went otherwise'? Maybe such a world includes the fact that 'Peter proposes to Mary', so Peter has two alternative pasts: one in which he proposes to Mary and another in which he does not. We can go on to imagine a chain of worlds describing Peter's alternative biography, but the more we reconstruct his biography, the more we lose sight

[14]See Kripke, [46]. For a detailed exposition of Kripke semantics we refer the reader to *First-order Modal Logic* by Fitting and Mendelsohn, [22].

of him. It seems that language produces an optical illusion, the sentence: 'here is the Peter that could have been happy' induces us to think that the possibly happy Peter is in front of us, whereas it actually proposes a mental experiment, whose hero is unkown to us in many respects.[15] What reason is there to think that between 'the unhappy Peter' and 'the happy Peter'; or between 'me now' and 'me tomorrow' there must in each case be the identity relation? We will favour a more liberal attitude and accept Lewis' view that in order to make sense of sentences like 'I'm in Paris now, but tomorrow I'll be in London' it is enough to assume that some special relation holds between 'me now in Paris' and 'me tomorrow in London' without assuming that it has to be the identity relation. Following this line of thought, we will consider frames endowed with a binary relation the *counterpart relation*, connecting individuals existing in different worlds and leave to this relation, however queer, the task of retracing them across the worlds. Of course depending on the different aims of a modal theory or on the different intended applications, various conditions will have to be imposed on the counterpart relation.

Lewis's counterpart theory

Counterpart theory was advanced by David Lewis in 1968, and he elaborates a first-order theory whose language contains special predicates for talking about possible worlds and individuals in those worlds.[16] In particular it contains the predicates 'to be a possible world', 'to be the actual world', 'to be a counterpart of', 'to be an individual existing in a given world'. A translation function is defined in order to reduce modal sentences to first-order sentences of such a theory. According to such a translation 'It is possible that someone will win the lottery' becomes: 'There is a possible world such that someone in that world that wins the lottery', and 'Someone will possibly win the lottery' becomes: 'There exists an individual of the actual world such that in some possible world his counterpart(s) wins the lottery'. As to the conditions that, according to D.Lewis, cannot be imposed on a counterpart relation, here are some of them:[17]

i. nothing in any world has more than one counterpart in any other world;

ii. no two things in any world have a common counterpart in any other world;

iii. for any two worlds, anything in one is a counterpart of something in the other;

[15]Pros and cons of both points of view have been discussed at length in the literature, see, for example, R.Chisholm [12], A.Plantinga [61], D.Kaplan [43], A.Hazen [33].

[16]See Lewis, [50].

[17]See Lewis, [50], p.116.

iv. for any two worlds, anything in one has some counterpart in the other;

v. anything in any world is counterpart of itself;

vi. nothing in a world is a counterpart of anything else in that world.

As A. Hazen says,[18] "Lewis does not present his theory as a model theory for modal languages. Such a model theory is, however, easily extracted from his work.". We can summarize in a few words the main tenets of counterpart semantics. In a counterpart frame the sets of individuals 'existing' in each world may be taken as disjoint from the sets of individuals 'existing' in any other world and a counterpart relation, $\mathfrak{C}$, may hold between any two individuals 'existing' in related worlds. If $a\mathfrak{C}b$ holds, then we say that b is a counterpart of a. No condition whatsoever is put on $\mathfrak{C}$, and so an individual of a world w can have none, one, or more than one counterpart in any related world v. A counterpart model is a family of classical (Tarskian) models related to each other by the accessibility relation and moreover the individuals of the domain of a model are related to the individuals of the domain of another (accessible) model by the counterpart relation. As Robert Stalnaker says,[19] "In counterpart semantics, the rules must be changed to make $\Diamond Gx$ true of a in w if Gx is true of a counterpart of a in some world w'. Hubert Humphrey, for example, will have the property of being possibly a president of the United States by virtue of the fact that in a different possible world, a different person who is a counterpart of Humphrey was president.".

Relational universes for multi-sorted temporal languages

Lewis' counterpart semantics can be viewed as a limit case of the semantics of relational universes introduced by Ghilardi and Meloni in [30] and developed subsequently in [26], [29] and [31]. The conceptual framework and the techniques used in their proofs come from category theory which appears to be very fruitful in the analysis of problems and in indicating the way to their solution; we refer the reader to Ghilardi [29] for an ample overview of the links between category theory and modalities. The universes of interpretation they introduce are *relational universes*, and, though we will make no use of them in what follows, we will briefly say what they are. "Given a (small) category $\mathbf{C}$, we [say] that a lax functor, here called a relational universe $D : \mathbf{C} \to \mathbf{Rel}$ is a map which associates with each object α of $\mathbf{C}$ a set D_α and with each arrow $k : \alpha \to \beta$ a relation $D_k \subseteq D_\alpha \times D_\beta$ in such a way that the following two conditions are met:

$$D_{1_\alpha} \supseteq 1_{D_\alpha} \quad \text{and} \quad D_k D_l \subseteq D_{k_l},$$

[18]See [33], p.324.
[19]See [74], p.123.

for every object α and for every pair of composable arrows k, l."[20]

In a relational universe the objects α of the category represent possible worlds, the arrows $k : \alpha \to \beta$, the accessibility relation, the set D_α the set of individuals existing at α and for each arrow k, the relation D_k represents the counterpart relation along k between individuals of D_α and D_β. Given a category, there are in general many (even infinitely many) arrows from an object α to an object β, and for each arrow $k : \alpha \to \beta$ the relation D_k can be interpreted as 'the counterpart relation along k'.

Relational universes can be generalized by starting with graphs $\mathbf{G}$ instead of categories: in this case a relational universe

$$D : \mathbf{G} \to \mathbf{Rel}$$

is a map which associates with each object α of $\mathbf{G}$ a set D_α and with each arrow $k : \alpha \to \beta$ a relation $D_k \subseteq D_\alpha \times D_\beta$. This move from categories to graphs is necessary if we do not want to impose any condition on the accessibility relation: categories are appropriate when such a relation is taken to be reflexive and transitive. As we shall see, a counterpart frame differs from a relational universe because there is at most one arrow between two elements α and β of $\mathbf{G}$ and so there is at most one counterpart relation, D_k, between the individuals of D_α and D_β.

Relational universes and, consequently, counterpart frames, provide a sound mathematical ground for the investigation of modal languages, and they are very important from a foundational point of view because natural and interesting answers to the supposed 'sins'[21] of quantified modal logic can be given in a uniform way, answers which, it is worth stressing, do not stem from artificial interpretations of the modal operators, but rather from a deeper understanding of the mechanisms of classical logic itself and their interplay with the modal operators. We believe that Ghilardi and Meloni's proposal finally dispels Ruth Marcus' worry:[22] "There is a normative sense in which it has been claimed that modal logic is without foundations". The rest of this paper is devoted to present counterpart semantics and to show that well known systems of quantified modal logic are complete with respect to that semantics.[23]

[20] See Ghilardi and Meloni, [31], p.82.

[21] See W.V.O.Quine, [66], the sin of confusing use and mention.

[22] See Barcan Marcus, [6], p.5.

[23] Generalizations of Kripke semantics have ben studied by many authors and with different aims: either to establish incompleteness/independence/non-axiomatizability results or to provide an alternative semantics with respect to which general completeness results can be achieved. It is beyond the scope of the present paper to provide even a short outline of the different approaches, it would imply the introduction of notions and techniques from category theory which we have been trying to dispense of. We refer the interested reader to S.Ghilardi [27] and [28], P. Skvortsov and V.Shehtman [70] and [71]. As an introduction to these topics, see Corsi and Ghilardi [16].

§2. Counterpart semantics and modal languages with types. By $\mathcal{L}^t$ we denote a modal language with types whose variables are all of one sort. The *alphabet* of $\mathcal{L}^t$ contains the unary connective $\Box$ in addiction to the boolean connectives and quantifiers. We will take as primitive $\neg$ (not), $\vee$ (or) and $\exists$ (there is). Moreover $\mathcal{L}^t$ contains a countable set, *Var*, of variables, $x_1, x_2, x_3, \dots$, the identity symbol $=$, and the following three sets, at most countable, J, F and P:

J is a set of individual constants $i, j, i_1, j_1, i_2, j_2, \dots$,
F is a set of function symbols, $f^n, g^n, h^n, \dots$, of arity n, $1 \leq n < \omega$,
P is a set of predicate symbols, $P^n, Q^n, R^n, \dots$ of arity n, $0 \leq n < \omega$.

Given the alphabet of $\mathcal{L}^t$, we can already say what a model for $\mathcal{L}^t$ is.

Counterpart frames and models for $\mathcal{L}^t$

A *counterpart frame* $\mathcal{F}$ for $\mathcal{L}^t$ is a quadruple $\langle W, R, D, \mathfrak{C}\rangle$, where $W \neq \emptyset$, $R \subseteq W^2$, D is a domain function such that D_w is a set for every $w \in W$ and $\mathfrak{C}$ is the counterpart relation such that $\mathfrak{C} = \biguplus_{w,v \in W}\{\mathfrak{C}_{\langle w,v\rangle}\}$, where for any $w, v \in W$ such that wRv, $\mathfrak{C}_{\langle w,v\rangle} \subseteq (D_w \times D_v)$.

A *counterpart model* $\mathcal{M}$ for $\mathcal{L}^t$ is a pair $\langle \mathcal{F}, I\rangle$, where $\mathcal{F}$ is a counterpart frame and I is a function that for each $w \in W$ determines an *interpretation function* I_w such that:

- for any predicate symbol P^n of $\mathcal{L}^t$, $I_w(P^n) \subseteq (D_w)^n$,
- $I_w(=) = \{\langle a, a\rangle : a \in D_w\}$,
- for any individual constant i of $\mathcal{L}^t$, $I_w(i) \in D_w$
- for any function symbol f^n of $\mathcal{L}^t$, $I_w(f^n) : (D_w)^n \to D_w$

Whenever $\mathcal{M} = \langle \mathcal{F}, I\rangle$, $\mathcal{M}$ is said to be *based on* $\mathcal{F}$. Interpretation functions in counterpart models are *local* interpretation functions in the sense that the interpretation at w of, say, a unary predicate is a subset of D_w and not as in Kripke's original semantics[24] a subset of the universe, i.e. the union of all domains. Hence counterpart models can be viewed as families of classical Tarskian models endowed with a (binary) relation R among the models of the family and a (binary) relation $\mathfrak{C}$ among the individuals of the domains of those models.

Towards finitary assignments

Before introducing the notions of well formed formula (wff) and of truth of a wff at a world an important point needs to be clarified. If the free variables occurring in a formula A are $x_1, \dots, x_m$, then the values of any assignment function σ relevant to A are those for $x_1, \dots, x_m$, and, as we will see, only counterparts of $\sigma(x_1), \dots, \sigma(x_m)$ should play a role in the

[24] See Kripke, [46].

satisfaction of A under σ. Consequently the notion of satisfaction with respect to an assignment function σ should be replaced by the notion of satisfaction with respect to a finite list $\sigma(x_1), \ldots, \sigma(x_m)$ of elements of the domain under consideration, thus: ${}^{\sigma} \models_w A(x_1, ..., x_m)$ should be replaced by $\langle \sigma(x_1), \ldots, \sigma(x_m) \rangle \models_w A(x_1, ..., x_m)$.

But observe that some care needs to be exercised since the number of the free variables occurring in a formula can be either greater than or less than the number of the free variables occurring in its subformulas. Just consider the formulas

$$P(x) \wedge Q(x, y)$$
$$\forall x Q(x, y).$$

One can even ask whether Boolean combinations of formulas with a different number of free variables do make sense. "Actually the (binary) propositional operators can only meaningfully be applied to (pairs of) relations having the same free variables. This may seem to prohibit such combinations as

$(*)$ $\quad A(x, y) \wedge A(y, z) \rightarrow A(x, z).$

Consider the actual meaning: A denotes some subobject of the square X^2 of some sort X, and $(*)$ denotes a certain subobject of the cube X^3."[25]

Now, let the projection functions $\pi_k^m : X^m \rightarrow X, 1 \leq k \leq m$, such that

$$\pi_k^m(u_1, ..., u_m) = u_k,$$

be at our disposal. Then $(*)$ is nothing but a shorthand for

$$A(\pi_1^3(x, y, z), \pi_2^3(x, y, z)) \wedge A(\pi_2^3(x, y, z), \pi_3^3(x, y, z)) \rightarrow A(\pi_1^3(x, y, z), \pi_3^3(x, y, z));$$

hence all three variables x, y and z *occur* either 'implicitly' or 'explicitly' in all subformulas of $(*)$. As a consequence, $(*)$ and all of its subformulas are satisfied or not satisfied, as the case may be, by triples of elements of the universe X. Again, if we consider the following list of formulas:

$$P(\pi_1^1(x))$$
$$P(\pi_1^2(x, y))$$
$$P(\pi_1^3(x, y, z))$$
$$\vdots$$

we recognize at once that all of them say the very same thing 'x is P', but they differ as to the number of free variables occurring in them, consequently they will be satisfied or not satisfied by m-tuples of the universe:

$$\langle a_1 \rangle \models_w P(\pi_1^1(x))$$
$$\langle a_1, a_2 \rangle \models_w P(\pi_1^2(x, y))$$
$$\langle a_1, a_2, a_3 \rangle \models_w P(\pi_1^3(x, y, z))$$

[25] See Lawvere [49].

$\vdots$

In order to see clearly right at the beginning the kind of problems we are going to encounter and discuss, we state in advance what the role of the counterpart relation is with respect to the satisfaction of modalized open formulas. The idea is that:

"an individual a existing at a world w satisfies at w the formula $\Box P(x)$ iff every counterpart a^* of a in any accessible world v, satisfies $P(x)$". In symbols:

$$\langle a \rangle \models_w \Box P(x)$$
iff for every v such that wRv and for every counterpart a^* of a in D_v,
$$\langle a^* \rangle \models_v P(x).$$

Therefore only the worlds where counterparts of a do exist are taken into account: to know if Mary satisfies now the open formula "x is necessarily good" we need to consider all and only the accessible worlds in which Mary or her counterparts exist. Analogously,

"an individual a of a world w satisfies at w a formula such as $\Diamond P(x)$ iff at least a counterpart a^* of a in an accessible world v, satisfies $P(x)$", in symbols:

$$\langle a \rangle \models_w \Diamond P(x)$$
iff for some v such that wRv and some counterpart a^* of a in D_v,
$$\langle a^* \rangle \models_v P(x).$$

Contrary to what happens in classical logic, we cannot expect that

$$\langle a_1 \rangle \models_w \Box P(\pi^1_1(x)) \qquad \text{iff} \qquad \langle a_1, a_2 \rangle \models_w \Box P(\pi^2_1(x,y))$$

for in the case of $\Box P(\pi^1_1(x))$ we take into consideration all the accessible worlds whose domains contain counterparts of a_1, whereas, in the case of $P(\pi^2_1(x,y))$ we take into consideration only the accessible worlds where there exist counterparts of both a_1 and a_2, and these, in general, are fewer. In details,

> $\langle a_1 \rangle \models_w P(\pi^1_1(x))$ iff for all v such that wRv and all counterparts a_1^* of a_1 in D_v, $\langle a_1^* \rangle \models_v P(\pi^1_1(x))$
>
> and
>
> $\langle a_1, a_2 \rangle \models_w P(\pi^2_1(x,y))$ iff for all v such that wRv and all counterparts a_1^* of a_1 and of a_2^* of a_2 in D_v, $\langle a_1^*, a_2^* \rangle \models_v P(\pi^2_1(x,y))$.

If we want to pursue the idea that only the worlds where an individual a exists are relevant to determining the modal properties of a, then otherwise valid inferences turn out to be no longer valid.

$$\frac{\langle a, b \rangle \models_w \Box Q(x,y) \qquad \langle a, b \rangle \models_w \Box (Q(x,y) \to D(y))}{\langle b \rangle \models_w \Box D(y)}$$

Suppose it is true in w that "a always quarrels with b" and that "every time that a quarrels with b, then b gets angry". From this it doesn't follows that "b is always angry", for b may not be angry in those worlds where a is absent.[26]

This leads us to maintain that each formula must contain information about the length of the lists of elements with respect to which it has to be evaluated, or, which is the same thing, the variables that occur in it (either explicitly or implicitly). As we shall see, this information is codified by a natural number, called the *type* of the formula. Briefly, a wff A is of type n iff the free variables occurring in A are $x_1, \ldots, x_n$ and consequently n-tuples of individuals do or do not satisfy A in a given world. The type of a wff can be seen as the *context* with respect to which the formula is meaningful. Undoubtedly, the formula expressing the transitivity of a relation is meaningful with respect to triples of individuals even if its proper subformulas contain (explicitly) only two variables. In classical logic we assume that the context with respect to which formulas are meaningful is the same for every formula: ω-sequences of elements of the domain. It is worth noticing that infinitary assignments are controversial in classical logic also. We read at p. 5 of Lawvere, [49]: "This traditional method (which by the way is probably one of the reasons why most mathematicians feel that a logical presentation of a theory is an absurd machine strangely unrelated to the theory or its subject matter) consists of declaring that there is one set I of variables on which all finitary relations depend, albeit vacuously on most of them; e.g. a binary relation on X is interpreted as $X^I \to 2$ depending vacuously on all but two of the variables in I."

Finitary assignments are functions defined on initial segments of the sequence $x_1, x_2, x_3, x_4, \ldots$ of the variables of $\mathcal{L}^t$. We take finitary assignments to be worldbound, so for each $w \in W$, we consider all the functions $\sigma : \{x_1, \ldots, x_m\} \to D_w$, $m \geq 1$. Consequently finitary assignments relative to a world w are just m-tuples $\langle a_1, ..., a_m \rangle$, $m \geq 1$, of elements of D_w, where a_i is the value for x_i.

Types and terms

Terms

1. For any individual variable x_i and $n \geq i$,
$$x_i^n$$
is a term of type n,
2. For any individual constant j and $n \geq 0$,
$$j^n$$

[26] See [31], p.78.

is a term of type n,

3. For any k-tuple of terms of type n, $t_1, ..., t_k$, and function symbol f^k of arity k,
$$f^k(n : t_1, ..., t_k)$$
is a term of type n,
4. Nothing else is a term.

Because of clause 1., an individual variable determines countably many terms. x_i^n can be seen, from an intuitive point of view, as the projection function π_i^n applied to $x_1, \ldots, x_n$ or as the variable x_i in the context $x_1, \ldots, x_n$. In clause 3. the number n at the head of the arguments is a reminder of the type of the terms $t_1, \ldots, t_k$. Clause 3. could have been phrased as

3'. For any k-tuple of terms of type n, $t_1^n, \ldots, t_k^n$ and function symbol f^k of arity k,
$$f^k(t_1^n, ..., t_k^n)$$
is a term of type n

We prefer to indicate the type of the arguments of a function only once. If we had function symbols of arity 0 instead of individual constants, then via clause 3, $f^0(n :)$ would have been a term for any n: this explains why individual constants have type greater than 0. We see at once from the definitions above that the type of a term is greater than or equal to the maximum index of variables occurring in it and then tells us what must be the length of the assignments 'entitled' to interpret such a term. In general, given a term t of type n, there are infinitely many terms that differ from t only as far as the type is concerned. By an easy induction, it can be shown that if t is a term of type n, then by replacing the type n with any m, $m > n$, we get a (well formed) term of type m.

m-tuples of terms

For any n and m-tuple of terms of type n, $s_1, \ldots, s_m$,
$$\langle n : s_1, ..., s_m \rangle$$
is a *complex term.* $\langle n : s_1, ..., s_m \rangle$ is said to be of type $n \to m$ or 'from type n to type m'.

For every n, the empty list of terms of type n, $\langle n : \rangle$ is a term of type $n \to 0$. Lists of terms $s_1, \ldots, s_m$ each of type n are in 1-1 correspondence with the complex term $\langle n : s_1, ..., s_m \rangle$; therefore the expressions 'the complex term $\langle n : s_1, ..., s_m \rangle$' and 'the terms $s_1, \ldots, s_m$ (in this order) of type n' are taken as interchangeable. To avoid a plethora of indices, we use the following conventions: we write $f^k(n : x_{i_1}, ..., x_{i_k})$ instead of $f^k(n : x_{i_1}^n, ..., x_{i_k}^n)$ and $\langle n : x_{i_1}, ..., x_{i_k} \rangle$ instead of $\langle n : x_{i_1}^n, ..., x_{i_k}^n \rangle$; $f^k(x_{i_1}, ..., x_{i_k})$

instead of $f^k(h : x_{i_1}, ..., x_{i_k})$ and $\langle x_{i_1} ... x_{i_k} \rangle$ instead of $\langle h : x_{i_1} ... x_{i_k} \rangle$, if $h = max\{i_1 \dots i_k\}$, moreover we write $\langle n : j \rangle$ instead of $\langle n : j^n \rangle$, $\langle j \rangle$ instead of $\langle 0 : j \rangle$, $\langle \, \rangle$ instead of $\langle 0 : \rangle$.

Explicit substitutions

As it is well known, substitution is an operation from terms and formulas to formulas which is defined by recursion on the length of the formulas. It commutes with respect to connectives and quantifiers and is performed on the atomic formulas: $(P(x_1) \wedge R(x_1))[x_1/x_2]$ stands for $(P(x_2) \wedge R(x_2))$. Instead of taking substitution as a defined operation, we can take it as a primitive logical operation, and state, e.g., that $P(x_1)[x_1/x_2]$ is the *molecular* formula obtained by applying the operation of substitution to $P(x_1)$ and x_2. In $P(x_1)[x_1/x_2]$ substitution is indicated, not performed: $P(x_1)[x_1/x_2]$ does not stand for $P(x_2)$ and we need a special axiom in order to state the equivalence between $P(x_1)[x_1/x_2]$ and $P(x_2)$. We believe that in modal contexts many features of the operation of substitution become more perspicuous if substitution is taken as a primitive logical operation. The notion of explicit substitutions has become quite central in the theory of λ-calculus since the work of Abadi *et al.*[27]

Let a formula A containing the variables $x_1, \dots, x_n$ be given. Substitution applies to all the variables $x_1, \dots, x_n$ although vacuously to some (none or all) of them (i.e. x_i will be substituted for x_i itself). Hence n-tuples of terms (of the same type) have to be considered. Moreover if each of the terms $t_1, \dots, t_n$ is of type m, the resulting formula will contain the free variables $x_1, \dots, x_m$ and so it will be of type m. By $\langle m : s_1, ..., s_n \rangle A$ we denote the formula of type m obtained by applying the operation of subsitution to the formula A of type n and to the complex term $\langle m : s_1, ..., s_n \rangle$ of type $m \to n$. We write $\langle m : s_1, ..., s_n \rangle$ at the left of A in the way we do for the operation of negation or quantification. We are now ready to give the definition of a well formed formula.

formulas and types

- If P^n is an n-ary predicate symbol then
$$P^n$$
is a *pure atomic formula* of type n,
- If A is a wff of type n and $\langle m : t_1, ..., t_n \rangle$ is a complex term of type $m \to n$, then
$$\langle m : t_1, ..., t_n \rangle A$$
is a wff of type m,
- If A and B are wffs of type n, then

[27] See [1]. Prof. Per Martin-Löf brought to my attention the literature on explicit substitutions.

$$\neg A, \quad \Box A, \quad A \vee B$$

are wffs of type n.

- If A is a wff of type $n+1$, then
$$\exists x_{n+1} A$$
is a wff of type n,
- Nothing else is a well formed formula.

Pure atomic formulas could be phrased as $P^n(n : x_1, ..., x_n)$. Given a pure atomic formula P^n of type n and a complex term $\langle m : t_1, ..., t_n \rangle$ of type $m \to n$, then $\langle m : t_1, ..., t_n \rangle P^n$ is said to be an *atomic formula* and, following the tradition, it will be written as $P^n(m : t_1, ..., t_n)$.

$\langle m : t_1, ..., t_n \rangle A$ is called a *substituted formula* and the complex term $\langle m : t_1, ..., t_n \rangle$ the *substitution term.* Atomic formulas are substituted formulas. The atomic formulas $Q^2(3 : x_1, x_3)$ and $Q^2(5 : x_1, x_3)$ are different, for they have different types, whereas $Q^2(2 : x_1, x_3)$ is not well formed because the type is less than the maximum index of the free variables.

Binary boolean connectives apply only to formulas of the same type, in accordance with the idea that sets of n-tuples of some universe X, (representing the meanings of formulas of type n) can be intersected or joined only with other sets of n-tuples.

Quantifying reduces the type by one, so from the pure atomic formula P^1 we get $\forall x_1 P^1$ of type 0 and from $Q^2(3 : x_1, x_3)$ we get $\forall x_3 Q^2(3 : x_1, x_3)$ of type 2, from $Q^2(3 : x_1, x_2)$ we get $\forall x_3 Q^2(3 : x_1, x_2)$ of type 2 (vacuous quantification). Note that we can bound only the variable with the maximum index that occurs (implicitly or explicitly) in the formula under consideration: in fact we can bound only the variable whose index coincides with the type of the formula we started with. As a consequence, $\forall x_1 Q^2(2 : x_1, x_2)$ is not well formed. This may appear to be an annoying limitation, but it has the advantage of eliminating any collision between free and bound variables: all bound variables have indices greater than the indices of the free variables.

Lemma 2.1. *If A is a wff of type n, then the free variables occurring in A have index at most n and any quantifier occurring in A bounds variables with index greater than n.*

The type of every formula is readily known, since the type of atomic, quantified and substituted formulas is explicitly indicated and any other formula is either one of these or a combination of them via application of $\neg, \vee$ or $\Box$, and these operators preserve the type of the formulas they apply to.

Interpretation of terms

Terms of type n are interpreted with respect to n-tuples of elements of the domain. It is only natural that to interpret the term x_3^4, we need to consider 4-tuples of elements of the domain and then choose the third component. Analogously a complex term $\langle n : s_1, ..., s_m\rangle$ has to be interpreted with respect to n-tuples $\langle a_1...a_n\rangle$ of elements of the domain. $\langle a_1, ..., a_n\rangle[n : s_1, ..., s_m]$ denotes the m-tuple of elements obtained by interpreting each s_i of type n with respect to $\langle a_1, ..., a_n\rangle$. This is given by the following definition.

Let $\mathcal{M}$ be a $\mathfrak{C}$-model for $\mathcal{L}^t$, $w \in W$ and $\vec{a} = \langle a_1, ..., a_n\rangle \in (D_w)^n$. For any complex term $\langle n : s_1, ..., s_m\rangle$,

$$\langle a_1, ..., a_n\rangle[n : s_1, ..., s_m]_w = \langle\langle a_1, ..., a_n\rangle[n : s_1]_w, \dots, \langle a_1, ..., a_n\rangle[n : s_m]_w\rangle,$$

where $\langle a_1, ..., a_n\rangle[n : s_i]_w$ is defined by induction on the terms:

$$\begin{aligned}
&\langle a_1, ..., a_n\rangle[x^n_i]_w = a_i,\\
&\langle a_1, ..., a_n\rangle[j^n]_w = I_w(j),\\
&\langle a_1, ..., a_n\rangle[f^k(n : t_1, ..., t_k)]_w =\\
&\qquad = I_w(f^k)(\langle a_1, ..., a_n\rangle[n : t_1]_w, \dots, \langle a_1, ..., a_n\rangle[n : t_k]_w).
\end{aligned}$$

Satisfaction in counterpart models

Let $\mathcal{M} = \langle W, R, D, \mathfrak{C}, I\rangle$ be a counterpart model. For any $w \in W$, n-tuple $\langle a_1, ..., a_n\rangle$ of elements of D_w and formula A of type n, we define when $\langle a_1, ..., a_n\rangle$ *satisfies* A *at* w *in* $\mathcal{M}$, $\langle a_1, ..., a_n\rangle \models_w A$. By induction on A.

$\langle a_1, ..., a_n\rangle \models_w P^n$	iff $\langle a_1, ..., a_n\rangle \in I_w(P^n)$
$\langle a_1, ..., a_n\rangle \models_w \langle n : s_1, ..., s_k\rangle B$	iff $\langle a_1, ..., a_n\rangle[n : s_1, ..., s_k]_w \models_w B$
$\langle a_1, ..., a_n\rangle \models_w \neg C$	iff $\langle a_1, ..., a_n\rangle \not\models_w C$
$\langle a_1, ..., a_n\rangle \models_w C \vee D$	iff $\langle a_1, ..., a_n\rangle \models_w C$ or $\langle a_1, ..., a_n\rangle \models_w D$
$\langle a_1, ..., a_n\rangle \models_w \exists x_{n+1} G$	iff for some $b \in D_w$, $\langle a_1, ..., a_n, b\rangle \models_w G$
$\langle a_1, ..., a_n\rangle \models_w \Box C$	iff for all v such that wRv and for all counterparts $a^*_1, ..., a^*_n$ in D_v of $a_1, ..., a_n$, respectively, $\langle a^*_1, ..., a^*_n\rangle \models_v C$.

¿From the given definitions it readily follows that

$\langle a_1, ..., a_n\rangle \models_w \forall x_{n+1} G$	iff for all $b \in D_w$, $\langle a_1, ..., a_n, b\rangle \models_w G$
$\langle a_1, ..., a_n\rangle \models_w \Diamond C$	iff for some v such that wRv and some counterparts $a^*_1, \dots, a^*_n$ in D_v of $a_1, \dots, a_n$, respectively, $\langle a^*_1, ..., a^*_n\rangle \models_v C$.

Truth and validity

A formula A of type n is *true at* w *in* $\mathcal{M}$, $\mathcal{M} \models^n_w A$, iff for any n-tuple $a_1, \dots, a_n$ of elements of D_w, $\langle a_1, ..., a_n\rangle \models_w A$. A formula A of type n is *valid on* $\mathcal{M}$, $\mathcal{M} \models^n A$, iff $\mathcal{M} \models^n_w A$ for all $w \in W$. A formula A of type n is *valid on a* $\mathfrak{C}$*-frame* $\mathcal{F}$, $\mathcal{F} \models^n A$, iff $\mathcal{M} \models^n A$ for every model $\mathcal{M}$ based on $\mathcal{F}$. A formula A of type n is $\mathfrak{C}$*-valid* iff $\mathcal{F} \models^n A$, for all counterpart frames $\mathcal{F}$.

Propositions are formulas of type 0, and so they will be satisfied or not satisfied, as the case may be, by 0-tuple, $\langle\rangle$. For example, if P is a unary predicate symbol and i is an individual constant of type 0, then $\langle\rangle \models_w P(i)$ iff $\langle\rangle \models_w \langle i\rangle P$ iff $\langle\rangle[i] \models_w P$ iff $\langle I_w(i)\rangle \models_w P$ iff $I_w(i) \in I_w(P)$.

Composition

Let the complex term $\langle m : s_1, ..., s_n\rangle$ of type $m \to n$ be given. The operation of composition with terms of type n is defined so:

- if x_i^n is a term of type n, then
$$\langle m : s_1, ..., s_n\rangle \circ x_i^n = \langle m : s_i\rangle,$$
- if j^n is a term of type n, then
$$\langle m : s_1, ..., s_n\rangle \circ j^n = j^m,$$
- if $f^k(n : t_1, ..., t_k)$ is a term of type n, then
$\langle m : s_1, ..., s_n\rangle \circ f^k(n : t_1, ..., t_k) = f^k(m : \langle m : s_1, ..., s_n\rangle \circ t_1, \ldots, \langle m : s_1, ..., s_n\rangle \circ t_k)$

Composition of complex terms

For any pair of complex terms $\langle m : s_1, ..., s_n\rangle$ of type $m \to n$ and $\langle n : t_1, ..., t_k\rangle$ of type $n \to k$,

- $\langle m : s_1, .., s_n\rangle \circ \langle n : t_1, .., t_k\rangle = \langle m : \langle m : s_1, .., s_n\rangle \circ t_1, ..., \langle m : s_1, .., s_n\rangle \circ t_k\rangle$.

Composition of terms is nothing but the operation of substitution of terms for individual variables. Example:
$\langle 3 : x_2, g^2(x_1, x_3)\rangle \circ f^3(2 : x_1, x_2, x_1) = f^3(3 : \langle 3 : x_2, g^2(x_1, x_3)\rangle \circ x_1, \langle 3 : x_2, g^2(x_1, x_3)\rangle \circ x_2, \langle 3 : x_2, g^2(x_1, x_3)\rangle \circ x_1) = f^3(3 : x_2, g^2(x_1, x_3), x_2)$.

When the type of a term or of a formula can be easily and unambiguously calculated, we often omit it.

Lemma 2.2. *Let $\mathcal{M}$ be a counterpart model for $\mathcal{L}^t$, $\langle a_1, ..., a_m\rangle \in (D_w)^m$, and $\langle m : s_1, ..., s_n\rangle$ be a complex term of type $m \to n$.*

1. *If t a term of type n, then*
$(\langle a_1, ..., a_m\rangle[s_1, ..., s_n]_w)[t]_w = \langle a_1, ..., a_m\rangle[\langle s_1, ..., s_n\rangle \circ t]_w$.
2. *If $\langle n : t_1, ..., t_k\rangle$ be a complex term of type $n \to k$, then*
$$(\langle a_1, ..., a_m\rangle[s_1, ..., s_n]_w)[t_1, ..., t_k]_w = \\ = \langle a_1, ..., a_m\rangle[\langle s_1, ..., s_n\rangle \circ \langle t_1, ..., t_k\rangle]_w.$$

PROOF (1.) By induction on the length of t.

As a consequence of the property that links composition and finitary assignments, stated in the lemma above, substitution with respect to substituted formulas (and so also with respect to atomic formulas) amounts to composition of terms. In fact the following equivalence is $\mathfrak{C}$-valid:

$$S^S \qquad \langle m : t_1, ..., t_n\rangle(\langle n : s_1, ..., s_k\rangle A) \leftrightarrow (\langle m : t_1, ..., t_n\rangle \circ \langle n : s_1, ..., s_k\rangle)A.$$

To wit: $\langle a_1, ..., a_m\rangle \models_w \langle m : t_1, ..., t_n\rangle(\langle n : s_1, ..., s_k\rangle A)$

iff $\langle a_1, ..., a_m\rangle[m : t_1, ..., t_n]_w \models_w \langle n : s_1, ..., s_k\rangle A$
iff $(\langle a_1, ..., a_m\rangle[m : t_1, ..., t_n]_w)[n : s_1, ..., s_k]_w \models_w A$
iff $\langle a_1, ..., a_m\rangle[\langle m : t_1, ..., t_n\rangle \circ \langle n : s_1, ..., s_k\rangle]_w \models_w A$
iff $\langle a_1, ..., a_m\rangle \models_w (\langle m : t_1, ..., t_n\rangle \circ \langle n : s_1, ..., s_k\rangle)A.$

Note that substitution is responsible for 'identifying variables' or 'changing types'. To take very simple examples, $R^3(3 : x_1, x_1, x_3)$ and $R^3(4 : x_1, x_2, x_3)$ are both substituted formulas.

SUBSTITUTION AND LOGICAL OPERATORS

Substitution commutes with respect to Boolean connectives:

$$S^{\neg} \qquad \langle m : t_1, ..., t_n\rangle(\neg C) \leftrightarrow \neg\langle m : t_1, ..., t_n\rangle C,$$
$$S^{\vee} \qquad \langle m : t_1, ..., t_n\rangle(C \vee D) \leftrightarrow \langle m : t_1, ..., t_n)C \vee (\langle m : t_1, ..., t_n\rangle D$$

are both $\mathfrak{C}$-valid.

When we came to quantifiers, things are a bit more complicated. Substitution with respect to quantified formulas presents delicate problems even for languages without types. In that context, there are two main ways to approach the problem. According to the first one, $\exists xA(x)[y/t]$ is taken to be equal to $\exists x(A(x)[y/t])$ provided "the term t is free for y in $\exists xA$", meaning that t does not contain variables that may fall under the scope of a quantifier once it is substituted for y in A, and only in that case can substitution be actually performed. According to the second approach, $\exists xA(x)[y/t]$ denotes the formula obtained from $\exists xA(x)$, by first taking a bound alphabetic variant $\exists xA(x)^*$ in such a way that all bound variables in $\exists xA(x)^*$ are different from the free variables occurring in $\exists xA(x)$ and from the variables occurring in t and then substituting t for all the occurences of y in $\exists xA(x)^*$. In the presence of types, it is this second approach that seems more natural. Let us start with an example: let the formula $\exists x_2Q^2(2 : x_2, x_1)$ be given and let us consider the substituted formula $\langle 2 : x_2\rangle\exists x_2Q^2(2 : x_2, x_1)$. What we intend to do is to substitute x_2 for x_1 in $Q^2(2 : x_2, x_1)$ without the occurrence of x_2 falling under the scope of the existential quantifier. A possible way out is to rename the bound variable, so obtaining $\exists x_3Q^2(3 : x_3, x_1)$, and then to substitute x_2 for x_1. Let us first show that $\langle 2 : x_2\rangle\exists x_2 Q^2(2 : x_2, x_1)$ is $\mathfrak{C}$-equivalent to $\exists x_3 Q^2(3 : x_3, x_2)$.[28]

$$
\begin{array}{ll}
\langle a_1, a_2\rangle \models_w \langle 2 : x_2\rangle\exists x_2 Q(2 : x_2, x_1) & \text{iff}\\
\langle a_1, a_2\rangle[2 : x_2] \models_w \exists x_2 Q(2 : x_2, x_1) & \text{iff}\\
\langle a_2\rangle \models_w \exists x_2 Q(2 : x_2, x_1) & \text{iff}\\
\text{for some } b \in D_w\ \langle a_2, b\rangle \models_w Q(2 : x_2, x_1) & \text{iff}\\
\text{for some } b \in D_w\ \langle a_1, a_2, b\rangle[3 : x_2, x_3] \models_w Q(2 : x_2, x_1) & \text{iff}\\
\text{for some } b \in D_w\ \langle a_1, a_2, b\rangle \models_w \langle 3 : x_2, x_3\rangle Q(2 : x_2, x_1) & \text{iff}\\
\langle a_1, a_2\rangle \models_w \exists x_3(\langle 3 : x_2, x_3\rangle Q(2 : x_2, x_1)) & \text{iff}\\
\langle a_1, a_2\rangle \models_w \exists x_3 Q(3 : x_3, x_2). &
\end{array}
$$

In general, it holds that

$\langle a_1, ..., a_k\rangle \models_w \langle k : s_1, ..., s_n\rangle\exists x_{n+1} C$ iff $\langle a_1, ..., a_k\rangle \models_w \exists x_{k+1}(\langle k+1 : s_1, ..., s_n, x_{k+1}\rangle C)$.

Consequently the following formula is $\mathfrak{C}$-valid:

$S^{\exists}$ $\quad\langle k : s_1, ..., s_n\rangle\exists x_{n+1} C \leftrightarrow \exists x_{k+1}(\langle k+1 : s_1, ..., s_n, x_{k+1}\rangle C)$.

Analogously,

$S^{\forall}$ $\quad\langle k : s_1, ..., s_n\rangle\forall x_{n+1} C \leftrightarrow \forall x_{k+1}(\langle k+1 : s_1, ..., s_n, x_{k+1}\rangle C)$

is $\mathfrak{C}$-valid.

In conclusion substitution commutes with the quantifiers, but requires, in general, the renaming of bound variables. E.g., given $\exists x_2\forall x_3\langle x_1, x_2, x_3\rangle B$, then

$$
\begin{array}{rcl}
\langle x_4\rangle\exists x_2\forall x_3\langle x_1, x_2, x_3\rangle B & \leftrightarrow & \exists x_5\langle x_4, x_5\rangle\forall x_3\langle x_1, x_2, x_3\rangle B\\
& \leftrightarrow & \exists x_5\forall x_6\langle x_4, x_5, x_6\rangle(\langle x_1, x_2, x_3\rangle B)\\
& \leftrightarrow & \exists x_5\forall x_6\langle x_4, x_5, x_6\rangle B
\end{array}
$$

As to substitution and modalities, note first that a simple syntactical tool is at our disposal to distinguish between *de re* and *de dicto* modalities. Here are two classical examples: $\langle i\rangle\Box P$ reads 'i is necessarily P' , whereas $\Box\langle i\rangle P$, (or, which is the same, $\Box P(i)$) reads 'it is necessary that i is P'. Of course this distinction can be just as well achieved by adding the λ-abstraction operator as in R. Thomason & R.Stalnaker, [76] or Fitting, [20]. $(\lambda x\Box P(x))(i)$ and $\Box[(\lambda x.P(x))(i)]$ are parallel to $\langle i\rangle(\Box P(x_1))$ and $\Box(\langle i\rangle P(x_1))$, respectively.

There is an intuitive sense according to which the truth conditions for $\langle i\rangle\Box P$ are different from those for $\Box P(i)$: in one case a 'necessitive'[29] property is asserted of an individual, in the other, the necessity of a sentence is asserted. In counterpart semantics we do justice to this difference in the following obvious way: in the first case, first we interpret i in the actual world (or the world we are in) and then see if all its counterparts in

[28] By this we mean that their equivalence is $\mathfrak{C}$-valid.

[29] The non-standard term 'necessitive' is in analogy with 'negative' or 'disjunctive' and was suggested to me by Tor Sandqvist.

all accessible worlds do satisfy the property P, in the second case, we first consider all worlds accessible from the actual one and then check if the interpretation of i in those worlds satisfies P. This semantical analysis parallels that of Fitting:[30] "In short, there are two basic actions: letting i designate, and moving to an alternative world. These two actions commute only if i is a rigid designator. Ordinary first-order modal syntax has no machinery to distinguish the two alternative readings of $\Box P(i)$. Consequently whenever non-rigid designators have been treated, one of the readings has been disallowed, thus curtailing expressive power.". According to Fitting, if i is a rigid designator then $\langle i\rangle\Box P \leftrightarrow \Box P(i)$ holds, or, in his notation, the equivalence $\lambda x(\Box P(x))(i) \leftrightarrow \Box[\lambda x.P(x)(i)]$ holds. We are going to disagree on this point, for we shall show that the failure of the equivalence $\langle i\rangle\Box P \leftrightarrow \Box P(i)$ does not depend on i being a non-rigid designator: in counterpart semantics this equivalence does not hold for rigid designators either.

Rigid designators

What can it mean for a constant i to be a rigid designator in counterpart semantics? Simply, that the following *rigidity condition* holds:

if wRv then the interpretation of i in v is a counterpart of the interpretation of i at w.

Formally, we say that an interpretation function I satisfies the *rigidity condition* iff

if wRv then $I_w(i)\mathfrak{C}I_v(i)$, and moreover
for all $a_1, \ldots, a_n \in D_w$ and all $b_1, \ldots, b_n \in D_v$, if $a_1\mathfrak{C}b_1$ and $\ldots$ and $a_n\mathfrak{C}b_n$ then $(I_w(f^n))(a_1, ..., a_n)\mathfrak{C}(I_v(f^n))(b_1, ..., b_n)$.

If i is a rigid designator, then $\langle i\rangle\Box P \to \Box P(i)$ is $\mathfrak{C}$-valid, whereas $\Box P(i) \to \langle i\rangle\Box P$ admits of countermodels. The $\mathfrak{C}$-validity of $\langle i\rangle\Box P \to \Box P(i)$ is shown as follows. $\langle\rangle \models_w \langle i\rangle\Box P$ iff $\langle I_w(i)\rangle \models_w \Box P$ iff for all v such that wRv and all counterparts c of $I_w(i)$ in D_v, $\langle c\rangle \models_v P$. Since i is a rigid designator, $I_v(i)$ is a counterpart of $I_w(i)$ in D_v, consequently for all $v.wRv$. $\langle I_v(i)\rangle \models_v P$, hence for all $v.wRv$. $\langle\,\rangle \models_v \langle i\rangle P$, and so $\langle\,\rangle \models_w \Box\langle i\rangle P$.
A countermodel for $\Box P(i) \to \langle i\rangle\Box P$ can be readily constructed: assume that v is the only world related to w and that $I_v(i) \in I_v(P)$, so for all $v.\,wRv.\,I_v(i) \in I_v(P)$, hence for all $v.\,wRv.\,\langle\rangle \models_v P(i)$, therefore $\langle\rangle \models_w \Box P(i)$. Assume moreover that $I_w(i)$ has two distinct counterparts in v,

[30]Fitting, [20], p.114.

namely $I_v(i)$ and c and that $c \notin I_v(P)$, consequently $\langle I_v(i)\rangle \not\models_w \Box P$, hence $\langle\rangle \not\models_w \langle i\rangle\Box P$.

The regidity condition for terms corresponds to the $\mathfrak{C}$-validity of

$$(R) \qquad \langle k : s_1, ..., s_n\rangle\Box B \to \Box\langle k : s_1, ..., s_n\rangle B$$

As we shall see, the so called anomalies of first-order modal logic do not depend at all on the assumption that constants are rigid designators (and everywhere-denoting, in accord to the given definition of the interpretation function). A natural approach would be that of taking two disjoint sets of individual constants, the rigid and the non-rigid ones and stating condition (R) only for the rigid ones. Observe that the two conditions together, denotation in every world and rigidity, make the behaviour of individual constants rather inadequate for a semantics intended to capture features of natural languages: an individual a of a particular domain D_w that happens to be "named", say by i, never dies, for in any accessible world v, it will have at least a counterpart, namely the interpretation of i at v.

Variables are rigid designators, in the sense that

$$(S^\Box) \qquad \langle k : x_{i_1}, ..., x_{i_n}\rangle\Box B \to \Box\langle k : x_{i_1}, ..., x_{i_n}\rangle B$$

is $\mathfrak{C}$-valid. In fact,

$\langle a_1, ..., a_k\rangle \models_w \langle k : x_{i_1}, ..., x_{i_n}\rangle\Box B$ iff
$\langle a_1, ..., a_k\rangle[k : x_{i_1}, ..., x_{i_n}] \models_w \Box B$ iff
$\langle a_{i_1}, ..., a_{i_n}\rangle \models_w \Box B$ iff
$\langle a^*_{i_1}, ..., a^*_{i_n}\rangle \models_v B$, for all v such that wRv and for all counterparts $a^*_{i_1}, \ldots, a^*_{i_n} \in D_v$ of $a_{i_1}, \ldots, a_{i_n}$, respectively, ONLY IF
$\langle a^*_1, ..., a^*_k\rangle[k : x_{i_1}, ..., x_{i_n}] \models_v B$ for all v such that wRv and for all counterparts $a^*_1, \ldots, a^*_k \in D_v$ of $a_1, \ldots, a_k$, respectively, iff
$\langle a^*_1, ..., a^*_k\rangle \models_v \langle k : x_{i_1}, ..., x_{i_n}\rangle B$ for all v such that wRv and for all counterparts $a^*_1, \ldots, a^*_k \in D_v$ of $a_1, \ldots, a_k$, respectively, iff
$\langle a_1, ..., a_k\rangle \models_w \Box\langle k : x_{i_1}, ..., x_{i_n}\rangle B$.

¿From the proof above we easily see that the $\mathfrak{C}$-validity of $(S^\Box)$ depends on the fact that the set of worlds where counterparts of $\langle a_1, ..., a_k\rangle[k : x_{i_1}, ..., x_{i_n}]$ exist includes the set of worlds were counterparts of $a_1 \ldots a_k$ exist. The vice-versa doesn't hold, and this explains the 'only if'.

We will come back to this principle when we will discuss the converse of the Barcan formula (CBF) which turns out to imply instances of $S^\Box$, see lemma 3.4.

Converse of rigidity

$$(CR) \qquad \Box\langle k : s_1, ..., s_n\rangle B \to \langle k : s_1, ..., s_n\rangle \Box B,$$

where $\langle k : s_1, ..., s_n\rangle : k \to n$.

This formula is a critical one and has different meanings depending on the complex term $\langle k : s_1, ..., s_n\rangle$.

- The simplest case is when $\langle k : s_1, ..., s_n\rangle$ is the identical substitution $\langle n : x_1, ..., x_n\rangle$ of type $n \to n$; it is easy to see that $\Box(\langle n : x_1, ..., x_n\rangle B) \to \langle n : x_1, ..., x_n\rangle(\Box B)$ is $\mathfrak{C}$-valid. More generally, let $\langle n : x_{i_1}, ..., x_{i_n}\rangle$ be a permutation of $\langle n : x_1, ..., x_n\rangle$, then
  $(P) \qquad \Box\langle n : x_{i_1}, ..., x_{i_n}\rangle B \to \langle n : x_{i_1}, ..., x_{i_n}\rangle \Box B$
  is $\mathfrak{C}$-valid.
- Suppose now that $\langle k : s_1, ..., s_n\rangle$ is $\langle n+1 : x_1, ..., x_n\rangle$ of type $n+1 \to n$, then as we shall prove in 2.3, the $\mathfrak{C}$-validity of
  $(D) \qquad \Box\langle n+1 : x_1, ..., x_n\rangle B \to \langle n+1 : x_1, ..., x_n\rangle \Box B,$
  is equivalent to the assumption that the counterpart relation is everywhere defined.

  A countermodel to D is readily constructed. Take the formula $\langle 2 : x_1\rangle P$ and consider the following model $\mathcal{M} = \langle W, R, \mathfrak{C}, I\rangle$ where $W = \{w, v\}$, wRv, $D_w = \{a, b\}$, $D_v = \{a^*\}$, $a\mathfrak{C}a^*$, $I_v(P) = \emptyset$. It follows that $\langle a, b\rangle \models_w \Box\langle 2 : x_1\rangle P$ iff for all v such that wRv and for all counterparts $a^*, b^* \in D_v$ of a and b, respectively, $\langle a^*, b^*\rangle \models_v \langle 2 : x_1\rangle P$, but this is trivially so because in v there are no counterparts of b. On the other hand, $\langle a, b\rangle \models_w \langle 2 : x_1\rangle \Box P$ iff $\langle a\rangle \models_w \Box P$ iff for all v such that wRv and for all counterparts $a^* \in D_v$ of a, $\langle a^*\rangle \models_v P$, but this condition is not met, since $a^* \notin I_v(P)$.
- Let $\langle k : s_1, ..., s_n\rangle$ be $\langle 0 : i\rangle$ of type $0 \to 1$. $\Box\langle 0 : i\rangle B \to \langle 0 : i\rangle \Box B$ can be falsified on counterpart models because the mere fact that the interpretation of i in all related worlds does fulfil the property B, is no guarantee that all counterparts of the interpretation of i in the present world have the property B. The validity of $\Box\langle 0 : i\rangle B \to \langle 0 : i\rangle \Box B$ requires the assumption that $I_w(i)$ has at most one counterpart in any related world v and that the counterpart (if any) in v of $I_w(i)$ coincides with $I_v(i)$. Since in counterpart semantics individual constants are everywhere defined, all this can be expressed by saying that $I_v(i)$ is the only counterpart of $I_w(i)$ in D_v.

Stability

Individual constants $i_1, \ldots, i_n$ of type k are said to be *stable* iff

$$\Box\langle k : i_1, ..., i_n\rangle B \to \langle k : i_1, ..., i_n\rangle \Box B$$

is $\mathfrak{C}$-valid, where B is a wff of type n.

Quantifiers and modalities

The notion of satisfaction for quantified formulas reflects the convention that the outmost quantifier binds the variable with maximum index:

$$\langle a_1, ..., a_n \rangle \models_w \exists x_{n+1} G \quad \text{iff} \quad \text{for some } b \in D_w,\ \langle a_1, ..., a_n, b \rangle \models_w G.$$

The element b comes after $a_1, ..., a_n$ and so it is a value for x_{n+1}.

Once the notion of satisfaction is defined as above, we see that the principles of particular and universal instantiation are $\mathfrak{C}$-valid:

1. $\langle a_1, ..., a_n, a_{n+1} \rangle \models_w G \rightarrow \langle n+1 : x_1, ..., x_n \rangle \exists x_{n+1} G$,
2. $\langle a_1, ..., a_n, a_{n+1} \rangle \models_w \langle n+1 : x_1, ..., x_n \rangle \forall x_{n+1} G \rightarrow G$.

The substitution term $\langle n+1 : x_1, ..., x_n \rangle$ plays the role of increasing by one the type of $\exists x_{n+1} G$ ($\forall x_{n+1} G$) and so of making it equal in type to G. The following forms of the particular and universal instantiation containing terms are $\mathfrak{C}$-valid:

1. $\langle h : \vec{t}, t \rangle A \rightarrow \exists x_{h+1} \langle h+1 : \vec{t}, x_{h+1} \rangle A$,
2. $\forall x_{h+1} \langle h+1 : \vec{t}, x_{h+1} \rangle A \rightarrow \langle h : \vec{t}, t \rangle A$,

where $\vec{t}$ is the n-tuple $t_1, ..., t_n$ and A is of type $n+1$.

The Barcan formula and the converse of the Barcan formula

$$(BF) \qquad \forall x_{n+1} \Box A \rightarrow \Box \forall x_{n+1} A,$$
$$(CBF) \qquad \Box \forall x_{n+1} A \rightarrow \forall x_{n+1} \Box A.$$

The Barcan formula is not $\mathcal{C}$-valid unless, as we shall see in lemma 2.4, the counterpart relation is surjective. This property paralles exactly the fact that in Kripke semantics BF is valid if it is assumed that $D_v \subseteq D_w$ whenever wRv. On the contrary, its converse $\Box \forall x_{n+1} A \rightarrow \forall x_{n+1} \Box A$ seems to be uncontroversial and unassuming. Let us consider CBF in the form CBF^*: $\exists x_{m+1} \Diamond B \rightarrow \Diamond \exists x_{m+1} B$. If someone existing now will be B in a related world, then there is a related world where someone is B. CBF^* is not valid in Kripke's original semantics,[31] and this depends on a basic property of the Kripke models: the domain of variation of the quantifiers is, in general, a proper subset of the domain of variation of the free variables. If we take the counterpart relation to be the identity relation, then we can speak of counterparts also in Kripke semantics and we see immediately that the non-validity of CBF^* depends on the fact that some a existing at w may be such that in all related worlds its counterpart (i.e. a) lies outside the range of the quantifiers. In counterpart semantics, as we know, the domain of variation of the quantifiers at a world w is exactly the same as the domain of variation of the free variables at w, therefore counterparts (if any) of an individual existing now, "do exist" in a related world v, in the sense that they belong to the range of the quantifiers at v. In counterpart semantics CBF corresponds to the principle that R.Stalnaker calls $QCBF$. "But $QCBF$, a qualified

[31] See Kripke, [46].

version of the converse of the Barcan formula does seem to be validated without any assumptions about the relationships between the domains of the different possible worlds:

$(QCBF)$ $$\Box\forall\hat{x}\phi \rightarrow \forall\hat{x}\Box(Ex \rightarrow \phi)$$

where E is the predicate of existence (defined as $\exists\hat{y}(x = y)$)). Whatever the relations between the domains, surely if in w it is necessary that everything must satisfy ϕ, then anything that exists in w must satisfy ϕ in every accessible possible world *in which that individual exists*."[32] We can rephrase this quotation by saying ' ... if in w it is necessary that everything must satisfy ϕ, then, of course, every counterpart of anything that exists in w must satisfy ϕ in every accessible possible world *in which that counterpart exists*.' But this is exactly the meaning of $\Box\forall x\phi \rightarrow \forall x\Box\phi$ in counterpart semantics, consequently CBF is synonymous of $QCBF$. Moreover counterparts share the same *kind* of existence as the individuals they are counterparts of, this feature manifests itself in the $\mathcal{C}$-validity of

$$\forall x_1\Box\exists x_2(x_1 = x_2).$$

In order to see the affinity between $S^{\Box}$ and CBF, consider the following two formulas:

(i) $\Diamond\langle 2 : x_1\rangle P \rightarrow \langle 2 : x_1\rangle\Diamond P$

and

(ii) $\exists x_2\Diamond\langle 2 : x_1\rangle P \rightarrow \Diamond\exists x_2\langle 2 : x_1\rangle P$

Both implications hold because of the same fact: 'counterparts of n-tuples of individuals are n-tuples of counterparts of individuals', and the set of worlds where there are counterparts of two individuals a and b, is a subset of the set of worlds where there are counterparts of one of them. CBF is a crucial formula and its validity hinges on basic structural features of the semantics more than other formulas such as GF or BF or NI which express assumptions about the counterpart relation (or the relationship between the domains of different possible worlds, as in Kripke's original semantics). It is a watershed for the semantics of quantified modal logics.[33]

The Ghilardi formula and its converse

(GF) $$\exists x_{n+1}\Box A \rightarrow \Box\exists x_{n+1}A,$$
(CGF) $$\Box\exists x_{n+1}A \rightarrow \exists x_{n+1}\Box A.$$

The Ghilardi formula is not $\mathcal{C}$-valid unless, as we shall see in lemma 2.3, the counterpart relation is everywhere defined. GF is equivalent to D: $\Box\langle m+1 : x_1, ..., x_m\rangle A \rightarrow \langle m+1 : x_1, ..., x_m\rangle\Box A$, as we shall see in lemma 3.3.1. It is worth noticing that the proof of $\exists x_1\Box P \rightarrow \Box\exists x_1 P$, where P is

[32] See [75], p.18.

[33] For the role of CBF in Kripke semantics, see also [15].

a unary predicate, given in section 1, hides an application of the schema D:

$$
\begin{array}{ll}
P \to \langle\rangle \exists x_1 P & \\
\Box P \to \Box\langle\rangle \exists x_1 P & \\
\Box P \to \langle\rangle \Box \exists x_1 P & \text{by schema } D, \\
\exists x_1 \Box P \to \Box \exists x_1 P. &
\end{array}
$$

As to the converse of the Ghilardi formula, it can be falsified in counterpart semantics along the same lines as in Kripke semantics.

Substitution and identity

Reflexivity of identity and substitutivity of equal terms in any formula *salva veritate* both hold true in counterpart semantics, whether or not the formula in question contains modal operators. In fact the following two formulas are $\mathfrak{C}$-valid:

I^* $\quad \langle m : t_1, t_1\rangle(x_1 = x_2)$

I^{**} $\quad (m : t_1 = s_1) \wedge \cdots \wedge (m : t_n = s_n) \to (\langle m : t_1, ..., t_n\rangle A \leftrightarrow \langle m : s_1, ..., s_n\rangle A)$.

We adopt the convention that $(x_1 = x_2)$ stands for the pure atomic formula " $=$ ", and that $(n : t = s)$ stands for the substituted formula $\langle n : t, s\rangle(x_1 = x_2)$. Let us start by examining a much debated formula, the so-called principle of necessity of identity, which asserts that if an identity is true, then it is necessarily so,

NI $\quad (x_1 = x_2) \to \Box(x_1 = x_2)$.

This formula is not valid in counterpart semantics and a countermodel for it is the case in which at least one individual of some domain D_w has two different counterparts in a related world. How is it possible that the validity of I^* and I^{**} is compatible with the failure of NI? Well, according to I^{**}, $(2 : x_1 = x_1) \wedge (x_1 = x_2) \to (\langle 2 : x_1, x_1\rangle\Box(x_1 = x_2) \to \langle 2 : x_1, x_2\rangle\Box(x_1 = x_2))$ is $\mathfrak{C}$-valid. Moreover, since I^* is $\mathfrak{C}$-valid $\langle 2 : x_1, x_1\rangle\Box(x_1 = x_2) \to ((x_1 = x_2) \to \langle 2 : x_1, x_2\rangle\Box(x_1 = x_2))$ is $\mathfrak{C}$-valid too (the antecedents have been permuted).
If $\langle 2 : x_1, x_1\rangle\Box(x_1 = x_2)$ were $\mathfrak{C}$-valid, then also $(x_1 = x_2) \to \langle 2 : x_1, x_2\rangle\Box(x_1 = x_2)$ would have been $\mathfrak{C}$-valid and consequently $(x_1 = x_2) \to \Box(x_1 = x_2)$ would have been $\mathfrak{C}$-valid because of S^I.

But $\langle 2 : x_1, x_1\rangle\Box(x_1 = x_2)$ is not $\mathfrak{C}$-valid and is not equivalent to $\Box\langle 2 : x_1, x_1\rangle(x_1 = x_2)$ (i.e. to $\Box(2 : x_1 = x_1)$ which is $\mathfrak{C}$-valid).

In the usual proof of $(x_1 = x_2) \to \Box(x_1 = x_2)$ we treat as 'equal' both wffs $\langle 2 : x_1, x_1\rangle\Box(x_1 = x_2)$ and $\Box\langle 2 : x_1, x_1\rangle(x_1 = x_2)$. Let us see how it goes:

$$\frac{\begin{array}{ll} x_1 = x_1 & \\ \Box(x_1 = x_1) & \Box(x_1 = x_1) \to (x_1 = x_2 \to \Box(x_1 = x_2)) \end{array}}{(ModusPonens) \quad (x_1 = x_2) \to \Box(x_1 = x_2)).}$$

If we phrase the above proof carefully, we get:

$$\frac{\begin{array}{l} \langle 2 : x_1, x_1\rangle(x_1 = x_2) \\ \Box\langle 2 : x_1, x_1\rangle(x_1 = x_2) \\ \langle 2 : x_1, x_1\rangle\Box(x_1 = x_2) \to (x_1 = x_2 \to \langle 2 : x_1, x_2\rangle\Box(x_1 = x_2)) \end{array}}{}$$

and *Modus Ponens* can no longer be applied.[34]

Analogously, $(i = j) \to \langle i, j\rangle\Box(x_1 = x_2)$ is not $\mathfrak{C}$-valid even when i and j are rigid designators.

As we will see in lemma 2.5, $(x_1 = x_2) \to \Box(x_1 = x_2)$ is not $\mathcal{C}$-valid unless the counterpart relation is a partial function, moreover it is equivalent to Meloni's formula (see section 3):

$M \qquad \Box\langle m + 1 : x_1, x_1, \vec{x}\rangle B \to \langle m + 1 : x_1, x_1, \vec{x}\rangle(\Box B),$

where $\vec{x} = x_2, \dots, x_{m+1}$ and B is a wff of type $m + 2$.

NI together with the principle D imply full commutativity of substitutions with respect to modal operators

$CR \qquad \Box\langle m : t_1, \dots t_n\rangle B \to \langle m : t_1, \dots t_n\rangle\Box B,$

see lemma3.7. Therefore $\mathcal{C}$-models with not-empty domains and with totally defined functions as counterpart relations are equivalent to Tarski-Kripke models with increasing domains, see [15].

A Quinean sentence: *'necessarily the number of planets is greater than 7'*

Let i denote 'the number of planets'. Then, according to Quine, we are bound to accept the following derivation:

1. $\Box(7 < 9)$
2. $i = 9$
3. $\Box(7 < i)$

This inference can be analysed along the same lines as before:

$\Box\langle 7, 9\rangle(x_1 < x_2)$	by 1.
$\langle 7, 9\rangle\Box(x_1 < x_2) \to (i = 9 \to \langle 7, i\rangle\Box(x_1 < x_2))$	by I^{**}

We need to assume that both 7 and 9 are stable designators to obtain

$$i = 9 \to \langle 7, i\rangle\Box(x_1 < x_2)$$

[34] See Ghilardi and Meloni, [31], p.87.

and so by 2.,

$$\langle 7, i\rangle \Box (x_1 < x_2).$$

Then by assuming that both i and 7 are rigid designators we obtain

$$\Box(7 < i)$$

but i can hardly be a rigid designator.

Quantifying into modal contexts leads us, according to Quine, from true statements, such as 'necessarily 9 is greater than 7', to meaningless ones such as 'There is something which is necessarily greater than 7', for is it 9 or the number of planets that is such a thing? But observe that the argument from $\Box(7 < 9)$ to $\exists x \Box(7 < x)$ is based on the assumption that 7 and 9 are both stable and rigid designators:

$\Box\langle 7, 9\rangle(x_1 < x_2)$,
$\langle 7, 9\rangle \Box(x_1 < x_2)$ (by stability of 7 and 9)
$\exists x_1 \langle 7, x_1\rangle \Box(x_1 < x_2)$
$\exists x_1 \Box \langle 7, x_1\rangle(x_1 < x_2)$ (by rigidity of 7)
$\exists x_1 \Box(7 < x_1)$

Quite different is the implication $\langle 9\rangle \Box(7 < x_1) \to \exists x_1 \Box(7 < x_1)$ which is perfectly acceptable: if being necesssarily greater than 7 is a property of 9, then there is something with that property.

Empty domains

$\forall x_1 P \to \exists x_1 P$ of type 0 admits of countermodels if for some w, the domain D_w is empty. Note however that $\forall x_2 P(2 : x_2) \to \exists x_2 P(2 : x_2)$ of type 1 is $\mathcal{C}$-valid for $\mathcal{M} \models \forall x_2 P(2 : x_2) \to \exists x_2 P(2 : x_2)$ iff for all w and for all $a \in D_w$, $\langle a\rangle \models_w \forall x_2 P(2 : x_2) \to \exists x_2 P(2 : x_2)$; therefore if D_w is empty, trivially, $\langle a\rangle \models_w \forall x_2 P(2 : x_2) \to \exists x_2 P(2 : x_2)$ holds. Again, $\langle 1 : \rangle \exists x_1(1 : x_1 = x_1)$ is $\mathfrak{C}$-valid, whereas $\exists x_1(1 : x_1 = x_1)$ is not.

Properties of the counterpart relation expressible by modal formulas

Lemma 2.3. (Ghilardi & Meloni, 1988) *The following two conditions are equivalent:*

1. $\exists x_{m+1} \Box A \to \Box \exists x_{m+1} A$ *is* $\mathfrak{C}$*-valid*
2. *the counterpart relation* $\mathfrak{C}$ *is* everywhere defined*:* wRv *only if for every* $a \in D_w$ *there exists an element* $b \in D_v$ *such that* $a\mathfrak{C}b$*.*

Proof $(i) \Rightarrow (ii)$. Consider a modal language containing a unary predicate letter P. Take a counterpart model $\mathcal{M}$, let $w \in W$, $a \in D_w$ and define $I_v(P) = \{b \in D_v : a\mathfrak{C}b\}$, for all v such that wRv. It obtains that $\langle\rangle \models_w \exists x_1 \Box P$; for, this is the case iff for some $a \in D_w$, $\langle a\rangle \models_w \Box P$ iff for all v such that wRv and for all $b \in D_v$ such that $a\mathfrak{C}b$, $\langle b\rangle \models_v P$, but this is so in virtue of the definition of $I_v(P)$. By hypothesis $\exists x_{m+1}\Box A \to \Box\exists x_{m+1}A$ is $\mathfrak{C}$-valid, whence $\langle\rangle \models_w \Box\exists x_1 P$. It follows that $\langle\rangle \models_v \exists x_1 P$ for all v such that wRv, whence for some $b \in D_v$, $\langle b\rangle \models_v P$, consequently $I_v(P) \neq \emptyset$, so $\{b \in D_v : a\mathfrak{C}b\} \neq \emptyset$ hence $\mathfrak{C}$ is everywhere defined.

$(ii) \Rightarrow (i)$. Exercise.

As we will see in section 3 the following formulas are equivalent:

(GF*)	$\Diamond\forall x_{m+1}A \to \forall x_{m+1}\Diamond A$
(D*)	$\langle m+1 : \vec{x}\rangle(\Diamond B) \to \Diamond\langle m+1 : \vec{x}\rangle B$
(GF)	$\exists x_{m+1}\Box A \to \Box\exists x_{m+1}A$
(D)	$\Box(\langle m+1 : \vec{x}\rangle B) \to \langle m+1 : \vec{x}\rangle(\Box B)$

where $\vec{x} = x_1, \ldots, x_m$, A is of type $m+1$ and B is of type m.

LEMMA 2.4. (Ghilardi & Meloni, 1988) *The following two conditions on counterpart models are equivalent:*

1. $\forall x_{m+1}\Box A \to \Box\forall x_{m+1}A$ *is $\mathfrak{C}$-valid*
2. *the counterpart relation is* surjective*: wRv only if for every $b \in D_v$ there exists an $a \in D_w$ such that $a\mathfrak{C}b$.*

PROOF $(i) \Rightarrow (ii)$. Consider a modal language containing just a unary predicate letter P. Take a counterpart model $\mathcal{M}$, a world $w \in W$ and for all $v \in W$ such that wRv, define $I_v(P) = \{b \in D_v : a\mathfrak{C}b$ for some $a \in D_w\}$.

It obtains that $\langle\rangle \models_w \forall x_1\Box P$, for this is the case iff for all $a \in D_w$, $\langle a\rangle \models_w \Box P$ iff for all v such such that wRv and for all $b \in D_v$ such that $a\mathfrak{C}b$, $\langle b\rangle \models_v P$, but this is so in virtue of the definition of $I_v(P)$. Therefore $\langle\rangle \models_w \Box\forall x_1 P$. Whence for all $v.wRv$. and for all $b \in D_v$, $\langle b\rangle \models_v P$. It follows that $D_v = I_v(P)$, hence for all $b \in D_v$ there is an $a \in D_w$ such that $a\mathfrak{C}b$. Consequently $\mathfrak{C}$ is surjective.

$(ii) \Rightarrow (i)$. Let $\mathfrak{C}$ be surjective and $\vec{a} \models_w \forall x_{m+1}\Box A$. Then for all $a' \in D_w$, $\langle\vec{a}, a'\rangle \models_w \Box A$, whence for all v, wRv, for all $\vec{c} \in (D_v)^m$ and for all $c' \in D_v$ such that $\vec{a}\mathfrak{C}\vec{c}$ and $a'\mathfrak{C}c'$, $\langle\vec{c}, c'\rangle \models_v A$. Since $\mathfrak{C}$ is surjective, then for all v, wRv, for all $\vec{c} \in (D_v)^m$ such that $\vec{a}\mathfrak{C}\vec{c}$ and for all $c' \in D_v$, $\langle\vec{c}, c'\rangle \models_v A$. So for all v, wRv, for all $\vec{c} \in (D_v)^m$ such that $\vec{a}\mathfrak{C}\vec{c}$, $\langle\vec{c}\rangle \models_v \forall x_{m+1}A$. Therefore $\vec{a} \models_w \Box\forall x_{m+1}A$.

LEMMA 2.5. (Ghilardi & Meloni, 1988) *The following two conditions on counterpart models are equivalent:*

1. $(x_1 = x_2) \to \Box(x_1 = x_2)$ *is $\mathfrak{C}$-valid*
2. *the counterpart relation is a partial function.*

PROOF $(i) \Rightarrow (ii)$. Take a counterpart model $\mathcal{M}$, a world $w \in W$ and let $a \in D_w$. For each $v.wRv$. consider the set $X_v = \{a^* \in D_v : a\mathfrak{C}a^*\}$. Since $\langle a, a\rangle \models_w (x_1 = x_2)$ it obtains that $\langle a, a\rangle \models_w \Box(2 : x_1 = x_2)$, so for all v such wRv either $X_v = \emptyset$ or $X_v = \{a^*\}$. Therefore the counterpart relation is a partial function. $(ii) \Rightarrow (i)$. Trivial.

As we shall see in section 3 the following formulas are equivalent, where $\vec{x} = x_2, \ldots, x_{m+1}$ and B is a formula of type $m + 2$.

(NI)	$(x_1 = x_2) \to \Box(2 : x_1 = x_2)$
(M)	$\Box\langle m+1 : x_1, x_1, \vec{x}\rangle B \to \langle m+1 : x_1, x_1, \vec{x}\rangle(\Box B)$
(NI*)	$\Diamond(x_1 \neq x_2) \to (x_1 \neq x_2)$
(M*)	$\langle m+1 : x_1, x_1, \vec{x}\rangle(\Diamond B) \to \Diamond\langle m+1 : x_1, x_1, \vec{x}\rangle B$

Lemma 2.6. (Ghilardi & Meloni, 1988) *The following two conditions on counterpart models are equivalent:*

1. $(x_1 \neq x_2) \to \Box(x_1 \neq x_2)$ *is* $\mathfrak{C}$*-valid*
2. *the counterpart relation is an injective relation, i.e. if* wRv, $a, b \in D_w$, $a^* \in D_v$, *then* $a\mathfrak{C}a^*$ *and* $b\mathfrak{C}a^*$ *implies that* $a = b$.

Proof $(i) \Rightarrow (ii)$. Take a counterpart model $\mathcal{M}$, a world $w \in W$ and consider two elements $a, b \in D_w$ such that $a \neq b$. For each $v.wRv$. define $X_v = \{a^* \in D_v : a\mathfrak{C}a^*\}$ and $Y_v = \{b^* \in D_v : b\mathfrak{C}b^*\}$. Then $\langle a, b\rangle \models_w (x_1 \neq x_2)$ so $\langle a, b\rangle \models_w \Box(x_1 \neq x_2)$, hence for all v such wRv either $X_v = \emptyset$ or $Y_v = \emptyset$ or $X_v \cap Y_v = \emptyset$. Therefore the counterpart relation is an injective relation.
$(ii) \Rightarrow (i)$. Trivial.

§3. Proof theory. First-order normal logics

Given a first-order language with types $\mathcal{L}^t$ based on the sets J, F and P, a first-order normal logic is defined to be any set $S \subseteq Fm\{J, F, P\}$ such that

- S includes the axioms on substitution, the axioms on the logical symbols, the axioms on identity,
- S is closed under the inference rules of modus ponens, $\exists$-introduction, substitution for variables and necessitation.

S is a first-order normal logic *with rigid terms* if it includes also the axiom (R) on rigid terms.

For any complex terms $\langle m : t_1, ..., t_k\rangle$, $\langle k : s_1, ..., s_n\rangle$, and formulas A, B of type n and C of type $n + 1$:

Axioms on substitution

S^I	$\langle n : x_1, ..., x_n\rangle A \leftrightarrow A$
S^S	$\langle m : t_1, ..., t_k\rangle(\langle k : s_1, ..., s_n\rangle A) \leftrightarrow (\langle m : t_1, ..., t_k\rangle \circ \langle k : s_1, ..., s_n\rangle)A$
$S^\neg$	$\langle k : s_1, ..., s_n\rangle(\neg A) \leftrightarrow \neg\langle k : s_1, ..., s_n\rangle A$
$S^\vee$	$\langle k : s_1, ..., s_n\rangle(A \vee B) \leftrightarrow \langle k : s_1, ..., s_n\rangle A \vee \langle k : s_1, ..., s_n\rangle B$
$S^\exists$	$\langle k : s_1, ..., s_n\rangle(\exists x_{n+1} C) \leftrightarrow \exists x_{k+1}\langle k+1 : s_1, ..., s_n, x_{k+1}\rangle C$

$S^{\Box}$ $\langle k+1 : x_{i_1}, ..., x_{i_n}\rangle \Box A \to \Box \langle k+1 : x_{i_1}, ..., x_{i_n}\rangle A$

Axiom on rigid terms

R $\langle k : s_1, ..., s_n\rangle \Box A \to \Box \langle k : s_1, ..., s_n\rangle A.$

Axioms on the logical symbols

Taut A, where A is any tautology
PI $C \to \langle n+1 : x_1, ..., x_n\rangle \exists x_{n+1} C$ (Particular instantiation)
K $\Box(A \to B) \to (\Box A \to \Box B)$

Axioms on identity

I1 $\langle 1 : x_1, x_1\rangle (x_1 = x_2)$
I2 $(n : x_1 = x_2) \to (f^{n-1}(n : x_1, x_3, ..., x_n) = f^{n-1}(n : x_2, x_3, ..., x_n))$, $n \geq 2$,
I3 $(n : x_1 = x_2) \to (P^{n-1}(n : x_1, x_3, ..., x_n) \leftrightarrow P^{n-1}(n : x_2, x_3, ..., x_n))$, $n \geq 2$.

Inference rules

Modus Ponens (MP) $$\frac{A \quad A \to B}{B}$$

$\exists$-Introduction($\exists$-I) $$\frac{A \to \langle n+1 : x_1, ..., x_n\rangle B}{\exists x_{n+1} A \to B}$$

Substitution for Variables (SV) $$\frac{A}{\langle k : s_1, ..., s_n\rangle A}$$

Necessitation rule (N) $$\frac{A}{\Box A}$$

Theorems

The members of a logic S are called *theorems*. We write $\vdash^n_S A$ to denote that the formula A of type n is a theorem of S.

Soundness and Completeness

Let H be a class of counterpart frames or of models. A logic S is *sound with respect to* H if for any type n and formula A of type n,

$$\vdash^n_S A \quad \text{implies} \quad H \models A.$$

S is *complete with respect to* H, if, for any n and formula A of type n,

$$H \models A \quad \text{implies} \quad \vdash^n_S A.$$

S is *determined by* H if it is both sound and complete with respect to H.

Deducibility
If $M \cup \{A\} \subseteq Fm\{J, F, P\}$, and every formula of $M \cup \{A\}$ is of the same type n, then

$$M \vdash^n_S A \quad \text{iff} \quad \vdash^n_S B_1 \wedge \cdots \wedge B_m \to A$$

for some $B_1, \ldots, B_m \in M$. When $M \vdash^n_S A$ we say that A is *S-deducible from M*. We write $M \nvdash^n_S A$ when A is not S-deducible from M.

The logics $Q.K^t$ and $R.K^t$

$$Q.K^t = \bigcap\{S \subseteq Fm(J, F, P) : S \text{ is a first-order modal logic}\}$$

$$R.K^t = \bigcap\{S \subseteq Fm(J, F, P) : S \text{ is a first-order modal logic with rigid terms}\}$$

We write $\vdash^n A$ to denote that the wff A of type n is a theorem of either $Q.K^t$ or $R.K^t$ instead of $\vdash^n_{Q.K^t} A$ or $\vdash^n_{R.K^t} A$. From the context it will be clear which system we refer to.

Lemma 3.1. *The following formulas are theorems of $Q.K^t$:*

1. $\langle m : \vec{s}, s\rangle A \to \exists x_{m+1}\langle m+1 : \vec{s}, x_{m+1}\rangle A$,
   where $\vec{s}$ stands for $s_1, \ldots, s_n$.
2. $\langle n+1 : \vec{x}\rangle \forall x_{n+1} A \to A$ $\qquad$ *UI:* Universal instantiation
   where $\vec{x}$ stands for $x_1, \ldots, x_n$.
3. $\forall x_{m+1}\langle m+1 : \vec{s}, x_{m+1}\rangle A \to \langle m : \vec{s}, s\rangle A$
   where $\vec{s}$ stands for $s_1, \ldots, s_n$.
4. $\langle k : s_1, ..., s_n\rangle \forall x_{n+1} C \leftrightarrow \forall x_{k+1}(\langle s_1, ..., s_n, x_{k+1}\rangle C)$.
5. $\langle i\rangle \Box A \to \exists x_1 \Box A$.
6. *The rule of $\forall$-Introduction is eliminable:*

$$\frac{\langle n+1 : x_1, ..., x_n\rangle A \to B}{A \to \forall x_{n+1} B}$$

Lemma 3.2. *If the language contains individual constants, then $\forall x_1 A \to \exists x_1 A$ is a theorem of $Q.K^t$, for any formula A of type 1.*

Proof. Let i be an individual constant of type 0.

$\vdash^1 \langle 1 : \rangle \forall x_1 A \to A$
$\vdash^0 \langle i\rangle\langle 1 : \rangle \forall x_1 A \to \langle i\rangle A$
$\vdash^0 \langle\,\rangle \forall x_1 A \to \langle i\rangle A$
$\vdash^0 \forall x_1 \langle 1 : x_1\rangle A \to \langle i\rangle A$
$\vdash^0 \forall x_1 A \to \langle i\rangle A$

Analogously, $\vdash^0 \langle i\rangle A \to \exists x_1 A$, therefore $\vdash^0 \forall x_1 A \to \exists x_1 A$.

If the language does not contain individual constants, then we can only prove that $\forall x_2\langle 2 : x_1\rangle A \to \exists x_2\langle 2 : x_1\rangle A$. Here is a proof:

$\vdash^1 \langle 1 : \rangle \forall x_1 A \rightarrow A$
$\vdash^1 A \rightarrow \langle 1 : \rangle \exists x_1 A$
$\vdash^1 \langle 1 : \rangle \forall x_1 A \rightarrow \langle 1 : \rangle \exists x_1 A$
$\vdash^1 \forall x_2 \langle 2 : x_2 \rangle A \rightarrow \exists x_2 \langle 2 : x_2 \rangle A$

Note. $\forall x_1 A \rightarrow \exists x_1 A$ can be falsified at any world w such that D_w is empty, whereas $\forall x_2 \langle 2 : x_1 \rangle A \rightarrow \exists x_2 \langle 2 : x_1 \rangle A$ is valid since it is satisfied in any world w by any unary sequence of elements of the domain of D_w.

LEMMA 3.3. (on identity) *The following formulas are theorems of* $Q.K^t$*:*

1. $(n : x_1 = x_2) \wedge \langle x_1, x_3, \ldots, x_n \rangle A \rightarrow \langle x_2, x_3, \ldots, x_n \rangle A.$
2. $(m : t_1 = s_1) \wedge \cdots \wedge (m : t_n = s_n) \rightarrow (\langle m : t_1, \ldots, t_n \rangle A \rightarrow \langle m : s_1, \ldots, s_n \rangle A).$
3. $(2 : x_1 = x_2) \rightarrow (2 : x_2 = x_1).$
4. $(3 : x_1 = x_2) \wedge (3 : x_2 = x_3) \rightarrow (3 : x_1 = x_3).$
5. $(n : x_1 = x_2) \rightarrow (\langle x_1, x_3, \ldots, x_n \rangle \circ t = \langle x_2, x_3, \ldots, x_n \rangle \circ t).$
6. $(m : t_1 = s_1) \wedge \cdots \wedge (m : t_n = s_n) \rightarrow (\langle m : t_1, \ldots, t_n \rangle \circ t = \langle m : s_1, \ldots, s_n \rangle \circ t).$
7. $\exists x_2 (x_1 = x_2).$
8. $\forall x_1 \Box \exists x_2 (x_1 = x_2).$

LEMMA 3.4. *(i)* $Q.K^t \vdash CBF$, *(ii)* $(Q.K^t - S^{\Box}) + CBF \vdash \langle m+1 : \vec{x} \rangle \Box B \rightarrow \Box \langle m+1 : \vec{x} \rangle B$*, and (iii)* $Q.K^t \vdash \langle n : \vec{y} \rangle \Box A \rightarrow \Box \langle n : \vec{y} \rangle A$*, where* $\vec{y} = x_{i_1}, ..., x_{i_n}$ *is a permutation of* $x_1, ..., x_n$.

PROOF. Let $\vec{x}$ be $x_1, ..., x_m$.

(i) $\vdash^{m+1} \langle m+1 : \vec{x} \rangle \forall x_{m+1} A \rightarrow A$ (UI)
$\vdash^{m+1} \Box \langle m+1 : \vec{x} \rangle \forall x_{m+1} A \rightarrow \Box A$
$\vdash^{m+1} \langle m+1 : \vec{x} \rangle \Box \forall x_{m+1} A \rightarrow \Box A$ (by $S^{\Box}$)
$\vdash^{m} \Box \forall x_{m+1} A \rightarrow \forall x_{m+1} \Box A$ (by $\forall$-I)

(ii) $\vdash^{m+1} \langle m+1 : \vec{x} \rangle B \rightarrow \langle m+1 : \vec{x} \rangle B$
$\vdash^{m} B \rightarrow \forall x_{m+1} \langle m+1 : \vec{x} \rangle B$ (by $\forall$-I)
$\vdash^{m} \Box B \rightarrow \Box \forall x_{m+1} \langle m+1 : \vec{x} \rangle B$
$\vdash^{m} \Box B \rightarrow \forall x_{m+1} \Box \langle m+1 : \vec{x} \rangle B$ (by CBF)
$\vdash^{m+1} \langle m+1 : \vec{x} \rangle \Box B \rightarrow \langle m+1 : \vec{x} \rangle \forall x_{m+1} \Box \langle m+1 : \vec{x} \rangle B$ (by SV)
$\vdash^{m+1} \langle m+1 : \vec{x} \rangle \Box B \rightarrow \Box \langle m+1 : \vec{x} \rangle B$ (by UI)

(iii) For every term $\langle n : \vec{y} \rangle$, where $\vec{y}$ is a permutation of $x_1, ..., x_n$, there is a k such that $\underbrace{\langle n : \vec{y} \rangle \circ \cdots \circ \langle n : \vec{y} \rangle}_{k-times} = \langle n : x_1, \ldots, x_n \rangle$.

$$\vdash^n \quad \langle n:\vec{y}\rangle\Box\langle n:\vec{y}\rangle A \to \Box\langle n:\vec{y}\rangle\langle n:\vec{y}\rangle A \qquad (S^{\Box})$$

$$\vdash^n \quad \underbrace{\langle n:\vec{y}\rangle\circ\cdots\circ\langle n:\vec{y}\rangle}_{(k-1)-times}\langle n:\vec{y}\rangle\Box\langle n:\vec{y}\rangle A \to$$

$$\underbrace{\langle n:\vec{y}\rangle\circ\cdots\circ\langle n:\vec{y}\rangle}_{(k-1)-times}\Box\langle n:\vec{y}\rangle\langle n:\vec{y}\rangle A \qquad (SV)$$

$$\vdash^n \quad \underbrace{\langle n:\vec{y}\rangle\circ\cdots\circ\langle n:\vec{y}\rangle}_{k-times}\Box\langle n:\vec{y}\rangle A \to$$

$$\langle n:\vec{y}\rangle\Box\underbrace{\langle n:\vec{y}\rangle\circ\cdots\circ\langle n:\vec{y}\rangle}_{(k-2)-times}\langle n:\vec{y}\rangle\langle n:\vec{y}\rangle A \qquad (S^{\Box})$$

$$\vdash^n \quad \Box\langle n:\vec{y}\rangle A \to \langle n:\vec{y}\rangle\Box\underbrace{\langle n:\vec{y}\rangle\circ\cdots\circ\langle n:\vec{y}\rangle}_{k-times} A$$

$$\vdash^n \quad \Box\langle n:\vec{y}\rangle A \to \langle n:\vec{y}\rangle\Box A \qquad (S^I)$$

Extensions of $Q.K^t$

$BF.K^t = Q.K^t + (\forall x_{m+1}\Box A \to \Box\forall x_{m+1}A)$.

$GF.K^t = Q.K^t + (\exists x_{m+1}\Box A \to \Box\exists x_{m+1}A)$.

$D.K^t = Q.K^t + (\Box\langle m+1:x_1,...,x_m\rangle B \to \langle m+1:x_1,...,x_m\rangle\Box B)$.

$M.K^t = Q.K^t + (\Box\langle m+1:x_1,x_1,x_2,...,x_{m+1}\rangle B \to \langle m+1:x_1,x_1,x_2,...,x_{m+1}\rangle\Box B)$. ($B$ is of type $m+2$.)

$NI.K^t = Q.K^t + ((x_1 = x_2) \to \Box(x_1 = x_2))$.

$ND.K^t = Q.K^t + ((x_1 \neq x_2) \to \Box(x_1 \neq x_2))$.

$CR.K^t = Q.K^t + (\Box\langle k:t_1,...,t_m\rangle B \to \langle k:t_1,...,t_m\rangle\Box B)$.

Lemma 3.5. $GF.K^t = D.K^t$

Proof. Let $\vec{x}$ be $x_1,...,x_m$.

(i) $\vdash^{m+1} \quad A \to \langle m+1:\vec{x}\rangle\exists x_{m+1}A$

$\vdash^{m+1} \quad \Box A \to \Box\langle m+1:\vec{x}\rangle\exists x_{m+1}A$

$\vdash^{m+1} \quad \Box A \to \langle m+1:\vec{x}\rangle\Box\exists x_{m+1}A$ (by D)

$\vdash^{m+1} \quad \exists x_{m+1}\Box A \to \Box\exists x_{m+1}A$

(ii) $\vdash^{m+1} \quad \langle m+1:\vec{x}\rangle B \to \langle m+1:\vec{x}\rangle B$

$\vdash^{m} \quad \exists x_{m+1}\langle m+1:\vec{x}\rangle B \to B$

$\vdash^{m} \quad \Box\exists x_{m+1}\langle m+1:\vec{x}\rangle B \to \Box B$ (by GF)

$\vdash^{m+1} \quad \langle m+1:\vec{x}\rangle\exists x_{m+1}\Box\langle m+1:\vec{x}\rangle B \to \langle m+1:\vec{x}\rangle\Box B$

$\vdash^{m+1} \quad \Box\langle m+1:\vec{x}\rangle B \to \langle m+1:\vec{x}\rangle\Box B$

Lemma 3.6. $NI.K^t = M.K^t$

PROOF. Without loss of generality, we consider the sentence M when $m = 1$ and B is a wff of type 3, i.e. $(\Box\langle 2 : x_1, x_1, x_2\rangle B \rightarrow \langle 2 : x_1, x_1, x_2\rangle \Box B)$.

(i) $\vdash^3 (3 : x_1 = x_2) \rightarrow (\langle 3 : x_1, x_1, x_3\rangle B \rightarrow \langle 3 : x_1, x_2, x_3\rangle B)$
$\vdash^3 (3 : x_1 = x_2) \rightarrow (\langle 3 : x_1, x_1, x_3\rangle B \rightarrow B)$
$\vdash^3 \Box(3 : x_1 = x_2) \rightarrow (\Box\langle 3 : x_1, x_1, x_3\rangle B \rightarrow \Box B)$
$\vdash^3 (3 : x_1 = x_2) \rightarrow (\Box\langle 3 : x_1, x_1, x_3\rangle B \rightarrow \Box B)$ (by NI)
$\vdash^3 \langle 2 : x_1, x_1, x_2\rangle(3 : x_1 = x_2) \rightarrow (\langle 2 : x_1, x_1, x_2\rangle \Box\langle 3 : x_1, x_1, x_2\rangle B \rightarrow \langle 2 : x_1, x_1, x_2\rangle \Box B)$
$\vdash^2 \langle 2 : x_1, x_1, x_2\rangle(3 : x_1 = x_2)$
$\vdash^2 \langle 2 : x_1, x_1, x_2\rangle \Box\langle 3 : x_1, x_1, x_2\rangle B \rightarrow \langle 2 : x_1, x_1, x_2\rangle \Box B$
$\vdash^2 \langle 2 : x_1, x_1, x_2\rangle \Box(\langle 3 : x_1, x_3\rangle \circ \langle 2 : x_1, x_1, x_2\rangle)B \rightarrow \langle 2 : x_1, x_1, x_2\rangle \Box B$
$\vdash^2 \langle 2 : x_1, x_1, x_2\rangle(\langle 3 : x_1, x_3\rangle \Box\langle 2 : x_1, x_1, x_2\rangle B) \rightarrow \langle 2 : x_1, x_1, x_2\rangle \Box B$
$\vdash^2 \langle 2 : x_1, x_2\rangle \Box\langle 2 : x_1, x_1, x_2\rangle B \rightarrow \langle 2 : x_1, x_1, x_2\rangle \Box B$
$\vdash^2 \Box\langle 2 : x_1, x_1, x_2\rangle B \rightarrow \langle 2 : x_1, x_1, x_2\rangle \Box B$

(ii) $\vdash^2 \langle 2 : x_1, x_1\rangle(x_1 = x_2)$
$\vdash^2 \Box\langle 2 : x_1, x_1\rangle(x_1 = x_2)$
$\vdash^2 \langle 2 : x_1, x_1\rangle \Box(x_1 = x_2)$ (by M)
$\vdash^2 \langle 2 : x_1, x_1\rangle \Box(x_1 = x_2) \rightarrow ((x_1 = x_2) \rightarrow \langle 2 : x_1, x_2\rangle \Box(x_1 = x_2))$
$\vdash^2 (x_1 = x_2) \rightarrow \langle 2 : x_1, x_2\rangle \Box(x_1 = x_2)$
$\vdash^2 (x_1 = x_2) \rightarrow \Box(x_1 = x_2)$

LEMMA 3.7. (Ghilardi, 1990) $NI.D.K^t = CR.K^t$.

PROOF. We show that $NI.D.K^t \vdash CR$. Without loss of generality, we show that $NI.D.K^t \vdash^3 \Box\langle 3 : t_1, t_2\rangle B \rightarrow \langle 3 : t_1, t_2\rangle \Box B$.
$\vdash^5 \langle 5 : t_1, t_2\rangle = \langle 5 : x_4, x_5\rangle \rightarrow (\langle 5 : t_1, t_2\rangle B \rightarrow \langle 5 : x_4, x_5\rangle B)$
$\vdash^5 \Box(\langle 5 : t_1, t_2\rangle = \langle 5 : x_4, x_5\rangle) \rightarrow (\Box\langle 5 : t_1, t_2\rangle B \rightarrow \Box\langle 5 : x_4, x_5\rangle B)$
$\vdash^5 \langle 5 : t_1, t_2\rangle = \langle 5 : x_4, x_5\rangle \rightarrow (\Box\langle 5 : x_1, x_2, x_3\rangle\langle 3 : t_1, t_2\rangle B \rightarrow \Box\langle 5 : x_4, x_5\rangle B)$
$\vdash^5 \langle 5 : t_1, t_2\rangle = \langle 5 : x_4, x_5\rangle \rightarrow (\langle 5 : x_1, x_2, x_3\rangle \Box\langle 3 : t_1, t_2\rangle B \rightarrow \langle 5 : x_4, x_5\rangle \Box B)$
$\vdash^3 \langle 3 : x_1, x_2, x_3, t_1, t_2\rangle(\langle 5 : t_1, t_2\rangle = \langle 5 : x_4, x_5\rangle) \rightarrow (\langle 3 : x_1, x_2, x_3, t_1, t_2\rangle\langle 5 : x_1, x_2, x_3\rangle \Box\langle 3 : t_1, t_2\rangle B \rightarrow \langle 3 : x_1, x_2, x_3, t_1, t_2\rangle\langle 5 : x_4, x_5\rangle \Box B)$
$\vdash^3 (\langle 3 : t_1, t_2\rangle = \langle 3 : t_1, t_2\rangle) \rightarrow (\Box\langle 3 : t_1, t_2\rangle B \rightarrow \langle 3 : t_1, t_2\rangle \Box B)$.

§4. Completeness results.

§4. Completeness results. As to the completeness results, here is a list of some of them.[35]

[35] A proof of these results can be found in S.Ghilardi, [26] and S.Ghilardi and G.Meloni, [30]. The novelty of the present proof is that it makes no use of category theory.

q.m.l.	is complete w.r.t. the class of counterpart frames in which the relation $\mathfrak{C}$ is
$Q.K^t$	any
$Q.T^t$	reflexive
$Q.S4^t$	reflexive and transitive
$Q.B^t$	symmetric
$BF.K^t$	surjective
$GF.K^t$	everywhere defined
$ND.K^t$	injective
$NI.K^t$	functional

Preliminaries

n-tuples of variables of type m, $\langle m : x_{i_1}, \ldots, x_{i_n} \rangle$, are called *projections* from m to n and are denoted by $\alpha : m \to n$, $\beta : m \to n$, ...

Given a set X, $\vec{a}$ is said to be *a list of length* m iff $\vec{a} \in X^m$. If $\vec{a} = \langle a_1, \ldots, a_m \rangle$ and $\vec{b} = \langle b_1, \ldots, b_r \rangle$, then $\vec{a} * \vec{b}$ is the list $\langle a_1, \ldots, a_m, b_1, \ldots, b_r \rangle$ of length $m + r$ and for any $c \in X$, $\vec{a} * c$ is the list $\langle a_1, \ldots, a_m, c \rangle$ of length $m + 1$. Any list whose elements are pairwise distinct is said to be *a base list.* Where $\alpha = \langle m : x_{i_1}, \ldots, x_{i_n} \rangle$ and $\vec{a} = \langle a_1, \ldots, a_m \rangle$, then

$$\vec{a}\,[m : x_{i_1}, \ldots, x_{i_n}] = \langle a_{i_1}, ..., a_{i_n} \rangle.$$

$\vec{a}\,[\alpha]$ denotes the *interpretation* of the projection α with respect to the list $\vec{a}$. When no confusion can possibly arise, we will write $\vec{a}\,[x_{i_1}, ..., x_{i_n}]$ instead of $\vec{a}\,[m : x_{i_1}, ..., x_{i_n}]$. In pictures,

$$\vec{a} \xrightarrow{\quad\alpha\quad} \vec{b}$$

where $\vec{b} = \langle a_{i_1}, \ldots, a_{i_n} \rangle$.

Lemma 4.1. (*a*) *If $\vec{a}\,[\alpha] = \vec{a}\,[\beta]$ and $\vec{a}$ is a base list, then $\alpha = \beta$.*
(*b*) *If $\langle a_{i_1}, \ldots, a_{i_n} \rangle$ is a list composed of elements taken from the base list $\langle a_1, \ldots, a_m \rangle$, then there is a unique projection $\alpha : m \to n$, such that*

$$\langle a_1, ..., a_m \rangle[\alpha] = \langle a_{i_1}, ..., a_{i_n} \rangle.$$

(*c*) *Suppose $\vec{a}\,[\tau] = \vec{b}$, where $\tau : m \to n$, $\vec{b}$ is a base list of length n and $\vec{a}$ is a list of length m. If any element of $\vec{a}$ is also an element of $\vec{b}$, then there is a unique projection $\pi : n \to m$, such that $\pi \circ \tau = \langle n : x_1, ..., x_n \rangle$.*

Proof. *(b)* Trivially $\alpha = \langle m : x_{i_1}, ..., x_{i_n} \rangle$. Suppose there is another projection $\langle m : x_{j_1}, ..., x_{j_n} \rangle$ such that $\langle a_1, ..., a_m \rangle[x_{j_1}, ..., x_{j_n}] = \langle a_{i_1}, ..., a_{i_n} \rangle$. Then $\langle a_{i_1}, ..., a_{i_n} \rangle = \langle a_{j_1}, ..., a_{j_n} \rangle$, i.e. $i_1 = j_1, ..., i_n = j_n$ and so $\langle m : x_{i_1}, ..., x_{i_n} \rangle = \langle m : x_{j_1}, ..., x_{j_n} \rangle$.

(c) Since $\alpha[\tau] = \vec{b}$, and any element of $\vec{b}$ is also an element of $\vec{a}$, then from *(b)* there is a unique π such that $\vec{b}\,[\pi] = \vec{a}$. So $\vec{b}\,[\pi \circ \tau] = (\vec{b}\,[\pi])\,[\tau] = \vec{a}\,[\tau] = \vec{b} = \vec{b}\,[n : x_1, ..., x_n]$, whence by *(a)*, $\pi \circ \tau = \langle n : x_1, ..., x_n \rangle$.

$$\begin{tikzcd}
\langle b_1, \dots, b_n \rangle \arrow[r, "\pi"] \arrow[rr, bend right, "\langle n : x_1, \dots, x_n \rangle"'] & \langle a_1, \dots, a_m \rangle \arrow[r, "\tau"] & \langle b_1, \dots, b_n \rangle
\end{tikzcd}$$

Definition 4.2. Let X be any set. Define $graph(X, \mathcal{L}^t) = \{\langle \vec{a}, A \rangle :$ for some $n \in N$, $\vec{a} \in X^n$ and A is a formula of $\mathcal{L}^t$ of type $n\}$.

Any subset of $graph(X, \mathcal{L}^t)$ is said to be an *X-graph*. Obviously, if $X = \varnothing$, then $graph(X, \mathcal{L}^t) = \{\langle \langle\rangle, A \rangle : A$ is a sentence of $\mathcal{L}^t\}$. In the following we will assume that a set X is fixed from the outset, and so we will speak just of *graphs* instead of X-graphs. A graph Γ is meant to represent a world w of a counterpart model because it describes the relation of satisfiability relative to w: $\langle \vec{a}, A \rangle \in \Gamma$ is intended to mean that $\vec{a} \models_w A$.

Now we define a property for graphs that corresponds to consistency for sets of formulas, and we call it *coherence*. To illustrate this property, suppose that Γ includes the subset

$$\{\, \langle \langle a, c, c \rangle, A \rangle, \quad \langle \langle a, b \rangle, B \rangle, \quad \langle \langle c, d \rangle, C \rangle \,\}.$$

So, according to the intended meaning,

$$\langle a, c, c \rangle \models_w A, \quad \langle a, b \rangle \models_w B \quad \text{and} \quad \langle c, d \rangle \models_w C.$$

Now consider the list $\langle a, b, c, d \rangle$ (unique up to the order of its elements) composed of all and only all the elements, without repetitions, occurring either in $\langle a, c, c \rangle$ or in $\langle a, b \rangle$ or in $\langle c, d \rangle$. Then

$$\langle a, b, c, d \rangle [4 : x_1, x_3, x_3] \models_w A, \qquad \langle a, b, c, d \rangle [4 : x_1, x_2] \models_w B,$$
$$\langle a, b, c, d \rangle [4 : x_3, x_4] \models_w C,$$

and so

$$\langle a, b, c, d \rangle \models_w \langle 4 : x_1, x_3, x_3 \rangle A, \qquad \langle a, b, c, d \rangle \models_w \langle 4 : x_1, x_2 \rangle B,$$
$$\langle a, b, c, d \rangle \models_w \langle 4 : x_3, x_4 \rangle C.$$

If it happens that

$$\vdash^4 (\langle 4 : x_1, x_3, x_3 \rangle A \wedge \langle 4 : x_1, x_2 \rangle B \wedge \langle 4 : x_3, x_4 \rangle C) \to \bot,$$

then Γ is said to be *incoherent*. In pictures,

$$\begin{tikzcd}
 & \langle a, c, c \rangle \quad A \\
\langle a, b, c, d \rangle \arrow[ur, "\langle 4 : x_1, x_3, x_3 \rangle"] \arrow[dr, "\langle 4 : x_1, x_2 \rangle"] \arrow[dd, "\langle 4 : x_3, x_4 \rangle"'] & \\
 & \langle a, b \rangle \quad B \\
\langle c, d \rangle \quad C &
\end{tikzcd}$$

Let S be any logic in the language of $\mathcal{L}^t$ such that $S \supseteq Q.K^t$, here is the full definition of S-incoherence:

Definition 4.3. A graph Γ is *S-incoherent* iff there are pairs

- $\langle \vec{a}_i, A_i \rangle \in \Gamma, \quad 1 \leq i \leq h$, where $\vec{a}_i$ is a list of length n_i and A_i is a wff of type n_i,
- and there are projections $\tau_i : m \to n_i$, $1 \leq i \leq h$, such that:
  - (*i*) $\quad {}_S \vdash^m \bigwedge \tau_i A_i \to \bot$, and
  - (*ii*) $\quad \vec{d}[\tau_i] = \vec{a}_i$, $1 \leq i \leq h$,

where $\vec{d}$ is *the* base list of length m composed of all and only all the elements occurring either in $\vec{a}_1$ or ... or in $\vec{a}_h$.

In what follows we refer to the conditions stated by the definition 4.3 by saying that Γ admits of a *critical diagram*:

$$\begin{tikzcd} & \vec{a}_1 & A_1 \\ \vec{d} \arrow[ur, "\tau_1"] \arrow[dr, "\tau_h"'] & & \cdots \\ & \vec{a}_h & A_h \end{tikzcd}$$

where ${}_S \vdash^m \bigwedge \tau_i A_i \to \bot$.

Definition 4.4. A graph is *S-coherent* iff it is not S-incoherent.

The order of the elements of $\vec{d}$ is immaterial since if $\vec{d}[\tau_i] = \vec{a}_i$ and $\vdash^m \bigwedge \tau_i A_i \to \bot$, then for any permutation $\vec{d^*}$ of $\vec{d}$ there are projections τ_i^* such that $\vec{d^*}[\tau_i^*] = \vec{a}_i$ and $S \vdash^m \bigwedge \tau_i^* A_i \to \bot$. To wit, by lemma 4.1(b) there is a unique projection α such that $\vec{d^*}[\alpha] = \vec{d}$, so let $\tau_i^* = \alpha \circ \tau_i$. This allows us to speak of *the* base list $\vec{d}$.

Definition 4.5. A set M of sentences is *S-consistent* iff for any finite subset of sentences $A_1 ... A_i ... A_n$ of M and any $m \in \mathbb{N}$, ${}_S \nvdash^m \langle m : \rangle \bigwedge A_i \to \bot$.

For simplicity's sake, we will write $\vdash^m A$ instead of ${}_S \vdash^m A$.

Lemma 4.6. *Let M be an S-consistent set of sentences. Then the graph* $\Gamma = \{\langle \langle \, \rangle, A \rangle : A \in M$ is of type $0\}$ *is S-coherent.*

Proof. Let Γ admit of a critical diagram

$$\begin{tikzcd}
& \langle\,\rangle & A_1 \\
\vec{d} \arrow[ur, "\tau_1"] \arrow[dr, "\tau_h"'] & & \cdots \\
& \langle\,\rangle & A_h
\end{tikzcd}$$

where $\vdash^m \bigwedge \tau_i A_i \to \bot$. Since each $\tau_i : m \to 0$ is just the projection $\langle m : \rangle$, then $\vdash^m \bigwedge \langle m : \rangle A_i \to \bot$, whence $\vdash^m \langle m : \rangle \bigwedge A_i \to \bot$, contrary to the S-consistency of M.

Lemma 4.7. *If Γ is an S-coherent graph, $\langle \vec{c}, C_j \rangle \in \Gamma$, $1 \leq j \leq k$, and $\vdash^p \bigwedge C_j \to B$, then $\Gamma + \langle \vec{c}, B \rangle$ is S-coherent.*

Proof. Let $\Gamma + \langle \vec{c}, B \rangle$ admit of a critical diagram

$$\begin{tikzcd}
& \vec{a}_1 & A_1 \\
\vec{d} \arrow[ur, "\tau_1"] \arrow[dr, "\tau_h"] \arrow[dd, "\pi"'] & & \cdots \\
& \vec{a}_h & A_h \\
\vec{c} & B &
\end{tikzcd}$$

where $\vdash^m \bigwedge \tau_i A_i \wedge \pi B \to \bot$. Therefore Γ admits of the following critical diagram

$$\begin{tikzcd}
& & \vec{a}_1 & A_1 \\
\vec{d} \arrow[urr, "\tau_1"] \arrow[drr, "\tau_h"] \arrow[dd, "\pi"] \arrow[dddd, "\pi"'] & & & \cdots \\
& & \vec{a}_h & A_h \\
& \vec{c}_1 & C_1 & \\
& & & \\
\vec{c}_k & C_k & &
\end{tikzcd}$$

where $\vdash^m \bigwedge \tau_i A_i \wedge \bigwedge \pi C_j \to \bot$, contrary to the hypothesis of the lemma. To wit,

$$\frac{\begin{array}{ll} \vdash^p \bigwedge C_j \to B & \\ \vdash^m \pi \bigwedge C_j \to \pi B & \\ \vdash^m \bigwedge \pi C_j \to \pi B & \vdash^m \bigwedge \tau_i A_i \wedge \pi B \to \bot \end{array}}{\vdash^m \bigwedge \tau_i A_i \wedge \bigwedge \pi C_j \to \bot.}$$

LEMMA 4.8. *Let Γ be an S-coherent graph. $\Gamma + \langle \vec{c}, \beta B \rangle$ is S-coherent iff $\Gamma + \langle \vec{c}[\beta], B \rangle$ is S-coherent, where $\beta : p \to q$ and B is of type q.*

PROOF. Let $\Gamma + \langle \vec{c}, \beta B \rangle$ admit of a critical diagram

$$\begin{tikzcd}
 & \vec{a}_1 & A_1 \\
\vec{d} \arrow[ur, "\tau_1"] \arrow[dr, "\tau_h"] \arrow[dd, "\pi"'] & & \cdots \\
 & \vec{a}_h & A_h \\
\vec{c} & \beta B &
\end{tikzcd}$$

where $\vdash^m \bigwedge \tau_i A_i \wedge \pi(\beta B) \to \bot$.
Since $\vec{d}[\pi] = \vec{c}$, it follows that $(\vec{d}[\pi])[\beta] = \vec{c}[\beta]$, and so $\vec{d}[\pi \circ \beta] = \vec{c}[\beta]$.
Therefore $\Gamma + \langle \vec{c}[\beta], B \rangle$ admits of the following critical diagram

$$\begin{tikzcd}
 & \vec{a}_1 & A_1 \\
\vec{d} \arrow[ur, "\tau_1"] \arrow[dr, "\tau_h"] \arrow[dd, "\pi \circ \beta"'] & & \cdots \\
 & \vec{a}_h & A_h \\
\vec{c}[\beta] & B &
\end{tikzcd}$$

where $\vdash^m \bigwedge \tau_i A_i \wedge (\pi \circ \beta) B \to \bot$. This last formula obtains from the fact that $\vdash^m \bigwedge \tau_i A_i \wedge \pi(\beta B) \to \bot$.
Let $\Gamma + \langle \vec{c}[\beta], B \rangle$ admit of a critical diagram

$$\begin{tikzcd}
 & & \vec{a}_1 & A_1 \\
\vec{d} \arrow[urr, "\tau_1"] \arrow[drr, "\tau_h"] \arrow[dd, "\pi"'] & & & \cdots \\
 & & \vec{a}_h & A_h \\
\vec{c}[\beta] & B & &
\end{tikzcd}$$

where $\vdash^m \bigwedge \tau_i A_i \wedge \pi B \to \bot$.

Let $\vec{s}$ be a base list of length k composed of all and only all the elements occurring in either $\vec{d}$ or $\vec{c}$ then, by the lemma 4.1(b), there exist and are unique, two projections $\gamma : k \to m$ and $\delta : k \to p$ such that
$\vec{s}[\gamma] = \vec{d}$, and $\vec{s}[\delta] = \vec{c}$. Now,

$$\dfrac{\dfrac{\genfrac{}{}{0pt}{}{\vec{s}[\gamma] = \vec{d}}{(\vec{s}[\gamma])[\pi] = \vec{d}[\pi]} \qquad \vec{d}[\pi] = \vec{c}[\beta]}{(\vec{s}[\gamma])[\pi] = \vec{c}[\beta]} \qquad \vec{s}[\delta] = \vec{c}}{(\vec{s}[\gamma])[\pi] = (\vec{s}[\delta])[\beta]}$$

$$\vec{s}[\gamma \circ \pi] = \vec{s}[\delta \circ \beta]$$

$$\gamma \circ \pi = \delta \circ \beta$$

Moreover,

$$\dfrac{\vec{d}[\tau_i] = \vec{a}_i \qquad \vec{s}[\gamma] = \vec{d}}{(\vec{s}[\gamma])[\tau_i] = \vec{a}_i}$$

$$(\vec{s}[\gamma \circ \tau_i] = \vec{a}_i$$

Therefore $\Gamma + \langle \vec{c}, \beta B \rangle$ admits of the following critical diagram

$$\begin{tikzcd}
 & & & \vec{a}_1 & A_1 \\
\vec{s} \arrow[rr, "\gamma"'] \arrow[ddr, "\delta"'] & & \vec{d} \arrow[ur, "\tau_1"] \arrow[dr, "\tau_h"] & & \cdots \\
 & & & \vec{a}_h & A_h \\
 & \vec{c} & \beta B & &
\end{tikzcd}$$

where $\vdash^k \bigwedge (\gamma \circ \tau_i) A_i \wedge \delta(\beta B) \to \bot$. To wit,

$$\begin{array}{l}\vdash^m \bigwedge \tau_i A_i \wedge \pi B \to \bot\\ \vdash^k \bigwedge(\gamma\circ\tau_i)A_i \wedge (\gamma\circ\pi)B \to \gamma\bot\\ \vdash^k \bigwedge(\gamma\circ\tau_i)A_i \wedge (\delta\circ\beta)B \to \bot\\ \vdash^k \bigwedge(\gamma\circ\tau_i)A_i \wedge \delta(\beta B) \to \bot.\end{array}$$

LEMMA 4.9. *If* Γ *is an* S*-coherent graph and* $\langle\vec{c}, C\vee D\rangle \in \Gamma$*, then either* $\Gamma + \langle\vec{c}, C\rangle$ *is* S*-coherent or* $\Gamma + \langle\vec{c}, D\rangle$ *is* S*-coherent.*

PROOF. Suppose by *reductio* that both $\Gamma + \langle\vec{c}, C\rangle$ and $\Gamma + \langle\vec{c}, D\rangle$ admit of critical diagrams. So we have

$$\begin{tikzcd}
 & \vec{a}_1 & A_1 \\
\vec{d} \arrow[ur, "\tau_1"] \arrow[dr, "\tau_h"] \arrow[dd, "\pi"'] & & \cdots \\
 & \vec{a}_h & A_h \\
\vec{c} & C &
\end{tikzcd}$$

where $\vdash^m \bigwedge \tau_i A_i \wedge \pi C \to \bot$; and

$$\begin{tikzcd}
\vec{c} & D & \\
 & \vec{b}_{h^*} & B_{h^*} \\
\vec{d^*} \arrow[uu, "\pi^*"] \arrow[ur, "\sigma_{h^*}"] \arrow[dr, "\sigma_1"'] & & \cdots \\
 & \vec{b}_1 & B_1
\end{tikzcd}$$

where $\vdash^m \bigwedge \sigma_i B_i \wedge \pi^* D \to \bot$.

Let $\vec{s}$ be a base list of length k containing all and only all the elements occurring either in $\vec{d}$ or in $\vec{d^*}$. Then, by lemma 4.1(b), there are and are unique two projections $\gamma : k \to m$ and $\delta : k \to m^*$ such that $\vec{s}[\gamma] = \vec{d}$ and $\vec{s}[\delta] = \vec{d^*}$. Now

$$\begin{array}{ll}
(\vec{s}[\gamma]) = \vec{d} & \vec{s}[\delta] = \vec{d^*}\\
(\vec{s}[\gamma])[\pi] = \vec{d}[\pi] & (\vec{s}[\delta])[\pi^*] = \vec{d^*}[\pi^*]\\
(\vec{s}[\gamma])[\pi] = \vec{c} & (\vec{s}[\delta])[\pi^*] = \vec{c}\\
\hline
\multicolumn{2}{c}{(\vec{s}[\gamma])[\pi] = (\vec{s}[\delta])[\pi^*]}\\
\multicolumn{2}{c}{\vec{s}[\gamma\circ\pi] = \vec{s}[\delta\circ\pi^*]}\\
\multicolumn{2}{c}{\gamma\circ\pi = \delta\circ\pi^*}
\end{array}$$

Let $\alpha = \gamma \circ \pi$, then $\vec{s}[\alpha] = \vec{c}$, since $\vec{c} = (\vec{s}[\gamma])[\pi]$. Moreover

$$\frac{(\vec{s}[\gamma]) = \vec{d} \qquad \vec{d}[\tau_i] = \vec{a}_i}{\vec{s}[\gamma \circ \tau_i] = \vec{a}_i} \qquad\qquad \frac{\vec{s}[\delta] = \vec{d^*} \qquad \vec{d^*}[\sigma_i] = \vec{b}_i}{\vec{s}[\delta \circ \sigma_i] = \vec{b}_i}$$

Therefore Γ admits of the following critical diagram

$$\begin{tikzcd}
 & & \vec{a}_1 & A_1 \\
 & \vec{d} \arrow[ur, "\tau_1"] \arrow[dr, "\tau_h"] \arrow[dd, "\pi"'] & \cdots & \\
 & & \vec{a}_h & A_h \\
\vec{s} \arrow[uur, "\gamma"] \arrow[r, "\alpha"] \arrow[ddr, "\delta"'] & \vec{c} & C \vee D & \\
 & & \vec{b}_{h^*} & B_{h^*} \\
 & \vec{d^*} \arrow[uu, "\pi^*"] \arrow[ur, "\sigma_{h^*}"] \arrow[dr, "\sigma_1"'] & \cdots & \\
 & & \vec{b}_1 & B_1
\end{tikzcd}$$

where $\vdash^k \bigwedge \gamma \circ \tau_i A_i \wedge \bigwedge \delta \circ \sigma_i B_i \wedge \alpha(C \vee D) \rightarrow \bot$. To wit

$$\frac{\begin{array}{ll} \vdash^m \bigwedge \tau_i A_i \wedge \pi C \rightarrow \bot & \vdash^{m*} \bigwedge \sigma_i B_i \wedge \pi^* D \rightarrow \bot \\ \vdash^k \bigwedge \gamma \circ \tau_i A_i \wedge \gamma \circ \pi C \rightarrow \gamma\bot & \vdash^k \bigwedge \delta \circ \sigma_i B_i \wedge \delta \circ \pi^* D \rightarrow \delta\bot \\ \vdash^k \bigwedge \gamma \circ \tau_i A_i \wedge \alpha C \rightarrow \bot & \vdash^k \bigwedge \delta \circ \sigma_i B_i \wedge \alpha D \rightarrow \bot \end{array}}{\begin{array}{c} \vdash^k \bigwedge \gamma \circ \tau_i A_i \wedge \bigwedge \delta \circ \sigma_i B_i \wedge (\alpha C \vee \alpha D) \rightarrow \bot \\ \vdash^k \bigwedge \gamma \circ \tau_i A_i \wedge \bigwedge \delta \circ \sigma_i B_i \wedge \alpha(C \vee D) \rightarrow \bot \end{array}}$$

Therefore Γ is S-incoherent, contrary to the hypothesis of the lemma.
Conventional notation Given a projection $\alpha = \langle x_{i_1}, ...x_{i_m} \rangle$, $\langle \alpha, x_k \rangle$ donotes the projection $\langle x_{i_1}, ..., x_{i_m}, x_k \rangle$.

LEMMA 4.10. *(a) If* $\langle a_1, ..., a_k, ..., a_n \rangle$ *is a base list and* $\sigma : n \rightarrow m$ *is such that* $a_k \notin \langle a_1, ..., a_k, \; ..., a_n \rangle[\sigma]$*, for some* $1 \leq k \leq m$*, then there is a* *unique projection* $\sigma^* : (n-1) \to m$, *such that* $\langle a_1, ..., a_{k-1}, a_{k+1}, ..., a_n\rangle[\sigma^*] = \langle a_1, ..., a_n\rangle[\sigma]$.

(*b*) *If* $\langle a_1, ..., a_n, a\rangle$ *is a base list and* $\delta : n+1 \to m+1$ *is such that* $\langle a_1, ..., a_n, a\rangle[\delta] = \langle a_{i_1}, ..., a_{i_m}, a\rangle$, *where* $a \notin \langle a_{i_1}, ..., a_{i_m}\rangle$, *then there is a unique projection* $\delta^* : n \to m$, *such that:*

(*i*) $\langle a_1, ..., a_n\rangle[\delta^*] = \langle a_{i_1}, ..., a_{i_m}\rangle$, *and*

(*ii*) $\langle a_1, ..., a_n, a\rangle[\langle n+1 : x_1, ..., x_n\rangle \circ \delta^*, x_{n+1}] = \langle a_{i_1}, ..., a_{i_m}, a\rangle$.

Proof.(a) σ^* differs from σ only because the variables of index $k+n, n \geq 0$, are replaced by the variables of index $(k+n)-1$. Since $\langle a_1, ..., a_{k-1}, a_{k+1}, ..., a_n\rangle$ is a base list, the uniqueness of σ^* follows from lemma 4.1.

(b) Trivially $\delta^* : n \to m$ is the projection $\langle n : x_{i_1}, ..., x_{i_m}\rangle$. We show that $\delta^* = \langle x_{i_1}, ..., x_{i_m}\rangle$ satisfies also the condition (ii):

$\langle a_1, ..., a_n, a\rangle[\langle x_1, ..., x_n\rangle \circ \langle x_{i_1}, ..., x_{i_m}\rangle, x_{n+1}] =$
$\langle\langle a_1, ..., a_n, a\rangle[\langle x_1, ..., x_n\rangle \circ \langle x_{i_1}, ..., x_{i_m}\rangle], \langle a_1, ..., a_n, a\rangle[x_{n+1}]\rangle =$
$\langle((\langle a_1, ..., a_n, a\rangle[x_1, ..., x_n])[x_{i_1}, ..., x_{i_m}], a\rangle =$
$\langle\langle a_1, ..., a_n[x_{i_1}, ..., x_{i_m}], a\rangle =$
$\langle a_{i_1}, ..., a_{i_m}, a\rangle$.

The uniqueness of δ^* follows from lemma 4.1(b), in virtue of the fact that $\langle a_1, ..., a_n, a\rangle$ is a base list.

Definition 4.11. Given an X-graph Γ, we say that an element b of X *doesn't occur in* Γ iff b doesn't occur in any n-tuple $\vec{a} \in X^n$ such that $\langle \vec{a}, A\rangle \in \Gamma$, for some A.

Lemma 4.12. *If* Γ *is an* S*-coherent* X*-graph and* $\langle \vec{c}, \exists x_{p+1} B\rangle \in \Gamma$, *then* $\Gamma \dot{+} \langle \vec{c} * b, B\rangle$ *is* S*-coherent, where* $b \in X$ *does not occur in* Γ.

Proof. Let $\Gamma + \langle \vec{c} * b, B\rangle$ admit of a critical diagram

$$\begin{tikzcd}
& \vec{a}_1 & A_1 \\
\vec{d} \arrow[ur, "\tau_1"] \arrow[dr, "\tau_h"] \arrow[dd, "\pi"'] & & \cdots \\
& \vec{a}_h & A_h \\
\vec{c} * b & B
\end{tikzcd}$$

where $\vdash^m \bigwedge \tau_i A_i \wedge \pi B \to \bot$. Since $\vec{d}[\pi] = \vec{c} * b$, $\vec{d}$ is $\langle e_1, ..., e_{j-1}, b, e_{j+1}, ..., e_m\rangle$. Let $\vec{s} = \langle e_1, ..., e_{j-1}, e_{j+1}, ..., e_m\rangle$, $\alpha = \langle x_1, ..., x_{j-1}, x_m, x_j, ..., x_{m-1}\rangle : m \to m$, then

$$(*) \qquad (\vec{s} * b)[\alpha] = \vec{d}.$$

It follows that

$$\frac{((\vec{s} * b)[\alpha])[\pi] = \vec{d}[\pi] \qquad \vec{d}[\pi] = \vec{c} * b}{\begin{array}{l} ((\vec{s} * b)[\alpha])[\pi] = \vec{c} * b \\ (\vec{s} * b)[\alpha \circ \pi] = \vec{c} * b \end{array}}$$

where $\alpha : m \to m$, $\pi : m \to p + 1$, $\alpha \circ \pi : m \to p + 1$.
By lemma 4.1(b), there is a projection $\gamma : (m - 1) \to p$ such that

$$(\#) \qquad \vec{s}[\gamma] = \vec{c}.$$

Then $(\vec{s} * b)[\langle x_1, ..., x_{m-1}\rangle \circ \gamma, x_m\rangle] = \vec{c} * b$, where $\langle\langle x_1, ..., x_{m-1}\rangle \circ \gamma, x_m\rangle : m \to p + 1$.
Whence $(\vec{s} * b)[\alpha \circ \pi] = (\vec{s} * b)[\langle\langle x_1, ..., x_{m-1}\rangle \circ \gamma, x_m\rangle]$, and so, since $\vec{s} * b$ is a base list,

$$\alpha \circ \pi = \langle\langle x_1, ..., x_{m-1}\rangle \circ \gamma, x_m\rangle.$$

But

$$\frac{(\vec{s} * b)[\alpha] = \vec{d} \qquad \vec{d}[\tau_i] = \vec{a}_i}{\begin{array}{l} ((\vec{s} * b)[\alpha])[\tau_i] = \vec{a}_i \\ (\vec{s} * b)[\alpha \circ \tau_i] = \vec{a}_i \end{array}}$$

Since $b \notin \vec{a}_i$, by lemma 4.1(a), there is a projection $\tau_i^* : (m - 1) \to n_i$ such that

$$\vec{s}[\tau_i^*] = \vec{a}_i$$

hence $((\vec{s} * b)[\beta])[\tau_i^*] = \vec{a}_i$, where $\beta = \langle m : x_1, ..., x_{m-1}\rangle$, so

$$(\vec{s} * b)[\beta \circ \tau_i^*] = \vec{a}_i.$$

Therefore $(\vec{s} * b)[\alpha \circ \tau_i] = (\vec{s} * b)[\beta \circ \tau_i^*]$ and so

$$\alpha \circ \tau_i = \beta \circ \tau_i^*.$$

$$\begin{tikzcd}
 & \vec{s} & \\
\vec{s} * b \arrow[ur, "\beta"] \arrow[r, "\alpha"] \arrow[rr, bend right, "\langle\langle x_1 \ldots x_{m-1}\rangle \circ \gamma, x_m\rangle"'] & \vec{d} \arrow[r, "\pi"] & \vec{c} * b
\end{tikzcd}$$

where

$$\vec{s} \xrightarrow{\gamma} \vec{c}$$

It follows that

$$
\begin{array}{ll}
\vdash^m & \bigwedge \tau_i A_i \wedge \pi B \rightarrow \bot \\
\vdash^m & \bigwedge \alpha \circ \tau_i A_i \wedge \alpha \circ \pi B \rightarrow \alpha \bot, \\
\vdash^m & \bigwedge \beta \circ \tau_i^* A_i \wedge \langle \langle x_1, ..., x_{m-1} \rangle \circ \gamma, x_m \rangle B \rightarrow \bot, \\
\vdash^{m-1} & \exists x_m (\beta (\bigwedge \tau_i^* A_i) \wedge \langle \langle x_1, ..., x_{m-1} \rangle \circ \gamma, x_m \rangle B) \rightarrow \bot, \text{ since} \\
 & \beta = \langle x_1, ..., x_{m-1} \rangle \\
\vdash^{m-1} & \bigwedge \tau_i^* A_i \wedge \exists x_m (\langle \langle x_1, ..., x_{m-1} \rangle \circ \gamma, x_m \rangle B) \rightarrow \bot, \\
\vdash^{m-1} & \bigwedge \tau_i^* A_i \wedge \langle \langle x_1, ..., x_{m-1} \rangle \circ \gamma \rangle \exists x_{p+1} B) \rightarrow \bot, \\
\vdash^{m-1} & \bigwedge \tau_i^* A_i \wedge \langle x_1, ..., x_{m-1} \rangle (\gamma (\exists x_{p+1} B)) \rightarrow \bot, \\
\vdash^{m-1} & \bigwedge \tau_i^* A_i \wedge \gamma (\exists x_{p+1} B) \rightarrow \bot.
\end{array}
$$

Therefore Γ is S-incoherent because it admits of the following critical diagram

$$
\begin{tikzcd}
 & \vec{a}_1 & A_1 \\
\vec{s} \arrow[ur, "\tau_1^*"] \arrow[dr, "\tau_h^*"'] \arrow[dd, "\gamma"'] & & \cdots \\
 & \vec{a}_h & A_h \\
\vec{c} & \exists x_{p+1} B &
\end{tikzcd}
$$

where $\vdash^{m-1} \bigwedge \tau_i^* A_i \wedge \gamma(\exists x_{p+1} B) \rightarrow \bot$.

Saturated graphs

DEFINITION 4.13. A graph Γ is *S-saturated* iff

(1) Γ is S-coherent,

(2) Γ is *complete*, i.e. for all $\vec{a} \in X^n$ and wff A of type n, either $\langle \vec{a}, A \rangle \in \Gamma$ or $\langle \vec{a}, \neg A \rangle \in \Gamma$,

(3) Γ is *rich*, i.e. $\langle \vec{a}, \exists x_{n+1} B \rangle \in \Gamma$ iff there is a $b \in X$ s.t. $\langle \vec{a} * b, B \rangle \in \Gamma$.

LEMMA 4.14. *If Γ is an S-saturated X-graph, then*

(a) *If $\langle \vec{a}, A_i \rangle \in \Gamma$, $1 \leq i \leq r$, and $\vdash_S^n A_1 \wedge ... \wedge A_r \rightarrow B$, then $\langle \vec{a}, B \rangle \in \Gamma$,*

(a') *If $\vdash_S^n B$, then for any list $\vec{a} \in X^n$, $\langle \vec{a}, B \rangle \in \Gamma$.*

(b) $\langle \vec{a}, \langle n : x_{i_1}, ..., x_{i_m} \rangle A \rangle \in \Gamma$ *iff* $\langle \vec{a}[n : x_{i_1}, ..., x_{i_m}], A \rangle \in \Gamma$.

(c) $\langle \vec{a}, A \vee B \rangle \in \Gamma$ *iff* $\langle \vec{a}, A \rangle \in \Gamma$ *or* $\langle \vec{a}, B \rangle \in \Gamma$.

PROOF.(a) If $\langle \vec{a}, B \rangle \notin \Gamma$, then $\langle \vec{a}, \neg B \rangle \in \Gamma$, hence, since $\vdash_S^n A_1 \wedge \ldots \wedge A_r \wedge \neg B \rightarrow \bot$, Γ would be S-incoherent.

(b) Let $\pi = \langle n : x_{i_1}, ..., x_{i_m} \rangle$. Suppose that $\langle \vec{a}, \pi A \rangle \in \Gamma$ and $\langle \vec{a}\,[\pi], A \rangle \notin \Gamma$, then $\langle \vec{a}\,[\pi], \neg A \rangle \in \Gamma$, hence by lemma 4.8 $\Gamma + \langle \vec{a}, \pi \neg A \rangle$ is S-coherent and by lemma 4.7, $\Gamma + \langle \vec{a}, \neg\pi A\rangle$ is S-coherent in contradiction with the fact that $\langle \vec{a}, \pi A\rangle \in \Gamma$. Suppose that $\langle \vec{a}\,[\pi], A\rangle \in \Gamma$ and $\langle \vec{a}, \pi A\rangle \notin \Gamma$ then $\langle \vec{a}, \neg\pi A\rangle \in \Gamma$, so $\Gamma + \langle \vec{a}, \pi\neg A\rangle$ is S-coherent by lemma 4.7, and by lemma 4.8, $\Gamma + \langle \vec{a}\,[\pi], \neg A\rangle$ is S-coherent in contradiction with the fact that $\langle \vec{a}\,[\pi], A\rangle \in \Gamma$.

(c) From lemma 4.9.

Lemma 4.15. *Let Γ be an S-coherent X-graph. Then there is a non-empty set Y and an S-saturated $(X \cup Y)$-graph Δ such that $\Delta \supseteq \Gamma$.*

Proof. Take a non-empty countable set Y disjoint from X and let $\langle \vec{a}_1, A_1\rangle, \langle \vec{a}_2, A_2\rangle, \ldots$ be an enumeration of $\{\langle \vec{a}, A\rangle$: for some $n \geq 0$, $\vec{a} \in (X \cup Y)^n$ and A is a formula of type n of $\mathcal{L}^t\}$. Define the following chain of $(X \cup Y)$-graphs:

$\Delta_0 = \Gamma$,

$$\Delta^*_{k+1} = \begin{cases} \Delta_k + \langle \vec{a}_{k+1}, A_{k+1}\rangle & \text{if } \Delta_k + \langle \vec{a}_{k+1}, A_{k+1}\rangle \text{ is } S\text{-coherent,} \\ \Delta_k + \langle \vec{a}_{k+1}, \neg A_{k+1}\rangle & \text{otherwise.} \end{cases}$$

$$\Delta_{k+1} = \begin{cases} \Delta^*_{k+1} + \langle \vec{a}_{k+1} * d,\ B\rangle & \text{if } \Delta^*_{k+1} = \Delta_k + \langle \vec{a}_{k+1}, A_{k+1}\rangle,\ A_{k+1} = \exists x_i B \text{ and} \\ & d \in X \text{ does not occur in } \Delta^*_{k+1}, \\ \Delta^*_{k+1} & \text{otherwise.} \end{cases}$$

$\Delta = \bigcup_{k \in N} \Delta_k$.

It is easy to see that Δ is S-saturated. The S-coherence of Δ follows from the fact that each Δ_k is S-coherent and this, in turn, follows from lemmas 4.9 and 4.12.

Lemma 4.16. *Let Γ be an S-saturated X-graph. If $\langle\langle a_1, ..., a_n\rangle, \Diamond A\rangle \in \Gamma$, then for any n-tuple $\langle b_1, ..., b_n\rangle$ such that $b_i \neq b_j$, if $i \neq j$,*

$$\Delta = \{\langle \vec{b}, A\rangle\} \cup \{\langle \vec{b}, B\rangle : \langle \vec{a}, \Box B\rangle \in \Gamma\}$$

is S-coherent, where $\vec{a} = \langle a_1, ..., a_n\rangle$ and $\vec{b} = \langle b_1, ..., b_n\rangle$.

Proof. Suppose by *reductio* that Δ admits of a critical diagram

$$\begin{tikzcd}
 & \vec{b}_1 & B_1 \\
\vec{d} \arrow[ur, "\tau_1"] \arrow[dr, "\tau_h"] \arrow[ddd, "\tau"'] & & \cdots \\
 & \vec{b}_h & B_h \\
 & & \\
\vec{b} & A &
\end{tikzcd}$$

where $\vdash^m \bigwedge \tau_i B_i \wedge \tau A \to \bot$.

Since $\vec{d}$ is a base list, by lemma 4.1(a), $\tau_i = \tau$. But $\vec{b}$ is a base list too and $\vec{d} \subseteq \vec{b}$, since $\vec{b}$ contains all the elements we started with, then by lemma 4.1(c) there is a π such that $\pi \circ \tau = \langle x_1, ..., x_n \rangle$. ¿From (i), by the rule of substitution, it obtains:

$\vdash^n\ \pi(\tau B_1) \wedge ... \wedge \pi(\tau B_h) \wedge \pi(\tau A) \rightarrow \pi\bot$;
$\vdash^n\ (\pi \circ \tau)B_1 \wedge ... \wedge (\pi \circ \tau)B_h \wedge (\pi \circ \tau)A \rightarrow\ \bot$;
$\vdash^n\ \langle x_1, ..., x_n\rangle B_1 \wedge ... \wedge \langle x_1, ..., x_n\rangle B_h \wedge \langle x_1, ..., x_n\rangle A \rightarrow \langle x_1, ..., x_n\rangle\bot$; i.e.
$\vdash^n\ B_1 \wedge ... \wedge B_h \wedge A \rightarrow \bot$; hence
$\vdash^n\ B_1 \wedge ... \wedge B_h \rightarrow \neg A$;
$\vdash^n\ \Box B_1 \wedge ... \wedge \Box B_h \rightarrow \Box\neg A$;
$\vdash^n\ \Box B_1 \wedge ... \wedge \Box B_h \rightarrow \neg\Diamond A$.

But, by hypothesis $\langle \vec{a}, \Box B_i\rangle \in \Gamma$, $1 \leq i \leq h$, so $\langle \vec{a}, \neg\Diamond A\rangle \in \Gamma$, contrary to the fact that $\langle \vec{a}, \Diamond A\rangle \in \Gamma$ and Γ is S-coherent.

4.1. Modal systems without individual constants, function symbols and identity. For reasons of clarity we first prove completeness theorems for modal logics whose languages contain neither individual constants nor function symbols nor identity symbol, i.e. languages whose only complex terms are projections. We denote such logics by S_*, $Q.K^t_*$, ...

Definition 4.17. Let $S_* \supseteq Q.K^t_*$ and U be an infinite set. The *canonical model* M^U for S_* is a quintuple $\langle W^U, R, D, \mathfrak{C}, I\rangle$, where

- W^U is the class of all S_*-saturated X-graphs w for some set $X \subset U$ such that $|U - X| \geq \aleph_0$,
- wRv iff if $\langle\langle\rangle, \Box B\rangle \in w$ then $\langle\langle\rangle, B\rangle \in v$,
- $D_w = X$, if w is an X-graph,
- $\mathfrak{C} = \biguplus\{\mathfrak{C}_{\langle w,v\rangle}\}_{w,v\in W}$, where $\mathfrak{C}_{\langle w,v\rangle} \subseteq D_w \times D_v$ is *admissible*, i.e. for every $n \geq 1$, if $\{\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle\} \subseteq \mathfrak{C}_{\langle w,v\rangle}$ then for all wff $\Box B$ of type m and projections $\langle n : x_{i_1}, ..., x_{i_m}\rangle$, if $\langle \vec{a}\,[n : x_{i_1}, ..., x_{i_m}], \Box B\rangle \in w$, then $\langle \vec{b}\,[n : x_{i_1}, ..., x_{i_m}], B\rangle \in v$, where $\vec{a} = \langle a_1, \ldots, a_n\rangle$ and $\vec{b} = \langle b_1, \ldots, b_n\rangle$.
- $I_w(P^n) = \{\langle a_1, \ldots, a_n\rangle : \langle\langle a_1, \ldots, a_n\rangle,\ P^n\rangle \in w\}$.

Conventional notation. If $\vec{a} \in (D_w)^n$ and $\vec{b} \in (D_v)^n$, then $\vec{a}\mathfrak{C}\vec{b}$ is an abbreviation for $\{\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle\} \subseteq \mathfrak{C}_{\langle w,v\rangle}$.

Note From the definitions of R and $\mathfrak{C}$ it follows that if for some $\vec{a} \in (D_w)^n$ there is a $\vec{b} \in (D_v)^n$ such that $\vec{a}\mathfrak{C}\vec{b}$, then wRv. To wit let $\vec{a} \in (D_w)^n$ and $\langle\langle\,\rangle, \Box B\rangle \in w$ for some sentence $\Box B$. Then $\langle \vec{a}[\langle\,\rangle], \Box B\rangle \in w$. Since w is S_*-saturated, $\langle \vec{a}, \langle\,\rangle\Box B\rangle \in w$, then, by axiom $S^{\Box}$, $\langle \vec{a}, \Box\langle\rangle B\rangle \in w$. Then by definition of $\mathfrak{C}$, $\langle \vec{b}, \langle\,\rangle B\rangle \in v$, whence, since v is S_*-saturated, $\langle \vec{b}[\langle\,\rangle], B\rangle \in v$, so $\langle\langle\rangle, B\rangle \in v$.

Note The following conditions on the counterpart relation $\mathfrak{C}$ of a canonical model are equivalent to one another:

(1) if $\vec{a}\,\mathfrak{C}\vec{b}$ and $\langle\vec{a},\Box B\rangle \in w$ then $\langle\vec{b},B\rangle \in v$,

(2) if $\vec{a}\mathfrak{C}\vec{b}$ and $\langle\vec{b},B\rangle \in v$ then $\langle\vec{a},\Diamond B\rangle \in w$.

LEMMA 4.18. *Let $M^U = \langle W^U, R, D, \mathfrak{C}, I\rangle$ be a canonical model for S_*. If $w \in W^U$, $\vec{a} \in (D_w)^n$ and $\langle\vec{a},\Diamond A\rangle \in w$, then there is a $v \in W^U$ and a list $\vec{b} \in (D_v)^n$, such that*

(a) $\langle\vec{b},A\rangle \in v$,

(b) $\mathfrak{C}_{\langle w,v\rangle} = \{\langle a_1,b_1\rangle,\dots,\langle a_n,b_n\rangle\}$ *is admissible.*

PROOF. (a) Let $\vec{b} = \langle b_1,...,b_n\rangle$ be a list of elements of U distinct from those of D_w. By lemma 4.16 the graph $\Delta = \{\langle\vec{b},A\rangle\} \cup \{\langle\vec{b},B\rangle : \langle\vec{a},\Box B\rangle \in w\}$ is S_*-coherent. Let X be a subset of $(U - D_w)$ such that $\{b_1\dots b_n\} \subseteq X$ and $|U - X| \geq \aleph_0$. Since $|U - D_w| \geq \aleph_0$, the set X exists. Then by lemma 4.15, there is an S_*-saturated graph Γ that extends Δ. Therefore Γ is an element of the canonical model, let us call it v.

(b) Suppose that for some $\langle n : x_{i_1},...,x_{i_m}\rangle$ and wff $\Box B$ of type m, $\langle\vec{a}\,[x_{i_1},...,x_{i_m}],\Box B\rangle \in w$, then $\langle\vec{a},\langle x_{i_1},...,x_{i_m}\rangle\Box B\rangle \in w$, and by axiom $S^\Box$, $\langle\vec{a},\Box\langle x_{i_1},...,x_{i_m}\rangle B\rangle \in w$, therefore by definition of Δ, $\langle\vec{b},\langle x_{i_1},...,x_{i_m}\rangle B\rangle \in \Delta$. Whence $\langle\vec{b}\,[x_{i_1},...,x_{i_m}],\,B\rangle \in v$, consequently $\{\langle a_1,b_1\rangle,\dots,\langle a_n,b_n\rangle\}$ is admissible.

LEMMA 4.19. *(of the canonical model) Let $M^U = \langle W^U, R, D, \mathfrak{C}, I\rangle$ be a canonical model for S_*. Then for any wff C of type n,*

$$\vec{c} \models^n_w C \quad \text{iff} \quad \langle\vec{c},C\rangle \in w.$$

PROOF. By induction on C. Let us just consider a few cases.
$\vec{c} \models^n_w P^n$ iff $\vec{c} \in I_w(P^n)$ iff $\langle\vec{c},P^n\rangle \in w$.
$\vec{c} \models^n_w \langle n : x_{i_1},...,x_{i_m}\rangle B$ iff $\vec{c}\,[n : x_{i_1},...,x_{i_m}] \models^m_w B$ iff, by induction hypothesis, $\langle\vec{c}\,[n : x_{i_1},...,x_{i_m}],B\rangle \in w$ iff $\langle\vec{c},\langle n : x_{i_1},...,x_{i_m}\rangle B\rangle \in w$.
$\vec{c} \not\models^n_w \Box B$ only if for some $\vec{d} \in D_v$ such that $\vec{c}\,\mathfrak{C}\vec{d}$, $\vec{d} \not\models^n_v B$ only if, by induction hypothesis, $\langle\vec{d},B\rangle \notin v$, then by definition of $\mathfrak{C}$, $\langle\vec{c},\Box B\rangle \notin w$. If $\langle\vec{c},\Box B\rangle \notin w$ then $\langle\vec{c},\Diamond\neg B\rangle \in w$, and by the lemma 4.18, there is a v and a list $\vec{d}$ of elements of D_v, such that $\vec{c}\,\mathfrak{C}\vec{d}$ and $\langle\vec{d},\neg B\rangle \in v$, therefore $\langle\vec{d},B\rangle \notin v$, so by induction hypothesis, $\vec{d} \not\models^n_v B$, consequently $\vec{c} \not\models^n_w \Box B$.

LEMMA 4.20. *Let $S_* \supseteq Q.K^t_*$ and M^U be a canonical model for S_*. Then M^U is a model for S_*.*

PROOF. We recall that M is a model for S_* iff every theorem of S_* is true at every world of M. Since every theorem of S_* belongs to any world of the canonical model for S_*, by the lemma of the canonical model, M^U is a model for S_*.

Lemma 4.21. *Let $S_* \supseteq Q.K^t_*$ and T be an S_*-consistent set of sentences. Then T has a model.*

Proof. Let M^U be a canonical model for S_*. By lemma 4.6 the $\emptyset$-graph $\Gamma = \{\langle\langle\,\rangle, A\rangle : A \in T\}$ is S_*-coherent. Let X be a denumerable set $X \subseteq U$ such that $|U - X| \geq \aleph_0$. By lemma 4.15 there is an S_*-saturated X-graph Δ such that $\Gamma \subseteq \Delta$. Therefore Δ is bound to be a point w of the canonical model M^U for S_* and so, by the lemma of the canonical model, all the sentences of T are true at w. Consequently M^U is a model for T.

Lemma 4.22. *Let $S_* \supseteq Q.K^t_*$ and A be a sentence such that $S_* \nvdash A$. Then A is false in some model for S_*.*

We recall that a counterpart frame is a *frame for a logic S* if any model based on that frame is a model for S.

Completeness theorem for $Q.K^t_*$. Take a denumerable set U and consider the canonical model M^U for $Q.K^t_*$. Trivially M^U is based on a counterpart frame for $Q.K^t_*$ since any counterpart frame is a frame for $Q.K^t_*$. The completeness of $Q.K^t_*$ follows from lemma 4.22.

Completeness theorem for $Q.S5^t_*$. Consider a canonical model M^U for $Q.S5^t_*$ for some infinite set U. We have to show that the counterpart relation is reflexive, transitive and symmetric. It is easy to show that $\mathfrak{C}$ is reflexive. Suppose that $\langle\vec{a}\,[\tau], \Box B\rangle \in w$ for some projection $\tau : n \to m$ and wff $\Box B$ of type m. Then by axiom T, $\langle\vec{a}\,[\tau], B\rangle \in w$, consequently $\vec{a}\mathfrak{C}\vec{a}$.

$\mathfrak{C}$ is symmetric. Suppose that $\vec{a}\mathfrak{C}\vec{b}$. If $\langle\langle\vec{b}\,[\tau], \Box B\rangle \in v$, then, since $\vec{a}\mathfrak{C}\vec{b}$, $\langle\vec{a}\,[\tau], \Diamond\Box B\rangle \in w$ obtains, so by axiom B, $\langle\vec{a}\,[\tau], B\rangle \in w$, consequently $\vec{b}\mathfrak{C}\vec{a}$.

$\mathfrak{C}$ is transitive. Suppose that $\vec{a}\mathfrak{C}\vec{b}$ and $\vec{b}\mathfrak{C}\vec{c}$. If $\langle\vec{a}\,[\tau], \Box B\rangle \in w$ then by axiom 4, $\langle\vec{a}\,[\tau], \Box\Box B\rangle \in w$, then $\langle\vec{b}\,[\tau], \Box B\rangle \in v$ and $\langle\vec{c}\,[\tau], B\rangle \in z$, consequently $\vec{a}\mathfrak{C}\vec{c}$.

Completeness theorems for $Q.T^t_*$, $Q.S4^t_*$ and $Q.B^t_*$ are straightforward.

Completeness theorem for $BF.K^t_*$ To this aim we show how to build canonical models in which the counterpart relation is surjective, i.e., if wRv then for all $b \in D_v$ there is an $a \in D_w$ such that $\langle a, b\rangle \in \mathfrak{C}_{\langle w,v\rangle}$. The following lemma is all we need.

Lemma 4.23. *Let Γ be a $BF.K^t_*$-saturated X-graph such that $\langle\vec{a}, \Diamond A\rangle \in \Gamma$, where $\vec{a} = \langle a_1, ..., a_n\rangle$. Then*

1. *there is set Y and a $BF.K^t_*$-saturated Y-graph Δ, such that $\langle\vec{b}, A\rangle \in \Delta$, for some $\vec{b} = \langle b_1, ..., b_n\rangle \in Y^n$,*

2. *there is a relation* $\mathfrak{C}_{\langle\Gamma,\Delta\rangle} \subseteq (X \times Y)$ *which is surjective, admissible and such that*
$\{\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle\} \subseteq \mathfrak{C}_{\langle\Gamma,\Delta\rangle}$.

PROOF. Let $Y = \{b_1, b_2, b_3, \ldots\}$ be a denumerable set disjoint from X. By lemma 4.16,

$$\Delta_0 = \{\langle \vec{b}, A\rangle\} \cup \{\langle \vec{b}, B\rangle : \langle \vec{a}, \Box B\rangle \in \Gamma\}$$

is $BF.K^t_*$-coherent.
Define $\langle b_1, ..., b_n\rangle$ to be the *generating list* of Δ_0, the formula A to be the *generating formula* of Δ_0 and $C_0 = \{\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle\}$.

Starting from Δ_0 we show how to construct an infinite sequence of $BF.K^t_*$-coherent Y-graphs Δ_1, Δ_2, ... Let $\langle \vec{e}_1, \exists x_{k_1} E_1\rangle, \langle \vec{e}_2, \exists x_{k_2} E_2\rangle, \ldots$ be an enumeration of all the pairs composed of a $(k_i - 1)$-tuple, $k_i \geq 1$, of elements of Y and an existential formula $\exists x_{k_i} E_i$ of $\mathcal{L}^t_*$.

Step 1.

Choose the first pair (and cancel it) $\langle \vec{e}_i, \exists x_{k_i} E_i\rangle$ in the enumeration above such that every element of $\vec{e}_i$ occurs in the generating list of Δ_0. Since $\langle \vec{a}, \Diamond A\rangle \in \Gamma$, then also $\langle \vec{a}, \Diamond(A \wedge \sigma\neg\exists x_{k_i} E_i) \vee \Diamond(A \wedge \sigma\exists x_{k_i} E_i)\rangle \in \Gamma$, where $\sigma : n \to k_i$ and $\vec{a} = \langle a_1, \ldots, a_n\rangle$. Then either $\langle \vec{a}, \Diamond(A \wedge \sigma\neg\exists x_{k_i} E_i)\rangle \in \Gamma$ or $\langle \vec{a}, \Diamond(A \wedge \sigma\exists x_{k_i} E_i)\rangle \in \Gamma$.
If the first is the case in point, define

$$\Delta_1 = \{\vec{b}, (A \wedge \sigma\neg\exists x_{k_i} E_i)\rangle\} \cup \{\langle \vec{b}, B\rangle : \langle \vec{a}, \Box B\rangle \in \Gamma\}.$$

By lemma 4.16, Δ_1 is $BF.K^t_*$-coherent.
Let the *generating list* of Δ_1 be the same as the generating list of Δ_0, $(A \wedge \sigma\neg\exists x_{k_i} E_i)$ be the *generating formula* of Δ_1 and $C_1 = C_0$.
If the second is the case in point, i.e. $\langle \vec{a}, \Diamond(A \wedge \sigma\exists x_{k_i} E_i)\rangle \in \Gamma$, then
$\langle \vec{a}, \Diamond(A \wedge \exists x_{n+1}\langle n+1 : \sigma, x_{n+1}\rangle E_i)\rangle \in \Gamma$,
$\langle \vec{a}, \Diamond\exists x_{n+1}(\tau A \wedge \pi E_i)\rangle \in \Gamma$, where $\tau = \langle n+1 : x_1, \ldots, x_n\rangle$ and $\pi = \langle n+1 : \sigma, x_{n+1}\rangle$. Then by BF,
$\langle \vec{a}, \exists x_{n+1}\Diamond(\tau A \wedge \pi E_i)\rangle \in \Gamma$. Hence for some $a^* \in X$
$\langle \vec{a} * a^*, \Diamond(\tau A \wedge \pi E_i)\rangle \in \Gamma$.
Define:

$$\Delta_1 = \{\langle \vec{b} * b^*, (\tau A \wedge \pi E_i)\rangle\} \cup \{\langle \vec{b} * b^*, B\rangle : \langle \vec{a} * a^*, \Box B\rangle \in \Gamma\}$$

where b^* is an element of Y not occurring in the generating list of Δ_0. Let $\vec{b} * b^*$ be the *generating list* of Δ_1, $(\tau A \wedge \pi E_i)$ be the *generating formula* of Δ_1 and $C_1 = C_0 \cup \{\langle a^*, b^*\rangle\}$.

Step n+1.

In the same way as above we construct Δ_{n+1} from Δ_n and C_{n+1} from C_n.

Let $Cl(\Delta_n)$ be the closure of Δ_n under deduction, substitution and disjunction. Since Δ_n is $BF.K^t_*$-coherent, by lemmas 4.7, 4.8 and 4.9, $Cl(\Delta_n)$ is $BF.K^t_*$-coherent too. Moreover $Cl(\Delta_n) \subseteq Cl(\Delta_{n+1})$.

Let $\Delta = \bigcup Cl(\Delta_n)$. It is easy to see that Δ is $BF.K^t_*$-saturated. Let $\mathfrak{C}_{\langle\Gamma,\Delta\rangle} = \bigcup C_n$. $\mathfrak{C}_{\langle\Gamma,\Delta\rangle}$ is trivially surjective, it remains to show that it is admissible. Suppose that $\langle\langle c_1,\dots,c_n\rangle[\tau],\Box B\rangle \in \Gamma$, where $\tau = \langle n : x_{i_1},\dots,x_{i_m}\rangle$. If $\langle c_1,d_1\rangle,\dots,\langle c_n,d_n\rangle \in \mathfrak{C}_{\langle\Gamma,\Delta\rangle}$, we have to show that $\langle\langle d_1,\dots,d_n\rangle[\tau],B\rangle \in \Delta$. Consider the first Δ_n whose generating list $\langle\langle e^*_1,\dots,e^*_r\rangle,E\rangle$ is such that it contains all the elements occurring in $\langle d_1,\dots,d_n\rangle$. Then there is a projection $\sigma : r \to n$ such that $\langle e^*_1,\dots,e^*_r\rangle[\sigma] = \langle d_1,\dots,d_n\rangle$. By the way in which each C_n has been defined, for each e^*_k there is exactly an e_k such that $\langle e_k,e^*_k\rangle \in \mathfrak{C}_{\langle\Gamma,\Delta\rangle}$, then $\langle e_1,\dots,e_r\rangle[\sigma] = \langle c_1,\dots,c_n\rangle$. Therefore

$\langle\langle e_1,\dots,e_r\rangle[\sigma])[\tau],\Box B\rangle \in \Gamma$,
$\langle\langle e_1,\dots,e_r\rangle[\sigma\circ\tau],\Box B\rangle \in \Gamma$,
$\langle\langle e_1,\dots,e_r\rangle,\langle\sigma\circ\tau\rangle\Box B\rangle \in \Gamma$, and by $S^\Box$
$\langle\langle e_1,\dots,e_r\rangle,\Box\langle\sigma\circ\tau\rangle B\rangle \in \Gamma$,
$\langle\langle e^*_1,\dots,e^*_r\rangle,\langle\sigma\circ\tau\rangle B\rangle \in \Delta$,
$\langle\langle e^*_1,\dots,e^*_r\rangle[\sigma\circ\tau],B\rangle \in \Delta$,
$\langle\langle e^*_1,\dots,e^*_r\rangle[\sigma])[\tau],B\rangle \in \Delta$,
$\langle\langle d_1,\dots,d_n\rangle[\tau],B\rangle \in \Delta$. So $\mathfrak{C}_{\langle\Gamma,\Delta\rangle}$ is admissible.

It follows that

Lemma 4.24. *Let $M^U = \langle W^U,R,D,\mathfrak{C},I\rangle$ be a canonical model for $BF.K^t_*$. If $w \in W^U$, $\vec{a} \in (D_w)^n$ and $\langle\vec{a},\Diamond A\rangle \in w$, then*

(a) *there is a $v \in W^U$ and a list $\vec{b} = \langle b_1,\dots,b_n\rangle \in (D_v)^n$, such that $\langle\vec{b},A\rangle \in v$,*
(b) *there is a relation $\mathfrak{C}_{\langle w,v\rangle} \subseteq D_w \times D_v$ which is surjective, admissible and such that $\{\langle a_1,b_1\rangle,\dots,\langle a_n,b_n\rangle\} \subseteq \mathfrak{C}_{\langle w,v\rangle}$.*

Completeness theorem for $GF.K^t_*$ To this aim we show how to build canonical models for $GF.K^t_*$ in which the counterpart relation is everywhere defined, i.e., if wRv then for all $a \in D_w$ there is an $b \in D_v$ such that $\langle a,b\rangle \in \mathfrak{C}_{\langle w,v\rangle}$.

Lemma 4.25. *Let Γ be a $GF.K^t_*$-saturated X-graph. If $\langle\vec{a},\Diamond A\rangle \in \Gamma$, then $\Delta = \{\langle\vec{a},A\rangle\} \cup \{\langle\vec{a}*\vec{e},B\rangle : \langle\vec{a}*\vec{e},\Box B\rangle \in \Gamma\}$ is $GF.K^t_*$-coherent, where $\Box B$ is of type $n+r$, $\vec{a} = a_1,\dots,a_n$ and $\vec{e} \in (X - \{a_1,\dots,a_n\})^r$ for some $r \geq 0\}$.*

Proof. Suppose by *reductio* that Δ admits of a critical diagram

$$\begin{tikzcd}
& \vec{a} * \vec{e}_1 & A_1 \\
\vec{d} \arrow[ur, "\tau_1"] \arrow[dr, "\tau_h"'] \arrow[dd, "\tau"] & & \cdots \\
& \vec{a} * \vec{e}_h & A_h \\
\vec{a} & A &
\end{tikzcd}$$

where $\vdash^m \bigwedge \tau_i B_i \wedge \tau A \to \bot$.

Since $\vec{d}[\tau] = \vec{a}$, $\vec{d}$ contains all the elements of $\vec{a}$, hence without loss of generality, we can assume that $\vec{d} = \vec{a}' * \vec{d}'$, where $\vec{a}'$ contains all and only the elements of a without repetitions and $\vec{d}'$ contains all and only the elements of $\vec{d}$ except those occurring in $\vec{a}$. Therefore $\tau = \langle m : x_{i_1}, \ldots, x_{i_n} \rangle$ and there are projections $\pi_1, \ldots, \pi_h$ containing variables with index $> n$, such that $\tau_i = \langle \tau, \pi_i \rangle : m \to r$, $1 \leq i \leq h$.

Then the above critical diagram becomes

$$\begin{tikzcd}
& \vec{a} * \vec{e}_1 & A_1 \\
\vec{a}' * \vec{d}' \arrow[ur, "\tau * \pi_1"] \arrow[dr, "\tau * \pi_h"'] \arrow[dd, "\tau"] & & \cdots \\
& \vec{a} * \vec{e}_h & A_h \\
\vec{a} & A &
\end{tikzcd}$$

Since $\vdash^m \bigwedge \tau_i B_i \wedge \tau A \to \bot$, it obtains:

$\vdash^m \bigwedge (\tau * \pi_i) B_i \wedge \tau A \to \bot$,
$\vdash^m \Box \bigwedge (\tau * \pi_i) B_i \to \Box \neg \tau A$,
$\vdash^m \bigwedge \Box (\tau * \pi_i) B_i \to \Box \tau \neg A$, by $S^\Box$
$\vdash^m \bigwedge (\tau * \pi_i) \Box B_i \to \Box \tau \neg A$, by GF
$\vdash^m \bigwedge (\tau * \pi_i) \Box B_i \to \tau \Box \neg A$.

But, $\langle \vec{a} * \vec{e}_i, \Box B_i \rangle \in \Gamma$, $1 \leq i \leq h$, hence $\langle \vec{a}' * \vec{d}'[\tau * \pi_i], \Box B_i \rangle \in \Gamma$, $1 \leq i \leq h$,
$\langle \vec{a}' * \vec{d}', (\tau * \pi_i) \Box B_i \rangle \in \Gamma$,
$\langle \vec{a}' * \vec{d}', \tau \Box \neg A \rangle \in \Gamma$,

$\langle \vec{a}' * \vec{d}'[\tau], \Box\neg A\rangle \in \Gamma$,
$\langle \vec{a}, \Box\neg A\rangle \in \Gamma$, in contradiction with the $GF.K^t_*$-coherence of Γ. Therefore Δ is $GF.K^t_*$-coherent.

LEMMA 4.26. *Let $M^U = \langle W^U, R, D, \mathfrak{C}, I\rangle$ be a canonical model for $GF.K^t_*$. If $w \in W^U$, $\vec{a} \in (D_w)^n$ and $\langle \vec{a}, \Diamond A\rangle \in w$, then*

(a) *there is a $v \in W^U$ and a list $\vec{b} \in (D_v)^n$, such that $\langle \vec{b}, A\rangle \in v$,*
(b) *$\mathfrak{C}_{\langle w,v\rangle} = \{\langle c, c\rangle : c \in D_w\}$ is admissible.*

¿From (b) it follows that $\mathfrak{C}_{\langle w,v\rangle}$ is everywhere defined.

PROOF. (a) By lemma 4.25, $\Delta = \{\langle \vec{a}, A\rangle\} \cup \{\langle \vec{a} * \vec{e}, B\rangle : \langle \vec{a} * \vec{e}, \Box B\rangle \in w$ } is $GF.K^t_*$-coherent, where $\Box B$ is of type $n + r$ and $\vec{e} \in (D_w - \{a_1, \ldots, a_n\})^r$, for some $r \geq 0\}$. By lemma 4.15, there is a $GF.K^t_*$-saturated graph Γ that extends Δ. Since $\langle \vec{a}, A\rangle \in \Delta$, let Γ be the element v of the canonical model we are looking for.

(b) Suppose that $\{\langle c_1, \ldots, c_n\rangle[\tau], \Box B\rangle \in w$, where $\tau = \langle n : x_{i_1}, \ldots, x_{i_m}\rangle$. To show that $\mathfrak{C}_{\langle w,v\rangle}$ is admissible, we have to prove that $\{\langle c_1, \ldots, c_n\rangle[\tau], B\rangle \in v$. Now,
$\langle c_1, \ldots, c_n\rangle = \langle \vec{a} * \vec{e}_i\rangle[\pi]$, where $\pi : p \to n$, for some p. Then
$\langle (\vec{a} * \vec{e}_i[\pi])[\tau], \Box B\rangle \in w$,
$\langle \vec{a} * \vec{e}_i[\pi \circ \tau], \Box B\rangle \in w$,
$\langle \vec{a} * \vec{e}_i, \langle \pi \circ \tau\rangle \Box B\rangle \in w$,
$\langle \vec{a} * \vec{e}_i, \Box(\pi \circ \tau)B\rangle \in w$, by $S^\Box$,
$\langle \vec{a} * \vec{e}_i, (\pi \circ \tau)B\rangle \in v$, since $\Delta \subseteq \Gamma = v$,
$\langle \vec{a} * \vec{e}_i[\pi \circ \tau], B\rangle \in v$,
$\{(\vec{a} * \vec{e}_i[\pi])[\tau], B\rangle \in v$,
$\{\langle c_1, \ldots, c_n\rangle[\tau], B\rangle \in v$.

Note We conclude this section by observing that the standard Henkin technique of adding to the language names for the individuals of the universe is not suitable in the present context. Take an S-saturated X-graph w and suppose that if $c \in X$. Then c is also a constant of type 0 of the language. Therefore the following fact holds for any formula A of type n:

$$(H) \qquad \langle\langle \underbrace{c \ldots c}_{n-times} \rangle, A\rangle \in w \quad \text{iff} \quad \langle\langle\,\rangle, \langle \underbrace{c \ldots c}_{n-times} \rangle A\rangle \in w.$$

But if so, w does not distinguish between *de re* and *de dicto* modalities. For, let A be a sentence of type 0, then
$\langle\langle c\rangle, \Box\langle 1 :\rangle A\rangle \in w$, iff, by H,
$\langle\langle\,\rangle, \langle c\rangle\Box\langle 1 :\rangle A\rangle \in w$, only if, by $S^\Box$,
$\langle\langle\,\rangle, \Box\langle c\rangle\langle 1 :\rangle A\rangle \in w$ iff
$\langle\langle\,\rangle, \Box A\rangle \in w$ iff
$\langle\langle c\rangle[1 :], \Box A\rangle \in w$ iff
$\langle\langle c\rangle, \langle 1 :\rangle\Box A\rangle \in w$.

Consequently $\langle\langle c\rangle, \Box\langle 1:\rangle A\rangle \in w$ iff $\langle\langle c\rangle, \langle 1:\rangle\Box A\rangle \in w$.

4.2. Modal systems with identity and non-rigid terms. Throughout this section, $S \supseteq Q.K^t$, therefore the language of S contains individual constants, function symbols and identity.

DEFINITION 4.27. Let Γ be an S-saturated X-graph.

$$a \sim b \text{ iff } \langle\langle a, b\rangle, (x_1 = x_2)\rangle \in \Gamma.$$

LEMMA 4.28. *Let Γ be an S-saturated X-graph. The relation $\sim$ is an equivalence relation.*

DEFINITION 4.29. An X-graph Γ is said to be *normal* iff

$$\langle\langle a, b\rangle, (x_1 = x_2)\rangle \in \Gamma \quad \text{iff} \quad a \text{ is identical with } b.$$

LEMMA 4.30. *For each S-saturated X-graph Δ, there is a Y-graph Γ such that*

(a) $Y \subseteq X$*,*

(b) Γ *is normal,*

(c) For all formulas A of type n and $\vec{a} \in X^n$, if $\langle\vec{a}, A\rangle \in \Delta$, then there is a $\vec{b} \in Y^n$, such that $\langle\vec{b}, A\rangle \in \Gamma$,

(d) Γ *is S-saturated,*

(e) Γ *is normal,*

(f) For all formulas A of type n and $\vec{a} \in X^n$, if $\langle\vec{a}, A\rangle \in \Delta$, $\vec{b} \in Y^n$ and $a_1 \sim b_1, \dots, a_n \sim a_n$, then $\langle\vec{b}, A\rangle \in \Gamma$.

PROOF. (a) Consider the equivalence classes of $X/_\sim$ induced by the relation $\sim$ and let Y consist of exactly one element from each equivalence class. Define

$$\Gamma = \{\langle\vec{a}, A\rangle : \langle\vec{a}, A\rangle \in \Delta \text{ and } \vec{a} \in Y^n\}.$$

(b) Γ is normal, for if $\langle\langle a, b\rangle, x_1 = x_2\rangle \in \Gamma$ then $\langle\langle a, b\rangle, x_1 = x_2\rangle \in \Delta$, so $a \sim b$ consequently a is the same as b since only one element from each equivalence class of X is in Y.

(c) Suppose that $\langle\vec{a}, A\rangle \in \Delta$, with $\vec{a} = \langle a_1 \dots a_n\rangle$. Let $b_1 \dots b_n$ be those elements of Y such that $a_1 \sim b_1$ and ... and $a_n \sim b_n$. Then $\langle\langle a_1, b_1\rangle, (x_1 = x_2)\rangle \in \Delta$ and ... and $\langle\langle a_n, b_n\rangle, (x_1 = x_2)\rangle \in \Delta$, therefore $\langle\vec{a} * \vec{b}[n+n : x_1, x_{n+1}], (x_1 = x_2)\rangle \in \Delta$ and ... and $\langle\vec{a} * \vec{b}[n+n : x_n, x_{n+n}], (x_1 = x_2)\rangle \in \Delta$.
Since Δ is S-saturated, $\langle\vec{a}*\vec{b}, x_1 = x_{n+1}\rangle \in \Delta$, $\langle\vec{a}*\vec{b}, x_2 = x_{n+2}\rangle \in \Delta$, ... , $\langle\vec{a}*\vec{b}, x_n = x_{n+n}\rangle \in \Delta$, and so $\langle\vec{a}*\vec{b}, (x_1 = x_{n+1}) \wedge \cdots \wedge (x_n = x_{n+n})\rangle \in \Delta$. Since $\langle\vec{a}, A\rangle \in \Delta$, then $\langle\vec{a} * \vec{b}[x_1 \dots x_n], A\rangle \in \Delta$, $\langle\vec{a} * \vec{b}, \langle x_1 \dots x_n\rangle A\rangle \in \Delta$. By lemma 4.7, $\langle\vec{a}*\vec{b}, \langle x_{n+1} \dots x_{n+n}\rangle A\rangle \in \Delta$, so $\langle\vec{a}*\vec{b}[x_{n+1} \dots x_{n+n}], A\rangle \in \Delta$, and consequently $\langle\vec{b}, A\rangle \in \Delta$. Then by definition of Γ, $\langle\vec{b}, A\rangle \in \Gamma$.

(d) Since $\Gamma \subseteq \Delta$, Γ is S-coherent. Moreover

(i) If $\langle\vec{b}, A\rangle \notin \Gamma$ and $\vec{b}$ is a list of elements of Y, then $\langle\vec{b}, A\rangle \notin \Delta$, whence $\langle\vec{b}, \neg A\rangle \in \Delta$ and so $\langle\vec{b}, \neg A\rangle \in \Gamma$.

(ii) $\langle\vec{b}, \exists x_{n+1}B\rangle \in \Gamma$ only if there is an $a \in X$ such that $\langle\vec{b} * a, B\rangle \in \Delta$. Take the (only) element a^* of Y such that $a \sim a^*$. Therefore $\langle\vec{b}*a^*, B\rangle \in \Delta$ and consequently $\langle\vec{b} * a^*, B\rangle \in \Gamma$.

Lemma 4.31. *Let Γ be an S-saturated and normal-Y-graph.*

(a) For each term t of type n there is exactly one element $b \in Y$ such that for any $\vec{a} \in Y^n$, $\langle\vec{a} * b, (\langle x_1, ..., x_n\rangle \circ t = x_{n+1})\rangle \in \Gamma$,

(b) $\langle\langle a, a\rangle, (x_1 = x_2)\rangle \in \Gamma$, for any $a \in Y$.

Proof. (a) Let $\vec{x}$ be $x_1, ..., x_n$. Since $\vdash^n \exists x_{n+1}(\langle\vec{x}\rangle \circ t = x_{n+1})$, then for any $\vec{a}$ of length n, $\langle\vec{a}, \exists x_{n+1}(\langle\vec{x}\rangle \circ t = x_{n+1})\rangle \in \Gamma$. Since Γ is rich, there is a $b \in Y$ such that $\langle\vec{a} * b, (\langle\vec{x}\rangle \circ t = x_{n+1})\rangle \in \Gamma$.

As to the uniqueness, suppose that there is a $c \in Y$, $b \neq c$, such that $\langle\vec{a}*c, (\langle\vec{x}\rangle \circ t = x_{n+1})\rangle \in \Gamma$. This amounts to say that $\langle\vec{a}*b*c[\vec{x}, x_{n+1}], (\langle\vec{x}\rangle \circ t = x_{n+1})\rangle \in \Gamma$ and that $\langle\vec{a} * b * c[\vec{x}, x_{n+2}], (\langle\vec{x}\rangle \circ t = x_{n+2})\rangle \in \Gamma$, and so, since Γ is S-saturated, that $\langle\vec{a}*b*c, (\langle\vec{x}\rangle \circ t) = x_{n+1}\rangle \in \Gamma$ and that $\langle\vec{a}*b*c, (\langle\vec{x}\rangle \circ t) = x_{n+2}\rangle \in \Gamma$. But $\vdash^n \forall x_{n+1}\forall x_{n+2}((\langle\vec{x}\rangle \circ t = x_{n+1}) \wedge (\langle\vec{x}\rangle \circ t = x_{n+2}) \to x_{n+1} = x_{n+2})$, whence $\langle\vec{a} * b * c, (x_{n+1} = x_{n+2})\rangle \in \Gamma$, therefore $\langle\vec{a} * b * c, \langle x_{n+1} = x_{n+2}\rangle(x_1 = x_2)\rangle \in \Gamma$, so $\langle\vec{a} * b * c[x_{n_1}, x_{n+2}], (x_1 = x_2)\rangle \in \Gamma$ and finally $\langle\langle b, c\rangle, (x_1 = x_2)\rangle \in \Gamma$. Consequently b is identical with c, since Γ is normal.

(b) If $\langle\langle a, a\rangle, (x_1 = x_2)\rangle \notin \Gamma$, then $\langle\langle a, a\rangle, \neg(x_1 = x_2)\rangle \in \Gamma$. But then there is a projection τ, i.e. $\langle x_1, x_1\rangle$ such that $\langle a\rangle[\tau] = \langle a, a\rangle$ and $\vdash^1 \tau\neg(x_1 = x_2) \to \perp$, consequently Γ is not S-coherent, contrary to the hypothesis.

Definition 4.32. Let Γ be an S-saturated and normal Y-graph, $\vec{a} \in Y^n$ and $\langle n : t_1, ..., t_m\rangle$ be a complex term. Define

$$\vec{a}\|t_1, ..., t_m\| = \langle b_1, ..., b_m\rangle,$$

where each b_k, $1 \leq k \leq m$, is the unique element of Y such that $\langle\vec{a} * b_k, (\langle x_1, ..., x_n\rangle \circ t_k = x_{n+1})\rangle \in \Gamma$.

¿From the given definition it follows that for each $\vec{a} \in Y^n$,

$$\vec{a}\|f^n(x_1, ..., x_n)\| = b \text{ iff } \langle\vec{a} * b, \langle x_1, ..., x_n\rangle \circ f^n(x_1, ..., x_n) = x_{n+1})\rangle \in \Gamma,$$

whence, $\|f^n(x_1, ..., x_n)\|$ is a function from X^n to X.

Lemma 4.33. *Let Γ be an S-saturated and normal Y-graph. The function $\|\ \|$ is an interpretation function:*

(a) $\vec{a}\|x_i\| = a_i$

(b) $\vec{a}\,\|\langle n : t_1, ..., t_m\rangle\| = \langle\vec{a}\,\|n : t_1\|, ..., \vec{a}\,\|n : t_m\|\rangle$

(c) $\vec{a}\,\|\langle n : t_1, ..., t_m\rangle \circ \vec{s}\,\| = (\vec{a}\,\|\langle n : t_1, ..., t_m\rangle\|)\|\vec{s}\,\|$

(d) $\langle\vec{a}, \langle n : t_1, ..., t_m\rangle A\rangle \in \Gamma$ *iff* $\langle\vec{a}\|\langle n : t_1, ..., t_m\rangle\|, A\rangle \in \Gamma$.

DEFINITION 4.34. Let $S \supseteq Q.K_t$ and U be an infinite set. The *canonical model* $M^U = \langle W^U, R, D, \mathfrak{C}, I\rangle$ for S is exactly as in definition 4.17 with the further condition that

$$I_w(f^n) = \|f^n(n : x_1, ..., x_n)\|.$$

LEMMA 4.35. *Let* $M^U = \langle W^U, R, D, \mathfrak{C}, I\rangle$ *be a canonical model for* S. *If* $w \in W^U$ *and* $\langle \vec{a}, \Diamond A\rangle \in w$, *then there is a* $v \in W^U$ *and a list* $\vec{c}$ *of elements of* D_v, *such that*

(a) $\langle \vec{c}, A\rangle \in v$,
(b) $\mathfrak{C}_{\langle w,v\rangle} = \{\langle a_1, c_1\rangle, \dots, \langle a_n, c_n\rangle\}$ *is admissible.*

PROOF. (a) By lemma 4.18 there is an S-saturated X-graph Δ (for some $X \subseteq U$) and a list $\vec{b}$ of elements of X such that $\langle \vec{b}, A\rangle \in \Delta$ and $\{\langle a_1, b_1\rangle, \dots, \langle a_n, b_n\rangle\}$ is admissible. By lemma 4.30 there is a Y-graph Γ which is normal and S-saturated and moreover for some list $\vec{c} \in Y^n$ such that $b_1 \sim c_1, \dots, b_n \sim c_n$, $\langle \vec{c}, A\rangle \in \Gamma$. Let $\Gamma = v$.

(b) Suppose that for some $\langle n : x_{i_1}, ..., x_{i_m}\rangle$ and wff $\Box B$ of type m, $\langle\langle \vec{a}\|x_{i_1}, ..., x_{i_m}\|\rangle, \Box B\rangle \in w$, so by lemma 4.33(d), $\langle \vec{a}, \langle n : x_{i_1}, ..., x_{i_m}\rangle \Box B\rangle \in w$, by Axiom $S^\Box$, $\langle \vec{a}, \Box\langle x_{i_1}, ..., x_{i_m}\rangle B\rangle \in w$, hence $\langle \vec{b}, \langle x_{i_1}, ..., x_{i_m}\rangle B\rangle \in \Delta$. Since $b_1 \sim c_1, \dots, n_n \sim c_n$, by definition of Γ (see lemma 4.30) $\langle \vec{c}, \langle x_{i_1}, ..., x_{i_m}\rangle B\rangle \in \Gamma$, so by lemma 4.33$(d)$, $\langle \vec{c}\|x_{i_1}, ..., x_{i_m}\|, B\rangle \in \Gamma$, therefore $\{\langle a_1, c_1\rangle, \dots, \langle a_n, c_n\rangle\}$ is admissible.

LEMMA 4.36. *(of the canonical model.) Let* $M^U = \langle W^U, R, D, \mathfrak{C}, I\rangle$ *be a canonical model for* S. *Then for any wff* C *of type* n,

$$\vec{c} \models^n_w C \quad \text{iff} \quad \langle \vec{c}, C\rangle \in w$$

PROOF. See lemma 4.19.
Completeness theorems for $Q.K^t$, $Q.T^t$, $Q.S4^t$, $Q.B^t$, $BF.K^t$ and $GF.K^t$ follow.

LEMMA 4.37. *Let* M^U *be a canonical model for* $NI.K^t$. *Then the counterpart relation is a partial function.*

PROOF. Take any $w \in W$ of the canonical model M^U. Because of axiom NI, $\langle\langle a, b\rangle, x_1 = x_2 \to \Box(x_1 = x_2)\rangle \in w$ for any $a, b \in D_w$. Consequently $\langle\langle a, a\rangle, \Box(x_1 = x_2)\rangle \in w$. Suppose that wRv, $a\mathfrak{C}b$ and $a\mathfrak{C}b^*$, then by the definitions of $\mathfrak{C}$ and R in the canonical model, $\langle\langle b, b^*\rangle, x_1 = x_2\rangle \in v$. Therefore $b = b^*$. So $\mathfrak{C}$ is a partial function.

LEMMA 4.38. *Let* M^U *be a canonical model for* $ND.K^t$. *Then the counterpart relation is injective.*

PROOF. Take any $w \in W$ of the canonical model M^U and an $a \in D_w$ such that $a \neq a^*$. Because of axiom ND, $\langle\langle a, a^*\rangle, x_1 \neq x_2 \to \Box(x_1 \neq x_2)\rangle \in w$ and so $\langle\langle a, a^*\rangle, \Box(x_1 \neq x_2)\rangle \in w$. Suppose, by *reductio* that for some

$v.wRv$. and for some $b \in D_v$, $a\mathfrak{C}b$ and $a^*\mathfrak{C}b$. Then $\langle\langle b, b\rangle, x_1 \neq x_2\rangle \in v$ which is impossible. Consequently $\mathfrak{C}$ is injective.
Completeness theorems for $NI.K^t$ and $ND.K^t$ follow.

4.3. Modal systems with identity and rigid terms. Throughout this section let $S \supseteq R.K^t$. Lemmas and definitions 4.33 - 4.41 hold trivially also for extensions of $R.K^t$.

DEFINITION 4.39. Let $S \supseteq R.K_t$ and U be an infinite set. The *canonical model* M^U for S is defined as in definition 4.17 except that

- $\mathfrak{C} = \biguplus\{\mathfrak{C}_{\langle w,v\rangle}\}_{w,v\in W}$, where $\mathfrak{C}_{\langle w,v\rangle} \subseteq D_w \times D_v$ which is *admissible*, i.e. for every $n, n \geq 1$, if $\{\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle\} \subseteq \mathfrak{C}_{\langle w,v\rangle}$, then for all wff $\Box B$ of type m and complex terms $\langle n : t_1, ..., t_m\rangle$, if $\langle\langle\vec{a}\,[n : t_1, ..., t_m]\rangle, \Box B\rangle \in w$, then $\langle\langle\vec{b}\,[n : t_1, ..., t_m]\rangle, \Box B\rangle \in v$, where $\vec{a} = \langle a_1, ..., a_n\rangle$ and $\vec{b} = \langle b_1, ..., b_n\rangle$.
- $I_w(f^n) = \|f^n(n : x_1, ..., x_n)\|$.

Given a canonical model for $S \supseteq R.K_t$, the analogue of lemma 4.35 can be readily proved: use axiom R instead of axiom $S^\Box$. Consequently completeness theorems for $R.K^t$, $R.T^t$, $R.S4^t$, $R.B^t$, $BF.R.K^t$, $GF.R.K^t$ and $NI.R.K^t$ and $ND.R.K^t$ follow.

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