Completing partial algebra models of term rewriting systems

MSc Thesis (*Afstudeerscriptie*)

written by

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Abstract

The study of partial algebras is the part of universal algebra which deals with structures whose operations are not defined everywhere. A natural question to ask if faced with a partial algebra is whether or not it can be completed, i.e. embedded into a total algebra. If furthermore we take partial algebras modelling a certain theory T, the question of finding a completion also modelling T may be very hard, depending on the strength of T.

In this thesis we investigate a special case of the difficult problem stated above. In particular, we show that partial algebras which model an orthogonal term rewriting system and which satisfy a certain condition on head-normal forms can be completed. Our result generalizes a previous result by Inge Bethke, Jan Willem Klop and Roel de Vrijer presented in [BKdV96]. In the process, we will prove an abstract confluence theorem which is independent from our particular choice of term rewriting system and therefore can be taken separately.

Our proof is preceded by a section on completions of partial algebras, describing the construction of the one-point completion and free completion of a given partial algebra, and showing the relations between these completions in category-theoretic terms.

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Introduction

In most of mainstream mathematics, there is not much attention given to partiality. One usually investigates structures that are closed under the operations defined on them, or equivalently one restricts functions or operators to the sets which are closed under them. However, as George Grätzer states in *Universal Algebra* [Grä79], the study of partial algebras is useful when discussing the behavior and properties of operations on subsets of algebras which are not closed under them. In practice, there are many areas where the use of partial structures is necessary or beneficial in order to understand the nature of the thing.

Much of the motivation for being able to handle partiality comes from computer science. In programming languages, functions are often defined for certain types, i.e. for a certain kind of input. However, a program may not be defined on every input, especially in borderline cases. For example, the program taking a list and returning the head of the list would have 'list' or 'array' as input type but would not be defined on the empty list.

Another example would be the study of natural language semantics. Kracht [Kra06] cites Dresner [Dre02] as saying that an algebraic approach to language acquisition is more plausible than a model-theoretic one. By language acquisition we mean here the process of learning the meaning of syntactic objects. The model-theoretic approach would be one in which the learner has, prior to learning the meaning of a word, many possible models of the world corresponding to the possibilities for the meaning, which she progressively eliminates as she gains more information. This is the approach behind dynamic epistemic logic where one maintains accessibility only to worlds compatible with one's epistemic state. The algebraic approach would be to see the meaning function as a partial function which gets progressively extended. So in the former approach, minimal knowledge would correspond to the most complicated structure (all models fitting the acquired knowledge), whereas seeing meaning as a partial function where a value is not (yet) assigned for certain arguments is arguably a more parsimonious and natural approach. Dresner [Dre02] defends this view for other reasons, namely that a holistic view of meaning (the view that meaning of expressions derive from their usage in connection to other expressions and the meaning of these expressions) is indefensible without partiality, as in order to learn a language we would need to already know the meaning of most of the expressions of the language.

If we have a partial natural language semantics, one natural question to ask is if one can extend it; for example if our meaning function is defined only on sentences one might ask if we can extend it to words. One may want to extend it in a certain way, say such that the resulting meaning function is compositional. A meaning function would be compositional on a partial algebra if and only if the equivalence relation it defines (synonymy) is a closed congruence on it (see [Hod01][Kra06]).

This brings us to the main subject of this thesis: how to complete partial structures and whether one can always do so. To complete a partial structure is to extend it to a total structure. For example, if we see a partial order as a partial binary function on the square of the domain, which of the two arguments returns the smallest, then the completion of a partial order is a total order. However, if our partial algebra is a model of a theory, we may want to ask the question of whether there is a completion which is a model of the same theory. While completing a model may be easy when the theory is weak, it may get more complicated the stronger the theory gets. It may even be impossible. As an example, consider the natural numbers $\langle \mathbb{N}, +_{\mathbb{N}}, -_{\mathbb{N}}, 0_{\mathbb{N}} \rangle$. The ordinary interpretation of minus in this structure is a partial operation. We can complete this partial structure to $\mathbb{Z}$ with the usual interpretations. However, if we now add the $<$ ordering to the signature and interpret it as the usual order, and if in our theory T we have the sentence "there is no element smaller than zero", we cannot complete the structure to a model of T.

We have said above that the main use of the study of partial algebras is to examine the behavior of operators on subsets of algebras when these subsets are not closed under these operators. However, we see now that not all partial models are necessarily such a restriction of a total model. In other words, the behavior of the partial operators on the structures may be so incompatible with one another or so chaotic that it is impossible to complete them in a satisfactory way. In Chapter 1 we will see how totalization or completion of partial algebras behave with respect to preserving the partial operations' values on subuniverses, and with respect to preserving congruence relations. We will see special kinds of completions such as the one-element completion and the free completion and how they can be seen as opposites in a category-theoretic view.

The main part of the thesis is Chapter 2. There, we will briefly describe a paper by I. Bethke, R. de Vrijer and J.W. Klop [BKdV96] which gives a condition under which a partial combinatory algebra has a completion. This condition is called having unique head normal forms. To this end, a term rewriting system is defined which provides the structure necessary to complete the partial combinatory algebra. In this thesis, we aim to generalize that result and show that the completion has little to do with the particular case of combinatory algebras and more to do with the underlying term rewriting system. We thus reformulate the condition on unique head normal forms, and show that a partial algebra with this condition modelling an orthogonal term rewriting system can be completed to a total algebra also modelling the term rewriting system.

To conclude, we will show that [BKdV96] is indeed a special case of our result, and mention its application to other combinatory systems, such as the completion of a partial model of the CL_I calculus.

Chapter 1

Completing partial algebras

Universal algebra is one of the areas that has most developed the study of partiality, together with category theory, and universal algebra is therefore the area we will be working in.

In this chapter we will study the completions of partial algebras of certain types, and the relations between these completions.

1.1 Definitions

Universal algebra can be thought of as a special case of model theory in which one only studies purely functional or algebraic signatures. A signature, henceforth denoted F, is a set of function or operation symbols. Each symbol is assigned an arity, a natural number denoting the number of arguments it takes when interpreted. Symbols in F will be called f, g and so forth. A universal algebra $\mathfrak{A}$ over the signature F will then be a set A together with an interpretation of each symbol in F. Below are the precise definitions of total algebras and partial algebras.

Definition 1. *Let A be a set, f an n-ary operation on A. f is called* **total** *if it is a function $A^n \to A$ whose domain is A^n. It is* **partial** *if its domain is a subset of A^n. If an n-tuple $\vec{a}$ is in the domain of f, we will say that $f(\vec{a})$ exists and write $f(\vec{a})\downarrow$; otherwise we write $f(\vec{a})\uparrow$. In expressions such as $f(\vec{a}) = b$, $f(\vec{a}) \neq b$ and $f(\vec{a}) \in A$ it will implicitly be assumed that $f(\vec{a})\downarrow$.*

Definition 2. *An* **algebra** *is a tuple $\mathfrak{A} = \langle A, \{f^{\mathfrak{A}}\}_{f\in F}\rangle$ where A is a set and F a set of function symbols, called the* **signature** *of $\mathfrak{A}$. To each function symbol $f \in F$ we assign a natural number which we call its* **arity** *and denote $\Omega(f)$. Now, to each $f \in F$ we assign an $\Omega(f)$-ary function on A, denoted $f^{\mathfrak{A}} : A^{\Omega(f)} \to A$ and called the* **interpretation** *of f in $\mathfrak{A}$. We say that $\mathfrak{A}$ is a* **partial algebra** *if $f^{\mathfrak{A}}$ is a partial function for all $f \in F$. If furthermore $f^{\mathfrak{A}}$ is a total function for all $f \in F$ we say that $\mathfrak{A}$ is* **total**.

For a total algebra, it is easy to define what a homomorphism is:

Definition 3. *A* **total algebra homomorphism** *from* $\mathfrak{A} = \langle A, \{f^{\mathfrak{A}}\}_{f \in F}\rangle$ *to* $\mathfrak{B} = \langle B, \{f^{\mathfrak{B}}\}_{f \in F}\rangle$ *is a total function* $\phi : A \to B$ *such that for all* $f \in F$*, for all* $\vec{a} \in A^{\Omega(f)}$*,*

$$f^{\mathfrak{B}}(\phi(\vec{a})) = \phi(f^{\mathfrak{A}}(\vec{a}))$$

where by $\phi(\vec{a})$ *we mean* $(\phi(a_1), \ldots, \phi(a_{\Omega(f)}))$*.*

However, the issue is not that simple when dealing with partial algebras, as applications to certain arguments on both sides may not be defined [Grä79].

There is in fact more than one way to define homomorphisms. For example, one can define:

Definition 4. *A* **closed partial algebra homomorphism** *from* $\mathfrak{A} = \langle A, \{f^{\mathfrak{A}}\}_{f \in F}\rangle$ *to* $\mathfrak{B} = \langle B, \{f^{\mathfrak{B}}\}_{f \in F}\rangle$ *is a total function* $\phi : A \to B$ *such that for all* $f \in F$*, for all* $\vec{a} \in A^{\Omega(f)}$*,*

1. $f^{\mathfrak{B}}(\phi(\vec{a}))\downarrow \iff f^{\mathfrak{A}}(\vec{a})\downarrow$ *and*
2. *If both exist, then* $f^{\mathfrak{B}}(\phi(\vec{a})) = \phi(f^{\mathfrak{A}}(\vec{a}))$*.*

Note that a total algebra homomorphism is a closed partial algebra homomorphism. However, one of our main aims is to investigate completions of (strictly) partial algebras, meaning embeddings of partial algebras into total ones. The above definition will certainly not do for this purpose as totality of the target algebra would immediately imply totality of the source algebra. So we need a weaker notion of homomorphism, which tolerates partiality and where the implication in condition 1 goes only in one direction.

Definition 5. *A* **weak partial algebra homomorphism** *between* $\mathfrak{A} = \langle A, \{f^{\mathfrak{A}}\}_{f \in F}\rangle$ *and* $\mathfrak{B} = \langle B, \{f^{\mathfrak{B}}\}_{f \in F}\rangle$ *is a total function* $\phi : A \to B$ *such that for all* $f \in F$*, for all* $\vec{a} \in A^{\Omega(f)}$*,*

1. $f^{\mathfrak{A}}(\vec{a})\downarrow \Rightarrow f^{\mathfrak{B}}(\phi(\vec{a}))\downarrow$ *and*
2. *If both exist, then* $f^{\mathfrak{B}}(\phi(\vec{a})) = \phi(f^{\mathfrak{A}}(\vec{a}))$*.*

Contrary to ordinary conventions, in this thesis if an algebra is not specified to be total it will assumed to be partial, i.e. the word "algebra" will, unless specified otherwise, mean "partial algebra" and "homomorphism" will mean "weak homomorphism". This will cause no confusion since for total algebras weak and closed homomorphisms coincide. Note that an algebra homomorphism as we define it here is only between algebras over the same signature F. Now we can define the notion of a completion or totalization of a partial algebra.

Definition 6. *Let* $\mathfrak{A} = \langle A, \{f^{\mathfrak{A}}\}_{f \in F}\rangle$ *be a partial algebra.* $\mathfrak{B} = \langle B, \{f^{\mathfrak{B}}\}_{f \in F}\rangle$ *is a* **completion** *of* $\mathfrak{A}$ *if*

1. $\mathfrak{B}$ *is a total algebra and*
2. *There exists an injective weak homomorphism* $\phi : A \to B$*, i.e. an embedding of* $\mathfrak{A}$ *into* $\mathfrak{B}$*.*

Just as there is more than one possible way to define a homomorphism, there is more than one way to define a subalgebra of a given algebra[1]. The following are those which correspond to weak and closed homomorphisms:

Definition 7. *Let $\mathfrak{A} = \langle A, \{f^{\mathfrak{A}}\}_{f \in F}\rangle$ and $\mathfrak{B} = \langle B, \{f^{\mathfrak{B}}\}_{f \in F}\rangle$ be algebras such that $A \subseteq B$. Then $\mathfrak{A}$ is a* **weak subalgebra** *of $\mathfrak{B}$ if the inclusion is a weak homomorphism, and it is a* **closed subalgebra** *of $\mathfrak{B}$ if the inclusion is a closed homomorphism.*

But there is also a concept in between the two:

Definition 8. *$\mathfrak{A} = \langle A, \{f^{\mathfrak{A}}\}_{f \in F}\rangle$ and $\mathfrak{B} = \langle B, \{f^{\mathfrak{B}}\}_{f \in F}\rangle$ be algebras such that $A \subseteq B$. Then $\mathfrak{A}$ is a* **relative subalgebra** *of $\mathfrak{B}$ if for any $f \in F$, for any $\vec{a} \in A^{\Omega(f)}$*

1. *$\mathfrak{A}$ is a weak subalgebra of $\mathfrak{B}$ and*
2. *If $f^{\mathfrak{B}}(\vec{a}) \in A$, then $f^{\mathfrak{A}}(\vec{a}) = f^{\mathfrak{B}}(\vec{a})$.*

One may think of these three types of subalgebra as follows: if $\mathfrak{A}$ is a weak subalgebra of $\mathfrak{B}$, functions interpreted on $\mathfrak{B}$ may take any value they like for arguments in A, as long as they agree with the interpretations in $\mathfrak{A}$ wherever these are defined. If it is a closed subalgebra, interpretations of functions in $\mathfrak{B}$ may not take values (anywhere) for arguments in A unless the function is defined in $\mathfrak{A}$. Lastly, if $\mathfrak{A}$ is a relative subalgebra, interpretations of functions in $\mathfrak{B}$ may take a value for arguments in A even when the function is not defined in $\mathfrak{A}$, but in this case the value must be outside of A. A closed subalgebra is a relative subalgebra, and a relative subalgebra is a weak subalgebra, and moreover these inclusions are strict [Grä79, page 81].

Now that we have these basic definitions concerning partial algebras, we can go on to look at some special cases of completions: the one-point completion and free completion of an algebra.

1.2 The One-element Completion and the Free Completion

Given no restriction on which properties to preserve, there are many ways to complete a given algebra $\mathfrak{A}$. One way to do this is to construct the one-point or one-element completion $\mathfrak{A}^*$ as in [Kra06]. This algebra is defined as follows: let $A^* = A \cup \{*\}$ be the underlying set and for any $f \in F$, $\vec{a} \in (A^*)^{\Omega(f)}$, define the interpretations as

$$f^{\mathfrak{A}^*}(\vec{a}) = \begin{cases} f^{\mathfrak{A}}(\vec{a}) & \text{if } \vec{a} \in A^{\Omega(f)} \text{ and } f^{\mathfrak{A}}(\vec{a})\downarrow \\ * & \text{otherwise} \end{cases}$$

[1] We will see in a bit that there is more than one way to define a congruence as well.

In short, one adds to A an element corresponding to "undefined" and postulates that any function given an undefined argument is undefined. This is clearly a total algebra and the inclusion map is clearly a weak partial algebra homomorphism. If $\mathfrak{A}$ is already total, define $\mathfrak{A}^*$ to be simply $\mathfrak{A}$.

Instead of adjoining a single point, one can go to the other extreme: adding one distinct point for every undefined application of a function. This is called the absolutely free completion of the partial algebra $\mathfrak{A}$ and is denoted $\mathfrak{F}(\mathfrak{A})$. We will need to define terms for what will follow.

Definition 9. *The set of* **terms** $Ter(F, V)$ *over a signature* F *and an arbitrary set of variables* V *is defined recursively as follows:*

i. Elements of V *are terms.*

ii. Nullary elements of F *are terms.*

iii. If $t_1, \ldots, t_k$ *are terms and* $f \in F$ *is of arity* k*, then* $f(t_1, \ldots, t_k)$ *is a term.*

iv. Nothing else is a term.

Note that terms as defined above are just sequences of symbols, and as such are uniquely expressible: if $f(t_1, \ldots, t_k) = g(s_1, \ldots, s_k)$ for terms t_i, s_i, $f, g \in F$, then $f = g$ and $t_i = s_i$ for all $1 \leq i \leq k$.

For a term t, we write $t = t(x_1, \ldots, x_n)$ to indicate that t has arity at most n (in the next chapter, we may want to write $t = t(x_1, \ldots, x_n)$ even though all the variables are not actually involved in t). If $\mathfrak{A}$ is an algebra with underlying set A and signature F, we can extend the notion of interpretation in $\mathfrak{A}$ from function symbols in F to terms, in the obvious way[2]:

$$(f(t_1, \ldots, t_k))^{\mathfrak{A}} = f^{\mathfrak{A}}(t_1^{\mathfrak{A}}, \ldots, t_k^{\mathfrak{A}})$$

The **subterms** of $t = f(t_1, \ldots, t_k)$ are t together with all the subterms of $t_1, \ldots, t_k$. If s is a subterm of t we write $s \leq t$. Strict inequality of course symbolizes the strict subterm relation.

It is easy to imagine how $\mathfrak{F}(\mathfrak{A})$ would be constructed inductively, first adding new elements as the values of functions with arguments in A which are undefined in $\mathfrak{A}$, then adding more elements as the values of functions that have the previously defined elements as arguments, and so forth. The construction described by Grätzer in [Grä79] follows this intuition. However, there is a slightly more elegant approach briefly mentioned without proof in the introductory part of Isidore Fleischer's *"On extending congruences from partial algebras"* [Fle], which we expand and prove explicitly here. It involves constructing the free completion as a subdirect product of an adequately chosen set of completions. The idea is as follows: let $\mathfrak{A}$ be a partial algebra over a signature F, and look

[2] We can either see the underlying set A as the set of variables or add them to the signature F as nullary function symbols: $Ter(F, A) = Ter(F \cup A, \emptyset)$. In any case, for $a_1, \ldots, a_n \in A$, $(t(a_1, \ldots, a_n))^{\mathfrak{A}} = t^{\mathfrak{A}}(a_1, \ldots, a_n)$. If t is a term over A and $t^{\mathfrak{A}}\downarrow$, then by definition $s^{\mathfrak{A}}\downarrow$ for any subterm s of t.

at all total algebras over the same signature completing $\mathfrak{A}$. We know that there is at least one, as the one-point completion can always be constructed. Unfortunately, these completions may be too numerous to form a set. However, we do not need all of them and we can restrict the number of completions to a set-sized collection. To achieve this, we look only at those completions $\mathfrak{B}$ of $\mathfrak{A}$ such that $\mathfrak{B}$ is **generated** by the image of A, i.e. such that for every $b \in B$ there is an n-ary term t and $a_1, \ldots, a_n \in A$ such that $t^{\mathfrak{B}}(a_1, \ldots, a_n) = b$. This restriction puts an upper bound on the cardinality of the completion $\mathfrak{B}$, a bound that depends on the cardinality of A and on the signature F. Some of these completions may be isomorphic to one another, but we take a representative from each isomorphism class, using the axiom of global choice. Because the number of possible completions is bounded, the representatives form a set, which we will denote $\{\mathfrak{B}_i : i \in I\}$. We now take the direct product $\prod_{i \in I} \mathfrak{B}_i$. This is also a total algebra over the signature F, with underlying set $\prod_{i \in I} B_i$ and with interpretations defined component-wise:

$$f^{\prod_{i \in I} \mathfrak{B}_i}((a_{1i})_{i \in I}, \ldots, (a_{ni})_{i \in I}) = (f^{\mathfrak{B}_i}(a_{1i}, \ldots, a_{ni}))_{i \in I}$$

Since for every $i \in I$ we by definition have an embedding $\phi_i : \mathfrak{A} \to \mathfrak{B}_i$, we can define an embedding $\phi : \mathfrak{A} \to \prod_{i \in I} \mathfrak{B}_i$ by $\phi(a) = (\phi_i(a))_{i \in I}$. Finally, call $\mathfrak{F}(\mathfrak{A})$ the closed subalgebra of $\prod_{i \in I} \mathfrak{B}_i$ generated by $\phi[A]$.

We should now show that this indeed corresponds to our usual idea of the free completion of an algebra $\mathfrak{A}$. The usual definition is the category-theoretical one below, here stated only for completions of partial algebras.

Definition 10. *$\mathfrak{B}$ is the* **free completion** *of $\mathfrak{A}$ if it is a completion of $\mathfrak{A}$ such that*

i. A generates $\mathfrak{B}$, and

ii. For any total algebra $\mathfrak{C}$, any homomorphism $\psi : \mathfrak{A} \to \mathfrak{C}$ extends uniquely to a homomorphism $\hat{\psi} : \mathfrak{B} \to \mathfrak{C}$.

It is easy to see that the free completion of a given partial algebra is unique up to isomorphism if it exists [Grä79]. Technically, we will show that $\mathfrak{F}(\mathfrak{A})$ is the free completion of $\phi(\mathfrak{A})$, which since ϕ is 1-1 can be seen as a partial algebra isomorphic to $\mathfrak{A}$[3].

Theorem 11. *$\mathfrak{F}(\mathfrak{A})$ is, up to isomorphism, the free completion of $\mathfrak{A}$.*

Proof. For simplicity, we identify $\phi(\mathfrak{A})$ with A.

First, from construction $\mathfrak{F}(\mathfrak{A})$ is a total algebra, since any closed subalgebra of a total algebra is total, and $\prod_{i \in I} \mathfrak{B}_i$ is total. Also by construction, A generates $\mathfrak{F}(\mathfrak{A})$.

[3]The underlying set will be $\phi[A]$, and the interpretations defined as

$$f^{\phi(\mathfrak{A})}(a_1, \ldots, a_n) = f^{\mathfrak{A}}(\phi^{-1}(a_1), \ldots, \phi^{-1}(a_n))$$

Let $\mathfrak{C}$ be a total algebra and let $\psi : \mathfrak{A} \to \mathfrak{C}$ be a homomorphism. We want to find a unique extension $\hat{\psi} : \mathfrak{F}(\mathfrak{A}) \to \mathfrak{C}$ of ψ. First we define a completion $\mathfrak{B}$ of $\mathfrak{A}$ such that the image of A generates $\mathfrak{B}$, and a map $\zeta : \mathfrak{B} \to \mathfrak{C}$ extending ψ. We want to define $\mathfrak{B}$ as the smallest total subalgebra of $\mathfrak{C}$ containing the image $\psi[A]$. So let $\{C_j\}_{j \in J}$ be all total subalgebras of $\mathfrak{C}$ containing $\psi[A]$. This family is nonempty; we know that at the very least $\mathfrak{C}$ is there. Now define $\mathfrak{B}$ to be the algebra with underlying set $B := \bigcap_{j \in J} C_j$, and with interpretations

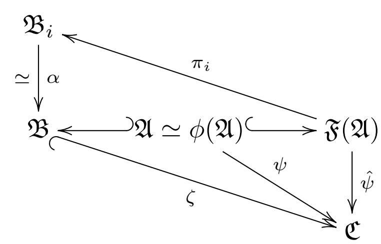
$$f^{\mathfrak{B}}(b_1, \dots, b_n) = b \iff f^{\mathfrak{C}_j}(b_1, \dots, b_n) = b \text{ for all } j \in J$$

for all $b_1, \dots, b_n, b \in B$. This is a total algebra: for any $i, j \in J$,

$$f^{\mathfrak{C}_i}(b_1, \dots, b_n) = f^{\mathfrak{C}}(b_1, \dots, b_n) = f^{\mathfrak{C}_j}(b_1, \dots, b_n)$$

as they are subalgebras, and $f^{\mathfrak{C}_j}(b_1, \dots, b_n) \in C_j$ for all j, as the algebras are all total. Our map ζ is just the inclusion.

Now that we have this completion $\mathfrak{B}$ generated by the image of A, we know that it is isomorphic to a certain $\mathfrak{B}_i$ in our set of representatives I via an isomorphism which we shall call α. Now we can define $\hat{\phi} := \zeta \circ \alpha \circ \pi_i$ where $\pi_i : \mathfrak{F}(\mathfrak{A}) \to \mathfrak{B}_i$ is the ith projection of the product. We have the diagram



$\hat{\psi}$ is a homomorphism because it is a composition of homomorphisms, and $\mathfrak{F}(\mathfrak{A})$ is generated by A. To see that it is the unique such homomorphism extending ψ, note that any other such homomorphism χ would agree with $\hat{\psi}$ on A. For any element b in $\mathfrak{F}(\mathfrak{A})$, there are $a_i, \dots, a_n \in A$ such that $b = t^{\mathfrak{F}(\mathfrak{A})}(a_1, \dots, a_n)$. But then

$$\hat{\psi}(b) = t^{\mathfrak{C}}(\hat{\psi}(a_1), \dots, \hat{\psi}(a_n)) = t^{\mathfrak{C}}(\chi(a_1), \dots, \chi(a_n)) = \chi(b)$$

□

We have now seen two opposing ways to complete an algebra. In the following sections we will further investigate the role that these completions play in the lattice of completions of a given partial algebra.

1.3 Completing an algebra preserving subuniverses

In the previous section we saw how to construct two completions of a partial algebra. The fact that these two completions are somehow opposed to one another is illustrated by the fact that they are the initial and final objects of a certain category. Let $\mathfrak{A}$ be a partial algebra and let $\mathbb{C}$ be the category whose objects are completions $\mathfrak{B}$ of $\mathfrak{A}$ such that

i. A generates $\mathfrak{B}$,

ii. $\mathfrak{A}$ is a relative subalgebra of $\mathfrak{B}$, and

iii. $B \setminus A$ is closed under unary polynomials[4] over $\mathfrak{B}$,

and the morphisms of $\mathbb{C}$ are homomorphisms preserving A pointwise.

Definition 12. *Let $\mathbb{C}$ be a category. We say that $C \in ob(\mathbb{C})$ is an* **initial object** *if for each $A \in ob(\mathbb{C})$ there is a unique morphism $C \to A$. We say that $C \in ob(\mathbb{C})$ is an* **terminal object** *if for each $A \in ob(\mathbb{C})$ there is a unique morphism $A \to C$.*

It is an elementary result that when they exist, initial and terminal objects are unique up to isomorphism.

Theorem 13. *$\mathfrak{A}^*$ is terminal in $\mathbb{C}$.*

Proof. First see that $\mathfrak{A}^*$ is indeed an object of $\mathbb{C}$. $\mathfrak{A}$ generates $\mathfrak{A}^*$ because if $*$ cannot be reached by application of functions from f then $\mathfrak{A}$ was already total in the first place and in that case $\mathfrak{A} = \mathfrak{A}^*$. Since we assign a value outside of A to any function application undefined in $\mathfrak{A}$, $\mathfrak{A}$ is a relative subalgebra of $\mathfrak{A}^*$. Lastly, $\{*\}$ is by definition closed under unary polynomials.
Let $\mathfrak{B} \in ob(\mathbb{C})$. We want to find a unique homomorphism from $\mathfrak{B}$ to $\mathfrak{A}^*$. We claim that this homomorphism is

$$\phi(b) = \begin{cases} b & \text{if } b \in A \\ * & \text{otherwise} \end{cases}$$

We first show that it is a morphism, then prove uniqueness. ϕ restricted to A is the identity, so it does preserve the subuniverse A.
Let $f \in F$ and $\vec{b} \in B^{\Omega(f)}$. We want to show that

$$\phi(f^{\mathfrak{B}}(\vec{b})) = f^{\mathfrak{A}^*}(\phi(\vec{b}))$$

We prove this case by case.

Case 1: $\vec{b} \in A^{\Omega(f)}$ and $f^{\mathfrak{A}}(\vec{b})\downarrow$.
In this case, $\phi(\vec{b}) = \vec{b}$. Also, $f^{\mathfrak{A}}(\vec{b}) \in A$ by definition so that $\phi(f^{\mathfrak{A}}(\vec{b})) = f^{\mathfrak{A}}(\vec{b})$. Lastly, $f^{\mathfrak{A}^*}(\vec{b}) = f^{\mathfrak{A}}(\vec{b}) = f^{\mathfrak{B}}(\vec{b})$ since both $\mathfrak{A}^*$ and $\mathfrak{B}$ are completions of $\mathfrak{A}$ (in fact it is enough that $\mathfrak{A}$ is a weak subalgebra of both). So

$$\phi(f^{\mathfrak{B}}(\vec{b})) = \phi(f^{\mathfrak{A}}(\vec{b})) = f^{\mathfrak{A}}(\vec{b}) = f^{\mathfrak{A}^*}(\vec{b}) = f^{\mathfrak{A}^*}(\phi(\vec{b})).$$

[4]If $\mathfrak{A} = \langle A, F\rangle$ is an algebra, a polynomial over $\mathfrak{A}$ is an element of $Ter(F \cup A, V)$, where we have added to the signature F the elements of A as constants, i.e. nullary function symbols.

Case 2: $\vec{b} \in A^{\Omega(f)}$ and $f^{\mathfrak{A}}(\vec{b}) \uparrow$.
Since $\vec{b} \in A^{\Omega(f)}$, $\phi(\vec{b}) = \vec{b}$. Also, since $\mathfrak{A}$ is a relative subalgebra of $\mathfrak{B}$, $f^{\mathfrak{B}}(\vec{b})$ cannot be in A because otherwise $f^{\mathfrak{A}}(\vec{b})$ would exist (and they would be equal). So $f^{\mathfrak{B}}(\vec{b}) \in B \setminus A$, therefore $\phi(f^{\mathfrak{B}}(\vec{b})) = * = f^{\mathfrak{A}^*}(\vec{b}) = f^{\mathfrak{A}^*}(\phi(\vec{b}))$.

Case 3: $\vec{b} \notin A^{\Omega(f)}$.
Say that $\vec{b} = (b_1, \dots, b_{\Omega(f)})$. If $\vec{b} \notin A^{\Omega(f)}$ then there is at least one b_i such that $b_i \in B \setminus A$. This means that $f^{\mathfrak{A}^*}(\phi(\vec{b})) = *$. It remains to show that $\phi(f^{\mathfrak{B}}(\vec{b})) = *$, so to show that $f^{\mathfrak{B}}(\vec{b}) \notin A$. If $b_i \in B \setminus A$, let $p(x) = f(b_1, \dots, b_{i-1}, x, b_{i+1}, \dots, b_n)$. Since $B \setminus A$ is closed under unary polynomials, $f^B(\vec{b}) = p^{\mathfrak{B}}(b_i) \in B \setminus A$. So $\phi(f^{\mathfrak{B}}(\vec{b})) = * = f^{\mathfrak{A}^*}(\phi(\vec{b}))$ and we have proved that ϕ is a morphism.

We will now prove uniqueness. Let $\psi : \mathfrak{B} \to \mathfrak{A}^*$ be another homomorphism which is the identity on A. For any $b \in B$, we will show that $\phi(b) = \psi(b)$. First note that this holds for any $b \in A$, because $\phi_{\restriction A} = id_A = \psi_{\restriction A}$. So let $b \in B \setminus A$. Since $\mathfrak{B}$ is generated by A, there is a term t over F and a tuple $\vec{a}$ of A such that $t^{\mathfrak{B}}(\vec{a}) = b$. Note that it is easy to show by induction that for any term t and morphism $\phi : \mathfrak{B} \to \mathfrak{A}^*$, $\phi \circ t^{\mathfrak{B}} = t^{\mathfrak{A}^*} \circ \phi$. So now

$$\begin{aligned}
\psi(b) &= \psi(t^{\mathfrak{B}}(\vec{a})) \\
&= t^{\mathfrak{A}^*}(\psi(\vec{a})) \text{ because } \psi \text{ is an algebra homomorphism} \\
&= t^{\mathfrak{A}^*}(\vec{a}) \text{ because } \psi \text{ is identity on } A \\
&= t^{\mathfrak{A}^*}(\phi(\vec{a})) \text{ because } \phi \text{ is identity on } A \\
&= \phi(t^{\mathfrak{B}}(\vec{a})) \text{ because } \phi \text{ is an algebra homomorphism} \\
&= \phi(b).
\end{aligned}$$

□

A remark is in order here: we haven't used the totality of $\mathfrak{B}$ at all in the above proof. So in reality $\mathfrak{A}^*$ is a terminal object in an even larger category, that of possibly partial algebras containing $\mathfrak{A}$, with the rest of the properties of objects and morphisms being the same. Also note that we needed the fact that A generates $\mathfrak{B}$ only to prove uniqueness of the homomorphism ϕ. This gives us

Corollary 14. *Any algebra containing $\mathfrak{A}$ as a weak subalgebra maps homomorphically into $\mathfrak{A}^*$ preserving A pointwise.*

Let us call the underlying set of $\mathfrak{F}(\mathfrak{A})$, $\bar{A}$. We will now prove that $\mathfrak{F}(\mathfrak{A})$ is an initial object in $\mathbb{C}$, but first we need a lemma.

Lemma 15. *If $f \in F$ be of arity n and $(b_1, \dots, b_n) \in \bar{A}^n \setminus A^n$. Then $f^{\mathfrak{F}(\mathfrak{A})}(b_1, \dots, b_n) \notin A$.*

Proof. Without loss of generality assume that $b_1 \notin A$ and $b_2, \dots, b_n \in A$. Since $\mathfrak{F}(\mathfrak{A})$ is generated by A, there is $g \in F$ and $a_1, \dots, a_m$ in A such that

$g^{\mathfrak{F}(\mathfrak{A})}(a_1, \ldots, a_m) = b_1$. Note that $g^{\mathfrak{A}}(a_1, \ldots, a_m) \uparrow$, because since $\mathfrak{A}$ is a weak subalgebra of $\mathfrak{F}(\mathfrak{A})$, b_1 would in that case be in A. But now define a term h such that $h(x_1, \ldots, x_m, y_2, \ldots, y_n) = f(g(x_1, \ldots, x_m), y_2, \ldots, y_n)$. Though its arguments are all in A, $h^{\mathfrak{A}}(a_1, \ldots, a_m, b_2, \ldots, b_n)$ cannot be defined; otherwise $g^{\mathfrak{A}}(a_1, \ldots, a_m)$ would also be defined. Since

$$h^{\mathfrak{F}(\mathfrak{A})}(a_1, \ldots, a_m, b_2, \ldots, b_n) = f^{\mathfrak{F}(\mathfrak{A})}(b_1, \ldots, b_2)$$

it is enough to show that $h^{\mathfrak{F}(\mathfrak{A})}(a_1, \ldots, a_m, b_2, \ldots, b_n) \notin A$.

We can assume that $\mathfrak{A}^*$ is in the product $\prod_{i \in I} \mathfrak{B}_i$, so there is a homomorphism $\beta : \mathfrak{F}(\mathfrak{A}) \to \mathfrak{A}^*$ preserving A pointwise (namely the projection). But then

$$\begin{aligned} \beta(h^{\mathfrak{F}(\mathfrak{A})}(a_1, \ldots, a_m, b_2, \ldots, b_n)) &= h^{\mathfrak{A}^*}(\beta(a_1), \ldots, \beta(a_m), \beta(b_2), \ldots, \beta(b_n)) \\ &= h^{\mathfrak{A}^*}(a_1, \ldots, a_m, b_2, \ldots, b_n) \end{aligned}$$

If $h^{\mathfrak{F}(\mathfrak{A})}(a_1, \ldots, a_m, b_2, \ldots, b_n) \in A$, then also $h^{\mathfrak{A}^*}(a_1, \ldots, a_m, b_2, \ldots, b_n)$, a contradiction. □

Theorem 16. *$\mathfrak{F}(\mathfrak{A})$ is initial in $\mathbb{C}$.*

Proof. $\mathfrak{F}(\mathfrak{A})$ is a completion of $\mathfrak{A}$ and is generated by A by definition 10. Now we want to show that $\bar{A} \setminus A$ is closed under unary polynomials over $\mathfrak{F}(\mathfrak{A})$. Let $b \in \bar{A} \setminus A$ and let $p(X)$ be a unary polynomial. We show by induction on p that $p^{\mathfrak{F}(\mathfrak{A})}(b) \in \bar{A} \setminus A$. First assume $p(X) = X$, i.e. p is the identity. Then $p^{\mathfrak{F}(\mathfrak{A})}(b) \notin A$ trivially. Now for the induction step: say $p(X) = f(p_1(X), \ldots, p_n(X))$ where f is n-ary and p_i are either unary or nullary polynomials. However, at least one, say p_i, is unary, as p is unary. By the induction hypothesis we know that $p_i^{\mathfrak{F}(\mathfrak{A})}(b) \in \bar{A} \setminus A$. But then $p^{\mathfrak{F}(\mathfrak{A})}(b)$ is a new element outside of A, by Lemma 15. So we have proven that $\mathfrak{F}(\mathfrak{A}) \in ob(\mathbb{C})$.

Now let $\mathfrak{B} \in ob(\mathbb{C})$. By definition 10 we know that the homomorphism $\mathfrak{A} \to \mathfrak{B}$ ($\mathfrak{B}$ being a completion of $\mathfrak{A}$) extends uniquely to a homomorphism $\phi : \mathfrak{F}(\mathfrak{A}) \to \mathfrak{B}$. This homomorphism is unique as an extension, but also as a morphism of $\mathbb{C}$, as we required that our morphisms preserve A pointwise. So we are done.

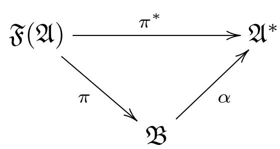
□

Note that we needed both the fact that $\mathfrak{A}$ was a relative subalgebra of the algebras of our category, and that $B \setminus A$ was closed under unary polynomials, in order to have a category where the free completion was initial and the one-element completion terminal (keeping the other conditions on the category). If $\mathfrak{A}$ were not a relative subalgebra of an object $\mathfrak{B}$ in our category, we could have a tuple $\vec{a}$ from A and $f \in F$ such that $f^{\mathfrak{A}}(\vec{a}) \uparrow$, but $f^{\mathfrak{B}}(\vec{a}) \in A$. But then if we had a homomorphism from $\mathfrak{B}$ to $\mathfrak{A}^*$ preserving A, we would get that $f^{\mathfrak{A}^*}(\vec{a}) \in A$, whence $f^{\mathfrak{A}^*}(\vec{a})\downarrow$ because A is a relative subalgebra of $\mathfrak{A}^*$. This is a contradiction, so there can be no such morphism. The case where $B \setminus A$ is not closed under unary polynomials is slightly more involved. Say that we had a unary polynomial over $\mathfrak{B}$ sending an element outside A to an element in A. In particular, assume there was an n-ary term t and a tuple $\vec{b} \in B^n \setminus A^n$ such that

$t^{\mathfrak{B}}(\vec{b}) \in A$. We know that there is a surjective homomorphism $\pi : \mathfrak{F}(\mathfrak{A}) \to \mathfrak{B}$, namely the projection. This homomorphism preserves A by definition; in other words, it is a morphism of our category. Because of surjectivity, there is a tuple $\vec{c} \in \bar{A}^n$ such that $\pi(\vec{c}) = \vec{b}$, and since π preserves A, it cannot be the case that $\vec{c} \in A^n$. However, we also know, since π is a morphism, that

$$\pi(t^{\mathfrak{F}(\mathfrak{A})}(\vec{c})) = t^{\mathfrak{B}}(\pi(\vec{c})) = t^{\mathfrak{B}}(\vec{b}) \in A$$

Since, as we have already seen, $\bar{A} \setminus A$ is closed under unary polynomials, $t^{\mathfrak{F}(\mathfrak{A})}(\vec{c}) \notin A$. But this is not possible. To see this, note that if $\mathfrak{F}(\mathfrak{A})$ is an initial object (and $\mathfrak{A}^*$ terminal), there is a unique morphism $\pi^* : \mathfrak{F}(\mathfrak{A}) \to \mathfrak{A}^*$ preserving A. So the diagram



commutes, where α is a homomorphism preserving A. Therefore if π sends an element outside of A into A, as we have assumed, then so does π^*. Call this element d. Since $\mathfrak{F}(\mathfrak{A})$ is generated by A, there is a term s and a tuple $\vec{a}$ from A such that $s^{\mathfrak{F}(\mathfrak{A})}(\vec{a}) = d$. Since $\mathfrak{A}$ is a relative subalgebra, $s^{\mathfrak{A}}(\vec{a}) \uparrow$. Using the definition of a homomorphism, and preservation of A, we have

$$s^{\mathfrak{A}^*}(\vec{a}) = s^{\mathfrak{A}^*}(\pi^*(\vec{a})) = \pi^*(s^{\mathfrak{F}(\mathfrak{A})}(\vec{a})) = \pi^*(d) \in A$$

Finally, since $\mathfrak{A}$ is a relative subalgebra of $\mathfrak{A}^*$, the statement above means that $s^{\mathfrak{A}}(\vec{a})\downarrow$, a contradiction. To sum up, conditions (*ii*) and (*iii*) on our category are a natural requirement if we are to have condition (*i*), universe-preserving morphisms, and initial and terminal objects $\mathfrak{F}(\mathfrak{A})$ and $\mathfrak{A}^*$ respectively.

1.4 Completing partial algebras with congruences

In this section, we will see how congruences behave with respect to the free completion and the one-element completion. We will see whether congruences can be extended to these completions (stated otherwise: for any congruence relation θ on the partial algebra, is there a congruence relation θ' of the completion such that θ is just θ' restricted to the domain of the partial algebra we started with?). We will also look at the question of extending congruences to certain types of extensions of algebras.
First, the definitions:

Definition 17. *Let $\mathfrak{A}$ be a partial algebra over a signature F. We call a binary relation θ on A a* **congruence relation** *on $\mathfrak{A}$ if*

i. θ is an equivalence relation on A.

ii. *For each $f \in F$ of arity n, for any n-tuples $\vec{a} = (a_1, \ldots, a_n)$, $\vec{b} = (b_1, \ldots, b_n)$ in A^n*

$$(\forall i \ (a_i, b_i) \in \theta) \wedge (f^{\mathfrak{A}}(\vec{a})\downarrow) \wedge (f^{\mathfrak{A}}(\vec{b})\downarrow) \Rightarrow (f^{\mathfrak{A}}(\vec{a}) \equiv f^{\mathfrak{A}}(\vec{b})(mod\ \theta))$$

Definition 18. *A congruence relation θ on $\mathfrak{A}$ is called* **closed** *if for each $f \in F$ of arity n, for any n-tuples $\vec{a} = (a_1, \ldots, a_n)$, $\vec{b} = (b_1, \ldots, b_n)$ in A^n*

$$(\forall i \ (a_i, b_i) \in \theta) \wedge (f^{\mathfrak{A}}(\vec{a})\downarrow) \Rightarrow (f^{\mathfrak{A}}(\vec{b})\downarrow) \wedge (f^{\mathfrak{A}}(\vec{a}) \equiv f^{\mathfrak{A}}(\vec{b})(mod\ \theta))$$

We denote by $Con(\mathfrak{A})$ the set of all congruences on $\mathfrak{A}$, by $ClCon(\mathfrak{A})$ the set of all closed congruences on $\mathfrak{A}$ and if $\theta \in Con(\mathfrak{A})$ then for each $a \in A$, $[a]_\theta := \{b \in A : (a, b) \in \theta\}$ is the equivalence class of a under θ.

Lemma 19. *If $\phi : \mathfrak{A} \longrightarrow \mathfrak{B}$ is a partial algebra homomorphism then the kernel of ϕ, $Ker(\phi) := \{(a, b) \in A^2 : \phi(a) = \phi(b)\}$, is a congruence of $\mathfrak{A}$. If ϕ is a closed homomorphism, then $Ker(\phi)$ is a closed congruence.*

Proof. Straightforward. □

We can ask the question of whether, for a given partial algebra $\mathfrak{A}$ with congruences, it is possible to find completions and congruences on them which extend the original congruences.
It would be expected that if θ were a congruence on $\mathfrak{A}$, we could extend that congruence to $\mathfrak{A}^*$ by adding $\{(*, *)\}$ to θ. However, this is not true if θ is not closed. To see this, let $\theta^* = \theta \cup \{(*, *)\}$ and let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be elements of A^* such that $(a_i, b_i) \in \theta^*$ for all $1 \leq i \leq n$. Let $f \in F$ be of arity n. Because $\mathfrak{A}^*$ is total, we already know that $f^{\mathfrak{A}^*}(a_1, \ldots, a_n)\downarrow$ and $f^{\mathfrak{A}^*}(b_1, \ldots, b_n)\downarrow$. If $a_i \in A$ for all i, then also $b_i \in A$ for all i and so in fact $(a_i, b_i) \in \theta$ for all i. However, it may be that $f^{\mathfrak{A}}(a_1, \ldots, a_n)\downarrow$ but $f^{\mathfrak{A}}(b_1, \ldots, b_n) \uparrow$. In this case $f^{\mathfrak{A}^*}(a_1, \ldots, a_n) \in A$ but $f^{\mathfrak{A}^*}(b_1, \ldots, b_n) = *$, by which it is impossible that the pair of these two terms be in θ^*. It is obvious by the above that we need a closed congruence on $\mathfrak{A}$ in order for this to work:

Proposition 20. *Let $\mathfrak{A}$ be a partial algebra. $\theta \in ClCon(\mathfrak{A})$ if and only if $\theta^* := \theta \cup \{(*, *)\} \in Cong(\mathfrak{A}^*)$.*

Proof. ($\Rightarrow$): Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be elements of A^* such that $(a_i, b_i) \in \theta^*$ for all $1 \leq i \leq n$. Let $f \in F$ be of arity n. Because $\mathfrak{A}^*$ is total, we already know that $f^{\mathfrak{A}^*}(a_1, \ldots, a_n)\downarrow$ and $f^{\mathfrak{A}^*}(b_1, \ldots, b_n)\downarrow$.
If $a_i \in A$ for all i, then also $b_i \in A$ for all i and so in fact $(a_i, b_i) \in \theta$ for all i. Since θ is a closed congruence, either both of $f^{\mathfrak{A}}(a_1, \ldots, a_n)$ and $f^{\mathfrak{A}}(b_1, \ldots, b_n)$ exist or neither does. In the first case

$$(f^{\mathfrak{A}^*}(a_1, \ldots, a_n), f^{\mathfrak{A}^*}(b_1, \ldots, b_n)) = (f^{\mathfrak{A}}(a_1, \ldots, a_n), f^{\mathfrak{A}}(b_1, \ldots, b_n)) \in \theta \subseteq \theta^*$$

because θ is a congruence. In the other case, both terms evaluate to $*$ in $\mathfrak{A}^*$, so the pair is in θ^* anyway.

So assume there is at least one $1 \leq i \leq n$ such that $a_i = *$. Then by definition of θ^*, $b_i = *$ as well. But then

$$(f^{\mathfrak{A}^*}(a_1, \ldots, a_n), f^{\mathfrak{A}^*}(b_1, \ldots, b_n)) = (*, *) \in \theta^*.$$

($\Leftarrow$): It is straightforward to see that the restriction of a congruence to the universe of a weak subalgebra is a congruence on the weak subalgebra. In this particular case, if $\vec{a} = (a_1, \ldots, a_n)$, $\vec{b} = (b_1, \ldots, b_n)$ are such that for all i, $(a_i, b_i) \in \theta \subseteq \theta^*$, $f^{\mathfrak{A}}(\vec{a})\downarrow$ and $f^{\mathfrak{A}}(\vec{a})\downarrow$, then

$$(f^{\mathfrak{A}}(\vec{a}), f^{\mathfrak{A}}(\vec{b})) = (f^{\mathfrak{A}^*}(\vec{a}), f^{\mathfrak{A}^*}(\vec{b})) \in \theta^* \cup A^2 = \theta$$

So theta is a congruence on $\mathfrak{A}$. If θ is not closed, then there are $f \in F$, $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$ such that $(a_i, b_i) \in \theta$ for all i, and $f^{\mathfrak{A}}(\vec{a})\downarrow$ but $f^{\mathfrak{A}}(\vec{b})\uparrow$. We prove that with these tuples $(f^{\mathfrak{A}^*}(\vec{a}), f^{\mathfrak{A}^*}(\vec{b})) \notin \theta^*$, making it impossible for θ^* to be a congruence. If $f^{\mathfrak{A}}(\vec{b})\uparrow$, then $(f^{\mathfrak{A}^*}(\vec{a}), f^{\mathfrak{A}^*}(\vec{b}))$ is of the form $(c, *)$ for some $c \in A$. But there is no pair of this form in θ^*. □

How about the free algebra? Does every congruence on $\mathfrak{A}$ extend to the free algebra $\mathfrak{F}(\mathfrak{A})$? It turns out that we do not need the congruence to be closed in this case. This has been proven in [Grä79, Theorem 1.15], but Isidore Fleischer gives a sketch of the following simpler proof in [Fle]. We first need to define, for a congruence θ on $\mathfrak{A}$, the quotient algebra $\mathfrak{A}/\theta$.

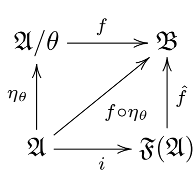
Definition 21. *Let $\mathfrak{A}$ be a partial algebra over F and θ a congruence on it. We define the* **quotient algebra** $\mathfrak{A}/\theta = \langle A/\theta, F\rangle$ *as follows: for each $f \in F$ of arity n, for any $b, b_1, \ldots b_n \in A/\theta$,*

$$f^{A/\theta}([b_1]_\theta, \ldots, [b_1]_\theta) = [f^{\mathfrak{A}}(a_1, \ldots, a_n)]_\theta$$

if there are $a_1, \ldots, a_n$ such that $a_i \in [b_i]_\theta$ for all i and $f^{\mathfrak{A}}(a_1, \ldots, a_n)\downarrow$. Otherwise, $f^{A/\theta}([b_1]_\theta, \ldots, [b_1]_\theta)$ is undefined.

Theorem 22. *For any partial algebra $\mathfrak{A}$ and any $\theta \in Con(\mathfrak{A})$, there is a $\hat{\theta} \in Cong(\mathfrak{F}(\mathfrak{A}))$ such that $\hat{\theta}_{\restriction A} = \theta$.*

Proof. Let $\mathfrak{B}$ be any total algebra into which $\mathfrak{A}/\theta$ embeds. We know there is at least one such algebra, namely $(\mathfrak{A}/\theta)^*$. Call the embedding f. We also have the canonical map η_θ from $\mathfrak{A}$ to $\mathfrak{A}/\theta$ sending a to its equivalence class $[a]_\theta$, so we have a map $f \circ \eta_\theta$ from $\mathfrak{A}$ to $\mathfrak{B}$ which is a composition of the two; it is a homomorphism because the composition of two homomorphisms yields a homomorphism. By our definition of the free completion, there is then a homomorphism $\hat{f} : \mathfrak{F}(\mathfrak{A}) \to \mathfrak{B}$ extending $f \circ \eta_\theta$. In other words, we have the diagram



Since $\hat{f}$ is a homomorphism, $Ker(\hat{f})$ is a congruence by Lemma 19.

To see that it extends θ, let $a, b \in A$. Then

$$\begin{aligned} \hat{f}(a) = \hat{f}(a) &\iff f \circ \eta_\theta(a) = f \circ \eta_\theta(b) \\ &\iff f([a]_\theta) = f([a]_\theta) \\ &\iff [a]_\theta = [b]_\theta \text{ because } f \text{ is 1-1} \\ &\iff (a, b) \in \theta \end{aligned}$$

□

As we saw in the last section, $\mathfrak{A}$ is a relative subalgebra of both $\mathfrak{A}^*$ and $\mathfrak{F}(\mathfrak{A})$ and the complements of A in their respective underlying sets are closed under unary polynomials. Are these conditions on completions of $\mathfrak{A}$ sufficient for congruence extendibility? We know that non-closed congruences need not be extendible, but we do have the following theorem, again from [Fle, Theorem S.C.]:

Theorem 23. *Let $\mathfrak{A}$ be a relative subalgebra of the (possibly partial) algebra $\mathfrak{B}$, and let $B \setminus A$ be closed under unary polynomials. If $\theta \in ClCon(\mathfrak{A})$, then θ extends to a congruence on $\mathfrak{B}$, i.e. there is $\theta' \in Cong(\mathfrak{B})$ such that $\theta' \cap A^2 = \theta$.*

Proof. We first prove that if θ extends to a congruence on a containing algebra $\mathfrak{C}$, then it also extends to a congruence on any containing algebra $\mathfrak{B}$ containing $\mathfrak{A}$ as a relative subalgebra and which maps homomorphically into $\mathfrak{C}$ preserving A pointwise. By Corollary 14 we know that any such $\mathfrak{B}$ will map in this way to $\mathfrak{A}^*$. Since we know that any closed congruence extends to the one-element completion, our theorem will be proven.

Say that $\theta \in ClCon(\mathfrak{A})$, and $\theta'' \in Cong(\mathfrak{C})$ such that $\theta''_{\restriction A} = \theta$. Let $\mathfrak{B}$ be an algebra containing $\mathfrak{A}$ as a relative subalgebra, and $\gamma : \mathfrak{B} \to \mathfrak{C}$ be a homomorphism such that $\gamma_{\restriction A} = id_A$. For every $b, d \in B$, define the congruence relation

$$(b, d) \in \theta' \iff (\gamma(b), \gamma(d)) \in \theta''$$

To see that this is a congruence, notice that $\theta' = Ker(\eta_{\theta''} \circ \gamma)$ where $\eta_{\theta''}$ is the canonical map from $\mathfrak{C}$ to $\mathfrak{C}/\theta''$. Since both $\eta_{\theta''}$ and γ are homomorphisms, so is their composition, and therefore by Lemma 19 θ' is a congruence. Since γ is the identity on A, θ' restricted to A is just the same as θ'' restricted to A: it is equal to θ. So θ' does indeed extend θ. □

This proof is also an alternative to the proof of Theorem 22 for closed congruences, as $\mathfrak{A}$ is a relative subalgebra of $\mathfrak{F}(\mathfrak{A})$.

We have defined an extension of a congruence in a certain way, but we could wish to strengthen that definition. For example, we could require that for θ' to extend θ on an algebra $\mathfrak{B}$ containing $\mathfrak{A}$, $[a]_\theta = [a]_{\theta'}$ for any $a \in A$. As it stands, we have only the left to right inclusion: θ' may very well relate elements outside of A to elements of A. With this stronger definition we exclude such a possibility. Call this sort of extension super-extension. Grätzer [Grä79, Theorem 2.16] proves that superextendibility to total algebras characterizes closed congruences:

Theorem 24. *Let $\mathfrak{A}$ be a partial algebra, $\theta \in Con(\mathfrak{A})$. Then θ is closed if and only if there is a total algebra $\mathfrak{B}$ containing $\mathfrak{A}$ as a relative subalgebra and such that θ superextends to a congruence on $\mathfrak{B}$.*

Chapter 2

Completing algebras with TRSs and HNFs

Let $\mathfrak{A}$ be a partial algebra over a signature F, and let T be an equational theory over the same signature.

Suppose that $\mathfrak{A}$ is a partial algebra model of this equational theory T, defined as

$$s^{\mathfrak{A}}\downarrow \wedge t^{\mathfrak{A}}\downarrow \rightarrow s^{\mathfrak{A}} = t^{\mathfrak{A}}$$

for any s and t such that $T \vdash s \approx t$. Our aim is to find a total algebra into which $\mathfrak{A}$ embeds, and which will also be a model of the same theory. A plausible methodology is as follows: take the free completion $\mathfrak{F}(\mathfrak{A})$ of $\mathfrak{A}$. Define on it the congruence θ_T induced by the theory T, so which will identify instances of T-convertible terms. More formally, if we call the underlying set of the free completion $\bar{A}$ as before, we define θ_T to be the smallest congruence relation containing all pairs $(s^{\mathfrak{F}(\mathfrak{A})}(a_1, \ldots, a_n), t^{\mathfrak{F}(\mathfrak{A})}(a_1, \ldots, a_n))$ where $s \approx t \in T$, s and t are n-ary terms over F and $a_1, \ldots, a_n \in \bar{A}$. Take the quotient algebra $\mathfrak{F}(\mathfrak{A})/\theta_T$. This is a total algebra by definition, and if the canonical map $\phi : \mathfrak{A} \rightarrow \mathfrak{F}(\mathfrak{A})/\theta_T$ is an embedding, then $\mathfrak{F}(\mathfrak{A})/\theta_T$ will be a completion of $\mathfrak{A}$, and it will model T.

So we are looking for conditions under which ϕ is an embedding. However, this is a difficult problem, and in general undecidable. For example, take T to be the theory of groups and G a model of it, seen as an algebra over the signature containing the group operations, the unit element and the generators of G. Then knowing when $[g]_{\theta_T} = [g']_{\theta_T}$ implies $g = g'$ for any given $g, g' \in G$ basically amounts to deciding when two terms over the group generators represent the same element of G. This problem is known as the word problem for groups and is undecidable. So in this thesis we tackle a simpler problem, where T has a nice proof theory. For us, this will be the situation where T is an orthogonal term rewriting system. We will show that under a certain condition (which we will call HNF) the constants of $\mathfrak{A}$ are normal forms, which will force ϕ to be an embedding.

We want $\mathfrak{A}$ to embed into the quotient of the free algebra, so the idea is to add to T the information contained in $\mathfrak{A}$ and require our completion to be a model of both. Formally, we add to T a new term rewriting system T_1 comprising the rules

$$q(\vec{a}) \longrightarrow q^{\mathfrak{A}}(\vec{a})$$

if $q^{\mathfrak{A}}(\vec{a})\downarrow$, where $q \in F$ is an n-ary function symbol and $\vec{a}$ is an n-tuple from A. If this new term rewriting system were to be confluent, and the constants from A were normal forms, then we would have proven that ϕ is an embedding, and we would be done. Unfortunately, this is not necessarily the case. For example, we could have an instance of a left-hand side T-rule with a subterm which evaluates to an element of A. Then we would have both a T-reduction $t \rightarrow s$ and a T_1-reduction $t \rightarrow r$ from our left-hand side t, whose results may not agree. In the worse case, s and r could be distinct normal forms of $T \cup T_1$, and we would not have confluence of this composite system. So we have to artificially add to $T \cup T_1$ rules which would prevent this situation.

The above is a sketch of the general idea behind our result, which proves the completability of a certain type of algebras modelling an orthogonal term rewriting system and satisfying the HNF condition. We will not go into more detail here as the definitions have not been given yet. In what follows we will start by giving some basic definitions, briefly describe the result which we will generalize, and then present this new result.

2.1 Term Rewriting Systems

Term rewriting systems play a fairly large role in theoretical computer science, as an abstraction of the process of computation. A term rewriting system consists of a set of terms and a set of rules for transforming one term into another; they embody the concept of term simplification and therefore normal forms. Combinatory logic and lambda calculus had a big hand in the development of their theory [BKdV03], as confluence, termination and normal forms are essential to these theories, and term rewriting systems provide an easy way to study these. The theory of term rewriting systems (henceforth denoted TRSs) also plays a big role in the investigation of the word problem of algebraic structures. The word problem consists in determining whether two terms denote the same element of the underlying set (for example, in an abelian group, ab and ba denote the same element). The word problem is easy to solve if one has normal forms: to see whether two terms refer to the same element, simply check whether their normal forms are the same.

Therefore, there is a strong link between universal algebra and term rewriting systems. Our present endeavor is a part of this link, and was based on a paper by I. Bethke, J.W. Klop and R. de Vrijer [BKdV96] linking universal algebra, combinatory logic and term rewriting systems. Before describing the content of this paper, let us give some definitions.

In what follows, signatures are defined in the same way as in Chapter 1. We first give the basic definitions of TRSs in order to provide some background for the subsequent description of [BKdV96]. More detailed definitions will follow in section 2.3.1.

Definition 25. *A* **term rewriting system** *consists of a set of terms $Ter(F, V)$ over a countably infinite set of variables V, together with a set of rewrite rules $\{l \to r : l, r \in Ter(F, V)\}$ such that*

i. $l \notin V$

ii. every variable occurring in r occurs in l as well.

The rules in question, in general denoted by the Greek letter ρ, are in reality rule schemata, where the variables may be replaced by terms. These replacements are called substitutions:

Definition 26. *A* **substitution** *is a map $\sigma : Ter(F, V) \to Ter(F, V)$ such that for any term $t = f(t_1, \ldots, t_k) \in Ter(F, V)$,*

$$\sigma(f(t_1, \ldots, t_k)) = f(\sigma(t_1), \ldots, \sigma(t_k))$$

This maps is completely determined by its action on variables, so we can consider it a map from V to $Ter(F, V)$. We will write t^σ for $\sigma(t)$. Now, in term rewriting systems we can rewrite terms to other terms in any context, using substitution instances of rules. Formally:

Definition 27. *Let $Ter(F, X)$ be the set of terms over a set X, and let $\Box$ be a nullary function symbol. A* **context** *over F and X is a term in $Ter(F \cup \{\Box\}, X)$.*

A context can be thought of as a term with holes in it, with the placement of the holes represented by $\Box$. We will use the following notation: if $C \in Ter(F \cup \{\Box\}, X)$, then $C[t_1, \ldots, t_n]$ means that we have replaced the n holes in C with $t_1, \ldots, t_n$, in that order. If C is a one hole context (there is only one occurrence of $\Box$ in it) then we will write $C[]$, if it is a two hole context we will write $C[,]$ and so forth. Now, the fact that we can rewrite in any context can be expressed formally as:

Definition 28. *A* **rewrite step** *according to the reduction rule $\rho : l \to r$ is the replacement*

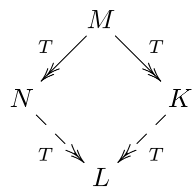
$$C[l^\sigma] \longrightarrow C[r^\sigma]$$

for some context $C[]$ and substitution σ.

If T is a term rewriting system, we will write $s \to_T t$ if $s \to t$ is a rewrite step according to a reduction rule in T. The transitive reflexive closure of the relation $\to_T$ will be denoted $\twoheadrightarrow_T$. A dotted arrow will signify that there exists such an arrow.

Definition 29. *A TRS $T = \langle Ter(F, V), \{\rho_i\}_{i \in I} \rangle$ is said to be* **confluent** *if $\twoheadleftarrow_T \cdot \twoheadrightarrow_T \subseteq \twoheadrightarrow_T \cdot \twoheadleftarrow_T$, or equivalently if for any terms M, N, $K \in Ter(F, V)$*

such that $M \twoheadrightarrow_T N$ and $M \twoheadrightarrow_T K$ there is a term $L \in Ter(F, V)$ and rewrite steps according to rules in T such that $N \twoheadrightarrow_T L$ and $K \twoheadrightarrow_T L$:



The above is sometimes called the Church-Rosser property. Lastly, if we have two term rewriting systems it may be that these two commute:

Definition 30. *If $T_1 = \langle Ter(F, V), \{\rho_i\}_{i \in I}\rangle$ and $T_2 = \langle Ter(F, V), \{\rho'_j\}_{j \in J}\rangle$ are two TRSs whose rewrite relations are $\to_1$ and $\to_2$ respectively, then T_1 and T_2 are said to* **commute** *if $\twoheadleftarrow_1 \cdot \twoheadrightarrow_2 \subseteq \twoheadrightarrow_2 \cdot \twoheadleftarrow_1$.*

This is enough background for the brief summary of [BKdV96], which we will henceforth refer to as BKV, in the next section.

2.2 Partial combinatory algebras and head-normal forms

The paper *Completing Partial Combinatory Algebras with Unique Head-Normal Forms* [BKdV96] deals with the completion of partial combinatory algebras. These partial combinatory algebras are a special kind of algebra modelling combinatory logic (and of course Lambda calculus) and its S and K operators. We will slightly modify the notation of the paper in order to make the subsequent transition to our new result smoother.

Definition 31. *A* **partial combinatory algebra** *is an algebra $\mathfrak{A}$ over the signature $F = \{s, k, \cdot\}$. s, k and $\cdot$ have arities 0, 0 and 2 respectively. The interpretations of these function symbols are such that*

i. For any $a, a' \in A$, $k^{\mathfrak{A}} \cdot^{\mathfrak{A}} a \cdot^{\mathfrak{A}} a' = a$,

ii. For any $a, a' \in A$, $s^{\mathfrak{A}} \cdot^{\mathfrak{A}} a \cdot^{\mathfrak{A}} a' \downarrow$,

iii. For any $a, a', a'' \in A$,

$$(a \cdot^{\mathfrak{A}} a') \cdot^{\mathfrak{A}} (a \cdot^{\mathfrak{A}} a'')\downarrow \Rightarrow s^{\mathfrak{A}} \cdot^{\mathfrak{A}} a \cdot^{\mathfrak{A}} a' \cdot^{\mathfrak{A}} a'' = (a \cdot^{\mathfrak{A}} a') \cdot^{\mathfrak{A}} (a \cdot^{\mathfrak{A}} a'')$$

Otherwise $s^{\mathfrak{A}} \cdot^{\mathfrak{A}} a \cdot^{\mathfrak{A}} a' \cdot^{\mathfrak{A}} a''$ is not defined.

iv. $s^{\mathfrak{A}}\downarrow$, $k^{\mathfrak{A}}\downarrow$ and $s^{\mathfrak{A}} \neq k^{\mathfrak{A}}$.

In the above, unparenthesized expressions associate to the left, and $\cdot$ is written infix as it symbolizes the application operation of the combinatory algebra;

often we will omit it altogether. So $(saa')^{\mathfrak{A}} = s^{\mathfrak{A}} \cdot^{\mathfrak{A}} a \cdot^{\mathfrak{A}} a'$. Lastly, condition *iii.* can be shortened to $(saa'a'')^{\mathfrak{A}} \simeq (aa''(aa'))^{\mathfrak{A}}$, where $\simeq$ is Kleene equality[1].

Not all partial combinatory algebras are completable. A nice example taken from I. Bethke's *On the Existence of Extensional Partial Combinatory Algebras* [Bet87] will be presented in the next section.

It is then proven that the following two conditions are sufficient for a partial combinatory algebra to be completable:

1. No two elements from two distinct sets among $\{s^{\mathfrak{A}}\}$, $\{k^{\mathfrak{A}}\}$,$\{(ka)^{\mathfrak{A}} : a \in A\}$,$\{(sa)^{\mathfrak{A}} : a \in A\}$ and $\{(saa')^{\mathfrak{A}} : a, a' \in A\}$ are equal. The five different forms, before evaluation in $\mathfrak{A}$, are called head-normal forms.

2. (Barendregt's Axiom) For any $a, a', b, b' \in A$, if $(saa')^{\mathfrak{A}} = (sbb')^{\mathfrak{A}}$ then $a = b$ and $a' = b'$.

If a partial combinatory algebra has these two properties we will say that it has unique head-normal forms.
The proof of completability involves the construction of two TRSs that will guide us in the completion of the algebra. The first, $T_1(\mathfrak{A})$, consists of terms from $Ter(F \cup A, \emptyset)$ and rules

$$\{(a \cdot a') \to (aa')^{\mathfrak{A}} : \quad (aa')^{\mathfrak{A}}\downarrow, a, a' \in A\}$$

The second, $T_2(\mathfrak{A})$, is a TRS over the terms $Ter(F \cup A, V)$, V a countably infinite set of variables, with the rules

- $k \cdot x \cdot y \to x$,
- $(ka)^{\mathfrak{A}} \cdot x \to a$,
- $s \cdot x \cdot y \cdot z \to (x \cdot z) \cdot (y \cdot z)$,
- $(sa)^{\mathfrak{A}} \cdot y \cdot z \to (a \cdot z) \cdot (y \cdot z)$, and
- $(saa')^{\mathfrak{A}} \cdot z \to (a \cdot z) \cdot (a' \cdot z)$.

for any $x, y, z \in V$ and $a, a' \in A$.

It is then shown that given the condition on head-normal forms and Barendregt's Axiom, both these term rewriting systems are confluent, and that they commute. The following theorem, traditionally called the Hindley-Rosen Lemma, gives us the confluence of the TRS consisting of the union of $T_1(\mathfrak{A})$ and $T_2(\mathfrak{A})$, which we will call $T(\mathfrak{A})$.

Theorem 32. *If R_1 and R_2 are two relations on a set X such that each is confluent and R_1 and R_2 commute with each other, the relation $R_1 \cup R_2$ is confluent.*

[1] The general definition is: for any terms s and t, $s \simeq t$ if $(s^{\mathfrak{A}}\downarrow \vee t^{\mathfrak{A}}\downarrow) \to (s^{\mathfrak{A}} = t^{\mathfrak{A}})$

We now take the reflexive symmetric transitive closure of the rewrite relation of $T(\mathfrak{A})$, making it an equivalence relation, and denote it $\sim$. This is called the convertibility relation of $T(\mathfrak{A})$. We construct the quotient algebra $\Gamma(\mathfrak{A})$, with underlying set $Ter(F \cup A, \emptyset)/\sim$. The interpretations are defined as

$$f^{\Gamma(\mathfrak{A})}([t_1], \ldots, [t_n]) = [f(t_1, \ldots, t_n)]$$

for any n-ary function symbol in $f \in F \cup A$ and terms $t_i \in Ter(F \cup A, \emptyset)$. It is then straightforward using the confluence of $T(\mathfrak{A})$ and the fact that constants of A are normal forms to prove that $\Gamma(\mathfrak{A})$ is total and that our original partial combinatory algebra embeds in it.

2.2.1 An Example

We present the promised example of a partial combinatory algebra which cannot be completed. Say that a partial combinatory algebra is extensional if for all $a, b \in A$,

$$(\forall c \in A((ac)^{\mathfrak{A}} \simeq (bc)^{\mathfrak{A}})) \Rightarrow a = b$$

Proposition 33. *A strictly partial extensional combinatory algebra cannot be completed, i.e. if $\mathfrak{A}$ is a strictly partial extensional combinatory algebra, it does not embed into a total extensional combinatory algebra.*

Proof. First, note that in every partial combinatory algebra $\mathfrak{A}$ there is an element $\bot_{\mathfrak{A}}$ such that for all $a \in A$, $(\bot_{\mathfrak{A}} a)^{\mathfrak{A}} \uparrow$. This element can be defined as $(s(ka)(ka'))^{\mathfrak{A}}$ for $a, a' \in A$ such that $(aa')^{\mathfrak{A}} \uparrow$. Since $\mathfrak{A}$ is strictly partial there are always such a and a' (application is the only operation that can be partial). $(s(ka)(ka'))^{\mathfrak{A}}$ always exists, because we know that $(ka)^{\mathfrak{A}}\downarrow$ for all a, and $(sab)^{\mathfrak{A}}\downarrow$ for all $a, b \in A$. Since $\mathfrak{A}$ is extensional, $\bot_{\mathfrak{A}} = (s(ka)(ka'))^{\mathfrak{A}}$ is unique.

Assume for a contradiction that $\mathfrak{A}$ embeds into a total extensional partial combinatory algebra $\mathfrak{B}$, via an injective homomorphism ϕ. Note that the fact that ϕ is a homomorphism forces $\phi(k^{\mathfrak{A}}) = k^{\mathfrak{B}}$ and $\phi(s^{\mathfrak{A}}) = s^{\mathfrak{B}}$.

Set $b := k(kk)$ and $b' := k(ks)$. Assume that $(sb\bot_{\mathfrak{A}} a)^{\mathfrak{A}}\downarrow$ for some $a \in A$. Then by definition $(ba(\bot_{\mathfrak{A}} a))^{\mathfrak{A}}$ also exists, wherefore $(\bot_{\mathfrak{A}} a)^{\mathfrak{A}}\downarrow$. But this contradicts the definition of $\bot_{\mathfrak{A}}$. So $(sb\bot_{\mathfrak{A}} a)^{\mathfrak{A}} \uparrow$ for all $a \in A$, and the same thing holds for $(sb'\bot_{\mathfrak{A}} a)^{\mathfrak{A}}$. By extensionality, $(sb\bot_{\mathfrak{A}})^{\mathfrak{A}} = (sb'\bot_{\mathfrak{A}})^{\mathfrak{A}}$. Since ϕ is a homomorphism, we obtain $(\phi(s^{\mathfrak{A}})\phi(b)\phi(\bot_{\mathfrak{A}}))^{\mathfrak{B}} = (\phi(s^{\mathfrak{A}})\phi(b')\phi(\bot_{\mathfrak{A}}))^{\mathfrak{B}}$ One can apply to $k^{\mathfrak{B}}$ on both sides to get

$$(s(k(kk))\phi(\bot_{\mathfrak{A}})k)^{\mathfrak{B}} = (s(k(ks))\phi(\bot_{\mathfrak{A}})k)^{\mathfrak{B}}$$

Simplifying on both sides (since $\mathfrak{B}$ is total, we have no problem with undefined terms) using the rules from Definition 31, we obtain $k^{\mathfrak{B}} = s^{\mathfrak{B}}$. In other words, $\phi(k^{\mathfrak{A}}) = \phi(s^{\mathfrak{A}})$. Since ϕ is injective, $k^{\mathfrak{A}} = s^{\mathfrak{A}}$ and we have obtained our contradiction. $\square$

2.3 Completing models of TRSs with HNF

2.3.1 Definitions

We will first need a few more definitions. We will define redexes, patterns, pattern overlap and nesting.

Definition 34. *Let $\rho : l \rightarrow r$ be a rule in a TRS T and σ a substitution. Then l^σ is a* **redex** *of ρ^σ (or of T) and r^σ is a* **contractum** *of ρ^σ. An instance of l^σ in a term t is called a* **redex occurrence***.*

The word "redex" comes from "reducible expression". Many of our proofs will involve the notion of the pattern of a redex.

Definition 35. *If $\rho : l \rightarrow r$ is a reduction rule, the* **pattern** *of ρ is l^ϵ where $\epsilon : V \rightarrow Ter(F \cup \{\Box\}, V)$ is the substitution which replaces all variables by the nullary symbol $\Box$. It is also the pattern of any substitution instance or redex occurrence $s = l^\sigma$ of l.*

Now, given two redex occurrences (possibly redexes according to different rules), there are different positions these can take relative to one another. They may be disjoint, or one may be a subterm of the other. However, we must make a distinction between whether one is a subterm of the other in some essential way or not. For example, it may be that t is a subterm of $s = l^\sigma$, but only in the sense that $x^\sigma = E[t]$ for some variable x involved in l and context $E[]$. In this case we say that t is **nested** in s. The more pernicious sort of subterm involves pattern overlap:

Definition 36. *We say that two redex occurrences in a term* **overlap** *if their patterns share at least one function symbol occurrence. Note that in this case the head symbol of one must occur in the other [BKdV03, Section 2.1.1]. The pattern overlap between a redex occurrence and itself does not count, unless the redex occurrence is a redex according to two different rewrite rules. This notion can be generalized to rewrite rules: two rules ρ_1 and ρ_2 overlap if in a term t there is a ρ_1-redex s_1 and a ρ_2-redex s_2 such that the patterns of s_1 and s_2 overlap.*

We call this kind of subterm relation pernicious because this is the kind of relation that can make or break the confluence of a term rewriting system, or the commutativity of two systems. The reason for this is that if ρ_2-redex s_2 is merely nested in ρ_1-redex s_1, one may effectuate a ρ_1-reduction even after a ρ_2-reduction has been made; the pattern of s_1 will not have been changed and the only thing to modify will be the substitution involved in the ρ_1-reduction. If the patterns overlap, however, the reduction of s_2 will modify s_1 to such extent that it will not be a ρ_1-redex any longer. For more on this subject see the Critical Pair Lemma in [BKdV03, Section 2.7.2].

We only need to define a few more general properties of TRSs.

Definition 37. *A TRS T is called* **non-ambiguous** *if none of the patterns of its reduction rules overlap.*

Definition 38. *A TRS T is called* **left-linear** *if for each of its rewrite rules, the left-hand side is such that each variable involved occurs only once.*

Definition 39. *A TRS T is called* **orthogonal** *if it is non-ambiguous and left-linear.*

We have the following theorem, which we will state here without proof. For a proof, see [BKdV03, Theorem 4.3.4].

Theorem 40. *Orthogonal TRSs are confluent.*

We can now start presenting our result. Our aim is to prove a general result concerning partial algebras and term rewriting systems: given a partial algebra $\mathfrak{A}$ modelling a TRS T, can we construct a completion of this algebra which is still a model of T? If so, how is this to be done? And what do we mean by "modelling"?

We have already given the definition of a partial algebra and of a term rewriting system, but it is still not clear in what sense the former should be considered to model the latter. Intuitively, we want the interpretation of two terms to be the same in the algebra $\mathfrak{A}$ if the two terms rewrite to each other in T.

Definition 41. *Let $T = \langle Ter(F, V), \{\rho_i\}_{i\in I}\rangle$ be a term rewriting system. We say that $\mathfrak{A} = \langle A, \{f^{\mathfrak{A}}\}_{f\in F}\rangle$ is a* **model** *of T if for any rule*

$$l(x_1, \ldots, x_n) \longrightarrow r(x_1, \ldots, x_n)$$

in T and any $a_1, \ldots, a_n \in A$,

$$l^{\mathfrak{A}}(a_1, \ldots, a_n){\downarrow} \Rightarrow r^{\mathfrak{A}}(a_1, \ldots, a_n) = l^{\mathfrak{A}}(a_1, \ldots, a_n)$$

Note that because of the way we defined a TRS, all of the a_i need not be used in r, but those which are should also be used in l.

We aim to show that partial algebras which model orthogonal term rewriting systems and which satisfy the following HNF condition on partial algebras can be completed to total algebras modelling the same term rewriting system.

HNF. *Say $s(x_1, \ldots, x_n)$ and $t(y_1, \ldots, y_m)$ are proper subterms of the left-hand side of a T-rule (not necessarily the same rule for both), such that the variables displayed are those that are actually used. Then for any tuples $\vec{a} = (a_1, \ldots, a_n) \in A^n$, $\vec{b} = (b_1, \ldots, b_m) \in A^m$,*

$$s^{\mathfrak{A}}(\vec{a}) = t^{\mathfrak{A}}(\vec{b}) \Rightarrow s = t \wedge \vec{a} = \vec{b}$$

Similarly to the approach used in BKV [BKdV96], we will define two new term rewriting systems $T_1(\mathfrak{A})$ and $T_2(\mathfrak{A})$. Unlike the result in BKV, these will not [2]both be confluent on their own, but the union of the two systems will be

[2]B.

shown to be confluent under the conditions stated above. We will then use this confluence to construct a completion of $\mathfrak{A}$.

Our aim is basically to synchronize our term rewriting system and our algebra: we will incorporate into the TRS the information contained in $\mathfrak{A}$ ($\mathfrak{A}$ has much more information contained in it than what is encoded in T, as is usually the case with a model of a theory), and then create new rules which will provide a scaffolding for our completion: in principle, these should be rules which incorporate the newly added information from $\mathfrak{A}$ into the rules of T. Our two new term rewriting systems will be defined as follows:

- The TRS $T_1(\mathfrak{A})$ consists of the set $Ter(F \cup A, V)$ together with the rules

  $$f(a_1, \dots, a_n) \longrightarrow a$$

  whenever $f^{\mathfrak{A}}(a_1, \dots, a_n) = a$, for every $f \in F$.

- The TRS $T_2(\mathfrak{A})$ is defined over the set $Ter(F \cup A, V)$. Its set of rules is the union of the rules of T and new rules defined inductively as described below. Let

  $$\rho : l(\vec{x}_1, \dots, \vec{x}_n, \vec{y}) \longrightarrow r(\vec{x}_1, \dots, \vec{x}_n, \vec{y})$$

  be a T-rule, where $\vec{x}_1, \dots, \vec{x}_n, \vec{y}$ are tuples in V. Suppose that there is l'' such that

  $$l(\vec{x}_1, \dots, \vec{x}_n, \vec{y}) = l''(q_1(\vec{x}_1), \dots, q_n(\vec{x}_n), \vec{y})$$

  for proper subterms $q_1, \dots, q_n$ and such that there are $\vec{a}_1, \dots, \vec{a}_n, b_1, \dots b_n$ in A satisfying

  $$q_i^{\mathfrak{A}}(\vec{a}_i) = b_i$$

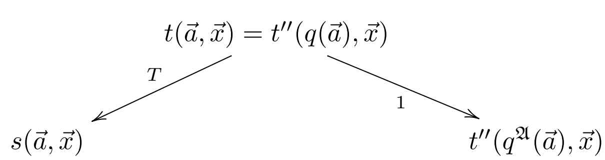
  for all i. Then we add to T the rule

  $$l''(b_1, \dots, b_n, \vec{y}) \longrightarrow r(\vec{a}_1, \dots, \vec{a}_n, \vec{y})$$

First note that because of the requirement that q_is have to be proper subterms, we will never end up with a $T_2(\mathfrak{A})$-rule whose left-hand side is just a constant.

Remark 42. *In the definition of $T_2(\mathfrak{A})$-rules, call the substitution which replaces $\vec{x}_i$ by $\vec{a}_i$ for all i, σ. Also, call $l''(b_1, \dots, b_n, \vec{y})$ l' and $r(\vec{a}_1, \dots, \vec{a}_n, \vec{y})$ r' for simplicity. Then $l^\sigma \twoheadrightarrow l'$ and $r^\sigma = r'$.*

To sum up, the TRS $T_1(\mathfrak{A})$ thus incorporates the information from $\mathfrak{A}$: we can now syntactically replace terms with their evaluations, or interpretations, in $\mathfrak{A}$. However, $T_1(\mathfrak{A})$ added to T is not sufficient for confluence: for example, we may have a T-rule $l \rightarrow r$, but also a subterm of l evaluating to a constant in A for certain arguments. It would then be possible to have rewrite steps



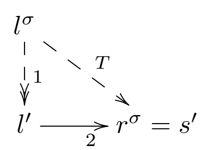
where $s(\vec{a},\vec{x})$ and $t''(q^{\mathfrak{A}}(\vec{a}),\vec{x})$ do not have a common reduct under $T \cup T_1(\mathfrak{A})$. This is the motivation behind our definition of $T_2(\mathfrak{A})$-rules: they allow us to add a rewrite step from $t''(q^{\mathfrak{A}}(\vec{a}),\vec{x})$ to $s(\vec{a},\vec{x})$. So the TRS $T_2(\mathfrak{A})$ allows for the replacements made via $T_1(\mathfrak{A})$: where there previously was a substitution instance of a rule where certain subterms could be evaluated in $\mathfrak{A}$, we have added a new rule with the corresponding evaluation replacing the subterm. This will allow us further on to prove the commutativity of $T_1(\mathfrak{A})$ and $T_2(\mathfrak{A})$.

Before we go any further, however, it will be useful to take a closer look at the construction of $T_2(\mathfrak{A})$ and to define the concept of an "original" counterpart in T to any rule in $T_2(\mathfrak{A})$.

2.3.2 Lifting a $T_2(\mathfrak{A})$-rule to a T-rule

From the description of $T_2(\mathfrak{A})$ rules, we see that a rule $l{\to}r$ in T and a substitution σ give rise to a multitude of new rules $l'{\to}r'$ such that $l^\sigma \twoheadrightarrow_1 l'$ and $r^\sigma = r'$ (which new rule is produced depends on the choice of subterms $q_i(\vec{a_i},\vec{b_i})$ to evaluate). Looking at it the other way, by definition any rule in $T_2(\mathfrak{A})\setminus T$ is a descendant of a rule in T via a substitution in exactly this fashion. If $l'{\to}r'$ is a descendant of $l{\to}r$, we call $l{\to}r$ an **original** of $l'{\to}r'$ in T, and we call the pair consisting of the rule and substitution, as well as the process of finding such a pair, a **lift** or **lifting**. We will now see that a lift is in fact unique.

Proposition 43. *For any rule $l'{\to}r'$ in $T_2(\mathfrak{A})$ there is a rule $l{\to}r \in T$ and a substitution σ such that $l^\sigma \twoheadrightarrow_1 l'$ and $r^\sigma = r'$. In sum, we have the diagram*



Furthermore, this original rule and substitution are unique (up to renaming of variables).

Proof. The first part of the theorem is obvious by the definiton of $T_2(\mathfrak{A})$-rules. If $l'{\to}r'$ is already in T, just take σ to be the empty substitution. So we must prove uniqueness.

Let $l_1{\to}r_1$ and $l_2{\to}r_2$ be rules in T and let σ, τ be substitutions such that $l_1^\sigma \twoheadrightarrow_1 l' \twoheadleftarrow_1 l_2^\tau$ and $r_1^\sigma = r' = r_2^\tau$. We want to show that $l_1 = l_2$, $r_1 = r_2$ and $\sigma = \tau$, up to renaming of variables. By definition a T-rule does not contain any occurrences of constants from A. So any constants from A occurring in l' must neccessarily have been brought about by $T_1(\mathfrak{A})$-reductions of substitution instances of subterms of l_1 and l_2. Say that the constants occurring in l' are $b_1,\ldots,b_n$ and say that the substitution instances of subterms they come from are $q_i(\vec{a}_i)$ in l_1^σ and $p_i(\vec{c}_i)$ in l_2^τ for all i (we will call such terms **expansions**: if $t^{\mathfrak{A}}(\vec{a}) = b$ for a term t and $\vec{a}, b$ in A, then $t(\vec{a})$ is an expansion of b). Thus for all i, there are proper subterms $q_i(\vec{x}_i)$ of l_1 and $p_i(\vec{y}_i)$ of l_2 and there are tuples

$\vec{a}_i$, $\vec{b}_i$ such that

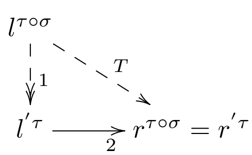
$$q_i^{\mathfrak{A}}(\vec{a}_i) = b_i = p_i^{\mathfrak{A}}(\vec{c}_i)$$

But then by our HNF condition, $p_i = q_i$ and $\vec{a}_i = \vec{b}_i$. Now note that l_1 is just l' with $q_i(\vec{x}_i)$ in the place of b_i, and similarly l_2 is just l' with $p_i(\vec{y}_i)$ in the same place. So by the above l_1 is just l_2 with $\vec{x}_i$ in place of $\vec{y}_i$. But since T-rules are left-linear, one can rename the variables to get the same term. So $l_1 = l_2$ up to renaming of variables.

Lastly, σ is the substitution which replaces $\vec{x}_i$ with $\vec{a}_i$ for all i, and τ replaces each $\vec{y}_i$ bu $\vec{c}_i$. Since $\vec{a}_i = \vec{c}_i$, if we rename all $\vec{y}_i$s to $\vec{x}_i$s, σ and τ are the same. So our proposition is proven. □

It is worth emphasizing the fact that any expansion involved in a lift will by definition be a substitution instance of a proper subterm of the left-hand side of the T-rule concerned.

However, we will normally be working with substitution instances of $T_2(\mathfrak{A})$-rules, not the rules themselves. This complicates matters a bit, but not too much; if τ is a substitution instance of a $T_2(\mathfrak{A})$-rule $l' \to r'$, then it works on variables which are also in the original rule $l \to r$ of $l' \to r'$; i.e. the variables which have not been modified by the substitution σ involved in the lift. Since T-rules are left-linear, we can define the substitution $\sigma \circ \tau$, and we get the diagram



So for any substitution instance $l'^{\tau} \to r'^{\tau}$ of a $T_2(\mathfrak{A})$-rule we still have a unique T-rule $l \to r$ and a unique substitution $\sigma \circ \tau$ such that the above diagram holds. We will want to distinguish, however, between occurrences of constants which come from the subtitution τ, and those which are actually part of the rule $l' \to r'$. The second kind will be the ones whose expansions are used in the lifting; we will call these occurrences of constants **essential** for the lift.

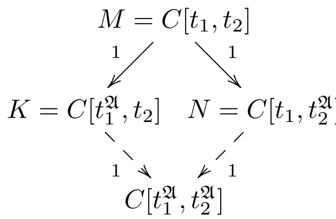
2.3.3 Confluence of $T_1(\mathfrak{A})$ and commutativity of the two systems

First, let us show that $T_1(\mathfrak{A})$ is by itself confluent. In fact, we will show a stronger theorem, namely that $T_1(\mathfrak{A})$ has the diamond property. It is easy to show by an induction argument that the diamond property implies confluence. We will from now on call the rewrite relations of $T_1(\mathfrak{A})$ and $T_2(\mathfrak{A})$ $\to_1$ and $\to_2$ respectively.

Theorem 44. *$T_1(\mathfrak{A})$ has the diamond property, i.e.* $\leftarrow_1 \cdot \to_1 \subseteq \to_1 \cdot \leftarrow_1$.

Proof. Let M, K and N be such that $K \leftarrow_1 M \to_1 N$, and let t_1, t_2 be the redex occurrences involved in each. $T_1(\mathfrak{A})$-rules are of the form $f(\vec{a}) \to a$ for $f \in F$,

$\vec{a}, a$ in A. Therefore the redexes t_1 and t_2 are either identical or disjoint. If they are identical the assertion is trivial. So let us assume they are disjoint. Then there is a context $C[]$ such that $M = C[t_1, t_2]$. Hence we have the following diagram:

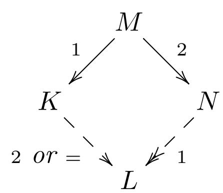


Therefore the system $T_1(\mathfrak{A})$ has the diamond property. □

Corollary 45. *$T_1(\mathfrak{A})$ is confluent.*

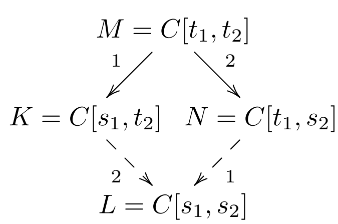
Now we will prove that $T_1(\mathfrak{A})$ and $T_2(\mathfrak{A})$ commute. But we first need a couple of auxiliary lemmas.

Lemma 46. *If M, N and K are terms such that $M \to_1 K$ and $M \to_2 N$, we have a term L such that $N \twoheadrightarrow_1 L$ and either $L = K$ or $K \to_2 L$:*



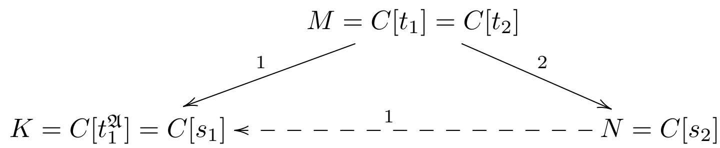
Proof. Let t_1 be the redex occurrence concerned in $M \to_1 K$ and t_2 the redex occurrence concerned in $M \to_2 N$. We will prove this lemma case by case, depending on the relative positions of t_1 and t_2 in M. Say that $t_1 \to s_1$ is a seesubstitution instance of the rule $\rho_1 : l_1 \to r_1 \in T_1(\mathfrak{A})$ via substitution σ, and that $t_2 \to s_2$ is a substitution instance of $\rho_2 : l_2 \to r_2 \in T_2(\mathfrak{A})$ via substitution τ.

Case 1: t_1 and t_2 are disjoint.
In this case we can write M as $C[t_1, t_2]$ for some context $C[]$. But then it is obvious that

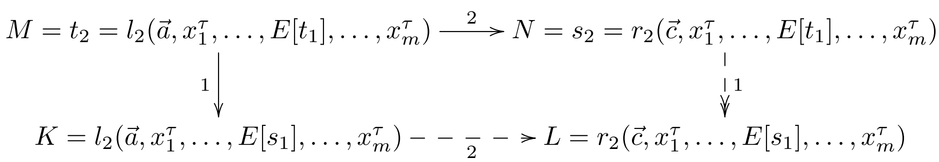


Case 2: $t_2 \leq t_1$.
Because of the shape of $T_1(\mathfrak{A})$-rules, and because it is never the case that a constant is the left-hand side of a $T_2(\mathfrak{A})$-rule, the only possible scenario is that

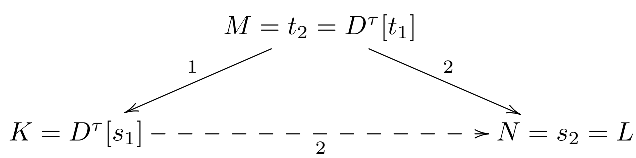
$t_1 = t_2$. We know that $t_1^{\mathfrak{A}}\downarrow$ because the $T_1(\mathfrak{A})$-transition is $t_1 \to s_1 = t_1^{\mathfrak{A}}$. So $t_2^{\mathfrak{A}}\downarrow$ as well. But since $\mathfrak{A} \models T$, $s_2^{\mathfrak{A}} = t_2^{\mathfrak{A}}$. So there is a $T_1(\mathfrak{A})$-rule $s_2 \to s_2^{\mathfrak{A}}$, and $s_2^{\mathfrak{A}} = t_2^{\mathfrak{A}} = t_1^{\mathfrak{A}} = s_1$. In sum, we have the diagram



Case 3.1: $t_1 < t_2$ and t_1 is nested in t_2.
As the left-hand side of a $T_2(\mathfrak{A})$ rule, we know that l_2 is of the form $l_2(\vec{a}, x_1, \ldots, x_m)$ for some tuples $\vec{a}$ of A, $\vec{x} = (x_1, \ldots, x_m)$ of V. If t_1 is nested in $l_2^\tau = t_2$, then there is an x_i in $\vec{x}$ such that $x_i^\tau = E[t_1]$ for some context $E[]$. Since r_2 is of the form $r_2(\vec{c}, x_1, \ldots, x_m)$, we can complete our diamond as below. We will again ignore the outer context and assume $M = t_2$.

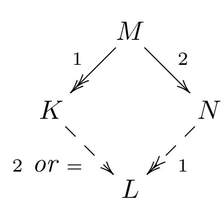


Case 3.2: $t_1 < t_2$, and t_1 and t_2 have overlapping patterns.
This case emerges directly from the definition of $T_2(\mathfrak{A})$. We can this time write $t_2 = D^\tau[t_1]$, inverting the roles of t_1 and t_2 in the previous case. So t_1 is a proper subterm of t_2 which evaluates to an element of A, namely s_1. Then by the inductive definition of $T_2(\mathfrak{A})$, there is a new $T_2(\mathfrak{A})$-rule $D^\tau[s_1] \to s_2$. So



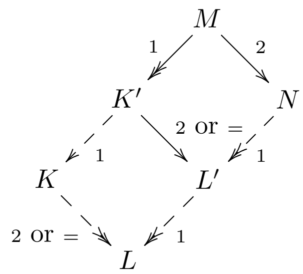
□

Lemma 47. *If M, N and K are terms such that $M \twoheadrightarrow_1 K$ and $M \to_2 N$, then there is a term L such that $N \twoheadrightarrow_1 L$ and either $K = L$ or $K \to_2 L$*



Proof. By induction on $length(M \twoheadrightarrow_1 K) =: m$. Lemma 46 gives us the base case directly. For the induction step, we assume that the lemma holds for m

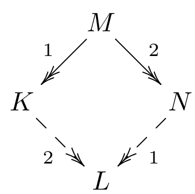
and show it holds for $m+1$. So assume $length(M \twoheadrightarrow_2 K) = m+1$. Let K' be the term such that $M \twoheadrightarrow_1 K' \to_1 K$ and $length(M \twoheadrightarrow_1 K') = m$. By the induction hypothesis, there is a term L' such that $K' = L'$ or $K' \to_2 L'$ and $N \twoheadrightarrow_1 L'$. Using Lemma 46 one more time gives us the term L as in the diagram below:



□

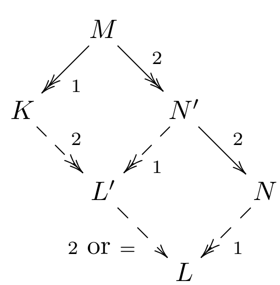
Now the proof of our theorem is almost trivial.

Theorem 48. *If M, N and K are terms such that $M \twoheadrightarrow_1 K$ and $M \twoheadrightarrow_2 N$, then there is a term L such that $N \twoheadrightarrow_1 L$ and $K \twoheadrightarrow_2 L$:*



where $length(K \twoheadrightarrow_2 L) \leq length(M \twoheadrightarrow_2 N)$.

Proof. Let $length(M \twoheadrightarrow_2 N) =: m$. We prove the theorem by induction on m. The base case is given by Lemma 47. Now we assume the theorem holds for m and show that it holds for $m+1$. Let N' be the term such that $M \twoheadrightarrow_2 N' \to_2 N$ and $length(M \twoheadrightarrow_1 N') = m$. By the induction hypothesis we have a term L' such that $N' \twoheadrightarrow_1 L'$, $K \twoheadrightarrow_2 L'$ and $length(K \twoheadrightarrow_2 L') \leq m$. Now by Lemma 47, we get the term L such that $L' = L$ or $L' \to_2 L$, $N \twoheadrightarrow_1 L$ and $length(K \twoheadrightarrow_2 L) = length(K \twoheadrightarrow_2 L') + length(L' \twoheadrightarrow_2 L) \leq m+1$. So we are done, with the diagram



□

2.3.4 $T_2(\mathfrak{A})$ pseudoconfluence

We saw in subsection 2.3.2 that we can lift a $T_2(\mathfrak{A})$-rule to a T-rule. But now how about a chain of two $T_2(\mathfrak{A})$-rules? Or a chain of n $T_2(\mathfrak{A})$-rules, for that matter? We will show that this can also be done.

However, we first need some terminology and notation which will allow us to talk about occurrences of subterms in terms rather than the subterms themselves. Occurrences of constants will from now on be denoted in bold font, and distinguished from each other by an overbar: for a term s, $\mathbf{s}, \bar{\mathbf{s}}, \ldots, \bar{\mathbf{s}}^{(\mathbf{i})}$ (where $\bar{\mathbf{s}}^{(\mathbf{i})}$ is $\mathbf{s}$ with i many bars over it and $\bar{\mathbf{s}}^{(\mathbf{0})}$ means $\mathbf{s}$ as expected) are occurrences of the term s. We will also need to deepen our notation for expansions. Given any chain of $T_1(\mathfrak{A})$-reductions $M \twoheadrightarrow_1 N$, for every occurrence $\mathbf{s}$ of a term s in N there is a corresponding subterm of M. We call this term the expansion of $\mathbf{s}$ in M and denote it $\mathbf{s}_M$. It is possible that $\mathbf{s}_M$ is just an instance of s (we will abuse notation and write $\mathbf{s}_M = s$): in this case we say that $\mathbf{s}_M$ is a trivial expansion.

Lemma 49. *$\twoheadrightarrow_1 \cdot \rightarrow_T \subseteq \rightarrow_T \cdot \twoheadrightarrow_1$. In fact, for any T-rule ρ, then $\twoheadrightarrow_1 \cdot \rightarrow_\rho \subseteq \rightarrow_\rho \cdot \twoheadrightarrow_1$.*

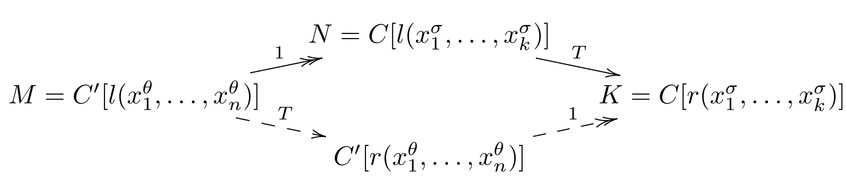
Proof. Let $M \twoheadrightarrow_1 N \rightarrow_\rho K$. Let ρ be the rule

$$l(x_1, \ldots, x_n) \rightarrow r(x_1, \ldots, x_n)$$

Then there is a ρ-redex in N, i.e. there is a substitution σ such that $N = C[l(x_1^\sigma, \ldots, x_n^\sigma)]$ for some context $C[]$. Now since $M \twoheadrightarrow_1 N$ is a $T_1(\mathfrak{A})$-reduction, there exist in M the expansions of certain constants occurring in N. If these constants all occur outside of $l(x_1^\sigma, \ldots, x_n^\sigma)$, then there is no problem and we can first do the T-reduction, then do the $T_1(\mathfrak{A})$-reductions necessary in $C[]$. So assume that certain of the constants occur in $l(x_1^\sigma, \ldots, x_n^\sigma)$. Say wlog that $\mathbf{a}$ occurs in x_j^σ for some $1 \leq j \leq n$, and that its expansion is $\mathbf{a}_M$. We can then write $x_j^\sigma = D^\sigma[\mathbf{a}]$ for some context $D[]$. But then we can define a new substitution θ acting just like σ except that $x_j^\theta = D^\sigma[(\mathbf{a})_\mathbf{M}]$. This means that $x_j^\theta \twoheadrightarrow_1 x_j^\sigma$. Note that we can define a single substitution θ to have the corresponding effect on all x_j^σs containing constants, because T is left-linear, so no two x_js are the same. So now, we have

$$C'[l(x_1^\theta, \ldots, x_n^\theta)] \rightarrow_\rho C'[r(x_1^\theta, \ldots, x_n^\theta)] \twoheadrightarrow_1 C[r(x_1^\sigma, \ldots, x_n^\sigma)]$$

where $C'[]$ is $C[]$ with the expansions of the rest of the constants, so that $C'[] \twoheadrightarrow_1 C[]$. Therefore we have the diagram



□

Corollary 50. $\twoheadrightarrow_1 \cdot \twoheadrightarrow_T \subseteq \twoheadrightarrow_T \cdot \twoheadrightarrow_1$.

Definition 51. *$M \twoheadrightarrow_1 N$ is called* uniform *if for all constants a and all occurrences $\mathbf{a}, \bar{\mathbf{a}}, \ldots, \bar{\mathbf{a}}^{(\mathbf{k})}$ of a in N, there is a term t such that $t \twoheadrightarrow_1 (\bar{\mathbf{a}}^{(\mathbf{i})})_{\mathbf{M}}$ for all $0 \leq i \leq k$. We will denote this term by $t_{(a,M)}$.*

In this definition, we have imposed that the expansions of all constant occurrences have a common $T_1(\mathfrak{A})$-ancestor. Quite satisfyingly, the same property for terms follows from this definition (for any term s, s_M is defined in the same way as for a constant).

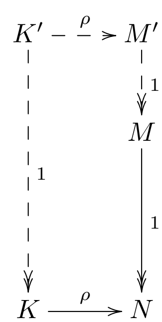
Lemma 52. *If $M \twoheadrightarrow_1 N$ is uniform, then for all subterms s of N and for all occurrences $\mathbf{s}, \bar{\mathbf{s}}, \ldots, \bar{\mathbf{s}}^{(\mathbf{k})}$ of s in N, there is a term $t_{(s,M)}$ such that $t_{(s,M)} \twoheadrightarrow_1 (\bar{\mathbf{s}}^{(\mathbf{i})})_{\mathbf{M}}$ for all $0 \leq i \leq k$.*

Proof. Say $s = s(a_1, \ldots, a_n, \vec{x})$ for $a_1, \ldots, a_n \in A$, $\vec{x}$ a tuple in V. Then for any occurrence $\mathbf{s}$ in N, $\mathbf{s} = s(\mathbf{a_1}, \ldots, \mathbf{a_n}, \vec{x})$ for some occurrences of $a_1, \ldots, a_n$. Therefore $\mathbf{s}_M = s((\mathbf{a_1})_M, \ldots, (\mathbf{a_n})_M, \vec{x})$. Because of uniformity, for all $1 \leq j \leq n$, for all occurrences $\bar{\mathbf{a}}_{\mathbf{j}}^{(\mathbf{i})}$ of a_j, there is a term $t_{(a_j,M)}$ such that $t_{(a_j,M)} \twoheadrightarrow_1 (\bar{\mathbf{a}}_{\mathbf{j}}^{(\mathbf{i})})_{\mathbf{M}}$. But then

$$s(t_{(a_1,M)}, \ldots, t_{(a_n,M)}, \vec{x}) \twoheadrightarrow_1 \mathbf{s_M}$$

so we can call this left-hand side term $t_{(s,M)}$. □

Lemma 53. *Let $M \twoheadrightarrow_1 N$ be uniform. Let $K \rightarrow_\rho N$ be a rewrite step corresponding to a T-rule ρ. Then there are terms K' and M' such that $K' \twoheadrightarrow_1 K$, $K' \rightarrow_\rho M'$, and $M' \twoheadrightarrow_1 M$. In other words, we have the diagram*



Proof. $K \rightarrow N$ is a reduction step corresponding to a T-rule, so it is of the form $C[l(x_1^\sigma, \ldots, x_n^\sigma)] \rightarrow C[r(x_1^\sigma, \ldots, x_n^\sigma)]$ for some context $C[]$ and substitution σ.

We can make occurrences of each variable in r explicit by writing

$$r = D[\mathbf{x_1}, \bar{\mathbf{x_1}}, \ldots, \bar{\mathbf{x_1}}^{(\mathbf{i})}, \ldots, \mathbf{x_n}, \bar{\mathbf{x_n}}, \ldots, \bar{\mathbf{x_n}}^{(\mathbf{j})}]$$

for some variable-free context $D[]$ (without loss of generality, up to reordering of the variables). Say that $x_i^\sigma = s_i$ for some term s_i, for all i. Then

$$N = C[D[\mathbf{s_1}, \ldots, \bar{\mathbf{s_1}}^{(\mathbf{i})}, \ldots, \mathbf{s_n}, \ldots, \bar{\mathbf{s_n}}^{(\mathbf{j})}]]$$

Since $M \twoheadrightarrow_1 N$, there is another context $C'[]$ satisfying $C'[] \twoheadrightarrow_1 C[]$, such that

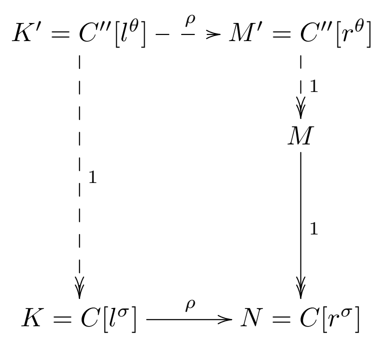
$$M = C'[D[(\mathbf{s_1})_\mathbf{M}, \ldots, (\bar{\mathbf{s_1}}^{(\mathbf{i})})_\mathbf{M}, \ldots, (\mathbf{s_n})_\mathbf{M}, \ldots, (\bar{\mathbf{s_n}}^{(\mathbf{j})})_\mathbf{M}]]$$

Now we define the term K' as follows: find all constants a occurring in N. Define K' as K with *all* occurrences of such constants a expanded to $t_{(a,M)}$. This generalizes to terms: if $x_i^\sigma = s = s(a_1, \ldots, a_m, \vec{x})$ and x_i is actually used in r, then every occurrence of $s = x_i^\sigma$ in l^σ is replaced in K' by $s(t_{(a_1,M)}, \ldots, t_{(a_n,M)}, \vec{x}) = t_{(s,M)}$. We can call this new substitution τ (we don't care what τ does on x_i's which are not actually used in r). This warrants the expression of K' as $C''[l^\theta]$ where $C''[]$ is $C'[]$ with all occurrences of constants a appearing in N and $C'[]$ expanded to $t_{(a,M)}$.

We can now apply the rule ρ to get $M' := C''[r^\theta]$. Note that with our previous conventions, we can write

$$M' = C''[D[t_{(s_1,M)}, \ldots, t_{(s_1,M)}, \ldots, t_{(s_n,M)}, \ldots, t_{(s_n,M)}]]$$

But this obviously $T_1(\mathfrak{A})$-reduces to M, and we are done!



□

Note that as a consequence of this proof, $K' \twoheadrightarrow_1 K$ is also uniform. To see this, let a be an arbitrary constant in K. $K' \twoheadrightarrow K$ only expands constants occurring in N, so if a does not occur in N then we have no problem. If on the other hand a occurs in N, then by construction $\mathbf{a}, \bar{\mathbf{a}}, \ldots, \bar{\mathbf{a}}^{(\mathbf{i})}$ are all replaced by the same term $t_{(a,M)}$, so we are done. We will call a transition which has this property totally uniform.

Definition 54. *We call a uniform transition* $M \twoheadrightarrow_1 N$ **totally uniform** *if for any constant a, for any two occurrences* $\bar{\mathbf{a}}^{(\mathbf{i})}$, $\bar{\mathbf{a}}^{(\mathbf{j})}$ *in* N,

$$(\bar{\mathbf{a}}^{(\mathbf{i})})_\mathbf{M} = (\bar{\mathbf{a}}^{(\mathbf{i})})_\mathbf{M}$$

It is easy to see that this induces the same property on terms.

Lemma 55. *Let* $M \twoheadrightarrow_1 K$ *and* $N \twoheadrightarrow_1 K$ *be such that for any constant a, any one-step (nontrivial) expansion of an occurrence of a which one encounters along*

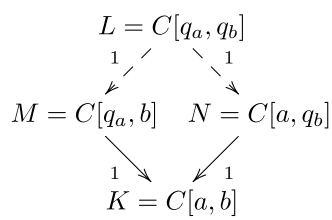
$K \twoheadleftarrow_1 M$ and $K \twoheadleftarrow_1 N$ can yield only one term[3]*. Then there is a term L such that $L \twoheadrightarrow_1 M$ and $L \twoheadrightarrow_1 K$ and*

i. *$length(L \twoheadrightarrow_1 M) = length(N \twoheadrightarrow_1 K)$ and $length(L \twoheadrightarrow_1 N) = length(M \twoheadrightarrow_1 K)$*

ii. *$L \twoheadrightarrow_1 M$ and $L \twoheadrightarrow_1 K$ are also such that for any constant a, any one-step (nontrivial) expansion of an occurrence of a can yield only one term.*

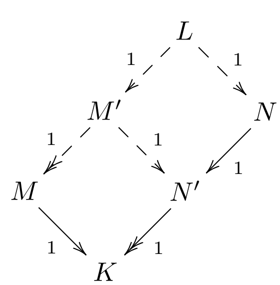
Proof. For each constant a, denote the unique term to which it can be expanded in a one-step expansion as q_a. Set $n := length(N \twoheadrightarrow_1 K)$ and $m := length(M \twoheadrightarrow_1 K)$. We prove the lemma by double induction on n and m. If either n or m is 0, then the proof is trivial.

First assume that $n = m = 1$. We have $M \rightarrow_1 K \leftarrow_1 N$, and we can distinguish two cases. The first is if $M \rightarrow K$ and $N \rightarrow K$ expand distinct constants, or two occurrences of the same constant. Then $K = C[a, b]$, for some context $C[]$ and where b is possibly another occurrence of a (the notation for occurrences is not necessary here since the two-hole context makes it clear that they are distinct occurrences). Then we have the diagram



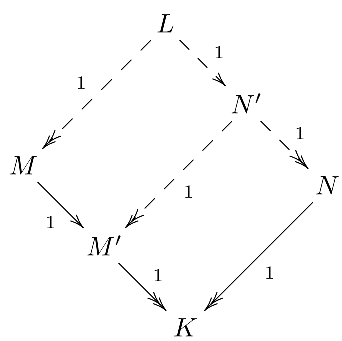
And it is easy to see that ($i.$) and ($ii.$) hold. The second case is if both expand the same occurrence of a constant. But in this case we know by assumption that $M = N$, and we are done.

Now we fix $m = 1$ and do the induction step for n: if $n \neq 1$ there is a term N' such that $N \rightarrow_1 N' \twoheadrightarrow_1 K$ and $length(N' \twoheadrightarrow K) = n - 1$. Then by induction there is a term M' such that $M' \twoheadrightarrow_1 M$ has length $n - 1$, $M' \rightarrow_1 N'$, and these transitions satisfy ($ii.$). But then we can use the base case again to get a term L such that $L \rightarrow_1 M'$, $L \rightarrow_1 N$ and ($ii.$) is again satisfied. Then it is easy to see that $L \twoheadrightarrow_1 M$ and $L \twoheadrightarrow_1 M$ satisfy ($i.$) and ($ii.$).



[3]Meaning that if $s \rightarrow_1 a$ and $t \rightarrow_1 a$, $s, t \neq a$, then $s = t$.

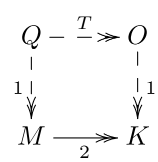
The induction step for m is similar: let M' be such that $M \to_1 M'$ and $M' \twoheadrightarrow_1 K$ has length $m-1$. Then by induction there is a term N' as in the diagram below, satisfying (*i.*) and (*ii.*). By the previous step, we get a term L such that $L \twoheadrightarrow_1 M$ and $L \to_1 N'$. Then $length(L \twoheadrightarrow M) = length(N' \twoheadrightarrow M') = length(N \twoheadrightarrow K)$, and $length(L \twoheadrightarrow N) = length(L \twoheadrightarrow N') + length(N' \twoheadrightarrow N) = 1 + (m-1) = m$. So (*i.*) is satisfied, and since (*ii.*) is a property which is preserved under composition of $T_1(\mathfrak{A})$-reductions, we are done.



□

We shall now prove that we can lift not only one $T_2(\mathfrak{A})$-transition to a T-transition, but a whole series of $T_2(\mathfrak{A})$ transitions to a series of T-transitions.

Theorem 56. *If T and $\mathfrak{A}$ satisfy the HNF condition, then we have the diagram*

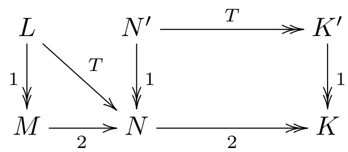


where $Q \twoheadrightarrow_1 M$ is uniform.

Proof. We will prove the theorem by induction on $length(M \twoheadrightarrow K) =: n$.

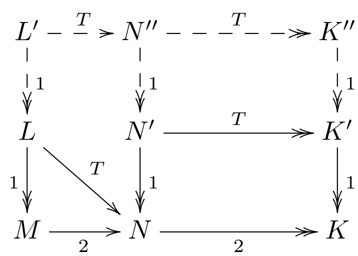
Base Case We know that $M \to_2 K$ has a lifting, call it $L \to_T K$. The fact that $L \twoheadrightarrow_1 M$ is uniform is guaranteed by the HNF condition: since we are dealing with a lift, any two nontrivial expansions $\mathbf{a}_L$ and $\bar{\mathbf{a}}_L$ of two occurrences of a constant $a \in A$ are proper subterms of the T-left-hand side concerned (because these occurrences have to be essential in order to be expanded in the lift), and so by the HNF condition $\mathbf{a}_L = \bar{\mathbf{a}}_L$. So take $Q = L$ and $O = K$.

Induction Step Let N be the term such that $M \to_2 N$ and $N \twoheadrightarrow_2 K$, and the latter transition has length n. By induction, we have the following diagram:



where $N' \twoheadrightarrow_1 N$ is uniform and $L \rightarrow_T N$ is a lifting of $M \rightarrow_2 N$.

Now by Lemma 53, we have terms L', N'' as in the diagram below, then a term K'' by Corollary 50.

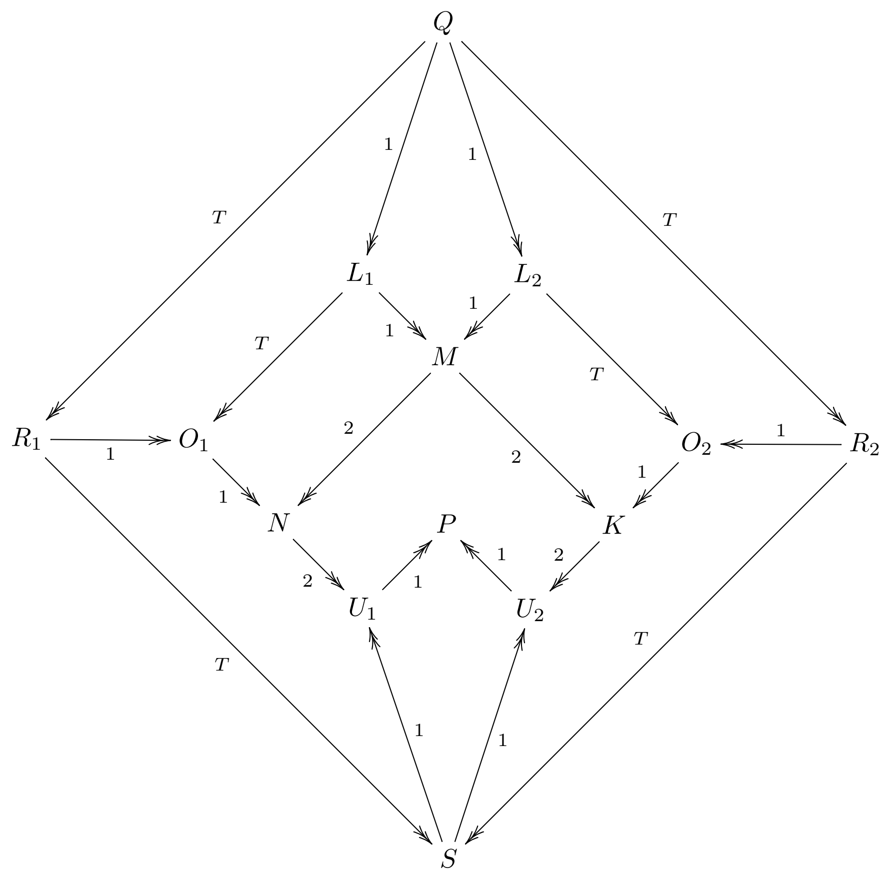


We will now show that that $L' \twoheadrightarrow_1 M$ is uniform. Let a be any constant, $\mathbf{a}, \bar{\mathbf{a}}, \ldots, \bar{\mathbf{a}}^{(\mathbf{k})}$ its occurrences in M. We want to show that $(\bar{\mathbf{a}}^{(\mathbf{i})})_{\mathbf{L'}}$ have a common expansion for all i. Note that by definition $(\bar{\mathbf{a}}^{(\mathbf{i})})_{\mathbf{L'}} = ((\bar{\mathbf{a}}^{(\mathbf{i})})_{\mathbf{L}})_{\mathbf{L'}}$. Suppose that no occurrences of a are expanded in L. Then we are done by uniformity of $L' \twoheadrightarrow L$. So assume at least one occurrence is expanded. Since $L \twoheadrightarrow_1 M$ is a lift, this means that that occurrence of a was essential: its expansion overlaps with the left-hand side of the T-rule concerned in $L \rightarrow_T N$. So for any $\bar{\mathbf{a}}^{(\mathbf{i})}$, $(\bar{\mathbf{a}}^{(\mathbf{i})})_{\mathbf{L}}$ is an occurrence of either a or q_a. Furthermore, $L' \twoheadrightarrow L$ is totally uniform, so we also know that all occurrences of a in L have the same expansion in L' (call it $a_{L'}$), and all occurrences of q_a likewise (call it $(q_a)_{L'}$). Now all we need to show is that $a_{L'}$ and $(q_a)_{L'}$ have a common $T_1(\mathfrak{A})$-expansion. But notice that $a_{L'} \twoheadrightarrow_1 a$, and $(q_a)_{L'} \twoheadrightarrow_1 q_a \twoheadrightarrow_1 a$, and by construction all constants nontrivially expanded in $L' \twoheadrightarrow_1 L$ are essential for some lift! So we can apply Lemma 55 and deduce that $a_{L'}$ and $(q_a)_L$ have a common $T_1(\mathfrak{A})$-expansion.

So $L' \twoheadrightarrow M$ is uniform and we have proven that any chain of $T_2(\mathfrak{A})$-transitions can be lifted to a chain of T-transitions. □

Theorem 57. *$T_2(\mathfrak{A})$ is pseudoconfluent, i.e.* $\twoheadleftarrow_2 \cdot \twoheadrightarrow_2 \subseteq \twoheadrightarrow_2 \twoheadrightarrow_1 \cdot \twoheadleftarrow_1 \twoheadleftarrow_2$.

Proof. Assume that there are terms M, N and K such that $M \twoheadrightarrow_2 N$ and $M \twoheadrightarrow_2 K$. We want to find a term P such that $N \twoheadrightarrow_2 \twoheadrightarrow_1 P$ and $K \twoheadrightarrow_2 \twoheadrightarrow_1 P$. We first give the diagram which solves this problem; we will then describe how the various terms mentioned in it are obtained.



By Theorem 56, there are terms L_1 and O_1, L_2 and O_2 such that $L_1 \twoheadrightarrow_T O_1$ is a lifting of $M \twoheadrightarrow_2 N$, and similarly $L_2 \twoheadrightarrow_T O_2$ is a lifting of $M \twoheadrightarrow_2 K$. By construction, we know that any expansions made from M to L_1 or L_2 are of constants which have an occurrence that is essential in a lift. Therefore, by the HNF condition there is only one choice for the one-step expansion of any such constant. So we can deduce using Lemma 55 that L_1 and L_2 have a common $T_1(\mathfrak{A})$-expansion Q.

Now by Corollary 50, we have terms R_1 and R_2 as in the diagram. But this means that $Q \twoheadrightarrow_T R_1$ and $Q \twoheadrightarrow_T R_2$, and since T is confluent, R_1 and R_2 have a common T-reduct, which we will call S.

But now we have that $R_1 \twoheadrightarrow_1 N$ and $R_1 \twoheadrightarrow_T S$. Since $T_1(\mathfrak{A})$ and $T_2(\mathfrak{A})$ commute, and since T is included in $T_2(\mathfrak{A})$, we have a term U_1 such that $S \twoheadrightarrow_1 U_1$ and $N \twoheadrightarrow_2 U_1$. On the other side we can use the same reasoning to get U_2 as in the diagram. Lastly, since $T_1(\mathfrak{A})$ is confluent, we get our term P such that $U_1 \twoheadrightarrow_1 P$ and $U_2 \twoheadrightarrow_1 P$, and pseudoconfluence of $T_2(\mathfrak{A})$ is proven!

□

2.3.5 Abstract Confluence

We now need to show that the confluence of $T_1(\mathfrak{A})$, the commutativity of $T_1(\mathfrak{A})$ and $T_2(\mathfrak{A})$, and the pseudoconfluence of $T_2(\mathfrak{A})$ together yield the confluence of the joint system $T_1(\mathfrak{A}) \cup T_2(\mathfrak{A})$. This proof does not use the properties of these TRSs, so we will state the following theorems for general TRSs T_1 and T_2.

Let T_1 and T_2 be two term rewriting systems over the same set of terms. We denote by $\rightarrow_1$ and $\rightarrow_2$ the respective transitive reflexive closures of the rewriting relations. Note that this differs from the previous sections where the reflexive transitive closures we denoted by a double-headed arrow. $\rightarrow_{1,2}$ denotes the transitive reflexive closure of the rewriting relation of $T_1 \cup T_2$. Our aim here is to show the following:

Theorem 58. *If we have the properties:*

1. *T_1 is confluent, i.e.* $\leftarrow_1 \cdot \rightarrow_1 \subseteq \rightarrow_1 \cdot \leftarrow_1$,
2. *T_1 and T_2 commute, i.e.* $\leftarrow_1 \cdot \rightarrow_2 \subseteq \rightarrow_2 \cdot \leftarrow_1$,
3. *T_2 is pseudoconfluent, i.e.* $\leftarrow_2 \cdot \rightarrow_2 \subseteq \rightarrow_2 \rightarrow_1 \cdot \leftarrow_1 \leftarrow_2$,

then $T_1 \cup T_2$ is confluent, i.e. $\leftarrow_{1,2} \cdot \rightarrow_{1,2} \subseteq \rightarrow_{1,2} \cdot \leftarrow_{1,2}$.

To prove this theorem, we will need some intermediary results. But first, a bit of terminology should be introduced.

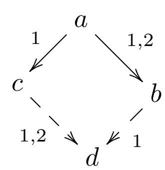
Let Σ be the set of all finite sequences over $\{1, 2\}$, and define over Σ the smallest equivalence relation $\equiv$ such that for all $a_1, \ldots, a_m \in \{1, 2\}$,

$$a_1 \ldots a_n 11 a_{n+1} \ldots a_m \equiv a_1 \ldots a_n 1 a_{n+1} \ldots a_m \text{ and}$$
$$a_1 \ldots a_n 22 a_{n+1} \ldots a_m \equiv a_1 \ldots a_n 2 a_{n+1} \ldots a_m$$

Now, let $\sigma \in \Sigma$. Define $\sigma 1$ to be σ with a 1 stuck on the end, and $\sigma 2$ similarly. Note that if σ ends with $i \in \{1, 2\}$ then $\sigma i \equiv \sigma$.

The purpose of this definition is to be able to speak about the patterns of $\rightarrow_{1,2}$-transitions without reference to the length of stretches of transitions with the same label. A 1 in the pattern σ, for example, symbolizes a stretch of T_1-transitions, i.e. a transition of the type $\rightarrow_1$. The fact that one can hold $a_1 \ldots a_n 11 a_{n+1} \ldots a_m$ and $a_1 \ldots a_n 1 a_{n+1} \ldots a_m$ to be equivalent is a reflection of the fact that $\rightarrow_1$ is transitive. One can translate any transition $a \rightarrow_{1,2} b$ into a sequence in Σ according to the pattern governing it. In fact, we can translate it into a sequence with no repeated 1's or 2's, by taking maximal stretches of T_1- and T_2-transitions at a time. We call this *the* pattern of $a \rightarrow_{1,2} b$ and denote the length of the pattern by $|a \rightarrow_{1,2} b|$. It is easy to see that the pattern of a transition $\sigma \in \Sigma$, is the unique shortest element of its equivalence class. For any $\sigma \in \Sigma$, $|\sigma|$ is defined to be the length of the pattern of σ.

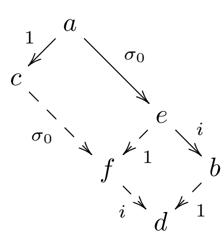
Lemma 59. *Assume that conditions 1, 2 and 3 of the theorem hold. Then for any terms a, b, c such that $a \rightarrow_{1,2} b$ and $a \rightarrow_1 c$ there is a term d such that $c \rightarrow_{1,2} d$ and $b \rightarrow_1 d$. So we have the diagram:*



Moreover, if the transition $a \rightarrow_{1,2} b$ has pattern σ, so does $c \rightarrow_{1,2} d$.

Proof. We prove the lemma by induction on $|\sigma|$. For the base case, say that $|\sigma| = 1$ (if it is zero, the statement is trivial). Then either $\sigma \equiv 1$ or $\sigma \equiv 2$. In fact, $\sigma = 1$ or $\sigma = 2$, since σ is the pattern of a transition, so by definition the shortest element in its equivalence class. In the first case we use confluence (1) and in the second we use commutativity (2); the statement is satisfied.

For the induction step, assume that $|\sigma| = n$ and that the statement holds for patterns of length $n - 1$. Assume that σ ends with symbol i, where $i \in \{1, 2\}$. Define σ_0 such that $\sigma_0 i = \sigma$ and call e the ending point of σ_0.



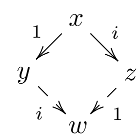
Then by induction, we have a term f as depicted, where $c \rightarrow_{1,2} f$ has pattern σ_0. Depending on whether i is 1 or 2, we have a term d as in the above diagram by either confluence of T_1 or commutativity of T_1 and T_2 respectively. Now $c \rightarrow_{1,2} d$ has pattern $\sigma_0 i = \sigma$ and we have proved the lemma. □

We will now prove our theorem. In fact, we will prove a slightly stronger version, namely

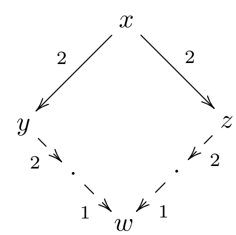
Theorem 60. *Assume that the conditions 1, 2 and 3 of the theorem hold. Let x, y and z be terms such that $y \leftarrow_\sigma x \rightarrow_\tau z$, where we use this notation to signify that σ is the pattern of $x \rightarrow_{1,2} y$ and τ is the pattern of $x \rightarrow_{1,2} z$. Then there is a term w such that $y \rightarrow_{\tau'} w \leftarrow_{\sigma'} z$ and σ' is either σ or $\sigma 1$, and τ' is either τ or $\tau 1$.*

Proof. We prove the lemma by induction on $|\sigma| + |\tau|$. For the base case, let $|\sigma| = |\tau| = 1$ (the case where one of the patterns is empty is trivial). Assume

$\sigma = 1$. Say $\tau = i$ where $i \in \{1, 2\}$. Then we have



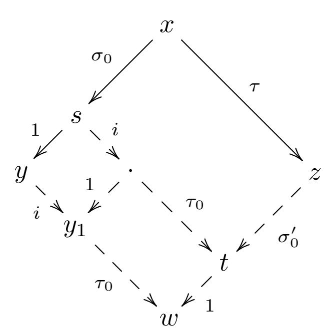
by Lemma 59, in which case $\tau = i = \tau'$ no matter what, and $\sigma = 1 = \sigma'$.
On the other hand, if $\sigma = 2$, we get



by pseudoconfluence, in which case $\tau' = \tau 1$ and $\sigma' = \sigma 1$.

Now for the induction step. We may assume without loss of generality that $|\sigma| > 1$. We assume the induction hypothesis for $n < |\sigma| + |\tau|$ and show for $|\sigma| + |\tau|$. We take a case-by-case approach.

Case 1: σ ends in 1.
We have the following diagram, to be explained in the text below it:



Define σ_0 such that $\sigma_0 1 = \sigma$. Note that by definition of the pattern of a transition we know that 1s and 2s alternate, hence σ_0 ends in 2. We have $|\sigma_0| < |\sigma|$ so that $|\sigma_0| + |\tau| < |\sigma| + |\tau|$. Therefore we can use the induction hypothesis to get a term t and transitions σ_0' and $i\tau_0$, where $\sigma_0' = (\sigma_0$ or $\sigma_0 1)$ and $i\tau_0 = (\tau$ or $\tau 1)$. Note that it could very well be the case that τ_0 is empty, in which case we would have $y_1 = w$ in the diagram above.

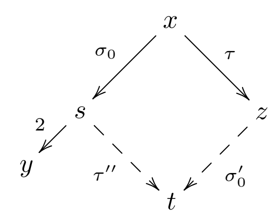
Now by confluence or commutativity (depending on whether i is 1 or 2), we have a term y_1 as in our diagram. We then use Lemma 59 to get the term w:

this lemma guarantees that the pattern from y_1 to w is exactly τ_0.
In sum, we have:

$$\sigma' := \sigma'_0 1 = (\sigma_0 1 \text{ or } \sigma_0 11) \equiv \sigma_0 1 = \sigma \text{ and}$$
$$\tau' = i\tau_0 = (\tau \text{ or } \tau 1)$$

Since σ and σ' are both patterns of transitions, they are both the shortest elements of their equivalence class. And since they are equivalent, they are equal. So the induction hypotesis is satisfied.

Case 2: σ ends in 2.
We then have the diagram

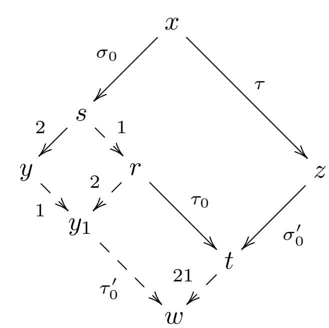


Again, we know that σ_0 ends with a 1. We get by induction the term t and patterns σ'_0 and τ'' such that $\tau'' = (\tau \text{ or } \tau 1)$ and $\sigma'_0 = (\sigma_0 \text{ or } \sigma_0 1)$. But now we again have to split into two cases.

Case 2.1: τ'' starts with a 1.
Then we have a term r such that $s \rightarrow_1 r$ is the first step of the pattern τ'' (so it is a maximal stretch of T_1-transitions). But then by commutativity we have a term y_1 completing the diamond formed by s, r and y (see diagram below). Call the pattern from r to t τ_0, so that $1\tau_0 = \tau''$, and by assumption τ_0 starts with a 2 (if it is nonempty - again, it could be the case that it is empty, but this does not change the proof).

It is easy to see that $|\tau_0| \leq |\tau|$. Since $|\sigma| > 1$, we have that $|\tau_0|+1 < |\tau|+|\sigma|$. So we can use the induction hypothesis on r, t and y_1 to get the term w as in the diagram below. Note that the pattern from t to w may be either 2 or 21 -

in the diagram we depict only the worst case scenario 21.

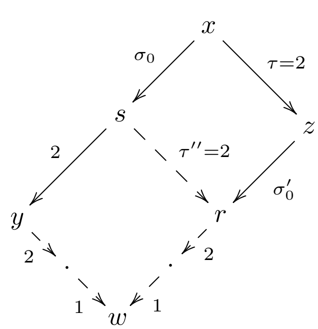


Also note that if τ_0 is empty then so is τ_0', so we can assume the worst and say that by the induction hypothesis we have $\tau_0' = (\tau_0 \text{ or } \tau_0 1)$ where the second possibility is simply nonexistent if τ_0 is empty. Now for the summing up:

$$\begin{aligned}
\tau' &:= 1\tau_0' = (1\tau_0 \text{ or } 1\tau_0 1) = (\tau'' \text{ or } \tau'' 1) \\
&= (\tau \text{ or } \tau 1 \text{ or } \tau 11) \equiv (\tau \text{ or } \tau 1). \\
\sigma' &:= (\sigma_0' 2 \text{ or } \sigma_0' 21) \\
&= (\sigma_0 2 \text{ or } \sigma_0 12 \text{ or } \sigma_0 21 \text{ or } \sigma_0 121) \\
&\equiv (\sigma_0 2 \text{ or } \sigma_0 21) \text{ because } \sigma_0 \text{ ends with } 1 \\
&= (\sigma \text{ or } \sigma 1).
\end{aligned}$$

Again, equivalence simplifies to equality since we are talking about patterns of transitions.

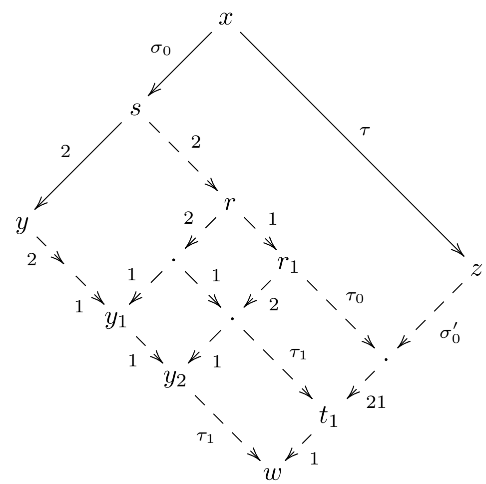
Case 2.2 τ'' starts with a 2.
If $\tau'' = 2$, then $\tau = 2$ and we get the diagram



So $\tau' = 21 = \tau 1$ and

$$\begin{aligned}
\sigma' &:= \sigma_0' 21 = \sigma_0 21 \text{ or } \sigma_0 121 \\
&\equiv \sigma_0 21 \text{ because } \sigma_0 \text{ ends with } 1 \\
&= \sigma 1
\end{aligned}$$

So we can assume now that τ'' starts with a 21. Then there are terms r and r_1 such that $s \rightarrow_2 r \rightarrow_1 r_1$. Using only our assumptions 1, 2 and 3 we can complete our diamonds to get terms y_1 and y_2 as depicted in the diagram below.



Define τ_0 such that $\tau'' = 21\tau_0$. Again, τ_0 is either empty or starts with a 2. In any case, clearly $|\tau_0| < |\tau|$, so we can use the induction hypothesis to get a term t_1 and a pattern τ_1 as in the diagram. Note that if τ_0 is empty then we can take τ_1 to be empty as well, so as in the last case we generalize to the worst possible scenario and say that $\tau_1 = (\tau_0 \text{ or } \tau_0 1)$. Lastly, we use Lemma 59 to get the term w, noting that the pattern from y_2 to w is exactly τ_1. So to sum up we have:

$$\begin{aligned}
\tau' &:= 211\tau_1 \equiv 21\tau_1 = (21\tau_0 \text{ or } 21\tau_0 1) = (\tau'' \text{ or } \tau'' 1) \\
&= (\tau \text{ or } \tau 1 \text{ or } \tau 11) \\
&\equiv (\tau \text{ or } \tau 1) \text{ and} \\
\sigma' &:= (\sigma'_0 21 \text{ or } \sigma'_0 211) \\
&\equiv \sigma'_0 21 = (\sigma_0 21 \text{ or } \sigma_0 121) \\
&\equiv \sigma_0 21 \text{ because } \sigma_0 \text{ ends in } 1 \\
&= \sigma 1
\end{aligned}$$

Once again equivalence reduces to equality. This concludes the proof of our theorem. □

Corollary 61. *Under conditions 1,2 and 3, $T_1 \cup T_2$ is confluent.*

2.3.6 Completion

Now that we know that the system $T_1(\mathfrak{A}) \cup T_2(\mathfrak{A})$ is confluent, we can complete our partial algebra $\mathfrak{A}$. We proceed exactly as in BKV, whose proof we will recapitulate here.

Let us denote by $\sim$ the convertibility relation generated by $T_1(\mathfrak{A}) \cup T_2(\mathfrak{A})$, i.e. the transitive symmetric reflexive closure of the union $\twoheadrightarrow_{1,2}$ of the two rewrite relations. This is an equivalence relation, whose equivalence classes we will denote by $[t]$. We now define an algebra $\Gamma(\mathfrak{A})$, whose underlying set is $Ter(F \cup A, \emptyset)/\sim$, i.e. the ground terms of $Ter(F \cup A, V)$ modulo $\sim$. Interpretations are again defined as

$$f^{\Gamma(\mathfrak{A})}([t_1], \ldots, [t_n]) = [f(t_1, \ldots, t_n)]$$

for any n-ary function symbol in $F \cup A$ and terms $t_i \in Ter(F \cup A, \emptyset)$. We will now go on to prove that $\Gamma(\mathfrak{A})$ is a completion of $\mathfrak{A}$ modeling T.

Theorem 62. *i.* $\Gamma(\mathfrak{A})$ *models* T

ii. The canonical map $\gamma : A \rightarrow Ter(F \cup A, \emptyset)$ *sending* $a \in A$ *to* $[a]$ *is an embedding of* $\mathfrak{A}$ *into* $\Gamma(\mathfrak{A})$*.*

Proof. **(i.)** Let $l(x_1, \ldots, x_n) \rightarrow r(x_1, \ldots, x_n)$ be a rule in T, and let $[t_1], \ldots, [t_n]$ be any elements of $Ter(F \cup A, \emptyset)/\sim$. Then $l(t_1, \ldots, t_n) \rightarrow_T r(t_1, \ldots, t_n)$. Therefore $[l(t_1, \ldots, t_n)] = [r(t_1, \ldots, t_n)]$, and we get

$$l^{\Gamma(\mathfrak{A})}([t_1], \ldots, [t_n]) = [l(t_1, \ldots, t_n)] = [r(t_1, \ldots, t_n)] = r^{\Gamma(\mathfrak{A})}([t_1], \ldots, [t_n])$$

(ii.) First we show that γ is injective. Let a and b be two constants in A such that $[a] = [b]$. Then there are $t_1, \ldots, t_k \in Ter(F \cup A, V)$ such that $a = t_1$, $t_k = b$, and for any $1 \leq i \leq k-1$, either $t_i \twoheadrightarrow_{1,2} t_{i+1}$ or $t_{i+1} \twoheadrightarrow_{1,2} t_i$. By a simple induction on k, it is easy to prove using the confluence of $\twoheadrightarrow_{1,2}$ that a and b have a common $\twoheadrightarrow_{1,2}$-reduct (for a proof, see [BKdV03] Proposition 1.1.10 page 11). However, a and b are already normal forms; a $T_1(\mathfrak{A})$-rule cannot further reduce a constant, and by definition a constant cannot be the left-hand side of a $T_2(\mathfrak{A})$-rule. Hence $a = b$ and we have proved injectivity.

Lastly, we show that γ is a weak homomorphism. Let $f \in F \cup A$ be n-ary, and let $a_1, \ldots a_n \in A$ be such that $f^{\mathfrak{A}}(a_1, \ldots, a_n)\downarrow$. Then there is a $T_1(\mathfrak{A})$-rule $f(a_1, \ldots, a_n) \rightarrow f^{\mathfrak{A}}(a_1, \ldots, a_n)$. As a result,

$$\begin{aligned} f^{\Gamma(\mathfrak{A})}(\gamma(\vec{a})) &= f^{\Gamma(\mathfrak{A})}([a_1], \ldots, [a_n]) \\ &= [f(a_1, \ldots, a_n)] \\ &= [f^{\mathfrak{A}}(a_1, \ldots, a_n)] \\ &= \gamma(f^{\mathfrak{A}}(\vec{a})) \end{aligned}$$

So we have constructed a completion of $\mathfrak{A}$! □

Discussion and Conclusion

Let us verify that the result concerning partial combinatory algebras [BKdV96] does indeed fit our result. In the case of partial combinatory algebras, our original TRS T would be the system consisting of the two rules

$$k \cdot x \cdot y \to x$$
$$s \cdot x \cdot y \cdot z \to (x \cdot z) \cdot (y \cdot z)$$

If $\mathfrak{A}$ is a partial combinatory algebra, then it definitely models the term rewriting system, according to our definition of modelling. The TRS T is certainly left-linear, and it is non-ambiguous. Furthermore, the fact that $\mathfrak{A}$ and T satisfy HNF is guaranteed by the conditions on unique head-normal forms and Barendregt's Axiom: any proper subterm of the left-hand side of a T-rule must be of one of the forms, $k \cdot a$, k, $s \cdot a \cdot b$, $s \cdot a$ or s (recall that the terms associate to the left by default). But in BKV the first condition on unique head-normal forms guarantees that no such terms of two different forms ever evaluate to the same element of A, and Barendregt's Axiom prevents two different terms of the same form from ever evaluating to the same element: Barendregt's Axiom implies that if $(sa)^{\mathfrak{A}} = (sb)^{\mathfrak{A}}$ then $a = b$, and it is already true that if $(ka)^{\mathfrak{A}} = (kb)^{\mathfrak{A}}$ then $a = b$. So we have indeed constructed a generalization of BKV.

It becomes apparent that the result in BKV is independent from the particular properties of s or k; in particular one could have chosen another set of combinators. A case in point would be the λ_I-calculus, which contrary to the λ_K-calculus, which is the one used in BKV, does not allow for lambda-abstraction over variables not occurring freely in the term that is abstracted over. In particular, $\lambda xy.x$ is not a valid term. So the corresponding CL_I calculus lacks the K combinator. It instead has three more combinators I, B, and C, where $Ix = x$, $Bxyz = x(yz)$ and $Cxyz = xzy$. So any partial algebra modelling the TRS T

$$Ix \longrightarrow x$$
$$Bxyz \longrightarrow x(yz)$$
$$Cxyz \longrightarrow xzy$$
$$Sxyz \longrightarrow xz(yz)$$

and satisfying the HNF condition will be completable.

This concludes our discussion of our result and its applications.

To sum up, in this thesis we started by giving some general results concerning partial algebras and their completions, with some attention given to completions preserving congruence relations. We then presented a generalization of a previous result by I. Bethke, J.W. Klop and R. de Vrijer concerning the completion of partial combinatory algebras.

Together with being strongly based on this last paper, our result ties in nicely with the previous chapter, as the convertibility relation generated by the rewrite relation of our term rewriting systems is a congruence relation (since rewriting is not context-sensitive), so the whole construction can be seen as a quotient of the free algebra over $\mathfrak{A}$. So we have in fact made the construction promised in our introduction to the second chapter.

In the future, it may be interesting to investigate what the consequences would be of taking a different notion of modelling of a term rewriting system by a partial algebra. Here we have postulated that if $l \rightarrow r$ is a rule,

$$l^{\mathfrak{A}}\downarrow \Rightarrow r^{\mathfrak{A}} = l^{\mathfrak{A}}$$

but we could have chosen

$$r^{\mathfrak{A}}\downarrow \Rightarrow l^{\mathfrak{A}} = r^{\mathfrak{A}}$$

or

$$l^{\mathfrak{A}}\downarrow \wedge\, r^{\mathfrak{A}}\downarrow \Rightarrow r^{\mathfrak{A}} = l^{\mathfrak{A}}$$

and so forth. We have used this definition in few but crucial places (see the commutativity of $T_1(\mathfrak{A})$ and $T_2(\mathfrak{A})$), and it may be interesting to see the significance of such a modification.

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