INQUISITIVE SEMANTICS AND THE PARADOXES OF MATERIAL IMPLICATION

MSc Thesis *(Afstudeerscriptie)*

written by

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In memory of my father,
who taught me perseverance
and developed my grit.

Abstract

The Paradoxes of Material Implication concern entailments which are valid according to Classical Propositional Logic but which contradict universal linguistic intuitions. These contradictions constitute one of the best-known objections to the classical truth-functional account of indicative conditionals.

In this thesis we give an Inquisitive Semantic account of the Paradoxes of Material Implication. We focus on the sixteen paradoxical inferences that can be found in the literature. We formalize, motivate and discuss two inquisitive systems: Basic Inquisitive Semantics and Radical Inquisitive Semantics. Further, we compare the Basic Inquisitive Semantic and the Radical Inquisitive Semantic account of the Paradoxes of Material Implication with the accounts given by Lewis' Strict Conditional Logic S2, Stalnaker's Conditional Logic C2, Update Semantics and Relevance Logic B. We also discuss the extent to which the inquisitive account of implication reflects the philosophical underpinnings of different non-classical accounts.

We demonstrate that Radical Inquisitive Semantics is the only system that allows us to account for all of the Paradoxes of Material Implication. We conclude that the account given by Inquisitive Semantics is better than the classical account and has certain advantages over other systems. Finally, we also suggest and discuss several possibilities for further research.

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Contents

"If and suppose - two small words,
but nobody has ever been able to explain them"
- Jack Johnson

Formal semantics attempts to model natural language and formalize the ways in which we communicate. Over the past several decades, many logical theories have been put forth with the aim of providing us with an adequate model of our language uses. Inquisitive Semantics is an example of one of the most recent developments. It departs from the classical tradition in semantics, in which the meaning of a sentence consists only in its informative content. Instead, it proposes to ameliorate the classical understanding of a proposition by acknowledging its inquisitive content. Such an "inquisitive twist" results in a new logical framework that provides us with new theorems and new insight into our uses of natural language.

A departure from a long-established tradition is generally met with a significant amount of doubt. There are many things that can go wrong and the prospective benefits of the new framework need to considerably outweight both its own disadvantages and the advantages of the framework we started with. In general, the deeper and the more varied the arguments for the new system, the better its chances of success and the bigger the probability of moving a step closer towards finding the optimal model of the phenomena we are interested in.

Inquisitive Semantics was developed mostly at the University of Amsterdam over the past decade. Despite its novelty, Inquisitive Semantics has managed to inspire researchers from several countries spread across Europe, America and Asia. Interest in the framework resulted in many developments, shedding light both on the nature of the system and the phenomenon it seeks to capture. To date, the semantic features of the propositional system are fully developed, the pragmatic underpinnings of the new framework have been specified and the system's algebraic features have been spelled out. The speedy development of Inquisitive Semantics and the attention it has attracted internationally are a testament to its viability and potential. All of the advantages over the classical semantics and the rival systems are not fully spelled out, though. The framework still needs additional motivation.

In this thesis, we will consider an as-of-yet unexplored branch of argumentation for the inquisitive enterprise; we will examine the inquisitive treatment of the paradoxical inferences involving material implication. In particular, we will compare the inquisitive approach to the Paradoxes of Material Implication with other non-classical approaches. We will argue that Inquisitive Semantics provides a better treatment of implausible material implications than classical semantics does. Furthermore, we will demonstrate that Radical Inquisitive Semantics allows one to account for more paradoxical inferences than any of the other systems. Last but not least, we will suggest that as Inquisitive Semantics gives an intuitive and non-*ad hoc* treatment of the Paradoxes of Material Implication, it can be seen as being advantageous over other systems considered.

In order to cogently resolve the matter in question, the thesis will be divided into five sections.

1. The first part of the thesis provides an introduction to propositional Inquisitive Semantics. We will restate, discuss and motivate inquisitive semantics. Furthermore, we will also discuss a not yet fully formalized extension of Inquisitive Semantics—Radical Inquisitive Semantics—and formalize additional notions which prove useful in realizing the objectives of the thesis.

2. The second part introduces paradoxical material implications and discusses different approaches towards these paradoxes. We do this by considering 16 paradoxical inferences. These inferences will be used as a *benchmark* that will allow us to compare different models of natural language implication. In this chapter we also introduce the semantics of S2, C2, US and B. Finally, we summarize the account of the Paradoxes of Material Implication given by these logics in a table.

3. The third part demonstrates the inquisitive treatment of the paradoxical inferences. We will prove that Inquisitive Semantics effectively accounts for some of the paradoxical entailments and discuss the extent to which Basic Inquisitive Semantics and Radical Inquisitive Semantics are successful in accounting for all of the problematic cases.

4. The fourth part compares the inquisitive approach to resolving the Paradoxes of Material Implication with classical and non-classical approaches. On the basis of this comparison, we will also analyze the role of different semantic definitions in the inquisitive treatment of the paradoxical implications in question.

5. In the final part of the thesis we will summarize the findings. Our discussion will make clear that the analysis provided by Inquisitive Semantics provides a strong case for the inquisitive enterprise.

CHAPTER 1

Inquisitive Semantics

Inquisitive Semantics enriches one of the most basic notions of classical semantics: that of a proposition. As pointed out in the introduction, this allows one to enrich the classical, purely informative meaning of a sentence with its inquisitive content. In order to fully grasp the subtleties of the inquisitive framework, it is useful to depart from the classical understanding of a proposition and then describe the semantics underlying the new framework. Such an approach highlights the key element of the new system and more clearly explicates its assumptions. After the description of Basic Inquisitive Semantics, we will consider a recent extension, *Radical Inquisitive Semantics*. This system proposes an even more fine-grained notion of meaning which characterizes positive, negative and issue-dispelling responses to a sentence uttered. Most importantly, RIS allows for differentiation between the rejection of a proposal made by uttering a sentence and the rejection of the supposition behind the sentence uttered.

1.1 The Classical Proposition

The classical notion of a proposition is exemplified by Stalnaker's 1978 article "Assertion" [41]. We will draw from this article in discussing the motivations for the classical treatment of propositions and natural language discourse. We will also contrast this classical view with the inquisitive one.

One of the most prominent views in semantics and logic is that a proposition represents the world as being a certain way. For instance, when one says that "The class is at 10AM", one communicates that it is the case that the class is at 10AM. Or, more specifically, one communicates that this sentence correctly describes the world as being this way and not the other, e.g., in which the class is at 11AM. Thus, a proposition can be seen as dividing the ways in which the world could be and could not be. By these means, it is classically assumed one understands a proposition expressed by a sentence when one knows when the sentence corresponding to it is true and when it is false. Consequently, when one engages in a conversation, one tries to distinguish between the ways things could have been and could not have been and decide upon different alternative descriptions of the world in order to arrive at the most plausible conclusion. In logic it is common to refer to these different alternative descriptions as possible worlds, viz. the ways the world could be. A proposition expressed by a sentence is then understood as a function from possible worlds to truth values. Thus, when one makes an assertion, one expresses a proposition and limits the range of possible worlds to the ones in which this proposition is true. So to speak, the proposition is a rule for picking a set of possible worlds such that the sentence corresponding to it is true in these worlds.

This account and understanding of a proposition gives rise to the following view on discourse. In a conversation assertions are made and accepted and thus the set of possible worlds compatible with propositions expressed by assertions made is reduced. As the conversation proceeds, individuals further continue to narrow down the set of alternative descriptions of the world with the intention to locate the actual state—the instantiation of how the world really is—among a set of alternative descriptions of the world that is narrow enough for their purposes.

Thus, the classical notion of a proposition can be seen as being grounded in the possible world paradigm. In this paradigm, a proposition is a representation of the world being a certain way. Crucially, for this representation there corresponds a set of possible states of the world which are in accord with it[39]. So to speak, the proposition is a characteristic function that gives a set of possible worlds in which the sentence corresponding to this proposition is true. This treatment of a proposition also implies that the meaning of a proposition is identified with informative content. That is, propositions are taken to embody the informative content of a sentence - they are a way of providing information about how the world is. Such an account seems to be strongly motivated for modeling valid reasoning, i.e., determining when we can conclude one piece of information from another one. In a straightforward fashion the classical account models when one assertion implies the other and seems to correctly account for the influence that a series of assertions have on the body of information that is being assumed at a given point of a conversation.

The inquisitive semantic enterprise does not question the appropriateness of classical theory in modeling these situations. It is rather based on the observation that argumentation makes a small class of our language use and it is by no means an exhaustive and paradigm example of it. More specifically, there are many other ways we use our language that cannot be modeled as argumentation, e.g., interviews, interrogations and even the majority of everyday conversations. These uses of natural language are rather an interplay between questions and answers and not only assertions. On the basis of this, inquisitive semantics postulates that natural language discourses are rather an interplay between the inquisitive content (requests for information) and assertive content and that this interplay allows one to proceed with the conversation and limits the set of alternative descriptions which correctly describe our actual world.[1]

1.2 The Inquisitive Twist

Based on the previous analysis of the classical view on discourse and proposition, inquisitive semantics aims at developing a model that would account for the shortcomings of the classical picture and thus, provide a better model of natural language discourse. There are two key observations made by Inquisitive Semantics. The first is comprised in the claim that there is "no sharp distinction between assertions, non-inquisitive sentences, and questions, non-informative sentences" ([22], pp. 2). The second observation is that propositions are better accounted for as *proposals* to change the information assumed by a discourse in one or more ways.

The first observation, that one can give a single *semantic* treatment of indicatives and interrogatives, can be seen as being rooted in observations made by Grice. For Grice notices that "a standard employment of 'or' is in the specification of possibilities (one of which is supposed by the speaker to be realized, although he does not know which one)" ([28], pp. 13). More specifically, as noticed by [22], it seems that certain indicatives invite the same responses as interrogatives:

[1]The fact that the class of natural language uses that can be correctly modeled by means of classical propositions is very limited, is one of the most striking problems of classical propositions, but certainly not the only one. Some of the other problems of the classical account of a proposition in modeling dialog are mentioned by [5], [21], [20]. For instance Ciardelli notices that under the classical notion of a proposition, it seems to be impossible to account for the coherence of a dialog and relations between utterances and it also seems impossible to account for the reactions of other participants to utterances made—e.g., disagreement or doubt.

The class is either at 10AM or at 11AM. (1.1)

Is the class either at 10AM or at 11AM? (1.2)

The class is at 10AM. (1.3)

In the above example it is possible for the inquisitive content of (1.1) and (1.2) to coincide. This is further visible if we imagine (1.1) and (1.2) as being uttered in a conversation; (1.3) seems to be a correct response to either of them. Thus, it seems to be that both (1.1) and (1.2) invite the same response from the conversational interlocutors. This points to the fact that natural language does not seem to have a separate semantic treatment for some of the questions and interrogatives. Consequently, this motivates postulating a single semantic object containing inquisitive and informative content. An example of such an object is (1.1), which can be viewed as a hybrid sentence (i.e., a sentence that both provides some information and invites some information).

On the other hand, the inquisitive treatment of propositions as proposals stems from the fact that Inquisitive Semantics considers a discourse not purely as an exchange of assertions, but rather as "a cooperative process of raising and resolving issues" ([14], pp. 1). This treatment is motivated by the intuitive observation that a conversation is an exchange of information in which generally one requests some information (i.e., raises an issue) and expects ones interlocutor to contribute to resolving an issue by providing some relevant information. To illustrate this point, consider the following example:

- Do we have the class at 10AM or 11AM? (1.4)
- The class is at 10AM. (1.5)

In this common situation the first speaker proposes 2 alternative ways of updating the common ground of the conversation[2] and requires his interlocutor to provide him with enough information to decide between them. The interlocutor, in order to answer the question, decides between the alternatives

[2]In line with Stalnaker, the common ground is understood as the body of information that has been assumed by the conversation so far[14]. More specifically the common ground of a conversation can be seen as a discourse context in which we utter sentences. Then, by uttering a sentence we express a proposition which proposes to change the common ground in certain ways. Notice that by uttering a sentence one may suggest to restrict the discourse context so that it is compatible with the information provided by it and one may also require a piece of information that would allow to decide between different restrictions of the common ground suggested by it.

suggested by the first speaker. To be more specific, it is intuitive to treat (1.4) as expressing a proposal to update the common ground of a conversation. The proposition expressed by (1.4) invites a response that would determine the time of the class. (1.4) is a question, since it only presupposes that the class is either at 10AM or at 11AM and simply requires information. That is, it asks for a response which would resolve whether the class is at 10AM or at 11AM and thus update the common ground of the conversation. (1.5) on the other hand, is an assertion: it provides exactly one update of the common ground and resolves the issue raised in (1.4).

What seems to follow from this analysis is that it seems to be very intuitive to model propositions as sets of sets of possible worlds, rather than only sets of possible worlds. This allows us to correctly account for questions exemplified by (1.4). This is because we can model the proposition expressed by (1.4) as just consisting of 2 sets of possible worlds: one in which it holds at every world that the class is at 10AM and the other in which it holds at every world that the class is at 11AM. It also allows us to correctly account for (1.5), as we can model the proposition expressed by it as a set of possible worlds in which at every world it holds that the class is at 10AM. Hence, by these means Inquisitive Semantics in its simplest form provides a very plausible and appealing interpretation of propositions as proposals to update the common ground of a conversation. Under this interpretation, questions propose at least two ways of updating the common ground and assertions give one proposal to update the common ground of a conversation and hence can be seen as simply suggesting to add their informative content to the common ground. Based on this alteration of the definition of a proposition, a proposition expressed by a sentence can be seen as capturing both the information it provides and the information it requests from other conversational participants.[7][3]

While the description of inquisitive propositions as pursued above can be seen as a plausible indication of motivations for the inquisitive treatment of a proposition, it does not specify which propositions are expressed by which sentences. Given the characterization of the new notion of a proposition, we need to give an account of how to determine propositions corresponding to

[3]More specifically we can think of the inquisitiveness of a proposition as its potential to raise some issues for consideration and responses that are invited by it. Thus, when someone raises a question, one can be seen as inviting other conversational participants to provide enough information to establish at least one of the alternatives proposed. Notice that, on this account, when one utters a sentence, one provides information that at least one of the pieces of information contained in the proposition the sentence expresses provides a correct description of the actual world and one requests enough information to establish which bit of information should be accepted.

more complex sentences than the ones mentioned in examples considered so far. Thus, we need to give a formal definition of Inquisitive Semantics.

1.3 Basic Inquisitive Semantics

In semantics, the standard way of accounting for the meaning of complex sentences is in terms of a recursive definition. This approach allows one to give an account of the meaning of every sentence in terms of its simpler constituents. Following this approach, Inquisitive Semantics defines the meaning of sentences recursively in terms of support at states. Then, the proposition expressed by a sentence θ, can be just taken to be the set of all states supporting θ. An example of one recursive way of accounting for the meaning of all inquisitive sentences and then propositions is given by [37]. Under this approach a proposition is a persistent set of sets of possible worlds, where persistence is a property of sets of sets of possible worlds which guarantees that whenever a set of possible worlds supports a sentence, so do all of its subsets.[4] We will proceed by outlining the formal framework of Inquisitive Semantics and then motivating and describing its features.

1.3.1 Issues and States

Two basic ingredients of Inquisitive Semantics are issues and states.

Definition 1 *(States) Let $\mathcal{P}$ be a finite set of propositional letters and ω be the set of all possible words, i.e., $\omega := \{0,1\}^{\mathcal{P}}$. A state is a subset of ω, i.e., any set of possible worlds $\sigma \subseteq \omega$.*

The idea behind the notion of a state σ is to encode the information that the actual world is among the possible worlds in σ. Notice that for a state σ, we can think of its subsets as enhancements of σ which give more detailed information concerning where the actual world might lie.

Definition 2 *(Enhancement) A state τ is called an enhancement of σ if and only if $\tau \subseteq \sigma$.*

Notice that given a context specified by σ, one can also try to locate the actual world within σ with more precision. Thus, one can suggest enhancements of σ that locate the actual world with sufficient precision and

[4] This section is based on [7] and [37]; the persistency requirement will be motivated in later subsections.

request an answer that would allow one to decide between them. The semantic content of such a request for information is referred to as an issue over σ - denoted $\mathcal{I}$, which corresponds to a non-empty set of enhancements of σ. In this setting it is also natural to think of an issue over σ as being *downward closed* and as forming a *cover* over σ. When considering the *downward closure* requirement, notice that if τ is an enhancement of σ then so is $\tau' \subseteq \tau$. This is because, if τ already locates the actual world with sufficient precision then an even more precise enhancement that also contains this world cannot fail to do so. When considering the requirement for $\mathcal{I}$ to form a *cover* of σ, notice that the information contained in σ does not preclude any possible world $w \in \sigma$ from being the actual world. Thus, it follows that for any possible world in σ, the issue raised by $\mathcal{I}$ must also contain an enhancement which contains it. Otherwise $\mathcal{I}$ precludes a possible world, which may well be the actual world given the information in σ.

Definition 3 *(Issues) Let σ be a state. $\mathcal{I}$ is an issue over σ if and only if:*

1. *$\mathcal{I}$ is a non-empty set of enhancements of σ;*
2. *$\mathcal{I}$ is downward closed, i.e., if $\tau \in \mathcal{I}$ and $\tau' \subseteq \tau$, then $\tau' \in \mathcal{I}$;*
3. *$\mathcal{I}$ forms a cover of σ, i.e., $\bigcup \mathcal{I} = \sigma$.*

Given the definition of an issue $\mathcal{I}$, it is also useful to define a response that allows us to decide between the enhancements suggested by $\mathcal{I}$, i.e., the situation in which one settles the issue raised by $\mathcal{I}$.

Definition 4 *(Settling an issue) Let σ be an information state, τ an enhancement of σ and $\mathcal{I}$ an issue over σ. Then, τ settles $\mathcal{I}$ if and only if $\tau \in \mathcal{I}$.*

1.3.2 Propositions

Given the definition of the basic ingredients of the inquisitive framework presented in the previous subsection, we are now in a position to state the definition of an inquisitive proposition. A more refined version of a proposition expressed by a sentence will be given in the next subsection.

Definition 5 *(Proposition) A proposition A over a state σ is an issue over an enhancement $\tau \subseteq \sigma$.*

Hence, it follows that the following fact holds for propositions:

Fact 1 *Proposition is a non-empty downward closed set of states.*

Notice that the definition of an inquisitive proposition allows us to encode two key ingredients of inquisitive meaning. For one, an inquisitive proposition A encodes the enhancement of the common ground of the conversation specified by $\bigcup A$, i.e., the information it proposes to enhance the common ground with. But, it also specifies an issue over $\tau = \bigcup A$. This is in line with our previous observations: when one utters a sentence, one proposes to update the common ground of the conversation *and* specifies an issue over the suggested update. In other words, in uttering a sentence one can provide and can request information. Furthermore, since an issue over a state σ is just a set of enhancements over σ and since every enhancement denotes a set of possible worlds, it follows that a proposition A is a set of states. By these means, we can take every state contained in a proposition to specify one piece of information; namely, that the actual world is among the possible worlds in that state. Every proposition can then be viewed as an invitation to specify which of these states correctly describes the actual world.

Let us discuss the downward closure requirement in greater detail in order to see why it is plausible from a philosophical point of view, to take propositions to have this feature.

Let A be a proposition over σ, $\mathcal{I}$ be the issue embodied by A, $R \in \mathcal{I}$ be an issue-settling piece of information. Then it follows that any issue-settling piece of information which is more informative than R also settles the issue raised in $\mathcal{I}$; for if R locates the actual world with sufficient precision, then so does its restriction. Thus, since by the definition $R \in \mathcal{I}$ corresponds to some state $\alpha \in A$, it follows that any subset of α is a set of possible worlds in A (i.e., A is a downward closed set of states).

More intuitively, this argument demonstrates that under the inquisitive assumption that propositions address certain issues and are specified by the range of responses that resolve those issues, it is natural to treat a proposition as a downward closed set of states. For, whenever a piece of information settles an issue, a more informative piece of information also settles that issue. By analogy, one can consider the following everyday example. If one asks "Is the car blue or red?", the issue raised by this person is fully settled e.g., by replying "The car is red". However, it is also fully settled by saying "The car is red and it is quite fast". The latter reply, however, is a more informative response since it gives strictly more information than necessary to resolve the issue in question. Hence, in everyday situations every more informative response also settles the issue raised. Similarly, the downward closure requirement guarantees that a proposition contains all possible issue resolving pieces of information.[5].

1.3.3 Support and Propositions Expressed by a Sentence

With the basic features and notions involved in Basic Inquisitive Semantics in place, we can proceed to the specification of which propositions are expressed by which sentences. Inquisitive Semantics defines the meanings of sentences recursively via the notion of support and evaluates sentences at states σ. Then, it takes the proposition expressed by a sentence θ to be the set of all states supporting θ.

Definition 6 *(Language) Let $\mathcal{P}$ be a finite set of propositional letters. We denote by $\mathcal{L}_{\mathcal{P}}$ the set of formulas built from letters in $\mathcal{P}$ using the connectives $\wedge, \vee, \neg, \rightarrow, \bot$.*

Definition 7 *(Support in BIS)*

$$\begin{array}{lll}
\sigma \models p & \textit{iff} & \forall v \in \sigma \ : \ v(p) = 1 \quad \textit{for atomic } p \\
\sigma \models \bot & \textit{iff} & \sigma = \emptyset \\
\sigma \models \neg\theta & \textit{iff} & \sigma \models \theta \rightarrow \bot \\
\sigma \models \theta \vee \sigma & \textit{iff} & \sigma \models \theta \ \textit{or} \ \sigma \models \theta \\
\sigma \models \theta \wedge \psi & \textit{iff} & \sigma \models \theta \ \textit{and} \ \sigma \models \psi \\
\sigma \models \theta \rightarrow \psi & \textit{iff} & \forall \tau \subseteq \sigma : \ \textit{if} \ \tau \models \theta \ \textit{then} \ \tau \models \psi
\end{array}$$

The first clause states that a state σ supports an atomic sentence p iff p holds at every possible world in this state.
The second clause states that a state σ supports $\bot$ iff σ is the empty set.
The third clause states that a state σ supports a negation $\neg\theta$ iff the only substate of σ that supports θ is the empty set.
The fourth clause states that a state σ supports a disjunction iff it supports one of the disjuncts.
The fifth clause states that a state σ supports a conjunction iff it supports both of the conjuncts.
The sixth clause states that a state σ supports an implication $\theta \rightarrow \psi$ iff every substate of σ that supports θ also supports ψ.

[5]Notice, however, that the requirement for a proposition to be constituted by *all* responses that resolve the issue raised by uttering a sentence is not a necessary one. For instance, [15] allows one to model only the compliant responses, i.e., the responses that are not-overinformative and which provide just enough information to decide between the alternatives proposed by uttering a sentence. See [15] for a more detailed account.

The above definition gives rise to the following fact about the support relation, which can be established by a straightforward induction on the complexity of θ.

Fact 2 *(Persistence) Let σ be a state and θ a formula in $\mathcal{L}$. If $\sigma \models \theta$, then for all $\tau \subseteq \sigma$ $\tau \models \theta$.*

The inquisitive semantic clauses allow us to define proposition expressed by a sentence.

Definition 8 *(Proposition) The proposition expressed by θ, denoted $[\theta]$, is the set of all states supporting θ.*

Notice that since $\emptyset$ supports every proposition, i.e $\forall\theta$, $\emptyset \models \theta$, it follows that propositions are non-empty set of states. Notice also that the definition of a proposition implies that propositions satisfy the persistency condition. This is because whenever a proposition contains a state σ, it also contains all of its substates and hence is a downward closed set of states. This matches our earlier observations concerning propositions in section 1.3.1. Hence, the semantic clauses involved in inquisitive semantics give an adequate model of propositions. Furthermore, properties of inquisitive propositions expressed by a sentence also exemplify the inquisitive interpretation of uttering a sentence. Namely, that when one utters a sentence, one provides information that at least one of the bits of information contained in the proposition contains the actual world; and one requests that the conversational participants establish which bit of information this is.

As pointed out before, in uttering a sentence θ one provides the information that the actual world is among the worlds supporting θ, i.e., worlds w s.t. $\{w\} \models \theta$. This demonstrates that the informative content of a proposition expressed by θ is embodied by the union of states supporting θ. We will denote this set of possible worlds $info([\theta])$.

Definition 9 *(Informative Content of a Proposition) Informative content of a proposition $[\theta]$ - $info(\theta)$ corresponds to the union of states supporting θ, i.e., $info([\theta]) = \bigcup[\theta]$.*

Furthermore, by uttering a sentence θ one also requests enough information to locate the possible world among one of the enhancements suggested by the issue raised by $[\theta]$. This is explicated by the following definitions.

Definition 10 *(Issue Raised by $[\theta]$) Let θ be a formula in $\mathcal{L}$. The issue $\mathcal{I}$ raised by the proposition $[\theta]$ is an issue over $info([\theta])$.*[6]

Definition 11 *(Inquisitive Content of a Proposition) Inquisitive content of a proposition is the issue raised by $[\theta]$.*

This also allows us to introduce a classification of propositions expressed by sentences:

Definition 12 *(Informative Proposition) We call a proposition $[\theta]$ informative iff its informative content does not coincide with ω, i.e., $info([\theta]) \neq \omega$.*

Definition 13 *(Inquisitive Proposition) We call a proposition $[\theta]$ inquisitive iff the issue raised by $[\theta]$ is not settled by its own informative content. That is iff $info([\theta]) \notin [\theta]$.*

Definition 14 *We call a proposition $[\theta]$ hybrid iff it is both informative and inquisitive.*

All of the above definitions give a way of determining the meaning of every proposition. For the purpose of clarity it is worth to discuss them in relation to the kinds of sentences that are in $\mathcal{L}$.

Atomic Sentences. The proposition expressed by an atomic sentence p corresponds to a persistent set of states in which p holds at every possible world. Thus, in order to determine the meaning of a proposition $[p]$ we need to determine the truth set $|p|$ and then construct $[p]$ by taking all subsets where p holds classically. Notice that the informative content of $[p]$ corresponds to $|p| \neq \omega$ and that $|p| \in [p]$. Hence, the proposition given by an atomic sentence is informative and non-inquisitive.

Negated Sentences. In order to determine the meaning of a proposition expressed by $\neg\theta$, we gather all states σ s.t. their only substate supporting θ is the empty set. Notice that a proposition expressed by $\neg\theta$ might contain only the empty state. This is the case if there are no states s.t. they do not support θ, i.e., θ is a tautology. Similarly as in the previous clause, it also follows that the informative content of a negated sentence $\neg\theta$ corresponds to its classical meaning $|\neg\theta|$, and $|\neg\theta| \in [\neg\theta]$. Hence, the proposition expressed

[6]Note that the specification of which issue is raised exactly by $[\theta]$ is determined by means of the support clauses in Definition 7. By these means the inquisitive content of a proposition $[\theta]$ corresponds simply to the proposition $[\theta]$ (since $[\theta]$ specifies a set of enhancements).

a negation of a sentence θ is non-inquisitive and as long as θ is not a contradiction it is also informative.

Disjunction. In order to determine the proposition expressed by a disjunction $\theta \vee \psi$, we need to collect all states that support θ and all states that support ψ (so to speak create one set of states in $[\theta]$ and one set of states in $[\psi]$). This guarantees that the set of states corresponding to $\theta \vee \psi$ is persistent. Furthermore, notice that in general the proposition corresponding to disjunction tends to be both inquisitive and informative. Consider, for instance, a disjunction of two atomic sentences $p \vee q$. Notice that the issue raised by $p \vee q$ corresponds to the set of downward closed sets $\{\mathfrak{P}(|p|), \mathfrak{P}(|q|)\}$[7] and does not contain the set $|p| \cup |q|$ (hence $p \vee q$ is inquisitive). Furthermore, it also follows that $p \vee q$ is informative, since $\bigcup[p \vee q] \neq \omega$. Following Grice's observation, in Inquisitive Semantics, disjunction is treated as the main semantic feature that introduces inquisitiveness. For notice that it gives us a straightforward way of modeling questions. Consider the following question: "Is the class at 10AM?"$\star$. Then, Inquisitive Semantics allows us to model this question as $p \vee \neg p$, where p corresponds to the sentence "Class is at 10AM". Furthermore, since the proposition expressed by $p \vee \neg p$ is constituted by the set of downward closed sets $\{\mathfrak{P}(|p|), \mathfrak{P}(|\neg p|)\}$ and hence is inquisitive (since $|p| \cup |\neg p| = \omega \notin \{\mathfrak{P}(|p|), \mathfrak{P}(|\neg p|)\}$), it correctly models the inquisitive nature of questions. That is, it models questions as requests for information and not only as proposals to add their informative content to the common ground. For notice that $info(p \vee \neg p) = \omega$ and hence by asking $\star$ one does not eliminate any possible worlds.

Conjunction. In order to define a proposition expressed by a conjunction of sentences $\theta \wedge \psi$ we take the states supporting both θ and ψ. Notice that if one of the conjuncts is inquisitive, then the conjunction might also be inquisitive. Intuitively, this can be seen as a sensible definition by considering the following simple example $(p \vee \neg p) \wedge (q \vee \neg q)$ (which e.g. corresponds to a sentence "Does Alexandra live in Amsterdam and does Ben live in Utrecht?"). Notice that the inquisitive content of this proposition corresponds to four downward closed sets of possible worlds: $\mathfrak{P}(|p \wedge q|)$ (Alexandra lives in Amsterdam and Ben lives in Utrecht), $\mathfrak{P}(|p \wedge \neg q|)$ (Alexandra lives in Amsterdam and Ben does not live in Utrecht), $\mathfrak{P}(|\neg p \wedge q|)$ (Alexandra does not live in Amsterdam and Ben lives in Utrecht) and $\mathfrak{P}(|\neg p \wedge \neg q|)$ (Alexandra does not live in Amsterdam and Ben does not live in Utrecht). Finally notice that the informative content of this conjunction corresponds to ω and hence an inquisitive conjunction may not be informative.

[7]Where $\mathfrak{P}(|p|)$ stands for the power set of $|p|$.

Implication. In order to determine the proposition expressed by an implication $\theta \rightarrow \psi$ we need to collect all states σ s.t., whenever any of their subsets supports the antecedent, it also supports the consequent. Hence, such a definition of implication guarantees that after an update with $\theta \rightarrow \psi$, whenever one updates the common ground of the conversation with a piece of information s.t. the antecedent is supported by it, then the consequent will be supported by it as well. Such a definition of implication reflects the fact that implication may behave inquisitively. In order to see this, consider the following sentence "If Pete plays the piano, Sue or Mary will sing." This sentence can be modeled as $p \rightarrow (q \vee r)$ and, by definition, corresponds to all states σ s.t. $\sigma \models p \rightarrow q$ or $\sigma \models p \rightarrow r$ $\star$. Notice that since $info([p \rightarrow (q \vee r)] = |p \rightarrow q| \cup |p \rightarrow r|$ and $|p \rightarrow q| \cup |p \rightarrow r| \notin [p \rightarrow (q \vee r)]$ (by $\star$), it follows that this sentence is inquisitive. This correctly reflects the intuition that when someone utters "If Pete plays the Piano, Sue or Mary will sing", one invites a response that would allow one to establish whether Sue will sing if Pete plays the piano or whether Mary will do so.[8]

Last but not least, it is important to define the notion of entailment. The semantic definitions involved in the Basic Inquisitive Semantics give rise to the following definition of entailment:

Definition 15 *(Entailment) Entailment is defined in terms of support. Namely,* $\theta \models \psi$ *iff for all states* σ*: if* $\sigma \models \theta$*, then* $\sigma \models \psi$*.*

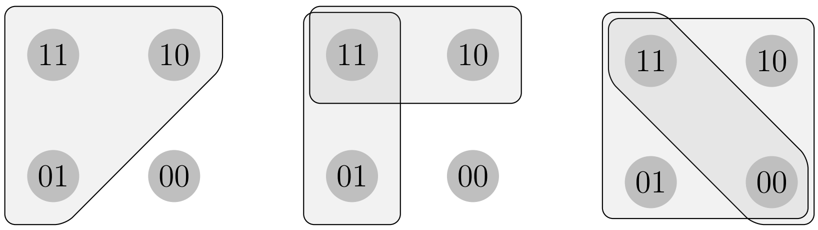
N.B. by these means the definition of inquisitive entailment gives rise to a new notion of logical entailment. Namely, the one in which $\theta \models \psi$ iff whenever θ is settled, so is ψ.

Thus, the inquisitive proposition suggested by inquisitive semantics, appears to account for all propositional sentences in our language not only in terms of their informative content, but also in terms of their inquisitive content. By these means, the shift from the classical picture of a proposition as sets of possible worlds to persistent sets of sets of possible worlds allows us to give a more complete account of the meaning expressed by sentences and the role that sentences play in natural language discourse.

[8]Basic Inquisitive Implication is also motivated by general algebraic concerns. As demonstrated in [37], similarly as in classical logic, implication in BIS corresponds to pseudo-complementation. In this sense BIS implication can be seen as an inquisitive counterpart of the material implication. In BIS, however, the pseudo-complementation concerns a richer notion of meaning, i.e., both inquisitive and informative content of a proposition. The algebraic motivation behind inquisitive implication and other connectives provides a proper foundation for the inquisitive semantics that is independent from intuitions about natural language examples. This can be seen as the algebraic motivation for inquisitive semantics and is discussed at length in [37].

1.4 BIS Examples

For the purpose of the clarity of exposition, it is useful to discuss an example of an inquisitive sentence in more detail. Consider the following figure, which schematically demonstrates the behavior of disjunction in classical semantics (a); and in Basic Inquisitive Semantics ((b) and (c)). For the sake of simplicity, we limit ourselves to the consideration of maximal states supporting a sentence in (b) and (c), i.e., states which are not properly included in any other state.



(a) (b) (c)

Figure 1.1: Disjunction

In the figure above, dots correspond to sets of possible worlds. Each dot contains two numbers. The first number denotes the truth value of p and the second number denotes the truth value of q. The shaded area indicates the extensional meaning of a given proposition, i.e., the sets of possible worlds that correspond to the possible worlds where the sentence corresponding to a given proposition holds.

Consider (a), which corresponds to the classical disjunction $p \vee q$. The meaning of the classical disjunction corresponds to the union of the truth sets of the disjuncts; hence $|p \vee q| = |p| \cup |q|$. Thus, the proposition expressed by the classical disjunction is constituted by a single set of possible worlds s.t. p or q holds at any of them. Clearly, this is reflected in figure 1, since the meaning of the proposition expressed by $p \vee q$ is represented as one set of possible worlds containing all (11)-worlds, (01)-worlds and (10)-worlds.

Consider (b), which corresponds to inquisitive disjunction. Notice that $p \vee q$ corresponds to two downward closed sets of states: one that contains all states s.t. p holds at them (i.e., all the (11)- and (10)-worlds), and the other that contains all states s.t. q holds at them (i.e., (11)- and (01)-worlds). Notice also that $p \vee q$ excludes one downward closed set of states, namely the one which contains all states s.t. neither p nor q holds at them. Thus, inquisitive disjunction can be informative; it can allow us to exclude some states and limit the set of possible worlds we are interested in. Notice that the inquisitive and informative aspects of the disjunction $p \vee q$ in BIS demonstrate its hybrid nature: on the one hand, it is inquisitive (it invites a response that would allow us to decide between the sets of possible worlds denoted by it) and on the other it also is informative (it suggests to reject some of the possible worlds.) Hence, this formalism is in line with our observations in Section 1.2. Last but not least, the requirement for the sets of states to be persistent guarantees that any more fine-grained response to an issue raised is also modeled. An example of a more fine-grained response, i.e., a response that provides more information than is sufficient to resolve the issue raised, is a state containing only (01)-worlds.

Consider (c), which is also an instantiation of inquisitive disjunction. It corresponds to $(p \to q) \vee (q \to p)$. Notice that the first disjunct corresponds to the downward closed set of states s.t. it is not the case that p holds at them and q does not and the second disjunct corresponds to the downward closed set of states s.t. it is not the case that q holds at them and p does not. Thus, the proposition expressed by $(p \to q) \vee (q \to p)$ invites a response that would allow one to decide between these two states. Similarly as in (b), the issue-resolving responses correspond to all of the subsets of states in (c).[9] Notice, however, that the disjunction corresponding to $(p \to q) \vee (q \to p)$ is no longer informative; it only invites a response that would allow to decide between the disjuncts and does not eliminate any set of possible worlds. This demonstrates that, depending on the sentence involved, inquisitive disjunction may or may not be informative.

1.5 The Radical Twist

One of the extensions of Basic Inquisitive Semantics is Radical Inquisitive Semantics. The key difference between the Radical and Basic Inquisitive System lies at the discourse level. RIS enriches the framework of Basic Inquisitive Semantics by allowing for a more detailed account of responses to a sentence uttered. Rather than understanding natural language discourse only as an exchange of information, Radical Inquisitive Semantics interprets it as a cooperative process in which issues are raised and resolved by means of positive, negative and *issue-dispelling* responses made to proposals to change the common ground. Issue-dispelling responses, however, correspond neither to Basic Inquisitive Semantic negations of sentences nor to positive responses to sentences; rather, they allow us to specify more fine grained distinctions, where one can negate the supposition behind an uttered sentence.

[9] Notice that in principle due to the persistency condition the number of states is exponential in the number of propositional letters and is upper bounded by $\sum_{i=0}^{|\mathcal{P}|} \binom{2^{|\mathcal{P}|}}{i}$.

As pointed out above, at the discourse level, the key postulate of RIS is that participants in a conversation raise and resolve issues by making negative, positive and issue-dispelling responses to proposals to update the common ground. Thus, conversational participants may propose to change the common ground in one or more ways and may react to suggested changes in a variety of ways. In order to see this in greater detail, consider the following conversation between four people: A, B, C and D.

- A: "If Pete plays the piano, will Sue sing?"
- B: "Yes, if Pete plays the piano, Sue will sing"
- C: "No, if Pete plays the piano, Sue will not sing."
- D: "Well, Pete will not play the piano"

In this conversation person A asks other participants a question whether it is the case that if Pete plays the piano, Sue will sing. Person B, then replies that this is the case, i.e., she claims that if Pete plays the piano, Sue will sing. Notice that person B simply gives a positive response, she accepts the proposal made by A. On the other hand, person C disagrees with person B. She still accepts the proposal made by A, however she does not think that Sue will sing if Pete plays the Piano. Last but not least, person D dispels the proposal made by A and the responses made by B and C. She thinks that the supposition behind A's utterance does not hold and dispels the issue raised by A. Importantly, D's issue-dispelling response does not contradict B's and C's responses; it only questions the supposition behind them. Interestingly, this reading of B's disagreement with C, can be seen as being further motivated by Ramsey's observation that "If two people are arguing "If p will q?" and are both in doubt as to p, they are adding p hypothetically to their stock of knowledge and arguing on that basis about q; so that in a sense "If p, q" and "If p, $\neg q$" are contradictories" ([35], pp. 155). This can be seen as a further motivation for RIS' enrichment of the responses to a sentence uttered, since neither in Classical Logic nor in BIS, B's and C's responses contradict each other. Furthermore as BIS allows us to model only positive and negative responses to a sentence, this also demonstrates the novelty of RIS's framework, i.e., the issue-dispelling responses that are not negations of positive nor negative responses and that do not correspond in any systematic way to the positive and negative responses.

Last but not least, in our example, depending on which response of which person is taken by conversational participants to be the most plausible one, an appropriate restriction of the common ground will then be chosen. Notice that as the conversation proceeds, raised and resolved issues as well as negative and positive responses made allow us to limit the set of possible words to the most plausible ones and thus reach the most likely conclusion.

1.6 Radical Inquisitive Semantics

In this section we will give a formal definition of RIS and discuss this system in detail. Similarly as in BIS, we will give recursive definition of the meaning of sentences in RIS. Furthermore, we will discuss semantic clauses and define additional notions that turn out to be vital in understanding this refinement of BIS. Most importantly we will highlight the new behavior of implication and the role of issue-dispelling responses.

Radical Inquisitive Semantics is stated recursively in terms of support and rejection.

Definition 16 *(Language) Let $\mathcal{P}$ be a finite set of propositional letters. We denote by $\mathcal{L}_{\mathcal{P}}$ the set of formulas built from $\mathcal{P}$ using the connectives $\div, \wedge, \vee, \rightarrow$.*

RIS is determined by the satisfaction pair $\models_+$ (support) and $\models_-$ (reject) and a recursive definition as:[10]

Definition 17 *(Support in RIS)*

$$
\begin{array}{lll}
\sigma \models_+ p & \textit{iff} & v(p) = 1 \textit{ for any } v \in \sigma \\
\sigma \models_- p & \textit{iff} & v(p) = 0 \textit{ for any } v \in \sigma \\
\sigma \models_+ \div\theta & \textit{iff} & \sigma \models_- \theta \\
\sigma \models_- \div\theta & \textit{iff} & \sigma \models_+ \theta \\
\sigma \models_+ \theta \wedge \psi & \textit{iff} & \sigma \models_+ \theta \textit{ and } \sigma \models_+ \psi \\
\sigma \models_- \theta \wedge \psi & \textit{iff} & \sigma \models_- \theta \textit{ or } \sigma \models_- \psi \\
\sigma \models_+ \theta \vee \psi & \textit{iff} & \sigma \models_+ \theta \textit{ or } \sigma \models_+ \psi \\
\sigma \models_- \theta \vee \psi & \textit{iff} & \sigma \models_- \theta \textit{ and } \sigma \models_- \psi \\
\sigma \models_+ \theta \rightarrow \psi & \textit{iff} & \forall \tau \subseteq \sigma.(\tau \models_+ \theta \textit{ implies } \tau \models_+ \psi) \\
\sigma \models_- \theta \rightarrow \psi & \textit{iff} & \exists \tau.(\tau \models_+ \theta \textit{ and } \forall \tau' \supseteq \tau.(\tau' \models_+ \theta \textit{ implies } \sigma \cap \tau' \models_- \psi))
\end{array}
$$

Notice that the support clauses for all of the connectives that RIS has in common with BIS, correspond to the respective clauses in BIS. Furthermore, the new inversion operator '$\div$' despite being RIS's equivalent of BIS's negation, is defined differently. Thus, in our discussion it is sufficient to focus on the rejection clauses for RIS and clauses for the new connective '$\div$'.

[10] The *Definition 17* in this section is motivated by the definition in [23].

A state σ rejects an atomic sentence p iff p does not classically hold at any possible worlds in σ.
A state σ supports a sentence $\div\theta$ iff σ rejects θ.
A state σ rejects a sentence $\div\theta$ iff σ supports θ.
A state σ rejects a sentence $\theta \wedge \psi$ iff σ rejects θ or σ rejects ψ.
A state σ rejects a sentence $\theta \vee \psi$ iff σ rejects both θ and ψ.
A state σ rejects a sentence $\theta \to \psi$ iff there is a set of possible worlds τ s.t. for every extension of it restricted to σ: if this extension supports the antecedent, then it rejects the consequent.

Support and rejection clauses allow us to determine the meaning of every proposition and counter-proposition.

Definition 18 *The proposition expressed by θ, denoted $[\theta]^+$, is the set of all states supporting θ.*

Definition 19 *The counter-proposition expressed by θ, denoted $[\theta]_-$, is the set of all states rejecting θ.*

Definition 20 *(Entailment$^+$) For any two sentences θ and ψ, $\theta \models_+ \psi$ iff every state that supports θ also supports ψ, i.e $\forall\sigma$: $(\sigma \models_+ \theta \Rightarrow \sigma \models_+ \psi)$.*

Definition 21 *(Entailment$_-$) For any two sentences θ and ψ, $\theta \models_- \psi$ iff every state that rejects ψ also rejects θ, i.e $\forall\sigma$: $(\sigma \models_- \psi \Rightarrow \sigma \models_- \theta)$.*

Definition 22 *(RIS Entailment) For any two sentence θ and ψ, $\theta \models_{RIS} \psi$ if and only if $\theta \models_+ \psi$ and $\theta \models_- \psi$.*

Note that, by the definition of the rejection clause for an atomic sentence, rejection of an atomic sentence corresponds to the negation of an atomic sentence in BIS. However, it is generally not the case that $[\theta]^+ = [\theta]$. For notice that $[\div(p \wedge q)]^+ \neq [\neg(p \wedge q)]$, since the former consists in a set of downward closed sets $\{\mathfrak{P}(|\neg p|), \mathfrak{P}(|\neg q|)\}$, whereas the latter in a set of downward closed sets $\{\mathfrak{P}(|\neg p| \cup |\neg q|)\}$.

The definition of *Entailment*$^+$ is a RIS-equivalent of entailment in BIS. As both of the entailments aim at modeling the positive responses, they are concerned with support relations between states and sentences. Similarly to the BIS case, $\theta \models_+ \psi$ holds if and only if ψ is settled whenever θ is.

The notion of *Entailment*$_-$ defines the meaning of entailment in terms of rejection. Notice that RIS defines the rejection entailment as a relation between the negative responses to two sentences. Furthermore, the standard direction in which one sentence entails the other is "flipped"; rather than requiring that every state which rejects the premises also rejects the conclusion, we require that every state that rejects the conclusion also rejects the premises. This reflects the intuition that a sentence θ *Entails*$_-$ a sentence ψ if and only if it is easier to reject θ than ψ. Notice that this is the case iff every state that rejects ψ also rejects θ, i.e., a set of states rejecting ψ corresponds to some set of states rejecting θ. Thus, we can define negative entailment also in terms of the positive entailment: $\div\psi \models_+ \div\theta$. The negative entailment gives rise to a new relation between sentences, which concerns the informativeness *and* inquisitiveness of their negative responses. Hence it requires that, in order to conclude that the rejection of a sentence θ entails the rejection of a sentence ψ, we need to guarantee that negative responses to ψ are at least as informative as negative responses to θ and that whenever a negative response settles the rejection of ψ, it also settles the rejection of θ.

RIS entailment is then defined in terms of these two entailment relations. One sentence radically entails the other iff the support and the reject entailment hold between them. As will become clear by our analysis of the paradoxical inferences in Chapter 3, RIS entailment seems to give rise to a weaker notion of validity than BIS, as it not only concerns positive responses to uttering a sentence, but also negative responses.

Definition 23 *(Informative Content of a Counter-Proposition) The informative content of a counter-proposition* $[\theta]_-$ *corresponds to the union of states rejecting* θ*, i.e.,* $info([\theta]_-) = \bigcup[\theta]_-$.

Definition 24 *(Informative Counter-Proposition) A counter-proposition is informative if and only if its informative content does not coincide with* ω.

Definition 25 *(Issue Raised by* $[\theta]_-$*) Let* θ *be a formula in* $\mathcal{L}$*. The issue* $\mathcal{I}$ *raised by a counter-proposition expressed by a sentence* θ *is an issue over* $info([\theta]_-)$.

Definition 26 *(Inquisitive Content of a Counter-Proposition) The inquisitive content of a counter-proposition* $[\theta]_-$ *is the issue raised by* $[\theta]_-$.

Definition 27 *(Inquisitive Counter-Proposition) A counter-proposition is inquisitive if and only if the issue raised by* $[\theta]_-$ *is not resolved by its own informative content, i.e.,* $info([\theta]_-) \notin [\theta]_-$.

Note that RIS definitions of inquisitive and informative content of a proposition are the same as in BIS. The RIS-specific definitions, together with the definitions from BIS, give a way of determining the meaning of every proposition and counter-proposition. For classificatory purposes, we will examine the new semantic features of RIS in greater detail. Since the semantic definitions of the support clauses are common between BIS and RIS for all connectives apart from '$\div$', we will focus on the description of the rejection clauses in RIS and clauses for '$\div$'.

Atomic Sentences. A counter-proposition for an atomic sentence p corresponds to a persistent set of states s.t. p is rejected by them. Thus, in order to determine the meaning of a counter-proposition corresponding to an atomic sentence p, we need to determine the classical truth set $|\neg p|$ and then construct $[p]_-$ by taking all subsets of this truth set. This definition mirrors the intuition that we can reject the claim that p, by pointing to any counter-possibility s.t. p does not classically hold at any possible world in it. E.g., when someone says "Sue will sing", we can reject that proposal by giving a negative response of the form: "Because of the weather Sue does not feel so good and she will not sing". Furthermore, it follows that the counter-proposition for an atomic sentence is always informative but never inquisitive.

Inverted Sentence. The inverted sentence in RIS is analogous to BIS's negation. The proposition expressed by an inverse of θ, $\div\theta$, consists in all negative responses to θ. Thus, in order to determine the proposition expressed by $\div\theta$ we gather all states rejecting θ. The key difference between the proposition expressed by an inverse of θ and the proposition expressed by $\neg\theta$ is that the former can be inquisitive whereas the latter is never inquisitive. This reflects the fact that the inquisitive content of the proposition $[\div\theta]^+$ may differ from the inquisitive content of the proposition $[\neg\theta]$. To illustrate this point, consider negative responses to a sentence "Sue will not sing and dance". Notice that if one claims that "Sue will not sing and dance" and expects that some people may reject this statement, one generally predicts that people would do so by saying "No, Sue will sing" or by saying that "No, Sue will dance" or by giving any more informative reply. This is correctly modeled in Radical Inquisitive Semantics since if p corresponds to "Sue will sing" and q corresponds to "Sue will dance", then $\div(p \wedge q)$ corresponds to the downward closed set $\{\mathfrak{P}(|\neg p|), \mathfrak{P}(|\neg q|)\}$. However, in BIS $\neg(p \vee q)$ corresponds to a downward closed set $\{\mathfrak{P}(|\neg p| \cup |\neg q|)\}$. This points towards the fact that RIS and BIS may give different results concerning the propositions expressed by sentences. For notice that in our example, BIS treats the rejection of the disjunction of two atomic sentences as non-inquisitive whereas RIS treats it as being inquisitive.

A counter-proposition for an inverted sentence $\div\theta$ corresponds to all states which support θ. This reflects the intuition that we can reject a sentence $\div\theta$ by specifying a state that supports θ. For instance, a negative response to a sentence that "It is not the case that you will clean your room or do the dishes.", is a positive response to it, e.g., "I will do the dishes and take the rubbish out, because my mother told me so". Notice that all possible positive responses to this sentence correspond to the proposition expressed by θ. Hence, negative responses to an inversion of a sentence are equivalent to positive responses to this sentence.

Disjunction. The counter-proposition for disjunction is constituted by all states s.t. they reject both of the disjuncts. This reflects the intuition that when we reject a disjunction, we do so by rejecting both of the disjuncts. For instance, when one says "The class is either at 10AM or at 11AM", we usually reject it by saying "No the class is neither at 10AM nor at 11AM", "No, the class is at 1PM", and so on.

Conjunction. In order to determine the counter-proposition for conjunction $\theta \wedge \psi$ we need to first collect all states that reject θ and all states that reject ψ (so to speak create one set of states $[\theta]_-$ and one set of states $[\psi]_-$). Notice that this shows that the counter-proposition to a conjunction $\theta \wedge \psi$ is inquisitive in nature. That is, the negative responses that are elicited by a conjunction invite a response that would allow us to determine which of the conjuncts is rejected. Notice, that the inquisitive treatment of negative responses to a sentence also demonstrates one of the key differences between the proposition corresponding to BIS's $\neg(p \wedge q)$ and the counter-proposition corresponding to $(p \wedge q)$. Namely, the former is not inquisitive, whereas the latter is.

Implication. The counter-proposition for an implication includes all states which contain at least one maximal enhancement supporting the antecedent[11], such that this enhancement also rejects the consequent. This does justice to the intuition that, if we update the common ground with a rejection of an implication, we want to make sure that there is one way of supporting the antecedent such that it always guarantees that the consequent is rejected. To see this consider the following example: "If you play football, you will be tired". Then one can reject this conditional statment by saying "No, if I play football, I will not be tired" or by giving any more informative response, e.g., "No, if I play football and my mom gives me a lift to the field, I will definitely not be tired". This is in line with the radical inquisitive modeling, since the first reply corresponds to the maximal enhancement supporting the an-

[11] Where, similarly as before, we can think of a state σ as being maximal w.r.t. a sentence θ, if it is not properly included in any other state supporting θ.

tecedent and rejecting the consequent; whereas the second reply corresponds to a more informative enhancement that supports the antecedent.

Such a definition of the rejection of an implication also gives a straightforward model of some additional phenomena, namely the *rejection of the supposition behind the conditional* and *the contradiction of the proposal made by the conditional.* To see this consider the following example: "If you play football or basketball, you will be tired". The ways in which one can reject this conditional fall into two main classes. On the one hand, one can give a reply along the following lines: "No. If I play football(/basketball), I will not be tired". On the other hand, one can reply "No. It is not the case that I will ever play football or basketball". Notice that the first reply rejects the proposal made by the conditional statement; it rejects that the claimed dependency between the antecedent and the consequent holds. Notice also that such a rejection of the proposal made by a conditional statement corresponds to all states which support the antecedent non-trivially and support an update of the common ground which is of the form 'one of the maximal enhancements supporting the antecedent ⇒ rejection of the consequent'. On the other hand, notice that the latter reply corresponds to an update of the common ground which is of the form 'rejection of the antecedent'. More specifically, this observation gives rise to a more refined ternary characterization of responses in RIS, where one can make a positive response and support the proposal made by the sentence uttered; e.g. reply

- "Yes, if I play football , I will be tired or if I play basketball I will be tired."

One can give a negative response to an issue raised by uttering a sentence; e.g. reply

- "If I play football, I will not be tired"

And one can also give an issue-dispelling response and reject the supposition behind the sentence; e.g. reply

- "No. It is not the case that I will ever play football or basketball"[12]

[12]Notice also that issue-dispelling responses both support and reject the proposal made by uttering a sentence. Hence, the range of responses definable in RIS allows us to give a more detailed classification of responses to sentences. More specifically, we can think of the issue-dispelling responses to a sentence θ, denoted $[\theta]^+_-$, as the ones s.t. they both support and reject the proposal made by uttering θ, i.e., $[\theta]^+_- = \{\sigma \in \omega | \ \sigma \in [\theta]^+ \text{ and } \sigma \in [\theta]_-\}$. We can also think of purely negative responses to a sentence θ as the ones that reject but do not support θ - i.e., the ones contained in $[\theta]^- \setminus [\theta]^+_-$; and purely positive responses to a sentence θ as the ones that support but do not reject θ - i.e., the ones contained in $[\theta]^+ \setminus [\theta]^+_-$

This analysis, coupled with the semantic features of inversion matches our earlier observations and allows us to reject the conditional statement by giving responses of the following kind: 'one of the enhancements supporting the antecedent ⇒ rejection of the consequent' and 'rejection of the antecedent'. Thus, we can reject an implication by either giving a negative response or by giving an issue-dispelling response. The fact that inquisitive semantics allows us to model these effectively can be seen as constituting yet another case for the inquisitive enterprise. This feature of inquisitive semantics will also turn out to be vital in accounting for some of the paradoxes of material implication that will be discussed in the coming sections.

1.7 RIS Examples

As in the Basic Inquisitive Semantics Section, for the purpose of the clarity of exposition, it is useful to discuss some further examples. We will consider 6 examples. In Figure 1.2, we will highlight the behavior of negative responses to a sentence and in Figure 1.3, we will discuss the issue-dispelling responses. For the purpose of simplicity and clarity of exposition, let us assume that there are no other propositional letters apart from p and q and let us limit ourselves to the consideration of maximal states supporting a sentence.

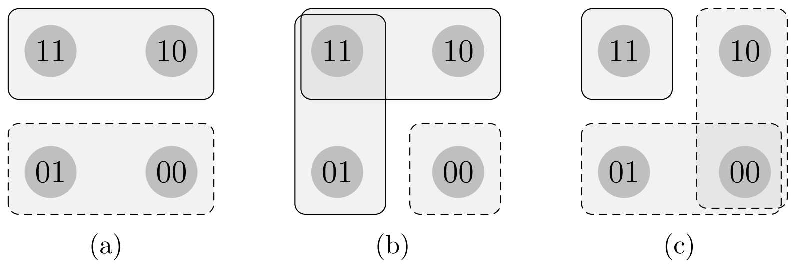


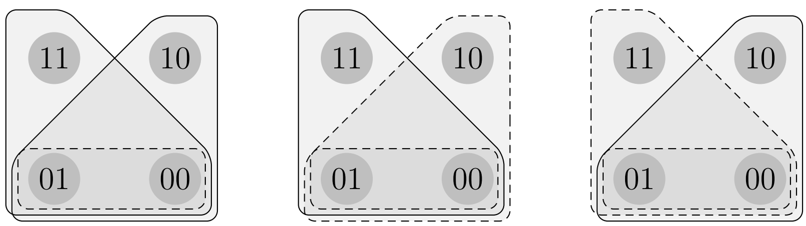
Figure 1.2: Support and Rejection in RIS.

The schematics of the figure are almost the same as in the previous figure on page 18. In comparison to the previous figure, the key difference concerns the set of possible worlds denoted by the dotted lines. This set of possible worlds corresponds to a negative response to a sentence in question.

Consider (a), which corresponds to a proposition expressed by an atomic sentence p and a counter-proposition to this sentence; the continuous area corresponds to a proposition expressed by p and the area given by the dotted line corresponds to a counter-proposition to p. Notice that the proposition expressed by p is constituted by the downward closed set of states s.t. p holds at them. The counter-proposition to p is constituted by the downward closed set of states s.t. p does not hold at them, i.e., a state containing (01)-worlds, a state containing (00)-worlds; and a state containing (01)- as well as (00)-worlds. Furthermore, since the informative content of a counter-proposition $[p]_-$ is equivalent to $|\neg p|$, it follows that the counter-proposition to an atomic sentence is not inquisitive.

Consider (b), which corresponds to a proposition expressed by a disjunction $p \vee q$, and a counter-proposition to this sentence. Notice that the proposition expressed by $p \vee q$ is still inquisitive and informative. On the other hand, the counter-proposition corresponding to $p \vee q$ is not inquisitive and is only informative. This is because the counter-proposition in (b) is constituted by the downward closed set of states s.t. both p is false and q is false, i.e., the states containing (00)-worlds.

Finally, consider (c), which corresponds to a proposition expressed by $p \wedge q$. In this case the proposition expressed by this sentence corresponds to the downward closed set of states, s.t. both p and q hold at them. Notice that the counter-proposition to an issue raised by $p \wedge q$ comprises two states: one that contains all sets of states s.t. p is false at them, and the other that contains all sets of states s.t. q is false at them. This demonstrates the key difference between the proposition expressed by a classical negation and the radical inquisitive proposition expressed by an inversion of a sentence: in RIS, negative responses do not correspond to the classical negation of a sentence. This is because a negative response to a sentence in RIS can correspond to several states, whereas classical negation of a sentence is always constituted by a single set of possible worlds. On the other hand, this also demonstrates a difference between supporting a negation of a conjunction and rejecting a conjunction. For notice that (c) demonstrates that the latter can be inquisitive in nature while the former is not.



(a) (b) (c)

Figure 1.3: Issue-Dispelling Responses

In Figure 1.3, (a) corresponds to the proposition and counter-proposition expressed by $p \to (q \vee \div q)$. Hence, we can think of (a) as corresponding to the situation resulting from uttering a sentence from the beginning of this section, i.e., "If Pete plays the piano, will Sue sing?". Notice that states supporting $p \to q$ and states supporting $p \to \div q$ correspond to two classes of positive responses that one can give to this sentence. Notice further, that the counter-proposition to this sentence is constituted by the downward closed set of states containing possible worlds s.t. they reject p. However, any state rejecting p is among the states supporting $p \to q$ and $p \to \div q$. Thus, $\div p$ both supports and rejects $p \to (q \vee \div q)$, i.e., $\div p \models_{+} p \to (q \vee \div q)$ and $\div p \models_{-} p \to (q \vee \div q)$. Hence, one of the features of the reject and support entailments in RIS is that they are not mutually exclusive. Notice also that in BIS $[\neg(p \to (q \vee \neg q))] = \{\emptyset\}$. This demonstrates more directly that negative responses in BIS are not always equivalent to negative responses in RIS. On the other hand, (b) and (c) correspond to propositions and the counter-propositions to $p \to q$ ("If Pete plays the piano, Sue will sing") and $p \to \div q$ ("If Pete plays the piano, Sue will not sing"). Notice that the issue-dispelling response in (a) is also an issue-dispelling response in (b) and (c). Furthermore, the proposition for $p \to q$ corresponds to a counter-proposition to $p \to \div q$, which correctly models the fact that $p \to q$ contradicts $p \to \div q$.

CHAPTER 2

Paradoxes of Material Implication and Non-Classical Logics

The Paradoxes of Material Implication concern entailments which are valid according to the classical framework but which contradict universal linguistic intuitions. These contradictions are based on very strong and general intuitions about how we use our language and are commonly agreed to constitute a significant problem for the classical semantics. In particular, by these means the Paradoxes of Material Implication raise concerns regarding the appropriateness of the classical recursive truth definition of a proposition and, more specifically, the understanding of English conditional in terms of material implication. Historically, the investigation of the Paradoxes of Material Implication lead to the development of many non-classical semantic systems, such as conditional semantics and relevance semantics. Given this very general understanding of these Paradoxes, in this section, we will introduce and consider approaches to these problematic inferences from Classical Logic as given by Conditional Logic, Update Semantics, Strict Conditional Logic and Relevance Logic. We will do so first of all by discussing the material implications in question, then by providing a short introduction to different non-classical systems, and finally by demonstrating which problematic inferences hold in which logic. We will summarize the results in a table; the proofs of all results can be found in the Appendix.

It will be clear throughout the discussion that our considerations focus on the issue of the correct representation of the English indicative conditional (as opposed to the English counterfactual conditional.) We will contrast other non-classical accounts of the conditional with the classical material interpretation of the conditional.

The difference between indicative and counterfactual conditionals is a very subtle one, with a significant number of "fuzzy cases" and no widespread philosophical agreement on the exact definitions to be employed (see [3]). For the purpose of our discussion it is sufficient to understand indicative conditionals as the ones in which both the antecedent and the consequent contain verbs in the indicative mood. Notice that by these means neither the antecedent, nor the consequent contains *would*, which occurs commonly in counterfactual conditionals. Following this very general definition, an example of English indicative conditional that we are concerned with is *"If you understand this introduction, then it was sufficiently clear."* and an example of conditional sentences that we will not be considering in our discussion is *"If this introduction were not clear enough, then I would jump of a cliff"*. Hence, the indicative conditionals usually concern only what *is/was* the case and not what *would have been*.

Before we proceed to the introduction of different non-classical logics, let us recap the classical material interpretation of the indicative conditional. The classical understanding of $\theta \rightarrow \psi$ is solely truth-functional. That is, it is characterized by the claim that the truth-values of θ and ψ are necessary and sufficient to determine the truth-values of English indicative conditionals. According to this account, the meaning of an indicative conditional $\theta \rightarrow \psi$ is equivalent to *it is not the case that* θ *and not* ψ and there are no other semantic properties which contribute to its meaning. Such a stance is referred to as the horse-shoe '$\supset$' account of English conditional.

It is generally accepted that any conditional with a true antecedent and false consequent is false (hence, θ being false or ψ not being false are necessary conditions for the conditional $\theta \rightarrow \psi$ to be true). However, over the past decades, there has been a significant debate concerning the sufficiency of the material interpretation of the English conditional. In particular, there are significant difficulties with holding the view that whenever it is not the case that θ and $\neg\psi$, the natural language implication "if θ, then ψ" is true. For instance, it is not clear that we should determine the truth value of the natural language implication "If am a European and non-European, then I am a ham sandwich" only in terms of the truth and falsity of the antecedent and the consequent, for this would deem the implication to be true. In this chapter we will demonstrate examples that cast doubt upon the sufficiency of the material interpretation of the English conditional and demonstrate that interpretations given by different systems yield better results.

2.1 Paradoxes of Material Implication

As pointed out in the introduction we assume a very broad understanding of the Paradoxes of Material Implication. Under this understanding we consider all material implications that lead to counterintuitive results as being instances of the Paradoxes of Material Implication. On the one hand, such a broad perspective allows us to highlight different problems of material implication, whereas on the other, it also allows us to highlight different properties of logics in question. By these means the paradoxes below will be used as a benchmark that will allow us to compare different models of natural language implication. For the purpose of clarity, in Chapter 4 we will specify which implications are originally referred to as instances of the Paradoxes of Material Implication, and which ones are referred to differently. In our discussion of the Paradoxes of Material Implication, we will consider the following paradoxes:

1. $p \models q \to p$
2. $\neg q \models p \to q$
3. $p \to s \models (p \wedge q) \to s$
4. $\models (p \wedge \neg p) \to q$
5. $\models p \to (q \vee \neg q)$
6. $\models p \to (q \to p)$
7. $p \wedge q \models p \to q$
8. $\models (p \to q) \vee (q \to p)$
9. $\neg p \models (p \to \neg p)$
10. $p \to q, q \to s \models p \to s$
11. $\models p \to (q \to q)$
12. $(p \wedge q) \to s \models (p \to s) \vee (q \to s)$
13. $(p \to q) \wedge (s \to t) \models (p \to t) \vee (s \to q)$
14. $\neg(p \to q) \models p$
15. $\neg(p \to q) \models \neg q$

16. $p \to q \models \neg q \to \neg p$

The implausibility of these paradoxes is demonstrated by the following natural language examples:
1) I will be the new president tomorrow. Therefore, if I die today, I will be the new president tomorrow.
2) I will not win the election today. Therefore, if I win the election today, I am a ham sandwich.
3) If the weather is nice tomorrow, I will play cricket. Therefore, if the weather is nice tomorrow and I die today, I will play cricket.
4) If I am European and non-European, then I am a ham sandwich.
5) If pigs can fly, then I am a logic student or not.
6) If I go to the cinema tomorrow, then if I die today, I will go to the cinema tomorrow.
7) It is sunny but windy. Hence, if it is sunny, then it is windy.
8) If I am in Europe then I am in America *or* if I am in America, then I am in Europe.
9) The class is not at 10AM. Hence, if the class is at 10AM, then it is not at 10AM.
10) If I win a million dollars, I will quit my job. If I quit my job, I will lose my apartment. Hence, if I win a million dollars, I will lose my apartment.
11) If I eat nuts, then if John won the election, he won the election.
12) If you press switch A and press switch B, then the light will go off. Hence, if you press switch A the light will go off *or* if you press switch B the light will go off.
13) If John is in Amsterdam, he is in the Netherlands and if John is in Warsaw, he is in Poland. Hence, if John is in Amsterdam he is in Poland or if John is in Warsaw he is in the Netherlands.
14) It is not the case that if there is a good God, then the prayers of evil people will be answered. Hence, there is a good God.
15) It is not the case that if I die today, I will see sunlight tomorrow. Hence, I will not see sunlight tomorrow.
16) If we take the car then it won't break down *en route*. Hence, if the car does break *en route*, we did not take it.[1]

It may be the case that among the above inferences some strike us as very counterintuitive whereas others are less counterintuitive. Thus, the categorization of some of the above inferences as being counterintuitive, might be viewed as being controversial. Out of the above list of inferences, the inferences (5), (10), (11) are sometimes viewed as being controversial. This

[1]The examples of Paradoxes (14), (16) are taken from [33] and (10) is taken from [9].

is because, the intuitions concerning the implausibility of these inferences are sometimes mixed, and it is not uncommon that people find these inferences quite intuitive. Nevertheless, it is worthwhile to include these inferences in our discussion as they highlight several aspects of logic and properties of implication. For instance, inferences (5) and (11) are the instances of natural language implications in which the antecedent is not *related* to the consequent in any way. Such an independence of the antecedent and the consequent is very often deemed to be undesirable. Similarly these two examples demonstrate a highly disputed rule, namely that a necessary proposition follows from anything at all. The instance (10) seem to be counterintuitive as it might have true premises and a false conclusion. In particular, this conditional statement seems to be counterintuitive, as it seems to be very implausible that I will lose the apartment despite the fact that I have enough money to afford it. (10) is also an instance of one of the properties of conditionals, namely transitivity.

2.2 Non-Classical Semantics

In this section we will introduce and discuss the non-classical semantic systems and their approaches to the Paradoxes of Material Implication from the previous section. We will give an introduction to Strict Conditional Logic S2, Conditional Logic C2, Update Semantics and Relevance Logic B. We will explain the semantics of these logics in detail with a special focus on implication. The description of these logics developed in this section will constitute the basis for the evaluation of the paradoxical inferences in the Appendix and the comparison in the Analysis Chapter.

2.2.1 Strict Conditional Logic

One of the logics developed to deal with the paradoxes of material implication is Clarence Irwing Lewis' non-normal modal logic S2 [24] [31]. In this logic Lewis suggests that paradoxes of material implication occur because material implication is too contingent upon the state of affairs. Thus, it is highly likely that material implication is not sufficient to model the behavior of English indicative conditionals effectively in situations involving uncertainty or logically contingent situations. Lewis suggests that, in modeling the conditional, we should also look at the alternatives to the actual world which we consider possible. Based on this observation, Lewis proposes that the proper modeling of implication is the one which involves some notion of necessity - namely $\Box(\theta \supset \psi)$, which is commonly referred to as strict implication. Hence by this logic, the truth value of the English conditional is partly determined by the truth values its antecedent and consequent take at possible worlds different from the actual world. In this section, following Priest [33], we will give an introduction to the semantics of Lewis' logic. This introduction will be later used to obtain the results concerning the problematic material conditionals. Throughout the section it is also important to keep in mind that S2 can be seen as an *ad hoc* solution to the Paradoxes of Material Implication. This section is based on [12], [4], [33].

Definition 28 *(Language) Let $\mathcal{P}$ be a finite set of propositional letters. We denote by $\mathcal{L}_{\mathcal{P}}$ the set of formulas built from letters in $\mathcal{P}$ using the connectives* $\wedge, \vee, \neg, \supset, \rightarrow, \Box, \Diamond$.

Definition 29 *(Model) A model for S2 is a quadruple $< W, N, R, v >$ where W is a set of possible worlds, $N \subseteq W$ is a set of normal possible worlds, R is a reflexive binary accessibility relation between the possible worlds, and v is a valuation function.*

Definition 30 *The semantics for S2 is given recursively in the following way:*

$M, w \models p$	*iff*	$v_w(p) = 1$ *(for atomic p)*
$M, w \models \neg\theta$	*iff*	$M, w \not\models \theta$
$M, w \models (\theta \wedge \psi)$	*iff*	$M, w \models \theta$ *and* $M, w \models \psi$
$M, w \models (\theta \vee \psi)$	*iff*	$M, w \models \theta$ *or* $M, w \models \psi$
for all normal worlds w $M, w \models \Box\theta$	*iff*	$\forall w'$ *s.t.* wRw' $M, w' \models \theta$
for all normal worlds w $M, w \models \Diamond\theta$	*iff*	$\exists w'$ *s.t.* wRw' $M, w' \models \theta$
for all non-normal worlds w		$v_w(\Box\theta) = 0$
for all non-normal worlds w		$v_w(\Diamond\theta) = 1$
$M, w \models \theta \supset \psi$	*iff*	$M, w \not\models \theta$ *or* $M, w \models \psi$
$M, w \models \theta \rightarrow \psi$	*iff*	$M, w \models \Box(\theta \supset \psi)$

Definition 31 *(Validity) Validity is defined at normal worlds:*
$\sum \models_{S2} \theta$ *iff for all models* $< W, N, R, v >$ *and all* $w \in N$*: whenever* $M, w \models_{S2} \psi$ *for all* $\psi \in \sum$*, then* $M, w \models_{S2} \theta$.

Before proceeding to the formal proofs it is useful to describe the semantics in more detail. The model for S2 consists of normal and non-normal possible worlds. Intuitively, normal possible worlds correspond to different ways the actual world could have been. The non-normal possible worlds are introduced to avoid the Rule of Necessitation, namely to guarantee that it is not the case that whenever a formula is a validity, then the prefixing of this formula with a $\Box$ is also a validity (i.e., *If* $\models \theta$*, then* $\models \Box\theta$). At the non-normal possible worlds, everything is possible and nothing is necessary. Thus, at non-normal possible worlds modalities are no longer defined recursively but are directly assigned their truth values. Since modality is defined only at the normal possible worlds, this guarantees that despite the fact that $\Box(\theta \vee \neg\theta)$ is a validity, $\Box\Box(\theta \vee \neg\theta)$ is no longer a validity in S2.

Possible worlds are related to each other by a binary accessibility relation $R \subseteq W \times W$. Hence, $w_1 R w_2$ means that a possible world w_1 is related to a possible world w_2. Intuitively, the accessibility relation encodes the notion of possibility i.e., that *"relative to* w_1 *situation* w_2 *is possible"*([33], pp. 21). Furthermore, based on the intuition that we always think of the actual world as possible, the binary relation R is also stipulated to be reflexive (i.e., $\forall w \in W,\ wRw$). The valuation function v assigns values from the set $\{1, 0\}$ to propositional atoms relative to a possible world. Intuitively, if $v_w(p) = 1$, then p holds at the possible world w and if $v_w(p) = 0$, then p does not hold at w. $\Diamond$ and $\Box$ encode notions of possibility and necessity. Thus, $M, w_1 \models \Diamond\theta$ means that θ is possible relative to w_1 and hence that w_1 accesses some possible world at which θ holds. Analogously, $M, w_1 \models \Box\theta$ means that every possible world we consider possible relative to w_1 is s.t. θ holds at it. As pointed out in the introduction, the motivation for the strict definition of the conditional is to make it more independent from the contingent states of affairs. Thus, the definition of the conditional as strict conditional $\Box(\theta \supset \psi)$ is meant to model it appropriately. This is because, by requiring $\theta \supset \psi$ to hold at all accessible possible worlds, whenever the conditional dependency is indeed contingent upon the states of affairs, the strict conditional does not hold. According to this analysis, despite the fact that I am a student of logic at the ILLC, on the strict interpretation of the conditional, the following conditional comes up false: *If there are infinitely many primes, then I am a student at the ILLC.* Notice, however, that on the material account of the conditional, since I am a student at the ILLC and there are infinitely many primes such a conditional is true.

The modeling of implication in terms of strict implication and the introduction of non-normal possible worlds allows one to account for many of the Paradoxes of the Material Implication. The role that the new notion of implication plays is exemplified well by the first Paradox: $p \models q \rightarrow p$. For consider the counter-model corresponding to an open branch in the tableaux proof in the Appendix:
The counter-model M is given by: $W = \{w_1, w_2\}$, $N = \{w_0\}$, $w_0 R w_0$, $w_0 R w_1$, $w_1 R w_1$, $v_{w_0}(p) = 1$, $v_{w_1}(p) = 0$ and $v_{w_0}(q) = v_{w_1}(q) = 1$. Notice that since $v_{w_0}(p) = 1$, it follows that $M, w_0 \models p$. Furthermore, since $v_{w_1}(q) = 1 \neq v_{w_1}(p)$, it follows by the definition of the clause for '$\supset$' that

$M, w_1 \not\models q \supset p$. Finally, since $w_0 R w_1$, it follows that $M, w_0 \not\models \Box(q \supset p)$. Thus, it follows by the definition of the validity that $p \not\models q \to p$.

In order to see the difference between the strict implication and material implication more clearly, notice that since $v_{w_0}(p) = 1$, it follows by the definition of '$\supset$' that $M, w_0 \models q \supset p$. Thus, it is the intensionality of strict implication that allows us to account for this paradox.

The definition of strict implication is also the key semantic feature that allows one to account for the Paradoxes (2), (6)-(9), (12)-(15). One can consider the Appendix to observe this.

The construction of non-normal possible worlds also allows us to account for some of the Paradoxes of Material Implication. For consider the counter-model to the inference (11) $\models p \to (q \to q)$:
The counter-model M is given by: $W = \{w_0, w_1\}$, $N = \{w_1\}$, $w_0 R w_0$, $w_0 R w_1$, $w_1 R w_1$, $v_{w_1}(p) = 1$, $v_{w_0}(p) = v_{w_1}(q) = v_{w_0}(q) = 1$.[2] Then, since $v_{w_1}(p) = 1$, it follows that $M, w_1 \models p \star$. Furthermore, since w_1 is a *non-normal possible world*, it follows by its definition that $v_{w_1}(\Box(q \supset q)) = 0$. Hence, it follows that $M, w_1 \not\models \Box(q \supset q)$. Thus, by $\star$ and the definition of the clause for '$\supset$', it follows that $M, w_1 \not\models p \supset \Box(q \supset q)$. Hence, since $w_0 R w_1$, it follows by the definition of the clause for '$\Box$' that $M, w_0 \not\models \Box(p \supset \Box(q \supset q))$. Importantly, notice that if w_1 was a normal possible world, then $\Box(q \supset q)$ holds at it. This is because $q \supset q$ is a tautology. Hence, the construction of non-normal possible worlds is vital to account for this paradox.

The semantic features of S2 do not lead only to desirable results. For whenever the antecedent of a conditional is necessarily false, or the consequent is necessarily true, the conditional is evaluated as being true. For consider (4) $\models \Box((p \wedge \neg p) \supset q)$. Let M and w be arbitrary. Then, by the definition of the clause for '$\wedge$', $p \wedge \neg p$ is false at every possible world accessible from w. Thus, by the definition of the clause for '$\supset$', $(p \wedge \neg p) \supset q$ holds at all of possible worlds. Hence, by the definition of '$\Box$', $\Box((p \wedge \neg p) \supset q)$ holds at w. For similar reasons (5) $\Box(p \supset (q \vee \neg q))$ is not accounted for in S2; i.e., since by the definition of the clause for '$\vee$' the consequent is necessarily true at every possible world, then by the definition of the clause for '$\to$' at every possible world accessible from w, $p \to (q \vee \neg q)$ holds.

2.2.2 Conditional Logic

A different approach towards modeling natural language implication was suggested by Stalnaker [40]. In the logical system developed by him (C2), he

[2] As will be clear by how the proof proceeds, the valuation function v can assign arbitrary truth values to p at w_0 and to q at w_0 and w_1.

suggested to account for some of the shortcomings of the material account of implication by reinterpreting the classical understanding of the implication so its truth value is not only determined by those of its antecedent and consequent at the actual world. Instead, Stalnaker argues that an indicative conditional's value should also depend on other possible worlds, together with some set of laws and truth statements. For instance, according to Stalnaker, when one utters a sentence of the form *If I go to the cinema, I will see a movie*, one implies that this conditional is dependent upon there not being any significant changes in the interim[3] and while evaluating it one is considering the most likely alternative at which she goes to the cinema. On the one hand, such an account takes into consideration the context of the conversation and the sentences which are not stated directly by the speaker but which are signaled by him. On the other hand, it makes the truth-conditions of $\theta \to \psi$ dependent upon a possible world which might be different from the actual world. On the basis of this analysis, Stalnaker proposes that the conditional $\theta \to \psi$ is true if and only if ψ is true at the most similar possible world to the actual world at which θ holds. Hence, when one evaluates the truth of the conditional $\theta \to \psi$, one considers the possible world which is essentially the same as the actual world, but at which θ holds. If ψ holds at this world as well, then the conditional is true, otherwise it is false. As Stalnaker puts it, when one is considering the indicative conditional *If* θ*, then* ψ, then "everything one is presupposing to hold in the actual situation is presupposed by one to hold in the hypothetical situation in which θ is true [...] [and where the] relevant respects of similarity are determined by the context" ([40], pp. 69).

This section has been based on [40], [29], [33] and [9]. For notational simplicity $|\ .\ |$ stands for a function which assigns to each sentence θ a subset $|\theta|$ of W (all those worlds $w \in W$ such that $v_w(\theta) = 1$) and α, β stand for subsets of possible worlds, i.e., $\alpha, \beta \subseteq W$.

Definition 32 *(Language) Let* $\mathcal{P}$ *be a finite set of propositional letters. We denote by* $\mathcal{L}_{\mathcal{P}}$ *the set of formulas built from letters in* $\mathcal{P}$ *using the connectives* $\wedge, \vee, \neg, \supset, \to$*.*

Definition 33 *(Model) A model for C2 is a quintuple* $< W, R, v, f, \lambda >$*, where* W *is a set of possible worlds,* R *is a binary reflexive accessibility relation on* W*,* v *is a valuation function*[4]*,* λ *is an absurd world (a world which accesses no possible worlds and which is inaccessible from all worlds, and at*

[3]For instance, in our example one presupposes that the projector is not broken at the cinema, that she will not have a heart-attack before the movie starts etc.

[4]Hence, $< W, R, v >$ is a reflexive Kripke model

which every sentence is true) and $f : \mathcal{P}(W) \times W \to W$ is a selection function satisfying:

$$f(\alpha, w) \in \alpha \tag{2.1}$$

$$f(\alpha, w) = \lambda \textit{ only if there is no } w' \textit{ s.t. } wRw' \textit{ and } w' \in \alpha \tag{2.2}$$

$$\textit{if } w \in \alpha, \textit{ then } f(\alpha, w) = w \tag{2.3}$$

$$\textit{if } f(\alpha, w) \in \beta \textit{ and } f(\beta, w) \in \alpha, \textit{ then } f(\alpha, w) = f(\beta, w) \tag{2.4}$$

$$\textit{if } f(\alpha, w) \neq \lambda, \textit{ then } f(\alpha, w) \in R(w) \tag{2.5}$$

Definition 34 *The semantics for C2 is given recursively in the following way:*

$$\begin{array}{lll}
M, w \models p & \textit{iff} & v_w(p) = 1 \quad \textit{(for atomic } p\textit{)} \\
M, w \models \neg\theta & \textit{iff} & M, w \not\models \theta \\
M, w \models (\theta \wedge \psi) & \textit{iff} & M, w \models \theta \textit{ and } M, w \models \psi \\
M, w \models (\theta \vee \psi) & \textit{iff} & M, w \models \theta \textit{ or } M, w \models \psi \\
M, w \models (\theta \supset \psi) & \textit{iff} & M, w \not\models \theta \textit{ or } M, w \models \psi \\
M, w \models (\theta \to \psi) & \textit{iff} & M, f(|\theta|, w) \models \psi
\end{array}$$

Definition 35 *(Validity) Validity is defined as truth preservation over all worlds of all models: $\sum \models_{C2} \theta$ iff for all models $M = < W, R, v, f, \lambda >$ and all $w \in M$: whenever $M, w \models \psi$ for all $\psi \in \sum$, then $M, w \models \theta$*

Let us now look at the semantics of C2 in more detail. Firstly notice that apart from using the standard Kripkean machinery, Stalnaker introduces two new features in C2: the absurd possible world λ and the selection function f. The absurd possible world is a possible world at which every sentence is true. Introduction of such a possible world allows Stalnaker to provide truth conditions for conditionals involving contradictions. For notice that (2.2) together with (2.5) guarantee that the selected possible world is absurd whenever the antecedent of a conditional is a contradiction.

It is also useful to discuss the requirements on the selection function f:

- Requirement (2.1) guarantees that the conditional $\theta \to \theta$ is always true.
- As pointed out above, requirements (2.2) and (2.5) determine when the absurd possible world is selected.
- Requirement (2.3) reflects the intuition that, since $f(|\theta|, w)$ relates w to a possible world which is essentially the same as w apart maybe from the fact that θ holds at it, then if θ holds at w, the world which we consider to be the most similar at which θ holds is the actual world.

- Requirement (2.4) guarantees that the ordering induced by the selection function is consistent. That is, whenever the most similar θ world is s.t. ψ holds at it, and the most similar ψ world is s.t. θ holds at it, then these are the same possible worlds.

- The last requirement is commonly referred to as Stalnakers Uniqueness Assumption. It guarantees that "there is always a unique possible world at which the antecedent is true and which is more like the actual world than is any other world at which the antecedent is true"([29], pp. 9). N.B. that one of the common critiques of C2 questions the Uniqueness Assumption (cf. [33]).

As noted earlier, Stalnaker's semantics gives a plausible model of our uses of conditionals, in which when we evaluate $\theta \to \psi$ we first add θ to our set of beliefs, alter this set of beliefs as little as possible in order to accommodate the new belief and verify whether ψ holds. This also points towards Stalnakers interpretation of possible worlds, in which they correspond to epistemically ideal situations [2].

As one can verify in the Appendix, the semantic features of C2 allow us to account for many of the Paradoxes of Material Implication considered. The semantic definitions involved in Stalnaker's logic lead, however, to some undesirable results. In particular Lewis' construction of the absurd world and some of the requirements on the selection function validate some of the inferences we would like to account for.

The construction of the absurd possible world can be seen as validating (4) $(p \wedge \neg p) \to q$. In order to see the contribution of an absurd world 'λ' consider the proof of (4).

Let M be an arbitrary model and let w be a possible world in this model. Since $p \wedge \neg p = \emptyset$, it follows that $f(|p \wedge \neg p|, w) = \lambda$. Now it follows that $\lambda \models q$. Hence, it follows by the definition of '$\to$', that $M, w \models (p \wedge \neg p) \to q$. Since, M and w were arbitrary, it follows that $\models (p \wedge \neg p) \to q$, as required.

Some of the requirements on the selection function also contribute to the fact that the undesirable classical validities hold in C2. For consider proofs below to see that requirement (2.3) is the key semantic factor behind the inference (7)$p \wedge q \models p \to q$, requirement (2.1) behind the inference (9)$\neg p \models (p \to \neg p)$ and (11)$\models p \to (q \to q)$.

In order to see why (7) holds, let M be an arbitrary model and let w be a possible world in this model. Suppose that $M, w \models p \wedge q$. Then it follows by the definition of '$\wedge$' that $M, w \models p$ and $M, w \models q$. Hence, it follows by *the property (2.3) of a selection function* that $f(|p|, w) = w$. Hence, it follows that $M, f(|p|, w) \models q$. Thus, it follows by the definition of the '$\to$'

that $M, w \models p \rightarrow q$. Thus, since M and w were arbitrary, it follows that $p \wedge q \models p \rightarrow q$, as required.

When considering (9), let M be an arbitrary model and let w be a possible world in this model s.t. $M, w \models \neg p$. Suppose for contradiction that $M, w \not\models p \rightarrow p$. Then, it follows that $\exists w'$ s.t. wRw', $f(|p|, w) = w'$ and $M, w' \not\models p$ ↯. This is a contradiction, since by *the property (2.1) of the selection function* it follows that $w' \in |p|$. Thus, $\neg p \models p \rightarrow p$, as required.

A proof by contradiction demonstrates why (11) holds in C2. For suppose for contradiction that there exists M and w s.t. $p \rightarrow (q \rightarrow q)$ does not hold at w. Then, it follows by the definition of '$\rightarrow$' that $M, f(|p|, w) \not\models q \rightarrow q$ $\star$. Wlog suppose $f(|p|, w) = w_1$. Then, it follows by $\star$ that $M, w_1 \not\models q \rightarrow q$. Hence, by the definition of '$\rightarrow$' $M, f(|q|, w_1) \not\models q$ ↯. This is a contradiction since *by property (2.1) of the selection function*, $f(|q|, w_1) \in |q|$ and hence $M, f(|q|, w_1) \models q$. Thus, since M and w were arbitrary it follows that $\models p \rightarrow (q \rightarrow q)$, as required.

It is also useful to consider the proofs for Paradox (3) $p \rightarrow s \models (p \wedge q) \rightarrow s$, (10) $p \rightarrow q, q \rightarrow s \models p \rightarrow s$ and (16) $p \rightarrow q \models \neg q \rightarrow \neg p$. This is because, as will become clear by the end of this chapter, the only non-classical system from the ones considered that correctly accounts for these paradoxes is C2. The consideration of the proofs of these inferences also further explicates the semantic features of C2.

In order to see why (3) holds, consider a model M s.t. $W = \{w_0, w_1\}$, w_0Rw_0, w_1Rw_1, w_0Rw_0, $|p| = \{w_0, w_1\}$, $|s| = \{w_0\}$, $|q| = \{w_1\}$, $f(|p|, w_0) = w_0$, $f(|p \wedge q|, w_0) = w_1$. Then, it follows by the definition of '$\rightarrow$' that $M, w_0 \models p \rightarrow s$, however since $f(|p \wedge q|, w_0) = w_1$ and $M, w_1 \not\models s$, it follows by the definition of '$\rightarrow$' that $M, w_0 \not\models (p \wedge q) \rightarrow s$. Thus, $p \rightarrow s \not\models (p \wedge q) \rightarrow s$, as required.

One can consider the following model M, in order to see why (10) does not hold. Let M be s.t. $W = \{w_0, w_1, w_2\}$, w_0Rw_0, w_1Rw_1, w_2Rw_2, w_0Rw_1, w_0Rw_2, $|p| = \{w_2\}$, $|q| = \{w_1, w_2\}$, $|s| = \{w_2\}$, $f(|p|, w_0) = w_1$, $f(|q|, w_0) = w_2$. Then, it follows that $M, w_1 \models q$ and hence $M, f(|p|, w_0) \models q$. Thus, by the definition of '$\rightarrow$' $M, w_0 \models p \rightarrow q$. Similarly, it follows that $M, w_2 \models s$ and hence $M, f(|q|, w_0) \models s$. Thus, by the definition of '$\rightarrow$' $M, w_0 \models q \rightarrow s$. Notice, however, that since $M, w_1 \not\models s$, i.e., $M, f(|p|, w_0) \not\models s$, it follows by the definition of '$\rightarrow$' that $M, w_0 \not\models p \rightarrow s$. Thus, $p \rightarrow q, q \rightarrow s \not\models p \rightarrow s$, as required.

In order to see why (16) holds, consider a model M s.t. $W = \{w_0, w_1\}$, w_0Rw_0, w_1Rw_1, w_0Rw_1, $|p| = \{w_0, w_1\}$, $|q| = \{w_0\}$, $f(|p|, w_0) = w_0$, $f(|\neg q|, w_0) = w_1$. Then, it follows that $M, w_0 \models q$ and hence that $M, f(|p|, w_0) \models q$. Thus, by the definition of '$\rightarrow$' $M, w \models p \rightarrow q$. Notice, however, that since $|p| = \{w_0, w_1\}$, it follows that $M, w_1 \models p$ and hence $M, f(|\neg q|, w_0) \models q$. Thus, it follows by the definition of '$\rightarrow$' that $M, w_0 \not\models \neg q \rightarrow \neg p$. Thus, $p \rightarrow q \not\models \neg q \rightarrow \neg p$, as required.

2.2.3 Update Semantics

In this section of the thesis we will consider the attempt to modeling natural language indicative conditionals as presented by Update Semantics [44]. In contrast to the accounts discussed above, the update semantic account is *dynamic*, i.e., it is no longer concerned solely with truth-preservation, but rather it is focused on the notion of information change and update. More specifically, the meaning of a sentence is no longer associated with its truth conditions, but it is an operation on information states, where information states are intuitively what an agent takes to be true. The difference between previous logics and the current logic is well reflected in the slogan of Update Semantics, namely: *"You know the meaning of a sentence if you know the change it brings in the information state of anyone who accepts the news conveyed by it"* ([44], pp. 1). As will be demonstrated in Section 2.3, Update Semantics gives rise to a system that allows one to account for some of the problems of the horse-shoe analysis of implication. Moreover, it is useful to keep in mind that this system is closely related to Inquisitive Semantics, which, to a certain extent, can be viewed as a static counterpart of this dynamic system. This section is based on [44] and [43].

Definition 36 *(Language). Let $\mathcal{P}$ be a finite set of propositional letters. We denote by $\mathcal{L}_{\mathcal{P}}$ the set of formulas built up from letters in $\mathcal{P}$ using the connectives $\neg, \vee, \wedge, \rightarrow, \Diamond$.*

Definition 37 *Let W be the powerset of $\mathcal{P}$. Then:*
1) σ is an information state iff $\sigma \subseteq W$
2) For every two states σ, τ, $\sigma + \tau := \sigma \cap \tau$

Definition 38 *The semantics for US is given recursively in the following way:*

$$\sigma[p] = \sigma \cap \{w \in W | p \in w\}$$
$$\sigma[\neg\theta] = \sigma \setminus \sigma[\theta]$$
$$\sigma[\theta \wedge \psi] = \sigma[\theta] \cap \sigma[\psi]$$
$$\sigma[\theta \vee \psi] = \sigma[\theta] \cup \sigma[\psi]$$
$$\sigma[\Diamond\ \theta] = \sigma \quad \textit{if} \quad \sigma[\theta] \neq \emptyset$$
$$\sigma[\Diamond\ \theta] = \emptyset \quad \textit{if} \quad \sigma[\theta] = \emptyset$$
$$\sigma[\theta \rightarrow \psi] = \sigma \quad \textit{if} \quad \sigma[\theta][\psi] = \sigma[\theta]$$
$$\sigma[\theta \rightarrow \psi] = \emptyset \quad \textit{if} \quad \sigma[\theta][\psi] \neq \sigma[\theta]$$

Definition 39 *(Support) A state σ supports a sentence θ , $\sigma \models \theta$, iff $\sigma[\theta] = \sigma$, where $\sigma[\theta]$ is an update of a state σ with information θ.*

Definition 40 *(Validity) An argument is valid iff updating a state σ with premises $\psi_1, \cdots, \psi_n$, yields an information state in which the conclusion θ is supported, i.e., $\psi_1, \cdots \psi_n \models \theta$ iff $\forall\sigma, \sigma[\psi_1]\cdots[\psi_n] \models \theta$.*

As there are significant differences between Update Semantics and the other systems we discussed, it is important to examine Update Semantics in greater detail. Notice that states σ can be viewed just as sets of possible worlds, whereas for a sentence θ an operation $\sigma[\theta]$ is a result of updating an information state σ with information encoded by θ. According to this framework, the support of a sentence θ at a state σ (i.e., $\sigma[\theta] = \sigma$) is equivalent to accepting θ in an information state σ. This can be thought of as reflecting the intuition that, if we already accept a sentence θ in our information state, then the update of this state with information θ does not change what we take to be true. The update clauses given by the semantics of US define how σ changes when somebody in a state σ accepts a sentence θ. Hence, for instance, updating an information state σ with a sentence $\neg\theta$ is equivalent to removing from σ all the possible worlds s.t. θ holds at them. Notice that in this framework, the semantics for all connectives apart from the conditional and '$\Diamond$' are dynamic, i.e., after their acceptance conversational participant modifies his information state. As for the conditional and '$\Diamond$', these are to be treated as *consistency tests*: by accepting them, a conversational participant only verifies if they hold, but does not update his information state. So to speak, the sentence involving *might* and the conditional correspond to performing a test on σ, rather than introducing some information to our information state.

The main benefit of pursuing update semantics is that it gives an intuitive and desirable account of our natural language as a dynamic process, it is not, however, the only one. Another key benefit of US relates to the Ramsey test[5]. For notice that, in order to verify if the conditional $\theta \to \psi$ holds, one checks if after updating his information state with θ, ψ holds. This is in line with the Ramsey Test, in which when one verifies the truth of a conditional, he appends θ to the set of his beliefs and checks if it is such that ψ holds in it.

[5]The Ramsey test is one of the first and most influential methods suggested to analyse conditionals. It defines a procedure for verifying whether an indicative conditional holds. As originally stated by Ramsey in his 1929 footnote: *"If two people are arguing 'If A will C?' and are both in doubt as to A, they are adding A hypothetically to their stock of knowledge and arguing on this basis about C... We can say they are fixing their degrees of belief in C given A."* ([3], pp. 28)

Indeed, the links between the update framework and the nature of natural language implication are visible in the analysis in Chapter 4.

Update Semantics allows us to account for 5 out of the 16 paradoxical inferences considered. Namely, inferences (8) $\models (p \to q) \vee (q \to p)$, (12) $(p \wedge q) \to s \models (p \to s) \vee (q \to s)$, (13) $(p \to q) \wedge (s \to t) \models (p \to t) \vee (s \to q)$, (14) $\neg(p \to q) \models p$ and (15) $\neg(p \to q) \models \neg q$. In all of these inferences, it is the modeling of implication as a test on states that can be attributed to be the key semantic feature that allows us to account for them. To explicate this point consider the counter-model for (10):

Let $\sigma = \{w_1, w_2\}$, $|p| = \{w_1\}$, $|q| = \{w_2\}$. Then, it follows that $\sigma[p] = \{w_1\} \neq \sigma[p][q] = \emptyset$. Hence, by the definition of '$\to$', it follows that $\sigma[p \to q] = \emptyset$. Similarly, it follows that $\sigma[q] = \{w_2\} \neq \sigma[q][p] = \emptyset$. Hence, by the definition of '$\to$', it follows that $\sigma[q \to p] = \emptyset$. Thus, it follows by the definition of '$\vee$', that $\sigma[(p \to q) \vee (q \to p)] = \emptyset \neq \sigma$. Thus, it follows that $\not\models (p \to q) \vee (q \to p)$, as required.

It is also worth to consider the counter-model for the inference (14):

Let $\sigma = \{w_1, w_2\}$, $|p| = \{w_1\}$ and $|q| = \{w_2\}$. Then, it follows by the definition of '$\neg$' and '$\to$' that $\sigma[\neg(p \to q)] = \sigma \setminus \sigma[p \to q] = \sigma \setminus \emptyset = \sigma$. Notice, however that $\sigma[p] = \{w_1\} \neq \sigma$. Hence, it follows that $\sigma[\neg(p \to q)] \not\models p$. Thus, $\neg(p \to q) \not\models p$. N.B. this example also demonstrates that that there is a close relation between '$\Diamond$' and '$\to$'. This is because when we test if a state σ supports $\neg(p \to q)$, in practice we also verify whether σ is consistent with $p \wedge \neg q$.

In similar fashion the Update Semantic modeling of implication allows us to account for (12), (13) and (15).

Unfortunately, the semantic features of US do not allow one to account for all the problematic inferences. Importantly, out of the systems considered it is the only system that validates (1) $p \models q \to p$, (2) $\neg q \models q \to p$ and (8) $p \to (q \to p)$. Let us consider the proofs for (1) and (2) to see how the semantic features of US validate these inferences.

Consider a proof by contradiction to see why (1) holds. Let σ be arbitrary. Suppose for contradiction that $\exists \sigma$ s.t. $\sigma[p] \not\models p \to q$. Then it follows that $\sigma[p][q][p] \neq \sigma[p][q]$ $\lightning$. This is a contradiction, since it follows by the definition of an update with an atomic sentence that for any state σ supporting an atomic sentence p, $\sigma = \sigma[p]$. Thus, it follows that $p \models q \to p$.

Similarly consider the following proof to see why (2) holds. Let σ be arbitrary. Suppose for contradiction that $\exists \sigma$ s.t. $\sigma[\neg p] \not\models \sigma[p][q]$. Then, it follows that $\sigma[\neg p][p][q] \neq \sigma[\neg p][p]$ $\star$. Notice that it follows by the support definition for atomic sentence and negation that $\sigma \models [p][\neg p]$ iff $\sigma = \emptyset$. Hence, by $\star$ $\emptyset \neq \emptyset[q] = \emptyset$ $\lightning$. Thus, it follows that $\neg p \models p \to q$.

2.2.4 Relevance Logic

One of the family of logics developed to deal with the Paradoxes of Material Implication are Relevance Logics. This family of logics take material implication to be problematic because it fails to model a tie between the antecedent and the consequent. More specifically, Relevance Logics aim to give a model that guarantees that there is always a connection between the antecedent and the consequent; in other words, that the content of the antecedent is relevant to the content of the consequent. The link between the content of the antecedent and the content of the consequent is established by utilizing the *variable sharing principle*: the requirement that, for the implication to be valid, the antecedent and the consequent should share at least one propositional variable. Thus, by these means Relevance Logic guarantees that there is *some* connection between the antecedent and the consequent; and that the antecedent and the consequent are to a (possibly very small) degree semantically relevant to each other. In this section we will look closer into one of the Relevance Logics, Logic B, and discuss its key semantic features.[6]

In the hierarchy of Relevance Logics, Logic B plays a similar role to that of Kripke's system K [12]. In short, logic B is a paraconsistent logic that utilizes a ternary accessibility relation. On the one hand, it uses Routley style semantics of negation to account for the inferences of the form $(\theta \wedge \neg\theta) \rightarrow \psi$, whereas on the other hand it uses a ternary accessibility relation to account for the inferences of the form $\theta \rightarrow (\psi \rightarrow \theta)$. The Routley Semantics for negation uses the $* - operator$. This operator defines the negation at a possible world w in terms of its mate w^*. Here, w and w^* can be thought of as "mirror images of one another, reversing 'in' and 'out' " ([12], pp. 191). By this means negation defines what is asserted at w in terms of what is not denied at w^*. Apart from accounting for some of the implausible inferences, the ternary relation also gives us a way of capturing the notion of relevance between the antecedent and the consequent. According to the interpretation

[6] N.B. The philosophical underpinnings of Relevance Logics are not fully developed yet. Indeed there does not exist a commonly agreed upon philosophical motivation for relevance semantics. In this section, we attempt to give the most charitable reading of Relevance Logic. One of the difficulties in providing a compact and perspicuous introduction to Relevance Logic, however, is that many of the notions involved are model theoretic notions with very tenuous explanatory power. Because of this, the current introduction might not be sufficient to *fully* explain the motivation behind Relevance Logic and provide the reader with enough insight. Nevertheless, even if the reader fails to find strong intuitions for relevance semantic definitions, it is plausible to treat Relevance Logic as a useful instrument for modeling English indicative conditionals because it avoids many of the implausible results yielded by the classical analysis. Should the reader wish to gain further insight into Relevance Logic, we direct you to [18].

given by Barwise [2], one can understand the ternary relation in a framework in which possible worlds are identified with "sites" and "channels". In this framework channels transmit the information from site to site. More intuitively a site corresponds to the context within which the information is obtained and a channel corresponds to the means by which we obtain the information. Then, the ternary relation $Rabc$ means that a is a channel between the sites b and c. On this account the conditional $\theta \rightarrow \psi$ is true at a possible world a if for all sites b and c connected by a channel a whenever information θ is obtained at b then information ψ is obtained at c. The idea is exemplified well by the Stanford Encyclopedia of Philosophy *"when the BBC news appears on the television in my living room, we can consider the living room to be a site and the wires, satellites, and so on, that connect my television to the studio in London to be channel. [...][Then we can] take Rabc to mean that a is an information-theoretic channel between sites b and c."*[27].

The semantic elements of Logic B will be described in greater detail below. While considering the semantics, one needs to take into account the fact that it has been developed *ad hoc* to deal with the problematic inferences, and this is strongly reflected in the semantic definitions and interpretations.

This section of the thesis is based on [2], [34], [27], [8], [12], [33] and [18].

Definition 41 *(Language) Let $\mathcal{P}$ be a finite set of propositional letters. We denote by $\mathcal{L}_{\mathcal{P}}$ the set of formula built up from letters in $\mathcal{P}$ using the connectives $\wedge, \vee, \neg, \rightarrow$.*

Definition 42 *(Model) A model for B is a structure $< W, N, R, *, v >$ where W is the set of all possible worlds, $N \subseteq W$ is the set of normal possible worlds, $*$ is a Routley Star, i.e., a function $* : W \rightarrow W$ requiring that $M, w \models \neg\theta$ iff $M, w^* \not\models \theta$, and R is a ternary relation on the sets of possible worlds, i.e., $R \subseteq W \times W \times W$ s.t. if $w \in N$ then $Rwxy$ iff $x = y$.*

Definition 43 *The semantics for B is given recursively in the following way:*

$$
\begin{array}{lll}
M, w \models p & \textit{iff} & v_w(p) = 1 \quad \textit{(for all atomic } p\textit{)} \\
M, w \models \neg\theta & \textit{iff} & M, w^* \not\models \theta \\
M, w \models (\theta \wedge \psi) & \textit{iff} & M, w \models \theta \textit{ and } M, w \models \psi \\
M, w \models (\theta \vee \psi) & \textit{iff} & M, w \models \theta \textit{ or } M, w \models \psi \\
M, w \models \theta \rightarrow \psi & \textit{iff} & \forall x, y \in W \textit{ s.t. } Rwxy \\
 & & \textit{if } M, x \models \theta, \textit{ then } M, y \models \psi
\end{array}
$$

Definition 44 *(Validity) Validity is defined as truth preservation over all normal worlds of all models: $\sum \models_B \theta$ iff for all models $< W, N, R, *, v >$: whenever $M, w \models_B \psi$ for all $\psi \in \sum$, then $M, w \models_B \theta$.*

As pointed out in the introduction to grasp the meaning of the $*$ operator, it is good to see it as determining a pair of worlds w and w^* such that what is asserted at w^* is exactly what is not denied at w and vice versa [18]. What one asserts at w is, so to speak, exactly what one fails to deny at w^*. Notice that defining the truth conditions by using the Routley Star makes it possible for contradictions to be true at a world. Hence, $\neg A \wedge A$ can be true at w. This example also demonstrates that w^* is negation incomplete, since by the definition at w^* both A and $\neg A$ can be false. As described by Dunn, "where a is inconsistent (containing both θ and $\neg\theta$), the other is incomplete (lacking both θ and $\neg\theta$), and vice versa (when $a = a^*$, a is both consistent and complete and we have a situation appropriate to classical logic)" ([12], pp. 191). This also points to a different treatment of the possible worlds in relevance framework. In this framework, possible worlds are still maximal, in the sense that the truth value of every atomic formula is determined at every possible world, but possible worlds are no longer such that all of the logical laws hold at them (as demonstrated, the possible worlds are negation incomplete). As pointed out in the introduction, such a treatment of possible worlds and negation was developed *ad hoc* to deal with the Paradoxes of Material Implication of the form $(p \wedge \neg p) \rightarrow q$. The requirement that for every normal possible world $Rwxy$ iff $x = y$ is commonly referred to as a *normality condition* [33]. It allows us to generalize the ternary accessibility relation to both normal and non-normal possible worlds. As with the other logics discussed, the non-normal possible worlds play mostly an instrumental role in the semantics.

It is difficult to give an intuitive justification for the semantic definitions involved in the logic given by B. This is caused mostly by the *ad hoc* character of this logic, where the sole purpose of this logic was to account for the problematic inferences of material implication. The fact that there is no strong and intuitive semantics developed for this logic is certainly one of its shortcomings. On the other hand, the *ad hoc* character of this logic makes it very successful at accounting for the problems of the material understanding of English indicative conditionals.

In order to exemplify the Relevance Logic approach let us consider a counter-model given by Relevance Logic B to the first paradox on our list, i.e., $p \models q \rightarrow p$. The counter-model for this paradox demonstrates especially the behavior of the ternary accessibility relation, the normality condition and the definition of implication:

Let the model M be given by $W = \{w_0, w_1, w_0^*, w_1^*\}$, $N = \{w_0\}$, $Rw_0w_1w_1$, $Rw_0w_0^*w_0^*$, $Rw_0w_1^*w_1^*$, $w_0 \rightarrow w_0^*$, $w_1 \rightarrow w_1^*$, $v_0(p) = 1$, $v_1(q) = 1$, $v_1(p) = 0$ and $v_{w_1^*}(p) = v_{w_1^*}(q) = v_{w_0^*}(p) = v_{w_0^*}(q) = 1$. Then, since $v_{w_0}(p) = 1$, it follows that $M, w_0 \models p$. Furthermore, since $Rw_0w_1w_1$ and $v_{w_1}(q) = 1$ while

$v_{w_1}(p) = 0$, it follows by the definition of the $\to$, that $M, w_1 \not\models q \to p$. Hence, $p \not\models q \to p$, as required.

In order to picture the semantic characteristics of negation and Routley Star consider the counter-model to Paradox (11) $\neg p \models p \to \neg p$:
Let the model M by given by $W = \{w_0, w_1, w_0^*, w_1^*\}$, $N = \{w_0\}$, $Rw_0w_1w_1$, $Rw_0w_0^*w_0^*$, $Rw_0w_1^*w_1^*$, $w_0 \to w_0^*$, $w_1 \to w_1^*$, $v_{w_0^*}(p) = 0$, $v_{w_1^*}(p) = 1$, $v_{w_1}(p) = 1$, $v_{w_0}(p) = 1$. Since $v_{w_1^*}(p) = 1$, it follows by the definition of negation that $M, w_1^* \models p$ and hence by the definition of negation that $M, w_1 \not\models \neg p$ †. Similarly, since $v_{w_0^*}(p) = 0$, it follows that $M, w_0^* \not\models p$ and hence by the definition of negation $M, w_0 \models \neg p$. Notice that since $v_{w_1}(p) = 1$, it follows that $M, w_1 \models p$ ‡. Now, since $Rw_0w_1w_1$ it follows by the definition of the clause for '$\to$' and by † and ‡ that $M, w_0 \not\models p \to \neg p$. Thus, since $M, w_0 \models \neg p$, $\neg p \not\models p \to \neg p$, as required.

The semantic definitions in Relevance Logic B allow us to account for almost all of the Paradoxes of Material Implication considered by us; for only 3 of them hold in this logic. Out of the inferences that are correctly accounted for by Relevance Logic B, it is useful to consider (4) $\models (p \wedge \neg p) \to q$ and (5) $\models p \to (q \vee \neg q)$. In particular, this is because Relevance Logic B is the only system that allows us to correctly account for these inferences.

In order to see why (4) does not hold in Relevance Logic B consider the following counter-model M given by: $W = \{w_0, w_1, w_0^*, w_1^*\}$, $N = \{w_0\}$, $w_0 \to w_0^*$, $w_1 \to w_1^*$, $Rw_0w_1w_1$, $Rw_0w_0^*w_0^*$, $Rw_0w_1^*w_1^*$, $v_{w_1}(q) = 0$, $v_{w_1}(p) = 1$, $v_{w_1^*}(p) = 0$, $v_{w_1^*}(q) = 1$ and at w_0, w_0^* valuation function assigns arbitrary values to p and q. Since $v_{w_1}(p) = 1$ and $v_{w_1^*}(p) = 0$, it follows by the definition of the clause for atomic sentences that $M, w_1 \models p$; and it follows by the definition of the clause for '$\neg$' that $M, w_1 \models \neg p$ †. Hence, by the definition of the clause for '$\wedge$' that $M, w_1 \models p \wedge \neg p$. Now, since $v_{w_1}(q) = 0$, it follows that $M, w_1 \not\models q$ ‡. Thus, by †, ‡ and the definition of the clause for '$\to$', it follows that $M, w_0 \not\models (p \wedge \neg p) \to q$. Hence, $\not\models (p \wedge \neg p) \to q$, as required.

The following counter-model demonstrates how Relevance Logic B accounts for (5):
Let M be given by: $W = \{w_0, w_1, w_0^*, w_1^*\}$, $N = \{w_0\}$, $w_0 \to w_0^*$, $w_1 \to w_1^*$, $Rw_0w_1w_1$, $Rw_0w_0^*w_0^*$, $Rw_0w_1^*w_1^*$, $v_{w_0}(p) = v_{w_1}(p) = 1$, $v_{w_1}(q) = 0$, $v_{w_1^*}(q) = 1$, $v_{w_1^*}(p) = 1$ and the values assigned by the valuation function at w_0 and w_0^* are arbitrary. Then, since $v_{w_1}(p) = 1$ and $v_{w_1}(q) = 0$, it follows that $M, w_1 \models p$ and $M, w_1 \not\models q$ †. Furthermore, since $v_{w_1^*}(q) = 1$, it follows by the definition of the clause for '$\neg$' that $M, w_1 \not\models \neg q$ ‡. Thus, it follows by † and ‡ and the definition of the clause for '$\vee$' that $M, w_1 \not\models q \vee \neg q$. Hence, it follows by the definition of the clause for '$\to$' that $M, w_0 \not\models p \to (q \vee \neg q)$. Thus, $\not\models p \to (q \vee \neg q)$, as required.

2.3 S2, C2, US, B: Results

In this Chapter we have specified the paradoxical inferences we will analyze in the thesis. We have also given examples that point out their counterintuitiveness. Importantly, the list of the problematic inferences from section 2.1 can be thought to be an exemplary list that constitutes a representative group of the inferences that can be found in the literature. Furthermore, we have also given a description of leading approaches towards modeling implication as given by S2, C2, US and B. Different semantic accounts of implication allow us to explain away a different number of the Paradoxes of Material Implication. We postpone the detailed discussion of the effectiveness of the systems introduced to Chapter 4. In the interim we summarize different accounts of the problematic inferences. For the formal proof of each of the results please consult the Appendix.

In the table below '✓' indicates that a given validity holds and '$\times$' indicates that it does not hold within the semantic system in question. Out of the logics considered, clearly classical propositional logic is the most problematic one. This is because all of the counterintuitive inferences hold in it. On the other hand Relevant Logic B is most successful at accounting for the paradoxes. The strict conditional logic S2 fails to invalidate five out of the 16 inferences considered. There are 13 paradoxes that are not accounted for by Update Semantics on the semantic level. Similarly to S2, the conditional logic C2 does not account for five of the problematic inferences.

	CPL	S2	C2	US	B
(1) $p \models q \to p$	✓	×	×	✓	×
(2) $\neg q \models q \to p$	✓	×	×	✓	×
(3) $p \to s \models (p \wedge q) \to s$	✓	✓	×	✓	✓
(4) $\models (p \wedge \neg p) \to q$	✓	✓	✓	✓	×
(5) $\models p \to (q \vee \neg q)$	✓	✓	✓	✓	×
(6) $\models p \to (q \to p)$	✓	×	×	✓	×
(7) $p \wedge q \models p \to q$	✓	×	✓	✓	×
(8) $\models (p \to q) \vee (q \to p)$	✓	×	×	×	×
(9) $\neg p \models (p \to \neg p)$	✓	×	✓	✓	×
(10) $p \to q, q \to s \models p \to s$	✓	✓	×	✓	✓
(11) $\models p \to (q \to q)$	✓	×	✓	✓	×
(12) $(p \wedge q) \to s \models (p \to s) \vee (q \to s)$	✓	×	×	×	×
(13) $(p \to q) \wedge (s \to t) \models (p \to t) \vee (s \to q)$	✓	×	×	×	×
(14) $\neg(p \to q) \models p$	✓	×	×	×	×
(15) $\neg(p \to q) \models \neg q$	✓	×	×	×	×
(16) $p \to q \models \neg q \to \neg p$	✓	✓	×	✓	✓

CHAPTER 3

BIS, RIS and the Paradoxes of Material Implication

In the first chapter we provided an introduction to the inquisitive semantic enterprise. We demonstrated that by enriching the meaning of a proposition, Inquisitive Semantics gives rise to a new semantic system that gives a plausible account of many of our language uses. Some of the key elements of the inquisitive enterprise are related to the enrichment of the classical meaning of a proposition with its inquisitive content and, in the case of Radical Inquisitive Semantics, to a more detailed modeling of the responses one can give to a sentence uttered. Following our discussion in Chapter 2 concerning different accounts of the Paradoxes of Material Implication, in this chapter we will give inquisitive semantic account of the problematic inferences in question. We will verify the paradoxical inferences within the framework of BIS and RIS. In RIS we will consider the entailment relation from two angles: support and rejection.

3.1 BIS and Paradoxical Inferences

The key element of Basic Inquisitive Semantics is the refinement of the definition of a proposition with its inquisitive meaning. One of the key semantic clauses that embodies the inquisitive aspect of BIS, is the inquisitive disjunction. It turns out that the inquisitive nature of disjunction also contributes to accounting for some of the problematic inferences involving material implication. Below we will demonstrate that among the implausible implications, all that involve disjunction are successfully accounted for by Basic Inquisitive

Semantics.

1. $p \models_{BIS} q \to p$
Proof by contradiction.
Let σ be arbitrary and suppose that $\sigma \models_{BIS} p$. Then, it follows by the definition of the support clause for atomic sentences that $\forall v \in \sigma$, $v(p) = 1 \star$. Suppose for contradiction that $\sigma \not\models_{BIS} q \to p$. Then it follows by the definition of '$\to$' that $\exists \tau \subseteq \sigma$ s.t. $\tau \models_{BIS} q$ and $\tau \not\models_{BIS} p$. Hence, $\exists v \in \tau$ s.t. $v(p) = 0 \lightning$. This is a contradiction to $\star$. Thus, $p \models_{BIS} q \to p$, as claimed.

2. $\neg p \models_{BIS} p \to q$
Proof by contradiction
Let σ be arbitrary and suppose that $\sigma \models_{BIS} \neg p$. Then it follows that $\forall v \in \sigma$, $v(p) = 0 \star$. Suppose for contradiction that $\sigma \not\models_{BIS} p \to q$. Then it follows by the definition of '$\to$' that $\exists \tau \subseteq \sigma$ s.t. $\tau \models_{BIS} p$ and $\tau \not\models_{BIS} q$. Hence, $\forall v \in \tau$ $v(p) = 1$ and $\exists v \in \tau$ $v(q) = 0$. Thus, since $\tau \subseteq \sigma$, $\exists v \in \sigma$ s.t. $v(p) = 1 \lightning$. This is a contradiction to $\star$. Thus, $\neg p \models_{BIS} p \to q$, as claimed.

3. $p \to s \models_{BIS} (p \wedge q) \to s$.
Proof by contradiction
Let σ be arbitrary and suppose that $\sigma \models_{BIS} p \to s$. Then it follows by the definition of '$\to$' that $\forall \tau \subseteq \sigma$ if $\tau \models_{BIS} p$, then $\tau \models_{BIS} s \star$. Suppose for contradiction that $\sigma \not\models_{BIS} (p \wedge q) \to s$. Then, it follows by the definition of $\to$ that $\exists \tau' \subseteq \sigma$ s.t. $\tau' \models_{BIS} p \wedge q$ and $\tau' \not\models_{BIS} s$. Thus, $\tau' \models_{BIS} p$ and $\tau' \models_{BIS} q$ but $\tau' \not\models_{BIS} s \lightning$. This is a contradiction since $\tau' \subseteq \tau \subseteq \sigma$ and hence by $\star$ $\tau' \models_{BIS} s$. Thus, $p \to s \models_{BIS} (p \wedge q) \to s$, as claimed.

4. $\models_{BIS} (p \wedge \neg p) \to q$
Proof by contradiction
Suppose for contradiction that $\not\models_{BIS} (p \wedge \neg p) \to q$. Then, it follows by the definition of the support clause for '$\to$; that $\exists \tau \subseteq \sigma$ s.t. $\tau \models_{BIS} p \wedge \neg p$ and $\tau \not\models_{BIS} q$. Thus, $\exists v \in \tau$ s.t. $v(q) = 0$ and by the definition of the support clause for "$\wedge$" s.t. $v(p) = 1$ and $v(p) = 0 \lightning$ Thus, $\models_{BIS} (p \wedge \neg p) \to q$, as claimed.

5. $\not\models_{BIS} p \to (q \vee \neg q)$
Proof
Let $\sigma = \{w_1, w_2\}$, $|p| = \{w_1, w_2\}$ and $|q| = \{w_2\}$. Then, it follows by the definition of the support clause for an atomic sentence $\sigma \models_{BIS} p$.

Notice, however, that since $|q| = \{w_2\}$, it follows by the definition of the support clause for atomic sentences and '$\neg$' that $\sigma \not\models_{BIS} q$ and $\sigma \not\models_{BIS} \neg q$. Thus, by the support definition for '$\vee$' $\sigma \not\models_{BIS} q \vee \neg q$. Thus, if follows by the definition of the support clause for '$\rightarrow$' that $\sigma \not\models p \rightarrow (q \vee \neg q)$. Hence, $\not\models_{BIS} p \rightarrow (q \vee \neg q)$, as required.

6. $\models_{BIS} p \rightarrow (q \rightarrow p)$
Proof by contradiction
Suppose for contradiction that $\not\models_{BIS} p \rightarrow (q \rightarrow p)$. Then it follows by the definition of the support clause for '$\rightarrow$' that $\exists \tau \subseteq \sigma$ s.t. $\tau \models_{BIS} p$ and $\tau \not\models_{BIS} q \rightarrow p$. $\lightning$. This is a contradiction by (1). Hence, $\models_{BIS} p \rightarrow (q \rightarrow p)$, as claimed.

7. $p \wedge q \models_{BIS} p \rightarrow q$
Proof by contradiction
Suppose that $\sigma \models_{BIS} p \wedge q$. Then, it follows by the definition of '$\wedge$' that $\sigma \models_{BIS} p$ and $\sigma \models_{BIS} q$. Hence, it follows by the definition of the support clause for an atomic sentence that $\forall v \in \sigma\ v(p) = v(q) = 1$ $\star$. Suppose for contradiction that $\sigma \not\models_{BIS} p \rightarrow q$. Then, it follows by the definition of '$\rightarrow$' that $\exists \tau \subseteq \sigma$ s.t. $\tau \models_{BIS} p$ and $\tau \not\models_{BIS} q$. Hence, $\exists v \in \tau$ s.t. $v(p) = 1$ and $v(q) = 0$ $\lightning$. This is a contradiction to $\star$. Hence, $p \wedge q \models_{BIS} p \rightarrow q$, as claimed.

8. $\not\models_{BIS} (p \rightarrow q) \vee (q \rightarrow p)$
Proof
Let $\sigma = \{w_1, w_2\}$, $|p| = \{w_1\}$, $|q| = \{w_2\}$. Then, it follows by the definition of '$\rightarrow$' that $\sigma \not\models_{BIS} p \rightarrow q$ and that $\sigma \not\models_{BIS} q \rightarrow p$. Thus, it follows by the definition of '$\vee$' that $\sigma \not\models_{BIS} (p \rightarrow q) \vee (q \rightarrow p)$. Thus, $\not\models (p \rightarrow q) \vee (q \rightarrow p)$, as claimed.

9. $\neg p \models_{BIS} p \rightarrow \neg p$
Proof by contradiction
Let σ be arbitrary and suppose $\sigma \models_{BIS} \neg p$. Then it follows by the definition of the support clause for '$\neg$' that $\sigma \models_{BIS} p \rightarrow \bot$ $\star$. Suppose for contradiction that $\sigma \not\models_{BIS} p \rightarrow \neg p$. Then, it follows by the definition of the support clause for '$\rightarrow$' that $\exists \tau \subseteq \sigma$ s.t. $\tau \models_{BIS} p$ and $\tau \not\models_{BIS} \neg p$. Hence, it follows by the definition of the support clause for '$\neg$' and the support clause for atomic sentences that $\exists v \in \tau$ s.t. $v(p) = 1$. $\lightning$ This is a contradiction to $\star$. Thus, $\neg p \models_{BIS} p \rightarrow \neg p$, as claimed.

10. $p \rightarrow q, q \rightarrow s \models_{BIS} p \rightarrow s$
Proof by contradiction

Let σ be arbitrary and suppose that $\sigma \models_{BIS} p \to q$ and $\sigma \models_{BIS} q \to s$. Then, it follows that $\forall\tau \subseteq \sigma$, if $\tau \models_{BIS} p$, then $\tau \models_{BIS} q$ † and if $\tau \models_{BIS} q$, then $\tau \models_{BIS} s$ ‡. Suppose for contradiction that $\sigma \not\models_{BIS} q \to s$. Then, it follows that $\exists\tau' \subseteq \sigma$ s.t. $\tau' \models_{BIS} p$ and $\tau' \not\models_{BIS} s$ ↯. This is a contradiction, since by † $\tau' \models_{BIS} q$, and hence by ‡ $\tau' \models_{BIS} s$. Hence, $p \to q, q \to s \models_{BIS} p \to s$, as claimed.

11. $\models_{BIS} p \to (q \to q)$
Proof by contradiction
Suppose for contradiction that $\sigma \not\models_{BIS} p \to (q \to q)$. Then it follows by the definition of '$\to$' that $\exists\tau \subseteq \sigma$ s.t. $\tau \models_{BIS} p$ and $\tau \not\models_{BIS} q \to q$. By the definition of '$\to'$ this is the case iff $\tau \models_{BIS} q$ and $\tau \not\models_{BIS} q$ ↯. Thus, $\models_{BIS} p \to (q \to q)$, as claimed.

12. $(p \wedge q) \to s \not\models_{BIS} (p \to s) \vee (s \to q)$
Proof
Let $\sigma = \{w_1, w_2\}$, $|p| = \{w_1\}$, $|s| = \{w_2\}$ and $|q| = \emptyset$. Then, it follows by the definition of the support clause for '$\to$' and the definition of the support clause for '$\wedge$' that $\sigma \models_{BIS} (p \wedge q) \to s$ vacuously. Notice, however, that since $\{w_1\} \models_{BIS} p$ and $\{w_1\} \not\models_{BIS} s$, it follows by the definition of the support clause for '$\to$' that $\sigma \not\models_{BIS} p \to s$. Similarly, since $\{w_2\} \models_{BIS} s$ and $\{w_2\} \not\models_{BIS} q$, it follows that $\sigma \not\models_{BIS} s \to q$. Hence, it follows by the definition of the support clause for '$\vee$' that $\sigma \not\models_{BIS} (p \to q) \vee (s \to t)$. Hence, $(p \wedge q) \to s \not\models_{BIS} p(p \to s) \vee (s \to q)$ as claimed.

13. $(p \to q) \wedge (s \to t) \not\models_{BIS} (p \to t) \vee (s \to q)$
Proof
Let $\sigma = \{w_1, w_2\}$, $|p| = \{w_1\}$, $|q| = \{w_1\}$, $|s| = \{w_2\}$, $|t| = \{w_2\}$. Hence, it follows by the definition of the support clause for atomic sentences that $\{w_1\} \models_{BIS} p$, $\{w_1\} \models_{BIS} q$, $\{w_1\} \not\models_{BIS} s$, $\{w_1\} \not\models_{BIS} t$, $\{w_2\} \not\models_{BIS} p$, $\{w_2\} \not\models_{BIS} q$, $\{w_2\} \models_{BIS} s$, $\{w_2\} \models_{BIS} t$. Then it follows by the definition of the support clause for '$\to$' that $\sigma \models_{BIS} p \to q$ and $\sigma \models_{BIS} s \to t$. Thus, it follows by the definition of the support clause for '$\wedge$' that $\sigma \models_{BIS} (p \to q) \wedge (s \to t)$. Notice, however, that since $\{w_1\} \models_{BIS} p$ and $\{w_1\} \not\models_{BIS} t$, it follows that $\sigma \not\models_{BIS} s \to q$. Similarly, since $\{w_2\} \models_{BIS} s$ and $\{w_2\} \not\models_{BIS} q$, it follows that $\sigma \not\models_{BIS} s \to q$. Thus, it follows by the definition of the support clause for '$\vee$' that $\sigma \not\models_{BIS} (p \to t) \vee (s \to q)$. Thus, it follows that $(p \to q) \wedge (s \to t) \not\models_{BIS} (p \to t) \vee (s \to q)$, as claimed.

14. $\neg(p \to q) \models_{BIS} p$
Proof by contradiction
Let σ be arbitrary and suppose that $\sigma \models_{BIS} \neg(p \to q)$. Then, it follows by the definition of '$\neg$' that $\sigma \models_{BIS} (p \to q) \to \perp$. Thus, it follows that the only set that supports $p \to q$ is the empty set $\star$. Suppose for contradiction that $\sigma \not\models p$. Then, it follows that $\exists v \in \sigma$ s.t. $v(p) = 0$. Take $\tau = \{v\}$. It follows by the definition of the support clause for '$\to$' that $\{v\} \models_{BIS} p \to q$ and $\{v\} \neq \emptyset$ $\lightning$. This contradicts $\star$. Hence, it follows that $\neg(p \to q) \models_{BIS} p$, as claimed.

15. $\neg(p \to q) \models_{BIS} \neg q$
Proof by contradiction
Let σ be arbitrary and suppose $\sigma \models_{BIS} \neg(p \to q)$. Then it follows by the definition of the support clause for '$\neg$', that $\forall \tau \subseteq \sigma$ if $\tau \models_{BIS} p \to q$, then $\tau \models_{BIS} \perp$. Hence, by the definition of the support clause for '$\to$', $\forall \tau \subseteq \sigma$. ($\forall \tau' \subseteq \tau$ if (if $\tau' \models_{BIS} p$, then $\tau' \models_{BIS} q$), then $\tau \models_{BIS} \perp$) $\star$. Suppose for contradiction that $\sigma \not\models_{BIS} \neg q$. Then it follows by the definition of the support clause for '$\neg$' that $\exists \tau \subseteq \sigma$ s.t. $\tau \models_{BIS} q$ and $\tau \not\models_{BIS} \perp$. Hence, it follows by the definition of '$\perp$' that $\exists v \in \tau$ s.t. $v(q) = 1$. Now notice that since $\{v\} \subseteq \sigma$, this implies that $\{v\} \models_{BIS} p$ and $\{v\} \models_{BIS} q$ but $\{v\} \not\models_{BIS} \perp$ $\lightning$. This is a contradiction to $\star$. Hence, $\neg(p \to q) \models_{BIS} \neg q$, as claimed.

16. $p \to q \models_{BIS} \neg q \to \neg p$
Proof by contradiction
Suppose $\sigma \models_{BIS} p \to q$. Then, it follows by the definition of the support clause for '$\to$' that $\forall \tau \subseteq \sigma$ if $\tau \models_{BIS} p$, then $\tau \models_{BIS} q$ $\star$. Suppose for contradiction that $\sigma \not\models_{BIS} \neg q \to \neg p$. Then it follows by the definition of the support clause for '$\to$' that $\exists \tau \subseteq \sigma$ s.t. $\tau \models_{BIS} \neg q$ and $\tau \not\models_{BIS} \neg p$. Hence, by the definition of the support clause for '$\neg$' and the definition of the support clause for '$\to$' that $\forall \tau' \subseteq \tau$ if $\tau' \models_{BIS} q$, then $\tau' \models_{BIS} \perp$ and $\exists \tau' \subseteq \tau$ s.t. $\tau' \models_{BIS} p$ and $\tau' \not\models_{BIS} \perp$. Hence, it follows by the definition of the support clause for atomic sentences that $\exists v \in \tau$ s.t. $v(p) = 1$ and $v(q) = 0$ $\lightning$. This is a contradiction to $\star$. Thus, $p \to q \models_{BIS} \neg q \to \neg p$, as claimed.

Thus, Basic Inquisitive Semantics allows one to account for 4 out of 16 problematic inferences in question. Namely, the inferences (5), (8), (12) and (13). Notice, that all of the problematic inferences that fail in Inquisitive Semantics involve inquisitive disjunction. More specifically, as indicated by counter-models in (5), (8), (12) and (13), it is mainly the inquisitive aspects of disjunction that allow us to reject implausible implications. This demonstrates that inquisitive nature of disjunction can play a role in accounting for some of the paradoxical inferences. Furthermore, it points towards the fact that inquisitive nature of disjunction can also be motivated by the behavior of natural language implication.

3.2 Support in RIS and Paradoxical Inferences

The previous subsection demonstrated that BIS allows us to account for four out of sixteen inferences we were considering. In this section we will consider the support entailment in Radical Inquisitive Semantics. Notice that, while considering the support notion, the only clause that leads to different semantic results between BIS and RIS is the clause for inversion, i.e., the RIS equivalent of negation. Hence, the only inferences that can be evaluated differently in terms of support in RIS involves '$\div$'. In this section we will evaluate these paradoxical inferences. As will be clear, the switch to inquisitive understanding of negation allows us to account for some of the problems in question.

2. $\div p \models p \rightarrow q$
   Proof
   Since for an atomic sentences p, $\div p \equiv \neg p$, it follows that the proof follows closely the corresponding proof in the previous section.

4. $\models_+ (p \wedge \div p) \rightarrow q$
   Proof by contradiction
   Let σ be arbitrary and suppose $\sigma \not\models_+ (p \wedge \div p) \rightarrow q$. Then, it follows by the definition of the support clause for '$\rightarrow$' that $\exists \tau \subseteq \sigma$ s.t. $\tau \models_+ p \wedge \div p$ and $\tau \not\models_+ q$ $\star$. Thus it follows by the definition of the support clause for '$\wedge$' that $\tau \models_+ p$ † and $\tau \models_+ \div p$. Thus, by the definition of the support clause for '$\div$', it follows that $\tau \models_- p$ ‡. Now notice that by $\star$ and the support definition for atomic sentences, it follows that $\exists v \in \tau$ s.t. $v(q) = 0$. Furthermore by † and ‡ it follows that $v(p) = 1$ and $v(p) = 0$ $\lightning$. Thus , it follows that $\models_+ (p \wedge \div p) \rightarrow q$, as claimed.

5. $\not\models_+ p \rightarrow (q \vee \div q)$
   Proof
   Let $\sigma = \{w_1, w_2\}$, $|p| = \{w_1, w_2\}$ and $|q| = \{w_2\}$. Then, it follows by the support definition for atomic sentences that $\sigma \models_+ p$. Notice,

however, that since $|q| = \{w_2\}$, it follows by definitions for the support and reject clauses for atomic sentences that $\sigma \not\models_+ q$ and $\sigma \not\models_- q$. Hence, by the definition of the support clause for '$\div$', $\sigma \not\models_+ \div\theta$. Thus, it follows that $\sigma \not\models_+ q \vee \div q$. Hence, since $\sigma \models_+ p$, it follows by the definition of the support clause for '$\rightarrow$' that $\sigma \not\models_+ p \rightarrow (q \vee \div q)$. Thus, $\not\models_+ p \rightarrow (q \vee \div q)$ as claimed.

9. $\div p \models_+ p \rightarrow \div p$
Proof by contradiction
Suppose $\sigma \models_+ \div p$. Then it follows by the definition of the support clause for '$\div$' that $\sigma \models_- p$. Thus, by the definition of the reject clause for atomic sentences, it follows that $\forall v \in \sigma\ v(p) = 0$. Suppose for contradiction that $\sigma \not\models_+ p \rightarrow \div p$. Then, it follows by the definition of '$\rightarrow$' that $\exists \tau \subseteq \sigma$ s.t. $\tau \models_+ p$ and $\tau \not\models_+ \div p$. Hence, $\exists v \in \tau$ s.t. $v(p) = 1$ $\lightning$. Thus, $\div p \models_+ p \rightarrow \div p$, as claimed.

14. $\div(p \rightarrow q) \not\models_+ p$
Proof
Let $\sigma = \{w_1\}$, $|p| = \emptyset$. Then it follows by the definition of the support clause for '$\rightarrow$' that $\sigma \models_- p \rightarrow q$ vacuously. Hence, by the definition of the support clause for '$\div$' $\sigma \models_+ \div(p \rightarrow q)$. Notice, however, that since $w_1(p) = 0$, it follows by the definition of the support clause for atomic propositions that $\sigma \not\models_+ p$. Thus, $\div(p \rightarrow q) \not\models_+ p$, as required.

15. $\div(p \rightarrow q) \not\models_+ \div q$
Proof
Let $\sigma = \{w_1\}$, $|p| = \emptyset$ and $|q| = \{w_1\}$. Then, since $|p| = \emptyset$ it follows that $\sigma \models_- p \rightarrow q$ holds vacuously. Hence, by the definition of the support clause for '$\div$' it follows that $\sigma \models_+ \div(p \rightarrow q)$. Notice, however since $w_1(q) = 1$ it follows by the definition of the reject clause for atomic sentences that $\sigma \not\models_- q$. Thus, by the definition of the support clause for '$\div$', it follows that $\sigma \not\models_+ \div q$. Hence, $\div(p \rightarrow q) \not\models_+ \div q$, as claimed.

16. $p \rightarrow q \models_+ \div q \rightarrow \div p$
Proof by contradiction
Suppose $\sigma \models_+ p \rightarrow q$. Then, it follows by the definition of the support clause for '$\rightarrow$' that $\forall \tau \subseteq \sigma.(\tau \models_+ p \Rightarrow \tau \models_+ q)$ $\star$. Suppose for contradiction that $\sigma \not\models_+ \div q \rightarrow \div p$. Then it follows by the definition of the support clause for '$\rightarrow$' that $\exists \tau \subseteq \sigma.(\tau \models_+ \div q$ and $\tau \not\models_+ \div p)$. Hence, by the definition of the support clause for '$\div$' $\exists \tau \subseteq \sigma.(\tau \models_- q$ and $\tau \not\models_- p)$. Thus, by the defini-

tion of the reject clause for atomic sentences, it follows that $\exists v \in \sigma$ s.t. $v(p) = 1$ and $v(q) = 0$. Take $\tau = \{v\}$. It follows by the definition of the support and reject clauses for atomic sentences that $\tau \models_+ p$ and $\tau \not\models_- q$. Thus, by the definition of the support clause for '$\rightarrow$' $\tau \not\models_+ p \rightarrow q$ $\lightning$. This is a contradiction to $\star$. Hence, $p \rightarrow q \models_+ \div q \rightarrow \div p$, as required.

In comparison to BIS, the positive entailment in RIS allows us to account for two additional paradoxical inferences. Namely, inferences (14) and (15). Notice that both of these inferences involve negation of the implication. This demonstrates that the radical inquisitive account of an implication, in which one can reject an implication by either rejecting the proposal made by it, or by rejecting the supposition behind it, allows us to further account for some of the paradoxical inferences. For notice that in inferences (14) and (15), the fact that we can reject an implication by rejecting the antecedent allows us to avoid the paradoxical commitment. This can be seen as demonstrating that a richer understanding of the ways in which we can reject a conditional statement allows us to model the behavior of natural language implication better.

3.3 Reject in RIS and Paradoxical Inferences

One of the remaining features that can be specified by means of Radical Inquisitive Semantics are the negative responses to a sentence. Negative responses to a sentence give rise to a new entailment relation. As will be demonstrated below, this entailment relation is very successful in accounting for problematic inferences as it allows us to account for all but three problematic inferences. Notice, that a sentence θ is a negative response to any sentence if and only if every negative response to θ is also a negative response to an arbitrary sentence. This is the case if and only if the only state which rejects θ is the absurd state.

Definition 45 *For any sentence θ, $\models_- \theta$ if and only if $[\theta]_- = \{\emptyset\}$, i.e., $\forall\sigma$: if $\sigma \models_- \theta$, then $\sigma = \emptyset$.*

1. $p \not\models_- q \rightarrow p$
   Proof
   Let $\sigma = \{w_1\}$, $|p| = \{w_1\}$ and $|q| = \emptyset$. Then, since $|q| = \emptyset$, it follows by the definition of the reject clause for '$\rightarrow$', that $\sigma \models_- q \rightarrow p$ holds vacuously. Furthermore, since $w_1(p) = 1$, it follows by the definition of the reject clause for atomic sentences that $\sigma \not\models_- p$. Hence, it follows that $p \not\models_- q \rightarrow p$, as claimed.

2. $\div p \not\models_- p \to q$
Proof
Let $\sigma = \{w_1\}$, $|q| = \{w_1\}$ and $|p| = \emptyset$. Then since $|p| = \emptyset$, it follows by the definition of the reject clause for '$\to$' that $\sigma \models_- p \to q$ holds vacuously. Furthermore, since $w_1(p) = 0$, it follows by the definition of the reject clause for atomic sentences that $\sigma \not\models_+ p$; and hence by the definition of the reject clause for '$\div$' that $\sigma \not\models_- \div p$. Thus, it follows that $\div p \not\models_- p \to q$, as claimed.

3. $p \to s \not\models_- (p \wedge q) \to s$
Proof
Let $\sigma = \{w_1\}$, $|p| = \{w_1\}$, $|q| = \emptyset$, $|s| = \{w_1\}$. Then, it follows by the definition of the support clause for '$\wedge$' that $\sigma \not\models_+ p \wedge q$ and hence it follows by the definition of the reject clause for '$\to$' that $\sigma \models_- (p \wedge q) \to s$ holds vacuously. However, since $\{w_1\} \models_+ p$ and $\{w_1\} \not\models_- s$, it follows by the definition of the reject clause for '$\to$' that $\sigma \not\models_- p \to s$. Thus, $p \to s \not\models_- (p \wedge q) \to s$, as claimed.

4. $\not\models_- (p \wedge \div p) \to q$
Proof by contradiction
Let σ be arbitrary and suppose for contradiction that $\sigma \not\models_- (p \wedge \div p) \to q$. Then, it follows by the definition of the reject clause for '$\to$' that $\exists \tau.(\tau \models_+ p \wedge \div p$ and $\forall \tau' \supseteq \tau.(\tau' \models_+ p \wedge \div p$ and $\tau' \cap \sigma \not\models_- q))$. Hence, by the definition of the support clause for '$\wedge$', this implies that $\exists \tau.(\tau \models_+ p \wedge \div p$ and $\forall \tau' \supseteq \tau.(\tau' \models_+ p$ and $\tau' \models_+ \div p$ and $\tau' \cap \sigma \not\models_- q))$. Now by the definition of reject for atomic sentences and since $\tau' \cap \sigma \not\models_- q$, it follows that $\exists v \in \sigma$ s.t. $v(q) = 1$. Similarly by the definition of the support clause for atomic sentences and since $\tau' \models_+ p$, it follows that $v(p) = 1$. Furthermore, by the definition of the support clause for '$\div$', it follows that $\tau' \models_- p$ and hence $v(p) = 0$ ↯. Thus, every state σ is s.t. $\sigma \models_- p \wedge \div p \to q$. Hence, $\not\models_- p \wedge \div p \to q$, as required.

5. $\not\models_- p \to (q \vee \div q)$
Proof
Let $\sigma = \{w_1\}$, $|p| = \emptyset$, $|q| = \emptyset$. Then, it follows that $\sigma \models_- p \to (q \vee \div q)$ vacuously. Hence, by the definition of the negative validity it follows that $\not\models_- p \to (q \vee \div q)$.

6. $\not\models_- p \to (q \to p)$
Proof
Let $\sigma = \{w_1\}$, $|p| = \emptyset$. Then, it follows that $\sigma \models_- p \to (q \to q)$

vacuously. Hence, by the definition of the negative entailment it follows that $\not\models_{-} p \to (q \to q)$.

7. $p \wedge q \not\models_{-} p \to q$
Proof
Let $\sigma = \{w_1, w_2\}$, $|p| = \{w_1\}$, $|q| = \{w_2\}$. Then, it follows by the definition of the reject clause for '$\to$' that $\sigma \models_{-} p \to q$. However, since $w_1(p) = 1$ and $w_2(q) = 1$, it follows by the definition of the reject clause for atomic sentences that $\sigma \not\models_{-} p$ and $\sigma \not\models_{-} q$. Thus, by the definition of the reject clause for '$\wedge$' it follows that $\sigma \not\models_{-} p \wedge q$. Thus, $p \wedge q \not\models_{-} p \to q$, as claimed.

8. $\not\models (p \to q) \vee (q \to p)$
Proof
Let $\sigma = \{w_1\}$, $|p| = \emptyset$, $|q| = \emptyset$. Then, it follows by the definition of the reject clause for '$\to$' that $\sigma \models_{-} p \to q$ vacuously and that $\sigma \models_{-} q \to p$ vacuously. Hence, by the definition of '$\vee$', it follows that $\sigma \models_{-} (p \to q) \vee (q \to p)$. Hence, since $\sigma \neq \emptyset$, it follows that $\not\models (p \to q) \vee (q \to p)$, as claimed.

9. $\div p \not\models_{-} (p \to \div p)$
Proof
Let $\sigma = \{w_1\}$, $|p| = \emptyset$. Since $|p| = \emptyset$, it follows by the definition of the reject clause for '$\to$' that $\sigma \models_{-} p \to \div p$ holds vacuously. However, since $w_1(p) = 0$, it follows by the definition of the reject clauses for atomic sentences that $\sigma \not\models_{+} p$ and hence by the definition of the reject clause for '$\div$' that $\sigma \not\models_{-} \div p$. Hence, it follows that $\div p \not\models_{-} p \to \div p$, as claimed.

10. $p \to q,\ q \to s \not\models_{-} p \to s$
Proof
Let $\sigma = \{w_1, w_2\}$, $|p| = \{w_1\}$, $|q| = \{w_1, w_2\}$, $|s| = \{w_2\}$. Then since $|p| = \{w_1\}$ and $|s| = \{w_2\}$, it follows by the definition of reject clause for '$\to$' that $\sigma \models_{-} p \to s$. Notice, however, that since $|q| = \{w_1, w_2\}$ and $|s| = \{w_1\}$, it follows by the reject clause for '$\to$' that $\sigma \not\models_{-} q \to s$. Similarly, since $|p| = \{w_1\}$ and $|q| = \{w_1, w_2\}$, it follows by the definition of the reject clause for '$\to$' that $\sigma \not\models_{-} p \to q$. Thus, $p \to q,\ q \to s \not\models_{-} p \to s$, as claimed.

11. $\not\models_{-} p \to (q \to q)$
Proof
Let $\sigma = \{w_1\}$, $|p| = \emptyset$. Then, it follows that $\sigma \models_{-} p \to (q \to q)$

vacuously. Hence, by the definition of the negative entailment, it follows that $\not\models_- p \to (q \to q)$, as required.

12. $(p \wedge q) \to s \models_- (p \to s) \vee (q \to s)$
Proof by contraposition.
Let σ be arbitrary and suppose that $\sigma \not\models_- (p \wedge q) \to s$. Then, by the definition of the reject clause for '$\to$' and since $\emptyset \models_+ p \wedge q$, it follows that $\exists \tau' \supseteq \emptyset$ s.t. $\tau' \models_+ p \wedge q$ and $\tau' \cap \sigma \not\models_- s$. Hence, it follows by the definition of the reject clause for atomic sentences and the definition of the support clause for '$\wedge$' that $\exists v \in \sigma$ s.t. $v(p) = v(q) = v(s) = 1$. Now notice that it follows by the definition of the reject clause for '$\to$' that $\sigma \not\models_- p \to s$ and $\sigma \not\models_- q \to s$. Thus, it follows by the definition of the reject clause for '$\vee$' that $\sigma \not\models_- (p \to s) \vee (q \to s)$. Hence, $(p \wedge q) \to s \models_- (p \to s) \vee (q \to s)$, as claimed.

13. $(p \to q) \wedge (s \to t) \not\models_- (p \to t) \vee (s \to q)$
Proof
Let $\sigma = \{w_1, w_2\}$, $|p| = \{w_1\}$, $|t| = \{w_2\}$, $|s| = \{w_2\}$, $|q| = \{w_1\}$. Then, since $|p| = \{w_1\}$ and $w_1(t) = 0$, it follows by the definition of the reject clause for '$\to$' that $\sigma \models_- p \to t$. Similarly since $|s| = \{w_2\}$ and $w_2(q) = 0$, it follows by the definition of the reject clause for '$\to$' that $\sigma \models_- s \to q$. Hence, by the definition of the reject clause for '$\vee$', it follows that $\sigma \models_- (p \to t) \vee (s \to q)$. Notice, however, that since $\{w_1\} \models_+ p$ and $\{w_1\} \not\models_- q$, it follows by the definition of the reject clause for '$\to$' that $\sigma \not\models_- p \to q$. Similarly, since $\{w_2\} \models_+ s$ and $\{w_2\} \not\models_- t$, it follows by the definition of the reject clause for '$\to$' that $\sigma \not\models_- s \to q$. Hence, by the definition of the reject clause for '$\wedge$', $\sigma \not\models_- (p \to q) \wedge (s \to t)$. Thus, it follows that $(p \to q) \wedge (s \to t) \not\models_- (p \to t) \vee (s \to q)$, as claimed.

14. $\div(p \to q) \models_- p$
Proof by contradiction
Let σ be arbitrary and suppose that $\sigma \models_- p$. Then it follows that $\forall v \in \sigma\ v(p) = 0\ \star$. Suppose for contradiction that $\sigma \not\models_- \div(p \to q)$. Then it follows by the definition of the reject clause for '$\div$' that $\sigma \not\models_+ p \to q$. Hence, by the definition of the support clause for '$\to$' it follows that $\exists \tau \subseteq \sigma$ s.t. $\tau \models_+ p$ and $\tau \not\models_+ q$. Hence, $\exists v \in \tau \subseteq \sigma$ s.t. $v(q) = 0$ and $v(p) = 1\ \lightning$. This is a contradiction to $\star$. Hence, it follows that $\div(p \to q) \models_- p$, as required.

15. $\div(p \to q) \models_- \div q$
Proof by contradiction
Let σ be arbitrary and suppose that $\sigma \models_- \div q$. Then it follows by the definition of the reject clause for '$\div$' that $\sigma \models_+ q$ and hence $\forall v \in \sigma$ $v(q) = 1\ \star$. Suppose for contradiction that $\sigma \not\models_- \div(p \to q)$. Then it follows by the definition of the reject clause for '$\div$' that $\sigma \not\models_+ p \to q$. Hence, by the definition of the support clause for '$\to$' it follows that $\exists v \in \sigma$ s.t. $v(p) = 1$ and $v(q) = 0\ \lightning$. This is a contradiction to $\star$. Thus, it follows that $\div(p \to q) \models_- \div q$, as claimed.

16. $p \to q \not\models_- \div q \to \div p$
Proof by contradiction
Let $\sigma = \{w_1, w_2\}$, $|q| = \{w_1, w_2\}$, $|p| = \{w_2\}$. Then, it follows that for all subsets of σ only $\emptyset$ is s.t. $\emptyset \models_- q$, i.e., only $\emptyset$ is s.t. $\emptyset \models_+ \div q$. Thus, it follows that $\div q \to \div p$ holds vacuously. Notice, however that since $|p| = \{w_2\}$ and $w_2(q) = 1$, it follows by the definition of the reject clause for '$\to$' that $\sigma \not\models_- p \to q$. Hence, it follows that $p \to q \not\models_- \div q \to \div p$, as claimed.

Thus, it follows that the negative radical entailment allows us to account for all but 3 problematic implications. Namely, only (12), (14) and (15) hold in it. Inferences (14) and (15) result because of the radical modeling of the negation of an implication. Namely, the fact that rejected antecedent always corresponds to a rejection of the supposition behind a conditional statement.

3.4 BIS and RIS: Results

In previous subsections we have analyzed whether the inferences in question hold in Basic and Radical Inquisitive Semantics. In this subsection, similarly as towards the end of Chapter 2, we summarize the results in a table. Notice that the inquisitive enrichment in BIS allows us to account for all paradoxical inferences that involve disjunction. Hence, our initial findings seem to suggest that the inquisitive treatment of disjunction contributes to a better account of natural language implication. Furthermore, when we consider RIS support entailment, the RIS richer account of responses to a sentence allows us to account for two additional inferences. Both of the accounted cases involve a negation of an implication. The reject entailment allows us to effectively account for majority of the problematic inferences. Only three inferences hold when we consider negative responses to a sentence. Last but not least, none of the problematic inferences holds when we consider RIS. On the one hand, this suggests that full fledged RIS entailment is significantly weaker than both RIS support entailment and RIS reject entailment. On the other hand, this demonstrates that RIS validity successfully accounts for all of the problematic inferences in question.

	BIS	RIS^+	RIS_-	RIS
(1) $p \models q \to p$	✓	✓	×	×
(2) $\neg q \models q \to p$	✓	✓	×	×
(3) $p \to s \models (p \wedge q) \to s$	✓	✓	×	×
(4) $\models (p \wedge \neg p) \to q$	✓	✓	×	×
(5) $\models p \to (q \vee \neg q)$	×	×	×	×
(6) $\models p \to (q \to p)$	✓	✓	×	×
(7) $p \wedge q \models p \to q$	✓	✓	×	×
(8) $\models (p \to q) \vee (q \to p)$	×	×	×	×
(9) $\neg p \models (p \to \neg p)$	✓	✓	×	×
(10) $p \to q, q \to s \models p \to s$	✓	✓	×	×
(11) $\models p \to (q \to q)$	✓	✓	×	×
(12) $(p \wedge q) \to s \models (p \to s) \vee (q \to s)$	×	×	✓	×
(13) $(p \to q) \wedge (s \to t) \models (p \to t) \vee (s \to q)$	×	×	×	×
(14) $\neg(p \to q) \models p$	✓	×	✓	×
(15) $\neg(p \to q) \models \neg q$	✓	×	✓	×
(16) $p \to q \models \neg q \to \neg p$	✓	✓	×	×

CHAPTER 4

Analysis

In this section we will analyse the inquisitive approach to the paradoxical material implications and compare it to the accounts given by the non-classical systems considered in Chapter 2. We will first restate results from the previous sections and then highlight the main semantic features that contributed to the success of different systems in accounting for particular implications. We will also consider possible criticisms of the analysis provided. Namely, we will consider two lines of argumentation: one which questions the implausibility of some of the sixteen inferences considered; and the other concerning the logical strength of Radical Inquisitive Semantics. More specifically, we will reject the criticism against our counterexamples to inferences (7), (10) and (16). Furthermore, we will discuss whether Modus Ponens and Modus Tollens hold in Radical Inquisitive Semantics.

4.1 Summary of the Results

The results from the previous sections can be summarized in the following table:

	S2	C2	US	B	BIS	RIS
(1) $p \models q \to p$	×	×	✓	×	✓	×
(2) $\neg q \models q \to p$	×	×	✓	×	✓	×
(3) $p \to s \models (p \wedge q) \to s$	✓	×	✓	✓	×	×
(4) $\models (p \wedge \neg p) \to q$	✓	✓	✓	×	✓	×
(5) $\models p \to (q \vee \neg q)$	✓	✓	✓	×	×	×
(6) $\models p \to (q \to p)$	×	×	✓	×	✓	×
(7) $p \wedge q \models p \to q$	×	✓	✓	×	✓	×
(8) $\models (p \to q) \vee (q \to p)$	×	×	×	×	×	×
(9) $\neg p \models (p \to \neg p)$	×	✓	✓	×	✓	×
(10) $p \to q, q \to s \models p \to s$	✓	×	✓	✓	✓	×
(11) $\models p \to (q \to q)$	×	✓	✓	×	✓	×
(12) $(p \wedge q) \to s \models (p \to s) \vee (q \to s)$	×	×	×	×	×	×
(13) $(p \to q) \wedge (s \to t) \models (p \to t) \vee (s \to q)$	×	×	×	×	×	×
(14) $\neg(p \to q) \models p$	×	×	×	×	✓	×
(15) $\neg(p \to q) \models \neg q$	×	×	×	×	✓	×
(16) $p \to q \models \neg q \to \neg p$	✓	×	✓	✓	✓	×

The above list includes the following paradoxes and properties of implication:

1. Paradoxes of Material Implication: (1), (2), (3), (6), (8), (9), (11), (12), (13), (14), (15)
2. Paradoxes of Strict Implication: (4), (5)
3. Centering: (7)
4. Transitivity: (10)
5. Contraposition: (16)

Moreover, inference (3) is also commonly referred to as antecedent strengthening and inference (6) as weakening. Last but not least, Paradoxes of Material Implication (12)-(15) are also referred to as Priest objections to material implication.[1]

As each of the logics considered allows one to account for some of the paradoxical inferences, it follows that all of the logics allow us to give a less

[1]Note that as the inferences we are discussing involve only atomic sentences, and not arbitrary formulas, we fix antecedent strengthening, weakening, centering, transitivity and contraposition as a way of referring to particular examples and not to general properties of the conditional. However, as the atomic cases constitute necessary conditions for these properties to hold, it follows that whenever the examples considered by us fail within any of the logics, the implication in these logics lacks the property considered.

counterintuitive account of the 16 paradoxical inferences than classical logic. Out of the logics discussed, RIS is the most successful. It allows us to account for all of the paradoxical inferences that can be formulated in its language.

Among others, Lewis' modal logic S2 allows us to effectively account for 11 out of 16 inferences considered. As the underlying notion of implication in S2 is a strict implication, not surprisingly, all of the Paradoxes of Strict Implication hold within this logic. As pointed out in Chapter 2 strict implication does not account for antecedent strengthening, transitivity and contraposition.

Stalnaker's Conditional Logic C2 is also very successful in accounting for the paradoxical inferences in question. Only 5 out of the 16 inferences considered hold within this logic. Notice also, that our findings demonstrate that conditional implication does not account for centering, but does go against the antecedent strengthening, transitivity and contraposition.

Except for Basic Inquisitive Semantics, Update Semantics allows us to account for the least number of the paradoxical inferences in question. Out of the inferences considered only five do not hold in this logic. Moreover, it follows that the Update Semantics' implication does not account for the antecedent strengthening, centering, weakening, transitivity and contraposition.

Basic Inquisitive Semantics allows us to account for only four out of the sixteen inferences. As in the case of Update Semantics, BIS implication also does not account for the antecedent strengthening, centering, weakening, transitivity and contraposition. In comparison to BIS, the refinement of the possible responses one can give to a sentence according to RIS^+, allows us to account for two additional paradoxes. Thus, positive entailment RIS^+ invalidates six out of the sixteen inferences considered. Only three inferences are not accounted for according to the RIS_- entailment. However, since all of these inferences are already accounted by the RIS^+ entailment, it follows that RIS allows us to account for all of the paradoxical inferences considered. Last but not least, as all of the three properties considered by us fail in RIS, it follows that the implication in RIS accounts for centering, transitivity and contraposition.

4.2 Approaches

In the previous section we have summarized the results of our analysis. In this section we will pinpoint and discuss the characteristics of different systems that allow us to account for the counterintuitive implications.

4.2.1 Strict Conditional Logic Approach

In order to account for some of the Paradoxes of Material Implication C.I. Lewis suggested strict implication as a solution. As pointed out in Section 2.2, Lewis claimed that material implication is too contingent and noticed that the horse-shoe analysis of indicative conditionals allows for two propositions with unrelated content to imply each other. In order to capture the conditional dependency between the sentences involved in an implication, Lewis suggests that we should treat implication as an intensional operator that is not only defined in terms of the truth of its antecedent and consequent at the actual world, but also at their truth values at other possible worlds. Hence, he suggests $\Box(p \rightarrow q)$ as a correct modeling of natural language indicative conditionals. Such a modeling is meant to guarantee that the conditional dependency between the antecedent and the consequent is a necessary one.

According to such an account only the implications which hold at every admissible alternative to the actual world are exemplifying valid indicative conditionals. As pointed out in [38],"the truth of '$p \rightarrow q$' requires not just the mere falsity of '$p \wedge \neg q$', but its impossibility. (And this impossibility is sufficient for the truth of '$p \rightarrow q$')" (pp. 69). Thus, the key element of Lewis' solution to the Paradoxes of Material Implication is the inclusion of the notion of necessity into the definition of material implication. Such an inclusion guarantees that the problematic implications as exemplified e.g., by (1) $p \models q \rightarrow p$ and (2) $\neg q \models q \rightarrow p$ do not hold.[2] Intuitively, this is because in all of the problematic cases exemplifying (1) and (2), we will be able to think of an alternative in which the antecedent will not imply the consequent and hence the desired implication will not hold.

The existence of suitable frames in which one of the possible worlds accessible from the actual world does not satisfy the implication allows us to explain away all but one of the examples accounted in S2. Hence, the intensionality of '$\rightarrow$' and the existence of suitable possible worlds in counter-models given for (1), (2), (6)-(9), (12)-(15) allows us to avoid the paradoxical commitment. The strict model of implication is not the only element, however, that allows S2 to account for the problematic implications in question. As demonstrated in Chapter 2.2, in example (11) $\models p \rightarrow (q \rightarrow q)$, it is the construction of the non-normal possible world that allows us to avoid this paradoxical commitment. It is also an easy exercise to demonstrate that (11) can hold in other modal logic systems that do not include non-normal possi-

[2]Notice that Lewis treated these two inferences as prime examples of the Paradoxes of Material Implication. Many of the examples of the Paradoxes of Material Implication considered by us are not directly discussed by Lewis.

ble worlds (for instance Kripke's logic K). As pointed out in the introduction, non-normal possible worlds are, however, introduced *ad hoc* and are not as such motivated by philosophical intuitions.

Despite the plausibility of S2, the Paradoxes of Strict Implication constitute a significant problem to S2. This is because they demonstrate that the strict conditional can be true irrespective of the semantic content of its antecedent and consequent. I.e., strict conditionals can be true if the $if-clause$ or the $main-clause$ of the conditional are of certain forms. More specifically, strict implication holds whenever the $if-clause$ is impossible (as exemplified by (4) $\models (p \wedge \neg p) \rightarrow q$). This is implausible as then the conditional holds *no matter* what the main clause is about. On the other hand, when considering the main clause, strict implication holds whenever the main clause is necessarily true (as exemplified by (5) $\models p \rightarrow (q \vee \neg q)$). This is implausible since then the indicative conditional holds no matter what the if-clause is.

4.2.2 Conditional Logic Approach

As in the case for S2, C2 suggests to interpret an implication as an intensional operator, i.e., as an operator that is not only dependent on the truth values of its antecedent and consequent at the actual world, but which also depends on their values at other possible worlds. Stalnaker's interpretation of a conditional sentence does not make it, however, dependent on all possible worlds accessible from the actual world, but instead makes it dependent only upon the most similar accessible world that satisfies its antecedent. Since a possible world can be seen as just an alternative to the actual world, Stalnaker's interpretation of the conditional sentence generates the following interpretation of indicative conditionals: "[When you evaluate a conditional] first, [you] add the antecedent hypothetically to your stock of beliefs; second, make whatever adjustments are required to maintain consistency (without modifying the hypothetical belief in the antecedent); finally, [you] consider whether or not the consequent holds" ([40], pp. 165). Such a view of conditionals captures the notion that in our evaluation of a conditional we also consider silent assumptions. This is because, the most similar accessible world is such that as many as possible of these assumptions hold.

One of the shortcomings of Stalnaker's logic is that all of the Paradoxes of Strict Implication still hold in it. When considering inference (4) $\models (p \wedge \neg p) \rightarrow q$, the reason why it holds can be attributed to the existence of the absurd world λ. For notice that it is the fact that for all sentences θ, $\lambda \models \theta$ that validates the paradoxical inference in question. (7) $p \wedge q \models p \rightarrow q$

holds in C2 because of the requirement (2.3) of the selection function.[3] This requirement captures the intuition underlying Stalnaker's system; i.e., that if the most similar possible world that satisfies the antecedent is the actual world, then the consequent needs to hold at this world as well (since clearly the actual world is at least as similar to itself as any other possible world). The fact that (5) $\models p \to (q \vee \neg q)$ and (11) $\models p \to (q \to q)$ hold in C2 demonstrates that C2 suffers from similar problems as S2. Namely, its modeling of indicative conditionals is such that no matter what the antecedent is, whenever the consequent is necessarily true, the conditional holds. The reason why (5) holds can be attributed to the fact that for any sentence q, $q \vee \neg q$ holds at every possible world. The reason why (11) holds in C2 can be attributed to the fact that the first requirement[4] on the selection function forces $q \to q$ to be necessarily true. Inference (9) $\neg p \models (p \to \neg p)$, on the other hand, highlights the fact that the requirements on the selection function give rise to some undesirable entailments. For notice that it is the first requirement on the selection function that validates (9).

Since all other occurrences are accounted for in C2, it follows that C2 is very successful in explaining away the problematic inferences. In particular, it allows us to account for all but two of the Paradoxes of Material Implication. Furthermore, the modeling of implication as suggested by Stalnaker allows one to account for antecedent strengthening, transitivity and contraposition.

The account given by C2 is not without its problems. One of the criticisms of C2 was given by [32]. This criticism points towards the fact that the modeling of indicative conditionals as suggested by Stalnaker fails to capture that the antecedent should not be irrelevant to the consequent. More specifically, it seems problematic that the joint truth of the antecedent and the consequent of a conditional at some possible world allows us to imply the consequent from the antecedent. Thus, [32] points at the implausibility of concluding that 'If p, then q' holds from the truth of arbitrary p and q, no matter how unrelated they are to each other. Notice, that Paradoxes of Strict Implication may be looked upon as an embodiment of this problem, where the necessary consequent is implied by anything and where the necessarily false antecedent implies any sentence.

[3] Namely the requirement that: if $w \in \alpha$, then $f(\alpha, w) = w$.

[4] I.e. $\forall w \; f(\alpha, w) \in \alpha$.

4.2.3 Update Semantics Approach

In contrast to the previous two systems, Update Semantics is an example of a dynamic system, i.e., its key focus is on information change and update. Within this setting, natural language implication is interpreted as a consistency test on an information state. When interpreting an implication, one checks if after updating one's information state with the antecedent of the conditional, the consequent holds. Hence, an implication is treated as an epistemic operator. As demonstrated in the summary of our results, Update Semantics allows us to account for five out of the 16 problematic inferences. The successful explanation of these inferences can be seen as being attributed to the fact that *might* and *implication* within the framework of Update Semantics are interpreted as consistency tests. More specifically, (12) $(p \wedge q) \to s \models (p \to s) \vee (q \to s)$ uses the fact that '$\to$' is just an epistemic operator that verifies whether the information encoded by a sentence is consistent with our information state. As the state σ in counterexample (12)[5] is such that after an update with $p \wedge q$, it supports s, it follows that it is consistent with $(p \wedge q) \to s$. However, σ is not consistent with '$p \to q$' and '$q \to s$'. Hence, it follows that σ does not support '$p \to q$' or '$q \to s$'.

For the consideration of examples (14) $\neg(p \to q) \models p$ and (15) $\neg(p \to q) \models \neg q$, notice that $\neg(p \to q)$ is equivalent to $\Diamond(p \wedge \neg q)$. Hence, these examples can also be viewed as consistency tests. The failure of these inferences can be attributed to the fact that consistency tests are only a way of verifying whether a sentence holds and not a way modifying the information state. Thus, a consistency test does not necessarily update our information state to a state which supports some other information; so to speak, it just verifies whether the information provided by one sentence is *consistent* with our information state. The conclusions, on the other hand, require that the information provided by it is *accepted* by every state that is *consistent* with the implications in question.

While considering inferences (8) $\models (p \to q) \vee (q \to p)$ and (13) $(p \to q) \wedge (s \to t) \models (p \to t) \vee (s \to q)$, similarly as in the previous example, they demonstrate that the modeling of '$\to$' as a consistency test allows us to avoid some of the implausible material implications.

Last but not least, it follows by our analysis that US implication does not account for antecedent strengthening, centering, transitivity and contraposition.

[5] The counterexample for inference (12) is given by $\sigma = \{w_1, w_2, w_3, w_4\}$, $|p| = \{w_1, w_2\}$, $|q| = \{w_1, w_3\}$, $|s| = \{w_1, w_4\}$.

4.2.4 Relevance Logic B Approach

Relevance Logic aims at capturing the fact that in order for an implication to hold the antecedent needs to be *relevant* to the consequent. In order to achieve this, Relevance Logic reformulates the truth conditions for '$\rightarrow$' by using a ternary accessibility relation. The account of implication thus obtained can be seen as being *engineered* to deal with the problematic inferences in question. Not surprisingly, the notion of relevance developed by these means is very successful in accounting for the problematic inferences. Because of the complex and *ad hoc* nature of the logical apparatus of B, it is, however, very difficult to pinpoint specific characteristics of this logic that contribute to accounting for specific paradoxical inferences. Having said this, the analysis given by relevance logicians demonstrates that inclusion of the notion of relevance by means of a ternary accessibility relation, its non-normal possible worlds and the assumption of Routley Semantics, defines a system that is very successful in accounting for the paradoxical inferences considered. Out of the problematic examples here discussed, only antecedent strengthening, transitivity and contraposition hold in Relevance Logic. The fact that out of the logics considered, Logic B was able to account for almost all problematic inferences, demonstrates that the notion of relevance and the variable sharing principle that is implied by it, are very effective in accounting for the paradoxes considered.

As pointed out in Chapter 2, one of the biggest problems of Relevance Logics is a methodological one. Namely, it seems to be the case that the logic given by B can be viewed as a proof-theoretic system with very limited explanatory power. Because of this reason, as noted before, it is very difficult to capture *the meaning* of the notion of relevance that is implied by it. As pointed out by [42] "It is dubious whether there are any advantages in lumping together these various ways in which arguments can be improper. The relevance logicians run the risk of turning logical validity into a clumsy thing" (pp. 43). The difficulties encountered with interpreting the semantics of Relevance Logic in the process of writing this thesis are a testimony to this opinion.

4.2.5 Inquisitive Semantics

The Inquisitive approach is characterized by enriching the notion of a proposition with its inquisitive content. As demonstrated by our analysis, this enrichment turns out as being crucial in accounting for the Paradoxes of Material Implication. Inquisitive Semantics defines implication via the support notion on states and treats implication as a requirement on a state that guarantees that whenever any of its enhancements supports the antecedent, it also supports the consequent.

Despite appearances, such a definition of implication makes it similar to material implication. For the informative content of an implication corresponds to its classical interpretation. This highlights the fact that it is the inquisitive enrichment that allows one to account for some of the paradoxes. For notice that all of the implausible inferences that are accounted for in BIS involve the main semantic feature that introduces inquisitiveness; i.e., inquisitive disjunction. More specifically (5) $\models p \to (q \vee \neg q)$, (8) $\models (p \to q) \vee (q \to p)$, (12) $p \wedge q \to s \models (p \to s) \vee (q \to s)$ and (13) $(p \to q) \wedge (s \to t) \models (p \to t) \vee (s \to q)$ are the only instances of the paradoxical inferences that involve disjunction and these inferences are also the only instances which are correctly accounted for in BIS. The counter-models we gave for (5), (8), (12) and (13) further demonstrate that it is the inquisitive meaning of a sentence that allows one to give a correct account of the corresponding indicative conditionals. For notice that the existence of suitable enhancements that invalidate the conclusion is due to the definition of inquisitive disjunction. Moreover, if we do not consider the inquisitive content of disjunctions in (5), (8), (12) and (13) and consider only their informative content, these inferences do hold. This further demonstrates, that it is the inquisitive enrichment that allows one to account for these paradoxical inferences. As the inferences that do not involve disjunction are not explained away in BIS, it follows that BIS implication does not account for antecedent strengthening, weakening, centering transitivity and contraposition.

As such BIS is not, however, very successful in accounting for the paradoxical inferences in question. Only four out of the 16 implausible inferences that can be formulated in its language fail in this system. The RIS^+ refinement of the responses that one can give to a sentence and modification of the notion of the rejection of an implication, allow one to account for two additional problematic inferences, namely (14) $\neg(p \to q) \models p$ and (15) $\neg(p \to q) \models \neg q$. These two inferences fail because of the reinterpretation of the definition of the negation of an implication in Radical Inquisitive Semantics. For the rejection of the negation of $p \to q$ in RIS is equivalent to the positive response '$p \to \neg q$' or the rejection of the antecedent '$\neg p$'. In both (14) and (15), such a definition of rejection of an implication allows us to provide suitable counterexamples. This demonstrates that the RIS analysis of an implication is such that the support of '$p \wedge \neg q$' is no longer a necessary condition for the rejection of '$p \to q$'. Hence, RIS does not only provide new sufficiency conditions for conditional sentences but also new necessity conditions. Namely the necessary and sufficient condition for a conditional to be supported at a state is that whenever any of its enhancements supports the antecedent, it also supports the consequent. Whereas the necessity and sufficiency condition for the rejection of the conditional '$\theta \to \psi$' is that either there is no non-trivial enhancement which supports θ or a maximal enhancement which supports θ also rejects ψ. Such a modeling does not only allow us to give a plausible characterization of our natural language uses, but also contributes to a more adequate account of the paradoxical inferences considered.

The negative part of the entailment given by RIS — RIS_- — allows us to account for all but three implausible inferences considered by us. The counter-models for more than half of the inferences that fail to hold in RIS_- — (1) $p \models q \to p$, (2) $\neg q \models q \to p$, (3) $p \to s \models (p \wedge q) \to s$, (5)$\models p \to (q \vee \neg q)$, (6) $\models p \to (q \to p)$, (8) $\models (p \to q) \vee (q \to p)$, (9) $\neg p \models (p \to \neg p)$, (11) $\models p \to (q \to q)$— reject the antecedent of the conditionals involved, i.e., correspond to issue-dispelling responses. This further demonstrates that the characterization of the responses which reject the supposition behind the conditional gives us one way of accounting for the paradoxical inferences. However, out of these inferences the issue-dispelling responses are necessary to account only for the inferences (5), (6) and (11). In example (6) this is because every non-empty enhancement τ that supports p, by the definition of the reject clause for '$\to$' cannot reject $q \to p$ if $\exists v \in \tau$ s.t. $v(q) = 1$. Whereas for inferences (5) and (11) this is because the consequent is rejected only by the absurd state $\emptyset$.[6]

Inferences (9), (10) and (13) further highlight the role that the clause for the rejection of an implication in RIS plays in accounting for the undesirable implications. We will use the counter-model to inference (10) $p \to q, q \to s \models p \to s$[7], to explicate this point. Notice that the premises in (10) do not imply the conclusion, because none of the maximal enhancements of the state σ is such that it supports their antecedents and rejects their consequents. This is the case because maximal enhancements that support their antecedents are constituted by enhancements that do not reject their consequents. For by the definition of the rejection clause for atomic sentences, their consequents would be only rejected, if these enhancements did not contain a possible world such that the consequent was supported by it.

Negative entailment, as such, does not invalidate all of the problematic

[6]For the remaining examples it is relatively easy to prove that the following states constitute suitable counterexamples that do not involve issue-dispelling responses: (1) & (2): $\sigma = \{w_1, w_2\}$, $|p| = \{w_1\}$, $|q| = \{w_2\}$, (3):$\sigma = \{w_1, w_2\}$, $|p| = \{w_1, w_2\}$, $|q| = \{w_1\}$, $|s| = \{w_2\}$, (8): $\sigma = \{w_1, w_2\}$, $|p| = \{w_1\}$, $|q| = \{w_2\}$, (9): $\sigma = \{w_1\}$, $|p| = \{w_1\}$.

[7]The counter-model for inference (10) is given by $\sigma = \{w_1, w_2\}$, $|p| = \{w_1\}$, $|q| = \{w_1, w_2\}$, $|s| = \{w_2\}$.

examples considered. In particular, it is the semantic features of RIS_{-} that lead to (12) $(p \wedge q) \to s \models (p \to s) \vee (q \to s)$, (14) $\neg(p \to q) \models p$ and (15) $\neg(p \to q) \models \neg q$ being valid entailments when we consider only the RIS_{-} part of the entailment in RIS. However, when one considers the negative responses to an implication none of the properties of the conditional hold. That is RIS_{-} allows us to account for antecedent strengthening, weakening, centering, transitivity and contraposition.

As pointed out in the discussion in Section 4.1, the full fledged Radical Inquisitive Semantics is very successful in accounting for the sixteen paradoxical inferences considered. From the systems considered, it is the only system that accounts for all of the problematic inferences. In the light of our enterprise, this definitely gives an argument for the new system.

RIS may not be without its problems, though. It may be the case that this system is too strong and while it allows us to account for many of the paradoxical instances considered, it might fail to validate many of the entailments concerning indicative conditionals that we would like to hold. This criticism will be discussed in detail in section 4.4.1.

4.3 Comparison

In this section we will provide a comparison between the approaches to the 16 paradoxical material implications as given by the logics considered. We will do so by analyzing whether the key elements highlighted in the discussion in the previous section are reflected by the Inquisitive Semantic modeling of implication. More specifically, we will verify whether the philosophical motivations for the approaches towards the 16 implications as described by US, C2, S2 and B can be also found in Inquisitive Semantics. We will argue that especially the approaches in Lewis' S2 and Veltman's Update Semantics can be seen as being reflected in the inquisitive account of implication. We will also suggest that *in principle* it is possible to utilize the claims concerning the approach towards the Paradoxes of Material Implication as given by C2. Moreover, we will discuss the notion of relevance within Inquisitive Semantics and suggest that it is *indirectly* reflected in inquisitive implication. Note that as the systems considered differ in formal frameworks assumed and logical machinery used, we treat the discussion in this section as an *indication* of the similarities rather than a detailed analysis of the approaches given by different systems. Because of this reason, we also request a charitable reading of the analysis below.

4.3.1 Strict Conditional Logic

As demonstrated in the discussion of S2 one of the key claims made by Lewis is that implication is not a contingent notion, i.e., a notion that depends only upon the actual world, but rather it is to be regarded as a necessary relation between two sentences. Interestingly, this observation to a significant extent can also be seen as being present in the inquisitive understanding of an implication. Without loss of generality we can consider BIS to explicate the point made here. For notice that BIS' definition of an implication implies that for an implication to hold at a state, *every* enhancement of this state that supports the antecedent, also supports the consequent. In principle, when one considers the inquisitive view on discourse, updating the common ground with an implication captures then the intuition behind S2. This is because, whenever an update with an implication is accepted, every enhancement of the common ground which supports the antecedent, will also support the consequent. Hence, for all further updates of the common ground, the conditional in question holds at them. So to speak, we can think of the requirements of S2 as being reflected in BIS locally at the level of enhancements of the common ground satisfying the conditional in question. Such an interpretation of Lewis' claim is also motivated since in Inquisitive Semantics a state can be seen as defining the admissible possible worlds.[8] RIS and BIS do not reflect the S2 modeling of necessity *globally*, i.e., at the level of the set of all states. This is because BIS does not model accessibility relation between states.[9] Thus, BIS cannot be seen as fully reflecting S2 intuitions about the characteristics of implication.

4.3.2 Update Semantics

The inquisitive semantic notion of implication can also be seen to capture a part of the motivation behind the Update Semantic interpretation of implication. In US implication corresponds to a test, i.e., it holds at a state,

[8]As one may notice we are using the word 'admissible' slightly sloppily here. This is a result of the differences in frameworks that make their formal comparison very difficult, if not impossible. Notice, however, that given the specifications of each of the frameworks, it is plausible to treat admissibility as corresponding to accessibility relation in S2 and as corresponding to the possible worlds in enhancements of states in BIS. This is because in both frameworks in this fashion we specify the alternatives of the actual world under consideration.

[9]So to speak, the necessity notion in BIS is epistemic, it operates at the state level and considers information in the common ground. It does not correspond directly to metaphysical or logical necessity which can be thought to be defined globally, as a relation between states.

if whenever a state updated with its antecedent, supports the consequent. Similarly in order to determine whether a state supports an implication in inquisitive semantics, we also need to perform a test. More specifically, in RIS one needs to verify that whenever the antecedent is supported by an enhancement of the state, so is the consequent. In comparison to US, RIS implication is, however, a "test" which verifies whether all of the enhancements of the state σ supporting the antecedent, support the consequent. Whereas in US, we only verify whether a single subset of σ that includes all the possible worlds such that the antecedent classically holds at them, also supports the consequent.

The similarity between US and the inquisitive approach to implication is also reflected at the discourse level. This is because whenever a state σ supports an implication and one updates the common ground with a piece of information that corresponds to an enhancement of σ that supports the antecedent, this enhancement also supports the consequent. Such an interpretation of implication gives a plausible model of the behavior of implication in natural language. I.e., it matches the intuition, that we take an implication to hold, if given the current ground of the conversation, no matter what further evidence we come across, this further evidence will guarantee the consistency between the antecedent and the consequent. Notice, however, that as US is a *dynamic system*, i.e., it encodes the result of updating a state with a particular sentence, the extent to which implication corresponds just to a consistency test in US is greater than the extent to which it does so in Inquisitive Semantics. Importantly, inquisitive implication is not an update function, it does not alter a state in any way.

4.3.3 Conditional Logic

Despite the fact that Inquisitive Semantics does not model the notion of similarity, there are also certain correspondences between the approach to implication as given by RIS and by the one given by Stalnaker. For an interpretation of an implication in which one adds the antecedent hypothetically and after suitable changes verifies whether the conclusion is consistent with it, can also be seen as being reflected in the inquisitive approach to implication. This is because, a proposal to update the common ground of the conversation with an implication is equivalent to restricting the common ground to states such that for their every enhancement, if this enhancement supports the antecedent, it also supports the consequent. The process of deciding whether or not the proposal to update of the common ground with an implication should be accepted, can be seen as corresponding to adding the antecedent hypothetically to the common ground. Consequently, restricting the common ground can be seen as aligning ones beliefs with the ones that support the antecedent. Last but not least, verifying whether the consequent is supported in the enhancement of the states in the common ground thus obtained can be seen as checking for consistency between the antecedent and the consequent. Hence, some of the key philosophical motivations behind the definition of conditional entailment seem to be also consistent with the inquisitive interpretation of the implication. Notice, however, that inquisitive implication does not *fully* reflect Stalnaker's implication. This is because, inquisitive implication does not allow one to alter one's beliefs so that they accommodates the antecedent; it is only a proposal that guarantees that whenever the antecedent is supported, so is the consequent.

4.3.4 Relevance Logic B

The Relevance logic requirement for the antecedent to be relevant for an implication to hold does not seem to be directly reflected in the inquisitive modeling of implication. For there is no *direct* requirement encoded within Inquisitive Semantics that would require that the antecedent is related to the consequent in the Relevance Logic sense, i.e., by sharing some variables. It follows, however, that the Inquisitive Semantic enrichment of the notion of the proposition can be also seen as capturing some aspects of the relevance between the antecedent and the consequent *indirectly*. This is because, now the antecedent needs to be relevant to the consequent in the inquisitive sense. In order to explain the point made consider Paradox (5) $\models p \to (q \vee \neg q)$ and compare it to the RIS and BIS validities (5$'$) $\models q \to (q \vee \neg q)$ and (5$''$) $\models \neg q \to (q \vee \neg q)$. The antecedents in all of these examples are non-inquisitive, they just provide information, whereas consequents are inquisitive, they request enough information to decide between their two disjuncts. (5$'$) and (5$''$) are valid entailments, because their antecedents can be seen as providing answers to the issue raised in their consequents. This is because for any state, every enhancement of this state that supports the antecedent, will also support one of the disjuncts in the consequent. This exemplifies the fact that if the consequent is not a tautology and the antecedent is not a contradiction, then the only valid implications in BIS and RIS, will be the ones for which the antecedent shares some variables with the consequent. So to speak, the ones for which the antecedent can be interpreted as an answer to the issue raised in the consequent.[10] The point made here, is also explicated

[10]An intuitive reason for this fact can be seen when we consider diagrams corresponding to a non-contradictory antecedent and a non-tautological consequent of any implication. Then, if the consequent does not share any variables with the antecedent, we will always be able to find an enhancement which spans over some maximal enhancements for the

by Paradox (5) $\models p \to (q \lor \neg q)$ where the implication considered does not hold in BIS and RIS. As demonstrated by the counter-model, this is because the antecedent does not settle the issue raised by the consequent. Thus, it seems to be the case that inquisitive implication to be valid in BIS or RIS, has to share at least one propositional variable, which resembles closely the variable sharing principle in Relevance Logic.

Furthermore, based on our results, it seems to be the case that despite the lack of focus on the notion of relevance, RIS manages to capture some of the aspects of relevance by means of more refined responses, as well as the entailment relation defined in terms of the negative and positive responses to a sentence. This is especially visible when we consider (7) $p \land q \models p \to q$. For notice that (7) is accounted in RIS and it is also one of the main arguments used by the relevance logicians to point out the importance of the notion of relevance between the antecedent and the consequent.

4.3.5 Concluding Remarks

Thus, it follows that the Inquisitive Semantic modeling of natural language implication and its approach towards the paradoxical inferences considered seem to a certain extent to reflect the philosophical intuitions behind the other approaches discussed. This can be seen as pointing towards the multi-faceted nature of inquisitive modeling of implication, where the initial philosophical motivation behind it also seems to be aligned with the philosophical observations concerning the behavior of natural language implication in the literature.

N.B. the comparison in this section is just an attempt to demonstrate that the philosophical motivations underlying other approaches to the paradoxical implications are not excluded by the inquisitive modeling of implication or are treated as giving an *implausible* account of how to account for the paradoxical inferences in question. Having said that, it is important to keep in mind that S2, C2 and B have been developed *ad hoc* to deal with the problem of the material account of conditionals. As Inquisitive Semantics was not *engineered* to account for any of the problematic inferences discussed and its notions are motivated not by the problems we want to account for, but rather by our language use, in our view it makes it more appealing than some of the other approaches discussed. Hence, despite the fact that BIS allows us to account for less of the paradoxical inferences than S2, C2 and B, one might still find the inquisitive approach more plausible.

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consequent and which is not contained in any of them. N.B. A formal proof of this fact is still needed.

4.4 Criticism

There are two potential points of criticism of the philosophical enterprise involved in this thesis. The first criticism is RIS specific and questions whether the notion of entailment involved in RIS is not too strong. Namely, one can argue that in order for RIS to give a plausible account of the paradoxical inferences in question, it also needs to be able to give a desirable account of standard properties attributed to indicative conditionals. That is, it cannot be such that it does not model correctly some of the widely accepted properties of the conditional. The second criticism considers the plausibility of the list of the paradoxical inferences considered by us. Namely, it questions whether it is desirable or not to treat transitivity, centering and contraposition as implausible properties of indicative conditionals.

4.4.1 The First Criticism

As pointed out above, RIS seems to give rise to a very strong entailment relation. Because of this reason, one may criticize the RIS account of the paradoxical inferences in question as being inadequate. For notice, that it is not only important to account for as many of the paradoxical inferences as possible, but it is also important to validate some of the desirable indicative conditionals and properties which are widely attributed to implication. It is commonly accepted that two of the key properties that implication is meant to respect are Modus Ponens (implication elimination) and Modus Tollens (denying the consequent). If RIS respects both of these properties of implication, then these can be seen as an indication, that the notion of implication developed by it embodies some of the desirable properties. Hence, the criticism concerning the strength of RIS is not decisive. Before proceeding to the discussion it is important to discuss the interpretation of inquisitive entailments that involve several premises in a greater detail.

Given the fact that inquisitive entailment is meant to preserve both the informative and the inquisitive content, throughout the thesis we have interpreted the entailment involving multiple premises $\theta_1, \cdots, \theta_n$ and a conclusion ψ as $\theta_1 \wedge \theta_2 \cdots \wedge \theta_n \models_+ \psi$ and on the negative side as $\theta_1 \wedge \theta_2 \cdots \wedge \theta_n \models_- \psi$[11]. The first definition, can be seen as saying that whenever all of the premises are jointly supported, so is the conclusion. The negative entailment requires that whenever one rejects the conclusion, one also specifies exactly which premises he rejects. Thus, such a notion captures the inquisitive meaning and embodies the intuition that when one rejects the conclusion, one also

[11] Which is equivalent to saying $\div\psi \models_+ \div\theta_1 \vee \div\theta_2 \cdots \vee \div\theta_n$.

points out explicitly what are the premises one disagrees with. RIS requires for both RIS_- entailment and RIS^+ entailment to hold for the premises to entail the conclusion.

The motivation for the RIS entailment is very transparent when we consider responses from the perspective of the common ground. For notice that then the positive entailment is motivated by an intuition that whenever we give a response that supports all of the premises, this response also supports the conclusion. On the other hand the negative entailment is motivated by an intuition that whenever we give a response that rejects the conclusion, it is also important to know what premises we disagree exactly. So to speak, when we disagree with the conclusion we also specify what premises we find implausible, and do not spend time on deliberations over premises we actually reject.

As proved in Chapter 3, such an interpretation of entailment between several premises and conclusion, contributes to the fact that transitivity of implication does not hold according to RIS_- entailment and hence does not hold in RIS. Furthermore, it follows that the atomic case of Modus Ponens holds in RIS whereas Modus Tollens fails in RIS. This can be demonstrated in the following way:

1. $p \to q, p \models q$
   Proof by contradiction
   Let σ be arbitrary and suppose that $\sigma \models_+ p \to q \wedge p$ and $\sigma \not\models_+ q$ $\star$. Then, it follows by the definition of the support clause for '$\wedge$' that $\sigma \models_+ p \to q$ and $\sigma \models_+ p$. Hence, by the definition of the support clause for '$\to$', $\forall \tau \subseteq \sigma$ if $\tau \models_+ p$, then $\tau \models_+ q$ †; and by the definition of the support clause for atomic sentences, $\forall v \in \sigma$, $v(p) = 1$. Thus, it follows by † that $\sigma \models_+ q$ $\lightning$. This is a contradiction to $\star$. Thus, $p \to q, p \models_+ q$.
   For the rejection entailment, let σ be arbitrary and suppose that $\sigma \models_- q$ and $\sigma \not\models_- (p \to q) \wedge p$. Then it follows by the definition of the reject clause for atomic sentences that $\forall v \in \sigma$, $v(q) = 0$ $\star$; and by the definition of the reject clause for '$\wedge$' that $\sigma \not\models_- p \to q$ and $\sigma \not\models_- p$. Thus, it follows by the definition of the reject clause for '$\to$' that $\exists v \in \sigma$ s.t. $v(p) = 1$ and $v(q) = 1$ $\lightning$ (this is a contradiction to $\star$). Hence, $p \to q, p \models_- q$.
   Thus, it follows that $p \to q, p \models q$ holds in RIS, i.e., Modus Ponens holds in RIS for atomic sentences.

2. $p \to q, \div q \not\models \div p$
   Proof by contradiction
   Let σ be arbitrary and suppose that $\sigma \models_+ p \to q \wedge \div q$ and $\sigma \not\models_+ \div p$. Then it follows by the definition of the support clause for '$\wedge$' that

$\sigma \models_+ p \rightarrow q$ and $\sigma \models_+ \div q$; and by the definition of the support clause for '$\div$' that $\sigma \not\models_- p$. Thus, it follows by the support clause for '$\rightarrow$' that $\forall \tau \subseteq \sigma$, if $\tau \models_+ p$, then $\tau \models_+ q \star$, and by the definition of the support clause for '$\div$' and reject clause for atomic sentences that $\forall v \in \sigma\ v(q) = 0$ and $\exists v \in \sigma$ s.t. $v(p) = 1$. Take a singleton set $\{v\}$, then it follows that $\{v\} \models_+ p$ and $\{v\} \not\models_+ q \lightning$. This is a contradiction to $\star$. Hence, $p \rightarrow q, \div q \models_+ \div p$.

Notice, however, that it is not the case that $p \rightarrow q, \div q \models_- \div p$. For consider a state σ s.t. $\sigma = \{w_1, w_2\}$, $|p| = \{w_1, w_2\}$, $|q| = \{w_1\}$. Then, by the definition of the reject clause for '$\div$', it follows that $\sigma \models_- \div p$. However, since $w_1(q) = 1$, it follows by the definition of the reject clause for '$\div$' that $\sigma \not\models_- \div q$ and since $\sigma \models_+ p$ and $\sigma \not\models_- q$, it follows by the definition of the reject clause for '$\rightarrow$' that $\sigma \not\models_- p \rightarrow q$. Thus, $p \rightarrow q, \div q \not\models_- \div p$, as required.

Hence, it follows that $p \rightarrow q, \div q \not\models \div p$ does not hold in RIS.

Thus, as claimed, the atomic case for Modus Ponens holds in RIS and Modus Tollens does not hold in RIS. As demonstrated in the proof, the failure of Modus Tollens in RIS can be attributed to the negative entailment. For notice that despite the fact that the state σ in the counter-model does not support any of the premises, Modus Tollens still fails. This can be seen as demonstrating the role that inquisitiveness plays in RIS. For it requires any negative response to the conclusion to specify *exactly* which premise is rejected. Consequently, the failure of specifying the premises can be attributed as the main reason why RIS_- entailment fails to model Modus Tollens.

Thus, it seems that there are some reasons to think that RIS may indeed be thought as a too strong system to model implication. Further research is, however, necessary to consider the strength of this system. For notice, that there are indeed other valid entailments that hold in RIS that may turn out to be interesting and useful from the philosophical perspective. For instance it is easy to show that out of the non-classical systems considered, Radical Inquisitive Semantics is the only systems for which $(p \rightarrow q), \div(p \rightarrow q) \models \div p$ holds non-trivially. Furthermore, there are also possibilities to define a weaker notion of radical entailment. In particular it is plausible to consider the negative entailment as a claim that whenever the conclusion is rejected, not all of the premises are supported. One can demonstrate that such a notion of entailment, still allows us to account for all of the paradoxical inferences considered, but respects the atomic cases for Modus Ponens and Modus Tollens.[12]

[12] Yet another definition of Radical Entailment is also motivated and discussed by [1]. As demonstrated in Aher's paper, a weaker notion of RIS entailment allows one to account

4.4.2 The Second Criticism

From the first formulation of the horse-shoe analysis in Principia Mathematica there has been a vivid discussion concerning the properties of natural language implication. This discussion to a significant extent is still alive and there is no wider philosophical agreement concerning *some* of the properties of implication. As noticed by [3] in our analysis of the paradoxical inferences the denial of transitivity (10) and contraposition (16) is not necessarily considered to be a merit to the theory. Furthermore, [32] also points out that the property of centering (7) might also be thought as a desirable property of indicative conditionals.

The general line of argumentation for the property of contraposition is exemplified well by [30] and is based on the observation that despite being counterintuitive in some cases, in general, contraposition is a very intuitive rule. In particular, in logic and mathematics, it is one of the most commonly used proof techniques. Similarly in natural language, we use it very often in our reasoning and, as such, this rule is plausible. Furthermore, as noticed by [3], in some problematic cases it is also possible to explain away the problematic implication involving contraposition by means of spelling out the meanings of indicative conditionals involved. Thus, the fact that a theory invalidates (16) is rather to be treated as its disadvantage.

With regards to transitivity, as noted by [3], one can argue that the purported cases that are meant to demonstrate that transitivity does not hold for indicative conditionals are all really *subjunctive conditionals* in disguise and do not, as such, constitute a problem for transitivity of indicative conditionals. This can, for instance, be exemplified when we consider our counterexample to transitivity in Chapter 2: *If I win a million dollars, I will quit my job. If I quit my job, I will lose my apartment. Hence, if I win a million dollars, I will lose my apartment.* For notice that in this case, it might be more appropriate to interpret such a conditional as a subjunctive conditional, i.e., what *would* be the case if I were to win a million dollars; and what *would* be the case, if I were to quit my job. Thus, according to this criticism, our alleged counterintuitive example to transitivity does not hold.

With regards to centering, along the lines of [25], one can argue that our counterexample to centering demonstrates only the fact that centering is a "dazzling" or "odd" property, and alone this is not sufficient to deem it implausible. Hence, despite the fact that $p, q \models p \rightarrow q$ strikes us as odd and counterintuitive, it is nevertheless true. As Lewis puts it "oddity is not falsity" ([26], pp. 28). Thus, we cannot reject an inference, from the mere fact that it seems odd. Hence, according to this line of argumentation, our

for many of the paradoxes in Deontic Logic.

counterexample does not give us enough reason to search for the ways to account for it.

The first criticism argues for the rule of contraposition based on the grounds of its general plausibility and the possibility of "spelling out" the meaning of contrapositions. Indeed, one may concede that in comparison to the Paradoxes of Material Implication, contraposition seem to be less counterintuitive and leads to a smaller number of problematic cases. Notice, also, that in all of the logics apart from C2 and RIS contraposition does indeed come up as a valid inference rule. Furthermore, in C2 the key motivation against contraposition concerns subjunctive conditionals.

Despite its initial plausibility such a criticism of our treatment of the rule of contraposition is inconclusive. First of all, notice that just omitting the problematic instances is not a satisfactory solution to a problem raised by us. This is because, such an approach does not give any reason *why* the problematic inferences involving contraposition are not uncommon. Rather, the fact that there are some plausible contrapositions can be taken to demonstrate that there might be a further distinction necessary, to separate the "good" cases of contraposition from the "bad" cases of contraposition. Furthermore, such a criticism would only work if the body of counterintuitive natural language conditionals was significantly smaller than the body of intuitive contrapositions. There are significant reasons, however, to think that this is not the case. There is a large number of arguments against contraposition in the literature, e.g. Adams, Jackson [3], and it is not uncommon to classify contraposition as an undesirable property because of the plethora of counter-examples, e.g. Egre, Cozic [9]. Furthermore, as noticed by [3], contraposition is "not a virtuous form in any theory giving primacy to the Ramsey test" (pp. 34).

The criticism of our treatment of transitivity questions the validity of the alleged counterexamples to transitivity. It is a claim that the counterexamples to transitivity of indicative conditionals are fictitious and are rooted in the fact that we wrongly interpret subjunctive conditionals as indicative conditionals. There are two ways to respond to such a criticism.

First of all, notice that the claim that our counterexample to transitivity *is* a subjunctive conditional in disguise can itself be questioned. For given that one has got a strong belief that he *will* win the lottery, the indicative reading of the implications involved is permissible. Furthermore, even granted that the counterexample produced by us and standard counterexamples in the literature are subjunctive conditionals in disguise, one can still point at a family of different counterexamples. For consider the following example originally stated in [3]:

If the cows are in the turnip field, the gate has been left open.
If the gate has been left open, then the cows have not noticed the gate's condition.

Therefore, if the cows are in the turnip field, then the cows have not noticed the gate's condition.

If uttered by a farmer who has got a strong belief that the gate is closed and that the cows are not in the field, then he can plausibly hold the first two indicative conditionals to be true, whereas it would not be plausible for him to hold that the conclusion holds. Notice, that such an argument cannot be claimed to involve subjunctive conditionals in disguise. Hence, it seems to be the case that such a line of argumentation against our treatment of transitivity is not sufficient, and does not demonstrate that transitivity is a desirable property of indicative conditionals.

The argument against the family of counterexamples to centering is explicated well by Lewis' point [26]. As he notices, despite the fact that we find some of the natural language correspondents of centering odd, this does not give us sufficient reason to deem them *false*. Lewis' agrees that these examples do indeed correspond to things which are not good to say. He disagrees, however, that it is implausible to conclude *If* p*, then* q from the truth of p and q. As he puts it, "But oddity is not falsity; not everything true is a good thing to say. In fact the oddity dazzles us. It blinds us to the truth of the sentences, and we can make no confident judgment one way or the other" (pp. 28).

In response, we claim that it seems to be the case that what dazzles us is not the fact that we say something odd, but rather that we say something which is completely unrelated. As noticed by [32] "In fact most of the concurrent events are unrelated to each other, whereas the relatedness of p and q is what 'If p, then q' is supposed to express" (pp. 32). The argument made here is reflected in the fact that similar reasoning is also attributed to be fallacious. For consider the fallacy *cum hoc ergo propter hoc.* This fallacy concerns the fact that on the basis of the fact that two events occur together, one cannot conclude that one must occur because of the other. Analogously, it seems to be the case that indicative conditionals do assume some notion of relatedness. Hence, as there is nothing that guarantees that p and q are even remotely related to each other, it is not the case that centering is desirable. For notice that hardly anyone would disagree that to argue irrelevantly is a bad thing. That is why it is extremely easy to produce numerous examples questioning centering that strike us as very counterintuitive.

4.4.3 Final Remarks

We have considered two different criticisms of the analysis of the results presented in this thesis and tried to present reasons why we do not consider these criticisms as being decisive or as refuting the conclusions drawn. In our analysis as well as the criticism, we try to presume very limited assumptions concerning the nature of indicative conditionals. It is not our attempt to claim that one interpretation of the conditional is better than the other in itself. Rather, our enterprise can be seen as a study of the inferences that have often been attributed as being paradoxical in the literature and that have been widely argued against. Then, our claim is significantly weaker than one could have initially thought it had been. On the one hand, our claim states that the characteristics of Inquisitive Semantics allow us to account for many of the paradoxical inferences and model them better than classical logic. On the other hand, it points to the fact that the inquisitive semantic implication is not an *ad hoc* notion and that it allows us to account for many, if not all of the paradoxical inferences considered. Consequently, the inquisitive account can be seen as being more advantageous than many of the approaches considered.

As conditionals have been an area of an intense academic focus since the beginning of the 20th century, there is a vast literature concerning their behavior and, probably many different criticisms concerning the enterprise as developed along the lines in this thesis. For instance, there is a very interesting and stimulating branch of a defense of the horse-shoe analysis of indicative conditionals that concerns their *assertability* and *pragmatical correctness*.[13] In this thesis, we withhold the judgment concerning the arguments brought forward and against these approaches.

There is much more to say about conditionals and there are stimulating ways of extending this thesis. We believe, however, that the inquisitive take on the paradoxical inferences and, most importantly, the inquisitive approach towards the Paradoxes of Material Implication, fits in nicely with the other non-classical approaches. It also sheds a new light on how to account for the Paradoxes of Material Implication. For notice that the discussion in this thesis points towards the role that inquisitive content of a proposition can play in accounting for some of the key problems concerning indicative conditionals.

[13]For instance Adam Rieger presented an interesting article [36] that defends the horse-shoe analysis of the indicative conditional on the basis of *assertability*. Similarly the Gricean account uses natural language *conventional implicature* to deem many of the inferences, but not all of the inferences considered in this thesis, as being valid, but pragmatically incorrect.

CHAPTER 5

Conclusion

In this thesis we have discussed an inquisitive take on the Paradoxes of Material Implication. We have demonstrated that both of the inquisitive systems considered — BIS and RIS — allow us to account for more paradoxical inferences than Classical Logic. Furthermore, we have also suggested that the account given by BIS and RIS is not developed *ad hoc* to deal with the problematic inferences. Because of this BIS and RIS can be seen as being advantageous over S2, C2 and especially B. The results of the analysis demonstrate that out of the systems considered, RIS is the most successful and allows us to account for the largest amount of the problematic inferences.

In the first chapter of the thesis, we have provided an introduction to Inquisitive Semantics. We have motivated the inquisitive semantic understanding of a proposition and contrasted it with the classical account of a proposition. We have then provided a detailed introduction to two inquisitive systems: BIS and RIS. The introduction to these systems is the first fully formalized and motivated account that assumes that a proposition is a downward closed set of states. In Radical Inquisitive Semantics, we have provided a state-based semantics with a new notion of RIS entailment and an extensive discussion of issue-dispelling responses. We have further motivated the key features of both of the systems by discussing suitable examples.

In the second chapter of the thesis we have provided an introduction to the Paradoxes of Material Implication. We have specified the 16 paradoxical inferences and provided suitable examples to demonstrate their implausibility. Furthermore, we have also provided an introduction to S2, C2, US and B. We have discussed the semantics of these systems and highlighted motivations underlying their key semantic features. In our discussion, we focused in particular on the modeling of implication and described the motivations behind it in a greater detail. Finally, we have also summarized the extent to which each of these logics allows one to account for the problematic inferences.

In the third chapter we have accounted for the Paradoxes of Material Implication by means of Basic Inquisitive Semantics and Radical Inquisitive Semantics. We have proved whether each of the inferences holds. We have also briefly discussed the results and their significance. Our analysis considered the paradoxical inferences in terms of the BIS entailment as well as RIS negative and positive entailment.

In the fourth chapter of the thesis, we have analyzed the results obtained. We have discussed different features of implication in S2, C2, US, B, BIS and RIS that allowed us to account for different paradoxes. We have also provided a comparison between the modeling of implication in Inquisitive Semantics and non-classical logics discussed. We suggested that inquisitive implication embodies many of the philosophical motivations behind the properties attributed to the behavior of implication by other logics discussed. We have also considered two possible criticisms of the philosophical enterprise involved in this thesis. One that questioned the plausibility of RIS and the other that questioned the implausibility of some of the paradoxical inferences considered. Consequently, we have argued that these criticisms are inconclusive.

We consider the key contribution of this thesis to be the placement of Inquisitive Semantics within the genealogy of discourses concerning the English indicative conditional. That is, we believe that the analysis provided in this thesis exemplifies the inquisitive approach to the Paradoxes of Material Implication and places the inquisitive enterprise within a new philosophical tradition.

The inclusion of the inquisitive voice into the debate concerning indicative conditionals is for the mutual benefit.

On the one hand, it produces a strong case for the inquisitive enterprise. It demonstrates that the inquisitive enrichment and the semantics which is motivated by it, gives an intuitive and non-*ad hoc* account of one of the key problems of Classical Logic. The fact that Inquisitive Semantics allows us to account for some of the undesirable inferences involving indicative conditionals can thus be seen as further motivating the inquisitive enterprise and inquisitive notion of a proposition. By these means the body of arguments for Inquisitive Semantics has just been extended. Most importantly, we believe that the fact that RIS allows one to account for more of the paradoxical inferences than any of the rival systems indicates the viability and potential of the inquisitive enterprise.

On the other hand, the inquisitive approach to the Paradoxes of Material Implication also benefits the current discourse on indicative conditionals. For it specifies a well motivated and intuitive approach to modeling implication. More specifically, it not only gives a non-truth functional account of English indicative conditionals, but is also very effective in accounting for some of their undesirable properties. By these means, an inclusion of inquisitive elements enriches the level of current debate on conditionals and points towards the role that inquisitiveness can play in giving a more adequate model of implication. It is our hope that Inquisitive Semantics will 'stir things up' a bit and lead to new developments and insights into the nature of not only indicative conditionals, but also conditionals in general.

As we expected, the analysis of the inquisitive take on the paradoxical inferences involving material implication produced more questions than answers. This points towards the possibilities for further research.

Firstly, as pointed out in the Analysis Chapter, the characteristics of the RIS system require further investigation. As RIS is a new semantic development, its features are not fully known. From the point of this thesis, the specification of valid classes of entailments as well as different notions of radical entailments constitute a stimulating and interesting research prospect. On the one hand, a better understanding of Radical Inquisitive Semantics may allow us to fully reject the criticism of RIS in the Analysis Chapter. On the other hand, it can also give some new insights into different linguistic phenomena, or allow us to account for other problems of classical and non-classical semantics.

Secondly, the thesis raises one very important question that it does not purport to address. Namely, *what are the exact contributions of each of the aspects of Radical Inquisitive Semantics that allow us to account for the paradoxical inferences considered?* Notice that it is not clear from the analysis provided, to what extent the effectiveness of Radical Inquisitive Semantics is due to the definition of the RIS entailment, to what extent it is due to the definition of inquisitive implication and to what extent it is due to the downward-closure requirement. A better understanding of the contribution of these factors can shed new light on the behavior of inquisitive implication and can contribute to the creation of an Inquisitive Theory of Indicative Conditionals. The results provided suggest that such a theory is viable and may, indeed, turn out to provide a vital insight into our use of indicative conditionals.

One can also try to extend current analysis to the consideration of some of the problems concerning the modeling of *subjunctive conditionals*. In principle, Inquisitive Semantics already provides sufficient logical machinery for this task. Note, that similarly to Stalnaker, one may model the context change involved in uttering subjunctive conditionals by switching the context of the conversation. By these means, defining a suitable relation on states σ while utilizing the characteristics of inquisitive implication may provide new insight into the behavior of counterfactual conditionals.

Last but not least, extending the analysis provided in this thesis so that it takes into account the *pragmatic constraints* on the inferences considered constitutes a straightforward and interesting extension. Such an analysis would complement current analysis. It may also recalibrate the focus of the current analysis, so that only the inferences that cannot be accounted for by the means of pragmatics are considered.

Having said this, we have reached the end of our analysis concerning some aspects of one of the small words in English: '*if*'. Hope you enjoyed it. There is a long way to explain the meaning of this word. The inquisitive enterprise seems to be on the right track, though.

Appendix

A.1 Strict Conditional Logic Proofs

We will give the proofs of the Paradoxes of Material Implication in Lewis' strict logic S2 using the semantic tableaux as defined in Priest [33]. The tableaux method is sound and complete with regards to the semantics. Nodes of a tableaux are constituted either by a formula and a natural number, or by iRj, where i, j are natural numbers. Intuitively, different numbers correspond to different possible worlds, a node of a form θ, i means that θ holds at possible world i; and iRj means that a possible world i is related to a possible world j.

In order to check whether the premises imply the conclusion, we assume the premises and the negation of the conclusion and check for consistency. A branch of a tableaux closes when there is a pair of the form θ, i, $\neg\theta, i$ on it. Whenever all branches of a tableaux close ($\times$), the tested inference holds; and whenever one of the branches is open (O), then the inference tested does not hold. After finding an open branch, we can read the counter-model from it.

In order to model Lewis' non-modal worlds, the rule for '$\Diamond\theta, i$' is triggered only when $i = 0$ or there is a node of the form $\Box\psi, i$ on a branch. Furthermore, since the accessibility relation in S2 is reflexive, it follows that for every new possible world i on a branch, we introduce a node iRi.

The tableaux system for Strict Implication uses the following rules:

$\theta \supset \psi, w$ $\neg\theta, w \qquad \psi, w$	$\neg(\theta \supset \psi), w$ $\mid$ θ, w $\neg\psi, w$	$\neg\neg\theta, w$ $\mid$ θ, w
$\theta \vee \psi, w$ $\theta, w \qquad \psi, w$	$\neg(\theta \vee \psi), w$ $\mid$ $\neg\theta, w$ $\neg\psi, w$	$\theta \wedge \psi, w$ $\mid$ θ, w ψ, w
$\neg(\theta \wedge \psi), w$ $\neg\theta, w \qquad \neg\psi, w$	$\Box\theta, w$ wRw' $\mid$ θ, w'	$\neg\Box\theta, w$ $\mid$ $\Diamond\neg\theta, w$
$\Diamond\theta, w$ $\mid$ wRw' θ, w'	$\neg\Diamond\theta, w$ $\mid$ $\Box\neg\theta, w$	

Below we present the proofs of the 16 inferences we considered in Chapter 2. We will explain the first example in detail in order to demonstrate how a proof that uses the tableaux method proceeds.

1) $p \not\models \Box(q \supset p)$

$$
\begin{array}{c}
p,0 \\
\neg\Box(q \supset p), 0 \\
\mid \\
\Diamond(\neg(q \supset p)), 0 \\
\mid \\
0R1 \\
\neg(q \supset p), 1 \\
\mid \\
q,1 \\
\neg p,1 \\
\mid \\
0R0 \\
1R1 \\
O
\end{array}
$$

Thus, (1) does not hold in S2.

Explanation

As we are checking for the consistency of the premises with the conclusion, in the first node of the tableaux we assume the premises and the negation

of the conclusion at the actual world (i.e., the premise $p, 0$ and the negation of the conclusion $\neg\Box(q \supset p), 0)$. The next node—$\Diamond(\neg(q \supset p)), 0$—follows by the application of the '$\neg\Box\theta$' rule to $q \supset p$. Then, by the application of the '$\Diamond$' rule we obtain the node $0R1$, $\neg(q \supset p), 1$, where $0R1$ means that the possible world 1 is accessible from the possible world 0. The one before last node follows by the definition of the '$\neg(\theta \supset \psi)$' rule. The last node follows since S2 is reflexive. The branch is open, hence $p \not\models \Box(q \supset p)$.
Note that we can read of the counter-model directly from the open branch. The counter-model is given by $W = \{0, 1\}$, $0R0$, $0R1$, $1R1$, $v_1(p) = 0$, $v_1(q) = 0$, $v_0(p) = 0$ and $v_0(q)$ is arbitrary.

2) $\neg q \not\models q \rightarrow p$

$$
\begin{array}{c}
\neg q, 0 \\
\neg\Box(q \supset p), 0 \\
| \\
\Diamond(\neg(q \supset p)), 0 \\
| \\
0R1 \\
\neg(q \supset p), 1 \\
| \\
q, 1 \\
\neg p, 1 \\
| \\
0R0 \\
1R1 \\
O
\end{array}
$$

Thus (2) does not hold in S2.

3) $\Box(p \supset s) \models \Box((p \wedge q) \supset s)$

$$
\begin{array}{c}
\Box(p \supset s), 0 \\
\neg\Box((p \wedge q) \supset s), 0 \\
| \\
\Diamond\neg((p \wedge q) \supset s), 0 \\
| \\
0R1 \\
\neg((p \wedge q) \supset s), 1 \\
| \\
p \wedge q, 1 \\
\neg s, 1 \\
| \\
p, 1 \\
q, 1 \\
| \\
p \supset s, 1 \\
\swarrow \quad \searrow \\
\neg p, 1 \qquad s, 1 \\
\times \qquad\quad \times
\end{array}
$$

Thus (3) holds in S2.

4) $\models \Box(p \wedge \neg p \supset q)$

$$
\begin{array}{c}
\neg\Box((p \wedge \neg p) \supset q), 0 \\
| \\
\Diamond\neg((p \wedge \neg p) \supset q), 0 \\
| \\
0R1 \\
\neg((p \wedge \neg p) \supset q), 1 \\
| \\
p \wedge \neg p, 1 \\
q, 1 \\
| \\
p, 1 \\
\neg p, 1 \\
\times
\end{array}
$$

Thus (4) holds in S2.

5) $\models \Box(p \supset (q \vee \neg q))$

$$
\begin{array}{c}
\neg\Box(p \supset (q \vee \neg q)), 0 \\
| \\
\Diamond\neg(p \supset (q \vee \neg q)), 0 \\
| \\
0R1 \\
\neg(p \supset (q \vee \neg q)), 1 \\
| \\
p, 1 \\
\neg(q \vee \neg q), 1 \\
| \\
\neg q, 1 \\
\neg\neg q, 1 \\
| \\
q, 1 \\
\times
\end{array}
$$

Thus, (5) holds in S2.

6) $\not\models \Box(p \supset \Box(q \supset p))$

$$
\begin{array}{c}
\neg\Box(p \supset \Box(q \supset p)), 0 \\
| \\
\Diamond\neg(p \supset \Box(q \supset p)), 0 \\
| \\
0R1 \\
| \\
\neg(p \supset \Box(q \supset p)), 1 \\
| \\
p, 1 \\
\neg\Box(q \supset p), 1 \\
| \\
\Diamond\neg(q \supset p), 1 \\
| \\
0R0 \\
1R1 \\
O
\end{array}
$$

Thus, (6) does not hold in S2. Note that, as there is no node of the form '$\Box\theta, 1$', the '$\Diamond$' rule is not triggered.

7) $p \wedge q \not\models \Box(p \supset q)$

$$
\begin{array}{c}
p \wedge q, 0 \\
\neg\Box(p \supset q), 0 \\
| \\
p, 0 \\
q, 0 \\
| \\
\Diamond\neg(p \supset q), 0 \\
| \\
0R1 \\
\neg(p \supset q), 1 \\
| \\
p, 1 \\
\neg q, 1 \\
| \\
0R0 \\
1R1 \\
O
\end{array}
$$

Thus, (7) does not hold in S2.

8) $\not\models \Box(p \supset q) \vee \Box(q \supset p)$

$$\neg(\Box(p \supset q) \vee \Box(q \supset p), 0$$
$$|$$
$$\neg\Box(p \supset q), 0$$
$$\neg\Box(q \supset p), 0$$
$$|$$
$$0R1$$
$$\neg(p \supset q), 1$$
$$|$$
$$p, 1$$
$$\neg q, 1$$
$$|$$
$$0R2$$
$$\neg(q \supset p), 2$$
$$|$$
$$q, 2$$
$$\neg p, 2$$
$$|$$
$$0R0$$
$$1R1$$
$$2R2$$
$$O$$

Thus, (8) does not hold in S2.

9) $\neg p \models \Box(p \supset \neg p)$

$$\neg p, 0$$
$$\neg\Box(p \supset \neg p)$$
$$|$$
$$\Diamond\neg(p \supset \neg p)$$
$$|$$
$$0R1$$
$$\neg(p \supset \neg p), 1$$
$$|$$
$$p, 1$$
$$\neg\neg p, 1$$
$$|$$
$$p, 1$$
$$|$$
$$0R0$$
$$1R1$$
$$O$$

Thus, (9) holds in S2.

10) $\Box(p \supset q), \Box(q \supset s) \models \Box(p \supset s)$

$$\begin{array}{c}
\Box(p \supset q), 0 \\
\Box(q \supset s), 0 \\
\neg\Box(p \supset s), 0 \\
| \\
\Diamond\neg(p \supset s), 0 \\
| \\
0R1 \\
\neg(p \supset s), 1 \\
| \\
p, 1 \\
\neg s, 1 \\
| \\
p \supset q, 1
\end{array}$$

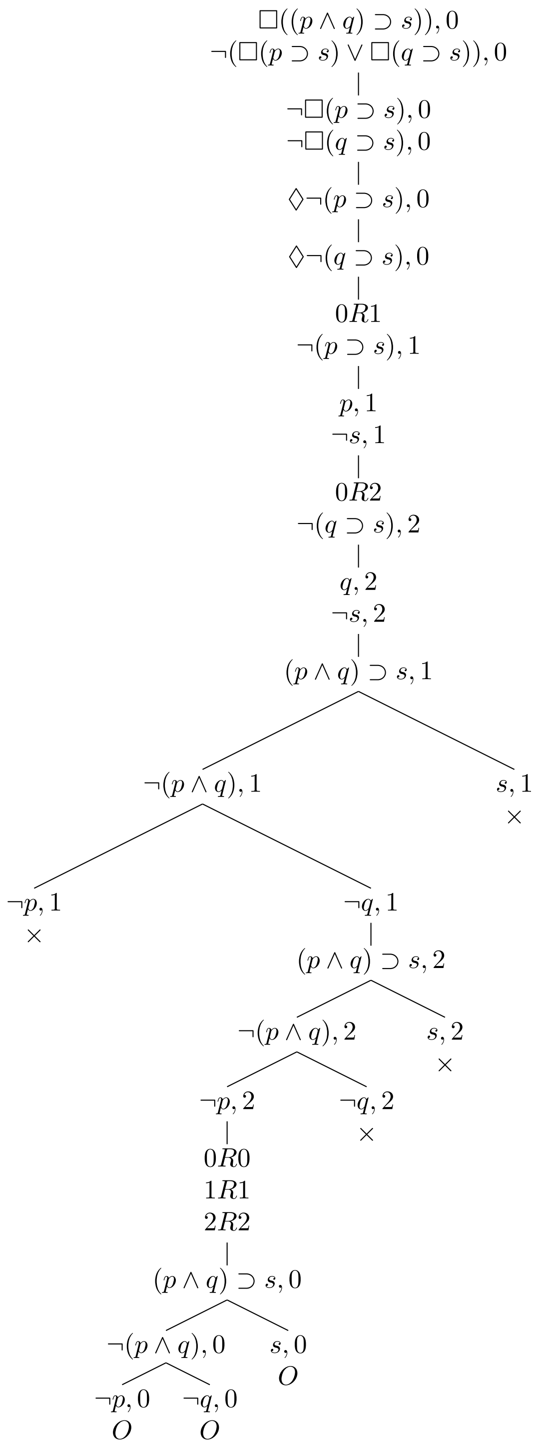
$$\begin{array}{ccc}
\neg p, 1 & & q, 1 \\
\times & & | \\
 & & q \supset s, 1 \\
 & & \begin{array}{cc} \neg q, 1 & s, 1 \\ \times & \times \end{array}
\end{array}$$

Thus, (10) holds in S2.
11) $\not\models \Box(p \supset \Box(q \supset q))$

$$\begin{array}{c}
\neg\Box(p \supset \Box(q \supset q)), 0 \\
| \\
\Diamond\neg(p \supset \Box(q \supset q)), 0 \\
| \\
0R1 \\
\neg(p \supset \Box(q \supset q)), 1 \\
| \\
p, 1 \\
\neg\Box(q \supset q), 1 \\
| \\
\Diamond\neg(q \supset q), 1 \\
| \\
0R0 \\
1R1 \\
O
\end{array}$$

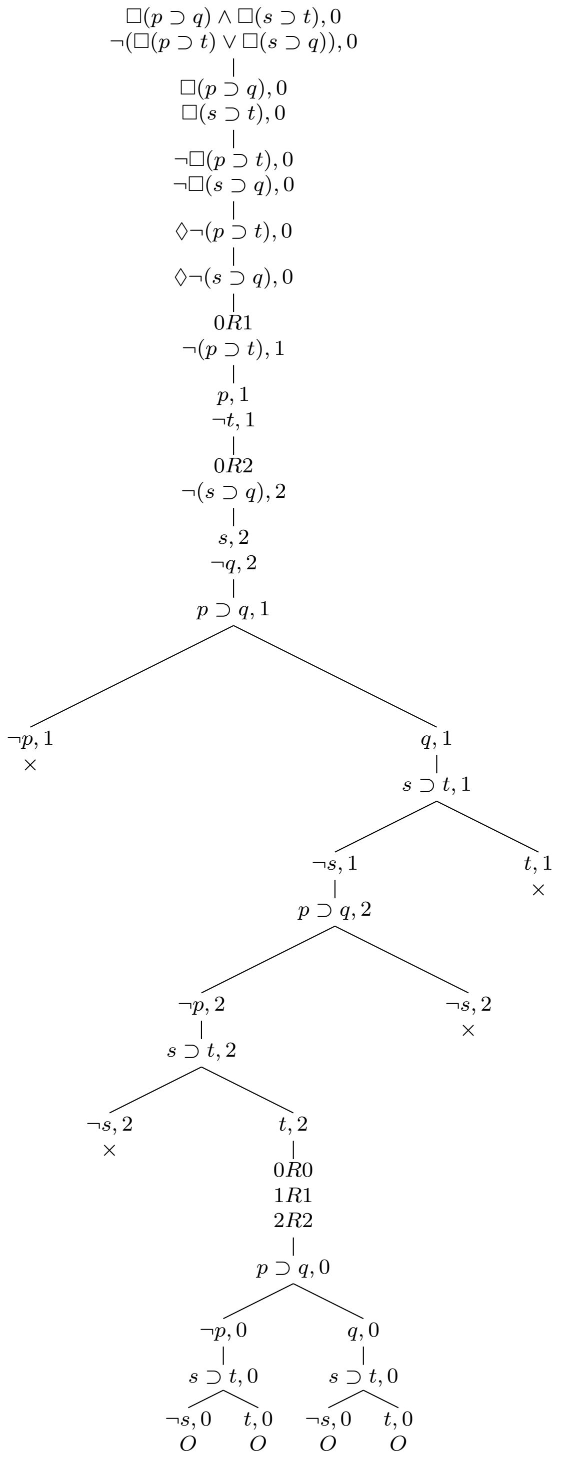
Thus, (11) does not hold in S2.

12) $\Box((p \wedge q) \supset s) \not\models \Box(p \supset s) \vee \Box(q \supset s)$



Thus, (12) does not hold in S2.

13) $\Box(p \supset q) \land \Box(s \supset t) \not\models \Box(p \supset t) \lor \Box(s \supset q)$



Thus, (13) does not hold in S2.

14) $\neg\Box(p \supset q) \not\models p$

$$\begin{array}{c} \neg\Box(p \supset q), 0 \\ \neg p, 0 \\ | \\ \Diamond\neg(p \supset q), 1 \\ | \\ 0R1 \\ \neg(p \supset q), 1 \\ | \\ p, 1 \\ \neg q, 1 \\ | \\ 0R0 \\ 1R1 \\ O \end{array}$$

Thus, (14) does not hold in S2.

15) $\neg\Box(p \supset q) \not\models \neg q$

$$\begin{array}{c} \neg\Box(p \supset q), 0 \\ \neg\neg q, 0 \\ | \\ q, 0 \\ | \\ \Diamond\neg(p \supset q), 1 \\ | \\ 0R1 \\ \neg(p \supset q), 1 \\ | \\ p, 1 \\ \neg q, 1 \\ | \\ 0R0 \\ 1R1 \\ O \end{array}$$

Thus, (15) does not hold in S2.

16) $\Box(p \supset q) \models \Box(\neg q \supset \neg p)$

$$\Box(p \supset q), 0$$
$$\neg\Box(\neg q \supset \neg p), 0$$
$$|$$
$$\Diamond\neg(\neg q \supset \neg p), 0$$
$$|$$
$$0R1$$
$$\neg(\neg q \supset \neg p), 1$$
$$|$$
$$\neg q, 1$$
$$\neg\neg p, 1$$
$$|$$
$$p, 1$$
$$|$$
$$p \supset q, 1$$
$$\neg p, 1 \qquad q, 1$$
$$\times \qquad\quad \times$$

Thus, (16) holds in S2.

A.2 Conditional Logic Proofs

In this section we present the results concerning Conditional Logic.

1. $p \not\models q \rightarrow p$
   Proof
   Consider a model M s.t. $W = \{w_0, w_1\}$, w_0Rw_0, w_1Rw_1, w_0Rw_1, $|p| = \{w_0\}$, $|q| = \{w_1\}$, $f(|q|, w_0) = w_1$. Then, it follows that $M, w_0 \models p$. However, since $f(|q|, w_0) = w_1$ and $M, w_1 \not\models p$, it follows by the definition of '$\rightarrow$' that $M, w_0 \not\models q \rightarrow p$. Thus, it follows that $p \not\models q \rightarrow p$, as required.

2. $\neg p \not\models p \rightarrow q$
   Proof
   Consider a model M s.t. $W = \{w_0, w_1\}$, w_0Rw_0, w_1Rw_1, w_0Rw_1, $|p| = \{w_1\}$, $|q| = \emptyset$ and $f(|p|, w_0) = w_1$. Then, it follows that $M, w_0 \models \neg p$, however, since $f(|p|, w_0) = w_1$ and $M, w_1 \not\models q$, it follows by the definition of '$\rightarrow$' that $M, w_0 \not\models p \rightarrow q$. Thus, it follows that $\neg p \not\models p \rightarrow q$, as required.

3. $p \rightarrow s \not\models (p \wedge q) \rightarrow s$
   Proof
   Consider a model M s.t. $W = \{w_0, w_1\}$, w_0Rw_0, w_1Rw_1, w_0Rw_0, $|p| = \{w_0, w_1\}$, $|s| = \{w_0\}$, $|q| = \{w_1\}$, $f(|p|, w_0) = w_0$, $f(|p \wedge q|, w_0) = w_1$. Then, it follows by the definition of '$\rightarrow$' that $M, w_0 \models p \rightarrow s$, however since $f(|p \wedge q|, w_0) = w_1$ and $M, w_1 \not\models s$, it follows by the definition of '$\rightarrow$' that $M, w_0 \not\models (p \wedge q) \rightarrow s$. Thus, $p \rightarrow s \not\models (p \wedge q) \rightarrow s$, as required.

4. $\models (p \wedge \neg p) \rightarrow q$
   Proof
   Let M be an arbitrary model and let w be a possible world in this model. Since $p \wedge \neg p = \emptyset$, it follows that $f(|p \wedge \neg p|, w) = \lambda$. Now it follows that $\lambda \models q$. Hence, it follows by the definition of '$\rightarrow$', that $M, w \models (p \wedge \neg p) \rightarrow q$. Since, M and w were arbitrary, it follows that $\models (p \wedge \neg p) \rightarrow q$, as required.

5. $\models p \rightarrow (q \vee \neg q)$
   Proof
   Let M be an arbitrary model and let w be a possible world in this model. Then it follows that there are two cases to consider:
   Case 1 $f(|p|, w) = \lambda$.
   Then, it follows that $f(|p|, w) = \lambda$ and $M, \lambda \models q \vee \neg q$ vacuously. Hence,

it follows by the definition of '$\rightarrow$' that $M, w \models p \rightarrow (q \vee \neg q)$.
Case 2 $f(|p|, w) \neq \lambda$.
Then, it follows by the definition of the selection function that $\exists w_1$ s.t. wRw_1 and $f(|p|, w) = w_1$. Thus, it follows by the definition of the selection function that $v_{w_1}(p) = 1$. Now since $w_1 \neq \lambda$, it follows that $v_{w_1}(q) = 1$ or $v_{w_1}(q) = 0$. Hence, $M, w_1 \models q$ or $M, w_1 \models \neg q$. Thus, it follows by the definition of '$\vee$' that $M, w_1 \models q \vee \neg q$. Now since $f(|p|, w) = w_1$, it follows by the definition of '$\rightarrow$' that $M, w \models p \rightarrow q \vee \neg q$.
Since, cases 1 and 2 were exhaustive; and M and w were arbitrary, it follows that $\models p \rightarrow (q \vee \neg q)$, as required.

6. $\not\models p \rightarrow (q \rightarrow p)$
   Proof
   Consider a model M s.t. $W = \{w_0, w_1, w_2\}$, w_0Rw_0, w_1Rw_1, w_2Rw_2, w_0Rw_1, w_1Rw_2, $|p| = \{w_1\}$, $|q| = \{w_2\}$, $f(|p|, w_0) = w_1$ and $f(|q|, w_1) = w_2$. Then, it follows that $M, f(|q|, w_1) \not\models p$ and hence by the definition of '$\rightarrow$', $M, w_1 \not\models q \rightarrow p$. Hence, since $f(|p|, w_0) = w_1$, it follows by the definition of '$\rightarrow$' that $M, w_0 \not\models p \rightarrow (q \rightarrow p)$.
   Thus, it follows that $\not\models p \rightarrow (q \rightarrow p)$, as required.

7. $p \wedge q \models p \rightarrow q$
   Proof
   Let M be an arbitrary model and let w be a possible world in this model. Suppose that $M, w \models p \wedge q$. Then it follows by the definition of '$\wedge$' that $M, w \models p$ and $M, w \models q$. Hence, it follows by property (2.3) of the selection function that $f(|p|, w) = w$. Hence, it follows that $M, f(|p|, w) \models q$. Thus, it follows by the definition of the '$\rightarrow$' that $M, w \models p \rightarrow q$.
   Thus, since M and w were arbitrary, it follows that $p \wedge q \models p \rightarrow q$, as required.

8. $\not\models (p \rightarrow q) \vee (q \rightarrow p)$
   Proof
   Consider a model M s.t. $W = \{w_0, w_1, w_2\}$, w_0Rw_0, w_1Rw_1, w_2Rw_2, $|p| = \{w_1\}$, $|q| = \{w_2\}$, $f(|p|, w_0) = w_1$, $f(|q|, w_0) = w_2$. Then, since $f(|p|, w_0) = w_1$, and $M, w_1 \not\models q$, it follows by the definition of '$\rightarrow$' that $M, w_0 \not\models p \rightarrow q$. Similarly since $f(|q|, w_0) = w_2$ and $M, w_2 \not\models p$, it follows by the definition of '$\rightarrow$' that $M, w_0 \not\models q \rightarrow p$. Thus, it follows by the definition of '$\vee$' that $M, w_0 \not\models (p \rightarrow q) \vee (q \rightarrow p)$. Hence, it follows that $\not\models (p \rightarrow q) \vee (q \rightarrow p)$, as required.

9. $\neg p \models p \to p$
Proof
Let M be an arbitrary model and let w be a possible world in this model s.t. $M, w \models \neg p$. Suppose for contradiction that $M, w \not\models p \to p$. Then, it follows that $\exists w'$ s.t. wRw', $f(|p|, w) = w'$ and $M, w' \not\models p$ $\lightning$. This is a contradiction, since by the properties of the selection function it follows that $w' \in |p|$. Thus, $\neg p \models p \to p$, as required.

10. $p \to q, q \to s \not\models p \to s$
Proof
Consider a model M s.t. $W = \{w_0, w_1, w_2\}$, w_0Rw_0, w_1Rw_1, w_2Rw_2, w_0Rw_1, w_0Rw_2, $|p| = \{w_2\}$, $|q| = \{w_1, w_2\}$, $|s| = \{w_2\}$, $f(|p|, w_0) = w_1$, $f(|q|, w_0) = w_2$. Then, it follows that $M, w_1 \models q$ and hence $M, f(|p|, w_0) \models q$. Thus, by the definition of '$\to$' $M, w_0 \models p \to q$. Similarly, it follows that $M, w_2 \models s$ and hence $M, f(|q|, w_0) \models s$. Thus, by the definition of '$\to$' $M, w_0 \models q \to s$. Notice, however, that since $M, w_1 \not\models s$, i.e., $M, f(|p|, w_0) \not\models s$, it follows by the definition of '$\to$' that $M, w_0 \not\models p \to s$.
Thus, $p \to q, q \to s \not\models p \to s$, as required.

11. $\models p \to (q \to q)$
Proof by contradiction
Suppose for contradiction that there exists M and w s.t. $p \to (q \to q)$ does not hold at w. Then, it follows by the definition of '$\to$' that $M, f(|p|, w) \not\models q \to q$ $\star$. Wlog suppose $f(|p|, w_0) = w_1$. Then, it follows by $\star$ that $M, w_1 \not\models q \to q$. Hence, by the definition of '$\to$' $M, f(|q|, w_1) \not\models q$ $\lightning$. This is a contradiction since by property (2.1) of the selection function $f(|q|, w_1) \in |q|$ and hence $M, f(|q|, w_1) \models q$.
Thus, since M and w were arbitrary it follows that $\models p \to (q \to q)$, as required.

12. $(p \wedge q) \to s \not\models (p \to s) \vee (s \to q)$
Proof
Consider a model M s.t. $W = \{w_0, w_1, w_2, w_3\}$, w_0Rw_0, w_1Rw_1, w_2Rw_2, w_3Rw_3, w_0Rw_1, w_0Rw_2, w_0Rw_3, $|p| = \{w_1, w_3\}$, $|q| = \{w_2, w_3\}$, $|s| = \{w_3\}$, $f(|p|, w_0) = w_1$, $f(|q|, w_0) = w_2$, $f(|p \wedge q|, w_0) = w_3$. Then, it follows that $M, w_3 \models s$ and hence $M, f(|p \wedge q|, w_0) \models s$. Thus, by the definition of '$\to$' $M, w_0 \models (p \wedge q) \to s$. Notice, however, that since $M, w_1 \not\models s$ and $M, w_2 \not\models s$, it follows that $M, f(|p|, w_0) \not\models s$ and $M, f(|q|, w_0) \not\models s$. Thus, it follows by the definition of '$\to$' that $M, w_0 \not\models p \to s$ and $M, w_0 \not\models q \to s$. Hence, it follows by the definition

of '$\vee$' that $M, w_0 \not\models (p \to s) \vee (q \to s)$.
Thus, it follows that $(p \wedge q) \to s \not\models (p \to s) \vee (s \to q)$, as required.

13. $(p \to q) \wedge (s \to t) \not\models (p \to t) \vee (s \to q)$
Proof
Consider a model M s.t. $W = \{w_0, w_1, w_2\}$, $w_0 R w_0$, $w_1 R w_1$, $w_2 R w_2$, $w_0 R w_1$, $w_0 R w_2$, $|p| = \{w_1\}$, $|q| = \{w_1\}$, $|s| = \{w_2\}$, $|t| = \{w_2\}$, $f(|p|, w_0) = w_1$, $f(|s|, w_0) = w_2$. Then, it follows that $M, w_1 \models q$ and $M, w_2 \models t$. Hence, it follows that $M, f(|p|, w_0) \models p$ and $M, f(|q|, w_0) \models t$. Thus, by the definition of '$\to$' it follows that $M, w_0 \models p \to q$ and $M, w_0 \models s \to t$. Thus, it follows by the definition of '$\wedge$' that $M, w_0 \models (p \to q) \wedge (s \to t)$. Now notice that, since $M, w_1 \not\models t$ and $M, w_2 \not\models q$, it follows by the definition of '$\to$' that $M, w_0 \not\models p \to t$ and $M, w_0 \not\models s \to q$. Thus, it follows by the definition of '$\vee$' that $M, w_0 \not\models (p \to t) \vee (s \to q)$. Thus, $(p \to q) \wedge (s \to t) \not\models (p \to t) \vee (s \to q)$, as required.

14. $\neg(p \to q) \not\models p$
Proof
Consider a model M s.t. $W = \{w_0, w_1\}$, $w_0 R w_0$, $w_1 R w_1$, $w_0 R w_1$, $|p| = \{w_1\}$, $|q| = \emptyset$, $f(|p|, w_0) = w_1$. Then, it follows that $M, w_1 \not\models q$ and hence that $M, f(|p|, w_0) \not\models q$. Thus, by the definition of '$\to$' $M, w_0 \not\models p \to q$, which implies, by the definition of '$\neg$' that $M, w_0 \models \neg(p \to q)$. Notice, however, that since $|p| = \{w_1\}$, it follows that $M, w_0 \not\models p$. Thus, $\neg(p \to q) \not\models p$, as required.

15. $\neg(p \to q) \not\models \neg q$
Proof
Consider a model M s.t. $W = \{w_0, w_1\}$, $w_0 R w_0$, $w_1 R w_1$, $w_0 R w_1$, $|p| = \{w_1\}$, $|q| = \{w_0\}$, $f(|p|, w_0) = w_1$. Then, it follows that $M, w_1 \not\models q$ and hence that $M, f(|p|, w_0) \not\models q$. Thus, by the definition of '$\to$' $M, w_0 \not\models p \to q$, which implies, by the definition of '$\neg$' that $M, w_0 \models \neg(p \to q)$. Notice, however, that since $|q| = \{w_0\}$, it follows that $M, w_0 \models q$ and hence $M, w_0 \not\models \neg q$.
Thus, $\neg(p \to q) \not\models \neg q$, as required.

16. $p \to q \not\models \neg q \to \neg p$
Proof
Consider a model M s.t. $W = \{w_0, w_1\}$, $w_0 R w_0$, $w_1 R w_1$, $w_0 R w_1$, $|p| = \{w_0, w_1\}$, $|q| = \{w_0\}$, $f(|p|, w_0) = w_0$, $f(|\neg q|, w_0) = w_1$. Then, it follows that $M, w_0 \models q$ and hence that $M, f(|p|, w_0) \models q$. Thus, by the definition of '$\to$' $M, w \models p \to q$. Notice, however, that since

$|p| = \{w_0, w_1\}$, it follows that $M, w_1 \models p$ and hence $M, f(|\neg q|, w_0) \models q$.
Thus, it follows by the definition of '$\rightarrow$' that $M, w_0 \not\models \neg q \rightarrow \neg p$.
Thus, $p \rightarrow q \not\models \neg q \rightarrow \neg p$, as required.

A.3 Update Semantics Proofs

In this section we present the results concerning Update Semantics. In our consideration of the results we will use following facts.

Fact 3 *For any sentence θ and state σ, $\sigma[\theta] = \sigma[\theta][\theta]$*

This fact is easily derivable within Update Semantics. Intuitively, it states that accepting the information encoded by θ once is equivalent to accepting it many times (i.e., if we already accept a sentence θ in our information state, then any further update with θ will not change our information state).

Fact 4 *For any two atomic sentences p, q and state σ, $\sigma[p][q] = \sigma[p \wedge q]$.*

That is, updating a state with with a conjunction of atomic sentences is equivalent to updating this state with p and q. This fact follows directly from the semantic definitions for atomic sentences and conjunction.

With these observations, we can proceed to proving the desired inferences.

1. $p \models q \rightarrow p$
   Proof by contradiction.
   Let σ be arbitrary. Suppose for contradiction that $\exists\sigma$ s.t. $\sigma[p] \not\models p \rightarrow q$. Then it follows that $\sigma[p][q][p] \neq \sigma[p][q]$ ↯. This is a contradiction, since it follows by the definition of an update with an atomic sentence that for any state σ supporting an atomic sentence p, $\sigma = \sigma[p]$. Thus, it follows that $p \models q \rightarrow p$.

2. $\neg p \models p \rightarrow q$
   Proof by contradiction
   Let σ be arbitrary. Suppose for contradiction that $\exists\sigma$ s.t. $\sigma[\neg p] \not\models \sigma[p][q]$. Then, it follows that $\sigma[\neg p][p][q] \neq \sigma[\neg p][p]$ $\star$. Notice that it follows by the support definition for atomic sentences and negation that $\sigma \models [p][\neg p]$ iff $\sigma = \emptyset$. Hence, by $\star$ $\emptyset \neq \emptyset[q] = \emptyset$ ↯. Thus, it follows that $\neg p \models p \rightarrow q$.

3. $p \rightarrow s \models (p \wedge q) \rightarrow s$
   Proof by contradiction
   Suppose for contradiction that $\sigma[p \rightarrow s] \not\models (p \wedge q) \rightarrow s$. Then, it follows that $\sigma[p \rightarrow s][(p \wedge q) \rightarrow s] \neq \sigma[p \rightarrow s]$. Hence, it follows by the definition of '$\rightarrow$' that $\sigma[p \rightarrow s][(p \wedge q) \rightarrow s] = \emptyset$ and $\sigma[p \rightarrow s] \neq \emptyset$ $\star$. Let $\tau = \sigma[p \rightarrow s]$. Then, it follows by $\star$ and the definition of '$\rightarrow$' that $\tau[p][q][s] \neq \tau[p][q]$. Hence, $\exists w \in \tau$ s.t. $w(p) = w(q) = 1$ and $w(s) = 0$. ↯. This is a contradiction, since $\tau = \sigma[p \rightarrow s]$ and hence since $w(p) = 1$,

it follows that $w(s) = 1$. Thus, it follows that $p \to s \models (p \wedge q) \to s$, as required.

4. $\models (p \wedge \neg p) \to q$
Proof by contradiction
Let σ be arbitrary and suppose for contradiction that $\sigma \not\models (p \wedge \neg p) \to q$. Then $\sigma[p \wedge \neg p] \not\models q \star$. Now it follows by the definition of '$\wedge$' that $\sigma[p \wedge \neg p] = \sigma[p] \cap \sigma[\neg p] = \emptyset$. Hence, it follows by $\star$ that $\emptyset \not\models q$ $\lightning$. Hence, since σ was arbitrary it follows that $\models (p \wedge \neg p) \to q$, as required.

5. $\models p \to (q \vee \neg q)$
Proof by contradiction
Suppose for contradiction that $\not\models p \to (q \vee \neg q)$. Then, it follows by the definition of entailment that $\exists \sigma$ s.t. $\sigma \not\models p \to (q \vee \neg q)$. Hence, it follows by the definition of the clause for implication that $\sigma[p][q \vee \neg q] \neq \sigma[p]$. Notice, however, that it follows by the definition of disjunction that $\sigma[p][q \vee \neg q] = \sigma[p][q] \cup \sigma[p][\neg q] = \sigma[p]$ $\lightning$. Thus, it follows that $\models p \to (q \vee \neg q)$, as required.

6. $\models p \to (q \to p)$
Proof by contradiction
Suppose for contradiction that $\sigma \not\models p \to (q \to p)$. Then $\exists \sigma$ s.t. $\sigma \not\models p \to (q \to p)$. Thus, it follows by the definition of the support clause for '$\to$' that $\sigma[p] \neq \sigma[p][q \to p]$. Let $\tau = \sigma[p]$. Then it follows by the definition of the support clause for '$\to$' that $\tau[q] = \tau[q][p]$ (since $\tau = \sigma[p]$). Thus, $\tau[q \to p] = \tau$. Hence, it follows that $\sigma[p] = \sigma[p][q \to p]$ $\lightning$. Hence, it follows that $\models p \to (q \to p)$, as required.

7. $p \wedge q \models p \to q$
Proof by contradiction
Suppose for contradiction that $p \wedge q \not\models p \to q$. Then, it follows by the definition of validity that $\exists \sigma$ s.t. $\sigma[p \wedge q] \not\models p \to q$, i.e., by the definition of the clause for '$\wedge$' $\sigma[p][q] \not\models p \to q$. Hence, it follows by the definition of the clause for '$\to$' that $\sigma[p][q][p] \neq \sigma[p][q][p][q]$ $\lightning$. This, is a contradiction since by the definition of the support clause for atomic sentences $\sigma[p][q][p] = \sigma[p][q][p][q]$. Thus, $p \wedge q \models p \to q$, as required.

8. $\not\models (p \to q) \vee (q \to p)$
Proof
Let $\sigma = \{w_1, w_2\}$, $|p| = \{w_1\}$, $|q| = \{w_2\}$. Then, it follows that $\sigma[p] = \{w_1\} \neq \sigma[p][q] = \emptyset$. Hence, by the definition of '$\to$', it follows

that $\sigma[p \to q] = \emptyset$. Similarly, it follows that $\sigma[q] = \{w_2\} \neq \sigma[q][p] = \emptyset$. Hence, by the definition of '$\to$', it follows that $\sigma[q \to p] = \emptyset$. Thus, it follows by the definition of '$\vee$', that $\sigma[(p \to q) \vee (q \to p)] = \emptyset \neq \sigma$. Thus, it follows that $\not\models (p \to q) \vee (q \to p)$, as required.

9. $\neg p \models p \to \neg p$
Proof by contradiction
Suppose for contradiction that $\neg p \not\models p \to \neg p$. Then it follows that $\exists \sigma$ s.t. $\sigma[\neg p] \not\models p \to \neg p$. Now notice that $\sigma[\neg p][p][\neg p] = \emptyset = \sigma[\neg p][p]$. Hence, it follows by the definition of the '$\to$' that $\sigma[\neg p][p \to \neg p] = \sigma[\neg p]$ and hence $\sigma[\neg p] \models p \to \neg p$ $\lightning$. Thus, it follows that $\neg p \models p \to \neg p$, as required.

10. $p \to q, q \to s \models p \to s$
Proof by contradiction
Let σ be arbitrary and consider $\sigma[p \to q][q \to s]$. It follows by the definition of '$\to$' that there are two cases to consider: $\sigma[p \to q][q \to s] = \emptyset$ or $\sigma[p \to q][q \to s] = \sigma$.
Case 1
Notice that $\sigma[p \to q][q \to s] = \emptyset$ if $\sigma[p][q] \neq \sigma[p]$ or $\sigma[p \to q][q][s] \neq \sigma[p \to q][q]$. If any of these holds, then $\sigma[p \to q][q \to s] \models p \to s$ vacuously.
Case 2
Suppose for contradiction that $\sigma[p][s] \neq \sigma[p]$. Then, it follows that $\exists v \in \sigma$ s.t. $v(p) = 1$ and $v(s) = 0$. Now since $\sigma[p][q] = \sigma[p]$ (otherwise we are in case 1), it follows that $v(q) = 1$. Since $\sigma[p \to q][q \to s] = \sigma[p \to q]$ (otherwise we are in case 1), it follows that $v(s) = 1$ $\lightning$.
Thus, since cases 1 and 2 are exhaustive and σ is arbitrary, it follows that $p \to q, q \to s \models p \to s$, as required.

11. $\models p \to (q \to q)$
Proof by contradiction
Suppose for contradiction that $\not\models p \to (q \to q)$. Then, it follows that $\exists \sigma$ s.t. $\sigma[p] \not\models q \to q$. Hence, it follows that $\sigma[p] \neq \sigma[p][q \to q]$. Hence, it follows by the definition of '$\to$' that $\sigma[p][q][q] \neq \sigma[p][q]$ $\lightning$. Thus, it follows that $\models p \to (q \to q)$, as required.

12. $(p \wedge q) \to s \not\models (p \to s) \vee (s \to q)$
Proof
Let $\sigma = \{w_1, w_2, w_3, w_4\}$, $|p| = \{w_1, w_2\}$, $|q| = \{w_1, w_3\}$, $|s| = \{w_1, w_4\}$. Then, it follows by the definition of '$\wedge$' that $\sigma[p \wedge q] = \sigma[p][q] = \{w_1\}$. Thus, it follows that $\sigma[p \wedge q][s] = \{w_1\}[s] = \{w_1\} = \sigma[p \wedge q]$. Hence, by

the definition of the clause for '$\to$', it follows that $\sigma[(p \wedge q) \to s] = \sigma \star$. Notice, however, that $\sigma[p][s] = \{w_1\} \neq \{w_1, w_2\} = \sigma[p]$ (thus, by the definition of the clause for '$\to$', $\sigma[p \to s] = \emptyset$), and that $\sigma[q][s] = \{w_1\} \neq \{w_1, w_3\} = \sigma[q]$ (thus, by the definition of the clause for '$\to$', $\sigma[q \to s] = \emptyset$). Hence, by the definition of '$\vee$', it follows that $\sigma[(p \to s) \vee (q \to s)] \neq \sigma$. Therefore, by $\star$, it follows that $\sigma[(p \wedge q) \to s] \not\models (p \to s) \vee (q \to s)$, as required.

13. $(p \to q) \wedge (s \to t) \not\models (p \to t) \vee (s \to q)$
Proof
Let $\sigma = \{w_1, w_2, w_3, w_4\}$, $|p| = \{w_1, w_2\}$, $|s| = \{w_1, w_3, w_4\}$, $|q| = \{w_1, w_2\}$ and $|t| = \{w_1, w_3, w_4\}$. Then it follows by the definition of '$\wedge$' and '$\to$' that $\sigma[(p \to q) \wedge (s \to t)] = \sigma[(p \to q)] \cap \sigma[(s \to t)] = \sigma \cap \sigma = \sigma \star$. Notice, however, that $\sigma[p][t] = \{w_1\} \neq \sigma[p] = \{w_1, w_2\}$ (and hence $\sigma[p \to t] = \emptyset$) and that $\sigma[s][q] = \{w_1\} \neq \sigma[s] = \{w_1, w_3, w_4\}$ (and hence $\sigma[s \to q] = \emptyset$). Thus, it follows by the definition of '$\vee$' that $\sigma[(p \to t) \vee (s \to q)] = \emptyset \neq \sigma$. Hence, $\sigma \not\models [(p \to t) \vee (s \to q)]$. Therefore, by $\star$ $\sigma[(p \to q) \wedge (s \to t)] \not\models (p \to t) \vee (s \to q)$, as required.

14. $\neg(p \to q) \not\models p$
Proof
Let $\sigma = \{w_1, w_2\}$, $|p| = \{w_1\}$ and $|q| = \{w_2\}$. Then, it follows by the definition of '$\neg$' and '$\to$' that $\sigma[\neg(p \to q)] = \sigma \setminus \sigma[p \to q] = \sigma \setminus \emptyset = \sigma$. Notice, however that $\sigma[p] = \{w_1\} \neq \sigma$. Hence, it follows that $\sigma[\neg(p \to q)] \not\models p$, as required.

15. $\neg(p \to q) \not\models \neg q$
Proof
Let $\sigma = \{w_1, w_2\}$, $|p| = \{w_1\}$ and $|q| = \{w_2\}$. Then, it follows by previous example that $\sigma[\neg(p \to q)] = \sigma$. Notice, however, that $\sigma[\neg q] = \sigma \setminus \sigma[q] = \sigma \setminus \{w_2\} = \{w_1\} \neq \sigma$. Hence, it follows that $\neg(p \to q) \not\models \neg q$, as required.

16. $p \to q \models \neg q \to \neg p$
Proof
Let σ be arbitrary and consider $\sigma[p \to q]$. There are two cases to consider:
Case 1 $\sigma[p][q] \neq \sigma[p]$
Then, it follows by the definition of '$\to$' that $\sigma[p \to q] = \emptyset$. Hence, $\sigma[p \to q] \models \neg q \to \neg p$ vacuously.
Case 2 $\sigma[p][q] = \sigma[p] \star$

Then, it follows by the definition of '$\rightarrow$' that $\sigma[p \rightarrow q] = \sigma$. Now suppose for contradiction that $\sigma[\neg q][\neg p] \neq \sigma[\neg q]$. Then, it follows by the definition of '$\neg$' and the definition of an update with an atomic sentence that $\exists w \in \sigma[\neg q]$ s.t. $w(q) = 0$ and $w(p) = 1$ ↯. This is a contradiction to $\star$, since by $\star$, it follows that $w(q) = 1$. Thus, $\sigma[p \rightarrow q] \models \neg q \rightarrow \neg p$, as required.
Since cases 1 and 2 were exhaustive, it follows that $p \rightarrow q \models \neg q \rightarrow \neg p$, as required.

A.4 Relevance Logic B Proofs

In this section we will give proves of the paradoxical inferences in Relevant Logic B. We will use the semantic tableaux methods defined in Priest[33]. The tableaux method is sound and complete with regards to the semantics.

In order to test whether the premises imply the conclusion, we assume that the premises hold and conclusion does not; and check for consistency. Whenever all branches of a tableaux close ($\times$), then tested inference holds; and whenever one of the branches is open (O), then the inference tested does not hold. A branch of a tableaux closes when there is a pair $\theta, +x$ and $\theta, -x$ on this branch. If one of the branches is open, the counter-model for the inference tested can be directly read from it.

In comparison to the tableaux for S2, one of the new features is that nodes may now be of the form $\theta, +x$ or $\theta, -x$, where x is either i or i^* and whichever of these it is, $\bar{x}$ is the other (for a natural number i). i^* denotes the star world of i. In order to model the ternary relation some nodes of the form $Rxyz$ are introduced.

Notice that in the rule for '$\theta \rightarrow \psi, +x$', y and z are anything of the form j or j^*, whereas in the rule for '$\theta \rightarrow \psi, -x$', j and k are new. Moreover, as required by the normality condition, if $x = 0$, i.e., is a normal world, then j and k are the same. Notice that the last rule completes the normality condition, where x is of the form j or j^*.

As noticed by Priest, the complete rule for the normality condition causes much clutter, so whenever there are no nodes of the form '$\theta \rightarrow \psi, +0$', we will not apply the last tableaux rule. This is legitimate since nodes of the form '$\theta \rightarrow \psi, +0$' are the only nodes that trigger the application of the last rule. Furthermore, in more complicated tableaux examples, for the purposes of simplicity, we will focus on demonstrating that there exists an open branch and omit other derivations. Whenever we omit any derivations, we will use '$\cdots$' to indicate that there are still some tableaux rules that need to be applied for the branch in question to be complete. Clearly, this is legitimate, since an occurrence of a single open branch is sufficient to specify a counter-model.

The tableaux system for Relevance Logic B uses the following rules:

$\theta \to \psi, +x$ $Rxyz$ $\neg\theta, -y \quad \psi, +z$	$(\theta \to \psi), -x$ $\mid$ $Rxjk$ $\theta, +j$ $\psi, -k$	$\neg\theta, +x$ $\mid$ $\theta, -\bar{x}$
$\theta \lor \psi, +x$ $\theta, +x \quad \psi, +x$	$(\theta \lor \psi), -x$ $\mid$ $\theta, -x$ $\psi, -x$	$\theta \land \psi, +x$ $\mid$ $\theta, +x$ $\psi, +x$
$(\theta \land \psi), -x$ $\theta, -x \quad \psi, -x$	$\neg\theta, -x$ $\mid$ $\theta, +\bar{x}$	. $\mid$ $R0xx$

Below we present the proofs of the 16 inferences we considered in chapter 2. Similarly as in S2, we will explain the tableaux method using the proof for the first inference.

1) $p \not\models q \to p$

$$p, +0$$
$$q \to p, -0$$
$$\mid$$
$$R011$$
$$q, +1$$
$$p, -1$$
$$O$$

Thus, (1) does not hold in B.

Explanation

Similarly as in S2, at the first node we assume the premises and the negation of the conclusion at the actual world (i.e., $p, +0$ and $q \to p, -0$). Then, by the normality condition and the definition of '$\to$', we introduce another node $R011$, $q, +1$, $p, -1$. Since there are no nodes of the form '$\theta \to \psi, +0$', we can skip the application of the last tableaux rule. The branch is open, so $p \not\models q \to p$.

We can read a counter-models from the open branch. The counter-model is given by: $W = \{0, 1, 0^*, 1^*\}$, $N = \{w_0\}$, $R011$, $R00^*0^*$, $R01^*, 1^*$ (the last two accessibility relations follow by the normality condition), $w_0 \to w_0^*$, $w_1 \to w_1^*$, $v_0(p) = 1$, $v_1(q) = 1$, $v_1(p) = 0$ and the values of p and q at 0^* and 1^* are arbitrary.

2) $\neg p \not\models p \to p$

$$\begin{array}{c} \neg q, +0 \\ q \to p, -0 \\ | \\ q, -0^{*} \\ | \\ R011 \\ p, +1 \\ q, -1 \\ O \end{array}$$

Thus (2) does not hold in B.

3) $p \to s \models (p \wedge q) \to s$

$$\begin{array}{c} p \to s, +0 \\ (p \wedge q) \to s, -0 \\ | \\ R011 \\ p \wedge q, +1 \\ s, -1 \\ | \\ p, +1 \\ q, +1 \\ \swarrow \quad \searrow \\ p, -1 \qquad s, +1 \\ \times \qquad\quad \times \end{array}$$

Thus (3) holds in B.

4) $\not\models p \wedge \neg p \rightarrow q$

$$p \wedge \neg p \rightarrow q, -0$$
$$|$$
$$R011$$
$$p \wedge \neg p, +1$$
$$q, -1$$
$$|$$
$$p, +1$$
$$\neg p, +1$$
$$|$$
$$p, -1^*$$
$$O$$

Thus (4) does not hold in B.

5) $\not\models p \rightarrow (q \vee \neg q)$

$$p \rightarrow (q \vee \neg q), -0$$
$$|$$
$$R011$$
$$p, +1$$
$$q \vee \neg q, -1$$
$$|$$
$$q, -1$$
$$\neg q, -1$$
$$|$$
$$q, +1^*$$
$$O$$

Thus (5) does not hold in B.

6) $\not\models p \rightarrow (q \rightarrow p)$

$$p \rightarrow (q \rightarrow p), -0$$
$$|$$
$$R011$$
$$p, +1$$
$$q \rightarrow p, -1$$
$$|$$
$$r123$$
$$q, +2$$
$$p, -3$$
$$O$$

Thus, (6) does not hold in B.

7) $p \wedge q \not\models p \rightarrow q$

$$
\begin{array}{c}
p \wedge q, +0 \\
p \rightarrow q, -0 \\
| \\
p, +0 \\
q, +0 \\
| \\
R011 \\
p + 1 \\
q, -1 \\
O
\end{array}
$$

Thus, (7) does not hold in B.

8) $\not\models (p \rightarrow q) \vee (q \rightarrow p)$

$$
\begin{array}{c}
(p \rightarrow q) \vee (q \rightarrow p), -0 \\
| \\
p \rightarrow q, -0 \\
q \rightarrow p, -0 \\
| \\
R011 \\
p, +1 \\
q, -1 \\
| \\
R022 \\
p, +2 \\
q, -2 \\
O
\end{array}
$$

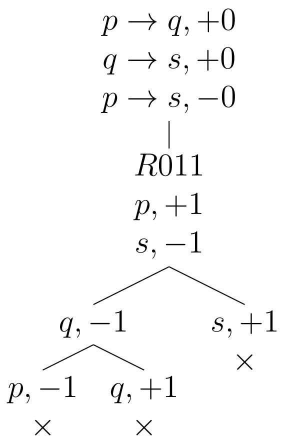
Thus, (8) does not hold in B.

9) $\neg p \not\models p \to \neg p$

$$\neg p, +0$$
$$p \to \neg p, -0$$
$$|$$
$$p, -0^*$$
$$|$$
$$R011$$
$$p, +1$$
$$\neg p, -1$$
$$|$$
$$p, +1^*$$
$$O$$

Thus, (9) does not hold in B.

10) $p \to q, q \to s \models p \to s$



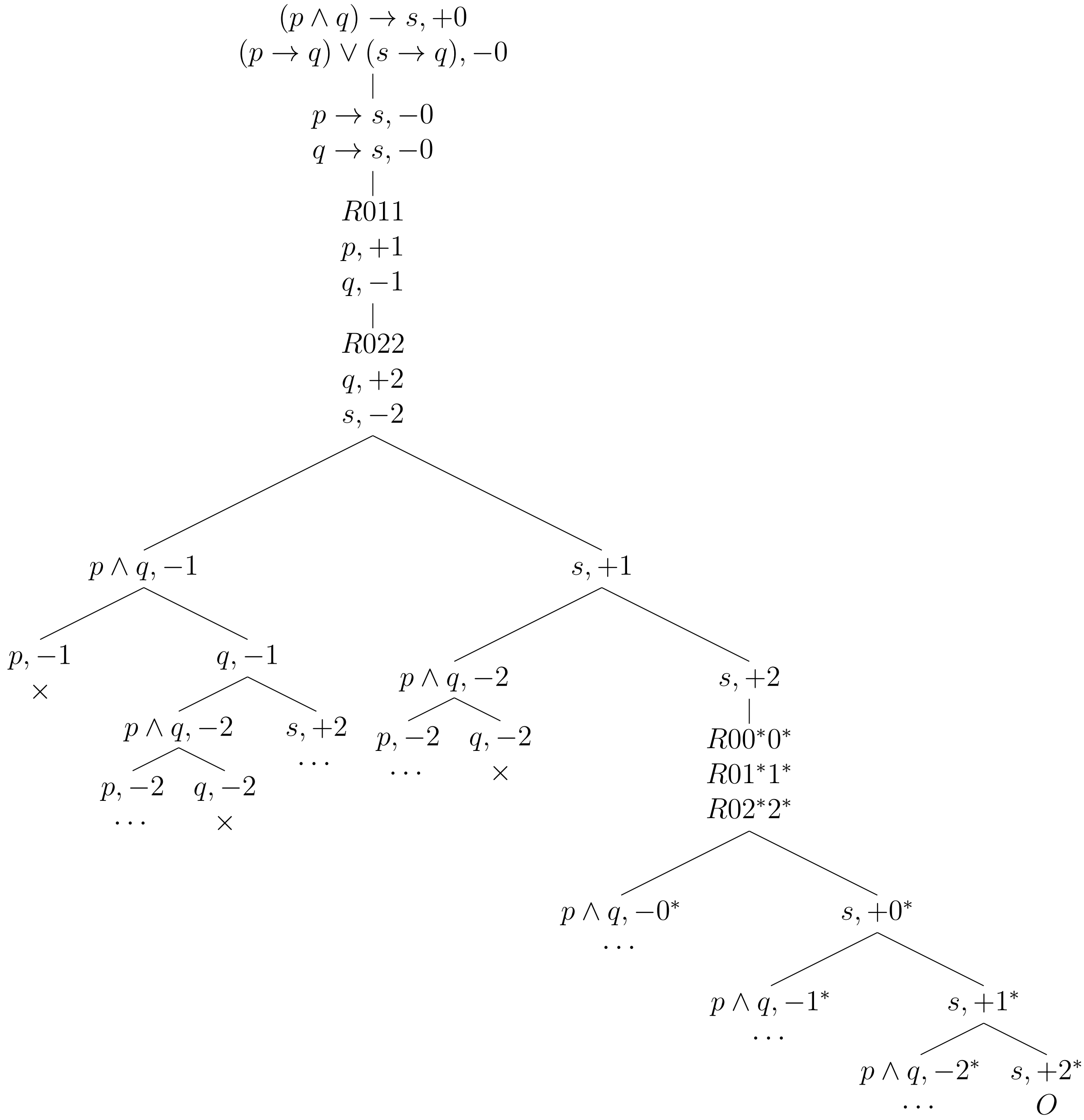
Thus, (10) holds in B.

11) $\not\models p \to (q \to q)$

$$
\begin{array}{c}
p \to (q \to q), -0 \\
| \\
R011 \\
p, +1 \\
q \to q, -1 \\
| \\
R123 \\
q, +2 \\
q, -3 \\
O
\end{array}
$$

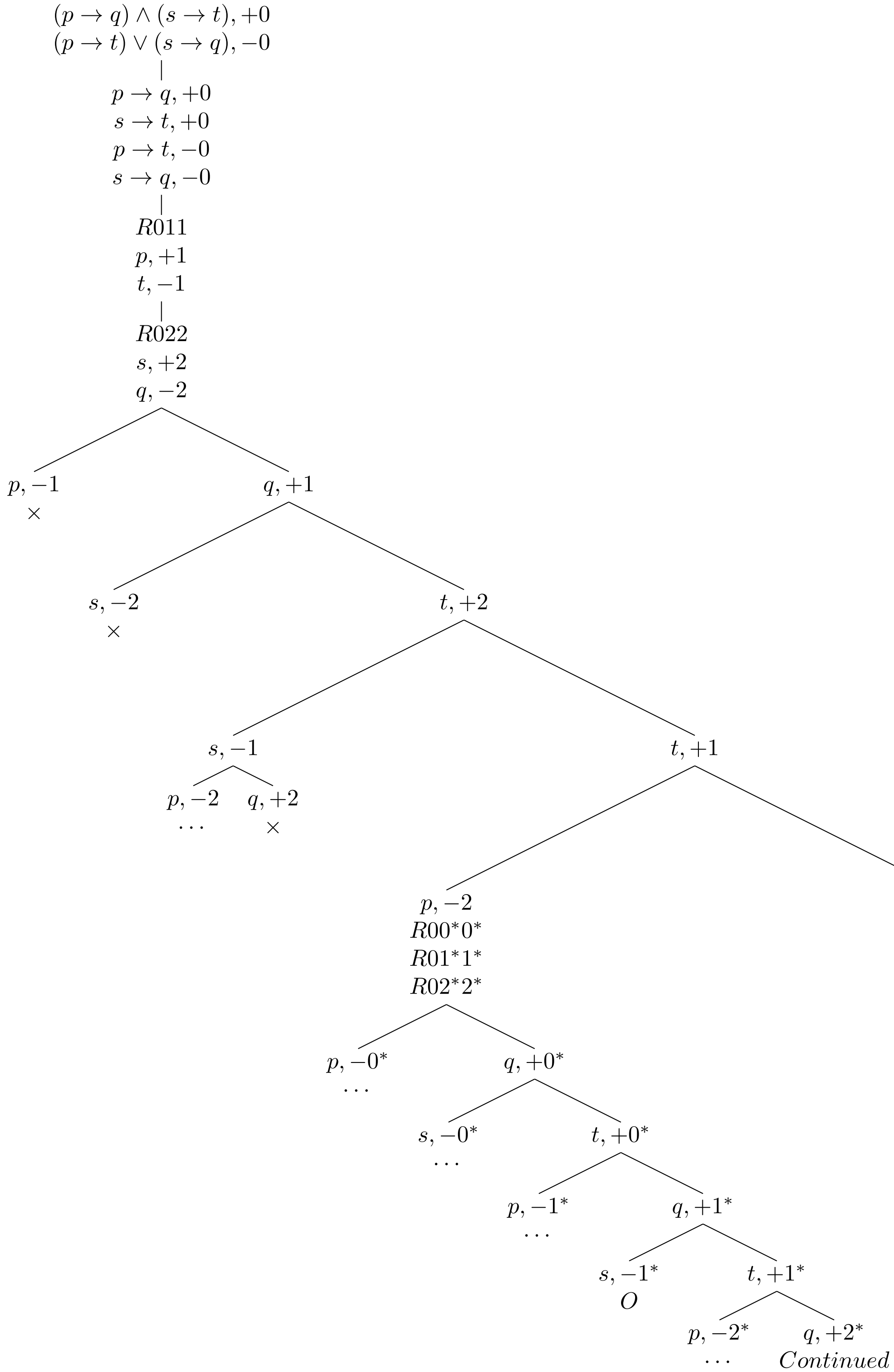
Thus, (11) holds in B.

12) $(p \wedge q) \to s \not\models (p \to q) \vee (s \to q)$



Thus, (12) does not hold in B.

13) $(p \to q) \wedge (s \to t) \not\models (p \to t) \vee (s \to q)$


(p → q) ∧ (s → t), +0
(p → t) ∨ (s → q), −0
p → q, +0
s → t, +0
p → t, −0
s → q, −0
R011
p, +1
t, −1
R022
s, +2
q, −2
p, −1
×
q, +1
s, −2
×
t, +2
s, −1
t, +1
p, −2
. . .
q, +2
×
p, −2
R00*0*
R01*1*
R02*2*
q, +2
×
p, −0*
. . .
q, +0*
s, −0*
. . .
t, +0*
p, −1*
. . .
q, +1*
s, −1*
O
t, +1*
p, −2*
. . .
q, +2*
Continued

Where the branch named *Continued* splits into two branches: one with the node $s, -2^*$ on it and the other with the node $t, +2^*$ on it. Both of these branches are open. Thus, (13) does not hold in B.

14) $\neg(p \rightarrow q) \not\models p$

$$\begin{array}{c} \neg(p \rightarrow q), +0 \\ p, -0 \\ | \\ (p \rightarrow q), -0^* \\ | \\ R0^*12 \\ p, +1 \\ q, -2 \\ O \end{array}$$

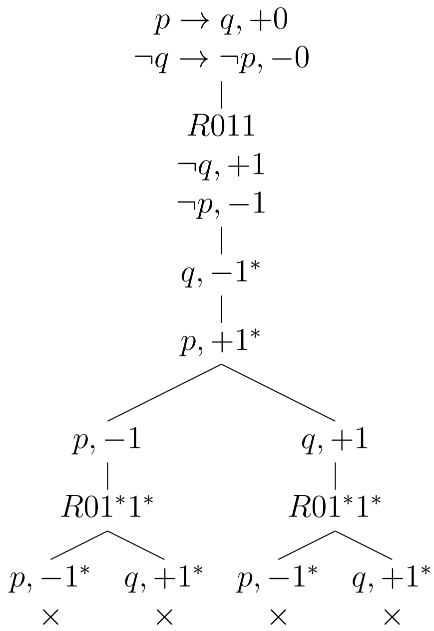
Thus, (14) does not hold in B.

15) $\neg(p \rightarrow q) \not\models \neg q$

$$\begin{array}{c} \neg(p \rightarrow q), +0 \\ \neg q, -0 \\ | \\ q, +0^* \\ | \\ p \rightarrow q, +0^* \\ | \\ R0^*12 \\ p, +1 \\ q, -2 \\ O \end{array}$$

Thus, (15) does not hold in B.

16) $p \to q \models \neg q \to \neg q$



Thus, (16) holds in B.

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