THE CYLINDRIC ALGEBRAS OF 4-VALUED LOGIC

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written by

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Abstract

In this thesis the syntax and semantics of four-valued first-order predicate logic are introduced. When we define the semantics, we use *4-cylindric set algebras*. Then we define *4-cylindric algebras* which are supposed to reflect the algebraic properties of this logic. We give a method for constructing 4-cylindric algebras out of cylindric algebras and prove that in fact every 4-cylindric algebra is isomorphic to a 4-cylindric algebra that is constructed in this way. It will turn out that every locally finite 4-cylindric algebra is a subdirect product of a family of 4-cylindric set algebras. This result will be used in order to prove a completeness theorem with respect to a proof system we introduce. At last, we compare 4-cylindric algebras to 3-cylindric algebras. It turns out that every 4-cylindric algebra contains a 3-cylindric algebra as a subreduct. Moreover, every 3-cylindric algebra is isomorphic to a subreduct of some 4-cylindric algebra.

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Contents

Introduction

The origin of many-valued logic can be traced back to antiquity. Already in those ancient times it was questioned whether a statement must necessarily be true or false, and whether there is no third truth status possible. This is the problem of the law of excluded middle, in Latin *tertium non datur*. In *De Interpratione, Chapter 9*, Aristotle proposed the introduction of a third truth value as a solution to the problem of future contingents, a philosophical problem that I will not explain here[1]. Whether one accepts or rejects the law of excluded middle is a matter of philosophical standpoint.

The first many-valued logic devised as a formal system was created by the Polish logician and philosopher Jan Łukasiewicz in 1920. He invented a three-valued logic and his intention was to use the third truth value for contingent propositions, in order to model the modality 'possible'.

Soon after that, working independently, the American mathematician Emil Post presented his many-valued logical system. His approach to many-valued logic was purely mathematical. He seems to have paid little attention to the logical interpretation of particular logical values. Apparently, philosophical aspects had no relevance to his considerations[2].

Later on, the American logician Kleene came up with some three-valued logics for foundational purposes. Kleene thought of the third truth value as undefined or undetermined, rather than as contingent[3].

[1] For a treatise of this topic, see e.g. the Stanford Encyclopedia of Philosophy, under the lemmas *future contingents* and *fatalism*:
`http://plato.stanford.edu/entries/future-contingents`;
`http://plato.stanford.edu/entries/fatalism`.

[2] see [Bol92].

[3] see [Fit94].

In [Bel77] Belnap extended Kleene's logic to a four-valued version with intended database applications. In a database it is not only possible that we cannot find an answer to a question, it is also possible that we will find inconsistent information on a given question. This leads to a fourth truth value $\top$ which could be thought of as 'overdefined'.

Belnap's logic was generalized by M.L. Ginsberg in [Gin88] who had applications in artificial intelligence in mind. He introduced the notion of a bilattice, an algebraic structure that contains two partial orders simultaneously. The intuition behind these two orderings is that one of them compares the amount of truth of two elements and the other compares the amount of knowledge (information). Nowadays there are several definitions of bilattice in use. Bilattices were further examined by Melvin Fitting. For a general introduction to the theory of bilattices, the reader is referred to [Fit94] and [Ari96].

Feldman considered the three-valued propositional logic used by Kleene in the theory of partial recursive functions. In [Kle52] Kleene gives truth tables for the propositional connectives of a this logic. Feldman extended this to three-valued first-order predicate logic. He also introduces the notion of a 3-cylindric algebra, an analogue of the cylindric algebra, that was invented by Alfred Tarski who intensively studied it with Leon Henkin and Donald Monk. Cylindric algebras are the result of algebraization of first-order predicate logic with equality. Just as Boolean algebras reflect the algebraic properties of propositional logic, cylindric algebras reflect the algebraic properties of first-order predicate logic with equality. Therefore a cylindric algebra is indeed a Boolean algebra equipped with additional cylindrification operations that model quantification, and constants that model equality. They may be regarded as a kind of multi-dimensional set algebra; associated with each cylindric algebra is an ordinal number α, which indicates the dimensionality of the algebra and corresponds to the number of variables in the logic.

In 1936, Marshall Stone showed that every Boolean algebra is isomorphic to a Boolean set algebra (see [Sto39]). Moreover, every locally finite cylindric algebra is isomorphic to a subalgebra of a product of cylindric set algebras (see [HMT2]). In [Fel98] Feldman proves a representation theorem for 3-cylindric algebras. He uses this representation theorem in order to prove completeness of the proof system he gives.

In this thesis the same is done with Belnap's four-valued propositional logic as Feldman did with Kleene's three-valued logic. This means that a four-valued first-order predicate logic is introduced and instead of 3-cylindric algebras, we will define 4-cylindric algebras. In addition, in the last chapter the relationship between 3-cylindric algebras and 4-cylindric algebras will be investigated.

Chapter 1

Four-valued Logic

The intuition behind four-valued logic is as follows: The truth value of a proposition depends on the information we have about it. There can be information in favor of a proposition, that is information indicating that the proposition is true, and there can be information against a proposition, that is information indicating that the proposition is false. For a given proposition, we thus obtain four possible situations, corresponding to four truth values: there is information in favor and no info against it (true, 1), there is information against and no info in favor (false, 0), there is both information in favor and against (inconsistent, $\top$), or there is no information about it at all (undefined, $\perp$). The set of these four truth values is denoted by 4 and we think of it as a partial ordering in which 0 is the least element, 1 is the greatest element and $\perp$ and $\top$ are incomparable, see figure 1.1.

1.1 Four-valued Predicate Logic

Fix a set of relation symbols (a vocabulary) $\mathcal{L} = \{R_i \mid i \in I\}$, where the arity of R_i is $n_i \in \omega$. The syntax of four-valued first-order predicate logic is the syntax of the usual first-order predicate logic, extended with the constant symbols for the two additional truth values and with a symbol for the *defined and consistent part of*-operator. We thus have ω many variables v_0, v_1, $\ldots$; symbols for negation ($\neg$), disjunction ($\vee$), conjunction ($\wedge$), existential quantification ($\exists v_m$), for the

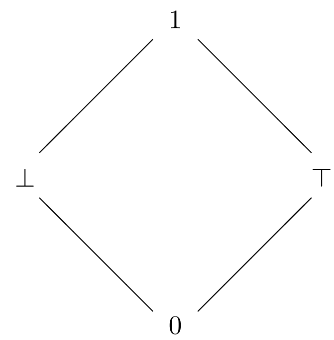

Figure 1.1: The ordered set 4

defined-and-consistent-part-of-operator (Δ), for equality ($\approx$) and n_i-ary relation symbols R_i for $i \in I$ for some index set I. Moreover, we have the logical constants 1 (true), 0 (false), $\top$ (inconsistent) and $\perp$ (undefined). The set of ($\mathcal{L}$-)formulas is defined by induction: Atomic formulas are: $v_i \approx v_j$ for $i, j \in \omega$ and $R_i v_{j_0} \dots v_{j_{n_i - 1}}$ for $i \in I$. If φ and ψ are formulas, then so are

$$\neg\varphi, \quad \varphi \vee \psi, \quad \varphi \wedge \psi, \quad \exists v_i \varphi \quad \text{and} \quad \Delta\varphi.$$

We let Fm be the set of ($\mathcal{L}$-)formulas. In order to define the semantics of four-valued predicate logic, we will introduce the concept of 4-cylindric set algebra of dimension α:

1.2 4-Cylindric Set Algebras of dimension α

Definition 1. Let A be a set and α an ordinal. We denote by ${}^{\alpha}A$ the set of all functions from α to A (also called *assignments*), that is the set of all α-sequences in A. We will define the following operations on the set $\mathcal{P}({}^{\alpha}A) \times \mathcal{P}({}^{\alpha}A)$. For (X, X') and $(Y, Y') \in \mathcal{P}({}^{\alpha}A) \times \mathcal{P}({}^{\alpha}A)$:

- $\neg(X, X') = (X', X)$;
- $(X, X') \vee (Y, Y') = (X \cup Y, X' \cap Y')$;
- $(X, X') \wedge (Y, Y') = (X \cap Y, X' \cup Y')$;

- $E_\kappa(X, X') = (C_\kappa X, Q_\kappa X')$;
- $\delta(X, X') = (X + X', X \leftrightarrow X')$

where C_κ is the usual cylindrification in a cylindric set algebra and Q_κ is its dual, that is

$$C_\kappa(X) = \{s \in {}^\alpha A \mid s[a/\kappa] \in X \text{ for some } a \in A\}$$

and

$$Q_\kappa(X) = \{s \in {}^\alpha A \mid s[a/\kappa] \in X \text{ for all } a \in A\}$$

where $s[a/\kappa](\kappa) = a$ and $s[a/\kappa](\lambda) = s(\lambda)$ for $\lambda \neq \kappa$. Furthermore $X + X'$ is the symmetric difference of X and X', that is $(X - X') \cup (X' - X)$ and $X \leftrightarrow X'$ is its complement. Next, we define the constants

$$1 = ({}^\alpha A, \emptyset); \quad 0 = (\emptyset, {}^\alpha A); \quad \top = ({}^\alpha A, {}^\alpha A); \quad \bot = (\emptyset, \emptyset)$$

and for all $\kappa, \lambda < \alpha$ the constants

$$\mathbf{D}_{\kappa\lambda} = (D_{\kappa\lambda}, \overline{D_{\kappa\lambda}})$$

where $D_{\kappa\lambda} = \{s \in {}^\alpha A \mid s(\kappa) = s(\lambda)\}$. Finally, we let A_i abbreviate $\neg E_i \neg$, hence $A_i(X, X') = (Q_i X, C_i X')$.

A set $C \subseteq \mathcal{P}({}^\alpha A) \times \mathcal{P}({}^\alpha A)$ that contains all these constants and which is closed under the operations defined above, is called a *4-cylindric set algebra of dimension* α (4-CSAα). If C is a 4-CSAα and $(X, X') \in C$, then $\mathcal{T}(X, X') = X$ is called the true part of (X, X') and $\mathcal{F}(X, X') = X'$ is called the false part of (X, X'). The set A is called the *base* of the 4-cylindric set algebra C. If $C = \mathcal{P}({}^\alpha A) \times \mathcal{P}({}^\alpha A)$, we say that C is the full 4-cylindric set algebra over A.

Let A be a set and n a natural number. A four-valued n-ary relation P on A is pair (P^T, P^F) where both P^T and P^F are ordinary (two-valued) n-ary relations on A, that is subsets of ${}^n A$. Now suppose we are given a vocabulary $\mathcal{L} = \{R_i \mid i \in I\}$ where the arity of R_i is n_i. A four-valued ($\mathcal{L}$-)structure is a tuple $\mathfrak{A} = (A, P_i)_{i \in I}$ where P_i is a four-valued n_i-ary relation on A for all $i \in I$. The interpretation $\varphi^{\mathfrak{A}}$ of a formula φ in $\mathfrak{A}$ will be an element of the full cylindric set algebra over A, having a true and a false part. It is defined by induction:

$$\begin{aligned}(R_i v_{j_0} \ldots v_{j_{n_i-1}})^{\mathfrak{A}} = (&\{s \in {}^\omega A \mid (s(j_o), \ldots, s(j_{n_i-1})) \in P_i^T\},\\ &\{s \in {}^\omega A \mid (s(j_o), \ldots, s(j_{n_i-1})) \in P_i^F\});\end{aligned}$$

$$(v_i \approx v_j)^{\mathfrak{A}} = \mathbf{D}_{ij};$$

$$1^{\mathfrak{A}} = 1; \quad 0^{\mathfrak{A}} = 0; \quad \top^{\mathfrak{A}} = \top; \quad \bot^{\mathfrak{A}} = \bot;$$

$$(\neg\varphi)^{\mathfrak{A}} = \neg\varphi^{\mathfrak{A}}; \quad (\varphi \vee \psi)^{\mathfrak{A}} = \varphi^{\mathfrak{A}} \vee \psi^{\mathfrak{A}}; \quad (\varphi \wedge \psi)^{\mathfrak{A}} = \varphi^{\mathfrak{A}} \wedge \psi^{\mathfrak{A}};$$

$$(\exists v_i \varphi)^{\mathfrak{A}} = E_i \varphi^{\mathfrak{A}} \quad \text{and} \quad (\Delta\varphi)^{\mathfrak{A}} = \delta\varphi^{\mathfrak{A}}.$$

Like the symbols $\neg$, $\vee$, and $\wedge$, the symbols for the logical constants are used ambiguously. However, at each occurrence it will be clear from the context which meaning they have.

For φ a formula, $\mathfrak{A}$ a structure and s an assignment in $\mathfrak{A}$, the truth value of φ in $\mathfrak{A}$ under the assignment s is defined as follows:

$$\varphi^{\mathfrak{A}}[s] = \begin{cases} 1 & \text{if } s \in \mathcal{T}(\varphi^{\mathfrak{A}}) - \mathcal{F}(\varphi^{\mathfrak{A}}); \\ 0 & \text{if } s \in \mathcal{F}(\varphi^{\mathfrak{A}}) - \mathcal{T}(\varphi^{\mathfrak{A}}); \\ \top & \text{if } s \in \mathcal{T}(\varphi^{\mathfrak{A}}) \cap \mathcal{F}(\varphi^{\mathfrak{A}}); \\ \bot & \text{if } s \notin \mathcal{T}(\varphi^{\mathfrak{A}}) \cup \mathcal{F}(\varphi^{\mathfrak{A}}). \end{cases}$$

We can explain the definition of $\varphi^{\mathfrak{A}}$: The intuition behind four-valued logic is that the truth value of a formula depends on the kind of information we have about it, where no information at all, and contradictory information are both possible. Now we interpret $s \in \mathcal{T}(\varphi^{\mathfrak{A}})$ as: In $\mathfrak{A}$, under the assignment s, there is information in favor of φ. Likewise, we interpret $s \in \mathcal{F}(\varphi^{\mathfrak{A}})$ as: In $\mathfrak{A}$, under the assignment s, there is information against φ. A structure contains the information we have about the predicates. For a predicate R_i and a tuple $(a_0, \ldots, a_{n_i-1})$ the structure tells whether there is information in favor and whether there is information against $R_i a_0 \ldots a_{n_i-1}$. From these considerations, the definition of $\varphi^{\mathfrak{A}}$ follows naturally. For instance, if there is, in a given structure $\mathfrak{A}$ and under a given assignment s, information in favor of φ or information in favor of ψ, then there is certainly information in favor of the weaker statement $\varphi \vee \psi$, hence $\mathcal{T}((\varphi \vee \psi)^{\mathfrak{A}}) = \mathcal{T}(\varphi^{\mathfrak{A}}) \cup \mathcal{T}(\psi^{\mathfrak{A}})$. On the other hand, mere information against φ is not yet information against the weaker statement $\varphi \vee \psi$. However, if we have information against φ *and* against ψ, then we have information against $\varphi \vee \psi$, hence $\mathcal{F}((\varphi \vee \psi)^{\mathfrak{A}}) = \mathcal{F}(\varphi^{\mathfrak{A}}) \cap \mathcal{F}(\psi^{\mathfrak{A}})$.

A formula φ is *true* in $\mathfrak{A}$ if $\varphi^{\mathfrak{A}} = 1$ and φ is *valid* if φ is true in every structure. If φ is true in $\mathfrak{A}$, we say that $\mathfrak{A}$ is a *model* of φ. If Σ is a set of formulas and $\mathfrak{A}$

a structure, we say $\mathfrak{A}$ is a model of Σ if it is a model of every element of Σ, and φ is a logical consequence of Σ, notation $\Sigma \models \varphi$, if φ is true in every model of Σ.

Definition 2. The set $Fv(\varphi)$ of free variables of a formula φ is defined by induction:

$$\begin{array}{ll} Fv(1) = Fv(0) = Fv(\top) = Fv(\bot) = \emptyset; & Fv(\varphi \vee \psi) = Fv(\varphi) \cup Fv(\psi); \\ Fv(R_i v_{j_0} ... v_{j_{n_i - 1}}) = \{v_{j_0}, ..., v_{j_{n_i-1}}\}; & Fv(\varphi \wedge \psi) = Fv(\varphi) \cup Fv(\psi); \\ Fv(v_i \approx v_j) = \{v_i, v_j\}; & Fv(\Delta\varphi) = Fv(\varphi); \\ Fv(\neg\varphi) = Fv(\varphi); & Fv(\exists v_i \varphi) = Fv(\varphi) - \{v_i\}. \end{array}$$

A formula φ is called a *sentence* if $Fv(\varphi) = \emptyset$.

Instead of interpreting formulas in a structure, if we consider atomic and existentially quantified formulas as propositional variables, we can also assign truth values to formulas by means of a valuation, as in propositional logic. We make use of the following truth tables:

$\vee$	0	$\bot$	$\top$	1
0	0	$\bot$	$\top$	1
$\bot$	$\bot$	$\bot$	1	1
$\top$	$\top$	1	$\top$	1
1	1	1	1	1

$\wedge$	0	$\bot$	$\top$	1
0	0	0	0	0
$\bot$	0	$\bot$	0	$\bot$
$\top$	0	0	$\top$	$\top$
1	0	$\bot$	$\top$	1

x	$\neg x$
0	1
$\bot$	$\bot$
$\top$	$\top$
1	0

x	δx
0	1
$\bot$	0
$\top$	0
1	1

Note that in the truth tables above, $\vee$ coincides with the join in the partial order $\mathbf{4}$ and $\wedge$ coincides with the meet, see Figure 1.1.

Definition 3. A prime formula is a formula that is atomic or of the form $\exists v_i \varphi$. A valuation is a function from the set of prime formulas to $\mathbf{4} = \{1, \top, \bot, 0\}$. If v is a valuation, let $T_v : Fm \to \mathbf{4}$ be defined as follows:

$$T_v(1) = 1; \quad T_v(\top) = \top; \quad T_v(\bot) = \bot; \quad T_v(0) = 0;$$

$$\begin{array}{ll} T_v(\varphi) = v(\varphi) & \text{if } \varphi \text{ is another prime formula;} \\ T_v(\varphi \vee \psi) = T_v(\varphi) \vee T_v(\psi); & T_v(\neg\varphi) = \neg T_v(\varphi); \\ T_v(\varphi \wedge \psi) = T_v(\varphi) \wedge T_v(\psi); & T_v(\Delta\varphi) = \delta T_v(\varphi). \end{array}$$

A formula φ is called a *tautology* if $T_v(\varphi) = 1$ for all valuations v.

Theorem 1. *Every tautology is valid.*

Proof. Let $\mathfrak{A}$ be a structure with base A and fix an assignment s. Define a valuation v as follows: For φ a prime formula set

$$v(\varphi) = \varphi^{\mathfrak{A}}[s].$$

By induction it is now easily seen that $T_v(\varphi) = \varphi^{\mathfrak{A}}[s]$ for any formula φ: The base case follows directly from the definiton. The induction steps are easy but tedious. We will write out the case of the disjunction: Remember that

$$\mathcal{T}((\varphi \vee \psi)^{\mathfrak{A}}) = \mathcal{T}(\varphi^{\mathfrak{A}}) \cup \mathcal{T}(\psi^{\mathfrak{A}}) \quad \text{and} \quad \mathcal{F}((\varphi \vee \psi)^{\mathfrak{A}}) = \mathcal{F}(\varphi^{\mathfrak{A}}) \cap \mathcal{F}(\psi^{\mathfrak{A}}).$$

Now suppose $s \in \mathcal{T}((\varphi \vee \psi)^{\mathfrak{A}}) - \mathcal{F}((\varphi \vee \psi)^{\mathfrak{A}})$. Without loss of generality we assume $s \in \mathcal{T}(\varphi^{\mathfrak{A}})$, hence by the induction hypothesis $T_v(\varphi) \geq \top$. If $s \notin \mathcal{F}(\varphi^{\mathfrak{A}})$, then by induction $T_v(\varphi) = 1$, so $T_v(\varphi \vee \psi) = T_v(\varphi) \vee T_v(\psi) = 1 \vee T_v(\psi) = 1$. If $s \notin \mathcal{F}(\psi^{\mathfrak{A}})$, then by induction $T_v(\psi) \geq \bot$, and hence $T_v(\varphi \vee \psi) = T_v(\varphi) \vee T_v(\psi) \geq \top \vee \bot = 1$. If $s \in \mathcal{F}((\varphi \vee \psi)^{\mathfrak{A}}) - \mathcal{T}((\varphi \vee \psi)^{\mathfrak{A}})$, then $s \in \mathcal{F}(\varphi^{\mathfrak{A}}) - \mathcal{T}(\varphi^{\mathfrak{A}})$ and $s \in \mathcal{F}(\psi^{\mathfrak{A}}) - \mathcal{T}(\psi^{\mathfrak{A}})$, hence by induction $T_v(\varphi) = T_v(\psi) = 0$ hence $T_v(\varphi \vee \psi) = 0$. Suppose $s \in \mathcal{T}((\varphi \vee \psi)^{\mathfrak{A}}) \cap \mathcal{F}((\varphi \vee \psi)^{\mathfrak{A}})$. Without loss of generality, we may assume $s \in \mathcal{T}(\varphi^{\mathfrak{A}})$. Since $s \in \mathcal{F}(\varphi^{\mathfrak{A}})$, by induction $T_v(\varphi) = \top$. Since $s \in \mathcal{F}(\psi^{\mathfrak{A}})$, by induction $T_v(\psi) \leq \top$, hence $T_v(\varphi \vee \psi) = \top$. Finally, suppose s is nor in the true part, nor in the false part of $(\varphi \vee \psi)^{\mathfrak{A}}$. Without loss of generality we may assume $s \notin \mathcal{F}(\varphi^{\mathfrak{A}})$. Since $s \notin \mathcal{T}(\varphi^{\mathfrak{A}})$, $T_v(\varphi) = \bot$ by induction, and since $s \notin \mathcal{T}(\psi^{\mathfrak{A}})$, $T_v(\psi) \leq \bot$ by induction and hence $T_v(\varphi \vee \psi) = \bot$. The cases of the other connectives are left to the reader. $\square$

Chapter 2

4-Cylindric Algebras of dimension α

In this chapter we will introduce 4-cylindric algebras. They are the result of algebraization of four-valued first-order predicate logic with equality, just as cylindric algebras are the result of algebraization of ordinary (two-valued) first-order predicate logic with equality. In the next section, we will present some elementary properties of 4-cylindric algebras, among which a partial ordering, the truth ordering. Furthermore, we prove that the class of all 4-cylindric algebras (of a fixed dimension) is a variety. Then we introduce two new operators from which we will derive a new partial ordering, the knowledge ordering. It will appear that the knowledge ordering is strongly connected with the truth ordering.

2.1 Definition and elementary propositions

Definition 4. Let α be an ordinal. Then an algebra of the form $\mathfrak{A} = (A, \vee, \wedge, \neg, 1, 0, \top, \bot, \delta, c_\kappa, d_{\kappa,\lambda})_{\kappa\lambda<\alpha}$ is a 4-cylindric algebra of dimension α if it satisfies the following axioms:

BA1.	$a \vee b = b \vee a$	C1.	$c_\kappa 0 = 0$
BA2.	$a \wedge b = b \wedge a$	C2.	$a \vee c_\kappa a = c_\kappa a$
BA3.	$(a \vee b) \vee c = a \vee (b \vee c)$	C3.	$c_\kappa(a \wedge c_\kappa b) = c_\kappa a \wedge c_\kappa b$
BA4.	$(a \wedge b) \wedge c = a \wedge (b \wedge c)$	C4.	$c_\kappa c_\lambda a = c_\lambda c_\kappa a$
BA5.	$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$	C5.	$d_{\kappa\kappa} = 1$
BA6.	$a \vee 0 = a$	C6.	$d_{\lambda\mu} = c_\kappa(d_{\lambda\kappa} \wedge d_{\kappa\mu}) (\kappa \neq \lambda, \mu)$
BA7.	$a \wedge 1 = a$	C7.	$c_\kappa(d_{\kappa\lambda} \wedge \delta a) \wedge c_\kappa(d_{\kappa\lambda} \wedge \neg\delta a)$
BA8.	$\delta a \vee \neg\delta a = 1$		$= 0$ (for $\kappa \neq \lambda$)

A1.	$\neg(a \vee b) = \neg a \wedge \neg b$	A6.	$c_\kappa a \vee \bot = c_\kappa(a \vee \bot)$
A2.	$\neg\neg a = a$	A7.	If $\delta a = \delta 1$, then $a = \delta(a \vee \bot)$.
A3.	$\delta d_{\kappa\lambda} = 1$	A8.	$\delta(a \wedge \top) = \neg\delta(a \vee \bot)$
A4.	$\delta c_\kappa(a \vee \bot) = c_\kappa \delta(a \vee \bot)$	A9.	$\delta(a \vee \neg a) = \delta a$
A5.	$\delta(c_\kappa a \wedge \bot) = q_\kappa \delta(a \wedge \bot)$	A10.	$a \wedge \delta a = \delta(a \vee \bot) \wedge \delta(a \vee \top)$

A11. $\delta(a \vee b \vee \bot) = \delta(a \vee \bot) \vee \delta(b \vee \bot)$

A12. $\delta((a \wedge b) \vee \bot) = \delta(a \vee \bot) \wedge \delta(b \vee \bot)$

A13. If $\delta(a \vee \bot) = \delta(b \vee \bot)$ and $\delta(a \wedge \bot) = \delta(b \wedge \bot)$, then $a = b$.

The reader easily checks that 4-CSAαs satisfy these axioms. For a 4-CAα $\mathfrak{A}$, we let $|\mathfrak{A}|$ denote the universe of $\mathfrak{A}$. We now prove a series of elementary propositions about 4-CAαs; Pn means Proposition n.

P 1. $\delta a = \delta\neg a = \delta(a \wedge \neg a)$.

Proof. $\delta\neg a = \delta(\neg a \vee \neg\neg a) = \delta(\neg a \vee a) = \delta a$ and $\delta(a \wedge \neg a) = \delta(\neg(\neg a \vee a)) = \delta(\neg a \vee a) = \delta a$. □

Axiom A1 is one of the two laws called after the British mathematician Augustus De Morgan. We can prove the other one:

P 2. $\neg(a \wedge b) = \neg a \vee \neg b$.

Proof. $\neg(a \wedge b) = \neg(\neg\neg a \wedge \neg\neg b) = \neg(\neg(\neg a \vee \neg b)) = \neg a \vee \neg b$. □

P 3. $\neg 0 = 1$ and $\neg 1 = 0$.

Proof. By BA6 and A1 we have $\neg 0 \wedge \neg a = \neg a$. Now substitute $\neg 1$ for a and use A2 and BA7 to obtain $\neg 0 = 1$. One more application of A2 yields $\neg 1 = 0$. □

Note that by A2, the two De Morgan laws (A1 and P2) and P3, $\vee$ and $\wedge$, and 0 and 1 are duals. Moreover, $\neg$ is self-dual. As a consequence, if we interchange all occurrences of $\vee$ and $\wedge$, and 0 and 1 in a valid equation in which only $\vee$, $\wedge$, 0, 1 and $\neg$ occur, the equation we obtain is still valid. Hence we obtain:

P 4. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

P 5. $\delta a \wedge \neg\delta a = 0$.

Proof. Negate both terms in BA8. □

The following proposition is known as the modular law. It follows directly from P4.

P 6. If $a \leq c$, then $a \vee (b \wedge c) = (a \vee b) \wedge c$.

P 7. $\delta 0 = \delta 1$ and hence $\delta \bot = 0$.

Proof. The first equality follows from P1 and P3. The second then follows from the first, BA6 and A7. □

P 8. $\delta a \wedge \delta a = \delta a$ and $\delta a \vee \delta a = \delta a$.

Proof. $\delta a = 1 \wedge \delta a = (\delta a \vee \neg\delta a) \wedge \delta a = (\delta a \wedge \delta a) \vee (\neg\delta a \wedge \delta a) = (\delta a \wedge \delta a) \vee 0 = \delta a \wedge \delta a$. The argument for $\delta a \vee \delta a = \delta a$ is similar. □

P 9. $0 \wedge 0 = 0$ and $1 \vee 1 = 1$.

Proof. By the second equality of P7 and P8 we obtain $0 \wedge 0 = 0$. By duality, $1 \vee 1 = 1$. □

P 10. $a \wedge a = a$ and $a \vee a = a$.

Proof. Since $1 \vee 1 = 1$ we have $a = a \wedge 1 = a \wedge (1 \vee 1) = (a \wedge 1) \vee (a \wedge 1) = a \vee a$. By duality we obtain $a \wedge a = a$. □

P 11. $\neg\bot = \bot$.

Proof. Follows directly from A9, P1, P10 and A13. □

P 12. $\delta a \wedge 0 = 0$ and $1 \vee \delta a = 1$.

Proof. By P8 and P5 we have $\delta a \wedge 0 = \delta a \wedge (\delta a \wedge \neg\delta a) = \delta a \wedge \neg\delta a = 0$. The second claim follows from the dual argument. $\square$

P 13. $a \wedge 0 = 0$ and $a \vee 1 = 1$.

Proof. By A12, BA6, P7 and P12 we have $\delta((a \wedge 0) \vee \bot) = \delta(a \vee \bot) \wedge \delta(0 \vee \bot) = \delta(a \vee \bot) \wedge \delta\bot = 0 = \delta(0 \vee \bot)$. Furthermore, since $\delta(1 \vee \bot) = 1$ by A7, by P1, P11, A11 and P12 we have

$$\delta(a \wedge 0 \wedge \bot) = \delta(\neg a \vee 1 \vee \bot) = \delta(\neg a \vee \bot) \vee \delta(1 \vee \bot) = \delta(\neg a \vee \bot) \vee 1 =$$

$$1 = \delta(1 \vee \bot) = \delta(0 \wedge \bot)$$

hence by A13 we obtain $a \wedge 0 = 0$. From duality it follows that $a \vee 1 = 1$. $\square$

P 14. $\delta 0 = \delta 1 = 1$.

Proof. Subsitute 1 for a in A7 and use P13 in order to get $\delta 1 = 1$. We already established $\delta 0 = \delta 1$ in P7. $\square$

The following equations are known as the absorption laws:

P 15. $a \wedge (a \vee b) = a = a \vee (a \wedge b)$.

Proof. $a \wedge (a \vee b) = (a \vee 0) \wedge (a \vee b) = a \vee (0 \wedge b) = a = a \wedge (1 \vee b) = (a \wedge 1) \vee (a \wedge b) = a \vee (a \wedge b)$. $\square$

P 16. $\top \wedge \bot = 0$.

Proof. By absorption, $\delta((\top \wedge \bot) \vee \bot) = \delta\bot = 0 = \delta(0 \vee \bot)$. Moreover, by A8, $\delta(\top \wedge \bot \wedge \bot) = \delta(\top \wedge \bot) = \neg\delta\bot = 1 = \delta(0 \wedge \bot)$. Hence $\top \wedge \bot = 0$ by A13. $\square$

P 17. $\delta\top = 0$.

Proof. By P16 and P11, $\neg\top \vee \bot = 1$, so by P1 and A8, $\delta\top = \delta(\neg\top \wedge \top) = \neg\delta(\neg\top \vee \bot) = \neg\delta 1 = 0$. $\square$

P 18. $\neg\top = \top$ and hence $\top \vee \bot = 1$.

Proof. By A8, P1 and P17 we have $\delta(\top \vee \bot) = \neg\delta\top = 1 = \delta 1 = \delta(\neg\top \vee \bot)$ and $\delta(\neg\top \wedge \bot) = \delta(\top \vee \bot) = \neg\delta\top = 1 = \delta(\top \wedge \bot)$ hence by A13, $\neg\top = \top$. It now follows from P11 and P16 that $\top \vee \bot = 1$. $\square$

P 19. $a \vee b = b$ iff $a \wedge b = a$.

Proof. Follows directly from the absorption laws (P15). $\square$

Definition 5. $a \leq b$ iff $a \vee b = b$ (or equivalently $a \wedge b = a$).

P 20. $\leq$ is a partial order in which 0 is the least element and 1 is the greatest element.

Proof. Straightforward checking. $\square$

P 21. $a \leq b$ implies $\neg b \leq \neg a$.

Proof. Use De Morgan and the equivalence in the definition of $\leq$. $\square$

P 22. $a \leq a \vee b$.

Proof. Follows directly from the first absorption law. $\square$

P 23. If $a \leq b$, then $a \vee c \leq b \vee c$ and $a \wedge c \leq b \wedge c$.

Proof. Trivial. $\square$

P 24. $\delta\delta a = 1$.

Proof. $\delta\delta a = \delta(\delta a \vee \neg\delta a) = \delta 1 = 1$. $\square$

P 25. If $0 \neq 1$, then $|\{1, 0, \top, \bot\}| = 4$.

Proof. Assume $0 \neq 1$ and suppose $\top = \bot$. Then by P10, P16 and P18 we have $1 = \top \vee \bot = \top \wedge \bot = 0$, contradiction. Now let $x \in \{0, 1\}$ and $y \in \{\top, \bot\}$ and suppose $x = y$. Then by P14, P7 and P17 we have $1 = \delta x = \delta y = 0$, contradiction. $\square$

P 26. $c_\kappa c_\kappa a = c_\kappa a$.

Proof. Substitute $c_\kappa b$ for a in C3. $\square$

P 27. If $\kappa \neq \lambda, \mu$, then $c_\kappa d_{\lambda\mu} = d_{\lambda\mu}$.

Proof. Apply c_κ to both sides in C6 and use P26. $\square$

P 28. If $\kappa \neq \lambda, \mu$, then $d_{\lambda\mu} \wedge c_\kappa a = c_\kappa(d_{\lambda\mu} \wedge a)$.

Proof. By C3 and P27, we have

$$d_{\lambda\mu} \wedge c_\kappa a = c_\kappa d_{\lambda\mu} \wedge c_\kappa a = c_\kappa(c_\kappa d_{\lambda\mu} \wedge a) = c_\kappa(d_{\lambda\mu} \wedge a).$$

□

As is clear from the axioms for 4-CAαs, the class of 4-CAαs is a quasivariety. We will now show that it is in fact a variety.

Definition 6. A structure $\mathfrak{B} = (B, \vee, \wedge, \neg, 0, 1, \bot, \top, \delta)$ is called a 4-Boolean algebra if it satisfies the axioms BA1–BA8, A1, A2 and A7–A13.

We will show that the class of 4-Boolean algebras is a variety, from which it follows directly that the class of 4-CAαs is a variety.

Theorem 2. *The class of 4-Boolean algebras is a variety.*

Proof. Let T be the set of equations consisting of the axioms BA1–BA8, A1, A2, A8–A12 and the axioms

(I1) $\delta 1 = 1$;

(I2) $\neg\bot = \bot$ and $\neg\top = \top$;

(I3) $\delta\top = 0$ and $\delta\bot = 0$;

(I4) $\top \vee \bot = 1$ and $\top \wedge \bot = 0$;

(I5) $a \vee a = a$ and $a \wedge a = a$;

(I6) $a \wedge 0 = 0$ and $a \vee 1 = 1$.

As we have seen, all the axioms of T follow from the axioms of 4-Boolean algebras. It remains to show that all the axioms of 4-Boolean algebras follow from T. We only need to verify A7 and A13. So assume the axioms of T. From distributivity and I6 one can derive the absorption laws, hence one can define the partial order $a \leq b$ iff $a \vee b = b$ iff $a \wedge b = a$ and it is easy to see that this ordering satisfies Proposition 21 ($\neg a \leq \neg b$ if $a \leq b$) and Proposition 23 (monotonicity of $\vee$ and $\wedge$ with respect to $\leq$). One can easily check that the items (i)–(iv) of

Lemma 3 (in Chapter 3) also follow from T. Now suppose $\delta a = \delta 1$. Then by I1, $\delta a = 1$, so by A10, $a \leq \delta(a \vee \perp)$. But if $\delta a = 1$, then by P1, which follows from T, $\delta \neg a = 1$ as well, hence by A10 again, $\neg a \leq \delta(\neg a \vee \top) = \neg\delta(a \vee \perp)$, hence $\delta(a \vee \perp) \leq a$ by Proposition 21, so $a = \delta(a \vee \perp)$, thus A7 follows from T. Now suppose

$$\delta(a \vee \perp) = \delta(b \vee \perp) \quad \text{and} \quad \delta(a \wedge \perp) = \delta(b \wedge \perp).$$

Then by A8 we must also have

$$\delta(a \wedge \top) = \delta(b \wedge \top) \quad \text{and} \quad \delta(a \vee \top) = \delta(b \vee \top).$$

But by Lemma 3(i),(ii) and Proposition 23 we have that

$$\delta(a \vee \perp) \vee \perp \leq a \vee \perp \quad \text{and} \quad \delta(a \vee \top) \vee \top \leq a \vee \top$$

and also (by Lemma 3(iv),(iii) and Proposition 23)

$$a \vee \perp = (a \wedge \top) \vee \perp \leq \delta(a \vee \perp) \vee \perp \quad \text{and} \quad a \vee \top = (a \wedge \perp) \vee \top \leq \delta(a \vee \top) \vee \top$$

hence

$$\delta(a \vee \perp) \vee \perp = a \vee \perp \quad \text{and} \quad \delta(a \vee \top) \vee \top = a \vee \top$$

and these equalities clearly hold for b as well, so we have

$$a \vee \perp = b \vee \perp \quad \text{and} \quad a \vee \top = b \vee \top.$$

Now since $a = a \vee (\top \wedge \perp) = (a \vee \top) \wedge (a \vee \perp)$, we obtain $a = b$ which completes the proof. $\square$

Corollary 1. *For any α, the class of 4-CAαs is a variety.*

2.2 The knowledge ordering

As was mentioned in the introduction, Belnap's four-valued logic involves two partial orderings: one that compares degrees of truth and one that compares degrees of knowledge. The ordering we defined in the previous section is the one that compares degrees of truth. Hence 0 is the falsest truth value and 1 the truest. We will now define the knowledge ordering in 4-CAαs. First we introduce two new operators, $\oplus$ and $\otimes$, from which we will derive the ordering. We can think of $\otimes$ as a consensus operator: $P \otimes Q$ is the most information that

P and Q agree on. Likewise $P \oplus Q$ combines the intelligence of both P and Q. It will turn out that our two orderings are strongly connected.

Definition 7. In a 4-CAα, the strong disjunction is defined as

$$a \oplus b = (a \vee b) \wedge (\top \vee (a \wedge b))$$

and the weak conjunction is defined as

$$a \otimes b = (a \wedge b) \vee (\perp \wedge (a \vee b)).$$

Proposition 29. The operations $\oplus$ and $\otimes$ are both commutative and associative.

Proof. Commutativity follows directly from the commutativity of both $\vee$ and $\wedge$. Associativity:

$$\begin{aligned}(a \oplus b) \oplus c &= ((a \oplus b) \vee c) \wedge (\top \vee ((a \oplus b) \wedge c)) \\ &= (((a \oplus b) \vee c) \wedge \top) \vee (((a \oplus b) \vee c) \wedge (a \oplus b) \wedge c) \\ &= ((a \oplus b) \wedge \top) \vee (c \wedge \top) \vee ((a \oplus b) \wedge c).\end{aligned}$$

Now note that

$$\begin{aligned}(a \oplus b) \wedge c &= (a \vee b) \wedge (\top \vee (a \wedge b)) \wedge c \\ &= (((a \vee b) \wedge \top) \vee ((a \vee b) \wedge a \wedge b)) \wedge c \\ &= ((a \wedge \top) \vee (b \wedge \top) \vee (a \wedge b)) \wedge c \\ &= ((a \wedge \top \wedge c) \vee (b \wedge \top \wedge c) \vee (a \wedge b \wedge c).\end{aligned} \tag{$*$}$$

If we substitute $\top$ for c in ($*$), we get $(a \oplus b) \wedge \top = (a \wedge \top) \vee (b \wedge \top)$, and combining the last three results, we obtain

$$(a \oplus b) \oplus c = (a \wedge \top) \vee (b \wedge \top) \vee (c \wedge \top) \vee (a \wedge b \wedge c).$$

By commutativity of $\oplus$ and symmetry in the term on the right side, we now have $(a \oplus b) \oplus c = a \oplus (b \oplus c)$. Since we have used no particular property of $\top$ in the reasoning above, this reasoning would still hold if there had been $\perp$ instead of $\top$ in the definition of $\oplus$. This implies that duality yields associativity of $\otimes$, i.e. in the valid equation $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ we substitute $\wedge$ for $\vee$, $\vee$ for $\wedge$ and $\perp$ for $\top$, thus obtaining the equation $(a \otimes b) \otimes c = a \otimes (b \otimes c)$ and we conclude that this equation must still be valid, since $\vee$ and $\wedge$ are duals. $\square$

P 30. $a \otimes (a \oplus b) = a$.

Proof. Expanding yields $a \otimes (a \oplus b) = [a \wedge (a \oplus b)] \vee [\bot \wedge (a \vee (a \oplus b)]$. From $(*)$ we derive $a \wedge (a \oplus b) = (a \wedge \top) \vee (a \wedge b)$. Also, from the first two lines in the calculation of $(*)$, we know

$$(a \oplus b) \vee c = (a \wedge \top) \vee (b \wedge \top) \vee (a \wedge b) \vee c$$

hence $a \vee (a \oplus b) = a \vee (b \wedge \top)$, so by $(*)$ again, we obtain

$$\bot \wedge (a \vee (a \oplus b)) = (\bot \wedge a) \vee (a \wedge b \wedge \bot) = \bot \wedge a$$

and hence we have

$$a \otimes (a \oplus b) = (a \wedge \top) \vee (a \wedge b) \vee (\bot \wedge a) = (a \wedge (\top \vee \bot)) \vee (a \wedge b) = a \vee (a \wedge b) = a.$$

□

P 31. $a \oplus (a \otimes b) = a$.

Proof. We obtain this equation after interchanging $\bot$ and $\top$, and $\vee$ and $\wedge$ in P30. Since the only feature of $\top$ and $\bot$ used in the proof of P30, is $\top \wedge \bot = 0$, and since $\wedge$ and $\vee$ are duals, this equation must still be valid. □

P 32. $\neg(a \oplus b) = \neg a \oplus \neg b$.

Proof. Remember that $\neg\top = \top$.

$$\begin{aligned}
\neg(a \oplus b) &= \neg[(a \vee b) \wedge (\top \vee (a \wedge b))] \\
&= \neg[(a \wedge b) \vee (\top \wedge (a \vee b))] \qquad \text{(by the modular law)} \\
&= \neg(a \wedge b) \wedge \neg(\top \wedge (a \vee b)) \\
&= (\neg a \vee \neg b) \wedge (\top \vee (\neg a \wedge \neg b)) \\
&= \neg a \oplus \neg b.
\end{aligned}$$

□

P 33. $\neg(a \otimes b) = \neg a \otimes \neg b$.

Proof. In the proof of the previous proposition, the fact that $\neg\top = \top$ is the only feature of $\top$ that is used. Since $\neg\bot = \bot$ and $\vee$ and $\wedge$ are duals, we can substitute $\bot$ for $\top$, and interchange $\vee$ and $\wedge$ in P32, obtaining $\neg(a \otimes b) = \neg a \otimes \neg b$ and we conclude that this equation is still valid. □

P 34. $a \oplus (b \wedge c) = (a \oplus b) \wedge (a \oplus c)$.

Proof. Expanding yields

$$\begin{aligned} a \oplus (b \wedge c) &= [a \vee (b \wedge c)] \wedge [\top \vee (a \wedge b \wedge c) \\ &= (a \vee b) \wedge (a \vee c) \wedge (\top \vee [(a \wedge b) \wedge (a \wedge c)]) \\ &= (a \vee b) \wedge [\top \vee (a \wedge b)] \wedge (a \vee c) \wedge [\top \vee (a \wedge c)] \\ &= (a \oplus b) \wedge (a \oplus c). \end{aligned}$$

□

P 35. $a \otimes (b \vee c) = (a \otimes b) \vee (a \otimes c)$.

Proof. Again, no feature of $\top$ was used in the proof of P34, so we can substitute $\bot$ for $\top$. Then by duality, we obtain the desired equation. □

P 36. $a \vee (b \oplus c) = (a \vee b) \oplus (a \vee c)$.

Proof. Expanding yields

$$\begin{aligned} a \vee (b \oplus c) &= a \vee [(b \vee c) \wedge (\top \vee (b \wedge c))] \\ &= (a \vee b \vee c) \wedge (a \vee \top \vee (b \wedge c))] \\ &= [(a \vee b) \vee (a \vee c)] \wedge (\top \vee [(a \vee b) \wedge (a \vee c)]) \\ &= (a \vee b) \oplus (a \vee c). \end{aligned}$$

□

P 37. $a \wedge (b \otimes c) = (a \wedge b) \otimes (a \wedge c)$.

Proof. Substitute P36 for P34 in the proof of P35. □

Theorem 3. *For all distinct* $\Cap$, $\Cup \in \{\vee, \wedge, \oplus, \otimes\}$ *we have that*

$$a \Cap (b \Cup c) = (a \Cap b) \Cup (a \Cap c).$$

Proof. We have already established six of the twelve distributivity laws. The next four follow easily:

$$\begin{aligned} a \oplus (b \vee c) &= \neg[\neg a \oplus (\neg b \wedge \neg c)] \\ &= \neg[(\neg a \oplus \neg b) \wedge (\neg a \oplus \neg c)] \\ &= \neg[\neg(a \oplus b) \wedge \neg(a \oplus c)] = (a \oplus b) \vee (a \oplus c). \end{aligned}$$

If we substitute $\otimes$ for $\oplus$ and $\vee$ for $\wedge$ in the above proof, we get a proof for $a \otimes (b \wedge c) = (a \otimes b) \wedge (a \otimes c)$. Then we have

$$\begin{aligned} a \vee (b \otimes c) &= \neg[\neg a \wedge (\neg b \otimes \neg c)] \\ &= \neg[(\neg a \wedge \neg b) \otimes (\neg a \wedge \neg c)] \\ &= \neg[\neg(a \vee b) \otimes \neg(a \vee c)] = (a \vee b) \otimes (a \vee c). \end{aligned}$$

Again, if we substitute $\vee$ for $\wedge$ and $\oplus$ for $\otimes$ in the above proof, we get $a \wedge (b \oplus c) = (a \wedge b) \oplus (a \wedge c)$. For the last two distributivity laws, note that

- $(a \oplus b) \vee \bot = a \vee b \vee \bot$;
- $(a \otimes b) \vee \bot = (a \wedge b) \vee \bot$;
- $(a \oplus b) \wedge \bot = a \wedge b \wedge \bot$;
- $(a \otimes b) \wedge \bot = (a \vee b) \wedge \bot$.

Hence we have

$$\begin{aligned} (a \otimes (b \oplus c)) \vee \bot &= (a \wedge (b \oplus c)) \vee \bot \\ &= ((a \wedge b) \oplus (a \wedge c)) \vee \bot \\ &= ((a \wedge b) \vee \bot) \oplus ((a \wedge c) \vee \bot) \\ &= ((a \otimes b) \oplus (a \otimes c)) \vee \bot \end{aligned}$$

and

$$\begin{aligned} (a \otimes (b \oplus c)) \wedge \bot &= (a \vee (b \oplus c)) \wedge \bot \\ &= ((a \vee b) \oplus (a \vee c)) \wedge \bot \\ &= ((a \vee b) \wedge \bot) \oplus ((a \vee c) \vee \bot) \\ &= ((a \otimes b) \oplus (a \otimes c)) \wedge \bot \end{aligned}$$

hence $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$ by A13. Finally, by this last result and by P30 and P31 we get

$$\begin{aligned} (a \oplus b) \otimes (b \oplus c) &= ((a \oplus b) \otimes a) \oplus ((a \oplus b) \otimes c) \\ &= a \oplus ((a \oplus b) \otimes c) \\ &= a \oplus (a \otimes c) \oplus (b \otimes c) \\ &= a \oplus (b \otimes c). \end{aligned}$$

$\square$

The validity of the equations in the next propositions follows easily.

P 38. $a \oplus \bot = a$ and $a \otimes \top = a$.

P 39. $a \oplus \top = \top$ and $a \otimes \bot = \bot$.

P 40. $0 \oplus 1 = \top$ and $0 \otimes 1 = \bot$.

P 41. $a \oplus a = a \otimes a = a$.

P 42. $a \oplus b = b$ if and only if $a \otimes b = a$, hence the order $\leq_k$ defined by $a \leq_k b$ iff $a \oplus b = b$ is a partial order.

P 43. $a \leq_k b$ implies $\neg a \leq_k \neg b$.

P 44. $a = (a \vee \bot) \oplus (a \wedge \bot)$.

Proof. Expanding yields

$$\begin{aligned}
(a \vee \bot) \oplus (a \wedge \bot) &= [(a \vee \bot) \vee (a \wedge \bot)] \wedge [\top \vee ((a \vee \bot) \wedge (a \wedge \bot))] \\
&= [a \vee \bot] \wedge [\top \vee (a \wedge \bot)] \\
&= [a \vee \bot] \wedge [\top \vee a] = a \vee (\bot \wedge \top) = a.
\end{aligned}$$

□

2.2.1 Bilattices

We will now see that a 4-CAα can be seen as a bilattice according to the definition in [Ari96]:

Definition 8. A *bilattice* is a structure $\mathcal{B} = (B, \leq_t, \leq_k, \neg)$ such that $|B| \geq 2$, $(B, \leq_t)$ and $(B, \leq_k)$ are lattices and $\neg$ is a unary operation that has the following properties:

(i) if $a \leq_t b$, then $\neg b \leq_t \neg a$;

(ii) if $a \leq_k b$, then $\neg a \leq_k \neg b$;

(iii) $\neg\neg a = a$.

We let $\wedge$ and $\vee$ denote the greatest lower bound and least upper bound operations with respect to $\leq_t$ and we let $\otimes$ and $\oplus$ denote the greatest lower bound and least upper bound operations with respect to $\leq_k$.

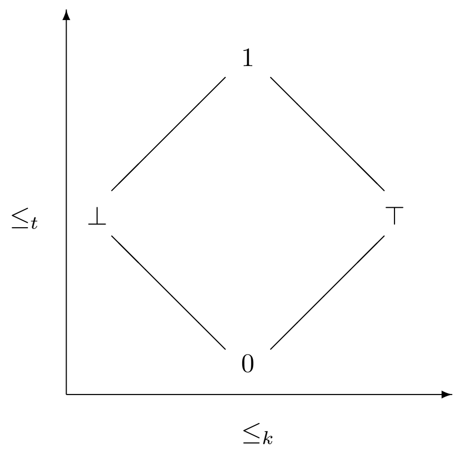

Figure 2.1: The double Hasse diagram of $\mathcal{4}$

Example 1. As we wrote earlier, we can think of the set of truth values $\mathcal{4} = \{0, 1, \top, \perp\}$ as a partial ordering in which 0 is the minimum, 1 the maximum, and $\top$ and $\perp$ are incomparable. The intuition behind his ordering is that it compares degrees of truth. We can also order the truth values according to their degrees of knowledge: $\top$ is the most informative, $\perp$ the least, and 0 and 1 are equally informative. These orderings are illustrated in Figure 2.1.

Lemma 1. Let $\mathfrak{A}$ be a 4-CAα and let $\le$ and $\le_k$ be as defined in the previous section and in this section. Then $(|\mathfrak{A}|, \le, \le_k, \neg)$ is a bilattice.

Proof. Follows directly from axiom A2 and the propositions P21 and P43. $\square$

Definition 9. A bilattice is called *distributive* if all of the twelve possible distributive laws concerning $\wedge$, $\vee$, $\otimes$ and $\oplus$ hold. It is called interlaced if each one of $\wedge$, $\vee$, $\otimes$ and $\oplus$ is monotonic with respect to both $\le_t$ and $\le_k$.

Lemma 2. Every distributive bilattice is interlaced.

Proof. See [Ari96]. $\square$

Corollary 2. *If $\mathfrak{A}$ is a 4-CAα, then $(|\mathfrak{A}|, \le, \le_k, \neg)$ is an interlaced bilattice.*

Proof. It is distributive by Theorem 3, hence it is interlaced by Lemma 2. $\square$

Definition 10. A unary operation $-$ on a bilattice is called a *conflation* if it has the following properties:

(i) if $a \leq_k b$, then $-b \leq_k -a$;

(ii) if $a \leq_t b$, then $-a \leq_t -b$;

(iii) $- - a = a$;

(iv) $-\neg a = \neg - a$.

Definition 11. A bilattice $\mathcal{B}$ with a conflation is called *classical* if for all $b \in \mathcal{B}$, $b \vee -\neg b = 1$.

Theorem 4. *Let $\mathfrak{A} = (A, \vee, \wedge, \neg, 1, 0, \top, \bot, \delta, c_\kappa, d_{\kappa\lambda})_{\kappa,\lambda<\alpha}$ be a 4-CAα and let $\leq$ be the ordering as defined in Definition 5. There is a unary operation $-$ on A such that $-$ is a conflation on $(|\mathfrak{A}|, \leq, \leq_k, \neg)$ and moreover, $(|\mathfrak{A}|, \leq, \leq_k, \neg, -)$ is a classical bilattice.*

Proof. We will prove this theorem after the main result in the next section: the decomposition theorem. $\square$

Chapter 3

Decomposition of 4-CAαs

In this section we study the relationship between four-cylindric algebras and cylindric algebras. In the first section we defined four-cylindric set algebras. The elements of 4-CSAαs are pairs of elements of CSAαs and the operations in 4-CSAαs were derived from the operations in CSAαs. We will now generalize this and give a method for constructing 4-CAαs out of CAαs. Then we will show that in fact every 4-CAα is the result of applying this method to some CAα.

In the following, the same symbol will sometimes have a different meaning, depending on the context. For instance, when we write $c_\kappa(x, x') = (c_\kappa x, q_\kappa x')$, the first c_κ means cylindrification in a 4-CAα while the second c_κ means cylindrification in a CAα.

Definition 12. Let $\mathcal{C} = (C, \vee, \wedge, \neg, 0, 1, c_\kappa, d_{\kappa\lambda})_{\kappa,\lambda<\alpha}$ be a cylindric algebra of dimension α. We now define $\mathbb{T}(\mathcal{C})$ to be the structure $(C \times C, \vee, \wedge, \neg, \mathbf{0}, \mathbf{1}, \bot, \top, \delta, c_\kappa, \mathbf{d}_{\kappa\lambda})_{\kappa,\lambda<\alpha}$ where $\mathbf{0} = (0, 1)$, $\mathbf{1} = (1, 0)$, $\bot = (0, 0)$, $\top = (1, 1)$, $\mathbf{d}_{\kappa\lambda} = (d_{\kappa\lambda}, \neg d_{\kappa\lambda})$ and for (x, x') and $(y, y') \in C \times C$ we have

- $(x, x') \vee (y, y') = (x \vee y, x' \wedge y')$;
- $(x, x') \wedge (y, y') = (x \wedge y, x' \vee y')$;
- $\neg(x, x') = (x', x)$;
- $\delta(x, x') = (x + x', x \leftrightarrow x')$;

- $c_\kappa(x, x') = (c_\kappa x, q_\kappa x')$,

where $x + x'$ is the symmetric difference of x and x' in the cylindric algebra $\mathcal{C}$, $x \leftrightarrow x' = \neg(x + x')$ and q_κ is the dual of c_κ in the cylindric algebra. Similarly, ignoring the cylindrifications and diagonal elements, one can define $\mathbb{T}(\mathcal{B})$ for an arbitrary Boolean algebra $\mathcal{B}$.

Theorem 5. *If $\mathcal{C}$ is a cylindric algebra, then $\mathbb{T}(\mathcal{C})$ is a 4-cylindric algebra. Moreover, if $\mathcal{C}$ is a cylindric set algebra, then $\mathbb{T}(\mathcal{C})$ is a 4-cylindric set algebra. If $\mathcal{B}$ is a Boolean algebra, then $\mathbb{T}(\mathcal{B})$ is a 4-Boolean algebra.*

Proof. Using some elementary theory of cylindric algebras one easily checks that $\mathbb{T}(\mathcal{C})$ satisfies all the axioms. $\square$

We will now show that every 4-CAα is isomorphic to $\mathbb{T}(\mathcal{C})$ for some cylindric algebra $\mathcal{C}$. First we prove some useful observations:

Lemma 3. In a 4-CAα we have:

(i) $\delta(a \vee \perp) \leq a \vee \perp$ hence $\neg\delta(a \vee \perp) \vee a \vee \perp = 1$.

(ii) $\delta(a \vee \top) \leq a \vee \top$ hence $\neg\delta(a \vee \top) \vee a \vee \top = 1$.

(iii) $a \wedge \perp \wedge \delta(a \wedge \perp) = 0$ hence $a \wedge \perp \leq \neg\delta(a \wedge \perp) = \delta(a \vee \top)$.

(iv) $a \wedge \top \wedge \delta(a \wedge \top) = 0$ hence $a \wedge \top \leq \neg\delta(a \wedge \top) = \delta(a \vee \perp)$.

(v) $a \wedge \neg a \wedge \delta a = 0$.

(vi) $\delta a \leq a \vee \neg a$.

(vii) $c_\kappa \perp = \perp$.

(viii) $c_\kappa(a \wedge \perp) = c_\kappa a \wedge \perp$.

(ix) $c_\kappa \top = \top$.

Proof. Items (i)–(iv) follow easily by substitution of $a \vee \perp$, $a \vee \top$, $a \wedge \perp$ and $a \wedge \top$ in A10 respectively. (v):

$$\begin{aligned} a \wedge \neg a \wedge \delta a &= a \wedge \delta a \wedge \neg a \wedge \delta \neg a \\ &= \delta(a \vee \perp) \wedge \delta(a \vee \top) \wedge \delta(\neg a \vee \perp) \wedge \delta(\neg a \vee \top) = 0. \end{aligned}$$

The last equality holds since $\delta(\neg a \vee \top) = \neg\delta(a \vee \bot)$. (vi): From (v) we get $a \wedge \neg a \leq \neg\delta a$, hence $\delta a \leq a \vee \neg a$. Substituting 0 for a in A6 yields (vii). In order to obtain (viii), substitute $\bot$ for b in C3 and apply (vii). (ix): By C2, $c_\kappa 1 = 1$, so $c_\kappa\top \vee \bot = c_\kappa(\top \vee \bot) = 1$. By (viii), $c_\kappa\top \wedge \bot = 0$ hence by the propositions P44 and P40, $c_\kappa\top = (c_\kappa\top \vee \bot) \oplus (c_\kappa\top \wedge \bot) = 1 \oplus 0 = \top$. $\square$

Theorem 6. *Assume* $\mathfrak{A} = (A, \vee, \wedge, \neg, 1, 0, \top, \bot, \delta, c_\kappa, d_{\kappa\lambda})_{\kappa,\lambda<\alpha}$ *is a 4-cylindric algebra of dimension* α*. Then there is a cylindric algebra* $\mathcal{C} = \mathfrak{A}^\delta$ *(also of dimension* α*) such that* $\mathfrak{A} \cong \mathbb{T}(\mathcal{C})$*.*

Proof. Given $\mathfrak{A}$ as in the theorem, let $A^\delta = \{\delta a \mid a \in A\}$. First note that $1 = \delta 1$, $0 = \delta\bot$, and $d_{\kappa\lambda} = \delta(d_{\kappa\lambda} \vee \bot)$. Then note that A^δ is closed under the operations $\vee$ (since $\delta a \vee \delta b = \delta(\delta a \vee \bot) \vee \delta(\delta b \vee \bot) = \delta(\delta a \vee \delta b \vee \bot)$ by Axiom 11), $\wedge$ (by a similar argument using Axiom 12), $\neg$ (since $\neg\delta a = \delta(\neg\delta a \vee \bot)$) and c_κ (since $c_\kappa\delta a = c_\kappa\delta(\delta a \vee \bot) = \delta c_\kappa(\delta a \vee \bot)$). From the axioms of 4-CAαs, it is now easily seen that $\mathfrak{A}^\delta = (A^\delta, \vee, \wedge, \neg, 1, 0, c_\kappa, d_{\kappa\lambda})_{\kappa,\lambda<\alpha}$ is a CAα. We now claim that

$$\theta : a \mapsto (\delta(a \vee \bot), \delta(\neg a \vee \bot))$$

is an isomorphism from $\mathfrak{A}$ onto $\mathbb{T}(\mathfrak{A}^\delta)$. First we will show it is a homomorphism: It is easy to check that θ preserves all the constants. $\vee$:

$$\begin{aligned}
\theta(a \vee b) &= (\delta(a \vee b \vee \bot), \delta((\neg a \wedge \neg b) \vee \bot)) \\
&= (\delta(a \vee \bot) \vee \delta(b \vee \bot), \delta(\neg a \vee \bot) \wedge \delta(\neg b \vee \bot)) \text{ (by A11 and A12)} \\
&= (\delta(a \vee \bot), \delta(\neg a \vee \bot)) \vee (\delta(b \vee \bot), \delta(\neg b \vee \bot)) \\
&= \theta(a) \vee \theta(b).
\end{aligned}$$

The argument for $\wedge$ is similar. Preservation of $\neg$ follows easily from the fact that $\neg\neg a = a$. Furthermore, we have

$$\begin{aligned}
\theta(c_\kappa a) &= (\delta(c_\kappa a \vee \bot), \delta(\neg c_\kappa a \vee \bot)) \\
&= (c_\kappa\delta(a \vee \bot), \delta(\neg c_\kappa a \vee \bot)) \text{ (by axioms A6 and A4)} \\
&= (c_\kappa\delta(a \vee \bot), \delta(c_\kappa a \wedge \bot)) \text{ (by P1)} \\
&= (c_\kappa\delta(a \vee \bot), q_\kappa\delta(\neg a \vee \bot)) \text{ (by axiom A5 and P1 again)} \\
&= c_\kappa\theta(a).
\end{aligned}$$

Finally, we have to show that $\theta(\delta a) = \delta\theta(a)$. Note that since $\theta(\delta a) = (\delta a, \neg\delta a)$ it suffices to show that the first component of $\delta\theta(a)$ equals δa. The first component of $\delta\theta(a)$ is $\delta(a \vee \bot) + \delta(\neg a \vee \bot)$

$$
\begin{aligned}
&= (\delta(a \vee \bot) \wedge \neg\delta(\neg a \vee \bot)) \vee (\delta(\neg a \vee \bot) \wedge \neg\delta(a \vee \bot)) \\
&= (\delta(a \vee \bot) \wedge \delta(a \vee \top)) \vee (\delta(\neg a \vee \bot) \wedge \delta(\neg a \vee \top)) \\
&= (a \wedge \delta a) \vee (\neg a \wedge \delta(\neg a)) \\
&= (a \vee \neg a) \wedge \delta a \\
&= \delta a \text{ (by lemma 3(vi))}.
\end{aligned}
$$

hence θ is a homomorphism. To show that it is injective, note that $\delta(\neg a \vee \bot) = \delta(a \wedge \bot)$, so if $\theta(a) = \theta(b)$, then $a = b$ by axiom A13. In order to show that θ is surjective, let $(\delta a, \delta b) \in \mathbb{T}(\mathfrak{A}^\delta)$ be arbitrary. Let $c := (\delta a \vee \bot) \oplus (\neg\delta b \wedge \bot)$. Now the first component of $\theta(c)$ equals

$$
\begin{aligned}
\delta(c \vee \bot) &= \delta([(\delta a \vee \bot) \oplus (\neg\delta b \wedge \bot)] \vee \bot) \\
&= \delta((\delta a \vee \bot) \oplus \bot) \text{ (since } \vee \text{ distributes over } \oplus\text{, see Theorem 3)} \\
&= \delta(\delta a \vee \bot) \text{ (by P38)} \\
&= \delta a.
\end{aligned}
$$

By P32, $\neg c = (\delta b \vee \bot) \oplus (\neg\delta a \wedge \bot)$, hence by symmetry we obtain that $\delta(\neg c \vee \bot) = \delta b$, hence $\theta(c) = (\delta a, \delta b)$, which completes the proof. $\square$

In the proof of Theorem 6 we saw that in any 4-CAα $\mathfrak{A}$, the subuniverse $A^\delta = \{\delta a \,|\, a \in A\}$ together with the original operations, turns out to be a CAα and we can identify a with the pair $(\delta(a \vee \bot), \delta(\neg a \vee \bot))$. Now note that since $\delta a = \delta(\delta a \vee \bot)$, $A^\delta = \{\delta(a \vee \bot) \,|\, a \in A\}$. Also note that if $\delta(a \vee \bot) = \delta(b \vee \bot)$, then by A13 it is easily seen that $a \vee \bot = b \vee \bot$. This suggests that we might as well identify a with the pair $(a \vee \bot, \neg a \vee \bot)$. The reason this is not done in the proof of Theorem 6, is that $A^\perp := \{\, a \vee \bot \mid a \in A \,\}$ is not closed under negation and does not contain 0 and the diagonal elements, so we would have to redefine negation, 0 and the diagonals in that subuniverse. However, since δ is a bijection from $A^\perp$ onto A^δ we can define them in such a way that δ becomes an isomorphism, hence we must have that if $f : A^\perp \to A^\perp$ is the negation in $A^\perp$, then $\delta(f(a \vee \bot)) = \neg\delta(a \vee \bot)$. Now it's easily seen that $f(a \vee \bot)$ must equal $\neg\delta(a \vee \bot) \vee \bot$. Similarly, if $\mathbf{d}_{\kappa\lambda} = a \vee \bot$ is a diagonal element in $A^\perp$, then we must have $\delta(a \vee \bot) = d_{\kappa\lambda}$, hence $a = d_{\kappa\lambda}$ hence $\mathbf{d}_{\kappa\lambda} = d_{\kappa\lambda} \vee \bot$. Finally, it is easy to see that we must let $\bot$ be the 0-element in $\mathfrak{A}^\perp$. Since δ preserves

1, $\vee$, $\wedge$ and c_κ, we now know that $\mathfrak{A}^\perp := (A^\perp, \vee, \wedge, f, 1, \perp, \mathbf{d}_{\kappa\lambda}, c_\kappa)_{\kappa,\lambda<\alpha}$ is a CAα such that $\delta : \mathfrak{A}^\perp \cong \mathfrak{A}^\delta$. This means we have an alternative decomposition $\mathfrak{A} \cong \mathbb{T}(\mathfrak{A}^\perp)$ given by $a \mapsto (a \vee \perp, \neg a \vee \perp)$. The proof of the next corollary is contained in the proof of Theorem 6.

Corollary 3. *For every 4-Boolean algebra $\mathfrak{B}$ there is a Boolean algebra $\mathcal{B}$ such that $\mathfrak{B}$ is isomorphic to $\mathbb{T}(\mathcal{B})$.*

The cylindric algebra which is constructed out of a 4-CAα $\mathfrak{A}$ the way it was done in the proof of Theorem 6 will be denoted by $\mathfrak{A}^\delta$.

Theorem 7. *If $\mathcal{C}$ is a CAα, then $\mathbb{T}(\mathcal{C})^\delta$ is isomorphic to $\mathcal{C}$. Moreover, if $\mathfrak{A}$ is a 4-CAα, then $\mathfrak{A} \cong \mathbb{T}(\mathfrak{A}^\delta)$.*

Proof. It is easily seen that the function $f : \mathcal{C} \to \mathbb{T}(\mathcal{C})^\delta$ defined by $c \mapsto (c, \neg c)$ is an isomorphism. The second assertion is Theorem 6. $\square$

Theorem 8. *Let $\mathcal{C}$ be a CAα such that $\mathcal{C}$ is embedded in $\prod_{j\in J} \mathcal{C}_j$, where each $\mathcal{C}_j$ is a CAα. Then $\mathbb{T}(\mathcal{C})$ is embedded in $\prod_{j\in J} \mathbb{T}(\mathcal{C}_j)$. Moreover, if the composition of the embedding and the projection onto $\mathcal{C}_j$ is surjective for all $j \in J$, then the same holds for the embedding $\mathbb{T}(\mathcal{C}) \hookrightarrow \prod_{j\in J} \mathbb{T}(\mathcal{C}_j)$.*

Proof. Let $f : \mathcal{C} \hookrightarrow \prod_{j\in J} \mathcal{C}_j$ be an embedding. Define $g : \mathbb{T}(\mathcal{C}) \to \prod_{j\in J} \mathbb{T}(\mathcal{C}_j)$ by $(a, b) \mapsto (f(a)_j, f(b)_j)_{j\in J}$. It is straightforward to check that g is a homomorphism. Suppose $g(a, b) = (f(a)_j, f(b)_j)_{j\in J} = (f(a')_j, f(b')_j)_{j\in J} = g(a', b')$. Then for all $j \in J$ we have $f(a)_j = f(a')_j$ and $f(b)_j = f(b')_j$, hence $f(a) = f(a')$ and $f(a') = f(b')$, hence $a = a'$ and $b = b'$ since f is an embedding, hence g is injective. Now let p_j be the projection from $\prod_{j\in J} \mathcal{C}_j$ onto $\mathcal{C}_j$ and q_j be the projection from $\prod_{j\in J} \mathbb{T}(\mathcal{C}_j)$ onto $\mathbb{T}(\mathcal{C}_j)$. Suppose $p_j \circ f$ is surjective for all $j \in J$. Let $(d, d') \in \mathbb{T}(\mathcal{C}_j)$. Then there are c and $c' \in C$ such that $p_j(f(c)) = d$ and $p_j(f(c')) = d'$. But now it is clear that $q_j(g(c, c')) = (d, d')$, hence $q_j \circ g$ is surjective for all $j \in J$. $\square$

Theorem 9. *Let $\mathfrak{A}$ be a 4-CAα such that $\mathfrak{A}$ is isomorphic to a subalgebra of $\prod_{j\in J} \mathfrak{A}_j$, where each $\mathfrak{A}_j$ is a 4-CAα. Then $\mathfrak{A}^\delta$ is isomorphic to a subalgebra of $\prod_{j\in J} \mathfrak{A}_j^\delta$.*

Proof. Let $f : \mathfrak{A} \hookrightarrow \prod_{j\in J} \mathfrak{A}_j$ be an embedding. We claim that the restriction of f to $\mathfrak{A}^\delta$ embeds $\mathfrak{A}^\delta$ into $\prod_{j\in J} \mathfrak{A}_j^\delta$. Clearly, this restriction is still injective. It also preserves all the operation of $\mathfrak{A}^\delta$ since they are operations of $\mathfrak{A}$ too. We only need to verify that $f \upharpoonright \mathfrak{A}^\delta$ indeed maps into $\prod_{j\in J} \mathfrak{A}_j^\delta$. But since f is a homomorphism,

$$f(\delta a) = \delta f(a) = (\delta_j f(a)_j)_{j\in J} \in \prod_{j\in J} \mathfrak{A}_j^\delta.$$

$\square$

Theorem 10. *If $\mathfrak{A}$ is a 4-CSAα, then $\mathfrak{A}^\delta$ is isomorphic to a CSAα.*

Proof. By definition, $\mathfrak{A}$ is a subalgebra of $\mathbb{T}(\mathcal{C})$ for some CSAα $\mathcal{C}$. By Theorem 9 and Theorem 7, $\mathfrak{A}^\delta$ is isomorphic to a subalgebra of $\mathbb{T}(\mathcal{C})^\delta \cong \mathcal{C}$. $\square$

We conclude this chapter with a proof of Theorem 4, as was promised in section 2.2.1.

Proof of Theorem 4. Let $\mathcal{C}$ be a cylindric algebra. By Theorem 6 it suffices to show that the theorem holds for

$$\mathfrak{A} = \mathbb{T}(\mathcal{C}) = (C \times C, \vee, \wedge, \neg, \mathbf{0}, \mathbf{1}, \perp, \top, \delta, c_\kappa, \mathbf{d}_{\kappa\lambda})_{\kappa,\lambda<\alpha}$$

as defined in Definition 12. Let C be the universe of $\mathcal{C}$. Then $C \times C$ is the universe of $\mathbb{T}(\mathcal{C})$. Define the conflation $-$ as follows: For $(a, b) \in C \times C$, let

$$-(a, b) = (\neg b, \neg a).$$

We have to show that $-$ is indeed a conflation. Recall the characteristic properties of a conflation from Definition 10. First of all, it is easy to see that $--(a, b) = (a, b)$ and that $\neg-(a, b) = -\neg(a, b)$. For the last two properties, we make two observations: First we observe that for any (a, b) and $(c, d) \in C \times C$,

$$(a, b) \leq (c, d) \quad \text{iff} \quad (a, b) = (a \wedge c, b \vee d) \quad \text{iff} \quad (a \leq c \quad \text{and} \quad d \leq b).$$

The second observation is that $(a, b) \oplus (c, d) = (a \vee c, b \vee d)$ and $(a, b) \otimes (c, d) = (a \wedge c, b \wedge d)$ and hence

$$(a, b) \leq_k (c, d) \quad \text{iff} \quad (a, b) = (a \wedge c, b \wedge d) \quad \text{iff} \quad (a \leq c \quad \text{and} \quad b \leq d).$$

Now suppose $(a,b) \leq (c,d)$. Then $a \leq c$ and $b \geq d$ in the cylindric algebra $\mathcal{C}$, hence $\neg b \leq \neg d$ and $\neg a \geq \neg c$ and , hence $-(a,b) \leq -(c,d)$. Next, suppose $(a,b) \leq_k (c,d)$. Then $a \leq c$ and $b \leq d$, hence $\neg c \leq \neg a$ and $\neg d \leq \neg b$, hence $-(c,d) \leq_k -(a,b)$ and hence $-$ is a conflation. Furthermore we have that

$$(a,b) \vee -\neg(a,b) = (a,b) \vee (\neg a, \neg b) = (1,0) = \mathbf{1} \tag{1}$$

hence $(C \times C, \leq, \leq_k, \neg, -)$ is a classical bilattice. Now for all $(a,b) \in C \times C$, let $\sim (a,b) = -\neg(a,b) = (\neg a, \neg b)$. We will now show that

$$\mathfrak{A}' = (C \times C, \vee, \wedge, \sim, 0, 1, c_\kappa, d_{\kappa\lambda})_{\kappa,\lambda<\alpha}$$

is a CAα. We already know that the commutativity, associativity and distributivity laws hold. Furthermore, it is easy to see that we have $\sim\sim (a,b) = (a,b)$, $\sim \mathbf{0} = \mathbf{1}$, $\sim \mathbf{1} = \mathbf{0}$ and

$$\sim ((a,b) \wedge (c,d)) = \sim (a \wedge c, b \vee d) = (\neg a \vee \neg c, \neg b \wedge \neg d) = \sim (a,b) \vee \sim (c,d)$$

and hence by (1) we have $(a,b) \vee \sim (a,b) = \mathbf{1}$ and $(a,b) \wedge \sim (a,b) = \mathbf{0}$, so $(C \times C, \vee, \wedge, \sim, \mathbf{0}, \mathbf{1})$ is a Boolean algebra. Now it follows that $\mathfrak{A}'$ is a CAα, if we can show that it satisfies the axioms C1-C6 and the axiom

$$c_\kappa(d_{\kappa\lambda} \wedge a) \wedge c_\kappa(d_{\kappa\lambda} \wedge \sim a) = 0 \text{ (for } \kappa \neq \lambda). \tag{C7$'$}$$

But the axioms C1–C6 are trivially satisfied, since $\mathfrak{A}$ is a 4-CAα, so it only remains to verify (C7$'$). Recall that $\mathbf{d}_{\kappa\lambda} = (d^{\mathcal{C}}_{\kappa\lambda}, \neg d^{\mathcal{C}}_{\kappa\lambda})$. Suppose $\kappa \neq \lambda$. The first coordinate of

$$c_\kappa((d_{\kappa\lambda}, \neg d_{\kappa\lambda}) \wedge (a,b)) \wedge c_\kappa((d_{\kappa\lambda}, \neg d_{\kappa\lambda}) \wedge (\neg a, \neg b))$$

is $c_\kappa(d_{\kappa\lambda} \wedge a) \wedge c_\kappa(d_{\kappa\lambda} \wedge \neg a) = 0$ since $\mathcal{C}$ is a cylindric algebra. For the same reason, the second coordinate equals

$$q_\kappa(\neg d_{\kappa\lambda} \vee b) \vee q_\kappa(\neg d_{\kappa\lambda} \vee \neg b) = \neg(c_\kappa(d_{\kappa\lambda} \wedge \neg b) \wedge c_\kappa(d_{\kappa\lambda} \wedge b)) = 1$$

hence we are done. $\square$

Chapter 4

Representation theory

In this section we study the relationship between 4-CAαs and 4-CSAαs. In [Sto39], Stone proved that every Boolean algebra is isomorphic to a field of sets. Likewise, in [HMT2] it is shown that every locally finite CAα is isomorphic to a subdirect product of CSAαs. In [Fel98], Feldman showed this for 3-CAαs. We will prove representation theorems for 4-Boolean algebras and 4-CAαs.

Definition 13. An algebra $\mathfrak{A}$ of type τ is said to be a *subdirect product* of a family $(\mathfrak{A}_j)_{j\in J}$ of type τ if there exists an embedding $f : \mathfrak{A} \hookrightarrow \prod_{j\in J} \mathfrak{A}_j$ such that $p_j \circ f$ is surjective for all $j \in J$, where p_j is the projection from $\prod_{j\in J} \mathfrak{A}_j$ onto $\mathfrak{A}_j$. An algebra of type τ is *subdirectly irreducible* if (i) $|\mathfrak{A}| > 1$ and (ii) if $\mathfrak{A}$ is a subdirect product of $(\mathfrak{A}_j)_{j\in J}$ with embedding f, then $p_j \circ f$ is an isomorphism for some $j \in J$.

Theorem 11. *An algebra $\mathfrak{A}$ is subdirectly irreducible if and only if it has a least congruence relation that strictly contains the identity relation.*

Let $\mathbf{2}$ be the two element Boolean algebra and let $\mathit{4} = \mathbb{T}(\mathbf{2})$.

Theorem 12. *The 4-Boolean algebra $\mathit{4}$ is subdirectly irreducible.*

Proof. We will show that the only congruence relation which is not the identity, is $\mathit{4} \times \mathit{4}$: Suppose $\sim$ is a congruence relation on $\mathit{4}$ and suppose $x \sim y$ for $x \neq y$. If $0 \sim 1$, then for all a, $a = a \wedge 1 \sim a \wedge 0 = 0$, hence $\sim$ is $\mathit{4} \times \mathit{4}$ by symmetry and transitivity. If $\perp \sim \top$, then $0 = \delta(\perp \vee \perp) \sim \delta(\top \vee \perp) = 1$, hence we are done.

If $x \in \{0,1\}$ and $y \in \{\top, \bot\}$, then $0 = \delta y \sim \delta x = 1$, hence we are done again. Now apply Theorem 11. $\square$

Theorem 13. *Let $\mathcal{B}$ be a Boolean algebra and suppose $\mathbb{T}(\mathcal{B})$ is subdirectly irreducible. Then $\mathcal{B}$ is subdirectly irreducible.*

Proof. Suppose $\mathcal{B}$ is a subdirect product of $(\mathcal{B}_j)_{j\in J}$ with embedding f. Let $g : \mathbb{T}(\mathcal{B}) \to \prod_{j\in J} \mathbb{T}(\mathcal{B}_j)$ as in the proof of Theorem 8, i.e. $g(a,b) = (f(a)_j, f(b)_j)_{j\in J}$. Let p_j be the projection from $\prod_{j\in J} \mathcal{B}_j$ onto $\mathcal{B}_j$ and q_j the projection from $\prod_{j\in J} \mathbb{T}(\mathcal{B}_j)$ onto $\mathbb{T}(\mathcal{B}_j)$. By Theorem 3, g is an embedding and $q_j \circ g$ is surjective for all $j \in J$. Since $\mathbb{T}(\mathcal{B})$ is subdirectly irreducible, there is a $j \in J$ such that $q_j \circ g : \mathbb{T}(\mathcal{B}) \cong \mathbb{T}(\mathcal{B}_j)$. Now suppose $a \neq b \in \mathcal{B}$. Then $(f(a)_j, f(a)_j) = g(a,a) \neq g(b,b) = (f(b)_j, f(b)_j)$ hence $f(a)_j \neq f(b)_j$. This means that $p_j \circ f$ is injective. Since it is a surjective homomorphism, it is an isomorphism between $\mathcal{B}$ and $\mathcal{B}_j$, hence $\mathcal{B}$ is subdirectly irreducible. $\square$

Corollary 4. $\mathfrak{4}$ *is the only subdirectly irreducible 4-Boolean algebra.*

Proof. This follows directly from Corollary 3, Theorem 12, Theorem 13, and the fact that $\mathfrak{2}$ is the only subdirectly irreducible Boolean algebra. $\square$

Corollary 5. *Every 4-Boolean algebra is a subdirect product of copies of* $\mathfrak{4}$.

Proof. The class of 4-Boolean algebras is an equational class by Theorem 2. The corollary follows immediately from the subdirect product theorem of G. Birkhoff (see [Bal74]). $\square$

Corollary 6. *An equation in the language of 4-Boolean algebras is valid in every 4-Boolean algebra if and only if it is valid in* $\mathfrak{4}$.

Proof. One direction is trivial. For the other, suppose it is valid in $\mathfrak{4}$. Then it is valid in every direct product of copies of $\mathfrak{4}$ and hence it is valid in every subdirect product of copies of $\mathfrak{4}$. By Corollary 6, it is valid in every 4-Boolean algebra. $\square$

Definition 14. A 4-CAα is *representable* if it is isomorphic to a subdirect product of 4-CSAαs.

The above definition is similar to the one in the theory of cylindric algebras. Suppose $\mathfrak{A}$ is a subalgebra of $\prod_{j\in J}\mathfrak{A}_j$ where $\mathfrak{A}_j$ is a 4-CSAα for all $j \in J$. Let i be the inclusion map and for all j, let $\mathfrak{B}_j = \{i(a)_j \mid a \in \mathfrak{A}\}$. It is easy to check that $\mathfrak{B}_j$ is a subalgebra of $\mathfrak{A}_j$. Hence, since a subalgebra of a 4-CSAα is again a 4-CSAα, $\mathfrak{A}$ is representable if and only if it is isomorphic to a subalgebra of a direct product of 4-CSAαs.

Definition 15. Let $\mathfrak{A}$ be a (4-)CAα and $a \in \mathfrak{A}$. The *dimension set* of a is defined by $dim(a) = \{\,\kappa < \alpha \mid c_\kappa a \neq a\,\}$. A (4-)CA$\alpha$ $\mathfrak{A}$ is *locally finite* if the dimension set of each of its members is finite.

Theorem 14. *Let $\mathcal{C}$ be a CAα and let $a, a' \in \mathcal{C}$. Then the dimension set of (a, a') in $\mathbb{T}(\mathcal{C})$ equals $dim(a) \cup dim(a')$.*

Proof. First note that in a cylindric algebra we have $c_\kappa a = a$ iff $q_\kappa a = a$: If $c_\kappa a = a$, then

$$0 = c_\kappa 0 = c_\kappa(a \wedge \neg a) = c_\kappa(c_\kappa a \wedge \neg a) = c_\kappa a \wedge c_\kappa \neg a = a \wedge c_\kappa \neg a$$

hence $a \leq \neg c_\kappa \neg a = q_\kappa a$. Since $q_\kappa a \leq a$ is valid in any cylindric algebra, we obtain $q_\kappa a = a$. Now suppose $q_\kappa a = a$, then $\neg a = \neg q_\kappa a = c_\kappa \neg a$. After applying the first direction to $\neg a$, we get $q_\kappa \neg a = \neg a$ and hence $a = c_\kappa a$. Now it's easily seen that $c_\kappa(a, a') \neq (a, a')$ iff $c_\kappa a \neq a$ or $c_\kappa a' \neq a'$. □

Definition 16. An element A of a CSAα with base U *depends* on a subset $\Gamma \subseteq \alpha$ if for all s and $t \in {}^\alpha U$, $s \restriction \Gamma = t \restriction \Gamma$ implies $s \in A$ iff $t \in A$. An element (A, A') of a 4-CSAα depends on Γ iff both A and A' depend on Γ. A (4-)CSAα is called *regular* if each of its members depends on its dimension set.

Theorem 15. *Let $\mathcal{C}$ be a regular CSAα. Then $\mathbb{T}(\mathcal{C})$ is a regular 4-CSAα.*

Proof. Note that whenever A depends on Γ, A depends on every superset of Γ. Now apply Theorem 14. □

Theorem 16. *Let $\mathfrak{A}$ be a 4-CAα such that $\mathfrak{A}^\delta$ is isomorphic to a subalgebra of a product of (regular) CSAαs. Then $\mathfrak{A}$ is isomorphic to a subalgebra of a product of (regular) 4-CSAαs, hence if $\mathfrak{A}^\delta$ is a representable CAα, then $\mathfrak{A}$ is representable.*

Proof. Suppose $\mathfrak{A}^\delta \hookrightarrow \prod_{j\in J} \mathcal{C}_j$, where each $\mathcal{C}_j$ is a CSAα. Then by Theorem 8 and Theorem 7 we have $\mathfrak{A} \cong \mathbb{T}(\mathfrak{A}^\delta) \hookrightarrow \prod_{j\in J} \mathbb{T}(\mathcal{C}_j)$ and by Theorem 5 the $\mathbb{T}(\mathcal{C}_j)$s are 4-CSA$\alpha$s. Moreover, if the $\mathcal{C}_j$s are regular, then so are the $\mathbb{T}(\mathcal{C}_j)$s by Theorem 15. $\square$

Theorem 17 (Representation Theorem)**.** *Every locally finite 4-CAα is isomorphic to a subalgebra of a product of regular 4-CSAαs, hence every locally finite 4-CAα is representable.*

Proof. Let $\mathfrak{A}$ be a locally finite 4-CAα. Remember that the universe of $\mathfrak{A}^\delta$ is $C = \{\delta(a \vee \bot) \mid a \in A\}$ (see the proof of theorem 6 and the discussion proceeding it). By the axioms A4 and A6, we have $c_\kappa \delta(a \vee \bot) = \delta(c_\kappa a \vee \bot)$, so if

$$c_\kappa \delta(a \vee \bot) \neq \delta(a \vee \bot)$$

then clearly $c_\kappa a \neq a$ hence the dimension set of $\delta(a \vee \bot)$ in $\mathfrak{A}^\delta$ is a subset of the (finite) dimension set of $a \in \mathfrak{A}$. Therefore $\mathfrak{A}^\delta$ is locally finite, hence $\mathfrak{A}^\delta$ is isomorphic to a subalgebra of a product of regular CSAαs (see Theorem 3.2.8 of [HMT2], p. 63), hence by Theorem 16, $\mathfrak{A}$ is isomorphic to a subalgebra of a product of regular 4-CSAαs and hence representable. $\square$

Chapter 5

Proof system

In this section we will introduce a proof system for our logic. The system uses a lot of axioms and a small set of rules. We will show that this proof system is complete in the sense that every logical consequence can be proven in the system.

Before we give the axioms and the rules, we introduce some abbreviations:

φ^t	abbreviates	$\Delta(\varphi \vee \bot)$.
φ^f	abbreviates	$\Delta(\varphi \wedge \bot)$.
$S_j^k \varphi$	abbreviates	$\exists v_k(v_k \approx v_j \wedge \varphi)$.
$\varphi \to \psi$	abbreviates	$\neg\varphi \vee \psi$.
$\varphi \Rightarrow \psi$	abbreviates	$(\varphi^t \to \psi^t) \wedge (\psi^f \to \varphi^f)$.
$\varphi \Leftrightarrow \psi$	abbreviates	$(\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi)$.

As usual we stipulate that $\to$ is right-associative. The first and second formula are referred to as the true part of φ and the false part of φ respectively. Note that $\varphi \to \varphi$ is not in general a validity. It is only valid when φ is defined and consistent. However, it is true (and easy to check) that whenever $\varphi^{\mathfrak{A}} = (\varphi \to \psi)^{\mathfrak{A}} = 1$, then $\varphi^{\mathfrak{A}} = 1$. Note also that $(\varphi \Leftrightarrow \psi)^{\mathfrak{A}} = 1$ if and only if $\varphi^{\mathfrak{A}} = \psi^{\mathfrak{A}}$. We will now give the axioms and the rules of inference:

Axioms: Any tautology is an axiom and the following formulas are also axioms:

1. $\varphi^t \to \exists v_k \varphi^t$

2. $\Delta \exists v_k \varphi^t$

3. $(\exists v_k \varphi^t \to \exists v_k \psi^t) \to (\forall v_k \psi^f \to \forall v_k \varphi^f) \to (\exists v_k \varphi \Rightarrow \exists v_k \psi)$

4. $\forall v_k \varphi^f \to \varphi^f$

5. $\Delta \forall v_k \varphi^f$

6. $\exists v_k \varphi \Leftrightarrow \varphi$ if $v_k \notin Fv(\varphi)$

7. $(\varphi \vee \exists v_k \varphi) \Leftrightarrow \exists v_k \varphi$

8. $\exists v_k (\varphi \wedge \exists v_k \psi) \Leftrightarrow \exists v_k \varphi \wedge \exists v_k \psi$

9. $\exists v_k \exists v_j \varphi \Leftrightarrow \exists v_j \exists v_k \varphi$

10. $v_k \approx v_k \Leftrightarrow 1$

11. $v_k \approx v_j \Leftrightarrow \exists v_\ell (v_k \approx v_\ell \wedge v_\ell \approx v_j)$ for $k, j \neq \ell$

12. $(S^k_j \Delta \varphi \wedge S^k_j \neg \Delta \varphi) \Leftrightarrow 0$

13. $\Delta(v_k \approx v_j) \Leftrightarrow 1$

14. $\Delta \exists v_k (\varphi \vee \bot) \Leftrightarrow \exists v_k \Delta(\varphi \vee \bot)$

15. $\Delta(\exists v_k \varphi \wedge \bot) \Leftrightarrow \forall v_k \Delta(\varphi \wedge \bot)$

16. $\exists v_k (\varphi \vee \bot) \Leftrightarrow \exists v_k \varphi \vee \bot$

17. $R_i v_{j_0} \dots v_{j_{n_i - 1}} \Leftrightarrow S^{k_0}_{j_0} \dots S^{k_{n_i-1}}_{j_{n_i-1}} S^0_{k_0} \dots S^{n_i - 1}_{k_{n_i-1}} R_i v_0 \dots v_{n_i - 1}$ (for $k_0, \dots,$ k_{n_i-1} all different and all outside $\{j_0, \dots, j_{n_i-1}\} \cup \{0, \dots, n_i - 1\}$).

18. $\Delta \forall v_k \varphi \to \forall v_k \varphi \to \varphi$

Rules of inference

- Modus Ponens: From φ and $\varphi \to \psi$ infer ψ.
- Δ-rule: From φ infer $\Delta\varphi$.

- $\exists$-rule: From $\varphi \to \psi$ infer $\exists v_k \varphi \to \psi$ if $v_k \notin Fv(\psi)$.

Definition 17. Let φ be a formula and let Σ be a set of formulas. We say φ is derivable from Σ, notation $\Sigma \vdash \varphi$, if there is a finite sequence of formulas $\chi_0, \ldots, \chi_n$ such that $\varphi = \chi_n$ and every element of the sequence is either an axiom, an element of Σ, or the result of an application of one of the rules on previous elements of the sequence.

Theorem 18 (Soundness)**.** *If* $\Sigma \vdash \varphi$*, then* $\Sigma \models \varphi$*.*

Proof. By Theorem 1 every tautology is valid, so true in every structure. It is easy to check that the other axioms are valid as well. We will give proofs for the validity of the Axioms 1, 2, 3, 17 and 18: Let $\mathfrak{A}$ be an arbitrary structure with base A. Recall that a 4-CSA is a 4-CA. Let C_k be the usual cylindrification in a cylindric set algebra, let Q_k be its dual and let $\overline{X}$ be the set theoretic complement of X. For Axioms 1 and 2, note that $(\varphi^t)^{\mathfrak{A}}$ is of the form $(X, \overline{X})$. Now

$$(\varphi^t \to \exists v_k \varphi^t)^{\mathfrak{A}} = (\overline{X} \cup C_k X), X \cap Q_k \overline{X})$$

and since $X \subseteq C_k X$ and $Q_k \overline{X} \subseteq \overline{X}$, $(\varphi^t \to \exists v_k \varphi^t)^{\mathfrak{A}} = ({}^{\omega}A, \emptyset) = 1^{\mathfrak{A}}$. Next we have $(\Delta \exists v_k \varphi^t)^{\mathfrak{A}} = (X + \overline{X}, X \leftrightarrow \overline{X}) = ({}^{\omega}A, \emptyset) = 1^{\mathfrak{A}}$. In order to see that Axiom 3 is valid, note that by A4, A5 and A6 we have that

$$((\exists v_k \varphi)^t)^{\mathfrak{A}} = (\exists v_k \varphi^t)^{\mathfrak{A}} \quad \text{and} \quad ((\exists v_k \psi)^f)^{\mathfrak{A}} = (\forall v_k \psi^f)^{\mathfrak{A}}.$$

Axiom 17: First we compare the true parts of the interpretations. By repeated application of P28, the true part of the right-hand side is
$X := C_{k_0} \ldots C_{k_{n_i-1}}[D_{k_0 j_0} \cap \ldots \cap D_{k_{n_i-1} j_{n_i-1}} \cap$

$$C_0 \ldots C_{n_i-1}(D_{0k_0} \cap \ldots \cap D_{n_i-1 k_{n_i-1}} \cap \mathcal{T}((R_i v_0 \ldots v_{n_i-1})^{\mathfrak{A}})].$$

Now it's easy to see that for any sequence s we have $s \in X$ if and only if

$$s[s(j_0)/0, \ldots, s(j_{n_i-1})/n_i - 1] \in \mathcal{T}((R_i v_0 \ldots v_{n_i-1})^{\mathfrak{A}})$$

if and only if

$$(s(j_0), \ldots, s(j_{n_i-1})) \in P_i^T$$

if and only if

$$s \in \mathcal{T}((R_i v_{j_0} \ldots v_{j_{n_i-1}})^{\mathfrak{A}}).$$

We will now compare the false parts of the interpretations. The false part of the right-hand side is
$Y := Q_{k_0} \dots Q_{k_{n_i-1}}[\overline{D}_{k_0 j_0} \cup \dots \cup \overline{D}_{k_{n_i-1} j_{n_i-1}} \cup$

$$Q_0 \dots Q_{n_i-1}(\overline{D}_{0k_0} \cup \dots \cup \overline{D}_{n_i-1 k_{n_i-1}} \cup \mathcal{F}((R_i v_0 \dots v_{n_i-1})^{\mathfrak{A}})].$$

But now it's easy to see that Y is the complement of
$C_{k_0} \dots C_{k_{n_i-1}}[D_{k_0 j_0} \cap \dots \cap D_{k_{n_i-1} j_{n_i-1}} \cap$

$$C_0 \dots C_{n_i-1}(D_{0k_0} \cap \dots \cap D_{n_i-1 k_{n_i-1}} \cap \overline{\mathcal{F}((R_i v_0 \dots v_{n_i-1})^{\mathfrak{A}})}]$$

hence for any sequence s we have $s \notin Y$ if and only if

$$s[s(j_0)/0, \dots, s(j_{n_i-1})/n_i - 1] \in \overline{\mathcal{F}((R_i v_0 \dots v_{n_i-1})^{\mathfrak{A}})}$$

if and only if

$$(s(j_0), \dots, s(j_{n_i-1})) \notin P_i^F$$

if and only if

$$s \notin \mathcal{F}((R_i v_{j_0} \dots v_{j_{n_i-1}})^{\mathfrak{A}})$$

which completes the proof of the validity of Axiom 17. Axiom 18: Let $\varphi^{\mathfrak{A}} = (X, X')$ and let s be an arbitrary sequence. Recall that

$$\Delta\forall v_k\varphi \to \forall v_k\varphi \to \varphi = \neg\Delta\forall v_k\varphi \vee \neg\forall v_k\varphi \vee \varphi.$$

If $s \in Q_k X \leftrightarrow C_k X'$, $s \notin Q_k X + C_k X'$ and we are done. Otherwise either $s \in Q_k X$ or $s \in C_k X'$. If $s \in C_k X'$, then $s \notin Q_k X$ and we are done. Finally, if $s \in Q_k X$, then $s \in X$ and moreover, since $s \notin C_k X'$, $s \notin X'$ and we are done again.
We also have that $(\exists v_k\varphi \to \psi)^{\mathfrak{A}} = 1$ whenever $(\varphi \to \psi)^{\mathfrak{A}} = 1$ and $v_k \notin Fv(\psi)$. Finally, we already noted that whenever $(\varphi \to \psi)^{\mathfrak{A}} = \varphi^{\mathfrak{A}} = 1$, $\psi^{\mathfrak{A}} = 1$ and it is clear that $(\Delta\varphi)^{\mathfrak{A}} = 1$ whenever $\varphi^{\mathfrak{A}} = 1$. □

Note that the variables $v_{k_0}, \dots, v_{k_{n_i-1}}$ are really needed in Axiom 17, since, for example, the formula $R_i v_1 v_0 \Leftrightarrow \exists v_0(v_0 \approx v_1 \wedge \exists v_1(v_1 \approx v_0 \wedge R_i v_0 v_1))$ is not valid, since the right-hand side is equivalent to $R_i v_1 v_1$.

In order to prove completeness, there is some preliminary work to do. First we will derive some additional rules of inference:

Lemma 4. The following rules can be derived:

(i) If $\Sigma \vdash \forall v_k \varphi$, then $\Sigma \vdash \varphi$.

(ii) If $k \notin Fv(\psi)$ and $\Sigma \vdash \psi \to \varphi$, then $\Sigma \vdash \psi \to \forall v_k \varphi$.

(iii) If $\Sigma \vdash \varphi^t \to \psi^t$, then $\Sigma \vdash \exists v_k \varphi^t \to \exists v_k \psi^t$.

(iv) If $\Sigma \vdash \psi^f \to \varphi^f$, then $\Sigma \vdash \forall v_k \psi^f \to \forall v_k \varphi^f$.

(v) If $\Sigma \vdash \varphi \Rightarrow \psi$, then $\Sigma \vdash \exists v_k \varphi \Rightarrow \exists v_k \psi$.

(vi) If $\Sigma \vdash \varphi \Leftrightarrow \psi$, then $\Sigma \vdash \exists v_k \varphi \Leftrightarrow \exists v_k \psi$.

Proof. (i) If $\Sigma \vdash \forall v_k \varphi$, then $\Sigma \vdash \Delta \forall v_k \varphi$ by the Δ-rule. Now use axiom 18 and apply Modus Ponens twice.
(ii) By the Δ-rule $\Sigma \vdash \Delta(\psi \to \varphi)$. Now use the tautology $\Delta(\psi \to \varphi) \to (\psi \to \varphi) \to (\neg\varphi \to \neg\psi)$ and MP to obtain $\neg\varphi \to \neg\psi$ and use the $\exists$-rule to get $\exists v_k \neg\varphi \to \neg\psi$. Apply the Δ-rule to the latter and use MP and the tautology $\Delta(\exists v_k \neg\varphi \to \neg\psi) \to (\exists v_k \neg\varphi \to \neg\psi) \to (\psi \to \neg\exists v_k \neg\varphi)$. To see that this is indeed a tautology, note that in any valuation $\Delta(\exists v_k \neg\varphi \to \neg\psi)$ is assigned a classical truth value, that is 0 or 1. If it is 0, we are done. If it is 1, the formula $\exists v_k \neg\varphi \to \neg\psi$ is assigned a classical truth value and since $\psi \to \neg\exists v_k \neg\varphi$ is clearly equivalent to $\exists v_k \neg\varphi \to \neg\psi$, we must have that $(\exists v_k \neg\varphi \to \neg\psi) \to (\psi \to \neg\exists v_k \neg\varphi)$ is assigned 1.
(iii) Use the axioms $\psi^t \to \exists v_k \psi^t$ (axiom 1) and $\Delta \exists v_k \psi^t$ (axiom 2) and the tautology $\Delta \exists v_k \psi^t \to (\varphi^t \to \psi^t) \to (\psi^t \to \exists v_k \psi^t) \to (\varphi^t \to \exists v_k \psi^t)$ along with three times MP and then apply the $\exists$-rule to obtain the result.
(iv) Use the axioms $\forall v_k \psi^f \to \psi^f$ (axiom 4) and $\Delta \forall v_k \psi^f$ (axiom 5) and the tautology $\Delta \forall v_k \psi^f \to (\psi^f \to \varphi^f) \to (\forall v_k \psi^f \to \psi^f) \to (\forall v_k \psi^f \to \varphi^f)$ along with three times MP and apply part (ii) to obtain the result.
(v) Use the tautologies $(\varphi \Rightarrow \psi) \to (\varphi^t \to \psi^t)$ and $(\varphi \Rightarrow \psi) \to (\psi^f \to \varphi^f)$ and then use axiom 3 and part (iii) and (iv).
(vi) Put $A = \varphi \Rightarrow \psi$ and $B = \psi \Rightarrow \varphi$. If $\Sigma \vdash A \wedge B$, use the Δ-rule to get $\Delta(A \wedge B)$ and use the tautology $\Delta(A \wedge B) \to (A \wedge B) \to A$ and MP to get A. Similarly, we can get B. By part (v) we obtain $C = \exists v_k \varphi \Rightarrow \exists v_k \psi$ and $D = \exists v_k \psi \Rightarrow \exists v_k \varphi$. Apply the Δ-rule on C and D and use the tautology $\Delta C \to \Delta D \to C \to D \to (C \wedge D)$ to obtain the result. $\square$

Theorem 19 (Deduction Theorem)**.** *Let ψ be a formula, φ a sentence and Σ a set of formulas. If $\Sigma \cup \{\varphi\} \vdash \psi$ and $\Sigma \vdash \Delta\varphi$, then $\Sigma \vdash \varphi \to \psi$.*

Proof. Let $\psi_0, \ldots, \psi_n = \psi$ be a witness of $\Sigma \cup \{\varphi\} \vdash \psi$ and $\chi_0, \ldots, \chi_m = \Delta\varphi$ be a witness of $\Sigma \vdash \Delta\varphi$. We will show how to prove $\varphi \to \psi_i$ by induction on i. Since ψ_i can be proven from $\Sigma \cup \{\varphi\}$, we distinguish cases:
If ψ_i is a tautology, then so is $\varphi \to \psi_i$ and hence it can be proven in one step.
If $\psi_i = \varphi$, then use the tautology $\Delta\varphi \to \varphi \to \varphi$ and MP.
If $\psi_i \in \Sigma$, then by the Δ-rule $\Sigma \vdash \Delta\psi_i$. Now use the tautology $\Delta\varphi \to \Delta\psi_i \to \psi_i \to \varphi \to \psi_i$ and three times MP.
If ψ_i is the result of application of MP on $\psi_j = \psi_k \to \psi_i$ and ψ_k with $j, k < i$, then by induction we know that we can prove $\varphi \to \psi_k$ and $\varphi \to \psi_k \to \psi_i$. Now use $\Delta\varphi$, $\Delta(\varphi \to \psi_k)$ and $\Delta(\varphi \to \psi_k \to \psi_i)$ (by the Δ-rule) and the tautology $\Delta\varphi \to \Delta(\varphi \to \psi_k) \to \Delta(\varphi \to \psi_k \to \psi_i) \to (\varphi \to \psi_k) \to (\varphi \to \psi_k \to \psi_i) \to \varphi \to \psi_i$ along with MP five times to obtain $\varphi \to \psi_i$.
If $\psi_i = \Delta\psi_k$ for some $k < i$, then use $\Delta\varphi$, $\Delta(\varphi \to \psi_k)$ (induction hypothesis and Δ-rule) and the tautology $\Delta\varphi \to \Delta(\varphi \to \psi_k) \to (\varphi \to \Delta\psi_j)$.
If $\psi_i = \exists v_k \chi \to \theta$, $k \notin Fv(\theta)$, and $\chi \to \theta = \psi_j$ for some $j < i$, then first use $\Delta\varphi$, $\Delta(\varphi \to \chi \to \theta)$ (induction hypothesis and Δ-rule) and the tautology $\Delta\varphi \to \Delta(\varphi \to \chi \to \theta) \to (\varphi \to \chi \to \theta) \to (\chi \to \varphi \to \theta)$ to obtain $\chi \to \varphi \to \theta$. Since φ is a sentence, we can use the $\exists$-rule to obtain $\exists v_k \chi \to \varphi \to \theta$. Finally, use $\Delta\varphi$, $\Delta(\exists v_k \chi \to \varphi \to \theta)$ and the tautology $\Delta\varphi \to \Delta(\exists v_k \chi \to \varphi \to \theta) \to (\exists v_k \chi \to \varphi \to \theta) \to (\varphi \to \exists v_k \chi \to \theta)$ along with three times MP to obtain the result. □

Definition 18. A set of formulas Σ is consistent if $\Sigma \nvdash 0$.

Note that by the Δ-rule, if $\Sigma \vdash \top$, then $\Sigma \vdash \Delta\top$ and hence $\Sigma \vdash 0$ by the tautology $\Delta\top \to 0$. Likewise, Σ is inconsistent if it proves $\bot$.

Theorem 20. *If φ is a sentence, $\Sigma \vdash \Delta\varphi$ and $\Sigma \nvdash \varphi$, then $\Sigma \cup \{\neg\varphi\}$ is consistent.*

Proof. Suppose $\Sigma \cup \{\neg\varphi\} \vdash 0$. Since $\Sigma \vdash \Delta\varphi$ and since $\Delta\varphi \to \Delta\neg\varphi$ is a tautology, $\Sigma \vdash \Delta\neg\varphi$. By the Deduction Theorem $\Sigma \vdash \neg\varphi \to 0$. But $\Delta\neg\varphi \to (\neg\varphi \to 0) \to \varphi$ is a tautology, hence by two applications of MP, we have $\Sigma \vdash \varphi$ which contradicts our assumption. □

Definition 19. Let Σ be a set of formulas. The relation $\equiv_\Sigma$ is defined as follows:

$$\varphi \equiv_\Sigma \psi \text{ if and only if } \Sigma \vdash \varphi \Leftrightarrow \psi.$$

We will usually omit the subscript Σ since we only consider one Σ at the time. Let Fm be the set of $\mathcal{L}$-formulas for some vocabulary $\mathcal{L}$ and let $\mathfrak{Fm} = (Fm, \vee, \wedge, \neg, 1, 0, \top, \bot, \exists v_k, \Delta, v_k \approx v_j)_{k,j<\omega}$ be the algebra of formulas, that is the formula $\varphi \vee \psi$ is the result of applying the operation $\vee$ to φ and ψ and likewise for the operations $\wedge$, $\neg$, $\exists v_k$ and Δ. The formulas 1, 0, $\top$, $\bot$ and $v_k \approx v_j$ $(k, j < \omega)$ are the constants in this algebra.

Theorem 21. *For each set of formulas Σ, the relation $\equiv_\Sigma$ is a congruence relation on the algebra $\mathfrak{Fm}$.*

Proof. First we have to show that $\equiv$ is an equivalence relation. It is easy to check that the formulas

$$\varphi \Leftrightarrow \varphi;$$

$$(\varphi \Leftrightarrow \psi) \to (\psi \Leftrightarrow \varphi);$$

$$(\varphi \Leftrightarrow \psi) \to (\psi \Leftrightarrow \chi) \to (\varphi \Leftrightarrow \chi)$$

are tautologies, hence by MP we obtain reflexivity, symmetry and transitivity. We let $[\varphi]$ denote the equivalence class of φ.
We now have to show that $\equiv$ is a congruence. Suppose $\varphi \equiv \psi$ and $\varphi_1 \equiv \psi_1$. Now it is easy to see that the following formulas are tautologies:

$$(\varphi \Leftrightarrow \psi) \to (\varphi_1 \Leftrightarrow \psi_1) \to (\varphi \vee \varphi_1 \Leftrightarrow \psi \vee \psi_1);$$

$$(\varphi \Leftrightarrow \psi) \to (\varphi_1 \Leftrightarrow \psi_1) \to (\varphi \wedge \varphi_1 \Leftrightarrow \psi \wedge \psi_1);$$

$$(\varphi \Leftrightarrow \psi) \to (\neg\varphi \Leftrightarrow \neg\psi);$$

$$(\varphi \Leftrightarrow \psi) \to (\Delta\varphi \Leftrightarrow \Delta\psi).$$

The derivability of the equivalences $\varphi \vee \varphi_1 \Leftrightarrow \psi \vee \psi_1$, $\varphi \wedge \varphi_1 \Leftrightarrow \psi \wedge \psi_1$, $\neg\varphi \Leftrightarrow \neg\psi$ and $\Delta\varphi \Leftrightarrow \Delta\psi$ follows from these tautologies. Finally, by Lemma 4(vi), we get that $[\exists v_k \varphi] = [\exists v_k \psi]$ hence $\equiv$ is a congruence. $\square$

Definition 20. For Σ a set of formulas, $\mathfrak{Fm}_\Sigma = \mathfrak{Fm}/\equiv_\Sigma$.

Theorem 22. *If Σ is consistent, then $\mathfrak{Fm}$ has at least two elements.*

Proof. Suppose $[1] = [0]$, that is $\Sigma \vdash 1 \Leftrightarrow 0$. Then since $(1 \Leftrightarrow 0) \rightarrow 0$ is a tautology, $\Sigma \vdash 0$, hence Σ is inconsistent. $\square$

Theorem 23. *For all $\varphi \in \Sigma$, $[\varphi] = [1]$.*

Proof. If $\varphi \in \Sigma$, then $\Sigma \vdash \varphi$ and hence by the Δ-rule, $\Sigma \vdash \Delta\varphi$. Now use the tautology $\Delta\varphi \rightarrow \varphi \rightarrow (\varphi \Leftrightarrow 1)$ along with two times MP. $\square$

Theorem 24. *$\mathfrak{Fm}_\Sigma$ is a locally finite 4-CAω.*

Proof. We need to show that $\mathfrak{Fm}_\Sigma$ satisfies all the axioms that define a 4-CAω. The axioms BA1-BA8 as well as the additional axioms A1, A2 and A7-A13 follow from appropriate tautologies. Take for example A7: Suppose $[\Delta\varphi] = [\Delta 1]$, that is $\Sigma \vdash \Delta\varphi \Leftrightarrow \Delta 1$. Then by MP and the tautology $(\Delta\varphi \Leftrightarrow \Delta 1) \rightarrow (\varphi \Leftrightarrow \Delta(\varphi \vee \bot))$ we obtain $\Sigma \vdash \varphi \Leftrightarrow \Delta(\varphi \vee \bot)$, hence $[\varphi] = [\Delta(\varphi \vee \bot)]$. For $i = 1, \ldots, 7$, Ci follows from Axiom $i + 5$ and additional axiom Ai follows from Axiom $i + 10$ for $i \in \{3, 4, 5, 6\}$. Clearly, by Axiom 6, $\mathfrak{Fm}_\Sigma$ is locally finite. This completes the proof. $\square$

Theorem 25 (Model Existence Theorem)**.** *If Σ is a consistent set of formulas, it has a model.*

Proof. Suppose Σ is consistent. By Theorem 24, $\mathfrak{Fm}_\Sigma$ is a locally finite 4-CAω, hence by the Representation Theorem (Theorem 17) there is an an embedding $\iota : \mathfrak{Fm}_\Sigma \hookrightarrow \prod_{j \in J} \mathcal{C}_j$ where each $\mathcal{C}_j$ is a regular 4-CSAω. By Theorem 22, $\mathfrak{Fm}_\Sigma$ has at least two elements, hence the product $\prod_{j \in J} \mathcal{C}_j$ has at least two elements and hence there must be a $j \in J$ such that $\mathcal{C}_j$ has at least two elements. Now put $h = p_j \circ \iota$ where p_j is the projection map from the product onto $\mathcal{C}_j$. Let C_j have base A. For an n_i-ary relation symbol R_i we define the four-valued n_i-ary relation P_i as follows: For $a_0, \ldots, a_{n_i-1} \in A$, $(a_0, \ldots, a_{n_i-1}) \in P_i^T$ if and only if there is an $s \in \mathcal{T}(h([R_i v_0 \ldots v_{n_i-1}]))$ such that $s \restriction n_i = (a_0, \ldots, a_{n_i-1})$. Note that since $dim(h([R_i v_0 \ldots v_{n_i-1}])) \subseteq \{0, \ldots, n_i - 1\}$ and $\mathcal{C}_j$ is regular, there is such an s if and only if for all s with $s \restriction n_i = (a_0, \ldots, a_{n_i-1})$ we have that $s \in \mathcal{T}(h([R_i v_0 \ldots v_{n_i-1}]))$. Similarly we define $(a_0, \ldots, a_{n_i-1}) \in P_i^F$ if and only if there is an $s \in \mathcal{F}(h([R_i v_0 \ldots v_{n_i-1}]))$ such that $s \restriction n_i = (a_0, \ldots, a_{n_i-1})$. Now let $\mathfrak{A} = (A, P_i)_{i \in I}$. We will show that $\mathfrak{A}$ is a model of Σ. First we claim that

$$(R_i v_{j_0} \ldots v_{j_{n_i}-1})^{\mathfrak{A}} = h([R_i v_{j_0} \ldots v_{j_{n_i}-1}]).$$

Proof of the claim: By definition of P_i, $s \in \mathcal{T}((R_i v_{j_0} \dots v_{j_{n_i}-1})^{\mathfrak{A}})$ if and only if $(s(j_0), \dots, s(j_{n_i-1})) \in P_i^T$ if and only if there is a $t \in {}^\omega A$ such that

$$t \upharpoonright n_i = (s(j_0), \dots, s(j_{n_i-1})) \text{ and } t \in \mathcal{T}(h([R_i v_0 \dots v_{n_i-1}]).$$

Since h is a homomorphism, by Axiom 17 we have that
$h([R_i v_{j_0} \dots v_{j_{n_i}-1}]) = C_{k_0} \dots C_{k_{n_i-1}} [D_{k_0 j_0} \wedge \dots \wedge D_{k_{n_i-1} j_{n_i-1}} \wedge$

$$C_0 \dots C_{n_i-1} (D_{0k_0} \wedge \dots \wedge D_{n_i-1 k_{n_i-1}} \wedge h([R_i v_0 \dots v_{n_i-1}]))]$$

so $s \in \mathcal{T}(h([R_i v_{j_0} \dots v_{j_{n_i}-1}]))$ if and only if

$$s[s(j_0)/0, \dots, s(j_{n_i-1})/n_i - 1] \in \mathcal{T}(h([R_i v_0 \dots v_{n_i-1}])).$$

But since $h([R_i v_0 \dots v_{n_i-1}])$ depends on $\{0, \dots, n_i - 1\}$, this yields that $s \in \mathcal{T}(h([R_i v_{j_0} \dots v_{j_{n_i}-1}]))$ if and only if there is a $t \in {}^\omega A$ such that

$$t \upharpoonright n_i = (s(j_o), \dots, s(j_{n_i-1})) \text{ and } t \in \mathcal{T}(h([R_i v_0 \dots v_{n_i-1}]))$$

hence

$$\mathcal{T}((R_i v_{j_0} \dots v_{j_{n_i}-1})^{\mathfrak{A}}) = \mathcal{T}(h([R_i v_{j_0} \dots v_{j_{n_i}-1}])).$$

One can give a similar proof for the false parts, which proves the claim. Now since h is a homomorphism, it is easy to prove by induction that $\varphi^{\mathfrak{A}} = h([\varphi])$ for all formulas φ. If $\varphi \in \Sigma$, then $[\varphi] = [1]$ by Theorem 23, hence $\varphi^{\mathfrak{A}} = h([\varphi]) = h([1]) = 1$ since h is a homomorphism, hence $\mathfrak{A}$ is a model for Σ. $\square$

Theorem 26. *If φ is a sentence and Σ is a set of formulas such that $\Sigma \models \varphi$, then $\Sigma \vdash \Delta\varphi$.*

Proof. Suppose it is not the case that $\Sigma \vdash \Delta\varphi$. Since $\Delta\Delta\varphi$ is a tautology, $\Sigma \vdash \Delta\Delta\varphi$. By Theorem 20, $\Sigma \cup \{\neg\Delta\varphi\}$ is consistent and hence it has a model $\mathfrak{A}$. But since $\Sigma \models \varphi$, we must also have $\Sigma \models \Delta\varphi$, hence we have a contradiction. $\square$

Theorem 27. *If φ is a sentence and $\Sigma \models \varphi$, then $\Sigma \vdash \varphi$.*

Proof. By Theorem 26, $\Sigma \vdash \Delta\varphi$. Now suppose $\Sigma \nvdash \varphi$. By Theorem 20, $\Sigma \cup \{\neg\varphi\}$ is consistent and hence it has a model, contradicting $\Sigma \models \varphi$. $\square$

Theorem 28. *Let φ be a formula and Σ a set of formulas. If $\Sigma \models \varphi$, then $\Sigma \vdash \varphi$.*

Proof. Let $Fv(\varphi) = \{v_{k_0}, \ldots, v_{k_{n-1}}\}$. Now since $\Sigma \models \varphi$, it is obvious that $\Sigma \models \forall v_0 \ldots \forall v_{n-1}\varphi$, and since $\forall v_0 \ldots \forall v_{n-1}\varphi$ is a sentence, by Theorem 27, $\Sigma \vdash \forall v_0 \ldots \forall v_{n-1}\varphi$. By multiple applications of Lemma 4(i), we obtain $\Sigma \vdash \varphi$. $\square$

Chapter 6

3-CAαs

In this section we are going to investigate the relationship between 4-CAαs and Feldman's 3-CAαs in [Fel98].

Definition 21. An algebra $\mathfrak{A} = (A, \vee, \wedge, \neg, 1, 0, \perp, \delta, c_\kappa, d_{\kappa\lambda})_{\kappa,\lambda<\alpha}$, with α some ordinal, is called a 3-CAα if and only if it satisfies all of the axioms of a 4-CAα except that instead of A8 and A10, it satisfies

$$\begin{array}{ll} \text{A8}' & a \vee \neg a \vee \perp = a \vee \neg a; \\ \text{A10}' & \delta(a \vee \perp) \wedge a = \delta(a \vee \perp). \end{array}$$

Definition 22. Given a 4-CAα $\mathfrak{A}$ and an element $a \in \mathfrak{A}$, we call a *consistent* if $a \wedge \neg a \leq \perp$.

Note that in a 4-CAα, $a \wedge \neg a \leq \perp$ is equivalent to A8$'$.

Theorem 29. *Let $\mathfrak{A}$ be a 4-CAα. The set of consistent elements of $\mathfrak{A}$ contains all of the constants except for $\top$ and it is closed under all the operations.*

Proof. The claim about the constants is easily verified. In particular, note that $d_{\kappa\lambda} = \delta(d_{\kappa\lambda} \vee \perp)$. Now suppose a and b are consistent, that is

$$a \wedge \neg a \leq \perp \qquad \text{and} \qquad b \wedge \neg b \leq \perp.$$

Then

$$(a \vee b) \wedge \neg(a \vee b) = (a \vee b) \wedge \neg a \wedge \neg b \leq (a \wedge \neg a) \vee (b \wedge \neg b) \leq \perp.$$

The case for $\wedge$ is similar. The cases for negation and δ are trivial. To see that the set of consistent elements is closed under cylindrification, by theorem 6 recall that $\mathfrak{A}$ is of the form $\mathbb{T}(\mathcal{C})$ for some cylindric algebra $\mathcal{C}$. Let $a = (a_1, a_2)$. Now a being consistent means that in the cylindric algebra $\mathcal{C}$ we have $a_1 \wedge a_2 = 0$, hence $c_\kappa(a_1 \wedge a_2) = 0$. Now note that in cylindric algebras we have $c_\kappa x \wedge \neg c_\kappa y \leq c_\kappa(x \wedge \neg y)$ (see [HMT1] on p. 176). Substitute a_1 for x and $\neg a_2$ for y in order to obtain $c_\kappa a_1 \wedge q_\kappa a_2 \leq c_\kappa(a_1 \wedge a_2) = 0$. But this means that in $\mathfrak{A}$ we have that $c_\kappa a \wedge \neg c_\kappa a \leq \perp$. □

Definition 23. Let $\mathfrak{A}$ be a 4-CAα and let K be the set of all consistent elements of $\mathfrak{A}$. We call the algebra $\mathfrak{K}(\mathfrak{A}) = (K, \vee, \wedge, \neg, 1, 0, \perp, \delta, d_{\kappa\lambda}, c_\kappa)_{\kappa,\lambda<\alpha}$ the *consistent part of* $\mathfrak{A}$.

Theorem 30. *The consistent part of a 4-CAα is a 3-CAα.*

Proof. Let $\mathfrak{A}$ be a 4-CAα. That $\mathfrak{K}(\mathfrak{A})$ satisfies all the axioms of 3-CAαs is trivial except for the axiom $\delta(a \vee \perp) \leq a$. To see that it is satisfied, note that since a is consistent we have $0 = \delta\perp = \delta((a \wedge \neg a) \vee \perp) = \delta(a \vee \perp) \wedge \delta(\neg a \vee \perp)$, hence

$$\delta(a \vee \perp) \leq \neg\delta(\neg a \vee \perp) = \delta(a \vee \top) \leq a \vee \top$$

by Lemma 3(ii). By Lemma 3(i) we have $\delta(a \vee \perp) \leq a \vee \perp$ hence

$$\delta(a \vee \perp) \leq (a \vee \perp) \wedge (a \vee \top) = a.$$

□

Theorem 31. *Every 3-CAα is isomorphic to the consistent part of a 4-CAα.*

Proof. Let $\mathfrak{A} = (A, \vee, \wedge, \neg, 1, 0, \perp, \delta, d_{\kappa\lambda}, c_\kappa)_{\kappa,\lambda<\alpha}$ be a 3-CAα. Just as we did with the 4-CAα in the proof of theorem 6, we take the cylindric algebra $\mathcal{C} = (\{\delta a \,|\, a \in A\}, \vee, \wedge, \neg, 1, 0, d_{\kappa\lambda}, c_\kappa)_{\kappa,\lambda<\alpha}$ and embed $\mathfrak{A}$ in $\mathbb{T}(\mathcal{C})$ by

$$\theta : a \mapsto (\delta(a \vee \perp), \delta(\neg a \vee \perp))$$

To see that $\mathcal{C}$ is a CAα is trivial again and the proof for the fact that θ is a homomorphism is exactly the same as the one we already gave for the case of a 4-CAα, except for δ. Again, since $\theta(\delta a) = (\delta a, \neg\delta a)$, it suffices to show that

$\delta(a \vee \perp) + \delta(\neg a \vee \perp) = \delta a$: We have $\delta(a \vee \perp) + \delta(\neg a \vee \perp)$

$$\begin{aligned}
&= (\delta(a \vee \perp) \wedge \neg\delta(\neg a \vee \perp)) \vee (\delta(\neg a \vee \perp) \wedge \neg\delta(a \vee \perp)) \\
&= (\delta(a \vee \perp) \vee \delta(\neg a \vee \perp)) \wedge (\delta(a \vee \perp) \vee \neg\delta(a \vee \perp)) \\
&\quad \wedge (\neg\delta(\neg a \vee \perp) \vee \delta(\neg a \vee \perp)) \wedge (\neg\delta(\neg a \vee \perp) \vee \neg\delta(a \vee \perp)).
\end{aligned}$$

The second and the third conjuncts clearly equal 1. Since $a \vee \neg a \vee \perp = a \vee \neg a$, the first conjunct equals $\delta(a \vee \neg a \vee \perp) = \delta(a \vee \neg a) = \delta a$. Likewise, since $(a \wedge \neg a) \vee \perp = \perp$, the last conjunct equals $\neg\delta\perp = 1$, hence we have shown that θ is homomorphism.

Injectivity is proved in exactly the same way as we did in the decomposition of a 4-CAα. We will now show that the image of $\mathfrak{A}$ under θ is exactly the consistent part of $\mathbb{T}(\mathcal{C})$. Remember that an element of $\mathbb{T}(\mathcal{C})$ is consistent if and only if the conjunction of its components equals 0 in $\mathcal{C}$. Let $a \in \mathfrak{A}$. Then $\theta(a) = (\delta(a \vee \perp), \delta(\neg a \vee \perp))$. But

$$\delta(a \vee \perp) \wedge \delta(\neg a \vee \perp) = \delta((a \wedge \neg a) \vee \perp) = \delta\perp = 0$$

hence $\theta(a)$ is a consistent element of $\mathbb{T}(\mathcal{C})$. On the other hand, let $(\delta a, \delta b)$ be an arbitrary consistent element of $\mathbb{T}(\mathcal{C})$, i.e. in $\mathcal{C}$ we have $\delta a \wedge \delta b = 0$. We have to show that $(\delta a, \delta b)$ is in the image of θ. In $\mathfrak{A}$, let $c := (\delta a \vee \perp) \wedge \neg\delta b$. Then

$$\begin{aligned}
\delta(c \vee \perp) &= \delta(((\delta a \vee \perp) \wedge \neg\delta b) \vee \perp) \\
&= \delta(\delta a \vee \perp) \wedge \delta(\neg\delta b \vee \perp) \\
&= \delta a \wedge \neg\delta b \\
&= (\delta a \wedge \neg\delta b) \vee (\delta a \wedge \delta b) \\
&= \delta a \wedge (\neg\delta b \vee \delta b) \\
&= \delta a
\end{aligned}$$

and

$$\delta(\neg c \vee \perp) = \delta(\delta b \vee (\neg\delta a \wedge \perp) \vee \perp) = \delta(\delta b \vee \perp) = \delta b$$

hence $\theta(c) = (\delta a, \delta b)$ hence $\mathfrak{A} \cong \mathfrak{K}(\mathbb{T}(\mathcal{C}))$. $\square$

Bibliography

[HMT1] L. Henkin, J.D. Monk, A. Tarski, Cylindric Algebras Part I, Volume 64 of *Studies in Logic and the Foundations of Mathematics*, North-Holland Publishing Co., Amsterdam, 1971.

[HMT2] L. Henkin, J.D. Monk, A. Tarski, Cylindric Algebras Part II, Volume 115 of *Studies in Logic and the Foundations of Mathematics*, Elsevier Science Publishers B.V., Amsterdam, 1985.

[Fel98] Norman Feldman, The Cylindric Algebra of Three-Valued Logic, *The Journal of Symbolic Logic*, vol. 63, no. 4 (Dec., 1998), pp. 1201-1217.

[Kle52] S.C. Kleene, Introduction to metamathematics, North-Holland Publishing Co., Amsterdam, 1952.

[Bel77] N.P. Belnap, A useful four-valued logic, *Modern Uses of Multiple-Valued Logic*, J.M. Dunn and G. Epstein, Eds. D. Reidel, 1977.

[Fit94] M.C. Fitting, Kleene's three-valued logic and their children, *Fundamenta Informaticae*, vol. 20 (Nov., 1994), pp. 113-131.

[Fit94] M.C. Fitting, Bilattices are nice things, *Self-reference*, T. Bolander, V. Hendricks and S.A. Pedersen eds., 2006

[Bol92] L. Bolc, P. Borowik, Many-Valued Logics (1. Theoretical foundations), Springer-Verlag, Berlin, 1992

[Gin88] M.L. Ginsberg, Multivalued logics: a uniform approach to reasoning in AI, *Computer Intelligence*, vol. 4, 1988, pp. 256-316.

[Ari96] O. Arieli, A. Avron, Reasoning with logical bilattices, *Journal of Logic, Language and Information*, vol. 5, 1996, pp. 25-63.

[Sto39] M.H. Stone, The theory of representations for Boolean algebras, *Transactions of the American Mathematical Society*, vol. 40, no. 1 (July 1936), pp. 37-111.

[Bal74] R. Balbes, P. Dwinger, Distributive lattices, University of Missouri Press, Columbia (Missouri), 1974.