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Frege, Russell and Wittgenstein on the Judgment Stroke

MSc Thesis (*Afstudeerscriptie*)

written by

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Abstract

Frege is highly valued as a logician by Russell and Wittgenstein, the latter nonetheless concludes in his *Tractatus* that one of Frege's central notions, the judgment stroke, is "logically quite meaningless". In order to see why Wittgenstein thinks so, we will investigate the 'indirect interpretation thesis', which says that Wittgenstein's interpretation of Frege was strongly influenced by the reading Russell gives of the Begriffsschrift in *Principia Mathematica* and *Principles of Mathematics*. This is done by analyzing the different conceptions of logic, focusing on the representations of judgment and assertion in Frege, Russell and the early Wittgenstein. Stong similarities can be found between the interpretations of Russell and Wittgenstein, this makes the indirect interpretation thesis plausible, although Russell's influence cannot be the only reason why Wittgenstein rejected the judgment stroke as a logical symbol.
Keywords: Begriffsschrift, Frege, indirect interpretation, judgment stroke, Russell, Wittgenstein, Tractatus.

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Contents

Chapter 1

Introduction

Three logicians are discussed in this thesis: Gottlob Frege, Bertrand Russell, and Ludwig Wittgenstein. The goal is to investigate the representation of Frege's ideas in the works of the other two. Both Russell and Wittgenstein admired Frege for his logical analysis and express their gratitude towards him in the prefaces of their major works.[1] Russell and Whitehead say that "[i]n all questions of logical analysis, our chief debt is to Frege"[2] and Wittgenstein owes to him the "stimulation of [his] thoughts."[3] Despite the admiration, a central notion of Frege's Begriffsschrift is considered problematic by both Russell and Wittgenstein; this is the 'judgment stroke'. Why is this vertical stroke, representing the act of judgment, so important to Frege? And what are the opinions of Russell and Wittgenstein about this? These questions will be considered in order to formulate an answer to the main question of this thesis: Why does Wittgenstein, despite his admiration of Frege, say in sentence 4.442 of the *Tractatus* that 'the judgment stroke is logically altogether quite meaningless"?

The hypothesis that will be investigated is that Russell's interpretation of the Begriffsschrift (Frege's two-dimensional concept language of pure thought) has influenced Wittgenstein's understanding of it. G.E.M. Anscombe was one of the first to suggest that Wittgenstein might have been following Russell's interpretation of Frege, rather than formulating his own.[4]

[1] These are *Principles of Mathematics* and *Principia Mathematica* by Russell and Wittgenstein's *Tractatus*. Solely the works of the early Wittgenstein are considered here, since these can be placed in the tradition of mathematical logic, of which Frege is considered to be one of the founding fathers.

[2] *Principia Mathematica*, p. ix

[3] *Tractatus*, preface

[4] Anscombe (1959), p. 104

This is based on the similarities in terminology in Russell and Wittgenstein, and on the fact that both (falsely) attribute to Frege the idea that the symbol '⊦' can be used to predicate the truth or falsity of a sentence.[5] These arguments will be discussed in more detail later on. The view that Wittgenstein's interpretation of Frege was largely based on the works of Bertrand Russell will be labelled the '*indirect interpretation thesis*'. Investigating the indirect interpretation thesis is of interest not only to be able to answer the question whether Wittgenstein's rejection of the judgment stroke has anything to do with Russell's intrepretation of it, but in doing so the transition from nineteenth century mathematical logic to twentieth century logic is illustrated. It will be explained in which aspects these conceptions of logic differ, and what has changed in the years between 1879, when Frege's *Begriffsschrift* was published, and 1921, when the *Tractatus* was released. While searching for arguments that support or reject the indirect interpretation thesis, it will be considered what Frege, Russell and Wittgenstein consider to be the goal of logic, its subject-matter and what one is *doing* while practicing logic.

Let's start with a historical note, before proceeding to the plan of action of this thesis. Frege tried to give a foundation for the theorems of mathematics. In his *Begriffsschrift*[6] he intended to provide a set of axioms from which all mathematical theorem could be derived via gapfree proofs, in order to leave nothing merely assumed in mathematics. He needed a more precise means of expression than natural language, for this purpose Frege invented the Begriffsschrift: his two-dimensional "formula language of pure thought."[7,8] Neither Frege's philosophical ideas, nor his notation prevailed among logicians in the 19th century. In 1903 in Cambridge, however, Russell had read some of Frege's books while writing his *Principles of Mathematics*. He was impressed by Frege's ideas, and added an Appendix to this book, in which he discussed Frege's analysis of 'assertion' (as Russell rebaptized Frege's concept of 'judging'). In a later attempt to give the principles of mathematics, *Principia Mathematica*, a joint effort with Alfred Whitehead, Russell even decided to add Frege's symbol for 'judgment', or, as he calls it, 'assertion', to his logical language. The symbol '⊢' is nowadays still part of the logical language (more precisely: of the metalanguage), and it expresses

[5] In Frege the '⊦' can only be used to acknowledge the *truth* of a statement, not its falsity.

[6] Frege, (1879)

[7] Subtitle of *Begriffsschrift*.

[8] In reference to the book published in 1879 I will use italics (*'Begriffsschrift'*), and when referring to Frege's concept script 'Begriffsschrift' will be used.

that a sentence can be derived syntactically from another sentence or a set of sentences. This use of the symbol will not be discussed here, the latest version of the symbol to be considered is the appearance in Wittgenstein's *Tractatus*.

Now, the starting point of this thesis is Frege's judgment stroke. In *Begriffsschrift* as well as in his later works, Frege stresses the importance of this symbol in his concept script. To understand what the judgment stroke means to him, and why it is such a crucial notion, a major part of this thesis will cover Frege. To the analysis of his Begriffsschrift, the first half of the thesis is devoted: what is the judgment stroke according to Frege, and why does he introduce it? This is discussed elaborately, because what Frege calls 'logic' is everything that is relevant for inferences, which appears to be quite distinct from what most logicians nowadays consider to be logic. To begin chapter 2, Frege's definitions of the judgment stroke are discussed and, to get a full grasp of its meaning, the judgment stroke is related to Frege's famous concepts '*Sinn*' and '*Bedeutung*', as well as to his conception of logic and what Frege thinks the *practice* of logicians (and mathematicians) entails. Concerning this practice, I highlight how making a judgment can be conceived of as an *act*, and how the judgment stroke brings about this act, rather than just representing a process that took place in the mind of the subject. This is an interesting property of the judgment stroke, and a related question about what a notation can *do*, will be considered in chapter 3.

The second part of chapter 2 is about the question *why* Frege has a judgment stroke. This is considered independently of the indirect interpretation thesis, and is also meant to provide a better understanding of the background of the Begriffsschrift and the role of the judgment stroke in it. First, I consider the relation between Frege's logicist goals and the judgment stroke. Did he come up with the judgment stroke, because he considered the *practice* of mathematicians? Apart from that, the logicist programme of giving the foundations of mathematics in logic is something both Frege and Russell participate in, but Wittgenstein considers this nonsense; might this explain why Wittgenstein fiercely rejects the judgment stroke, and Russell is still looking for a 'logical notion of assertion' (to be represented by the judgment stroke)? This is followed by a discussion of the Kantian aspects of important themes in Frege, such as judgments and the normativity of logic. The relation between logic, thinking, and the normativity of logic is also treated by the early Wittgenstein. Placing Frege and his logic in a broader context is also the topic of the final section of chapter 2, where Frege is compared to other logicians whose goal it was to axiomatize mathematics,

as well as to their twentieth century descendants.

Frege takes an interesting place in the history of logic. His notation did not become widely adopted, but he is nonetheless considered to be an important logician, which might be because of Russell's discussion of his ideas in *Principia Mathematica*. Russell's discussion of 'assertion' is the topic of the third chapter. Assertion is the term that is used to refer to what Frege called 'judging', and the judgment stroke is conceived of as part of the 'assertion sign' $\vdash$, which is Russell's transcription of $\vdash$. Russell's interpretation of Frege will be discussed chronologically and this is complemented by a section on the different notations Frege and Russell use.

Those aspects of Russel's representation of Frege that may provide evidence for the indirect interpretation thesis will be evaluated in chapter 4, after discussing the logic in *Tractatus* and Wittgenstein's comments on the judgment stroke. Which arguments can be found against the judgment stroke as a logical symbol? And what reasons could be identified that support the idea that Wittgenstein's interpretation was based on Russell's representation of Frege rather than on Frege's own works? Or might there be other reasons to explain Wittgenstein's rejection of the judgement stroke?

Altogether this will lead to the conclusion that there are very convincing similarities between the interpretations of Frege as they are presented by Russell and Wittgenstein. Both use the same terminology, and consider the judgment stroke to be part of the simple sign $\vdash$, which can be used to indicate that a sentence 'is false' as well as that an author holds it to be true. This supports the indirect interpretation thesis. The question that remains is whether this is the only way to explain these similarities. Wouldn't that be an underestimation of Wittgenstein? The rejection might also be explained by the differences between Frege's epistemologic conception of logic on the one hand, and Wittgenstein's subject-free logic on the other. In the latter, there is no place for acts of judgment and coming to know something, this is similar to Russell's ideas about logic. There are many differences between Russell and Wittgenstein, but both are very cautious about letting a subject play a role in logic. The similarities between Russell and Wittgenstein regarding their interpretation of Frege are striking, which indicates that the indirect interpretation thesis isn't that implausible at all.

Chapter 2

Frege and the Judgment Stroke

2.1 What is the Judgment Stroke?

2.1.1 Three introductions of the Judgment Stroke

The Judgment Stroke appears in the majority of Frege's logical works, it was first introduced in *Begriffsschrift* in 1879 and since then used in many derivations proving mathematical theorems. During his life Frege refined and changed his opinion on several issues. A famous example is the *Sinn* and *Bedeutung* distinction he explicated in the paper named hereafter. This disctinction and the idea to consider concepts as functions mark the transition from the early to the mature Frege.[1] Following Nicholas J.J. Smith we will look at three introductions of the judgment stroke in order to discuss what Frege meant and whether his conception of the judgment stroke evolved.

Frege introduces the judgment stroke in two different ways, the first version appears in *Begriffsschrift* and a later, altered definition can be found in the 1891 paper *Funktion und Begriff* and in *Grundgesetze der Arithmetik*. In *Begriffsschrift* he writes:

> I 1:
> A judgement will always be expressed by means of the sign
>
> ⊢

[1] Terminology is Macbeth's, Macbeth (2005)

which stands to the left of the sign, or combination of signs, indicating the content of the judgement. If we *omit* the small vertical stroke at the left end of the horizontal one, the judgement will be transformed into a *mere combination of ideas* [eine blosse Vorstellungsverbindung], of which the writer does not state whether he acknowledges it to be true or not. For example, let

$$\vdash A$$

stand for the judgement "Opposite magnetic poles attract each other"; then

$$-A$$

will not express this judgement; it is to produce in the reader merely the idea of the mutual attraction of opposite magnetic poles, say in order to derive consequences from it and to test by means of these whether the thought is correct. When the vertical stroke is omitted, we express ourselves paraphrastically, using the words "the circumstance that" or "the proposition that" ... *The horizontal stroke* that is part of the sign $\vdash$ *combines the signs that follow it into a totality, and the affirmation expressed by the vertical stroke at the left end of the horizontal one refers to this totality.* Let us call the horizontal stroke the *content stroke* and the vertical stroke the *judgement stroke.*[2]

Introductions of the judgment stroke that represent a more mature perspective can be found in *Funktion und Begriff* (I2) and *Grundgesetze* (I3). Both will be given here.

I 2:
If we write down all equation or inequality, e.g. $5 < 4$, we ordinarily wish at the same time to express a judgement; in our example, we want to assert that 5 is greater than 4. According to the view I am here presenting, '$5 < 4$' and '$1 + 3 = 5$', just give us expressions for truth-values, without making any assertion. This separation of the act from the subject matter of judgement seems to be indispensable; for otherwise we could not express a mere supposition [blosse Annahme] - the putting of a case without a simultaneous judgement as to its arising or not. We thus need a special sign in order to be able to assert something. To

[2] *Begriffsschrift*, p. 1-2

this end I make use of a vertical stroke at the left end of the horizontal, so that, e.g., by writing

$$\vdash 2 + 3 = 5$$

we assert that $2 + 3$ equals 5. Thus here we are not just writing down a truth-value, as in

$$2 + 3 = 5,$$

but also at the same time saying that it is the True.[3]

I 3:
We have already said that in a mere equation there is as yet no assertion; '$2 + 3 = 5$' only designates [bezeichnet] a truth-value, without its being said which of the two it is. Also when I write '$(2 + 3 = 5) = (2 = 2)$' and, presupposed that one knows that '$2 = 2$' is the True, then I would not yet have asserted that the sum of 2 and 3 equals 5, instead I have only designated its truth-value; that '$2 + 3 = 5$' means [bedeute] the same as '$2 = 2$'. We therefore require another special sign to be able to assert something as true. For this purpose I let the sign '$\vdash$' precede the name of the truth-value, so that for example in

$$\text{'}\vdash 2^2 = 4'$$

it is asserted that the square of 2 is 4. [...] I regard this '$\vdash$' as composed of the vertical line, which I call the *judgement-stroke*, and the horizontal line, which I will now simply call the *horizontal* [...] Of the two signs of which '$\vdash$' is composed, only the judgement-stroke contains the act of assertion.[4]

In the next sections these three definitions will be compared on several aspects. First it will be discussed whether $\vdash$ is an act or a sign, then the corresponding conception of logic will be explained and the final subsection will be about a development in Frege's works regarding the content of the judgment; to answer the question why the mature Frege preferred *horizontal* over *content stroke*.

[3] *Funktion und Begriff* in: Beaney (1997) p. 149
[4] *Grundgesetze*, p. 9

2.1.2 Act or sign?

Introductions I2 and I3 are quite similar, as they refer to expressions without judgment stroke as "expressions for truth-values" or "a mere equation [which] only designates a truth-value", where I1 calls this "a mere combination of ideas". The judgment stroke "expresses the affirmation" of this (I1), together with the content stroke the $\vdash$-sign is formed to express judgment. In the later introductions it is emphasized that the judgment stroke is not a mere sign for judging, but that "by writing $\vdash 2 + 3 = 5$ we *assert* that $2 + 3$ equals 5" (I2, emphases mine). The vertical part of the symbol is not just signifying some act of judgment it seems, rather "the judgment stroke contains the act of assertion"(I3). This raises the question whether the judgment stroke is a sign signifying an act or whether writing it actually effects the assertion.

This ambiguity exists not only in the quoted introductions of the judgment stroke, but throughout Frege's work. Sometimes he talks about the signs of the Begriffsschrift as a linguistic representation of an act that takes place in our cognitive system. For instance in his *Collected Papers* when he says that "this divergence of expressive symbol and expressed thought is an inevitable consequence of the difference between spatio-temporal phenomena and the world of thoughts."[5] But the judgment stroke is not a sign like any other symbol of the Begriffsschrift. In *Grundgesetze*, Frege distinguishes it from names and marks: "it is a sign of its own special kind. A sign consisting of a judgment stroke and a name of a truth value preceded by a horizontal is called a proposition."[6,7] It is a special sign, namely one that signifies an act of assertion, instead of an object. This is said in a footnote of I2[8] as well: the judgment stroke cannot be used as a functional expression "for it doesn't serve, in conjunction with other signs, to designate [bezeichnet] an object, [...] it asserts something." The judgment stroke does not only signify that an act of assertion has taken place, it actually *is* an act of assertion. Putting the judgment stroke in front of a sentence is the act of acknowledging the sentence to be true, it effects the assertion. Something similar is said in *Grundgesetze*, namely that *by* a proposition it *is asserted* that the name is the true, like in I2 and I3.

But which one of these readings should one choose? Is the judgment

[5] *Collected Papers*, p. 393

[6] Note that the term proposition has a different meaning here than in the modern usage, already in Russell and Whiteheads *Principia Mathematica* something else is meant by 'proposition'.

[7] *Grundgesetze*, p. 26

[8] *Funktion und Begriff*, p. 22

stroke a sign referring to the act of judging, or should we adopt the interpretation that the writing of the judgment stroke is in itself the act of judging? Here the latter interpretation will be followed, not because this is the more recent one, for the ambiguity exists in the later books as well. But I do think that the conception of the judgment stroke as an act within the logical language reflects Frege's actual practice as a logician. In the preface of *Begriffsschrift* Frege says that the goal of logic is to give a foundation of arithmetic. In order to do so he has to explicate the 'formula language of pure thought', i.e. the Begriffsschrift, in which a mathematician can make gapfree proofs of all mathematical sentences. It is only within the Begriffsschrift that these proofs are made. There might be a cognitive realm of thoughts outside of the concept language, but, similarly as the foundational proofs of mathematicians, the act of saying that something is true happens within this logical language. This act is made by a subject, by the person writing the judgment stroke, but it is not something psychological or subjective in the sense that it depends on the subject whether a sentence might be asserted. Macbeth explains this very clearly in her book *Frege's Logic* when she distinguishes an active and a passive counterpart of judging. The passive part is that the thought needs to be true and the active part is that one acknowledges it to be so. This will be discussed in more detail in a next section, for now the main conclusion is that the judgment stroke *effects* the judgment. In the next section Frege's conception of logic will be discussed, a logic of inferences in which the acts of judgement, made by the judgment stroke, play a crucial role.

> The judgment stroke in front of the whole presents this proposition[9] as an assertion.[10]

2.1.3 Inference and Consequence

In this section the difference between a logic of consequences and a logic of inferences will be discussed, to explain Frege's general conception of logic.

In his 1998 paper *Inference versus Consequence* Goran Sundholm explains the difference between the concepts of inference and consequence. The relation of logical consequence holds between sentences A_1 ... A_n and C[11] iff whenever A_1 ... A_n is true, then so is C.

[9] 'Proposition' as it is used by Frege: referring to a sentence together with the judgment stroke.

[10] *Begriffsschrift*, p. 93

[11] C is a logical consequence of A_1 ... A_n.

> This is the notion of logical consequence that is explained in the passage: whenever all the antecedent propositions[12] are true, the consequent proposition C is true also. One should note however, that propositions and judgments are conflated in the passage. The relata in logical consequence are propositions, whereas inference effects a passage from known judgments to a novel judgment that becomes known in virtue of the inference in question.[13]

A proposition is the logical consequence of another proposition if and only if their truth-values function according to the rule expressed by the relation of consequence. Logical consequence holds independently of a subject drawing the consequence, which makes it a relation rather than a process or an act. One can make a derivation (in any logic) to prove that a logical consequence holds (in that particular logic), but the consequence relation exists independent of that derivation. As can be read in the quoted passage, an inference is something else. The relata in an inference are not propositions but judgments. In Fregean terms this means that the relata in inference are not sentential contents; it is the relation of logical consequence that holds if the truth values of different thoughts are related in a certain way. An inference is not about a relation between truth-values, but about judgments, i.e. asserted sentences. A judgment presupposes an act of a subject; the act of asserting a sentence as true and therefore inferences do not exist without subjects. The "passage from known judgments to a novel judgment" Sundholm mentions, is done by a subject knowing the judgments that function as premises, deriving from them other judgments, that become known through the process of inferring. By means of an inference one does not prove relations that were already 'out there', rather one is inferring something new; a judgment gets known on the basis of other judgments that were already made. According to Sundholm an inference is "an act of intermediate judgment"[14], rather than a formal relation. An inference is a judgment and as such it is really an act. Moreover, it is an act of judgment of which the product is yet another judgment.

How do we know that Frege's conception of logic is one of inferences rather than consequences? There are several arguments supporting this, first of all a linguistic reason: Frege uses the German word *Schluss* when talking about logical derivations, the Eglish translation of this is 'inference'. The translation of 'consequence' is *Folge* or *Schlussfolgerung*. Frege refers

[12] Note that these are propositions in the modern (Russellian) sense.
[13] Sundholm (1998), p. 27
[14] Sundholm (1998), p. 28

to these inferences as acts: "einem Schluss wird vollzogen"[15], an inference is carried out. 'Proposition' had a different meaning to Frege than it does to logicians nowadays, as was mentioned in footnote 9, something similar might be the case with the concepts '*Schluss*' and '*Folge*'. The question is now; are there other reasons to assume that Frege upholds a logic of inferences?

As early as in the *Begriffsschrift* Frege formulates the condition that inferences can only be drawn from true judgments. If the premises of a derivations are not 'judgments acknowledged to be true', the only thing one gets is a pseudo-inference:

> It is necessary to recognize the truth of the premises. When we infer [schliessen], we recognize a truth on the basis of other previously recognized truths according to a logical law. Suppose we have arbitrarily formed the propositions
>
> $$'2 < 1'$$
>
> 'If something were smaller than 1, then it is greater than 2' Without knowing whether these propositions are true. We could derive[16]:
>
> $$'2 < 2'$$
>
> from them in a purely formal way; but this would not be an inference because the truth of the premises is lacking. And the truth of the conclusion is no better grounded by means of this pseudo-ineference than without it. And this procedure would be useless for the recognition of any truths.[17]

Nothing can be inferred from false premises, because this does not lead to a new judgment, i.e. a sentence that is known to be true. The relata are judgments in Frege's logic, not propositions which can be either true or false. Only in this development from judgments to judgments can one expand knowledge and get new judgments, instead of consequences that were already 'out there'.[18] In this logic the subject plays a role; what kind

[15] *Grundgesetze*, p. 17

[16] Note that Frege is using the word 'schliessen' (to infer) here, rather than 'ableiten', which would mean something like: 'to deduce'.

[17] *Wissenschäftliche Briefwechsel* (1917), Dingler, IX/2

[18] A problematic feature of this conception of logic is that it is difficult to account for sentences that are in fact already 'out there'; tautologies or other generally known truths. This will be addressed in the discussion of Kantian aspects in the Begriffsschrift, and is also dealt with in *Tractatus* by Wittgenstein.

of subject this is will be discussed in more detail later, for now it is enough to note that there is a subject that acknowledges judgments to be true and that makes the steps from one judgment to the next. Together with the perception of judging as an act, one cannot but conclude that Frege's general conception of logic is one of inferences.

2.1.4 Sinn and Bedeutung

So far it is discussed how the judgment stroke should be perceived and what to think of Frege's conception of logic. When reading the three different introductions of the judgment stroke in section 2.1.1 one also notices another difference: in the *Begriffsschrift*-definition (I1) of the judgment stroke the horizontal part of the assertion sign is called *content stroke* and in I2 and I3 this is renamed the *horizontal*. This is related to a development in Frege's thinking which was described explicitly in *Über Sinn und Bedeutung*, but it was already apparent in the earlier paper *Funktion und Begriff* from which I2 originates.

Immediately after I1 comes the following sentence:

> What follows the content stroke must always have a *judgeable content*.[19]

This judgeable content is further explained as the thing that influences the possible consequences[20] of a judgment. The judgeable content is not related to terms 'Subject' and 'Predicate', this distinction only has a grammatical meaning. Since it is irrelevant for 'what may be derived from a sentence', the subject-predicate distinction has no place in *Begriffsschrift*. Instead of Subject-Predicate, we should conceive sentences as having a functional structure; the dichotomy that is related to the 'conceptual content' is the pair consisting of 'Function' and 'Argument'. Sentences of the Begriffsschrift are built from a function and its argument, similar as mathematical sentences, that this is relevant for the content, becomes clear when either of the two is indetermined. How this is more related to the content of a sentence than the subject-predicate distinction is not exactly clear, but that we should adopt this functional perception of sentences is stressed throughout *Begriffsschrift*.

What the content of a judgment is, and what is asserted, is explained in more detail in *Funktion und Begriff*, where Frege relates the terms 'Object' and 'Concept' to the functional sentences. The concept-term in a sentence

[19] Beaney (1997), p. 53, emphasis mine.
[20] Here the word *Folgerungen* is used.

should be taken as the function, and the argument of this function corresponds to the object. As a result, the value of the function is the truth-value of the sentence. The value is either True or False, depending on whether the object falls under the concept, only together with an object term the concept can be assigned a truth value. Similar with a function: an argument can saturate a function and only this combination (which makes up a complete whole) has a value. The arithmetical function '$2x^3 + 2$' only has a solution if we take some number for x, just like a concept such as '*is blue*' only has a truth value when it is said *of* some object. In the paper the comparison goes even further, when Frege explains that assertoric sentences work like functions and concepts might function as arguments as well, but some things remain unclear even in this reading: What is exactly the meaning of a function? Or of a sentence? Sometimes this seems to be the truth value:

> The value of our function is a truth value. Consequently '$2^2 = 4$' means the True, just as '2^2' means '4' and '$2^2 = 1$' means the False. [...] Indeed all equations mean the same thing, e.g. the True. Thus the following is a correct equation:
>
> $$'(2^2 = 4) = (2 > 1)'^{21}$$

But this is not all there is, as becomes clear immediately after these sentences:

> The objection might be that the two sentences tell us different things. But also '$2^4 = 2^2$' and '$4 \times 4 = 4^2$' express different thoughts, and yet we can replace '2^4' by '4×4', since both have the same Bedeutung.[22]

This referential aspect of meaning is worked out in more clarity and detail in the paper *Über Sinn und Bedeutung*, where Frege discusses the two counterparts of meaning: 'Sense' and 'Reference', on the basis of the Morningstar-Eveningstar example. The terms *morningstar* and *eveningstar* are both names for the planet Venus as we see it in the morning respectively evening sky. The object to which these names refer is the *Bedeutung* of the names, i.e. their reference. Since that is the planet Venus in both cases, the terms morningstar and eveningstar have the same reference. One could even be tempted to use an identity sign here: *morningstar* = *eveningstar*. But this

[21] *Funktion und Begriff*, p. 13
[22] *Funktion und Begriff*, p. 13

sentence does not have the same epistemic value as the formally similar sentence $a = a$, because there is more to meaning than just reference. The *Sinn* of a name is its objective representation, the way in which it is presented. The Venus-example illustrates this clearly: *morningstar* is the luninious dot we see in the morning sky and *eveningstar* is presented as something seen in the nightly sky. The Sinn is the sense of a term, which is not to be confused with subjective ideas associated with concepts. Rather, it is something objective in the sense that it is shared between individuals, it is really the mode of presentation of an object.

The distinction between sense and reference does not replace the distinction between object and concept, nor that between function and argument. Sinn and Bedeutung are additional phenomena that can be distinguished for each concept- or object-term, as well as for each function or argument. Sentences are considered to be similar to functions, and also seem to have a sense and a reference: the Bedeutung of a sentence is its truth-value and the Sinn of a sentence is the Thought it expresses. This interpretation is consistent with the explanation Frege gives of truth-values in *Funktion und Begriff*, where he says that truth values are *objects*, just like planets (Venus) and other phenomena we refer to. The Thought or *Gedanke* would be the conceptual content of a sentence, '$-A$', which is called '*objective content*' in *Über Sinn und Bedeutung*. Without the other neither the truth-value nor the Gedanke yields knowledge. The combination of the two is effected in judging, which can be considered the "advancement from thought to truth value."[23] A judgment is not the mere grasping of a thought, but the assertion of its truth.

In the two papers discussed here Frege adds the pair Sinn & Bedeutung to the terminology of the *Begriffsschrift* and refined the Object & Concept, respectively Function & Argument distinction. This means that Frege is able to give a more precise definition of what it is that is asserted to be true, when a subject is judging. Both concepts and objects have a sense *and* a reference. The concepts take objects (or other concepts) as their arguments in order to form a a saturated function: a sentence. Sentences also have a Sinn *and* a Bedeutung; the Sinn is the Gedanke (Thought) it expresses and the Bedeutung is the truth-value of that 'Thought'. Such sentences replace the notion of 'assertible content' and function like saturated functions. If the truth-value is the True, then the sentence might be asserted.

The notion of '*assertible content*' has developed into Sinn and Bedeutung in the works of the mature Frege. In *Begriffsschrift*, the assertible content

[23] *Über Sinn und Bedeutung*, p. 50

refered to what is acknowledged as true in the act of judgment, and in his later works the meaning has shifted towards Sinn and Bedeutung. This development raises the question whether the act itself has changed, and, correspondingly, its representation in Begriffsschrift: the judgment stroke. Already in *Begriffsschrift* the aspect of combining things in order to gain knowledge was apparent, but what it is that is synthesized to come to knowledge, is made explicit in Frege's later writings. Even though this notion has been altered, the fact that the judgment stroke signifies and effects an act (as discussed in section 2.1.2) hasn't changed. This is in line with the conclusion N.J.J. Smith draws in his paper *Frege's Judgment Stroke*[24], where he says that the judgment stroke is always put forward by Frege as the sign effecting and indicating *assertion*. According to him, it is significant that Frege does not mention anything about changing the meaning or explanation of the vertical part of this sign, whereas he explicitly discusses the rebaptizing of the horizontal part. The vertical stroke still refers to judging, the meaning of the horizontal has evolved. The following passage brings this all together:

> I say: when one grasps a thought, one is thinking; when one acknowledges a thought to be true, one is judging, when one expresses a judgment, one is asserting.[25]

2.2 Why have a Judgment Stroke?

In section 2.1 it is discussed what the judgment stroke is, this next part will be about the question why Frege needs this symbol. Why does Frege need a symbol related to the act of assertion in his conceptual language? This is a relevant question for the comparison with Wittgenstein and Russell, since Frege stresses the importance of this particular sign, whereas Wittgenstein considers it meaningless. The hypothesis is that the significance of the judgment stroke stems from Frege's very conception of logic. Which part of 'Frege's conception of logic' entails that he requires something like the judgment stroke? Is it his logicist programme, the fact that he has designed his Begriffsschrift in order to give a foundation of mathematics? It is emphasized in the the prefaces of *Begriffsschrift* and *Grundgesetze* that Frege's main goal is to prove all theorems of mathematics in logic, it may be the case that the most remarkable sign in his logic is related to the goal it was designed for. Apart from that, logicism is something Wittgenstein rejects, he

[24] Smith (2000), p. 164

[25] *Wissenschäftliche Briefwechsel*, Dingler IX/4, translation mine.

does not consider it meaningful to attempt to reduce mathematics to logic. Another aspect which will be discussed is the relation to Kant; some of the central notions in Frege's works we've encountered before in Kant's transcendental logic. Kant also considered the relation between thought, judgment and the subject in reasoning; do these terms mean the same for Frege? This historical perspective will be continued by a comparison between Frege and other practitioners of a logicist programme. It will be discussed why Frege has a judgment stroke, in contrast to other logicists, and whether Frege was practicing meta-logic while proving the foundations of mathematics.

2.2.1 Frege's Logicist Programme

In the introductions of *Begriffsschrift* and *Grundgesetze* Frege explains why he had developed the Begriffsschrift in the first place. This was to give an axiomatization of mathematics, more specifically of arithmetic. He was looking for a tool allowing for the formulation of "gapfree chains of inferences"[26], in order to investigate how far one could come in arithmetic solely via inferences. However, he encountered some difficulties:

> In striving to fulfill this requirement in the strictest way, I found an obstacle in the inadequacy of language: however cumbersome the expressions that arose, the more complicated the relations became, the less the precision was attained that my purpose demanded. Out of this need came the idea of the present Begriffsschrift. It is thus intended to serve primarily to test in the most reliable way the validity of a chain of inference [Schlusskette] and reveal every presupposition [Voraussetzung] that tends to slip in unnoticed, so that its origin can be investigated.[27]

And so the idea for the Begriffsschrift arose. Frege wanted to give a foundation for arithmetic on the basis of logic. In order to do so, he had to show that the arithmetical inferences are reducible to logical modes of inference and to isolate the substantive, non-logical presuppositions sneaking in. A first (and incomplete) attempt is made in *Begriffsschrift* and this project was continued in *Grundgesetze der Arithmetik* in 1895. In this book the logicist programme was given another chance, after improving the Begriffsschrift in *Über Sinn und Bedeutung* and *Funktion und Begriff*. In the preface of the first volume he acknowledges that

[26] *Begriffsschrift*, iv
[27] *Begriffsschrift*, iv

> [i]t is frequently said that arithmetic is only a more highly developed logic; but that remains disputable so long as transitions occur in the proofs that do not take place in accordance with recognized logical laws, but appear to rest on intuitive knowledge. Only if these transitions are analysed into simple logical steps can one be convinced that nothing but logic lies at the base.[28]

A gapfree inference has not been made for all modes of inference of arithmetic, despite the various attempts, but here Frege thinks to have given it.[29]

The Logicist Programme is exactly this: from the idea that mathematics is an extension of logic it follows that all mathematical principles may be reduced to logic. In practice this means for Frege that he is giving a proof in Begriffsschrift for every theorem of arithmetic. These proofs are inferences from judgment to judgment, but is this logicist goal the reason that Frege has a judgment stroke at all? It seems natural, since there is a subject proving things, expanding its knowledge while giving these foundations, and the judgment stroke appears in every proof in the hundreds of pages of *Grundgesetze*. But Frege hasn't been the only one pursuing the logicist goal, *Principia Mathematica* was written with similar intentions. Whitehead and Russell also wanted to

> show that all pure mathematics follows from purely logical premises and uses only concepts definable in logical terms.[30]

But they do this without the judgment stroke in the Fregean sense. That fact alone, that Whitehead and Russell have the same goal as Frege, but are able to do so using a different conceptual apparatus, is not enough to say that Frege could have done without the judgment stroke. In order to give a foundation of mathematics, one does not need to have a logic of inferences, nor is it necessary to separate the act of assertion from the truth of a sentence. Many attempts[31] have been made to prove all the presuppositions

[28] *Grundgesetze*, p. xx

[29] A minor disclaimer is made, namely that if there is to be found something defective in this framework, it can only be in the Basic Law V. This law concerns value ranges of functions and expresses that different functions with the same value for every argument have the same range of value (Wertverlauf). Since a concept whose value is a truth value, is a function, the range of value of a function can be thought of as the extension of a concept (as was already introduced in *Funktion und Begriff*. Russell's paradox showed that Basic Law V is indeed problematic.

[30] Russell (1959), p. 74

[31] By Hilbert, Dedekind and Peano, among others, as will be discussed in section 2.2.3 on meta-logic.

of the theory of number -or any other mathematical subdiscipline, without reference to a judging subject. The fact that Frege pursues logicist goals alone, explains nor justifies his use of the judgment stroke.

2.2.2 Kantian aspects of the Begriffsschrift

Judgment plays a central role in Kantial philosophy, especially in his philosophy of mathematics and logic, but also in his writings on perception via the categories. Related to the latter is his description of judging "as the subsumption of an object under a concept of pure understanding."[32] Judging is a process of recognizing that objects of experience belong to certain concepts, a process done by a *subject* and nonetheless leading to *objective* knowledge. In this we already recognize elements apparent in the works of Frege. The Kantian perspective on logic was the main paradigm in the 19th century, that is why we must examine in which sense (if any) and to what extent Frege's views are in line with Kants.

2.2.2.1 Kant on Judging

When saying that judging is the subsumption of objects under concepts, Kant is talking about what we empirically perceive. A similar process of synthesis in intuition is responsible for generating the objects of an a priori science, like mathematics:

> This is not at all because mathematical thinking merely reflects and generalizes what we empirically perceive, but, on the contrary, because the mental acts by means of which we are capable of perceiving a homogeneous multiplicity -which is never merely given, but the result of a successive synthesis of apprehension, governed by a *synthesis speciosa* in accordance with the category of quantity -these mental acts are also what enables us to generate an a priori science of multiplicities and their relations. In other words, Kant considers the same mental capacities that generate appearances as *quanta* (things in which there is quantity) also generates the objects of mathematical sciences.[33]

Here it is explained that the objects of mathematics are presented a priori in pure intuition. But, how can one gain new knowledge in an a priori manner? Kant solved this with his explanation of judgments that are both synthetic

[32] Longuenesse (1998), p. 244
[33] Longuenesse (1998), p. 274

and a priori. Such synthetic a priori judgments can be made in mathematics only. They are a priori because of their independance of sensory impressions. A priori judgments are solely based on reason and not on the perception of external phenomena. Although, as Longuenesse describes, the construction of mathematical objects happens via the same capacities as the construction of objects of experience, the mathematical concepts are a priori, "because they are nothing other than concepts of homogeneous multiplicities."[34] The adjective 'synthetic' refers to the process of synthesis that was mentioned in the quoted passage above. The synthesis of apprehension shows that we *do* something when judging: one *constructs* the mathematical concepts in pure intuition. Synthesis is a capacity of the faculty of Reason which takes place in a subject. It is the subject that is constructing mathematical concepts in its intuition and is able to reason from them in order to discover new truths.

An example of an a priori synthetic judgment in mathematics (more specifically: in arithmetic) is the simple addition of 5 and 7. In his *Critique of Pure Reason* Kant says about this:

> The concept of '12' is by no means already thought in merely thinking this union of 7 and 5; and I may analyze my concept of such possible sum as long as I please, still I shall never find the '12' in it. We have to go outside these concepts, and call in the aid of the intuition.[35]

The concept of '12' is not equivalent to the sum of 5 and 7, we have to go through a process of intuition in order to find: whenever 5 and 7 should be added, we will get the number 12. The sentence '$5 + 7 = 12$' is a judgment, an objectively true one, but arrived at via a subjective process.

> The judgment '$3 + 4 = 7$' does seem to be a purely theoretical judgment, and objectively regarded that is what it is; but subjectively, the sign '+' signified the synthesis involved in getting a third number out of two other numbers, and it signifies a task to be done.[36]

Statements of arithmetic are theoretical judgments, they are objectively true, but reached via an 'endeavor' done by a subject. The result was not already contained in the (combinations of) symbols on the other side of the equality sign, but discovered by the subject. '$5+7 = 12$' is a judgment yielding knowledge. We saw something similar in Frege's *Grundlagen*, where he

[34] Longuenesse (1998), p. 275
[35] *CoPR*, B15-16
[36] Kant, Letter to Schultz, November 25, 1788 in: Longuenesse (1998), p. 281

said that the sentence '$2^2 = 4$' makes an epistemic claim. The Bedeutungen of '2^2' and '4' are the same number, but their Sinn is distinct, it is therefore that the judgment '$2^2 = 4$' is not trivial, but actually contains information.

In *Grundlagen* Frege described numbers in a Kantian manner,[37] and also the idea that a subjective mental process may lead to knowledge, can be found in Frege. Judgments have an epistemic value, they produce knowledge; on this, Kant and Frege agree. How is the objective truth of these judgments guaranteed? In order to be genuine knowledge, the judgments need to supersede the subjective process and hold objectively, independent of whoever is making the judgment. Kant and Frege have their own explanation to bridge the gap between the reasoning subject and the objective truth of a judgment: Kant needs a transcendental subject and Frege a realm of thoughts that may be grasped. For now it is enough to note that true judgments aren't just true for a subject, or according to a subject, but *objectively* true and acknowledged to be so by a subject.[38] The synthesis within a Kantian subject is a capacity of the faculty of reason. He describes how we have the potential to judge by means of a synthetic process. Frege goes one step further: in order to judge one really has to *do* something. Judging is not a capacity but an act. Judgment does not exist *in potentia*, its actualization is what's important: when judging, one is acting, and a judgment can only be made through this act.

Macbeth makes this comparison as well:

> As the point is put in the *Grundlagen*, such theorems *are contained in the definitions, but as plants are contained in seeds, not as beams are contained in a house* (*GL*, 88). The theorems proven are analytic, grounded in logic alone, but also ampliative in Kant's sense because the judgment is not proven solely on the basis of an analysis of the concept of the subject. They are, then, extensions of our knowledge, properly speaking - or so the author of *Grundlagen* argues.[39]

[37] Not only because of the epistemic value of identity claims, other aspects will be discussed in the next section on Arithmetics and the Normativity of Logic.

[38] In the SEP entry *Kants Theory of Judgment* Robert Hanna speaks of true judgments, as opposed to false judgments and judgments without truth value, of the latter category are (among others) judgments about noumenal phenomena. In a Fregean paradigm a true judgment is a bit pleonastic: only true statements may be asserted, and therefore judgments can only be true. Negated sentences may be asserted, but that does not render a false judgment either, as will be discussed later.

[39] Macbeth (2005), p. 103

So far, we have discussed the objective truth of judgments. In order to address the ampliative proofs mentioned in the passage above, some other issues need to be considered first. Kant and Frege have a different perception of logic and arithmetic. They agree on the close connection between those sciences and thinking in general. Closely related to this is the important role of logic and arithmetic in the extension of knowledge, which is done via reasoning from judgment to judgment.[40]

2.2.2.2 Arithmetics and the Normativity of Logic

Frege and Kant disagree about the question which one is the most fundamental of all sciences, mathematics or logic. As described by John MacFarlane, the disagreement lies in the fact:

> [...] that Frege endorsed:
> (F) Arithmetic is reducible to logic
> and that Kant endorsed:
> (K) Arithmetic is not reducible to logic.[41]

Because of their different views on logic, Kant and Frege hold opposite opinions about logicism. According to the former logic is a mere set of rules and not a genuine science. As a result it cannot yield an extension of knowledge about *reality*, as mathematics can. It is, therefore, both impossible and insensible to try to reduce mathematics (or more specifically arithmetic) to logic. Logic is purely formal in the sense that it

> abstracts away from all content of cognition, i.e. from any relation of it to the object, and considers only the logical form in the relation of cognitions to one another, i.e., the form of thinking in general.[42]

Because of this and in contrast to mathematics, logic cannot be used to make claims about existence. It seems as if Freges Begriffsschrift is not this kind of logic. First of all, it is not formal in the Kantian sense of the term; there is no abstraction from all content, because of Freges restrictions to substitution. Not only that; for Frege logic can be about actual objects, mathematical objects. As a consequence Frege considers logic to be a real science, not just a set of rules (or the scaffolding of the world), but "a body

[40] As Frege explained in the *Grundgesetze*.
[41] MacFarlane (2002), p. 25
[42] *CoPR*, A 55 in: MacFarlane (2002), p. 48

of truths".[43] In such a 'real science' one can make existence claims, obtain new knowledge by means of inference, and, as Frege says in several places, it can even be used to give the foundations of arithmetic.

The term 'logic' in Kant does not mean the same as it does in the works of Frege, nonetheless, there are similarities in their explanation of how arithmetic relates to thinking as such. Kant writes:

> [A]rithmetic is the most universal of all mathematical sciences because it is the most closely related to the laws of pure thought [...] the rules of arithmetical thought are the flip side of the rules of general logic (as Kant delimits it).[44]

MacFarlane was cited above saying that general logic concerns "the form of thinking in general." It doesn't merely concern the forms of thinking, general logic is the science of the necessary laws of thought. As such it gives us the conditions of its correct use: "in logic, [...] it is not about how we do think, but how we ought to think."[45] Logic gives us the rules of thinking, it prescribes how one ought to think. It is not a science like any of the others, for it is not related to a specific object, about which it may extend our cognition. Logic is "merely for passing judgment and correcting cognition, but not for expanding it."[46] It is normative in the sense that it tells us how to think: how to reason and how to make judgments. But logic is neither a descriptive nor an ampliative science, like the natural science or mathematics.[47] How this is to be brought in agreement with the earlier statement about arithmetics and logic is discussed elaborately in MacFarlane (2002). For now the most important conclusions are the connection between arithmetic and thinking in general, the normativity of logic for thinking and the necessity of this normativity. According to Kant thinking cannot proceed but in accordance with logical rules.

These aspects can be found in Frege as well, the following passage entails all three of these characteristics:

> Here, we have only to try denying any one of them [of the principles of arithmetic], and complete confusion ensues. Even to think at all seems no longer possible. The basis of arithmetic lies deeper, it seems than any of the empirical sciences, and even

[43] MacFarlane (2002), p. 32
[44] Longuenesse (1998), p. 282
[45] *Jäsche Logik*, p. 14
[46] *Jäsche Logik*, p. 13
[47] According to Kant.

> than that of geometry. The truths of arithmetic govern all that is numerable. This is the widest domain of all; for it belongs not only to the actual, not only to the intuitable, but everything thinkable. Should not the laws of number, then, be connected very intimately with the laws of thought?[48]

Arithmetic is closely related to thinking *tout court*; without the principles of arithmetic thinking is not even possible. Is this meant to imply that the laws of number are connected to the laws of thought? In her book *Kant and the Capacity to Judge* Beatrice Longuenesse remarks that the focus on arithmetic in relation to thinking is due to to a similar concept of number in Kant and Frege.[49] Both take this to be a property of a concept: "the content of a statement about number is an assertion about a concept."[50] The goals differ, as Kant emphasizes the transcendental nature of it and Frege uses the concept if number to make objective existence claims. But both philosophers consider number to be an a priori phenomenon without which thinking is not possible at all, and thus consider the laws of arithmetic to be closely connected to the laws of thought. However, Frege's conception of the logical laws differs from Kant's prescriptive interpretation:

> The word law is used in two senses. When we speak of moral or civil laws we mean prescriptions, which ought to be obeyed but with which actual occurences are not always in conformity. Laws of nature are general features of what happens in nature, and occurences in nature are always in accordance with them. It is rather in this sense that I speak of laws of truth. Here of course it is not a matter of what happens but of what is. From the laws of truth there *follow* prescriptions about asserting, thinking, judging, inferring.[51]

Laws of logic are essentially descriptive, like the laws of any other science are. A law of a natural science describes what happens in nature, that things in the world will always happen accordingly, this is the lawlike regularity of nature. The "laws are descriptive in their content, [but] they imply prescriptions."[52] Laws as such are not presciptive, they do not tell us what to do nor how to think, but we can derive norms from them.

[48] *Grundlagen*, §14 in: MacFarlane (2002), p. 38
[49] Longuenesse (1998), p. 257-8
[50] Longuenesse (1998), p. 257
[51] *Collected Papers*, p. 351, emphasis mine
[52] MacFarlane (2002), p. 36

Logic is normative in the sense that we can only reason in conformity with logical laws. It renders us the rules for judging, inferring and thinking *an Sich*, for reasoning from judgment to judgment. Such an act is done by a subject and can only depart from true judgments. When one reasons in accordance with the rules, the resulting inference is a true judgment as well, and, as such, may be an extension of our knowledge. Frege stands in a Kantian tradition with respect to his views on the a priori expansion of knowledge, and the crucial role he assigns to the subject in logic. The dispute is on how this role needs to be fulfilled, but both Kant and Frege consider the logic to be normative in a very strict sense. So strict that they seem to rule out the possibility of making mistakes, and in that sense idealize the subject: it cannot infer judgments that aren't objectively true.

Even though both presuppose that a subject can come to objective knowledge, it seems as if in Frege the potential to reason and to judge, which is also apparent is Kant, is actualized. There are more factors in which Frege distances himself from the 18th century philosopher: the logicist ideas are the complete opposite of Kants, who doesn't consider logic to be a genuine science, but just a set of rules. Let alone that mathematics might be reduced to this! Another difference in opinion arises from the formality of logic; the Begriffsschrift does not abstract away from all content,[53] but it is computable, and therefore considered to be formal by Frege. For Kant the substitutivity is a decisive feature of formality. Since the Begriffsschrift is only allowing this in a restricted form, Kant would not think of it as a proper logic.

The elements of a logic of inferences as opposed to one of consequences, judgments and an acting subject, can be traced back to some extent to Kant, as well as the idea that logic renders us norms for thinking. Logic is a science of the necessary laws of thought, "without which no use of the understanding or reason takes place at all."[54]

2.2.3 Logic and Meta-logic

In their paper on *Completeness and Categoricity* Steve Awodey and Erich Reck investigate the development from 19th century axiomatics to 20th century metalogic.[55] The Greek mathematician Euclid is often considered the

[53] Frege doesn't mean it to either, as becomes particularly apparent in the debate between Frege and Hilbert, on formalism and logicism, which will be addressed in the next section.

[54] *Jäsche Logic*, 13

[55] As is stated in the subtitle of the paper.

first to apply the axiomatic method; he gave five postulates from which all geometric truths were supposed to be derivable. Ever since, axiomatization had been a way of organizing a scientific discipline and increasing the certainty of its theorems. The science to be organized is usually an area of mathematics, such as geometry or arithmetic. In the 19th century, Dedekind, Peano and Hilbert (among others) aimed to give a logical foundation of numbers and geometry. In their paper, Awodey and Reck sketch the development from these early examples of axiomatizations towards what they refer to as '*formal axiomatics*'. These 20th century practitioners of formal axiomatics wanted to characterize a scientific discipline *completely*. Besides stating the axioms, formal axiomatics also involves the use of a *formal* language and the specifiation of a *formal* deductive system. One has to be careful when employing terms as 'formal' or 'complete' without defining them, for it can be confusing what is meant by them exactly, as we already saw in section 2.2.2. According to Awodey and Reck, a formal language is "a language that is taken to be uninterpreted and for which various different interpretations can be considered and compared."[56] And a formal deductive system "makes explicit which logical inferences between sentences of the language are permitted."[57] In 19th century axiomatics neither an uninterpreted language, nor a specification of inference rules were defined. By means of five examples Awodey and Reck sketch the development towards formal axiomatics. From this 20th century approach to logicism, a more meta-theoretical perspective emerged. In the meta-semantics the issue of completeness of an axiomatic framework became significant as requiring actual proof. Hilbert, Dedekind, Peano and other logicians weren't able to do so, they merely *stated* that their set of axioms did the things it was supposed to: characterize the intended subset of mathematics. Although the notions of completeness, categoricity and a distinction between semantic and syntactical consequence gradually evolved, it was only after the *Principia Mathematica* that the more formal approach to logical deduction became customary. This paved the way for meta-proofs in the 1920s and later; the first completeness proof for propositional logic was made by Paul Bernays in 1918. The isolation of subsystems of logic in order to study them, was not yet done in axiomatics, because of the lack of a purely formal notion of deductive consequence. In the paper, Awodey and Reck are "ignoring the work of Gottlob Frege, as was in effect done at the time,"[58] because

[56] Awodey & Reck (2002), p. 5
[57] Awodey & Reck (2002), p. 5
[58] Awodey & Reck (2002), p. 19

of his traditional and anti-formalist views on axiomatics, as they explain. Subject of this section will be where in this development the works of Frege should be positioned. Despite the lack of influence, the Begriffsschrift does not seem to be that traditional at all.

In comparison to the 19th century axiomaticians, Frege pursued a similar goal: to give the foundation of arithmetics. The goal of the Begriffsschrift was to give a gapfree chain of inferences in order to prove the intuition that arithmetic was "only a more highly developed logic."[59] Like his contemporaries Frege aims to formulate a system of arithmetic, i.e., "a complete and adequate axiomatization of arithmetic."[60] This is one of the characteristics Awodey and Reck attribute to the formal axiomatics, in stating its goals clearly, Frege's *Begriffsschrift* from 1879 was actually quite modern.

Other features of formal axiomatics are an uninterpreted language and a specification of deduction rules. Frege's Begriffsschrift is not an uninterpreted language; the meaning of the terms plays and therefore it is not unresrictedly free for substitution, that is, not every interpretation is allowed. Frege himself considers the Begriffsschrift to be a formal language, because it is abstracting away from the ambiguities of natural language and allows for mechanical calculations. Awodey and Reck do not consider it to be an example of the formal axiomatic approach, for it is not fully uninterpreted. However, a deductive system is apparent in the works of Frege. Already in *Grundlagen*, published in 1884, Frege acknowledges the need to state inference rules in advance. A logician should not let anything through that wasn't explicitly presupposed: "every jump must be barred from our deductions."[61] Also, in *Begriffsschrift* a detailed explaination of the inference rules is given, prescribing how to reason from judgment to judgment. Those rules are similar to the one he gives in §§14-17 of *Grundgesetze*, although the latter appear in a different order. The deduction rules do not apply to an uninterpreted language, and the truth of the premises is necessary in order to make an inference, but the rules *an Sich* make explicit which inferences between sentences are permitted, exactly as Awodey and Reck described. The reason why they referred to Frege's views as anti-formalist, is probably because his Begriffsschrift is not an uninterpreted language, but it cannot be set aside as traditional without further ado. The fact that he has stated the goals of his axiomatization clearly, as well as his advanced set of inference rules, is rather untraditional for a 19th century logician, if not quite modern.

[59] *Grundgesetze*, preface
[60] Macbeth (2005), p. 9
[61] *Grundlagen*, 91

Awodey and Reck explained how *Principia Mathematica* marked a significant step forward towards a meta-theoretical investigations. In the meta-logical approach theorems are proven *about* logical (sub-)systems, such as the completeness or categoricity of a system. As advanced as the specifiation of the inference rules may be, there is nothing meta-theoretical in Frege's Begriffsschrift. The axioms of Frege's systems are the Basic Laws I, II (a, b), III, IV, V and VI. For each of these laws a justification is given in Begriffsschrift; it is explained what the laws mean and why they are necessarily true, but Frege does not formulate systematic proofs about the system as a whole. In the introduction of *Grundgesetze* Frege explains his foundational goals: he is going to prove that all the theorems of arithmetic can be derived from laws of the Begriffsschrift. In doing so

> every 'axiom', every 'assumption' [Voraussetzung], 'hypothesis' or whatever you want to call it, upon which a proof [Beweis] is based is brought to light. [...]
> If anyone should found something defective, he must be able to state precisely where, according to him, the error lies; in the Basic Laws, in the Definitions, in the Rules, or in the application of the Rules at a particular point.[62]

The idea of stating where the error lies, is internal to the system, i.e. it is done within the Begriffsschrift and thus not meta-logical. In the last section, on Kantian aspects of Frege's logic, it was discussed that Frege considers logic to be foundational for thinking as such. Arithmetic applies to everything numerable, which is the widest domain there is, and as such there is a connection between the laws of logic and the laws of arithmetic. This justification of the claim that the Begriffsschrifft is foundational to artithmetic is extra-logical, but it is not meta-logical, for it is not a proof *about* a logic. Frege makes the reduction of arithmetic to logic within the Begriffsschrift. The meta-perspective is not part of logic as Frege conceives it. His agenda is to actually *make* the reduction, in this respect he is clearly concerned with axiomatics, and not with meta-semantics. The few remarks on the justification of the logical laws or the adequacy of the system, are no more than that; remarks. He does not prove anything *about* logic, he works *within* logic to prove theorems of mathematics. This could be why Awodey and Reck take Frege to be traditional; he organizes the science of arithmetic and tries to increase its certainty by giving gapfree proofs. Quite advanced are his formalizations of the Begriffsschift, the inference rules and

[62] *Grundgesetze*, vii

the logical laws, as well as the explication of the goals of his works, but it is not metamathematics what Frege is doing.

Chapter 3

Russell's Interpretation of the Judgment Stroke

Russell speaks of the *assertion sign* ' $\vdash$ ' as if it is a simple symbol, whereas Frege considers it to be a complex of signs, consisting of the horizontal and the judgment stroke. Two books in which Russell discussed the symbol will be considered, in order to sketch his interpretation of Frege: *Principles of Mathematics* (1903) and *Principia Mathematica* (first edition, 1910, second edition 1927). Both works are systematic introductions to Russell's mathematical logic, and known to have been read by Wittgenstein.[1]

3.1 Assertion in the *Principles of Mathematics*

3.1.1 A Pre-Fregean Notion of Assertion

Principles of Mathematics (*Principles*) is Russell's first major work on the foundations of mathematics. In this book he develops his own logicist programme and argues that all mathematical propositions are deducible from a small number of fundamental logical principles. In the light of the current project *Principles* is a peculiar book, since Russell only read the works of Frege for the first time when the manuscript had already gone to press. The pre-Fregean notion of assertion, as laid out in the body of *Principles*, is discussed prior to the investigation of the adapted account of assertion as it is given in its Appendix. In this way we can track changes and notice the original emphasis of Russell. The logical language that is used in *Principles*

[1] According to Proops (2002)

is adopted from Peano.[2]

The distinction between asserted and unasserted propositions is treated in §38 of *Principles*. Here, it is discussed whether the assertion of "q" is implied, whenever the propositions "p" and "$p \supset q$" are asserted. It is not, says Russell, since assertion and implications are two distinct notions:

> In grammar, the distinction is that between a verb and a verbal noun, between, say, "A is greater than B" and "A's being greater than B". In the first of these the proposition is actually asserted, whereas in the second it is merely considered. But these are psychological terms, whereas the difference which I desire to express is genuinely logical. It is plain that, if I may be allowed to use the word *assertion* in a non-psychological sense, the proposition "p implies q" *asserts* an implication, though it does not *assert* p or q. The p and the q which enter into this proposition are not strictly the same as the p or the q which are separate propositions, at least, if they are true. The question is: How does a proposition differ by being actually true from what it would be as an entity if it were not true? It is plain that true and false propositions alike are entities of a kind, but that true propositions have a quality not belonging to false ones, a quality which, in a non-psychological sense, may be called *asserted*. There are grave difficulties giving a consistent theory on this, for if assertion would in any way change a proposition, no proposition which can possibly in any context be unasserted could be true, since when asserted it would become a different proposition. But this is plainly false; for in "p implies q" p and q are not asserted, and yet they may be true. Leaving this puzzle to logic, however, we must insist that there is a difference of some kind between asserted and unasserted propositions.[3]

In this lengthy passage several features of Russell's account of assertion can be identified: (1) propositions occurring within assertions are themselves not asserted, (2) the verb effects that something is asserted and (3) it is difficult to formulate a genuinely logical notion of assertion. Like Frege, Russell distances himself from a subject-predicate distinction within propositions. Instead he divides the proposition into a logical subject (term) and something which is said about the term. A proper name is always the subject a

[2] Russell's choice of notation will be discussed in more detail in section 3.2.1.

[3] *Principles*, §38

proposition is about, and only adjectives or verbs can say something about the term, i.e., function as the assertion *in* the proposition. The assertion is part of the proposition and at the same time effecting its assertion. This confusing claim is illustrated by the following example: Consider "Caesar died" and "The death of Caesar". Only the former sentence makes an assertion: it asserts the death of Caesar. The latter statement, containing a verbal noun instead of a verb, does not assert anything. Russell admits that he cannot give a satisfactory account for the difference between "Caesar died" and "the truth of Caesar's death", despite the fact that only the former contains an assertion. "The truth of Caesar's death" expresses that the assertion that Caesar died is true, without asserting it. Russell fails to define the logical counterpart of assertion and is not able to distinguish true propositions from asserted ones. This issue remains unsolved:

> I leave this question to the logicians with the above brief indication of a difficulty.[4]

In the Appendix Russell reopens the disscussion of this difficulty, when analyzing Frege's conception of assertion.

3.1.2 Appendix A: Discussing Frege

The logical sense of assertion is addressed in the works of Frege. Russell reads *Begriffsschrift* and *Grundgesetze* in the summer of 1902, when his *Principles* had already gone to press:

> My book is already in the press: I shall discuss your work in an appendix because it is now too late to talk about it in detail in the text. When I read your *Grundgesetze* for the first time, I could not understand your conceptual notation [Begriffsschrift]; I succeeded only when I began to notice the gaps in Peano's notation. Unfortunately my book was already completed at the time.[5]

The work was already completed, but Frege's writings were worth investigating, as appears from the Appendix A of *Principles*:

> The work of Frege, which appears to be far less known than it deserves, [...]
> Frege's work abounds in subtle distinctions, and avoids all the

[4] *Principles*, §52
[5] *Wissenschäftliche Briefwechsel*, XXXVI/5 Russell, p. 220

> ususal fallacies which beset writers on Logic. His symbolism, though unfortunately so cumbrous as to be very difficult to employ in practice, is based upon an analysis of logical notions much more profound than Peano's, and is philosophically superior to its more convenient rival.[6]

It can be asked whether Frege's analysis of judgment influenced Russell's conception of assertion, despite its inaccessible notation. In §477 Russell discusses truth values and judgments, concepts related to the difference between asserted and unasserted propositions. "Frege's position on this question is more subtle than mine, and involves a more radical analysis of judgment."[7] The subtlety lies in the distinction between Sinn and Bedeutung, which applies to terms as well as to sentences.[8] Russell explains that a Fregean judgment has three elements: (1) the recognition of truth, (2) the Gedanke, and (3) the truth value. What Russell has called an unasserted proposition covers both the Gedanke and the Gedanke together with its truth value. The Gedanke alone will be called the 'propositional concept' from now on, and the truth value of a Gedanke the 'assumption'.[9] In judgment one has, according to Russell, a propositional concept, its truth or falsity and the assertion of the truth or falsity of the propositional concept. The following remark sums this up nicely:

> This theory is connected in a very curious way with the theory of meaning and indication. It is held that every assumption indicates the true or the false (which are called truth values), while it means [its Sinn is] the corresponding propositional concept. When a sentence has a truth value, this is its indication [Bedeutung]. Thus every assertive sentence is a proper name, which indicates the true or the false. The sign of judgment does not combine with other signs to denote an object. A judgment indicates nothing, but asserts something. Frege has a special symbol

[6] *Principles*, §475

[7] *Principles*, §477

[8] Russell correctly remarks that the German term 'bedeuten' as Frege uses it, does not correspond to what he calls 'denoting'. 'Bedeuten' is therefore translated with 'to indicate'. To denote, in Russell's *Principles*, means to refer to a unique object by means of a phrase of the form 'the so-and-so'. The Bedeutung of a term in Frege's Begriffsschrift is nothing like a denoting phrase, since a description of the object referred to is lacking and it does not necessarily refer to a unique object.

[9] It is not exactly clear at this stage whether this is just the truth-value or the truth-value and the Gedanke. In *Principia Mathematica* it appears to be the Gedanke together with its truth value, but here Russell remarks that it is not formally required that the content of an assumption is a propositional content.

> for judgment which is something distinct from and additional to the truth value of a propositional concept.[10]

So far Russell's representation of Frege's theory of judgments. He does not completely agree with it, but after reading Frege's books, Russell revises his own account in some aspects. He no longer considers the assertion to be a constituent of a(-n asserted) proposition and he adds the demand that only true propositions can be asserted. This results in a conception in which negation belongs to the proposition. If p were false, then 'not-p' would be true and may be asserted. It is difficult to give a logical account of what the assertion consists in, when is is neither identical to the truth of a proposition nor appearing as a part of it:

> It is almost impossible for me, to divorce assertion from truth, as Frege does. An asserted proposition, it would seem, must be the same as a true proposition. [...] To divorce assertion from truth seems only possible by taking assertion in a psychological sense.[11]

It remains unclear what 'assertion' adds to the truth of a proposition. We saw earlier that Frege considered assertion to be an act of the subject, the act of committing oneself to the truth of the Gedanke. Russell does not mention a subject, nor does he explicitly deny its existence, but he does seem to regard this part of Frege's theory of judgment as something pychological rather than logical:

> [I]t seems quite sufficient to say that an asserted proposition is one whose meaning is true, and that to say the meaning is true is the same as to say the meaning is asserted.[12]

Combined with what he said in the quote from §478 above, divorcing truth and assertion is only possible when taking assertion in a psychological sense. Russell is searching for a logical notion of assertion and presupposes that this exists, but he fails to formulate what it consists of, other than a true proposition. He makes clear that the logical notion of assertion, appearing in the works of Frege, is too psychological and as such not logical. In Russell's conception of assertion there is no longer room for a judging subject to acknowledge the truth of a proposition. Logically correct, according to Russell in *Principles*, would be a solution in which an asserted proposition is a true proposition:

[10] *Principles*, §477, [...]-bracketed remarks mine
[11] *Principles*, §478
[12] *Principles*, §479

> The case of asserted propositions is difficult, but is met, I think, by holding that an asserted proposition is merely a true proposition, and is therefore asserted wherever it occurs, even when grammar would lead to the opposite conclusion.[13]

3.2 *Principia Mathematica*

Principia Mathematica (*PM*) was written in collaboration with Alfred N. Whitehead, and the first edition was published about a decade after *Principles of Mathematics*. It is a continuation of the project that Russell started in *Principles*: to derive all mathematical truths from a set of logical principles. It is known that Wittgenstein read *PM* and this could very well have been a source for him to get in touch with the philosophy of Frege, since his analyses are discussed throughout *PM*. In the Preface, Russell and Whitehead already express their gratitude towards Frege: "In all questions of logical analysis, our chief debt is to Frege."[14] Before discussing how assertion and the judgment stroke are represented in this book, the issue concerning the notation needs to be addressed. Also in this respect *PM* is a continuation of *Principles*, since Peano's linear notation is preferred over Frege's two-dimensional Begriffsschrift-notation. Even though the former has its shortcomings (section 3.1.2), it is better than the "cumbrous" Begriffsschrift, which is "very difficult to employ in practice."[15]

3.2.1 Notation in *Principia Mathematica*

> In the matter of notation, we have as far as possible followed Peano, supplementing his notation, when necessary, by that of Frege or by that of Schöder.[16]

As in *Principles*, the logic of *Principia Mathematica* is a based on Peano's linear notation. In the Preface and in the Introduction Russell and Whitehead explain why they have chosen this notation.

> [S]ymbolic logic, which, after a period of growth, has now, thanks to Peano and his followers, acquired the technical adaptability

[13] *Principles*, §483
[14] *PM*, vii
[15] *Principles*, §475
[16] *PM*, viii

> and logical comprehensiveness that are essential to a mathematical instrument for dealing with what have hitherto been beginnings of mathematics.[17]

It has proven to be a good instrument for mathematical analysis, which is what Whitehead and Russell need in *PM*. Furthermore, they praise the details of Peano's notation and the fact that is freed from the "forms of ordinary algebra."[18] These properties make it a suitable instrument for research, and the fact that (at least) one of the authors was already familiar with the notation would not have been a negative argument either.

Russell and Whitehead's project consists to a certain extent in the transcription of Frege's logical analyses into the Peano notation, but the question arises; is it possible to do so without further implications for the analysis? According to Danielle Macbeth, it is not. There is an aspect that is lost when moving from the Begriffsschrift to the linear notation, and that is the *operative* aspect of the logic. This terminology comes from Sybille Krämer and in this section the differences between the linear and the 2-D notation are discussed.

In her paper *Writing, Notational Iconicity, Calculus: On Writing as a Cultural Technique*[19] Sybille Krämer distinguishes three aspects of written languages: the structural, the referential and the performative aspect. The structural aspect refers to the fact that writing is a notational medium, in which language can be symbolized in a structured way. The representational aspect refers to the property of script to make a reference to 'objects', in the most general conception of the term: terms or symbols may refer to words of a language, concepts, objects, persons, ideas, and so on. One can even make up new things while referring, and in this way language may be constitutive. The third aspect is the most crucial one for our purposes. The performative aspect refers to the use of writing as a *technique*. In the performative sense writing is no longer just a *medium*, but also a *tool* for manipulating signs and making calculations. This function appears, for instance, in the place-value Hindu-Arabic numerical system: one can make calculations within it, by just mechanically applying the rules, without thinking about the meaning of the terms and operations.

> We can thus see that operative writing is not only a tool for describing, but also a tool for cognizing, a technique for thinking

[17] *PM*, v
[18] *PM*, vii
[19] Krämer (2003)

> that enhances intelligence. Long before the computer became a universal medium and a programmable machine, we developed the computer 'in ourselves', which is understood here as the cognitive use of algorithmic sign-languages that are freed of the constraints of interpretation.[20]

In various publications[21] Danielle Macbeth claims that this performative aspect is a distinctive feature of Frege's Begriffsschrift, which is not apparent in Wittgenstein's Tractarian logic nor the Standard Notation[22]. She clarifies this claim with the help of an analogy with the differences between the Roman and the Arabic numeral system. The Roman numeral system is first and foremost representational, it expresses quantities in a direct manner: with the 'i' referring to the amount '1', and 'v' and 'x' being merely convenient abbreviations for sets of five, respectively, ten units. The Roman system is designed to record 'how many' and is called a *lingua characterica*, because language shares characteristics with the phenomena it represents. The Arabic numeral system is suited for different purposes: making calculations. The Arabic numerals do not refer directly (the structural analogy between '6' and the amount it refers to is nil), but, more importantly, it is possible to compute with them.

> Unlike the system of Roman numeration, the Arabic numeration system is essentially written. It is a positional numeration system that utilizes the expanse of the page, and thereby the relative locations of the signs, to exhibit arithmetical content. Although one can use Arabic numeration to record how many, that system's primary purpose is the quite different one of providing a written system of signs within which to perform paper-and-pencil arithmetical calculations.[23]

The Arabic numeration system is essentially performative, whereas the Roman numbers are essentially representational. Analogously is the Begriffsschrift designed for reasoning, making inferences, i.e., for *performative* purposes, and the linear notation for *representational* purposes. If so, what does this matter? It needs to be explained what the difference is between

[20] Krämer (2003), p. 534

[21] Macbeth (2002), Macbeth (2005) and Macbeth (forthcoming)

[22] As Macbeth calls "for instance, the logic of Aristotle, of Leibniz, of Boole and of our textbooks" in Macbeth (fortcoming), p. 10. The linear notation falls under the standard notation, since it is a predecessor of the 'logic of our textbooks'.

[23] Macbeth (forthcoming), p. 32-33

'6' and '*vi*'. Both refer to the same quantity, and that explains why they can be translated into each other. As any Roman numeral can be translated into an Arabic numeral, so can sentences of the Begriffsschrift into the linear notation and the other way around. The truth-conditions expressed by the sentences remain the same, therefore the translations Russell and Whitehead make are allowed. But there is something that gets 'lost in translation', according to Macbeth, and that is the operative or performative aspect. She says that one cannot really perform 'hands-on' reasoning within the logic of *PM*. It is possible to record an inference, to describe it, but this representation of an inference does not work as a 'computer in ourselves'.

> Standard notation was not designed as a language within which to reason, and it cannot be so used. One can use the language to *record* what else is true if ones given premises are true, but the language does not enable reasoning *in* the system any more than Roman numeration enables arithmetical calculations.[24]

It is not impossible to make calculations within the Roman numeral system, but it is more easily done with Arabic numerals.

It has been mentioned before that Frege holds an epistemological conception of logic, in which he focuses on the process of making inferences, the construction of a proof. His purpose was to practice logic, and the Begriffsschrift was designed exactly for this. The writing of an inference and the cognitive process are inseparable. The 2D-notation plays an essential role in the construction of an inference: while mechanically applying the rules, one might arrive at new knowledge by means of the drawing on paper. Macbeth's analysis of this seems to comprise Frege's intentions, but investigation of *PM* is needed in order to see whether the Peano-notation is indeed mainly representational.

Russell and Whitehead state derivation rules in *PM*, that prescribe how to proceed from one proposition to another. Following Macbeth's line of arguing, a derivation or proof in the linear notation is not the construction of the proof itself, it is not the (process of) calculation, but a *record* of it, and as such not 'hands-on-reasoning'. The derivation is a transcription of a cognitive process, which could have taken place without the 'pen on paper'-component. Does Russell hold this opinion as well? He and Whitehead refer to the Peano-notation as an instrument in doing mathematical analysis, and even stronger is the following passage:

[24] Macbeth (forthcoming), p. 36

> The symbolic form of the work has been forced upon us by necessity: without its help we should have been unable to perform the requisite reasoning. It has been developed as the result of actual practice, and is not an excresence introduced for the mere purpose of exposition.[25]

The symbolic form enabled them to perform the requisite reasoning, does this mean that Russell and Whitehead do actually reason within the symbolic language? When reading the Introduction, which is dedicated to notational matters solely, one might be lead to the negative. Which is consistent with the interpretation of Macbeth, as presented above. The purpose of the symbolism is to "embody strictly accurate demonstrative reasoning."[26] The symbolism is a representation of the reasoning, a perfectly precise expression of it. When things get very complicated, the symbolic representation can be of great help, but the actual reasoning takes places within the imagination and not on a piece of paper, as appears the following passages:

> The adaptation of the rules of the symbolism to the process of deduction aids the intuition in regions too abstract for the *imagination* readily to present to the mind the true relation between the ideas employed. [...] And thus the mind is finally led to contruct trains of reasoning in regions of thought in which the *imagination* would be entirely unable to sustain itself without symbolic help.[27]

> Most mathematical investigation is concerned not with the analysis of the complete process of reasoning, but with the *presentation* of such an abstract of the proof as is sufficient to convince a properly instructed mind. For such investigations the detailed presentation of the steps in reasoning is of course unnecessary, provided that the detail is carried far enough to guard against error.[28]

The actual process of reasoning takes place in imagination, which is a faculty of the mind. The notation serves the purpose of presenting the proof, and in most mathematical investigation an abstract suffices. The language seems to be representational, but in the passage quoted from p. 2 is also said that

[25] *PM*, vii
[26] *PM*, p. 1
[27] *PM*, p. 2, emphasis mine
[28] *PM*, p. 3

the imagination may need symbolic help. This would hint at a performative aspect in the *PM*-notation. Furthermore, note that in this introduction Russell and Whitehead discuss what the symbolism of *PM embodies*; it can be concluded that they want to record inferences, but from this it cannot be concluded that there aren't performative features in their linear logic as well. The more careful conclusion is that it does not seem to be the primary goal of the *PM*-logic to enable calculations within the language. Rather, the purpose of the *PM* is to give the foundations of mathematics, and to prove that these are indeed the correct foundations.[29]

A translation from a Begriffsschrift sentence into linear notation is possible, the truth conditions that are given remain unchanged in the translation, even though they are presented in different ways. What is lost in translation is the *inference potential*, the performative aspect of the Begriffsschrift-sentences.[30] Macbeth's analogy with Arabic and Roman numerals explains that Peano's linear language is less suitable for making calculations. But that does not mean that the linear notation cannot be used for reasoning at all, nor that it cannot be used within a framework of reasoning. An example of the latter would be a system of natural deduction. When making natural deductions one is certainly calculating, while using (for instance) sentences of Standard Notation. Reasoning within logic is not as easy in the linear notation as in Begriffsschrift, nor does it seem to be the main goal of Russell and Whitehead, but that does not say that it is not possible to calculate within the *PM*-notation. The consequences of the linear notation for the *PM*-conception of assertion and the role of the judgment stroke in the symbolism will be discussed in the next section.

3.2.2 Assertion in *Principia Mathematica*

In *Principia Mathematica* the $\vdash$ symbol is introduced as an extension of Peano's symbolic logic. Russell and Whitehead call it the assertion sign and introduce it in the following way:

> The sign "$\vdash$", called the 'assertion sign', means that what follows is asserted. It is required to distinguish a complete proposition, which we assert, from any subordinate propositions contained in it but not asserted. In ordinary written language a sentence contained between full stops denotes an asserted proposition, and if

[29] Recall the analysis in section 2.2.3: such proofs would be an example of meta-logic rather than reasoning within logic.

[30] According to Macbeth it is this property of the Begriffsschrift that makes it ampliative, in contrast to derivations in standard notation, which are not informative.

> it is false the book is in error. The sign "$\vdash$" prefixed to a proposition serves this same purpose in our symbolism. For example, if "$\vdash (p \supset p)$" occurs, it is to be taken as a complete assertion convicting the authors of error unless the proposition "$p \supset p$" is true (as it is). Also a proposition without this sign "$\vdash$" prefixed is not asserted, and is merely put forward for consideration, or as subordinate part of an asserted proposition.[31]

In this definition many Fregean aspects can be found: there is a subject, the 'author', who is committed to the truth of the asserted proposition, there is a separation of the notions of truth and assertion, and the assertion is contrasted to a proposition that is "merely put forward for consideration."[32] The latter reminds of the "blosse Vorstellungsverbindung" we have encountered in Frege's *Begriffsschrift*, which is a mere combination of ideas, as opposed to a judgment which truth is acknowledged by the subject. This role of a subject is apparent in the *PM*-definition as well, and as such assertion is distinguished from truth. As in the first introduction in *Principles*, subordinate propositions are not immediately asserted when occurring in an assertion. Even though it is presented as a single sign rather than a composition of a horizontal and a judgment stroke (as Frege does), the Fregean analysis of the symbol is more or less preserved in this introduction.

With the help of the assertion sign, Russell and Whitehead are able to define the notion of inference within the framework of *PM*. This is a *process* taking place in time:

> The process of inference is as follows: a proposition 'p' is asserted, and a proposition 'p implies q' is asserted, and then as a *sequel* the proposition 'q' is asserted.[33]

This development from asserted propositions to the assertion of a conclusion, is consistent with Frege's epistemological analysis of inference. Unfortunately Russell and Whitehead do not persist in this manner, as appears in the passage immediately following the one above:

> The trust in inference is the belief that of the two former assertions are not in error, the final assertion is not in error. Accordingly whenever, in symbols, where p and q have of course special determinations,
>
> "$\vdash p$" and "$\vdash (p \supset q)$"

[31] *PM*, p. 8
[32] *PM*, p. 8
[33] *PM* p. 8-9, emphasis mine

> have occurred, then "$\vdash q$" will occur if it is desired to put it on record. The process of inference cannot be reduced to symbols. Its sole record is the occurrence of "$\vdash q$". It is of course convenient, even at the risk of repetition, to write "$\vdash p$" and "$\vdash (p \supset q)$" in close juxtaposition before proceeding to "$\vdash q$" as the result of an inference. When this is to be done, for the sake of drawing attention to the inference which is being made, we shall write instead
>
> "$\vdash p \supset \vdash q$"
>
> which is to be considered as a mere abbreviation of the threefold statement
>
> "$\vdash p$" and "$\vdash (p \supset q)$" and "$\vdash q$"[34]

Instead of writing the whole inference, consisting of the three assertions "$\vdash p$", "$\vdash (p \supset q)$" and "$\vdash q$" , Russell and Whitehead propose an abbreviation containing the assertions of the two atomic propositions connected by an implication: "$\vdash p \supset \vdash q$". Frege would consider this a category mistake; in the Begriffsschrift it is not possible to have a judgment stroke within the scope of a conditional. But in *PM*, where the assertion sign is treated as one of the functional operators instead of a symbol, this is allowed.*sui generis*[35] In the light of the last section on notation this makes sense: an abbreviation is possible in representational symbolism, when merely picturing a connection that was already established is the main purpose, rather than drawing the actual inference.

A reason why Russell and Whitehead consider this abbreviation acceptable can be found in their explanation of syllogisms:

> It should be observed that syllogisms are traditionally expressed with 'therefore', as if they asserted both premises and conclusion. This is, of course, merely a slipshod way of speaking, since what is really asserted is only the connection of premises with conclusion.[36]

Where in Frege the premises *and* the the conclusion, as well as the connection between them need to be asserted in order to constitute an inference, this

[34] *PM*, p. 9

[35] At least here, at p. 9 of *PM*, in contrast to p. 8 where the assertion sign was introduced.

[36] *PM*, p. 28

demand is dropped here. What is asserted in a syllogism is the connection between premises and the conclusion, not the sentences themselves. This seems to be an explanation for allowing for an abbreviated form, but in order to conclude so, it has to be considered whether a syllogism is an inference.

So far only the introductory chapters of *PM* are discussed, in part I on mathematical logic, Russell and Whitehead present the principles of mathematical logic. In this light they discuss the difference between asserted propositions and the mere presentation of propositions.

> In language, we indicate when a proposition is merely considered by *"if so-and-so"* or *"that so-and-so"* or merely by inverted comma's. In symbols, if p is a proposition, p by itself will stand for the unasserted proposition, while the asserted proposition will be designated by
>
> $$\text{“} \vdash p\text{”}$$
>
> The sign "$\vdash$" is called assertion sign; it may be read "it is true that" (although philosophically this is not exactly what it means). [...]
> Thus "$\vdash p \supset q$" means "it is true that p implies q", whereas "$\vdash p \supset \vdash q$" means "$p$ is true; therefore q is true." The first of these does not necessarily involve the truth either of p or of q, while the second involves the truth of both.[37]

As interesting as the first part of this passage is, the part in which $\vdash p \supset \vdash q$ is considered will be discussed first. The truth of the statements p and q does matter for the therefore-relation. A syllogism must be expressing something else, when it only asserts the connection between the propositions. The syllogism is a formal implication of the form $\vdash p \supset q$, rather than an inference. The difference between an inference and a formal implication is that the former is a relation between assertions, and the latter between propositions that are merely considered.

In contrast to *material* implications both inferences and formal implications serve similar purposes; they serve to make us know what was previously unknown to us.

> [S]uch implications[38] do not serve the purpose for which implications are chiefly useful, namely that of making us know,

[37] *PM*, p. 92
[38] Material implications, to which the therefore-relation is contrasted in this passage.

> by deduction, conclusions of which we were previously ignorant. Formal implications, on the contrary, do serve this purpose.[39]

The relata are different, but formal implications, as well as inferences, have an epistemological component. In this passage, the subject appears again, and it seems as if the logic of *PM* is ampliative after all. That "$\vdash q$" will occur whenever "$\vdash p$" and "$\vdash p \supset q$" occur, does not necessarily indicate that Russell holds an ontologic conception of logic. There is a process of inference, "which cannot be reduced to symbols"[40], in which there is something made known to us. The personal pronoun signifies what an epistemological conception presupposes: a subject. This also appeared in the first introduction of the assertion sign in *PM*, and these passages strongly indicate that a subject indeed plays a role in the logic of Russell and Whitehead.

The question is whether Russell and Whitehead present a Fregean notion of assertion here, even within a different notation. The fact that there is an epistemological component in the logic of *PM* is very Fregean and the same holds for the role of the subject. How this role is to be fulfilled, and whether Russell and Whitehead succeed in giving a logical notion of assertion is not explicated, as such some of the questions of *Principles* remain unanswered. Another confusing aspect of this work is that it is not always clear why there needs to be a distinction between formal implications and inferences, and when the different relations should be applied. The abbreviated form is used a lot, which makes it somewhat unclear when premises are asserted or merely put forward as propositions. That the assertion of a premise involves its truth is addressed in this work; it may be read "it is true that". In a bracketed clause it is mentioned that these two notions do not coincide, "although philosophically that is not exactly what it means", but what it means philosophically is not discussed here.

Altogether, Russell and Whitehead's adapted several Fregean ideas in *PM*. The epistemological aspect and the subject were already mentioned, furthermore they state that only true propositions may be asserted and that assertion is no longer part of the proposition. They reject many Fregean notions as well; the notions Sinn and Gedanke are not extensional enough, the 2D-notation isn't workable as mathematical instrument and the assertion sign has lost its '*sui generis*' status. Combined with the many abbreviations and the linear notation, it can be concluded that there is less emphasis on what Frege considers to be the core of logic: the act of drawing inferences.

[39] *PM*, p. 21
[40] *PM*, p. 9

Inferences are called processes which serves to make something known to us, but when exercising logic, Russell and Whitehead do not consider this process a lot. In *PM* they present the proofs they made, the inferences that were established; Russell and Whitehead give the principles of mathematics. These principles are presented in a structured way, but the presuppositions of their logical system is not always very clear. Russell and Whitehead use some Fregean aspects to pursue their own goal, which seems to be very similar to Frege's but is arrived at in a different way: by using a logic that is, rather than one enabling of acts of judgments, meant to record inferences. This is in line with the analysis of Macbeth in 3.2.1 and might explain why the interpretation of Fregean concepts is sometimes confusing in *PM*. Russell and Whitehead are using the assertion sign in another way than Frege; they are not so much using it while reasoning, they use it to mark the truth of statements.

In this section the representation of Frege's logic in the works of Russell is discussed. Russell respected the work done by Frege and highly valued it, but it took a while before he found a way to incorporate Frege's ideas in his own logical language. As a result, when reading *Principles of Mathematics* and *Principia Mathematica* some ambiguities can be found, as well as some non-Fregean applications of the act of asserting. Russell's works are known to be a source for Wittgenstein to come in contact with Frege's works, and that is the reason that they are considered here. In the next section it will be investigated which ideas in *Tractatus* can be traced back to the Russellian interpretation of the Begriffsschrift.

Chapter 4

Wittgenstein

Wittgenstein's *Tractatus Logico-Philosophicus (TLP)* is discussed in order to address the main question of this thesis; whether Russell's interpretation of the judgment stroke has influenced Wittgenstein's conception of it. In this early work Wittgenstein explains his perspective on logic and philosophy at the time. He tries to draw a limit to our thinking, or rather, to our expression of thoughts:

> The book will, therefore, draw a limit to thinking, or rather - not to thinking, but to the expression of thoughts; for, in order to draw a limit to thinking we should have to be able to think both sides of this limit (we should therefore have to be able to think what cannot be thought). [1][2]

To give a logical clarification of thought, that is what philosophy is about (4.112).[3] Within philosophy we can draw the limits of the thinkable and of the expressible, and as such delimit the unthinkable and unspeakable from the inside out. The assumption is that "that what can be said and what can be thought are in essence one and the same (that they coincide structurally and extensionally)."[4]

Since it concerns the drawing of boundaries, philosophy is an activity rather than a theory. It is the activity of elucidating what it is that is

[1] *Tractatus*, p. 8

[2] For the English translation I used Ogden's version (1922), but I made some alterations, for instance in the cases where Ogden uses 'proposition' as the translation of 'Satz', here 'sentence' will be used and also 'Sinn' and 'Bedeutung' are translated as 'meaning' respectively 'reference', instead of 'sense' and 'meaning'.

[3] All the references to *Tractatus* will be given in this way, by giving the number of the sentence.

[4] Stokhof (2002), p. 37

thinkable and which thoughts can be expressed. As such it is something "over, or under, but not beside, the natural sciences"[5], because in philosophy the limits of reasoning within the natural sciences are given. In this chapter it is investigated how Wittgenstein defines the limits of the expression of our thinkingand why he does not need a judgment stroke in order to do so.

4.1 Sentences in *Tractatus*

The vehicle for the expression of thoughts is the sentence:

> 3.1 In the sentence the thought is expressed perceptibly through the senses.

As the perceptible form of thoughts, the sentence is the most fundamental entity of *Tractatus*. There are three kinds of sentences: the *sinnvoll* (meaningful) ones, the *sinnlose* (meaningless), and the *unsinnige* (nonsense) sentences.[6] The meaningful sentences are an expression of a thought. The meaningless sentences have a logical form, but they lack meaning, these are the logical sentences. The third category are the nonsense sentences, they don't have a logical form or meaning, and as such no function at all. These nonsense-sentences will not be discussed further.

Before investigating the difference between *sinnvolle* and *sinnlose* sentences, it needs to be addressed what a sentence is. A sentence is a structured whole, built up from names that are combined in a particular *structure*. This structure is rather important, this is apparent in the quoted passage from Stokhof as well; where he said that the expressible and the thinkable coincide structurally. The structure of the sentence is similar to the structure of what it represents: the thought. The structure is formed by the constituents of a sentence, these are the names. Names have a logical form, which defines its combinatory possibilities. As in Frege and Russell, the sentence is perceived as a function of its parts (3.318), it is a connection or linking of names (4.22) in a structured way.

The names that constitute a sentence are simple signs, which cannot be analyzed any further, neither by giving a definition nor by means of a description (3.26). Names refer to objects and are the direct representatives of these objects in sentences; they are the linguistic counterparts of whatever

[5] Potter (2009), B 67

[6] The word 'Sinn' in *Tractatus* does have the same meaning as it has for Frege, therefore 'Sinn_w' will be used to refer to Wittgenstein's 'Sinn', to be able to distinguish between the two concepts.

they represent and do not 'describe' it in any way. What these objects are will become clearer in the section on meaningful sentences, for now it should be noted that names only have meaning, i.e. refer to an object, in the context of a sentence. The sentence is the smallest meaningful entity:

> 3.3 Only the sentence has meaning [Sinn$_w$], only in the context of a sentence a name has reference [Bedeutung].

Interpreters of *Tractatus* refer to this as the 'Fregean context-principle', because of the resemblance it bears to §60 of Frege's *Grundlagen*. I take it to express something else as well, namely that sentences are most important entities in Tractarian logic: the meaningful entities. Which brings us to the next issue: to describe the meaningful sentences.

4.1.1 Meaningful Sentences

Sentences are constructions of names: each name represents an object and the sentence as a whole pictures a state of affairs.

> 4.0311 One name stands for one thing, and another for another thing, and they are connected together. And so the whole, like a living picture, presents the state of affairs.

A sentence is compared to a 'tableau vivant' (a living picture). In a tableau vivant the actors, together with the props and costumes, represent a situation as a picture on stage, without moving and even without saying a word, the scene is presented. Sentences function as pictures of situations in a similar way.

> 4.01 The sentence is a picture of reality. The sentence is a model of reality as we think it.

Elementary sentences picture *states of affairs*.[7] States of affairs are the logically independent atoms of the world: the smallest 'things that are the case'. This makes sense: sentences have already been identified as the smallest meaningful entities of the Tractarian language and they are the representations of the smallest things that are the case in the world. Sentences are related to the world, at least *meaningful* sentences are, they are a picture of reality, even though, at first sight, the similarities may seem marginal. In 4.011 Wittgenstein addresses this issue:

[7] Elementary sentences depict states of affairs and complex sentences, which are built from elementary sentences, represent situations. This distinction will not be discussed here, but is made by Wittgenstein in *Tractatus*.

> 4.011 At the first glance the sentence - as it is printed on the paper - does not seem to be a picture of reality of which it treats. But nor does the musical score appear at first sight to be a picture of a musical piece; nor does our phonetic spelling (letters) seem to be a picture of our spoken language. And yet these symbolisms [Zeichensprachen] prove to be pictures - even in the ordinary sense of the word - of what they represent.

The musical notation does not look like music, but it is a sign language for music, and as such a *picture* of music. Sentences are a picture of reality, because things in reality can be represented by names, this representational property makes the language a *'Zeichensprache'* of reality.

All meaningful sentences are pictures of a state of affairs or a situation in the world. They contain names that represent objects in the world, and their structure is similar to the structure of the world.

> 2.15 That the elements of the picture are combined with another in a definite way, represents that the things are so combined with one another.

As in a 'regular' picture taken with a camera both the objects and their orientation are depicted. A sentence is not a jumble of words, "kein Wörtergemisch" (3.141), it is a structured whole in which the elements are related to another in a certain way. A sentence is articulated and, because of this, it has a meaning. A mere collection of words does not represent a *fact*, but a structured whole can, and only facts can express a Sinn_w(3.14): The Sinn_w is the meaning of a sentence, the state of affairs it depicts.

> 4.2 The meaning [Sinn_w] of a sentence is its agreement and visagreement with the possibilities of the existence and non-existence of states of affairs.

The state of affairs is the Sinn_w of the sentence, this is distinguished from its truth or falsity. The meaning can be understood without knowing whether the sentence is actually true or false, but only meaningful sentences can be true or false. The truth-value of sentences in Tractarian logic is an issue that needs further explanation.

Meaningful sentences are always contingent.

> The essence of meaning is contingency: meaningful sentences are pictures of contingent situations.[8]

[8] Stokhof (2002), p. 38

According to Stokhof, meaningful sentences are contingent because they are pictures of the contingent world. The *logical space* is the collection of all possible states of affairs, of all possible facts, and the ones that are actually the case make up the *world*. It is contingent which states of affairs are actually the case, every possibility may be true or false, there are no necessary possibilities. No state of affairs in the world is necessary true or false, every fact of reality is contingent. In *Tractatus* truth is defined in terms of the obtaining or not-obtaining of states of affairs: In sentence 4.3 is explained that the truth possibilities mean the existence and non-existence of states of affairs. Sentences are pictures of possible facts in the world, and if a fact is the realized, then the sentence is a representation of the fact.

So far we have discussed the meaning and the truth-value of sentences: the Sinn_w is the fact in the world a sentence depicts and its truth-value depends on the existence of that fact.

> 2.21 The picture agrees with reality or not; it is right or wrong, true or false.
> 2.221 What the picture represents is its meaning [Sinn_w].
> 2.222 In the agreement or disagreement of its meaning with reality, its truth or falsity consists.

Wittgenstein explains these two features of sentences in terms of what a sentence *says* and what it *shows*.

> 4.022 The sentence *shows* its meaning.
> The sentence *shows*, how things stand, *if* it is true. And it *says that* things do so stand.

A sentence *shows* its meaning. The Sinn_w is not contained in the sentence (3.13), but it is projected by the sentence. The sentence *shows* the state of affairs it represents; the names in it represent (*bedeuten*) the objects in the world and the structure of the sentence shows *how* these are related. Apart from a representation of a state of affairs, a sentence also *shows* its logical form. This is the form underlying the structure of the names, and it is what a sentence needs to have in common with reality in order to be able to depict it (4.12).[9]

> 4.121 A sentence cannot depict logical form: logical form is mirrored in it.

[9] It escapes the scope of this thesis to discuss the difference between the form and the structure of sentences in more detail, in Stokhof (2002) more can be read on this topic.

> What mirrors itself in language, language cannot depict. What expresses *itself* in language, *we* cannot express by means of language.
> A sentence *shows* the logical form of reality. It displays it.

A sentence cannot describe its logical form, nor can we: one cannot *say* what the logical form of a sentence is. The logical form is mirrored in the sentence, it can only be shown. In order to say what it is, we would have to go outside the language, outside the realm of the expressible, which would mean outside the world. Similar for the state of affairs a sentence depicts: this can only be *shown*. It cannot be explained what a situation looks like, such things are mirrored in the language and cannot be told:

> 4.1212 What *can* be shown, *cannot* be said.

What is it then, what a sentence *says*? A sentence says that the state of affairs it is a projection of, is actually the case. A sentence says of the possibility it shows, *that* this possibility obtains, i.e. that it is part of the world. A sentence *says* that it is true, and that is all that a sentence can say. It cannot express *that* it says something, nor that it *shows* something. Such meta-expressions, about sentences, the language, the world, or the logic, cannot be said *meaningfully*, because they are not pictures of facts of the world.

Sentences show what they depict: their Sinn_w, the state of affairs of which they say that it obtains. But in the beginning of this section Wittgenstein was quoted saying that the sentence a is an expression of a thought. How does the thought relate to the Sinn_w of a sentence, to the state of affairs it depicts?

> 3 The logical picture of the facts is the thought.
> 3.02 The thought contains the possibility of the state of affairs which it thinks. What is thinkable is also possible.
> 4. The thought is the meaningful sentence.
> 4.01 The sentence is a picture of reality.

The logical picture of a thought is a fact. A thought contains the possibility of the situation it thinks, and states of affairs are represented by sentences. A sentence is a picture of a state of affairs, the representation of the smallest fact in the world in the language, and the thought is the third counterpart in this connection: it is expressed by means of a meaningful sentence and contains the possibility of a state of affairs. The states of affairs are situated in reality and the sentences are part of the language, what is the medium

of thought? According to Wittgenstein, thinking is not something psychological, it is the recognizing of a sentence as something that has a certain structure, namely as depicting a state of affairs.

> 3.11 We use the sensibly perceptible sign of the sentence as a projection of the possible state of affairs.
> The method of projection is the thinking of the meaning of a sentence.

In meaningful sentences these three components are united: sentences picture a state of affairs and are the perceptible counterpart of a thought. Recall that the goal of *Tractatus* was to give the limits of our thinking or, more specifically, of the expression of our thought. In 3.02 it is said that the thinkable, is possible. Everything that is possible, that is every possible state of affairs, is expressible by means of a contingent sentence: everything possible may either be the case or it may not be the case. Meaningful sentences capture exactly this. The limits of (the expression of) our thought can be given from the inside out, by the collection of all meaningful sentences. All meaningful sentences form the logical space, thus all possible states of affairs or, as we now know, all that is thinkable.

4.1.2 Logical Sentences

In section 4.1.1 the central notions of *Tractatus* have been discussed: how thinking, language and the world come together in meaningful sentences. Apart from meaningful sentences there are also sentences that do not *show* a fact in the world: tautologies and contradictions. These necessary sentences are *sinnlos*, because they are true (respectively false) no matter what the world is like. Among them are the 'logical sentences' that are logically valid: the tautologies. The logical sentences do not have a Sinn_w, for they do not represent facts in the world, but they are not *unsinnig*. Logical sentences aren't nonsense; they belong to the symbolism (4.4611) and serve a purpose in logic. What this purpose is and what logical sentence say and show is discussed in the following.

Logical sentences do not *say* anything:

> 4.461 The sentence shows what it says, the tautology and the contradiction that they say nothing.

> 6.11 The sentences of logic therefore say nothing. (They are all analytical sentences.)

Tautologies do not represent a state of affairs and they are necessary. A tautology doesn't determine the world in any way, it leaves all the possibilities as they are, it does not create a partition in the logical space, and that is why it does not *say* anything. A tautology does not *say* that a particular possibility is the case, if it did, its truth wouldn't be necessary.

Can a tautology *show* something? It cannot depict a situation in the world, but does it show something else? Logical sentences *show* that they are tautologous (6.12), this can be derived from their structure alone, and is necessary so, since logical sentences cannot be confirmed by experience. Logical sentences are not necessarily true, shich is what the term tautology usually means, they are just necessary. Truth or falsity does not apply to sentences without meaning, because it is the meaning, the Sinn_w, that can be realized or not. In 4.461 is said that logical sentences are meaningless, and as such they cannot have a truth-value.

Every tautology shows that it is a tautology (6.127), is that all it shows? No, logical sentences serve another (more useful) purpose: they *show* the scaffolding of the world.

> 6.124 The logical sentences describe the scaffolding of the world, or rather they present it. They "treat" nothing.

Logical sentences show the regularity of the world. They do not give a description or a theory of the laws of the world, the logical sentences *show* the logic of the world. They are a "mirror of the world" (6.13). They show the properties of the logical space, for instance by showing that in certain combinations of meaningful signs the meaning is dissolved. Suppose p is a meaningful sentence, then $p \vee \neg p$ is no longer meaningful. This tautology shows that p cannot obtain and not obtain at the same time. In this way, logical sentences show *all* the laws of the world. Because, as Wittgenstein says, outside logic everything is coincidence. The only necessity there is, is logical necessity.

The function of the logical sentences is a clarifying one, they do not *explain* the lawlike regularities in the world, but they *show* them. The tautologies do not say that they are true and relate to reality, this can be read off from their structure. The logical sentences are necessary true, this makes them meaningless, but they are not *sinnlos*, for they show "the scaffolding of the world" (6.124).

4.2 Inferences and Proof

Meaningful sentences are distinguished from tautologies, but it has not yet been discussed what we can do with them. This section is about how sentences are proven and how they can be used in inferences, to illustrate how reasoning works in Tractarian logic.

In sentence 4.023 the connection is made with the last section:

> 4.023 The sentence determines reality to this extent, that one only needs to say "Yes" or "No" to it to make it agree with reality. It must therefore be completely described by the sentence. A sentence is the description of a state of affairs. As the description of an object describes it by its external properties so propositions describe reality by its internal properties. The sentence constructs a world with the help of a logical scaffolding, and therefore one can actually see in the proposition all the logical features possessed by reality if it is true. One can *draw inferences* [Schlüsse ziehen] from a false sentence.

The sentence expresses what the world looks like, if it is true. Here is said that, even if the sentence is not true, it can be used to draw inferences. What are inferences according to Wittgenstein?

A sentence can *follow* from another sentence, is described in 5.121-4.

> 5.121 The truth conditions of q are contained in those of p.
> 5.122 If p follows from q, the meaning of p is contained in that of q.
> 5.123 If a god creates a world in which certain propositions are true, he creates thereby also a world in which all sentences consequent to them are true. And similarly, he could not create a world in which the sentence 'p' is true, without creating all its objects. 5.124 The sentence confirms [bejaht] every sentence which follows from it.

If a sentence is true, that is, if the situation it depicts obtains, then every sentence following from this sentence is true as well. The meaning of the consequent is already contained in the meaning of the first sentence. This looks like a consequence-relation rather than one of inference, as these were distinguished in section 2.1.3. This impression is strengthened by the following passages:

> 5.131 If the truth of one sentence follows from the truth of others, this expresses itself in relations in which the forms of these sentences stand to another, and *we do not need to put them in these relations first* by connecting them with another in a sentence; *for these relations are internal and exist as soon as, and by the very fact that, the sentence exists.*
> 5.132 If p follows from q, I can infer p from q; conclude from q to p.
> The method of inference is to be understood from the two sentences alone.
> *Only they themselves can justify the inference.*
> Laws of inference, which - as in Frege and Russell - are to justify the conclusions are meaningless and would be superfluous.
> 5.133 All consequence [Folgern] take place a priori.[10]

The justification of a logical consequence can be given solely by the sentences themselves. There need to be no inference rules, in fact these are superfluous and not constructive. All that is needed in order to have a deduction are the sentences themselves. The logical consequence exists from the moment the sentences exist and is internal to them. One does not need to establish a consequence relation between sentences, this is already 'out there' in the logical space.

From this we can conclude that Wittgenstein holds an *ontologic* conception of logic, as Sundholm defined this in contrast to the epistemological conception which was held by Frege; the sentences are objects and any consequence relation between them holds independently of a subject discovering it. It does not need to be established that one sentence follows from the other, this is objectively the case. It doesn't even matter whether the sentences considered are true or not, even if their truth is not known, it is clear which sentences follow from them and which don't, this is contained in the sentences. On this conception of logic there is no such thing as an inference, a process in which one sentence is derived from another, since the consequence relations are already 'out there' and do not need to be discovered or acknowledged by a subject.

Wittgenstein nonetheless talks about 'proofs in logic' and what is meant by this is discussed in sentences 6.1 and further.

> 6.126 Whether a sentence belongs to logic can be determined by determining the logical properties of the *symbol.*

[10] Emphasis mine.

> And this we do when we prove a logical sentence. For without troubling ourselves about a meaning [Sinn_w] and a reference [Bedeutung], we form the logical sentence out of others by mere *symbolic rules* [blosse Zeichenregeln].
> We prove a logical sentence by creating it out of other logical sentences by applying in succession certain operations, which again generate tautologies out of the first. (And from a tautology only tautologies *follow*.)
> Naturally this way of showing that its sentences are tautologies is quite unessential to logic. Because the sentences, from which the proof starts, must show without proof that they are tautologies.

A logical sentence can be proven: by means of a successive application of certain operations it can be shown that only tautologies follow from it. The proof is a mechanical expedient to show that a sentence is necessary true, if that is not easily recognized. This proof is made without considering the Sinn_w or Bedeutung of a sentence, it is constructed via mechanical application of "symbolic rules".

This applies only to logical sentences, for which neither the meaning nor the truth value is particularly relevant, since they lack meaning and are always true. In 6.1263 Wittgenstein says that "it would be rather strange to prove a meaningful sentence *logically* from another sentence," as can be done with logical sentences. The logical proof of a meaningful sentence is thus something completely different from the proof of a logical sentence. A meaningful sentence *says* that a situation obtains, and its proof should show that this is indeed the case, is said in 6.1264, but that is the only reference to the proof of a *sinnvoll* sentence in *Tractatus*.

In the Tractarian logic there are no inferences as they appear in Frege's Begriffsschrift. Every (tauto-)logical sentence is its own proof, for tautologies are self-evident. One can give logically equivalent sentences by mechanically constructing a proof, in order to demonstrate that a sentences is indeed a tautology. This is just for clarification, it will never yield actual knowledge, because it should have already been apparent from the symbolism that the sentence is a necessary truth and all its consequences are already 'out there'. The conception of logic laid out in *Tractatus* is an ontologic one, the logical entities and relations are objects, that exist independent of a subject. In fact, there is no role for a subject at all: there is no process or action, nor are there any developments taking place in time by means of which one can come to know something. The subject has no place in reasoning, and it does not belong to logic. There are logical proofs, but these do not require

insight or creativity. This ontologic conception also explains why there are no surprises in logic:

> 6.1261 In logic process and result are equivalent. (Therefore no surprises.)

Whether one can 'prove' a meaningful sentence is addressed indirectly in the next section, where Wittgenstein's 'argument' against assertion will be discussed.

4.3 No Judgment Stroke in *Tractatus*

It is understandable that Wittgenstein does not need the judgment stroke, when one considers the Tractarian logic as described above: neither in the proofs nor anywhere else, is there a role for the subject, let alone for it to acknowledge the truth of a sentence. Besides, there is no epistemological aspect of the relation of consequence, nor is there an act of judging or proceeding from one sentence to another. There is no such thing as 'acknowledging the truth of a sentence', in fact, the truth of a meaningful sentence is solely dependent of the situation it represents. It is explicitly said *Tractatus* that the subject is "ein Unding" (5.5421) (an absurdity), and that it belongs to psychology and not to logic.

> 5.631 The thinking, presenting subject; there is no such thing.

If there is no subject, then one wouldn't need a symbol for 'acknowledging as true' either. Thus, Wittgenstein says:

> 4.442 [...] (Frege's 'judgment stroke' [Urteilstrich] '$\vdash$' is logically altogether meaningless; in Frege (and Russell) it only shows that these authors hold as true the sentences marked in this way. '$\vdash$' belongs therefore to the sentence no more than does the number of the proposition. A sentence cannot possible assert [aussagen] of itself that it is true.) [...]

The judgment stroke, or the 'assertion-sign' $\vdash$ as a whole, is logically meaningless. Logic is not about assertions. This can be found in the *Notes on Logic* as well:

> C40 There are only unasserted propositions. Assertion is merely psychological.
> C45 Judgment, question and command are all on the same level. What interest logic in them is only the unasserted proposition.

Wittgenstein's reasons for rejecting asserion as a logical notion need to be axplained. In sentence 4.063 an explicit objection against the assertion sign can be found, presented in the form of an analogy. This 'Fregean analogy'[11] to undermine Frege's own notion of truth, will be discussed next, followed by an investigation of the indirect interpretation thesis, which suggests that Wittgenstein's interpretation of Frege is based on Russell's representation of Frege's philosophy, rather than on the original works of Frege.

4.3.1 Wittgenstein's Fregean Analogy

> 4.063 An illustration to explain the concept of truth. A black spot on white paper; the form of the spot can be described by saying of each point of the plane whether it is white or black. To the fact that a point is black corresponds a positive fact; to the fact that a point is white (not black), a negative fact. If I indicate [andeuten] a point of the plane (a truth-value in Frege's terminology), this corresponds to the assumption [Annahme] proposed for judgement, etc. etc.
> But to be able to say that a point is black or white, I must first know under what conditions a point is called white or black; in order to be able to say 'p' is true (or false) I must have determined under what conditions I call 'p' true, and thereby I determine the meaning [Sinn_w] of the sentence. The point at which the simile breaks down is this: we can indicate [zeigen] a point on the paper, without knowing what white and black are; but to a sentence without a meaning corresponds nothing at all, for it signifies [bezeichnet] no thing (truth-value) whose properties are called "false" or "true"; the verb of the sentence is not "is true" or "is false" - as Frege thought - but that which "is true" must already contain the verb.
>
> 4.064 Every sentence must *already* have a meaning [Sinn_w]; the affirmation [Bejahung] cannot give it a meaning, for what it affirms is the meaning itself. And the same holds of denial, etc.

This objection is directed against a conception of assertion as predicating 'is true' of a sentence. Wittgenstein does this by means of an analogy that is inspired by one of Frege's own.[12] The analogy is made between two

[11] Terminology is Potter's, Potter (2009).
[12] Potter (2009), p. 89

processes: saying of a point on a piece of paper whether is it white or black, on the one hand, and the assertion of a sentence on the other.

The argument proceeds as follows: suppose, I have a white sheet of paper, with a black patch on it, what is involved in saying that a particular point on the sheet is white or black? At first I can indicate the point; I localize a particular point and name it P. This indication is analogous to the Fregean 'name of a truth-value' or 'assumption'.[13] This assumption is the mere utterance of a sentence A. When I say about P whether it is white or black, I would be asserting something. In saying 'P is black' I make a statement about the point that is either true or false, and in order to make such a statement I would have to know under what circumstances P is to be called 'black'. I would have to know what 'black' and 'white' are, at least to the extent that I know which points fall under which predicate. 'P is black' corresponds to the assertion of the sentence A, i.e. to $\vdash A$. Analogously, in order to judge that A is true, I have to know under which circumstances I can do so.

There are thus two phases in predicating a point to be black or white: the indication of the point and the predication. Only for predication knowledge of 'black' and 'white' is required. If the analogy were correct, it should be possible to name A, to merely utter a sentence, without knowing what truth and falsity are. But then, A would be a sentence without Sinn_w, says Wittgenstein, as is explained in 4.064: a sentence must already have a Sinn_w in order to be a sentence at all. It is inherent in sentences that they already have a meaning, the confirmation of its truth cannot give the sentence this. When predicating A with 'is true' or prefixing it with $\vdash$ it must already *show* its Sinn_w and *say* that this situation obtains: "in order to express a thought, I have to realize that thoughts aim at truth."[14] The thing that is true in A cannot be predicated of it, that which 'is true', the Sinn_w, must already a part of the sentence. In Frege's logic it is not, seems to be Wittgenstein's reproach in 4.063. There cannot be *sinnvoll* sentences to which the truth value is attached later. Indication of a truth value and predication are not two separate moments, as Frege suggests.

The issue of two phases in the assertion a sentence is also discussed in Wittgenstein's *Philosophische Untersuchungen* (*PU*). In this later work Wittgenstein no longer entertains his Tractarian logic, instead he describes the theory of language games, which is by far not as strict and formal as his

[13] The name of the truth value is called 'assumption' in *Tractatus* as well as in Russell's writings, as will be discussed in more detail in section 4.3.3.

[14] Potter (2009), p. 89

early ideas. This book falls outside the scope of our hypothesis, but in §22 of *PU* the Fregean assertion is discussed and it is interesting to notice that Wittgenstein still holds the same opinion about this: "it is a mistake if one thinks that assertion consists of two actions, entertaining and asserting."[15] Even in a completely distinct framework, he refutes that there is an act of assertion on top of that of expressing a sentences.

The line of reasoning in *PU* §22 is the following: Wittgenstein compares the assertion sign with a punctuation mark. In the same way as the full-stop marks the end of a sentence, the assertion sign signifies the beginning of a statement.[16] As such the assertion sign marks the distinction between a whole sentence from a clause within it. When a sentence is written down or said by someone who *means* it, then there need not to be an act of assertion on top of that. In a natural language example, a Fregean assertion would look like this: "It is asserted that such-and-such is the case."[17] The assumption contained in this assertion is "... that such-and-such is the case". What is left of the assertion without "*It is asserted*" is not a well-formed sentence in English, and as such it cannot be a *move* in our language game, or (as Frege would put it) the premise of an inference. Frege needs the assertion on top of the assumption, to complete the sentence and to signify which role it will play in the language game. An assumption could become a question, an assignment, a promise or a judgment, and the latter case is represented by the assertion sign.

Wittgenstein changes the natural language example slightly, to show that this function may be performed by a punctuation sign as well: "It is asserted: such-and-such is the case.". "Such-and-such is the case" is an acceptable sentence, which makes the words "It is asserted" superfluous. They aren't necessary to make the statement that something is the case; if a sentence said sincerely, one doesn't need additional symbols to signify that it is *meant*. One might use the assertion sign to do so, to contrast it with a question for instance, but according to Wittgenstein this is rather artificial; when uttering a sentence[18] one *means* a sentence, and there is no additional act needed to highlight that. The mistake Frege is making

[15] *Philosophische Untersuchungen*, §22

[16] Note that the comparison between the assertion sign and full-stop is made by Russell as well. This was discussed in section 3.2.2 in the introduction of the assertion sign in *PM*: "In ordinary written language a sentence contained between full stops denotes an asserted proposition, and if it is false the book is in error." (*PM*, p. 8

[17] *Philosophische Untersuchungen*, §22

[18] Or write down a statement, but let's stick with the utterance example to make things clearer.

is that he presupposes an 'assumption', a "proposition radical", that may become an assertion, or another move in a language game. This assumption, consisting of the Thought and the truth-value, need not to be acknowledged op top of being expressed.

That is what Wittgenstein tries to show by means of this analogy: the difference between (Wittgenstein's conception of) Fregean assertion and Wittgenstein's own idea of a sentence *saying* that its Sinn_w is true. In *Tractatus* a sentence cannot take a meta-position and compare the Sinn_w it *shows* with the world. Therefore, the sentence cannot acknowledge its own truth. A sentence cannot 'check' whether it corresponds to the world and decide upon the truth of the claim it makes, there are no extra-linguistic acts attributed to sentences. Nevertheless, every sentence *aims* at truth. A sentence is a picture of reality and as such it already makes a claim about what reality looks like. That is what 4.022 expresses: a sentence depicts a situation and *says* that this situation is the case. Sentences are not completely neutral phenomena; in *showing* a situation, they already 'support' that situation. It would be incorrect to ascribe intentions to sentences, but a sentence depicts a possibility and embodies that possibility in the same time: the possibility may in fact not obtain, but the 'proposal' the sentence makes is that it does obtain. In representing a possibility, the sentence more or less puts the truth of this possibility under consideration.

> 4.2 The meaning of the sentence is its agreement or disagreement of the possibilities of the existence and non-existence of states of affairs.

It is not asserted in the Fregean sense, for this agreement is not assertion. Whether it is actually true, or whether it corresponds to reality, seems to be irrelevant to Wittgenstein, at least for logical considerations. It may be part of the natural sciences or of interest to someone, but it is not relevant when formulating the limits of our thought. The actual truth of a sentence can only be established by a subject, and it is already mentioned that a subject does not belong to logic.

The argument against Frege is directed against the two 'phases' of truth he distinguishes: the acknowledged truth and the name of the truth-value. What Frege considers to be the central notion in logic, judgment, falls outside the scope of Tractarian logic. Neither the role of the subject, acknowledging the truth of a sentence, nor the actual truth of a statement, are part of logic in *Tractatus*. The subject as an acting and judging person belongs to psychology, and what the world actually looks like is topic for the natural sciences. In *Notes on Logic* and in *Tractatus*, the acknowledged truth is

considered to be psychological, which explains why Wittgenstein judges the related assertion sign to be logically meaningless. It has no meaning in logic for Wittgenstein, but Frege's definitions of the terms 'logic' and 'psychology' are different than Wittgenstein's. Therefore, the two come to different conclusions about the judgment stroke.

This argument against Fregean assertion does not indicate full comprehension of the fact that Frege's conception of logic differs substantially from the one Wittgenstein holds. Whether this is due to Russell's representation of Frege will be discussed in the next, when the arguments for and against the indirect interpretation thesis are investigated.

4.3.2 Is Russell to blame? Arguments in favor of the indirect interpretation thesis

One of the first to state that Russell's interpretation of Frege had influenced Wittgenstein's rejection of the judgment stroke, was G.E.M. Anscombe in her *Introduction to Wittgenstein's Tractatus* of 1959. In a footnote she discusses Wittgenstein's attribution of the term *Annahme* to Frege.[19] In his 'argument' against Frege, in 4.063 of *Tractatus*, Wittgenstein makes an analogy between a 'Fregean assumption' that will be judged, and a point on the paper. About this Anscombe says that 'assumption' is not a technical term in the works of Frege. It is used in *Funktion und Begriff*:

> This separation of the act from the subject matter of judgment seems to be indispensable; for otherwise we could not express *a mere supposition* [eine blosse Annahme] - the putting of a case, without a simultaneous judgment as to its arising or not.[20]

The mere assumption, the putting of a case, is to be distinguished from the act of judging it as true. The term 'assumption' is not introduced as a formal term here and does not reappear very often in Frege's writings. In *Principles* Russell does conceive 'Annahme' in this way, when identifying the three elements of Fregean judgments in Appendix A: "Frege, like Meinong, calls this an *Annahme*."[21] The truth-value of a thought is called 'assumption', which is similar to what Wittgenstein says in 4.063: "if I indicate [andeuten] a point of the plane (a truth-value in Frege's terminology), this corresponds to the assumption [Annahme] proposed for judgement." The truth-value is the assumption which may be judged, and therefore Anscombe concludes:

[19] Anscombe (1959), p. 105
[20] *Funktion und Begriff*, p. 21-22
[21] *Principles*, p. 503

> Russell's remarks, which mistakenly give special prominence to Frege's use of the word 'assumption', must be the source of Wittgenstein's references to it.[22]

The word assumption comes from Russell, and that is not the only terminological simililarity between Russell and Wittgenstein described by Anscombe in the same footnote. Another example can be found at the end of 4.063, where Wittgenstein writes:

> 4.063 [...] The verb of the sentence is not "is true" or "is false" - as Frege thought - but that which "is true" must already contain the verb.

Several non-Fregean aspects can be identified in this sentence, the first matter of terminology concerns the word 'verb'. Frege did not employ the term 'verb' in to describe assertion, only Russell did so: in *Principles of Mathematics* the verb' of a proposition effects the its assertion. In contrast to a verbal noun, a verb-construction was able to assert something, to say that something is so-and-so and that it is actually the case. Assertion as part of a proposition, does not reappear in Appendix A, nor in *PM*, which makes it remarkable that Witgenstein nonetheless attributes this term to Frege's analysis of judgment. However, when we conceive the verb as the predicating part of a sentence rather than effecting assertion, as Russell does in §52 of *Principles*,[23] this usage of Wittgenstein makes sense. He means to say that the assertion sign functions as a predicate and adds the predicate 'is true' to the proposition.

But not only in the works of Russell the assertion sign is called a truth-predicate, this comparison is also made by Frege himself. In *Begriffsschrift* he explains the working of the judgment stroke in terms of it predicating "is a fact" and "the sign $\vdash$ is their common predicate for all judgments."[24] Reading the context of these passages makes clear that Frege doesn't say that the judgment stroke *is* a predicate, but that he tries to explain the working of the judgment stroke by means of this (rather unfortunate) comparison. That this comparison may give rise to confusion is recognized by Frege and this explanation is explicitly rejected in his later works.

Wittgenstein's conception of the judgment stroke as predicating 'is true' can be traced back to Frege as well as to Russell. Had Wittgenstein read

[22] Anscombe (1959), p. 106, fn. 1

[23] In §52 Russell explains that he prefers the term-verb distinction of that between subject and predicate, but the verb is the 'predicating part' of the proposition, for it says that things are 'thus-and-so'.

[24] *Begriffsschrift*, p. 4

Frege's *Begriffsschrift* and his later works more attentively, then he would have understood that Frege rejected this predicate-explanation of the working of the judgment stroke and didn't even intend it in the first place. In the works of Russell, there is no explict rejection of this reading. In *Principles* it is mentioned that the assertion sign is predicating 'is true' or 'is false', but also that it is to be distinguished from truth. In this work the question of how to account for a logical notion of assertion, other than conceiving it as a truth predicate, remains unanswered. In *PM* it is also said that the assertion sign may be read as saying "it is true that", but this is followed by the warning that it is philosophically incorrect to define it in this way. Since Russell compares the assertion sign to a truth predicate in many places and does not offer a clear explanation of how to perceive it in a correct way, it makes sense to explain the remark in 4.063 about the verb of the sentence as based on Russell, rather than Frege.[25]

Another aspect that seems to be inspired by Russell appears in the same passage, "the verb of a sentence is not 'is true' or *'is false'*." This suggests that Frege has two kinds of assertion: a positive and a negative assertion, of which 'is true' would be represented by '$\vdash$' and 'is false' by '$\vdash\!\!\!\top$'. But the opposite is the case; in Frege's Begriffsschrift, only true sentences can be acknowledged as true. There is no such thing as acknowledging the falsity of a sentence in Begriffsschrift. This should not be confused with the assertion of a negative sentence, which is unproblematic. In asserting a negative sentence there is a negated sentence, like '$\neg A$', that is the name of the True, expressing that "A is not the case."[26] If '$\neg A$' is true, then this sentence can be asserted by putting the judgment stroke left of it. This is not a denial of A, but the assertion of the truth of a negative sentence about A.

In *Principles* Russell does not distinguish these two things when explaining the assertion sign as a truth-predicate. The assertion can either predicate 'is true' or 'is false' of a proposition, which is similar to 4.063, but absolutely not what Frege meant. In section 3.2.2 is mentioned that in *PM* the symbol '$\vdash$' involves the truth of the proposition it prefixes. But this does not imply that Russell considers it problematic to read '$\vdash\sim p$' as "it is false that p." Frege would not allow this, because of his constraint that only true judgments may be used when drawing inferences. The complex of symbols '$\vdash\!\!\!\top$' means "it is true that not...". Neither Russell nor Wittgenstein has a similar constraint, reasoning can proceed from true as well as from

[25] Especially on Russell's *Principles*.
[26] *Beggriffsschrift*, p. 10

false premises.

Related to this is the conception of the assertion sign as a one symbol '$\vdash$' or as a composed sign consisting of two symbols. Frege holds the latter perspective and considers it to be composed of the judgment stroke and the horizontal part, which was called 'content stroke' in his early works. In fact, he doesn't even have a name for the combination of the vertical and horizontal part, since it consists of two separate symbols.[27] Russell, in *Principia Mathematica*, only discusses the 'assertion sign $\vdash$'. This is a simple sign that effects assertion; it says of a proposition that it 'is true' or that it is acknowledged to be so. The judgment stroke and the horizontal are not distinguished anymore, probably because there is no individual role for the two parts in the logic of *PM* and *Principles*. Russell does not consider the 'name of a truth-value', which is prefix by a horizontal '-'. Assumptions in his logic do not have a prefix at all, unless in the case that the whole proposition is negated: when p is negated, this is expressed by '$\sim p$'. The negation functions as a sentential operator, rendering a new sentence $(\sim p)$ with opposite truth-conditions. However, this is not referred to a as 'name of a truth-value' and it is not relevant that the truth-conditions, and thus the name of the truth value, have changed. The negated sentence may be asserted, by putting the assertion sign in front of it: '$\vdash\sim p$', which may be read as "it is false that p" or as "it is true that not p".

In 4.442 Wittgenstein refers to '$\vdash$' as '*Urteilsstrich*', judgment stroke. He does not adopt Russell's terminology, but he does take it to be a simple sign, signifying that the authors hold the sentence it prefixes to be true. Wittgenstein employs a Fregean name, but he clearly uses the $\vdash$ in the same way Russell does: as a simple sign signifying assertion.

So far we have pointed out several Russellian features in the *Tractatus*: the terminology, the conception of the $\vdash$ as a *simple* sign that is able to predicate *both* the truth and the falsity of a proposition. In the next section arguments against the indirect interpretation thesis are discussed, in order to be able to say whether the similarities discussed here are enough to conclude that Russell is to blame for Wittgenstein's interpretaton of the Begriffsschrift and the role of the judgment stroke in it.

[27] On page 4 of *Begriffsschrift* Frege refers to $\vdash$ as "the sign" (as quoted above). This is a reference to it as 'one' thing, but throughout the *Begriffsschrift* as well as in other works, Frege stresses the fact that it are in fact two signs: a judgment stroke and the horizontal or content stroke.

4.3.3 Is Russell to blame? Arguments against the indirect interpretation thesis

Various authors responded to Anscombe's claim that Russell must be the source of Wittgenstein's misconception of Frege, among them are Ian Proops, Michael Potter en Thomas Ricketts. These three attribute to Wittgenstein a thorough understanding of Frege and ask the question whether Anscombe isn't underestimating Wittgenstein, when she says that he just followed Russell's erroneous interpretation of Frege. This is a complaint that is often heard: why would an appreciated philosopher as Wittgenstein fail to see the things others do notice?

An answer to this can be found in *Tractatus* itself: in the Preface Wittgenstein explains that *Tractatus* will perhaps only be understood by those who have already thought the thoughts expressed in it themselves, or at least similar thoughts. Reasoning from the fact that the thoughts Wittgenstein expresses in *Tractatus* presuppose a completely different conception of logic than that of Frege, it might be concluded that Wittgenstein did not understand Frege's writings. He may have read and interpreted *Begriffsschrift, Grundgesetze, Grundlagen*, and some of the shorter papers, but since Wittgenstein's ideas were very distinct from Frege's, he wasn't able to grasp Frege's intentions. And as such, failed to see what the judgment stroke was and why it was so important for Frege.

This is not the same as underestimating Wittgenstein, since it doesn't mean that he did not try hard enough, or didn't read as carefully as he should have, but that his conception of logic was so distinct that he was not capable of seeing what Frege meant. Central notions in this debate, such as logic, psychology, judgment and assertion have a completely different meaning for each of the three philosophers discussed here, which prevented them from fully understanding each other.

Thus far, this line of reasoning seems an easy way out, it needs to be investigated whether it would make sense to understand Frege in another way than the exposition of his work in chapter 2 of this thesis, when one attaches a different meaning to some of the central concepts of Frege. Ian Proops defends this position and gives some examples of passages from Frege that may have been the source for Wittgenstein's interpretation in *Tractatus*, which differs from the reading presented here. The first of these 'confusing passages' can be found in *Grundgesetze*. Proops identifies two phrases about "mere equations", such as '$2 + 3 = 5$', that only designate a truth-value and do not yet assert anything.[28]

[28] *Grundgesetze*, §2 and §5

> By drawing an analogy between names of truth values and complex numerals, Frege might seem to be suggesting that these expressions are alike in failing to express complete thoughts, rather than alike in failing to express judgments. The two passages thus work in tandem to create the impression that the assertion sign is invoked to enable a mere name to do linguistic work of an appropriately fact-presenting kind.[29]

Another example, in the same book, is Frege's explanation of the assertion of negative sentences. In §6 the judgment '$\vdash_{\top} 2^2 = 5$' is paraphrased as "$2^2 = 5$ is not the True." [30] And '... is not the True' may be read as a mere notational variant of 'is false', which could have been the reason to say that the verb is either 'is true' or 'is false' in 4.063, as we saw in the previous section. The special status Frege attached to the judgment stroke is a third example to explain why Wittgenstein got the wrong impression from Frege himself; this special status may be conceived of as extra-logical.

Proops has a point here: it makes more sense to look at the original work of Frege, instead of investigating secondary sources, such as Russell's *Principles* and *Principia Mathematica*. The examples he gives, can be found in *Grundgesetze*, and when put together like this, they might be confusing. In order to say whether these passages actually *caused* the Tractarian interpretation, further investigation is needed. Let us therefore take a closer look at the example of 'is not the True':

> According to our stipulation $_{\top} 2^2 = 5$ is the True, hence
>
> $$\vdash_{\top} 2^2 = 5$$
>
> in words: 2^2 is not the True; or: the square of 2 is not 5.[31]

What is the 'stipulation' referred to in this passage? I think this is the definition of negation that it is given in §6 of *Grundgesetze*. The negation stroke combined with the horizontal is a function '$_{\top}\,\xi$' rendering a name of a truth value in the following way: if '$-\xi$' means the True, then '$_{\top}\,\xi$' is the False, and in all other cases it is the True. One page earlier, in §5 of *Grundgesetze*, it was said that " '$-2^2 = 5$' refers to *[bedeutet]* the False" , and thus '$_{\top} 2^2 = 5$' is true and may be asserted. In this particular section it is not stated explictly that only true sentences may be asserted, and for

[29] Proops (1997), p. 132
[30] *Grundgesetze*, §5
[31] *Grundgesetze*, §6

this reason it may be unclear that Frege indeed maintains this demand, while allowing that *negated sentences* may become judgments, as in the example. Besides, since Wittgenstein seems to conceive $\vdash$ as a simple sign, it would make sense to take $\vdash\!\!\top$ as a simple sign as well. Keeping this in mind, it is understandable that Frege might the confusing in this example. As in section 4.3.2, when it was considered whether the judgment stroke was called a predicate in *Begriffsschrift*, Frege leaves out relevant information, which does not help in stating his case as clear as possible. The fact that he does not explicitly make the link with his demand that only true sentences may become judgments, and that only true premises can be the starting point of an inference, is certainly ground to classify this passage as unclear. However, I am not convinced that this justifies an interpretation that $\vdash\!\!\top$ means 'is false', because the demand that only true sentences may be asserted still holds: the sentence that is asserted in the example is in fact true. The square of 2 is not 5, thus its negation is true, and as such there is no motive to think that also false sentences may be asserted. The confusion is understandable when the passage is considered in isolation and not in the context of the whole Begriffsschrift-framework.

This is not the only confusing example in Frege. When discussing whether the judgment stroke represented an act or actually effected the act of judgment in section 2.1.2, I mentioned that there exists ambiguity on this matter in Frege's writings. And that is just one example where one one might run into trouble when interpreting it; Frege's ideas develop, that is the reason Macbeth speaks of 'early' and 'mature Frege', but he is not very keen on admitting his own mistakes. Frege can only rarely be caught explicitly denying his earlier views; most alterations to his framework are made without pointing out the differences with earlier ideas. A rare example where he is explicitly correcting himself can be found in *Funktion und Begriff*, when Frege introduces the horizontal:

> I used to call this horizontal stroke the content stroke - name that no longer seems to me appropriate. I now wish to call it simply the horizontal.[32]

When one has not read this particular paper, the shift from 'content stroke' to 'horizontal' in Frege's works, may be confusing. It is imaginable that one would take this to mean that the horizontal no longer has a special function, since 'horizontal' is a rather neutral term, as opposed to 'content stroke'. In the discussion of the term 'assumption' in the last section, we

[32] *Funktion und Begriff*, p. 21

already saw that it is not always clear in Frege what the technical terms are en which words are used in their every-day meaning. Since horizontal looks even less like a technical term then 'content stroke', it is understandable that one might interpreted this as just a stroke, without a special purpose in Begriffsschrift. Whereas, its function is in fact very feel described; the horizontal turns everything that is not yet a name of a truth value into one, namely the False (if its Bedeutung is not yet a truth value, otherwise the truth value remains as it is).

This quote is an example in which Frege explicitly corrects an earlier view, and as such rules out the 'imaginable confusion' that was sketched. In Sinn and Bedeutung similar explicit remarks can be found, regarding Frege's revision of the concept of 'content' into Sinn and Bedeutung. However, in *Grundgesetze* these explicit remarks do not reappear. The framework was altered in the abovementioned shorter papers, and in *Grundgesetze* Frege builds on these improved foundations, without listing all the reparations that were made. As a consequence, when one is not familiar with *Über Sinn und Bedeutung*, *Funktion und Begriff*, and other papers in which one particular aspect of the Begriffsschrift is addressed, passages from *Begriffsschrift* and *Grundgesetze* may give rise to some confusion. Proops counterargument, that Frege's own writings may have been the source for Wittgenstein interpretation of Frege, is justified to some extent. In my opinion it makes more sense to come to the interpretation as presented in chapter 2, than to the one Wittgenstein gives in *Tractatus*, when one considers Frege's mature conception of the Begriffsschrift and the notion of inference appearing in it.

The horizontal, the concepts of Sinn and Bedeutung, and the function-argument distinction are all discussed in shorter papers. Papers in which Frege explains his perspective and how this might have changed over the years. The example which lead us to investigate whether Frege might be confusing, was the question whether 'is false' could come from Frege. This example is not in all aspects similar to the horizontal. Apart from the fact that this constraint in Frege's logic is rather particular (not many logicians employ the demand that one can only reason from true sentences, but then again, not many logicians have a notion of a Fregean Sinn), it is also not explicated very clearly in his works. As early as in *Begriffsschrift*, Frege stresses the point that one can only reason from true judgments. He also says that only true sentences can be asserted and he does not use the phrase 'it is false that...' when referring to an asserted sentence, but the most obvious passage in which Frege explains this can be found in a letter to Dingler. Frege's personal correspondence was not published until 1979, and it is likely to assume that Wittgenstein did not have access to it. As a consequence,

Proops seems to be right in saying that confusion about whether one can assert that a sentence is false may come from the works of Frege himself.

Another counterargument comes from Michael Potter. In his book on *Wittgenstein's Notes on Logic*, he presents another argument against the indirect interpretation thesis. According to Potter, the terminology cannot be a decisive argument, since Wittgenstein was just using terminology in ways then current in Cambridge. The best argument, according to him, is that

> Russell's discussion[33] of Frege's conception of sentences as names of truth values is riddled with confusions and misunderstandings; Wittgenstein's, as I have outlined above, is not. That is enough to cast doubt on the idea that Wittgenstein approached Frege's writings via Russell: if he has, one would surely expect his criticisms of them to have been equally confused.[34]

Wittgenstein's argument is structured and shows a profound engagement with Frege's conception of logic. This is based on Wittgenstein's understanding of the relationship between thought and truth.

This calls for clarification: are the criticisms in *Tractatus* indeed that straightforward and do they reflect a profound understanding of Frege's conception of logic? In 4.442 it is said that the judgment stroke shows that something is acknowledged as true, which is indeed what Frege intended the judgment stroke to do. As such 4.442 reflects an understanding of the judgment stroke as an *act* of a subject, that - apart form being true - something is acknowledged as true. Frege has rather strict norms prescribing when something may be asserted, namely when it is objectively true. From the phrase 'that the authors held this to be true', it may be concluded that this objectivity is not fully grasped by Wittgenstein. That the authors held something to be true suggests that we are talking about an opinion or a mental state, but that is not what Frege had in mind, according to him only true sentences can become judgments. Suppose that Wittgenstein indeed interpreted him in this way, still, judgments would not belong to logic, as Wittgenstein defines this 'science'. From sentence 4.442 alone, it cannot be concluded whether Wittgenstein has understood the objective norms underlying the 'acknowledgement by a subject'. He did grasp the fact that the judgment stroke represents an act of a subject, and that it is not a truth predicate, rather an attitude of a subject towards a statement.

[33] In §477 of *Principles*.
[34] Potter (2009), p. 90

The same question, whether Wittgenstein fully understood Frege, is addressed by Ricketts. He agrees with Potter to the extent that he doesn't "attribute a full appreciation [to Wittgenstein] of the subtleties of Frege's position."[35] But he does think that Wittgenstein understood the difference between predication and assertoric force, even though he may not have been aware of the circumstances under which something may be judged as true. Another 'subtlety' that both Wittgenstein and Russell do not fully appreciate is the notion of Sinn. Russell dismisses Sinn in *PM*, because it is not an extensional concept, as everything in logic should be according to Whitehead and him. Wittgenstein uses the terms Bedeutung and Sinn throughout *Tractatus*, which makes perfect sense, they are normal German words; the confusing thing is that they mean something different to Wittgenstein and Frege. In Frege every linguistic entity has both a Sinn and a Bedeutung, whereas in 3.3 Wittgenstein says that:

> 3.3 Only the sentence has meaning [Sinn_w], only in the context of a sentence a name has reference [Bedeutung].

Names have a Bedeutung, the object they represent, and sentences have a Sinn, which is the state of affairs they depict. The notions Sinn and Sinn_w do not coincide, this is relevant because Wittgenstein does not distinguish between them very clearly in his comments on Frege. For instance in 6.232 when he argues against the identity-sign as a logical symbol, but also in the analogy in 4.063 it is unclear which 'Sinn' is supposed to be Frege's Sinn, and when 'Sinn' means Sinn_w. According to Proops the

> second paragraph [of 4.063] is not a continuation of the analogy. Rather, it presents Wittgenstein's own views about what it is to have a grasp of the notion of truth.[36]

The same holds for 4.064, which says that the affirmation cannot give a sentence a Sinn. This sentence is not to be conceived as a complaint to Russell and Frege, instead it reflects Wittgenstein's own ideas about Sinn_w. The fact that both Wittgenstein and Russell do not accept nor fully analyze the Fregean Sinn is not a argument in favor of the indirect interpretation thesis, but it raises the question to what extent the analogy in 4.063 can be conceived of as an argument against Frege's notion of Sinn. The analogy does explain that Wittgenstein rejects the two acts Frege recognizes in the assertion of a sentence, but it fails to reflect a full understanding of the notion of 'Sinn' in Begriffsschrift.

[35] Ricketts (2002), p. 242
[36] Proops (1997), p. 131

4.3.4 Evaluation of the Arguments

All the arguments for and against the indirect interpretation thesis have been collected in the sections above. The only remaining question is whether it is plausible to conclude form these arguments that Wittgenstein's interpretation of the judgment stroke has been affected by Russell's confusing representation of Frege in *PM* and *Principles*.

Arguments in favor of the indirect interpretation thesis are related to the notation and terminology of Wittgenstein in *TLP*, which is very similar to Russell's: The linear notation, the $\vdash$ as a simple sign instead of composed, referring to 'assumptions' and 'verbs' when discussion the judgment stroke, are examples of this. Some of these can be traced back to Frege's *Begriffsschrift*, but in later works, *Grundgesetze*, *Funktion und Begriff* and *Über Sinn und Bedeutung*, these notions are refined and this lead to the interpretation of the mature Frege as presented in chapter 2. In *Tractatus* similar mistakes are made concerning the interpretation of Frege's judgment stroke as in Russell's *Principles of Mathematics*. That both Russell and Wittgenstein take the assertion sign to function as truth predicate and are thereby neglecting the function of the horizontal, is an argument in favor of the indirect interpretation thesis. There are terminological examples showing that Wittgenstein has read Frege; for instance his use of the word 'Urteilsstrich' in 4.442 and the fact that he considers a Fregean 'Sinn' in contrast to 'Bedeutung' in 6.232. But the indirect interpretation thesis doesn't make the insensible claim that Wittgenstein had not read the works of Frege, rather it states that his interpretation has been influenced to large extent by Russell. A reproach to these arguments and the indirect interpretation thesis is that the terminology and notation were common practice at the time in Cambridge, but this is disqualified by the substantive arguments that the meaning of the Fregean signs and terms has been changed in Russell's interpretation; an assertion sign is not a truth predicate or verb, it is an *act*, according to Frege.

Both Russell and Wittgenstein recognize that the assertion sign represents a conviction of an author, that the asserting subject acknowledges the sentence as true. Where Russell is searching for a logical counterpart of this, Wittgenstein takes the more radical approach and decides that the assertion sign cannot be a logical symbol, for the conviction of a subject does not belong to logic. Frege's subject is not just an author. In the theory of the 'Gedanke', the *objective truth* of thoughts to be grasped is stressed. This idea is reflected in the notion of the subject as well; the Fregean subject cannot make falsely acknowledge something to be true, since the truth of

the statement is objective. The subject is idealized in a Kantian way, and this ideal subject does have a role in logic; the judgment of a subject is relevant for inference, and everything that is needed for inferences, belongs to logic, says Frege.

Wittgenstein's own argument against two acts in asserting, as it is given in the analogy in 4.063, does not show a thorough understanding of Frege's notion of Sinn, and may be perceived to disqualify itself for this reason. However, combined with what Wittgenstein says in *PU*22 it becomes clear what his real objection is against Frege: the fact that he presupposes an assumption as 'proposition radical' which needs to be entertained before it may be asserted. Wittgenstein considers it superfluous to have two separate acts in asserting; a sentences always aims at truth, this is what it *says*, one does not need an act of assertion on top of that.

There are reasons to reject the indirect interpretation thesis, as laid out in 4.3.3. Of these, the argument that the terminology was current at the time in Cambridge, makes a case, but it remains highly coincidental that Russell and Wittgenstein make the same 'mistakes' in their interpretations of Frege. It would make more sense to consider whether these flaws could be traced back to Frege, as it appears: to some extent they can. Especially in *Begriffsschrift* some confusion passages are found; remarks that leave room for another interpretation than the reading of the mature Frege given here. Frege made refined his Begriffsschrift over the years; he improved the function-argument structure, added the notions 'Sinn' and 'Bedeutung', renamed the horizontal and idealized his subject. In doing so he did not always make explicit what passages were revised, or which elements changed and which remained the same. Confusing examples from Frege himself, as discussed in 4.3.3, are potentially a very strong argument against the indirect interpretation thesis, and there certainly are ambiguities in his works. Especially concerning the possible predication of 'is false' this ambiguity is relevant for the hypothesis investigated here. This appears in Russell's *Principles* as well as in Wittgenstein's *Tractatus*, but from reading Frege's works one cannot come to the conclusion that the assertion sign expresses that the statement it prefixes 'is false'. A nuance to be made is that in *PM* Russell recognizes that an assertion always involves the *truth* of a statement, when we consider the 'is false' to be an argument in favor of the indirect interpretation thesis, this would have to be restricted to *Principles*. All in all, whether assertions may be false or how to paraphrase this is an issue that needs further investigation.

There is enough evidence to consider the indirect interpretation thesis plausible. Even more indications can be found when Witggenstein's

Philosophische Untersuchungen is considered as well, since the full-stop comparison seems to be copied from Russell. However, it can not be concluded (and I certainly do not want to do so) that Russell's interpretation is the only reason for Wittgenstein to reject the judgment stroke, nor that Wittgenstein merely copied Russell's interpretation of Frege. Wittgenstein came to a different conclusion than Russell and presented his own argument for it, but the similarities between their conceptions of logic and their interpretations of Frege are clearly present.

Chapter 5

Conclusion and Discussion

This project was an attempt to give a interpretation of Frege's judgment stroke and to investigate the indirect interpretation thesis. In order to do so Frege's conception of logic and the role of the judgment stroke in it were analyzed. The Begriffsschrift is an unusual logic, for instance because of its 2-dimensional notation and the fact that Frege was one of the first to specify rules of inference. In this project the focus was on Frege's *act of judgment*. The judgment stroke is an act of a subject that acknowledges a sentence to be true, what Frege exactly meant by this needed to be analyzed before investigating the interpretations of Russell and Wittgenstein.

Russell questions the existence of a logical notion of assertion and presents an interpretation of Frege's judgment stroke that differs in some aspects from the interpretation given in the first part of this thesis. Remarkably, many of those differences can be found in Wittgenstein's *Tractatus* as well. The similarities Wittgenstein's and Russell's interpretations of Frege can be conceived of as a ground for adopting the indirect interpretation thesis. It is not unlikely that Russell's representation of the judgment stroke was an important influence on Wittgenstein's interpretation of it, but a notable distinction between these two philosophers is the conclusion drawn in 4.442 of *Tractatus*: that the judgment stroke is not to be considered a *logical* notion. I doubt that Russell's representation of the judgment stroke is the only reason for this; the conceptions of logic of Frege and Wittgenstein differ in many more aspects. An important distinction between Wittgenstein and Frege concerns their idea of what logic *is*; what we need it for, what its subject matter is and whether it may be used to extend our knowledge.

Apart from these differences there are also many similarities between Frege and Wittgenstein, both presuppose a strong connection between think-

ing and logic. This may be one of the reasons Wittgenstein admired Frege's work for. Because of the focus of this project on the indirect interpretation thesis and the judgment stroke, this aspect has not been discussed elaborately. The same holds for many other related issues: would there be any contemporary applications of an act of judgment in logic? The different notations could be investigated further and also the Kantian aspects of Frege and Wittgenstein might be a line of further research, as well as the elements of Frege's Begriffsschrift that did survive in the way Frege intended.

In this thesis an interpretation of the judgment stroke was given, the indirect interpretation thesis and the three logicians involved were discussed. There are similarities between Russell and Wittgenstein and their interpretation of Frege; both lack a thorough understanding of the subtleties concerning the act of judgment and attach a different meaning to the judgment stroke than Frege. As a consequence the judgment stroke as representing and effecting an act of judgment has not survived in logic.

Bibliography

[1] Anscombe, G. E. M. (1959). *An Introduction to Wittgenstein's Tractatus*. Hutchinson University Library.

[2] Awodey, S. and Reck, E. H. (2002). Completeness and Categoricity. Part I: Nineteenth-century Axiomtics to Twentieth-century Metalogic. *History and Philosophy of Logic*, 23:1-30.

[3] Beaney, M. (1997). *The Frege Reader*. Blackwell Publishing Ltd.

[4] Blanchette, P. (2010). The Frege-Hilbert Controversy. In: Zalta E. N. (ed.) (2010). *The Stanford Encyclopedia of Philosophy*.

[5] Dutilh Novaes, C. (2010). Surprises in Logic. In: *LOGICA Yearbook 2009*. College Publications, London.

[6] Frege, G. (1879). *Begriffsschrift. Eine Arithmetische Nachgebildete Formelsprache des Reinen Denkens*. Georg Olms, Hildesheim.

[7] Frege, G. (1884). *Grundlagen der Arithmetik. Eine logisch mathematische Untersuchung über den Begriff der Zahl*. Georg Olms, Hildesheim.

[8] Frege, G. (1891). Funktion und Begriff. In: Beaney (1997).

[9] Frege, G. (1892). Über Sinn und Bedeutung. In: Beaney (1997).

[10] Frege, G. (1893). *Grundgesetze der Arithmetik*. Georg Olms, Hildesheim.

[11] Frege, G. (1969). *Nachgelassene Schriften*, eds. H. Hermes, F. Kambartel, and F. Kaulbach. Felix Meiner Verlag, Hamburg.

[12] Frege, G. (1976). *Wissenschäftliche Briefwechsel*, eds. G. Gabriel, H. Hermes, F. Kambartel, C. Thiel and A. Veraart. Felix Meiner Verlag, Hamburg.

[13] Frege, G. (1984). *Collected Papers on Mathematics, Logic, and Philosophy*, ed. B. McGuiness, transl. M. Black, V. H. Dudman, P. Geach, H. Kaal, E.-H. Kluge, B. McGuiness, and R. H. Stoothoff. Basil Blackwell, Oxford.

[14] Hanna, R. (2011). Kant's Theory of Judgment. In: Zalta E. N. (ed.) (2011). *The Stanford Encyclopedia of Philosophy*.

[15] Kant, I. (1965). *Critique of Pure Reason*, transl. N. Kemp Smith. St Martin's Press, New York.

[16] Kant, I. (1992). Jäsche Logic. In: J. Michael Young (ed. and transl.) (1992). *Lectures on Logic*. Cambridge University Press.

[17] Krämer, S. (2003). Writing, Notational Iconicity, Calculus: On Writing as a Cultural Technique. *MLN*, 118:518-537.

[18] Longuenesse, B. (1998). *Kant and the Capacity to Judge. Sensibility and Discursivity in the Transcendental Analytic of the* Critique of Pure Reason, transl. C. T. Wolfe. Princeton University Press.

[19] Macbeth, D. (2002). Frege and Early Wittgenstein on Logic and Language. In: Reck (2002).

[20] Macbeth, D. (2005). *Frege's Logic*. Harvard University Press.

[21] Macbeth, D. (2007). Striving for Truth in the Practice of Mathematics: Kant en Frege. *Grazer Philosophische Studien*, 75:65-92.

[22] Macbeth, D. (forthcoming). Diagrammic Reasoning in Frege's *Begriffsschrift*. Forthcoming in *Synthesis*.

[23] MacFarlane, J. (2002). Frege, Kant, and the Logic in Logicism. *The Philosophical Review*, 3:25-65.

[24] Potter, M. (2009). *Wittgenstein's Notes on Logic*. Oxford University Press.

[25] Proops, I. (1997). The Early Wittgenstein on Logical Assertion. *Philosophical Topics*, 25:121-144.

[26] Proops, I. (2002). The *Tractatus* on Inference and Entailment. In: Reck (2002).

[27] Reck, E. H. (ed.) (2002). *Perspectives on Early Analytic Philosophy*. Oxford University Press

[28] Reck, E. H. (2002). Wittgenstein's "Great Debt" to Frege; Biographical Traces and Philosophical Themes. In: Reck (2002).

[29] Ricketts, T. (2002). Wittgenstein against Frege and Russell. In: Reck (2002).

[30] Russell, B. (1903). *The Principles of Mathematics*. George Allan & Unwin Ltd., London.

[31] Smith, N. J. J. (2000). Frege's Judgment Stroke. *Australasian Journal of Philosophy*, 78:153-175.

[32] Smith, N. J. J. (2009). Frege's Judgement Stroke and the Conception of Logic as the Study of Inference not Consequence. *Philosophy Compass*, 4:639-665.

[33] Stokhof, M. (2002). *World and Life as One. Ethics and Ontology in Wittgenstein's Early Thought*. Stanford University Press.

[34] Sundholm, G. (1994). Ontologic versus Epistemologic: Some Strands in the Development of Logic. In: D. Prawitz and D. Westerstahl (eds.) (1994). *Logic and Philosophy of Science in Uppsala*. Kluwer Academic Publishers.

[35] Sundholm, G. (1998). Inference versus Consequence. In: T. Childers (ed.) (1998). *The Logica Yearbook 1997*. Filosofia.

[36] Whitehead, A. N. and Russell, B. (1910). *Principia Mathematica*. Cambridge Unversity Press.

[37] Wittgenstein, L. (1922). *Tractatus Logico-Philosophicus*, transl. C. K. Ogden. Routledge and Kegan Paul, London.

[38] Wittgenstein, L. (1953). *Philosophical Investigations / Philosophische Untersuchungen*, transl. G. E. M. Anscombe. Basil Blackwell, Oxford.

[39] Wittgenstein, L. (1975). *Tractatus Logico-Philosophicus*, transl. W. F. Hermans. Anthenaeum - Polak & van Gennep.