Kant's Transcendental Synthesis of the Imagination and Constructive Euclidean Geometry

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written by

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Abstract

ILLC

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Friedman [1, 2] claims that Kant's constructive approach to geometry was developed as a means to circumvent the limitations of his logic, which has been widely regarded by various commentators as nothing more than a *glossa* to Aristotelian subject-predicate logic. *Contra* Friedman, and building on the work of Achourioti and van Lambalgen [3], we purport to show that Kant's constructivism draws its independent motivation from his general theory of cognition. We thus propose an exegesis of the Transcendental Deduction according to which the consciousness of space as a formal intuition of outer sense (with its properties of, e.g., infinity and continuity) is produced by means of the activity of the transcendental synthesis of the imagination in the construction of geometrical concepts, which synthesis must be in thoroughgoing agreement with the categories. In order to substantiate these claims, we provide an analysis of Kant's characterization of geometrical inferences and of geometrical continuity, along with a formal argument illustrating how the representation of space as a continuum can be constructed from Kantian principles.

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Chapter 1

Groundwork

§1.1 Introduction

Friedman [1, 2] claims that Kant's constructive approach to geometry was developed as a means to circumvent the limitations of his logic, which has been widely regarded by various commentators as nothing more than a *glossa* to Aristotelian subject-predicate logic, containing moreover numerous imperfections[1]. Indeed, Friedman himself holds the view that Kant's logic is essentially monadic, and thus incapable of expressing relations or quantifier dependencies. This view on Kant's logic has been recently challenged by Achourioti and van Lambalgen [3]. The two authors, on the basis of the accurate exegesis of the forms of judgment presented in the *Critique of Pure Reason*[2] developed by Longuenesse [5], have developed a formalization of Kant's transcendental logic which provides strong evidence supporting the conclusion that these forms of judgment can be represented by the class of restricted $\forall\exists$ formulas of first order logic. Thus, Kant's logical forms seem to be much more complex than what the received philosophical wisdom implies.

The present work starts from these observations, and attemps to develop a consistent and coherent exegesis of Kant's theory of geometrical space. In particular, since theorems in Euclidean geometry can be expressed in the forms of restricted $\forall\exists$ statements, it then follows that the intrinsic poverty of logical tools is not an appropriate argument to explain Kant's constructive approach to geometry; a different analysis must then be developed to account for it, which does not consider it a mere expedient, but that grounds it on his very theory of cognition.

[1] A paradigmatic example of this view of Kant's logic is [4], pp. 74-82.

[2] Henceforth: CPR, or Critique.

We shall then propose an exegesis of the Transcendental Deduction according to which the consciousness of space as a formal intuition of outer sense (with its properties of, e.g., infinity and continuity) is produced by means of the activity of the transcendental synthesis of the imagination in the construction of geometrical concepts, which synthesis must be in thoroughgoing agreement with the categories. Construction of geometrical objects in pure intuition is then what first makes us aware of space itself and of its properties, and is the original ground of the applicability of geometrical concepts to the world of appearances, hence of their objective reality. This entire work can be considered as an explanation of this latter statement.

The work is divided into two chapters. The first chapter focuses on the construction of an exegesis of the Transcendental Deduction from the ground up, starting with the most basic notions of concepts and intuitions, and culminating with the transcendental synthesis of the imagination. The reader will notice that our rendition of the core argument of the Deduction is more in agreement with its exposition as it appears in the A edition than in the B edition. This is due to the fact that we consider the treatment in the A edition, in which one starts from the empirical world of appearances, and builds his way up to the consideration of the functions of the understanding which are necessary for it to be an experience, more perspicuous than the approach employed in the B edition, of starting from such necessary functions of the understanding, and only then investigating their application to the appearances.

The second chapter considers Kant's theory of geometry more in detail. We argue that system E, proposed by Avigad *et al.*[3] as a constructive formalization of Euclidean geometry, captures adequally numerous aspects of Kant's own theory of geometrical reasoning. We also argue, however, that it is only partially constructive, in that the interpretation of some of its axioms seems to presuppose a given infinite universe of points. This fact is in turn interpreted as an instance of a more general problem, i.e., that of giving an adequate constructive justification for the fact that we conceive of space as a continuous medium, in which we are allowed to choose freely among an infinity of points. We then analyse Friedman's exegesis more in detail, in particular with respect to the notions of the continuity of space and of geometrical point. We conclude the chapter with a formal argument illustrating how the representation of space as a continuum can be constructed from Kantian principles.

[3]See [6].

§1.2 The logical architecture of the Critique of Pure Reason

The Critique has at its core a complex philosophical architecture which is built on a limited set of fundamental notions. These notions constitute the backbone of Kant's theory of the mind and provide the technical apparatus necessary to its development. Since Kant's theory of mathematics, and in particular of geometrical space, is essentially a cognitive (albeit not psychological) theory, it follows that it is impossible to understand Kant's philosophy of mathematics without an appropriate analysis of his theory of the mind and of its core notions. In other words, if one wants to understand Kant's theory of geometry, one must first understand his treatment of time and space as forms of intuition in general, as that on which the former theory is developed; and this, in turn, involves coming to terms with the architecture of the mind expounded in the CPR, in particular in the most obscure *loci* of the Transcendental Deduction. Before we even try to unfold Kant's theory of geometry, in order to evaluate its relevance to various contemporary debates relating to the nature of space, we must then develop the appropriate philosophical analysis of his overall framework for human cognition.

I believe that the numerous difficulties that the interpreter faces in trying to construct a coherent hermeneutical framework for Kant's theory of the mind are due more to the complex and dynamic interplay between the relevant notions than to a lack of clarity in their definition. Kant himself openly acknowledged these obscurities in his work[4], which he justified by appealing to the intrinsic difficulty and profundity of the problems he treated. Be it as it may, it is undeniable that the construction of such an hermeneutical framework is a consuming enterprise, which needs to be carried out methodically rather than rhapsodically. The steps involved in the process are essentially three. First, one draws a *prima facie* interpretation of the notion at hand, relying on those passages which seem most relevant, and which are taken usually from the CPR. Second, one gathers ulterior evidence from other sources, such as, e.g., the lectures Kant gave to his students, and then revises the *prima facie* interpretation in light of this evidence. Third, one tries to accomodate the interpretation which has been achieved with that of the other parts of the system, in order to obtain a coherent and consistent whole. In the following sections we shall then try to develop an interpretative framework according to this method, as a basis for understanding Kant's theory of geometry. Notice that it is unfortunately impossible to give here an introductory account of Kant's philosophy; consequently,

[4]Consider, for instance, the letter from Kant to Garve dated August 7, 1783 ([7], 51, [205], (10:339)). All the letters cited in this work are taken from the Cambridge edition of Kant's Correspondence, and are cited with the number corresponding to them in that edition, along with that of the academy edition in square brackets.

acquaintance with the CPR, with the other Kantian writings of the critical period, and with the relevant commentary will be assumed without ceremony. The unfamiliar reader might thus want to consult [8], Kant's introduction to his own philosophy.

§1.3 The dichotomy concept-intuition

The fundamental distinction at the heart of the kantian theory of the mind is, of course, that between *concepts* and *intuitions*. Concepts are universal and discursive representations, i.e., representations that are thought as common to several objects. Intuition, on the contrary, are singular representations:

> An intuition is a *singular* representation, (*repraesentatio singularis*), a concept a universal (*repraesentatio per notas communes*) or *reflected* representation (*repraesentatio discursiva*)[5].

The dichotomy between concepts and intuitions is clarified, in a taxonomical fashion, in the CPR (the passage is known under the name *"stufenleiter"*, or ladder):

> The genus is representation in general (*repraesentatio*). Under it stands the representation with consciousness (*perceptio*). A perception that refers to the subject as a modification of its state is a sensation (*sensatio*); an objective perception is a cognition (*cognitio*). The latter is either an intuition or a concept (*intuitio vel conceptus*). The former is immediately related to the object and is singular; the latter is mediate, by means of a mark, which can be common to several things[6].

Thus, objective perceptions (cognitions) are either:

1. intuitions: immediately related to the object; singular
2. concepts: mediately related to the object by means of marks ; universal; discursive

[5] [9], 91. See also [9], 33, for a treatment of the notions of concept, intuition and representation. Another interesting characterization of this last term appears in [10], *Notes on logic*, R1676, (16:76), p. 34:

> Rapraesentatio est determinatio mentis (interna), quatenus ad res quasdam ab ipsa (nempe repraesentatione) diversas refertur.

From this definition, it is apparent that the fundamental feature of a representation, for Kant, is the capability of it to refer to external objects (different from itself). All the references to Kant's *marginalia* are taken from the Cambridge edition of Kant's notes and fragments.

[6] [11], A320/B377

Let us indicate with body^c the concept "body" and with body^i the intuition of a body. Then body^c is thus distinguished from the body^i in somewhat the same fashion as universal properties are distinguished from their instantiations. While body^i is singular and spatio-temporally located (*that* body *there*), and immediately refers to an object without the need to recur to other representations, body^c is instead a representation that can be predicated of other representations (hence discursive and universal), thus it is not spatio-temporally located and can relate to the object only through the medium of other representations, being either other concepts or singular intuitions[7]. Of course, intuitions and concepts are subjected to the usual categorization into pure and empirical.

The notion of a *mark*, which is mentioned in the above definition, is of complex interpretation. Kant defines a mark as:

> A mark is a partial representation, which as such is a ground of cognition. It is either intuitive (synthetic part): a part of the intuition, or discursive: a part of the concept, which is an analytical ground of cognition. *Vel intuitus vel conceptus partialis*[8].

Some commentators [12, 13] have held the view that marks are essentially universal properties, and that intuitions refer immediately to object in virtue of the fact that they do not refer to objects through marks. However, Smit [14] has argued for the existence of two different notions of mark, namely, *intuitive marks* and *discursive marks*. The textual evidence is in agreement with Smit's interpretation[9]. A mark, according to Kant, is a partial representation conceived as a ground of cognition. Since a representation is a *determinatio mentis* which refers to something external (see the footnote above), a representation can be a ground of cognition only if it refers to an external object. It follows that a mark, whether intuitive or discursive, is a partial representation which refers to an external object: a *cognitio partialis*. Intuitive marks are nothing other than singular instances of properties, representing the building blocks which make up the content of our determinate intuitions. For example, the computer screen in front of me has

[7] [11], A68/B93:

> Since no representation other than intuition goes immediately to an object, a concept is never related immediately to an object, but rather to some other representation of the same [object] (be it an intuitionor itself already a concept).

[8] [10], *Notes on logic*, R2286, (16:299), p. 41.

[9] Apart from the passage above, there is widespread evidence for the existence of this notion of intuitive marks. For instance, Kant writes at [10], *Notes on logic*, R2282 (16:298), p. 41:

> A mark is not always a concept of a thing, but often only part of a thing. E.g., the hand is a mark of a human; but having hands is only this mark as concept of a human [...]

Making use of our notation, we would have that hand^i is a mark of human^i, as has hands^c is part, as a mark, of human^c.

the property of being rectangular; this instance of the property constitutes my intuition (is rectangular)^i, which is a part of my intuition of the screen itself, and, since intuitions refer to objects immediately, it is an intuitive mark (partial intuitive representation) of the object $= X$ to which screen^i refers. Analogously, the general (universal) property of being rectangular is the content of (has a rectangular shape)^c, which is a discursive mark (partial discursive representation) by means of which I represent the screen object in front of me. Marks are then partial representations, either intuitive or discursive, by means of which I refer to objects.

The difference between the intuition red^i and the concept red^c resides then not that much in the content of the two representations, i.e., the "redness" property, but in the different form of these two representations; in the fact that while the former is singular and spatio-temporally located (space and time are the "forms of intuition"), the latter is universal and discursive, and can be predicated of many other different representations. That the distinction between concepts and intuitions is to be regarded as a formal distinction is confirmed by the analysis that Kant provides of the process which leads us to the formation of new concepts from other concepts or intuitions[10]. According to Kant, three functions are involved in the process:

1. comparison: different representations are compared and their difference is established in terms of their different intuitive/discursive marks
2. reflection: different representations are compared and their similarity is established in terms of their common intuitive/discursive marks
3. abstraction: a concept is formed by "abstracting away" from the marks that differ in the representations

It is the application of these logical functions that guides the formation of any concept, and therefore of any discursive mark, from other representations. In particular, it seems that this process guides the formation of concepts from intuitions: different manifolds of intuitions are compared and reflected in order to individuate their common properties, i.e., the similarities of their intuitive marks both in terms of content and in terms of form (spatio-temporal configuration), while abstracting away from their differences. In this fashion, properties that can be predicated of other representations are obtained, that go make up the content of concepts (discursive marks). Kant's example is instructive:

> I see, e.g., a spruce, a willow, and a linden. By first comparing these objects with one another I note that they are different from one another in regard to

[10] See the discussion at [9], 94.

> the trunk, the branches, the leaves, etc.; but next I reflect on that which they have in common among themselves, trunk, branches, and leaves themselves, and I abstract from the quantity, the figure, etc., of these; thus I acquire a concept of a tree[11].

What I take Kant to mean in this passage is that the three intuitions (the spruce, the linden, and the willow) undergo a process of comparison-reflection-abstraction with respect to their intuitive marks which isolates those that are common to all three, such as, e.g., the property of having a trunk: an intuitive mark with a certain shape and a certain spatial relation with respect to the other intuitive marks that constitute the intuition. These intuitive marks which have been thus individuated as similar are then turned into discursive marks by considering their content under the form of universal properties that can be predicated of different representations; the mark, e.g., (has a trunk)^c, is so produced. Finally, all these discursive marks together form the intension of the concept tree^c. Furthermore, as we shall see more in detail later, the concept tree^c, as any other concept, is nothing else than a rule which specifies, thanks to the discursive marks of which it is composed, the possible objects of experience to which it can be applied. In light of this mechanism for the production of new concepts from previous representations, and in particular from intuitions, we see that the difference between tree^i and the concept tree^c is mainly formal, in that the former is subjected to the forms of intuition (so that its intuitive marks are spatio-temporally located), while the latter has a discursive form, that is, it consists in a series of discursive marks, or *notae communes*. In other words, the crucial difference between an intuition and the corresponding concept is that the former is a sensible representation of a configuration of perceptions, which we could quite appropriately describe, using contemporary metaphysical jargon, as a configuration of "tropes", while the latter is the *thought*, by means of discursive marks, of such a configuration.

§1.4 Synthesis and acts of combination

The pivotal section of the CPR is the Transcendental Deduction, in which Kant tries to ground the applicability of the categories to the manifold of intuition, i.e., to justify our use of the categories by showing that human cognition is such that these must necessarily be applicable to the manifold of intuitions. The Transcendental Deduction B opens with the definition of acts of combination[12]. An *act of combination* is a spontaneous act of the understanding which:

[11] [9], 95.
[12] [11], B130.

1 involves the *synthesis in general* or putting together of representations

2 involves the *unification* of representations, where unification means that the manifold becomes a *synthetic unity*.

We see that this is a very general characterisation. In particular, if we just take into consideration the above two defining traits, it follows that an act of combination can involve the unification of concepts, but also of sensible intuitions alike (since both are representations)[13]. Given that for Kant the only way of unifying concepts is by means of a judgment, it also follows that a judgment must certainly be regarded as an act of combination according to the above definition. However, in order to achieve a full understanding of this notion, we need to know (I) what a synthesis in general is and how it is brought about, (II) to what kind of synthetic unity is Kant referring here, (III) what other (if any) types of combination exist, apart from combination of concepts in judgments. As far as point (I) and (III) are concerned, one finds the following:

> By **synthesis** in the most general sense, however, I understand the action of putting different representations together with each other and comprehending their manifoldness in one cognition.[14]

Synthesis in general consists in taking together a multiplicity of representations and comprehending them into *one* cognition. Now a representation, we have seen, can be either an intuition or a concept. Consequently, one might *prima facie* interpret the notion of combination as follows. Let $R = \{r_1, ..., r_n\}$ be a set of representations, and let us denote with $s(R)$ the synthesis in general of the representations contained in R. Without loss of generality, we can restrict us to the case in which $R = \{r_1, r_2\}$, since the cases in which $|R| > 2$ can be reduced to this by application of recursion[15]. The operation $s()$ of synthesis in general applied to the two representations r_1, r_2 unifies them into a new representation r_3, which is the product of this act of synthesis. Three cases might occur:

(a) r_1 and r_2 are concepts,

(b) r_1 and r_2 are intuitions,

(c) r_1 is an intuition and r_2 is a concept.

[13] Indeed, Kant states in the same passage that an act of combination can involve the putting together of a manifold of intuition, whether sensible or non-sensible, or of several concepts.

[14] [11], B103.

[15] Indeed, Kant uses recursion in the case of combination of motions: see [15], [489], p. 24.

If (a) holds, then the synthesis of r_1, r_2 is an act of judgment. Consider the following passage from Kant's *Reflexionen*, dated 1769:

> In all judgments of the understanding things are like this.[...] If anything x, which is cognized by means of a representation a, is compared with another concept b, as either including or excluding this concept, then this relation is in the judgment. This judgment is thus either the cognition of agreement or of opposition, so that in the thing x, which I know by means of the concept a, either b is contained as a partial concept and thus x, which is cognized by means of a, can also be cognized by means of b, or x negates the concept of b.[16]

A judgment unifies two concepts by subordinating one to the other with respect to a certain relation, and thereby produces a new partial singular concept of the object x. In other words, if we cognize object x through the discursive mark y^c, then by means of a judgment of the form; every y^c is z^c, we unify y^c and z^c in the representation of the object x, so that now whenever x is cognized through y^c it is also cognized through z^c. This means that the two representations y^c and z^c are unified, as discursive marks, in the partial representation of object x[17]. Consider, e.g., the following:

> If I say: a body is divisible, this means the same as: Something x, which I cognize under the predicates that together comprise the concept of a body, I also think through the predicate of divisibility[18]

What Kant seems to be saying here is that through the judgment "every body is divisible", body^c is subordinated to the concept divisible^c, in such a way that the partial concept of an object x to which body^c belongs as a discursive mark is enlarged so as to comprise the mark divisible^c; in other words, by means of the judgment the two concepts

[16] [10], *Notes on Metaphysics*, R 3920, (17:344), p. 94.

[17] This is why Kant insists (see [10], *Notes on Metaphysics*, R 3921, (17:346)) that with a predicate I do not represent a part of the thing or have a concept of the part, but I have a partial representation of the whole object x.

[18] [10], *Notes on Metaphysics*, R 4634, (17:616).

are connected in such a way that the partial concept of an object x containing the mark body^c is also "updated" with the mark divisible^c[19].

If (c) holds, then the process of synthesis must be that of the recognition of the intuition under a concept. The Kantian rendition of this process is a complicated one, which would require in itself an in-depth analysis that would bring us too astray. However, the suggested interpretation of the process can be delineated as follows. Given an intuition x^i and a concept x^c, x^i can be subsumed under x^c either through an act of judgment (that is, by means of a mediating concept m^c), or through the *schema* of the concept as hinted in the chapter on the Schematism of Pure Reason[20]. For instance, if x^i is the intuition of a body, then through the judgment "all bodies are divisible", x^i is subsumed under the concept divisible^c. However, the only sensible way to subsume an intuition x^i under the concept body^c is that of recurring to a schema, i.e., to a rule which exemplifies the concept: body^c is something which makes necessary the representation of the marks of extension, impenetrability, shape, and it is through these marks the intuition x^i can be subsumed under it:

> [...] Thus the concept of a body serves as the rule for our cognition of outer appearances by means of the unity of the manifold that is thought through it [...] thus in the case of the perception of something outside of us the concept of body makes necessary the representation of extension, and with it that of impenetrability, of shape, etc.[21]

In other words, a concept, such as body^c, is a rule by means of which we think the synthetic unity of a manifold of representations, i.e., that the representations in the manifold (extension, impenetrability, etc.) actually constitute one and relate to the

[19] The object x has a crucial role in the case of synthetic judgment, since if the judgment is analytic (as in the example above) then it is actually only the relation between the two concepts which is relevant, and not their connection in the concept of the object ([10], *Notes on Metaphysics*, R 4676, (17:655), p.165.):

> But if a and b are not identical [...] and x is not thought entirely determinately through the concept of a, then a and b are not in a logical but in a real relation (something different) of combination [...]. Thus their relation is not determined through their concepts themselves, but rather by means of the x, of which a contains the designation. How are such syntheses possible? x must be a *datum* of sensibility, in which a synthesis, i.e., a relation of coordination, takes place; for this contains more than is thought through its concept a, and is the representation of a *in concreto*.

In other words, if a^c is not the singular concept of the object x (so that it does not determine x completely), and b^c is not contained in a^c as a *pars*, then the combination of the two concepts relies on the object x, of which a^c is only a mark, a designation, a partial representation. The x is an intuition or an appearance, brought about by means of a synthesis of intuitions or appearances (which coordinates them in space and time), representing a^c *in concreto* - as we will see in the case of geometry, it is the sensible *correlatum* which gives a^c its objective validity.

[20] [11], A141.

[21] [11], A106.

same object x. Notice that, for Kant, if several cognitions are considered as one this means that when one of them is posited, then the others are also posited along with it.[22]. A concept then signifies just a unity of a set of representations and is useful in that it allows us to bring synthetic unity to the manifold of intuitions, by subsuming the latter under the former. This is the reason why Kant states that in order to cognize a line in space, I must draw it (which, as we will see in the sequel, is a procedure of construction) so as to combine a manifold (i.e., the manifold of its parts) in such a way that this manifold is unified (subsumed) in the concept of a line[23].

If (b) holds, then the synthesis involves a combination of two intuitions r_1, r_2 into a different, composite intuition r_3. For instance, the synthesis of my intuition of a rectangular shape and of this specific shade of white involves combining them together, as intuitive marks, in order to produce my (partial) intuition of this specific piece of paper. Analogously, my intuition of this body standing before me involves the combination of its shape and colours, along with the intuition of heaviness^i, which, since I previously combined its concept with that of body^c as always appearing together in the object, is now *anticipated* of the object itself by means of the empirical imagination (as we shall later investigate). This type of combination, however, does not necessarily correspond to a cognition, as for the latter a concept is always needed under which the intuition so combined can be subsumed.

It is clear that since the syntheses illustrated at (a) and (c) must involve a process of judgment or subsumption under a concept, they therefore require the full power of the understanding in thinking *per notas communes*, and hence they are also chiefly intellectual. This however is not necessarily the case as far as the synthesis presented in (b) is concerned, in which the combination in the manifold is brought about by recurring to neither judgments nor subsumption under concepts, although concepts, which as we have seen must essentially be conceived as rules, must play a role in guiding this synthesis, favouring certain combinations of intuitions in place of others. The synthesis at (b), then, can be characterized as a process of binding of intuitive or perceptual features. Notice that, of course, in those cases in which the manifold to be synthesized does not consist of empirical representations, but only of pure *a priori* ones (*a priori* intuitions in space and time), then in (a) to (c) above we must talk of *pure a priori* syntheses.

The notion of synthesis in general is thus multifaceted. In the Transcendental Deduction A we find the following passage:

[22][10], *Notes on Logic*, R 3044, (16:629), p. 59.
[23][11], B138.

> Synthesis in general is [...] a mere effect of the imagination, of a blind though indispensable function of the soul [...] yet to bring this synthesis to **concepts** is a function which pertains to the understanding [...][24]

I take it that Kant is here (albeit confusingly) pointing at the two senses (b) and (c) of synthesis in general which I have outlined above. Indeed, we are here told that the synthesis of intuitions, as a combination which is prior to subsumption under concepts, is a function of the imagination, while to bring this manifold of intuitions so synthesized to concepts, and therefore to the possibility of being reflected in a judgment, is a duty of the faculty of the understanding. This interpretation of the above passage is reinforced by examination of the following:

> The same function that gives unity to the different representations in a judgment also gives unity to the mere synthesis of different representations in an intuition, which, expressed generally, is called the pure concept of the understanding[25].

The function of the understanding which gives unity to different representations in a judgment is the same function that brings unity to the mere synthesis of a manifold consisting **only** of intuitions (since only intuitions are representations that can be synthesized into another intuition; i.e., the aforementioned synthesis of the imagination at point (b)), and this function is identified with the category. Notice that the categories appear as the crucial functions that give unity to the processing of representations by the imagination and understanding alike; it is clear, in particular, that a parallel is drawn between the synthesis of representations in a judgment - which proceeds only in accord with the understanding, and corresponds to the *synthesis intellectualis* of the Transcendental Deduction - and the synthesis or putting together of intuitive representations by the imagination. The fact that both syntheses receive their synthetic unity from the categories is one of Kant's fundamental insights, whose exact meaning we shall try to clarify in what follows. In the meantime, we remark that the above analysis shows that the notion of synthesis in general seems to be highly contextual, possibly referring to the synthesis of concepts by means of a judgment, to the subsumption of an intuition under a concept, or to the synthesis of the manifold of intuitions by the imagination; all of which must proceed in agreement with the categories.

Another important *locus* where Kant treats the notion of act of combination is the passage at [11], B201/B202, situated at the outset of the Axioms of Intuition[26]. There,

[24] [11], A78.
[25] [11], B105.
[26] Notice that Kant adds, in his copy of the first edition of the CPR, a note to the above definition of synthesis in general referring to "combination, composition and nexus" ([11], p. 210).

Kant says that every combination (*conjunctio*) is either composition (*compositio*) or connection (*nexus*). The former is "the synthesis of a manifold of what does not necessarily belong to each other", for instance the manifold of the two triangles obtained by dividing a square in tracing its diagonal, and is the "synthesis of the homogeneous in everything that can be considered mathematically". *Compositio* is itself classified into either aggregation or coalition. Aggregation consists in combining a multitude of antecedently given parts in order to produce a whole[27], and is characteristic of extensive magnitudes[28] (e.g., spaces), while coalition refers to combination of intensive magnitudes[29] (e.g., masses). *Nexus*, on the other hand, is the "synthesis of that which is manifold insofar as they necessarily belong to one another, as e.g., an accident belongs to some substance, or the effect to the cause", and hence it involves the synthesis of what can be unhomogeneous but that with this synthesis is represented as combined *a priori*. *Compositio* is thus a form of mathematical synthesis, while *nexus* is a form of dynamical synthesis. Notice that a *prima facie* interpretation of this passage might lead us to the erroneous conclusion that *nexus* is a form of synthesis which *must* involve a manifold which is not homogeneous, and thus is such that it cannot be applied to a manifold of geometrical objects. This interpretation is perfected by considering what Kant says in the Prolegomena[30]. There he specifies that *nexus* does not require the homogeneity of the manifold to which it is applied, but that it can also be applied to manifolds which are homogeneous:

> [...] in the connection of cause and effect homogeneity can indeed be found, but is not necessary; for the concept of causality (whereby through one thing,

[27] See, e.g., [11], B203/B204

[28] A magnitude is extensive if it is such that the representation of the whole is made possible by the representation of the parts, i.e., by the successive apprehension of these parts ([11], A162.).

[29] A magnitude is intensive if it is such that it can only be represented as a unity, and in which multiplicity can only be represented as approximation to negation ([11], B210). For instance, consider an observer o and a point source of light P radiating light in all directions equally. Assume that the luminosity of point P be L (notice that luminosity, defined as electromagnetic energy radiated by P over unit of time, is an absolute magnitude). Assume furthermore that the distance between o and P be r. Then the apparent brightness of source P at the observer's location is $B = \frac{L}{4\pi r^2}$, which is a unity that does not need successive apprehension in time to be synthesized, but whose apprehension is instantaneous; it "takes place by means of the mere sensation in an instant and not through successive synthesis of many sensations, and thus does not proceeds from the parts to the whole; it therefore has a magnitude, but not an extensive one." ([11], B210). If r (the distance between o and P) increases, then B decreases accordingly. However, there is no smallest unit by means of which B can decrease, since there is no smallest quantity ϵ by means of which r can increase. This is what Kant means when he states that ([11], *ibid.*):

> [...] between reality in appearance and negation there is a continuous nexus of many possible intermediate sensations, whose difference from one another is always smaller than the difference between the given one and zero, or complete negation [...] every reality in appearance, however small it may be, has a degree, i.e., an intensive magnitude, which can still always be diminished [...] every color, e.g., red, has a degree, which, however small it may be, is never the smallest [...]

[30] See [8], (4:343), §53, p. 133.

> something completely different from it is posited) at least does not require it[31].

We can use these remarks to begin understanding the relation, hinted at in the passages above, between acts of combination and the categories, which are themselves classified into mathematical (categories of quantity and quality) and dynamical (categories of relation). Consider, for instance, Kant's example of a square S which is divided by its diagonal into two congruent right triangles T_1 and T_2. Now, T_1 and T_2 are two spatial intuitions (extensive magnitudes) which constitute a manifold and can be thus synthesized, hence combined, in order to yield a third representation (the intuition of the square) in which they are unified. It is by successive aggregation of the two triangles that I produce the intuition of the square, i.e., by successive addition of homogeneous elements which come to be seen as parts of a constructed whole. The same procedure of synthesis I must employ in order to produce the intuition of a line segment, by "drawing in thought" all its parts starting from one point and aggregate them together[32] in a whole. More generally, any extensive magnitude can be cognized "through successive synthesis (from part to part) in apprehension"[33], and by subsuming this synthesis under a concept (e.g., the concept of a square) which gives it unity. A first understanding of the relation between the categories and these acts of combination can be obtained by considering that this process of synthesis seems to be in agreement with the category of quantity:

> Thus if, e.g., I make the empirical intuition of a house into perception through apprehension of its manifold, my ground is the necessary unity of space and of outer sensible intuition in general, and I as it were draw its shape in agreement with this synthetic unity of this manifold in space. This very same synthetic unity, however, if I abstract from the form of space, has its seat in the understanding, and is the category of the synthesis of the homogenous in an intuition in general, i.e., the category of quantity [...][34]

In the apprehension (empirical synthesis) of the shape of a house I synthesize successively its parts to produce the whole. However, as we shall see, according to Kant the empirical synthesis has as its ground an *a priori* (independent of experience) synthesis of intuitions, in the sense that the former synthesis is is an application of the latter to sense-data[35]. The ground for the empirical synthesis of appearances in space seems then to be the

[31] [8], *ibid.*
[32] [11], A162.
[33] [11], B204.
[34] [11], B162.
[35] See for instance [11], B161/B165.

synthesis of spatial magnitudes *a priori*, i.e., the combination of manifolds of *a priori* spatial intuitions into composite intuitions, in the same fashion in which we combine the intuitions of the parts of a line segment into the intuition of the whole segment. This synthesis, however, seems to be possible in virtue of the fact that the category of quantity allows me to conceive of the synthesis of a series of homogeneous parts, i.e., of a series of parts which, taken together, constitute one object. Thus, in combining T_1 and T_2 into the square S I am making use of the category of quantity, in thinking T_1 and T_2 (two unities) as a multitude (a multiplicity) that constitutes one (a totality).

This interplay between the categories, the synthesis of shapes in pure intuition, and the apprehension of the intuition of an object in appearance is of crucial importance in order to understand Kant's theory of the mind, and, consequently, his theory of space and geometry. The rest of this chapter will be devoted to make clearer and more explicit the exact nature of this interplay. At the present stage, the above analysis shows that acts of combinations are acts of synthesis in general through which a manifold of representations (whether of concepts or intuitions) is unified in one representation; they can be purely intellectual, as those that are simply thought in the combination of concepts in a judgment, or can be brought about in intuition (whether pure or empirical) by the faculty of imagination, whose activity is both grounded on sensibility and requires a spontaneous act of the understanding.

§1.5 The definition of category

Kant gives his definition of category in §14 of the Transcendental Deduction:

> [The categories] are concepts of an object in general, by means of which its intuition is regarded as determined with regard to one of the logical functions for judgments.[36]

The above definition can be glossed as follows. A category is a general concept of an object x, such that if a manifold of intuitions is subsumed under it, the logical role in a judgment of the concept with respect to which this manifold constitutes an object is fully determined. We can clarify this with a concrete example, following Kant, drawing from the category of substance and the corresponding categorical judgment. Thus, if I simply consider the judgment "all bodies are divisible", it is here left undetermined whether there is an objective and necessary relation between body^c and divisible^c in judgments, with respect to the logical roles of subject and predicate. That is, equally

[36][11], B128.

well I could say “some divisibles are bodies” (by application of the rules of general logic). At this stage, then, the manifolds of intuition which could be subsumed under the concepts body^c or divisible^c are not determined with respect to the logical role of their corresponding concepts in the judgment (subject or predicate). In other words, of an object x to which body^c can be applied, it is here not specified whether this object is to serve as a subject or a predicate in judgments; I could say “this x is divisible” as well as “this divisible is an x”[37]. The relation of the cognition of the object x with respect to my other cognitions is hence left undetermined.

If, however, I apply to body^c the category of substance, I thereby determine that every object x which falls under the extension of body^c is always, with respect to divisible^c, to act as a subject in judgments[38]. Now, since an object x is “that, in the concept of which the manifold of intuitions is united”[39], this boils down to saying that a manifold of intuitions, when subsumed (unified) under a concept c constituting a (partial) specification of the concept of the object[40], and such that a mark of c is body^c, is determined with respect to divisible^c objectively as *respective* the subject versus the predicate, and thereby its relation with respect to the cognition of divisibility is settled.

Notice that this *prima facie* explanation of the definition of a category does not show on what basis one could apply a category to a concept. The category of substance, for instance, signifies here only a pure concept of the understanding, which, if applied to body^c, determines any object x thought through this latter concept as what is objectively, with respect to divisible^c, the subject in a judgment. We do not have, however, a criterion which would explain why one applies the category of substance to body^c and not to divisible^c, i.e., why one would consider the object thought through body^c (its

[37]Consider what Kant says at [10], *Notes on Metaphysics*, R 4672, (17:635):

> [...] I will not regard whatever I want in the appearance as either subject or predicate, rather it is determined as subject or *respective* as ground. Thus what sort of logical function is actually valid of one appearance in regard to another, whether that of magnitude or of the subject, thus which function of judgments. For otherwise we could use logical functions arbitrarily, without making out or perceiving that the object is more suitable for one than another.

In the same passage, Kant states that only by means of subsumption under the categories can intuitions be subjected to rules, and hence refer to an object. Consider also [15], [475].

[38]Consider also what Kant says in a handwritten note to his copy of the first edition of the CPR ([11], B106):

> Logical functions are only forms for the relation of concepts in thinking. Categories are concepts, through which certain intuitions are determined in regard to the synthetic unity of their consciousness as contained under these functions; e.g., what must be thought as subject and not as predicate.

[39][11], B137.

[40]Note: a partial specification of the concept of the object. A concept c under which a manifold of intuitions can be subsumed (e.g., the concept “bycicle”) specifies the object only partially if c is only a mark of the complete concept of the object. A complete concept of an object is a singular concept (e.g., the concept “Socrates”).

corresponding intuitive manifold) as the subject of the judgment, in place of the intuitive manifold thought through the mark divisible^c. In order to find such a criterion, as we shall see, one must go beyond this definition, and consider the categories as original concepts or functions for the combination of appearances.

§1.6 The unity of apperception

The form of outer sense is space, while the form of inner sense is time. That time is the form of inner sense simply means that perceptions or intuitions are given to us in the form of a temporal succession of data. For instance, in drawing a line segment I represent in inner sense a temporal succession of parts in space. Analogously, in the synthesis of the intuition of a house, I apprehend the parts of it successively, i.e., as ordered in time. Inner sense is thus always changing and mutable, i.e., there is in it always a stream of different appearances, that do not offer a ground for the determination of a stable self[41]. Inner sense (empirical apperception, or empirical consciousness) is such that in it every (successive) sensation or intuition is accompanied by consciousness of this sensation or intuition. However, these consciousnesses can be totally unrelated one with respect to the other. I can have consciousness of x^i at time t, and of y^i at time $t' > t$, but this does not mean that I have consciousness of x^i at time t *and* of y^i at time t'; the two selves which have consciousness of, respectively, x^i and y^i, can be totally unrelated, and as far as the stream of representations in inner sense is concerned, I can be in a condition of not being able to think the identity of consciousness at time t and time t'.

It is, however, clear that in order for an objective experience to be possible, a stable unity of consciousness is demanded, which must *necessarily* be able to accompany all of my possible representations in such a way that they are all conceived as *mine*, i.e., that the identity of the consciousness accompanying these representations be thought. The source of this original unity of consciousness, which Kant denotes with various terms - such as "transcendental unity of apperception" and "original-synthetic unity of apperception" - is the original *I think*, or pure apperception, which must then itself precede every data of sense or intuition, since it could neve arise from these always fleeting representations:

[41] [11], A107 (The bold characters are Kant's):

> The consciousness of oneself in accordance with the determinations of our state in internal perception is merely empirical, forever variable; it can provide no standing or abiding self in this stream of inner appearances, and is customarily called **inner sense** or **empirical apperception**.

> That which should necessarily be represented as numerically identical cannot be thought of as such through empirical data[42].

This original self, which allows for the possibility of experience, must thus be logically prior to the never ending flux of sensations and intuitions that constitute the material for our cognition. In particular, a thoroughgoing and *a priori* necessary connection and unity among this material, by means of which all appearances are *my* appearances, is possible only according to the unity of this manifold in the pure apperception, this unchanging *I think* which "must be able to accompany all my representations"[43], and without which I would have "as multicolored, as diverse a self as I have representations"[44][45]. The necessary unity of apperception (which is *a priori*) is thus the transcendental ground for the unity of empirical consciousness, i.e., of inner sense; indeed, Kant states that:

> [...] even the purest objective unity, namely that of the *a priori* concepts (space and time), is possible through the relation of the intuitions to it [the transcendental (unity of) apperception][46]

I believe that we must intend here the reference to the "concepts" of time and space as a reference to time and space as *a priori* intuitions, since (i) it is evident that here Kant is not referring to specific, determined times or spaces, and (ii) in light of the metaphysical expositions of the concept of space and time, the *a priori* content of these concepts are exactly the corresponding intuitions. Hence, what Kant is saying here is that the unity of the intuitions of time and space (the fact that we have one time and one space of which we are conscious, in which intuitions are ordered) is possible because of the necessary relation of intuitions to the original apperception in which they are unified; which is a reasonable claim if we consider that lack of relation to the original unity of apperception implies lack of unity of inner sense and of outer sense, hence lack of unity even in terms of their forms. Without the necessary relation of all my intuitions to the apperception and its unity, I would have as many times and spaces as I have intuitions, since each intuition would have its own time and space not necessarily constituting a whole with that of all the others; empirical cognition, and with it experience in general, would thus be irremediably disrupted.

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[42] [11], A107.

[43] [11], B132.

[44] [11], B134.

[45] One does not have to make the mistake of considering the *I think* as a substance: it is to be understood as a mere logical self, which refers to a suitably connected set of representations. See the discussion in [16].

[46] [11], A107.

We can better understand why the original apperception and its synthetic unity are fundamentally needed by examining their relation to the notion of combination. The Transcendental Deduction B opens with this statement:

> Combination is the representation of the synthetic unity of the manifold. The representation of this unity cannot, therefore, arise from the combination; rather, by being added to the representation of the manifold, it first makes the concept of combination possible[47].

Kant then proceeds to claim that this unity is not the category of unity, since in this logical function for judging combination is already thought, but it is something higher, which "itself contains the ground of the unity of different concepts in judgment, and hence of the possibility of the understanding, even in its logical use"[48]. Considering the central role of the unity of apperception with respect to judgments[49], and the fact that the section that follows deals exactly with the *I think*, it is reasonable to suppose that Kant is here stating that the possibility of combination relies or presupposes the pure apperception, or *I think*. However, Kant claims at the same time that the transcendental unity of apperception is originally related to acts of combination, which, in synthesizing or binding together representations, allow us to become aware of the identity of consciousness in them[50]:

> Combination does not lie in the objects [...] but is rather only an operation of the understanding, which is itself nothing further than the faculty of combining a priori and bringing the manifold of given **representations** under **unity of apperception**, which principle is the supreme one in the whole of human cognition [...] the original **synthetic unity of apperception**, under which all representations given to me stand, but under which they must also be brought by means of a **synthesis** [51]

[47] [11], B131

[48] [11], B131.

[49] See [11], B142.

[50] The same interpretation of the pure apperception and its original unity we find in the letter sent by J.S. Beck to Kant, dated 1791 ([7], 129, [499], (11:311)):

> I understand the words "to connect" [verbinden] in the Critique to mean nothing more or less than to accompany the manifold with the identical "I think" whereby a unitary representation comes to exist. I believe that the Critique calls the original apperception the unity of apperception just because this apperception is what makes such a unitary representation possible [...]

It is through acts of combination that the manifold of representations is unified in such a way that all of these representations are accompanied with the same *I think*, or pure apperception, and thus are brought to the unity of consciousness; and it is this original apperception which makes this unity possible.

[51] [11], B135-B136.

The synthetic unity of apperception is, conceptually, prior to any act of combination of representations, as Kant makes clear in the aforementioned passage from §15 of the Transcendental Deduction; more, it is this unity that makes any thought of synthetic combination possible. On the other hand, it is only through combinations (syntheses) that we can bring a manifold of representations under the synthetic unity of apperception, thereby thinking the identity of the consciousness (apperception) in them. While the synthetic unity of apperception is the first ground that makes possible for us to combine representations, and in the process of cognition to synthesize them in agreement with the categories, on the other hand the only way in which we can bring a manifold to this unity of apperception, which *unifies* this manifold as *mine* in one consciousness, is by an act of combination or synthesis in general. Kant's insistence that combination is not presented to our cognitive system ready-made in perception, but that it is an operation of the understanding, goes to underline the role of active processing of raw representations by the mind, by means of which their manifold is brought the unity of apperception; indeed, in reference to the type (b) of act of combination expounded above, we find in the A edition:

> [...] since every appearance contains a manifold, thus different perceptions by themselves are encountered dispersed and separate in the mind, a combination of them, which they cannot have in sense itself, is therefore necessary[52].

In the absence of an active power that brings combination into appearances, these just reduce to a stream of perceptions lacking any order. We might perhaps have consciousness of each of these perceptions, but since these are dispersed and lack any order or connection among themselves, our empirical consciousnesses of them are also dispersed - we are in the situation described above of a disrupted self. Hence, we should infer that it is by means of an active faculty of the mind, which brings combination to the manifold of representations (and in particular of intuitions and appearances), that the transcendental unity of consciousness - the ground of the empirical unity of consciousness - is thought in these representations.

Two related issues spring from this interpretation, namely, (I) it must be better understood why it is necessary for representations to be combined together in order for them to be unified in one consciousness, and (II) we must understand the interplay between the original unity of apperception and the unity of empirical consciousness. We shall tackle both of these problems at once. Kant defines the faculty of the understanding in three ways which, he claims, are essentially equivalent: the understanding is a faculty of concepts, a faculty of judgments, and a faculty of rules[53]. We have already seen before

[52][11], A120.

[53]See, e.g., [17], (28:240).

that a concept is essentially a rule which makes necessary certain representations, and by means of which a manifold of representations can be unified. For example, body^c was seen as a rule for the cognition of outer appearances which makes necessary the representation of extension^c, shape^c, and so forth; since a rule "gives the relation of the particular to the general"[54], body^c thus relates all the particular representations contained under it (whether intuitions or subordinated concepts) to the general marks thought in it. A judgment, however, is itself also a rule. For instance, the judgment "Cicero is learned"[55] is a rule by means of which we think the mark of learnedness in every action performed by Cicero. Thus we cognize the particular (Cicero's actions) by means of the general (the *conceptus communis* learned^c). Now, according to Kant all our cognition must proceed according to rules. This means that appearances cannot be combined (synthesized) randomly, but have to be combined according to some sort of rule, which makes it possible for the mind to go from one appearance to the other even in the absence of the object. For instance, the apprehension of an empirical object would be impossible if:

> [...] were there not a subjective ground for calling back a perception, from which the mind has passed on to another, to the succeeding ones, and for exhibiting entire series of perceptions [...][56]

However, appearances cannot be combined together simply as they are taken up by the senses, since then there would be merely "unruly heaps of them"[57] from which cognition would never arise. Indeed, if I combined the concept red^c with every concept corresponding to an appearance that is apprehended by the senses at the same time in which the intuition of the colour red is apprehended, then red^c would be combined with almost anything else without any reason; hence these combinations would hardly have any cognitive value, since the perception of red^i would just inundate my mind with a heap of random appearances and their concepts. Instead, appearances must be combined together according to a specific rule, i.e., there must be a ground justifying the fact that a certain appearance is combined with some appearances and not combined with some other ones. Association is then defined by Kant as the combination of appearances according to a subjective rule, and the faculty which is devoted to the association of appearances is the empirical imagination, i.e., the reproductive imagination[58]. That association is carried out in agreement with a rule which is subjective means that the combination which is thought in the association of appearances is not considered as

[54] [17], *ibid.*
[55] [17], *ibid.*
[56] [11], A121.
[57] [11], *ibid.*
[58] [11], A121.

present in the object, but only in the subject's mind. It is the distinction, fundamental in the Kantian framework, between judgments of perceptions and judgments of experience. The association of appearances corresponds to the former, in that with it I assert the combination only according to a subjective rule. For instance, the judgment of experience "when I carry a body, I feel a pressure of weight"[59] is a combination of body^c and weight^c according to a subjective rule that the understanding has extracted from past appearances, in that it has established that in every past instance every intuition of a body was accompanied by the intuition of it being heavy. However, this combination is here being thought only as an association, and thus body^c and weight^c are not being thought as combined in the object itself. Thus:

> [...] the empirical unity of consciousness, through association of the representations, itself concerns an appearance, and is entirely contingent [...] One person combines the representation of a certain word with one thing, another with something else; and the unity of consciousness in that which is empirical is not, with regard to that which is given, necessarily and universally valid[60].

We come thus to understand that there must be two sorts of unity of consciousness, one which is empirical (unity of inner sense) and subjective, and one which is necessary and objective. In particular, by means of the association of representations I achieve a unity of empirical consciousness (of inner sense) which is non-objective and contingent, since it is based on a subjective rule which is thought as a subjective condition of perception, and is effected by the reproductive imagination[61].

In case, however, I connect appearances by means of a rule which is not thought as subjective, but as a rule that subsists in the object itself, then I obtain the real judgments in which the *copula* represents exactly this objective connection of representations[62]. Hence, the judgment "bodies are heavy" represents a combination of body^c and heavy^c

[59] [11], B142.

[60] [11], B140.

[61] Consider the following passage from [10], *Notes on Logic*, R 3051, (16:633):

> The representation of the way in which different concepts (as such) belong to one consciousness (in general (not merely mine)) is the judgment. They belong to one consciousness partly in accordance with laws of the imagination, thus subjectively, or of the understanding, i.e., objectively valid for every being that has understanding. The subjective connection pertains to the particular situation of the subject in experience.

I can combine together certain concepts according to a subjective rule, for instance I can combine the concept of the willow tree with that of my old grandfather who used to sit at the foot of such trees in the summer, but this would be merely a rule for a subjective combination of appearances by means of my imagination, and hence it would be neither universally valid nor objective. I would thus achieve a merely empirical and non-objective unity of consciousness, whose ground of possibility, though, remains the original unity of consciousness.

[62] Indeed ([11], B142):

which is thought of the object itself. It is only by means of this sort of combinations that, according to Kant, I achieve the *necessary* and *objective* unity of consciousness; since only in this way I think the connection of these representations, and hence the unity of consciousness in these representations, as necessary for every mind. However, it must be noticed that even by means of a judgment of this sort, the unity of consciousness which is thereby obtained is still contingent, in that it is founded on a rule which is empirical; hence, this rule does not suffice to ensure that *any* possible apperance will be such that it could be brought to the objective and necessary unity of consciousness[63]. We must then distinguish: (I) the empirical unity of consciousness obtained through association, which is contingent and not objective; (II) the necessary and objective unity of consciousness obtained through judgments of experience based on empirical rules, which, despite relating the appearances to the original apperception (the *I think*), with respect to all possible appearances that can be given is still contingent; and (III) the original and transcendental unity of consciousness which extends to all appearances *a priori*, and which is the ground of the possibility of (I) and (II).

We now reach a fundamental turning point in Kant's argumentation, one which is most apparent in the Transcendental Deduction A, but which is also present in the B Deduction. In order for us to be able to associate appearances, reproduce them, formulate judgments of perceptions and empirical judgments of experience, it is necessary that there be a regularity in the appearances which must be able to be understood *a priori* of all empirical rules of combination (which are the ground for (I) and (II) above). The argument seems to go as follows. If there was no ground that could be understood *a priori* for the combination of appearances according to rules, then it would be contingent whether appearances would always have to be given to us in such a way that a regularity could be found in them, which would render possible for us to associate them and bring them under empirical rules. For instance:

> If cinnabar were now red, now black, now light, now heavy [...] then my empirical imagination would never even get the to think of heavy cinnabar on the occasion of the representation of color red; or if a certain word were

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> For this word [the *copula*] designates the relation of the representations to the original apperception and its **necessary** unity, even if the judgment itself is empirical, hence contingent [...]

Notice that Kant claims here that even if the judgment is empirical, it nevertheless effects the necessary unity of consciousness of the representations.

[63]A clarification regarding the use of the term "necessary" is in order here. That in a judgment involving empirical concepts a necessary unity of consciousness is obtained simply means that the combination of the concepts appearing in the judgment is thought as necessary for every mind, i.e., as not subjective. This is different from the idea of a unity of consciousness which necessarily extends to all possible appearances, in that here what is thought is that every possible appearance must be such that it can be reflected in a judgment of experience. Thus the two notions of necessity are radically different.

> attributed now to this thing, now to that [...] without the governance of a certain rule to which the appearances are already subjected in themselves, then no empirical synthesis of reproduction could take place[64].

If there was no regularity to which all *possible* appearances must be subjected just in virtue of the fact of being given to us, hence entirely *a priori*, i.e., according to some transcendental principles which have not been drawn from experience, then it would remain a contigent fact whether appearances would be always, in any possible experience, be given in such a way that they could be subjected to a rule of association, and hence to an objective rule in a judgment of experience. It would then be possible for a set of appearances to be given such that (a) the empirical rules that the understanding extracted from past appearance be inapplicable to it, and (b) no other empirical rule of association could be found by the understanding in it. Consequently, the reproductive imagination would not be able to effect any combination of these appearances, lacking an appropriate rule connecting them; they would thus constitute only a set of disconnected perceptions, i.e., a set of disconnected consciousnesses of sense-data, lacking a proper unification in *one* empirical consciousness, and hence their connection by means of a judgment would be impossible:

> For even though we had the faculty for associating perceptions, it would still remain in itself entirely undetermined and contingent whether they were also associable; and in case they were not, a multitude of perceptions and even an entire sensibility would be possible in which much empirical consciousness would be encountered in my mind, but separated, and without belonging to one consciousness of myself, which, however, is impossible[65].

We thus come *en passant* to better understand why, for Kant, appearances have to be subjected to rules in order for the identity of consciousness to be thought in them. If heavy^{i} were not associable according to rules to other appearances, rules which allow me to go from the representation of heaviness to, e.g., that of body or of cinnabar, then my empirical consciousness of heavy^{i} would be totally unrelated to all the other empirical consciousnesses of perceptions which I had, or will have in the future. On what ground could I then think the identity of the *I* in the appearances, if I were not able, by means of combination according to rules, to submit them to a thoroughgoing connection? Without the latter, I would not be able to think of heaviness when I see a body, or any other appearance, and the *I* which intuits or thinks heaviness would have

[64] [11], A101.
[65] [11], A122.

nothing to do with that which intuits or thinks, e.g., a body. Kant states this concept as follows (notice that he is referring here to the *necessary* unity of consciousness):

> Thus the original and necessary consciousness of the identity of oneself is at the same time a consciousness of an equally necessary unity of the synthesis of appearances in accordance concepts, i.e., in accordance with rules that not only make them necessarily reproducible, but also thereby determine an object for their intuition [...][66]

Let us summarize the above argument as follows. Granted:

1 that a necessary (= for any understanding = objective) unity of consciousness, which thus extends to all *possible* appearances, and hence can be thought *a priori*, is necessary in order for an objective experience to be possible; and that necessary unity of consciousness is obtained by combining together by means of judgments of experience the concepts under which appearances are subsumed (operation of the understanding);

2 that a combination of appearances by means of judgments of experience requires beforehand their association, i.e., their combination according to judgments of perception (operation of the imagination)[67];

3 that appearances cannot be combined according to subjective rules if they are not, beforehand, subjected to some rules; and that appearances are nothing else than reprentations of our sensibility, and thus they cannot be perceived as already combined together;

[66][11], A108. A *locus* in which Kant is explicit regarding the relation between rules and unity of consciousness is [18], (28:449):

> [...] we can also think a reproduction (*reproduction*), anticipation (*praevision*), without the least self-consciousness, but such a being could not prescribe rules to itself, for the possibility of a rule requires making consciousness of oneself the object of one's intuition, one must be conscious of what different beings agree in; if many beings exhibit a large degree of the effects which can arise in human beings through reason, it still does not at all follow from that that they also would have reason, for, if they are lacking consciousness, then they are also missing understanding and reason, and sensibility alone reigns.

A rule is possible only if we make the consciousness of ourselves the object of our own intuition, i.e., we intuit our self. This statement is suggestive, and its consequences would be a most interesting topics of investigation not only for Kantian critical philosophy, but for cognitive neuroscience alike. Unfortunately we cannot go in depth on the matter here; still, we notice that Kant draws an essential connection between self consciousness and being able to combine appearances according to rules. Without the former, we could still be able to associate appearances together, but we would not be able to do this according to rules which we have previously produced. Thus a being who could display all the exterior signs of human understanding would not necessarily be in possess of the understanding itself, since he could lack consciousness. This amounts to say that he would be a zombie.

[67]A provisional judgment by the understanding preceeds judgments of experience: see [17], (28:234).

It then follows: that an objective experience is possible only if there is a synthesis *a priori* which subjects all *possible* appearances to rules, bringing them to the necessary original unity of consciousness, and making them associable *in themselves*.

Notice that this argument hinges on the crucial point that the necessary unity of consciousness must extend to all possible appearances, not only to those that have already been encountered. If this were not the case, then disconnected appearances would be possible, and hence we would not have a truly genuine empirical consciousness, i.e., experience. This in turn means that this unity of consciousness must be achieved *a priori*, for only if it is *a priori* can it necessarily extend also to appearances which I have not yet had. Hence it cannot be a mere unity of consciousness obtained by means of empirical rules or concepts (such as those appearing in (I) and (II) above), since these are by no means achieved *a priori*. Hence also the naïve realist is confuted, since it cannot be the case that this combination of appearances is picked up from the object itself by means of the senses; for first of all, appearances are only representations that come from our senses, but we do not sense the rule or the connection in them, and secondly, sense could only afford us with combinations *a posteriori*, unable to guarantee the requirement of necessity. Of course, this argument holds only if one accepts the premise that a necessary condition for experience is that all *possible* appearances must admit of objective combination. However, this premise seems quite plausible. Indeed, if we reject it, then this amounts to claim that it would be possible to have an experience in which certain appearances, e.g., that of heaviness, would be totally disconnected and in no relation with all the others. But then, according to the above, (a) we would not be able to think the identity of the self in the representations, and (b) all of the laws of physics, and all of the applications of the facts of arithmetic and geometry to the physical world, would not be truths or even approximations to the truth, but mere illusions, as a new batch of appearances could always show up that not only does not conform to these laws, but does not conform to any law at all[68].

It is by means of this elaborated transcendental argument from conditions of possible experience that Kant establishes the necessity of a synthesis combining appearances *a priori*. We then must affix our gaze to this synthesis if we wish to understand Kant's theory of cognition, and with it his conception of space and geometry, more fully.

[68] See, for instance, the footnote added by Kant to his copy of the A edition at [11], A126.

§1.7 The transcendental synthesis of the imagination

If an objective experience is to be possible, all appearances must be subjected to a synthesis of combination according to *a priori* rules, by means of which they are related to the original apperception, and are thus subjected to a law that connects them together and makes it possible for them to be associated and combined according to empirical rules. Clearly, since this synthesis is *a priori*, it cannot involve any quality of appearances, such as the content of sensations, which are empirical and thus *a posteriori*. However, since the concept of synthesis demands necessarily a manifold which is to be synthesized, it must be the case that this synthesis be exercised on a manifold of representations which must be given *a priori*, and in such a way that it can then be applied to appearances. This implies that it must be exercised on a manifold of pure *a priori* intuitions, i.e., of space and time. For space and time, being the form of outer and inner sense, are the condition of possibility of appearances on the side of sensibility. Every appearance must thus be given to us as sensation in the *a priori* forms of space and time. Therefore, a synthesis which combines together intuitions, whether they are pure (such as geometrical spaces) or empirical, only according to *a priori* objective rules based on spatio-temporal conditions, will be such that its application to appearances will produce their connection into a whole and their original regularity and relation to the objective unity of consciousness.

The faculty which, according to Kant, brings this original regularity and combination to appearances is the faculty of imagination, and in particular what from a *prima facie* analysis he seems to term alternatively the figurative synthesis of the productive imagination (*synthesis speciosa*) or the transcendental synthesis of the imagination:

> This synthesis of the manifold of sensible intuition, which is possible and necessary a priori, can be called figurative (*synthesis speciosa*) [...] yet the figurative synthesis, if it pertains merely to the original synthetic unity of apperception, i.e., this transcendental unity, which is thought in the categories, must be called, as distinct from the merely intellectual combination, the transcendental synthesis of the imagination[69].

In the same passage, the transcendental synthesis of the imagination is characterized as a synthesis which "can thus determine the form of sense *a priori* in accordance with the unity of apperception", and hence is a synthesis "for determining the sensibility *a priori*" which proceeds in accordance with the categories and is then an "effect of the

[69] [11], B151.

understanding on sensibility". What this means is slightly clarified in the passage that follows, where Kant states that what affects sensibility is the faculty of the understanding of combining the manifold of intuitions[70], and that:

> [the synthesis of the understanding] is nothing other than the unity of the action of which it [the understanding] is conscious as such even without sensibility, but through which it is capable of itself determining sensibility internally with regard to the manifold that may be given to it in accordance with the form of its intuition [transcendental synthesis of the imagination][71]

We shall draw forth from these passages a *prima facie* interpretation of the transcendental synthesis of the imagination. We have established that this synthesis is supposed to bring unity to appearances by means of *a priori* rules which involve spatio-temporal conditions. Since these rules are *a priori*, moreover, they must also be objective, i.e., they must be such that in them a necessary unity of consciousness be thought; for their being *a priori* means that they apply necessarily to appearances, and, as we have established before, it is in the rule that the objective and necessary unity of consciousness is thought. We have also seen, though, that rules are either concepts or judgments (which in turn involve concepts). Hence, it follows that the transcendental synthesis of the imagination must subject appearances to rules which involve concepts that are *a priori*. These concepts are, of course, the categories. As we combine appearances *a posteriori* by means of empirical rules which we have formed, and which involve the subsumption of these appearances under empirical concepts, we must analogously combine appearances *a priori* by means of rules which are not empirical, and which thus involve their subsumption under the categories. However, while in the case of the empirical synthesis appearances are subsumed under concepts by means of empirical schemata[72], in the case of the transcendental synthesis of the imagination (and its intellectual counterpart, the synthesis of the understanding) appearances are subsumed under these pure *a priori* concepts by means of transcendental schemata, which are a product of the transcendental synthesis of the imagination itself and are nothing else than determinations of time[73].

[70] [11], B153.
[71] [11], *ibid.*
[72] [11], B180.
[73] Thus Kant writes ([11], A142.):

> The schema of a pure concept of the understanding [...] is rather only the pure synthesis, in accord with a rule of unity according to concepts in general, which the category expresses, and is a transcendental product of the imagination, which concerns the determination of the inner sense in general, in accordance with conditions of its form (time) in regard to all representations, insofar as these are to be connected together *a priori* in one concept in accord with the unity of apperception.

What Kant seems to be saying, then, is this: the categories prescribe rules to appearances *a priori*, i.e., they originally combine appearances, bringing them to the original, objective and necessary unity of consciousness. Since, however, the understanding is only a faculty for thinking, and not intuiting, it cannot take the intuitions onto itself and combine them[74]. It needs the transcendental faculty of the imagination, which produces the schema, i.e., determines appearances objectively in terms of their mutual temporal relations and thus mediates their subsumption under the categories. This is why Kant claims that the synthesis of the understanding (= combination of appearances by means of the categories) brings unity to the action by means of which it determines sensibility internally (i.e., time); this "action" is exactly the transcendental synthesis of the imagination at work, and is an effect of the spontaneous faculty of the understanding towards the combination of intuitions.

The reliabilty of this *prima facie* interpretation seems to be supported by a wealth of textual data. For instance, Kant states in his letter to Beck dated 1792 that:

> [...] a manifold must be given a priori for those a priori concepts. And because it is given a priori, it must be given [...] in just the form of intuition [...] it is therefore in conformity with the merely sensible intuition, whose synthesis through the imagination, under the rule of the synthetic unity of consciousness, the concept expresses; for the rule of the schematism of concepts of the understanding is then applied to perceptions [...][75].

A manifold must be given to the categories in order for them to be real concepts, i.e., to achieve objective reality and application to appearances; however, since it is to be *a priori* it can only be given in the forms of intuition without empirical perception. The categories thus express the synthesis of appearances by means of the imagination, which combines them according to determinations of time and allows for their subsumption. Notice that this is in agreement with what Kant says at [11], B151, where he remarks that only by means of the transcendental synthesis of the imagination (which produces the schemata) categories acquire objective reality, i.e., application to objects. Indeed, the A edition had been even more explicit in this regard:

Notice how Kant states that the transcendental schema concernes the *determination* of inner sense in accordance with its form (time). This is the same terminology that Kant used when he spoke of the transcendental synthesis of the imagination and said that this determined the form of sense in accordance with the categories; it does not come as a surprise, then, that the transcendental schemata are a product of the transcendental imagination.

[74] See, e.g., [11], B135 and B153.

[75] [7], 130, [500], (13:316).

> The unity of apperception in relation to the synthesis of the imagination is the understanding, and this very same unity, in relation to the transcendental synthesis of the imagination, is the pure understanding [...][76].

The categories thus represent concepts that think the unity of the combination which is brought about by the transcendental synthesis of the imagination. Indeed, the definition of the categories given in the Critique as concepts of an object in general, by means of which its intuition is determined with respect to the logical function in a judgment, just means that the categories are original functions for thinking a necessary combination of a manifold of intuitions in general[77]. Kant, as we have seen, repeatedly claims that the concept of a combination of representations cannot be taken up from sense-data[78], but has to be a product of a spontaneous activity of the understanding. Combinations based on empirical rules, for the reasons we have seen, must have as a ground of possibility combinations which are brought about according to rules *a priori*. The categories are exactly these original functions in which the concept of a necessary combination of representations in an object is thought:

> [the unity of apperception] in virtue of the diversity of intuitable representations of objects in space and time, requires different functions to combine them; these are called categories [...][79]

The categories are thus what makes possible objective judgments such as "All As are Bs", since in this judgment an objective combination of the appearances designated by the marks A and B is thought, and this is possible only if the relation of A to B is determined necessarily, which happens by means of the category of substance and accident. Analogously, in the judgment "If A, then B" a necessary relation is thought which determines A as the gound on which the existence of B depends (A and B are combined as cause and effect). The categories are then rules for the necessary combination of

[76] [11], A119. The passage continues by stating that:

> [the pure understanding (the categories)] is related to all objects of the senses, though only by means of intuition, and to their synthesis by means of imagination, under which, therefore, all appearances as data for a possible experience stand.

Confirming the interpretation that the pure understanding is related to appearances by means of the transcendental synthesis of the imagination.

[77] It is in this spirit that Kant writes, in his letter to Beck dated 1792 ([7], 130, [500)], (11:316)), that:

> Since composition, either through the object or through its representation in intuition, cannot be given but must be produced, it must rest on the pure spontaneity of the understanding in concepts of objects in general (of the composition of the given manifold).

[78] See the aforementioned passage in the critique, and the letter from Kant to Beck dated 1792 ([7], 130, [500], (11:376)).

[79] [19], (20:276).

representations, and by their means these representations are unified, and an object (as that in which the representations are necessarily connected) is thought. However, being nothing more than concepts of a necessary combination of representations, they are also purely intellectual, and therefore are in need of a faculty able to mediate the subsumption of appearances under them. This is the transcendental function of the imagination, which produces the schema, i.e., a rule of transcendental time-determination, by means of which this subsumption takes place. The category of cause is then realized by means of the transcendental imagination, which produces the schema that determines the relation of appearances in the order of time in terms of their existence; A and B are then subsumed under the category of cause and effect if the existence of B always follows A in the *order* of time (which does not depend on duration[80]). It is ultimately from this original fact, that all given appearances must be subsumed under the categories in order for them to be reflected in an objective judgment, i.e., in order for objective experience to be possible, that stem all of the synthetic *a priori* principles which are conditions of the possibility of experience[81].

This *prima facie* interpretation is to be supplemented with an account of the role of the *a priori* imagination with respect to the intuition of space and the science of geometry. For the transcendental synthesis of this faculty has been related only to the categories (as original functions for thinking a combination with respect to appearances) and to the form of inner sense, disregarding an eventual role of the *a priori* imagination with respect to space. However, the *a priori* synthesis of the imagination is also characterized as a *productive* and figurative synthesis (*synthesis speciosa*). I argue that this definition has to be understood as follows: (I) the productive synthesis of the imagination is a synthesis whose purpose is to produce (the form of) an object for a given concept, that is, to construct it in pure intuition; (II) the theorems of geometry depend on the construction of their objects by means of this synthesis; (III) our consciousness of time and space (as forms of sensibility) and of their properties depends on the acts of consctruction performed by this synthesis. I shall prove these points in this order. Kant states explicitly in the Critique that:

> On this successive synthesis of the productive imagination, in the generation of shapes, is grounded the mathematics of extension (geometry) with its axioms, which express the conditions of sensible intuition a priori, under

[80][11], A203/B248.

[81]Thus, the synthetic *a priori* principle: "every event has a cause", is a condition of possibility of experience, whose necessity stems from the fact that all appearances have to be determined according to the law of causality in order for objective experience to be possible. The discussion of this principles in the *prolegomena* is particularly enlightening ([8], §23 and following.).

> which alone the schema of a pure concept of outer appearance can come about[82].

It is also stated elsewhere that the productive imagination draws shapes in space[83], and that it produces the schemata of the pure sensible concepts (e.g., the concept of a triangle), which signify "a rule for the synthesis of the imagination with respect to pure shapes in space"[84]. Moreover, the imagination in general is defined as the faculty for representing an object without its presence in intuition[85], and at some point Kant also speaks of a "formative synthesis by means of which we construct a figure in imagination"[86].

These data are consistent with the characterization Kant gives of the faculty of imagination throughout his lectures. There, he claims that intuitions can either come from sense in the presence of the object, or can come from the power of imagination in the absence of the object with respect to time (reproductive imagination and anticipation), or can be produced by the fictive faculty (*facultas fingendi*) with respect to no time[87], these last two being "imitated representations of the senses"[88]. He writes:

> The faculty of imagination is the faculty for producing images from oneself, independent of the actuality of objects, where the images are not borrowed from experience, e.g., an architect pretends to build a house which he has not yet seen. One calls this faculty the faculty of fantasy, and [it] must not be confused with the reproductive imagination[89].

In other words, the faculty of the productive imagination seems to be explained as a faculty that allows for the construction of an intuition of the object for a given concept (e.g., the concept house[c]) without this exact empirical intuition having been given in some past or present time. In his more mature lectures, Kant also specifies that the productive imagination (*facultas fingendi*) can fabricate things only in terms of their form, and this fabrication has to proceed, in order to be disciplined, according to the analogy of experience[90]. It follows that the productive imagination is a faculty for constructing the form of an object corresponding to a concept, and, since concepts of geometrical entities do not involve anything empirical, that the productive synthesis

[82] [11], B204.
[83] [11], A157/B196.
[84] [11], A140/B179.
[85] [11], B151.
[86] [11], B271.
[87] [20], (28:881).
[88] [17], (28:230)
[89] *ibid.*, (28:237).
[90] [20], (29:885).

with respect to these concepts is devoted to the construction of pure shapes in space. Indeed, Kant remarks that the construction of the object corresponding to the concept of a triangle depends on the activity of the productive imagination, which "draws the lines greater or smaller, thus allowing them to abut at any arbitrary angle"[91].

We must make here an important *caveat*, however. Construction in pure intuition just means that the definition of the concept admits of a possible constructive procedure, which can be carried out as far as one desires, and by means of which an object instantiating that concept is given. It does not mean that this construction has to be fully carried out, either on paper or in the imagination itself, since this would be absurd. That a chiliagon (a polygon with one thousand sides) can be constructed in pure intuition means that one knows the rule by means of which it can be constructed, and can carry on this construction *ad libitum*[92]. Analogously, the possibility of constructing a circle does not follow from the fact that we can practically construct it by means of the motion of, e.g., a rope around a fixed point; it follows instead from its very definition, by means of which I can exhibit the concept in the intuition of a curve with the property that any two points lying on it are equidistant from a given point, and such that the end-point of the curve is also its starting point[93].

Let us now consider point (II). Kant wrote in one of his notes, dating around 1775:

> In the (through the) construction x of the concept a (triangle) the equality of the three corners of the triangle etc. is determined. Through the specification x of the concept a the relation b is at the same time determined in this a[94].

Here the notion that synthetic *a priori* geometrical statements (theorems) flow directly from the construction of the concept is made explicit. This is consistent with what Kant states in the first section of the first chapter of the Doctrine of Method in the Critique. There, the distinguishing feature of mathematical cognition is claimed to be the fact that mathematics:

[91] [11], A164/B205.

[92] [21], (8:212). The example of the chiliagon was used by Eberhard as a counter-example to Kant's theory of construction in pure intuition. See also his letter to Karl Leonard Reinhold dated 1789 ([7], 94, [359], (11:46)).

[93] See the letter from Kant to Herz dated 1789 ([7], 96, [362], (11:53)).

[94] [10], *Notes on metaphysics*, R 4678, (17:662). Consider also the similar passage at R 4676, (17:654), where it is claimed that the relation in a synthetic *a priori* judgment between two concepts a and b can be obtained by means of the construction of $a = x$; which statement is clearly to be interpreted as implying that the relation in the judgment "the sum of the internal angles of a triangle is equal to two right angles" is established by means of the construction of the triangle. This is in turn consistent with the more recent note R 5924 (18:387), where it is claimed that through the construction of concepts synthetic *a priori* judgments are possible.

> [...] considers the universal in the particular, indeed even in the individual, yet nonetheless *a priori* and by means of reason, so that just as this individual is determined *under certain general conditions of construction*, the object of the concept, to which this individual corresponds only as its schema, must likewise be thought as universally determined[95].

In other words, the geometer brings forth his theorems by reasoning on an individual schema of an object which has been constructed according to the concept (e.g., the concept of a triangle), with the specification that what must be considered in this reasoning are only those properties which pertain to the individual in virtue of the procedure by means of which it is constructed, which as we have seen is already contained in the definition (if this definition is not nominal but real, i.e., it admits of exhibition in intuition). Indeed, Kant adds in the paragraph that follows[96] that the individual schema, representing the construction of the concept, must express "universal validity for all the possible intuitions which belong under the same concept", i.e., what is proven for this individual schema must be valid for any object instantiating the concept. Thus, in proving that the sum of the internal angles of a triangle is equal to two right angles, the geometer will consider the concept *in concreto* in an individual triangle, and will draw a series of inferences based on the properties that follow from the constructions that have been effected, according to the definition. For instance, she extends one side of the triangle, and infers that the sum of the two angles obtained in this fashion is equal to that of two right angles; which inference must hold of every triangle, since it relies only on the general properties which belong to the figure exclusively from the way it was constructed (exhibited) in intuition.

The problem which arises, however, is that of distinguishing the properties that follow from the construction (and thus hold of every intuition instantiating the concept) from those that are only contingent to the particular individual. Here, Kant seems to suggest that the properties which hold universally are either (i) mereotopological properties that follow from the construction of the schema, (ii) metric properties that follow from the concept (definition), (iii) metric properties that follow from the established mereotopological properties of the figure[97]. An example of (ii) would be the inference that two

[95] [11], A714/B742 (my emphasis).
[96] [11], A714/B742.
[97] We refer the reader to the enlightening discussion by Shabel ([22], pp. 212-213):

> [...] neither Euclid's elements nor its eighteenth-century analogs offer formal axioms but rather definitions and postulates which, if taken seriously, provide a mereotopological description of the relations among the parts of the euclidean plane. The content of these relations is, I claim, precisely what Kant alleges is accessible to us in pure intuition, prior to geometric demonstration.

Important are also the considerations at pages 211-212.

rays of a circle are the same in terms of magnitude, which follows from the definition of circle itself. An example of (i) would be that two lines, or two circles, intersect[98]. An example of (iii) would be that of the equality between the sum of the magnitudes of two right angles and the sum of the magnitudes of all the adjacent angles that can be constructed on a straight line. Of course, absolute metric considerations on the schema cannot be generalized[99].

Let us now consider point (III). We have seen at the beginning of the previous section that Kant claims that the unity of the intuitions of time and space, the fact that there is one space and one time in which appearances are ordered, is an effect of the combination of intuitions, by means of which these are related to the unity of apperception. However, for this unity to be necessary it must be established *a priori*, and thus cannot involve the combination of empirical appearances. Hence, we deduce that Kant must be referring to the combination which is brought about by the productive synthesis, which combines a manifold given *a priori* according to the schemata of geometrical concepts. One is then led to consider the idea that the unity of time and space (along with their other properties of infinity, continuity and so forth) might be a product of this productive synthesis of the imagination in the construction of pure sensible concepts.

This conjecture seems to be confirmed by a famous footnote to §26 of the Transcendental Deduction B, where Kant remarks that only by means of a synthesis through which the understanding determines sensibility (the productive synthesis of the imagination) concepts of space or time are possible; more, he claims that space and time as formal intuitions (not as mere forms of intuitions), along with their unity, are given through this synthesis[100]. In other words what Kant seems to be saying is that the picture given

[98]This of course involves the notion of continuity, which will be examined in the following chapter, where we shall consider Kant's theory of geometry *in concreto*.

[99]See [11], A714/B742.

[100]The full quote is as follows ([11], B160)

> Space, represented as object (as is really required in geometry), contains more than the mere form of intuition, namely the comprehension of the manifold given in accordance with the form of sensibility in an intuitive representation, so that the form of intuition merely gives the manifold, but the formal intuition gives unity of the representation. In the Aesthetic I ascribed this unity merely to sensibility, only in order to note that it precedes all concepts, though to be sure it presupposes a synthesis, which does not belong to the senses but through which all concepts of space and time first become possible. For since through it (as the understanding determines the sensibility) space or time are first given as intuitions, the unity of this a priori intuition belongs to space and time, and not to the concept of the understanding (§24).

Notice that Kant claims here that this unity, which presupposes a synthesis of productive imagination, precedes all concepts. Longuenesse has interpreted this as evidence that this synthesis must precede also all *a priori* concepts, and in particular the categories (see [23], p. 105-106). However, the analysis of the transcendental synthesis of the imagination which I provided above shows that the categories must play an important role in it. Hence I cannot share Longuenesse's interpretation; on the contrary, I would argue that when Kant is referring to concepts here he is doing it with respect to the Aesthetics, and therefore he is only considering *empirical* concepts. In my view, what Kant is claiming here is that the unity of time and space (as one space and one time), which in the Aesthetics was ascribed merely to in the Aesthetics, in which time and space, along with their defining properties (not only unity, but also infinity, continuity, and so forth), were taken as given, must be revised in light of the Transcendental Deduction. These properties of the pure intuitions are not given, but they require the activity of the productive synthesis of the imagination, whose role, as we have seen, is to construct the objects for the pure sensible concepts. Indeed, in his letter to Kieswetter dated 1790, discussing a proof of the ideality of space, Kant states:

> The consciousness of space, however, is actually a consciousness of the synthesis by means of which we construct it, or, if you like, whereby we construct or draw the concept of something that has been synthesized in conformity with this form of outer sense[101].

The above statement makes the relationship between the synthesis of the productive imagination and the intuition of space most transparent. The reason why we are conscious of space as a pure intuition and of its properties is that we are able to construct in it the form of objects instantiating the pure sensible concepts. It is by means of the productive synthesis of the imagination in the construction of *a priori* sensible concepts that we become conscious of space and its properties of unity, infinity, continuity; indeed, without this process of construction, we would never be able to bring these properties to light. Space, in the absence of such a synthesis, would just signify the form of outer sense, of which we do not necessarily need to have consciousness; only with this synthesis does it become a formal intuition, i.e., an intuition *a priori* with the aforementioned properties. This also explains why Kant claims that we cannot represent the dimensions of space without placing three lines perpendicular to each other; and why we cannot represent time without attending, in drawing a straight line, to "the action of the synthesis of the manifold through which we successively determine the inner sense, and thereby attending to the succession of this determination in inner sense."[102]. That space has three dimensions is a property of it that we cannot absolutely know unless we proceed to the construction of three lines perpendicular to each other, by means of which this property is first cognized. Analogously, we cannot cognize time unless we construct the intuition of a straight line, thereby becoming aware of the succession in inner sense, and thus of time as a linear order[103]. Indeed, we could not possibly cognize the infinite

sensibility to make it clear that it preceeds all our *empirical* concepts, presupposes instead the productive synthesis of the imagination, which constructs concepts that are *a priori* according to the categories (as we shall soon discover).

[101] [7], 102, [405a].

[102] [11], B154.

[103] Of course, the construction of the intuition of a straight line happens through motion, i.e., the description of a space, by means of which we affect inner sense; see [11], B155.

divisibility of space without, in bisecting a line iteratively, becoming conscious of the fact that this procedure can be repeated *ad libitum*.

We find an ulterior confirmation of these conjectures in a passage from "What Real Progress has Metaphysics Made in Germany since the Time of Leibniz and Wolff", where we find the following, important remark:

> Space and time, subjectively regarded, are forms of sensibility, but in order to frame a concept of them, as objects of pure intuition (without which we could say nothing at all about them), we require a priori the concept of a composite, and thus of the compounding (synthesis) of the manifold, and thus synthetic unity of apperception in combining this manifold [...][104]

This remark is consistent with the passages previously mentioned from the correspondence and the Critique. All these *loci* support the interpretation that, according to Kant, concepts of space and time as objects of pure intuition (as formal intuitions) are possible only in virtue of the acts of combination that we perform in constructing specific geometrical objects, which constitute the schemata of pure sensible concepts. It is by means of these acts of construction that we become aware of time and space along with their properties, and can thus frame a concept of them. I consider this sufficient evidence in order to prove point (III).

We shall end this section by considering very briefly a final point, namely, the relation between the productive and transcendental aspects of the pure *a priori* imagination. We have seen that the transcendental synthesis of the imagination combines appearances together according to the categories in order to bring them under an *a priori* regularity, while this very same synthesis, when qualified as "productive", refers to the process by means of which we construct the form of an object corresponding to the pure sensible concepts. For the analysis to be complete, we must now inquire into the relation that holds between these two functions of the pure imagination. We find a possible answer to the problem in Kant's notes, where we find a classification of the productive imagination in general into three kinds: (i) the empirical synthesis of the productive imagination, i.e., the synthesis of apprehension; (ii) the synthesis of the pure productive imagination; (iii) the transcendental synthesis of the productive imagination. It is claimed that (i) presupposes (ii), and in turn (ii) presupposes (iii), and that:

> The pure synthesis of the imagination is the ground of the possibility of the empirical synthesis in apprehension, thus also of perception. It is possible a

[104][19], (20:276). At the end of the same paragraph, Kant also states that since we do not perceive time and space, *a priori* principles according to the concepts of the understanding are necessary for this purpose, and to prove their reality.

> priori and produces nothing but shapes. The transcendental synthesis of the imagination pertains solely to the unity of apperception in the synthesis of the manifold in general through the imagination[105].

We can understand from this note that the synthesis of apprehension, which features centrally in the first Critique and appears only marginally in the second, is nothing else than the application of the pure productive synthesis of the imagination to empirical data in order to construct an image of the object. While the pure productive imagination constructs the form of the object corresponding to the pure sensible concepts, its application to the data of sense constitutes the synthesis of apprehension, by means of which the manifold of perception is combined and ordered into an image. When I perceive a house, for instance, I draw its shape[106], and thus I construct the image of the object in the same fashion in which I would construct the schema of a geometrical concept (e.g., a square); only, here the material is furnished by the senses, and is not given *a priori* by sensibility[107]. This explains why Kant, in the B edition of the Critique, proceeds flowlessly from the treatment of the *synthesis speciosa* to that of the empirical synthesis of apprehension: the latter is just the former applied *in concreto*. Furthermore, the fact that the synthesis of apprehension, in the construction of the image, is just an application of the pure synthesis of the imagination in the construction of geometrical objects constitutes the necessary bridge that guarantees the applicability of geometrical concepts to the world of appearances[108].

If the claim that the synthesis of apprehension relies on the synthesis of the pure productive imagination means, as we have seen, that the former is an application of the

[105] [10], *Notes on Metaphysics*, (23:18).

[106] [11], B162.

[107] The fact that the synthesis of apprehension is just the application of the productive synthesis of the pure imagination to empirical sense-data furnishes us with another indirect argument for claim (III) above, in that it allows us to understand the following passage at [11], A100:

> Now this synthesis of apprehension must also be exercised *a priori*, i.e., in regard to representations that are not empirical. For without it we could have a priori neither the representations of space nor of time since these can be generated only through the synthesis of the manifold that sensibility in its original receptivity provides. We therefore have a pure synthesis of apprehension.

In light of what we said above, this pure synthesis of apprehension cannot be anything else than the synthesis of the productive imagination in the construction of schemata of pure sensible concepts.

[108] Indeed, Kant writes at [11], A224 that:

> [...] this very same formative synthesis by means of which we construct a figure in imagination is entirely identical with that which we exercise in the apprehension of an appearance in order to make a concept of experience of it - it is this alone that connects with the concept the representation of the possibility of such a thing [...]

The synthesis of the productive imagination *a priori* is exactly the same as the synthesis of apprehension by means of which we construct the image of the empirical object.

latter to sense-data, it would be reasonable to suppose, as the pure synthesis of the productive imagination relies in turn on the transcendental synthesis of this faculty, that the former be an application of the latter to the manifold of pure intuition (time and space). This interpretation seems to be appropriate if we consider the *locus* in which, to my knowledge, Kant is most explicit regarding the interaction between the categories and the construction of mathemathical concepts: the letter sent to Johann Heinrich Tieftrunk dated 1797. There, after remarking that the categories are functions which are used to combine intuitions (as the concept of combination cannot be found among appearances), he claims the following:

> All the categories are directed upon some material composed a priori; if this material is homogeneous, they express mathematical functions, and if it is not homogeneous, they express dynamic functions. Extensive magnitude is a function of the first sort, for example, a one in many. Another example of a mathematical function is the category of quality or intensive magnitude, a many in one. [109].

It is instructive to compare the passage above with the axioms of intuition and the characterization of the notion of combination given in the footnote at [11], B201, which we have already treated in the last part of the section on the acts of combination. The construction of mathematical concepts by the pure synthesis of the productive imagination must proceed according to the categories, since these signify only general concepts of combination, which can then be applied or realized *in concreto* by means of construction in pure intuition or synthesis of appearances. For instance, the category of quantity in its three moments of unity, plurality and totality is the source of the *preadicamentum* of magnitude[110], whose concept is simply that of a plurality homogeneous elements (unities) which taken together constitute one element (totality)[111]. The concept of extensive magnitude, as we have seen, is obtained from that of magnitude by adding to this concept the additional specification that it is the representation of the parts which makes the representation of the whole possible, i.e., the whole is a multitude of antecedently given parts (e.g., an aggregate[112]). However, the concept of extensive magnitude is simply a

[109] [7], 202, [790], (12:223). The passage continues thus:

> An example of extensive magnitude would be a collection of similar things (for example, the number of square inches in a plane); an example of intensive magnitude, the notion of degree (for example, of illumination of a room). As for the dynamic functions, an example would be the synthesis of the manifold insofar as one thing's existence is subordinate to another's (the category of causality) or one thing is coordinated with another to make a unity of experience (modality as the necessary determination of the existence of appearances in time).

[110] See, for instance, [10], *Notes on Metaphysics*, R 6359, (18:687).
[111] See [24], (21:455), and [11], B203.
[112] [11], B204.

function of the understanding, by means of which one thinks the unification (combination) of many elements (a one in many) previously given, which concept follows from the category of unity *a priori*, but whose possibility (whether an object can be given which corresponds to it) is still to be shown. The pure productive synthesis of the imagination provides this concept with a corresponding intuition by means of the construction of, e.g., a line in space, which is effected through the generation (by means of motion) of all the parts of the line and their comprehension in an intuition as a whole[113].

The concept of extensive magnitude is thus realized through construction in pure intuition, but at the same time it is this concept which makes this construction possible in a unified consciousness, as in it a combination of representations is thought originally and related to one apperception through the category of quantity. Analogously, the division of, e.g., a line happens in agreement with the category of community. The category of community is the concept of a plurality of substances which are in a reciprocal relation of cause and effect with respect to their determinations. This means that any two substances are such that one is the cause of the determinations of the other and viceversa, and thus as far as their existence is concerned they are not subordinated (as the effect is subordinated to the cause), but coordinated[114]. From this original concept of combination it is straightforward to fabricate the concept of a substance (a "whole") consisting of parts which are in a relation of community. This concept finds its application in the division of a line (the whole)[115], for instance through bisection, in that here the pure synthesis of the imagination constructs two lines (parts) which partition the whole and are in a relation of community as far as their existence is concerned (the existence of one part is determined, with respect to the whole, by the existence of the other)[116].

[113][11], B138 and A162.

[114]Indeed, in the schematism chapter ([11], B184) we find:

> The schema of community (reciprocity), or of the reciprocal causality of substances with regard to their accidents, is the simultaneity of the determinations of the one with those of the other, in accordance with a general rule.

Of course, since we are dealing here with a schemata, the concept of community is expressed in term of a time-determination of simultaneity. A more abstract definition of community is given by Kant, e.g., at [11], B112 and at [7], 58, (221), (10:367).

[115]The reader will easily find that the concept of community is also necessary in the construction of the line itself, since all the parts of the line have to be considered as simultaneous, and simultaneity of events according to Kant can be determined only by means of their subsumption under the category of community (see [11], A213/B259.

[116] Indeed, Kant writes at [11], B113:

> The understanding follows the same procedure when it represents the divided sphere of a concept as when it thinks of a thing as divisible, and just as in the first case the members of the division exclude each other and yet are connected in one sphere, so in the latter case the parts are represented as ones to which existence (as substances) pertains to each exclusively of the others, and which are yet connected in one whole.

The representation of the divided sphere of a concept is obtained by means of the disjunctive judgment, which corresponds to the category of community, in that this category determines intuition with respect to it. Thus the concept of a divisible thing is obtained by means of the category of community, in

We conclude that the pure synthesis of the imagination in the construction of concepts proceeds in agreement with the categories, and indeed from them directly, in that it is just the application of the transcendental synthesis of the same faculty to a manifold of intuition which is given *a priori*. When we construct a geometrical object, we do so by means of concepts which are nothing else than rules for the combination of intuitions given *a priori*, and these rules are derived and must be in thoroughgoing agreement with the categories. The interested reader will be able to find other examples of the use of the categories in the construction of geometrical objects. In the next chapter, we shall investigate Kant's theory of geometry more closely.

the same fashion as the representation of the divided sphere of a concept is obtained by means of the disjunctive judgment.

Chapter 2

Space and geometry

§2.1 The Problem of Continuity

Consider the proof of the first Proposition of Book I of the elements[1]:

Proposition 2.1. *On a given finite straight line to construct an equilateral triangle*

Proof. Let AB be the given straight line. Thus it is required to construct an equilateral triangle on the straight line AB. With centre A and distance AB let the circle BCD be described [post. 3]; again, with centre B and distance BA let the circle ACE be described [post. 3]; and from point C, in which the circles cut one another, to the points A, B let the straight lines CA, CB be joined [post. 1]. Now, since the point A is the centre of the circle CDB, AC is equal to AB [def. 15]. Again, since the point B is the centre of the circle CAE, BC is equal to BA [def. 15]. But CA was also proved equal to AB; therefore each of the straight lines CA, CB is equal to AB. And things which are equal to the same thing are also equal to one another [C. N. I]; therefore CA is also equal to CB. Therefore the three straight lines CA, AB, BC are equal to one another. Therefore the triangle ABC is equilateral; and it has been constructed on the given finite straight line AB. $\square$

The diagram originally accompanying the proof of the proposition is reproduced in Figure 2.1. In his fifth century commentary on Euclid, Proclus divides the propositions of the Elements into those that are *problems*, which assert that a certain geometrical object can be constructed, or *theorems*, which prove that certain statements hold of a constructed geometrical object. Indeed, Euclid himself ended propositions such as the one above with the sentence "ὅπερ ἔδει δεῖξαι", i.e., *quod erat faciendum*, while theorems

[1] The translation from the elements is from [25].

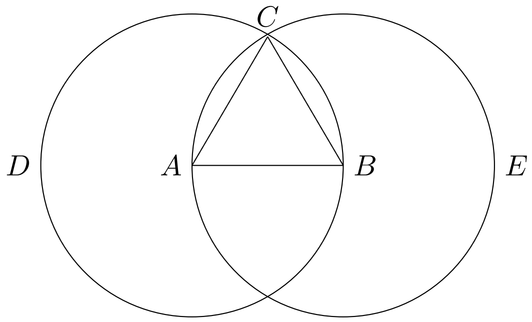

FIGURE 2.1: Construction of an equilateral triangle

were concluded with "ὅπερ ἔδει ποιῆσαι", *quod erat demonstrandum.* Proposition I above, then, purports to show that an equilateral triangle can be constructed on a given line segment, and this is effected by means of (I) a sequence of constructions of geometrical objects, where the construction of an object can rely on previously constructed objects, (II) a series of inferences, from the postulates or from previously proven theorems, regarding mereotopological or metric properties of the objects so constructed. Generally, these two phases of the proof are sequential: first all the relevant geometrical objects are constructed, and only then the inferences are drawn. For instance, in the above proof the first two steps consist in the construction of circle BCD and ACE, then their intersection point C is constructed (which depends on the previous construction of both circles), and finally line segments CA and CB are constructed. Only once all of the relevant objects have been produced, one infers the relevant metric relations between AB, AC and BC.

This method of proof employed in the Elements has recently been given a rigorous formalisation in a series of articles by Mumma ([26],[27]) and Avigad *et al.*[6], who proposed two strictly interrelated formal systems, E and Eu, not only with the aim of explaining the peculiar features of Euclid's geometrical reasoning, but also in order to at least partially defend it from its modern critics. Indeed, as already noted for instance by Heath [25] and Friedman ([1, 2]), from the perspective of the modern reader acquainted with the seminal works on the axiomatization of Euclidean geometry (e.g., Pasch [28], Hilbert [29], and Tarski [30]) the above proof of proposition I seems to not to be immune from criticism. First of all, we notice that the statement which the above proposition makes is a universal statement: for any given line segment, it is possible to construct an equilateral triangle with that segment as base. In other words, as noted in [6], the logical form of Euclid's propositions is something like the following:

$$\forall \vec{a}, \vec{L}, \vec{\alpha}(\phi(\vec{a}, \vec{L}, \vec{\alpha}) \rightarrow \exists \vec{b}, \vec{M}, \vec{\beta}(\psi(\vec{a}, \vec{b}, \vec{L}, \vec{M}, \vec{\alpha}, \vec{\beta}))$$

Where $\vec{a}, \vec{b}$ refer to vectors of points, $\vec{L}, \vec{M}$ to vectors of lines, and $\vec{\alpha}, \vec{\beta}$ to vectors of circles, and where the existential statement should be interpreted constructively: it is possible to *construct* (Kant: exhibit in pure intuition) $\vec{b}, \vec{M}, \vec{\beta}$ such that $\psi(...)$ holds. Notice that if $\vec{b}, \vec{M}, \vec{\beta}$ are not empty, as in the above proposition, the proposition is a problem in Proclus' sense, while if they are empty it is a theorem. If we look back to the proof above from a modern perspective, though, the customary criticism is that Euclid has not really proven that point C, at which the two circles allegedly intersect, exists. Better, he has not proven *from the postulates* that the point exists; he just infers its existence immediately from the fact that the two circles cross in the diagram, i.e., "cut one another". However, even the basic fact that the two cicles intersect, i.e., that they cross each other as presented in the figure, is itself not justified by any of the postulates, but is inferred from the figure itself. The reader must then reach the unavoidable conclusion that the only argument that Euclid seems to give for the existence of point C, in which the two circles intersect, simply relies on inspection of the accompanying diagram. If this is the case, however, it is difficult to see how the universal statement of the proposition is justified; for the fact that in *this* particular diagram the two circles intersect at point C is hardly a proof that for *any* finite segment, and *any* two circles having this segment as radius and, respectively, its two end points as a center, these two circles intersect. It seems then impossible to generalize the proposed proof to any arbitrarily given line segment.

The modern standpoint solves the aforementioned problem by specifying explicitly, i.e. axiomatically, the properties of the universe of points, which is seen as primitive and as underlying geometric constructions. For instance, Hilbert posits a primitive universe consisting in points and lines[2], while Tarski considers only points as primitive; both, however, ensure the existence of the intersection point C above by including specific axioms establishing the order and cardinality of the set of points on a line. Tarski, e.g., includes among his thirteen axioms the axiom schema of continuity (A13 in [30]), i.e., the infinite collection of sentences of the following form:

$$\exists z \forall xy(\phi(x) \wedge \psi(y) \rightarrow \beta(zxy)) \rightarrow \exists u \forall xy(\phi(x) \wedge \psi(y) \rightarrow \beta(xuy))$$

where $\beta(zxy)$ stands for the relation "x is between z and y" (the cases in which $x = z$ or $x = y$ are excluded), and $\phi(x)$ stands for a formula in which variables y, z and u do not occur free, and analogously for $\psi(y)$ (with x and y interchanged). We can see that this axiom schema is essentially a first order version of the Dedekind's cut construction, whose full formalization, requiring second order quantification, is hence not suitable for Tarski's purposes of axiomatizing elementary geometry. Indeed, Tarski's

[2]In the case of planar geometry; for three-dimensional geometry, planes are added.

axiomatization requires us to assume even more than what we would really need to carry out Euclidean constructions. It is a well known result that the constructions of Euclidean geometry (which are effected by means of straightedge and compass) have models which are isomorphic to the Cartesian space[3] over the Euclidean ordered field[4] $\mathbb{Q}^*$, obtained by closing the rationals under the operation of taking square roots. Tarski proves ([30], theorem 1), however, that every model of his axioms (A1-A13) is isomorphic to the Cartesian space over any real closed field[5], and thus his axiomatization commits to the existence of more points than is necessary for carrying out Euclidean geometric constructions (which only require the existence of the intersection points of lines and circles). However, he also notices that we can replace the continuity axiom schema with the circular continuity axiom, stating that a line segment joining two points, one inside and one outside a given circle, must intersect this circle in one point. If we make this substitution, we obtain a theory which is incomplete and finitely axiomatizable, whose models are exactly those isomorphic to the Cartesian space over an Euclidean ordered field. At the other end of the spectrum, of course, if we substitute the first order axiom schema of continuity with the full second-order axiom, then every model of our theory will be isomorphic to $\mathbb{R}^2$.

In light of these remarks, it seems then that Euclid's Elements are lacking some fundamental postulates, without which the proofs do not really go through, and that his system is thus fundamentally flawed. It would be a mistake, however, to reach this conclusion on the basis of the arguments expounded above. Indeed, these arguments presuppose a modern, classical view of geometry which seems very far from Euclid. In particular, they presuppose the existence of a universe of primitive geometrical elements, such as points and lines, which are not constructed, and whose specific properties are established via a set of axioms. On the contrary, Euclid's approach seems opposite, in that the existence of a certain geometrical object is not a consequence of a set of existential axiom, but of a concrete step by step sequence of straightedge and compass constructions. For instance, in Euclid the existence of a point B which splits line segment AC into halves must be proven through bisection: by means of a finite iterative process of construction with straightedge and compass, one exhibits a specific point B lying on segment AC and hence constructs the two segment AB and BC, whose metric properties are then proven from the postulates and common notions. Accordingly, one can think of point C in the above proof as arising by means of the sequence of mental acts involving the construction of the two circles BCD and ACE. The diagram accompanying the proof is then supposed to be an illustration of this mental process of construction, and it

[3] That is, set of pairs.

[4] An ordered field $\mathcal{K}$ is Euclidean when $\forall x \in \mathcal{K}(x > 0 \to \exists y \in \mathcal{K}(x = y^2)$.

[5] A real closed field F is an ordered field such that it is Euclidean and any polynomial of odd degree with coefficients in F has at least one root in F.

therefore suffices to prove that point C is really there. System E ([6]) models this process by introducing, along with construction rules and metric axioms, a set of so-called diagrammatic inferences: a set of axioms which allow for drawing certain conclusions from the merotopological configuration of the elements that have been constructed. It is illustrative to examine how the proof of Proposition I above is carried out in system E:

Proposition 2.2. *Assume a and b are distinct points, construct point c such that $ab = bc$ and $bc = ca$*

Proof. Let α be the circle with centre a passing through b
Let β be the circle with centre b passing through a
Let c be a point on the intersection of α and β
Have $ab = ac$ [since they are radii of α]
Have $ab = bc$ [since they are radii of β]
Hence $ab = bc$ and $bc = ca$. □

The above proof is expressed in natural language for the sake of clarity, but it could easily be translated into a formal sequent proof. If a and b are two different points, construction rule 2 of E allows us to contruct a circle α with centre a passing through b (having b on its circumference), and a circle β with centre b passing through a. After this construction has been carried out, one can apply diagrammatic axiom 5 in order to infer a diagrammatic consequence - namely, the intersection of the two circles:

> If a is on α, b is in α, a is in β, b is on β, then α and β intersect

Since the two circles intersect, construction rule 6 allows us to construct point c on this intersection. The diagrammatic facts that ab is the radius of α and ba the radius of β, along with some trivial metric axioms, give us the last three metric inferences. We can see that in system E we are able to prove that the two circles intersect in a point by first performing a diagrammatic inference concluding that the two circles must intersect, and then by constructing a point at this intersection thanks to one of the construction rules; existence of the intersection point is then inferred through a sequence of mental constructions and inferences from these constructions, which lead to the exhibition of a witness, point c above. The formalization of Euclidean proofs in system E also shows that the inferences which Euclid draws from his diagrams are fully general, in the sense that they hold for any diagram involving the construction of the same geometrical objects, regardless of how these constructions might differ in practice. Thus, system E does not only provide an accurate account of Euclid's proof method, but can also be taken to represent Kant's constructive stance on geometrical reasoning. For instance, the answer provided by system Eu [26, 27] to the sticky philosophical problem of pinning down what exactly it is that makes a diagrammatic inference general (i.e., an inference from the construction of the object in pure intuition) is analogous to that which we have seen in the previous chapter, when treating the relation between the construction of pure sensible concepts and the theorems of geometry. In Eu proofs of propositions are not a list of sequents as in E, but are based on particular diagrams (we could say, on particular *schemata* of pure *a priori* imagination); the prover, however, is not allowed to use in the proof all the features of the geometrical objects occurring in the diagram, but only a restricted set of these appropriately identified, which hold of the objects exclusively because of the general properties of their construction. This is in agreement with the analysis provided in the previous chapter regarding Kant's stance on theorems of geometry. There, we saw that according to Kant there exist properties which hold of geometrical objects only in virtue of the way they have been constructed, and which thus must hold generally for any object (intuition) that instantiates the relevant geometrical concept. On the same line, what is general in Eu (and in E) depends on how the geometrical objects under consideration have been constructed. The diagrammatic axioms, for instance, are supposed to capture exactly those mereotopological properties which hold of the object in virtue of its construction; consider for instance axiom 1 of the intersection axioms in [6]:

> If a and b are on different sides of L, and M is the line through a and b, then L and M intersect.

In light of the previous discussion, we can interpret this axiom as stating that given an object constructed by the pure productive imagination, consisting in a line L and two points a and b on opposite sides of L (a mereotopological condition which denotes the way the object has been constructed) and a line M through a and b, it then follows from this construction in the pure imagination that lines L and M intersect. Systems E and Eu seem then to adequately characterize the constructive aspect of Kant's (and Euclid's) conception of Euclidean geometry.

There is, however, a problem. In order for systems E and Eu to be fully constructive (also in the Kantian sense), every construction rule should be able to be represented logically by a function producing a singular geometrical object; in other words, we would need to be able to substitute, for the existential quantifiers in these construction rules, the corresponding Skolem functions exibiting the object. For instance, given the construction rule 2 for lines and circles:

$$\forall a \forall b (a \neq b \rightarrow \exists \alpha (Centre(a, \alpha) \land On(b, \alpha)))$$

We can replace the existential statement with the Skolem function $f_c(a, b)$ which takes two (different) points as input and produces the circle α with centre a and passing through b. However, both E and Eu contain construction rules which are interpreted as allowing the prover to pick an arbitrary point satisfying specific mereotopological or positional conditions with respect to objects that have been already constructed. For instance, we find in E the following (construction rule 8 for points):

$$\forall \alpha \exists a (Inside(a, \alpha))$$

This rule states that for any circle, there is a point inside that circle. We can substitute for the existential quantifier in the above rule a Skolem function which would pick a specific point inside α (for instance, the centre of the circle). However, the interpretation that Avigad *et al.* give of this axiom is that the prover is allowed to pick any arbitrary point inside the circle. Since the spatial location of this point inside the circle is not determined univocally, but has to be chosen by the prover somehow arbitrarily, it is impossible to express this interpretation of the axiom constructively by means of a function. Analogously, the prover in E is allowed to pick an *arbitrary* point on a line satisfying certain positional conditions with respect to other points on the same line. Even worse, E allows us to pick any arbitrary point different from any previously given point, which rule cannot of course be expressed constructively by means of a function yielding a specific geometrical object. In other words, system E seems to rely on an infinite universe of previously given points, among which the prover is allowed to choose those that satisfy determinate positional conditions. Mumma acknowledges this problem openly:

> In the construction stage, most steps produce unique geometric objects from given ones and so can be represented logically by functions. Yet one kind does not: the *free* choice of a point satisfying non-metric positional conditions [...] the natural logical representation for what licenses their introduction are thus existential statements [...] And so, though the geometric reasoning in Eu is always performed with a particular finite diagram, it still seems to *presuppose* a domain of geometric objects [...][6]

This situation is a symptom of a more general problem, i.e., that of giving a constructive account of the notions of continuity and infinity of space. When the prover in E is

[6] [27], pp. 117-118, my emphasis.

allowed to freely choose a point, he is choosing among an infinity of these points in a spatial medium which is conceived as continuous. As also noted by Mumma[7], the philosophical question which would need to be addressed in order to make E and Eu fully constructive then is: what constructive justification can be given for the fact that we conceive of geometrical space as a continuous medium, and hence on this basis we perform the cognitive act of freely choosing in it a (dimensionless) point ? An answer to this question can be obtained, I believe, if we combine the analysis of Kant's theory of mind as given in the previous chapter with his treatment of the notions of geometrical point and line, to which we now turn.

§2.2 Kant's constructivism: continuity, points and lines

Kant's conception of logic has certainly not been judged positively by logicians and philosophers, at least since Frege laid the foundations of our modern perspective on the discipline. A widely held view has been that what Kant calls general logic constitutes nothing more than a haphazard compilation of Aristotelian inferences and syllogisms, while his transcendental logic has been considered "terrifyingly narrow minded and mathematically trivial"[8]. This negative view of Kant's logic has been highly influential, in particular in the analytic world, and has insinuated itself, as a received fact, into the exegesis of Kantian philosophy as a whole, even in the works of those commentators who seem most sympathetic to Kant's philosophical efforts.

An example of this phenomenon is Friedman's influential work on Kant's theory of geometry, as has been presented in [1, 2]. Indeed, Friedman considers the expressive poverty of Kant's logic to be the core interpretative key in order for us to understand his notion of construction in pure intuition of geometrical objects. The argument runs roughly as follows. Contemporary classical axiomatizations of geometry, as we have seen them in the previous section, rely heavily on the tools of modern polyadic logic, whose development dates back to the end of the nineteenth century, with the seminal works of Frege and Russell. In particular, these axiomatizations require the capability of expressing quantifier dependencies such as $\forall\exists$ and $\forall\exists\forall$. We can, for instance, enforce density on our primitive universe of points by including the axiom $\forall a \forall b (a \neq b \rightarrow \exists c (c \neq a \wedge c \neq b \wedge \beta(acb)))$, whose logical form is exactly $\forall\exists$. The circular continuity axiom (A13') of Tarski's axiomatization[9] also has logical form $\forall ... \forall\exists$, and thus makes essential use of modern quantification theory. The continuity schema, of course, requires even more expressive strength, its

[7] *ibid.*, p. 118.
[8] See [31]
[9] [30], p. 174

logical form being ∀...∀∃∀∃∀. Even the possibility of expressing the existence of an infinity of different objects (e.g., points, or natural numbers in arithmetic) makes crucial use of the quantifier dependency ∀∃.

Kant's logic, however, is essentially Aristotelian, and thus monadic. It is therefore impossible to express with it any quantifier dependency. Indeed, it is a basic result from model theory that any set of sentences Σ in which a finite number k of different monadic predicates $(P_1, ..., P_k)$ occur has a model with at most 2^k objects; hence, with monadic logic alone one cannot even state the existence of an infinite number of different objects, as required by any serious mathematical theory. Friedman then argues that Kant resorted to his constructive approach in order to make up for the deficiencies of his logical system. Unable to express existential axioms by means of a judgment of the understanding, Kant resorted to a process of construction in pure intuition which corresponds to our use of Skolem functions:

> The notion of infinite divisibility or denseness, for example, cannot be represented by any such formula [...] the logical form simply does not exist. Rather, denseness is represented by a definite fact about my intuitive capacities: namely, whenever I can represent (construct) two distinc points a and b on a line, I can represent (construct) a third point c between them. [...] So we do not derive new points between A and B from an existential axiom, we construct a bisection function from our basic operations[10] and obtain the new points as the values of this function. In short, we are given what modern logic calls a Skolem function for the existential quantifier [...][11]

Analogously, infinity of space cannot be derived from judgments, but is founded on the (potential) infinite iterability of specific spatial constructions, which presuppose a pure intuition of space that is given and in which these constructions take place:

> This, I suggest, is why Kant gives priority to the singular intuition *space*, from which all parts or *spaces* must be "cut out" by intuitive construction ("limitation"). Only the unbounded iterability of such constructive procedures makes the idea of infinity, and therefore all "general concepts of space", possible[12].

Potentially infinite iteration of geometric constructions in intuition thus takes the place of classical existential axioms; density is not enforced through an existential statement,

[10]That is, the operations of construction by straightedge and compass.
[11][1], pp. 467-468
[12]*ibid.*, p. 474

but it is represented by iteration of a definite process which, given two distinct points, constructs a third between them. It is for this reason, according to Friedman, that Kant thought of geometry as a pure *a priori*, and not analytical, discipline.

Of course, while infinite iterability of finite constructions (such as bisection) is sufficient to generate all the points needed to carry out Euclidean geometry (namely, all the points in $\mathbb{Q}^*$), it is by no means enough for the needs of more advanced mathematical theories which were under development between the seventeenth and eighteenth centuries. The calculus, in particular, goes well beyond Euclidean geometry in considering arbitrary curves in the plane and limit operations, and thus requires the full continuity which only the continuum $\mathbb{R}^2$ can provide. According to Friedman, this is the reason which explains the importance of motion in Kant's theory of geometry. Infinite iteration of bisection generates an infinite number of points, but every point p_n is determined by a sequence of previously constructed points $p_1, ..., p_{n-1}$ which is finite. Generation of a point through the limit operation requires however an *actual* infinity of previously given points, as expressed by the Cauchy completeness axiom, which states that every infinite sequence of point $p_1, p_2, ...$ must converge to a given limit r. In other words, to generate all the real numbers one must be in possess of a much stronger notion of infinity than that involved in a mere process of iterated construction. Since Kant did not have at his disposal the logical tools to express a complicated continuity axiom such as Cauchy's, he resorted to motion in order to generate or construct the continuum of the real numbers:

> Thus, for example, we can easily "construct" a line of length π by imagining a continuous process that takes one unit of time and is such that at $t = \frac{1}{2}$ a line of length 3.1 is constructed, at $t = \frac{2}{3}$ a line of length 3.14 is constructed, and in general at $t = \frac{n}{n+1}$ a line of length s_n is constructed, where s_n again equals the decimal expansion of π carried out to n places. Assuming this process in fact *has* a terminal outcome, at $t = 1$ we have constructed a line of length π. In some sense, then, we can thereby "construct" any real number[13].

Thus a "dynamic" theory of motion, essentially derived from the theory of fluxions, does the work of our "static" continuity axioms. Every point undergoing a process of finite motion (i.e., motion in a finite time) on a line stops at a definite point on that line; and through this process this end point is thereby "constructed", and full continuity (i.e., existence of all the points of the real line) is ensured.

Despite the convincing aspects of Friedman's analysis, it seems to me to be based on the wrong assumption that Kant's logic is essentially Aristotelian, and hence monadic;

[13] [1], p. 477.

an assumption which, as I have remarked above, has been widely held in the analytic philosophical world. Recent works ([3, 5]), however, have finally started to show that this assumption is essentially inappropriate. In particular, Achourioti and van Lambalgen have convincingly argued in [3] that the logical form of Kantian judgments is far more complex than what can be achieved through simple subject-predicate Aristotelian logic. Consider, for instance, the following passage from Kant's treatment of causality:

> If I consider a ball that lies on a stuffed pillow and makes a dent in it as a cause, it is simultaneous with its effect[14].

The judgment underlying this passage is: "if a ball lies on a stuffed pillow, it makes a dent in that pillow". Achourioti and van Lambalgen argue that the appropriate formalization of this judgment would involve the use of the quantifier dependency $\forall\exists$. It would then be something like:

$$\forall x \forall y \forall t (B(x) \wedge P(y) \wedge LO(x, y, t) \rightarrow \exists z (D(z) \wedge In(z, y, t))$$

Where $B(x)$ stands for "x is a ball", $P(y)$ stands for "y is a pillow", $LO(x, y, t)$ stands for "x lies on y at time t", $D(z)$ stands for "z is a dent", $In(z, y, t)$ stands for "z is in y at time t". Of course, the above judgment involves both the use of relations and of the logical form $\forall...\forall\exists$. Notice also that the above judgment involves quantification over time instants; Kant was indeed very explicit in making the point that temporal conditions are crucial for hypotetical judgments:

> If I say "a person who is unlearned is not learned", the condition **at the same time** must hold; for one who is unlearned at one time can very well be learned at one other time [...][15].

In other words, the logical form of the judgment "every person who is unlearned is not learned" must be something like $\forall x \forall t (P(x) \wedge U(x, t) \rightarrow \neg L(x, t))$, which involves the use of two place relations and is thus not monadic[16]. Another, clear example of the

[14][11], A203/B246.

[15][11], A153/B192. The bold text is Kant's.

[16]The discussion in this passage regarding the principle of contradiction, and in particular the difference between a predicate contradicting a concept synthetically combined with that of the object, and it contradicting the concept of the object itself, is beautiful and highly recommended. Notice, in any case, that the above is a very clear example of a judgment which is clearly *not* monadic. Friedman writes ([1], p. 466):

> If we do not limit ourselves to the traditional forms of syllogistic logic, Kant's table of judgments makes no sense [...]

I think the reader will clearly see that there is no hope of expressing the judgment above, including the temporal constraint, without going beyond tranditional Aristotelian logic.

hidden formal complexity of Kantian judgments is again given by Achourioti and van Lambalgen. The following judgment:

> [...] if a body is illuminated by the sun for long enough, it becomes warm[17].

Can be formalized as follows:

$$B(x) \wedge I(x,y,s,t) \wedge (s-t>d) \wedge T(x,t,v) \rightarrow \exists w(T(c,s,v+w) \wedge (v+w>c)$$

Where variables x, t, s, v are universally quantified, $B(x)$ stands for "x is a body", $I(x,y,s,t)$ stands for "y illuminates x between times s and t", $T(x,t,v)$ stands for "the temperature of x at time t is v", d is a criterion for "long enough" and c is a criterion for "warm". Notice the logical complexity of this judgment, with a 4-ary relation I and the quantifier dependency $\forall\exists$.

From their analysis, Achourioti and van Lambalgen conclude that the logical forms of judgments in Kant's Transcendental Logic is that of restricted $\forall\exists$ formulas:

Definition 2.3. Let $\mathcal{L}$ be a first-order language. A formula is positive primitive in $\mathcal{L}$ if it is constructed from atomic formulas in $\mathcal{L}$ using $\vee, \wedge, \perp$ and $\exists$.

Definition 2.4. A formula in $\mathcal{L}$ is restricted $\forall\exists$ if it is of the form $\forall\vec{x}(\phi(\vec{x}) \rightarrow \psi(\vec{x}))$, where ϕ and ψ are positive primitive.

Notice that the general form of theorems in Euclidean geometry is exactly that of restricted $\forall\exists$ formulas. Achourioti and van Lambalgen have developed[18], on the basis of a well grounded interpretation of the Kantian notion of threefold synthesis[19], a formal semantics and a notion of validity[20], and proved that only restricted $\forall\exists$ formulas are valid according to the semantics. These results constitute a strong formal argument supporting the claim that not only Kantian judgments are much more complex than what can be obtained on the basis of simple Aristotelian logic, but also that Kant's table of judgments is in a sense sound and complete with respect to the semantics of distributed objects of synthesis which he had in mind.

This analysis has far reaching consequences if we consider the geometrical domain. As we have seen above, restricted $\forall\exists$ formulas are sufficient to express the axioms of density

[17] [8], (4:312).

[18] [3], section 9.1.

[19] Threefold synthesis refers to the synthesis of apprehension in intuition, of reproduction in imagination, and of recognition in a concept, as expounded by Kant starting at [11], A99.

[20] The notion of validity developed by Achourioti and van Lambalgen is intended as a formalization of the notion of objective validity in Kant; see [3], sections 8 and 9.

and circular continuity, which at least suffice to assert the existence of all the points needed for the purposes of Euclidean geometry. A question which naturally arises, and which we must address, is then the following: if Kant had at his disposal such complex logical forms, why did he decide to resort to a constructive approach in the first place, instead of augmenting the set of Euclid's axioms with some (existential) axioms for density or circular continuity? The answer to this fascinating problem, I believe, can be found in Kant's theory of cognition. Indeed, we must consider that, for Kant, any concept (and thus mathematical and geometrical concepts alike) is real and objectively valid only in virtue of the fact that it is applicable to objects outside us, i.e., it refers to external (empirical) objects:

> Thus all concepts and with them all principles, however *a priori* they may be, are nevertheless related to empirical intuition [...] without this they have no objective validity at all, but are rather a mere play, whether it be with representations of the imagination or of the understanding.[21].

In other words, any concept acquires its significance only because it refers to some possible objects of empirical intuition. This also applies to the concepts or axioms of pure mathematics, whose objective validity (which only determines their relation to an object, and hence their relation to the truth[22]) depends on the fact that they can be exhibited in empirical intuition. However, if mathematical concepts and principles are to be exhibited in empirical intuition, then they must in turn be made sensible first, i.e., they must be constructed in *a priori* intuition. To every mathematical concept or principle there must correspond a constructive procedure of a (form of) an object:

> Hence it is also requisite for one **to make** an abstract concept **sensible**, i.e., to display the object that corresponds to it in intuition, since without this the concept would remain (as one says) without **sense**, i.e., without significance. Mathematics fullfills this requirement by means of the construction of the figure, which is an appearance present to the senses (even though brought about *a priori*)[23].

[21] [11], A239/B298. The passage continues by considering the case of mathematical concepts:

> One need only take as an example the concepts of mathematics, and first, indeed, in their pure intuitions. Space has three dimensions, between two points there can be only one straight line, etc. Although all these principles, and the representation of the objects with which this science occupies itself, are generated in the mind completely *a priori*, they would still not signify anything at all if we could not always exhibit their significance in appearances (empirical objects)

[22] see [3], section 8.
[23] [11], A240/B299.

Construction in intuition is thus essential for a mathematical concept or principle to be able to refer to an object, whereby it acquires its objective reality and its applicability to the world of appearances. If we express density through an existential axiom, this is by itself only a judgment of the understanding, stating that for every two points a and b, there is a point c between them. This judgment could easily be a mere "play of representations", in the sense that it might very well be that no object could correspond to it, in the same way as no object can correspond to the judgment of the understanding stating that every unicorn is white. It is only through the process of construction by bisection that the density axiom appears no more as just a combination of representations by the understanding, but as a real condition of sensible intuition; and this is only because a general procedure is given through which we can instantiate the axiom (which is a judgment, and thus a rule) in intuition, i.e., construct the object (point c)[24]. Thus, an *a priori* concept or a principle (axiom) acquires objective reality only if it can be constructed (exhibited) in intuition[25]. Since empirical intuition is possible only through pure intuition, if a concept can be given an object in the latter through a general construction process, then it also must apply to the former[26]. Kant's constructive approach seems then to be dictated more by cognitive reasons than by intrinsic limitations of his logic.

[24]Consider also what Kant says at [11], B287:

> Now in mathematics a postulate is the practical proposition that contains nothing except the synthesis through which we first give ourselves an object and generate its concept, e.g., to describe a circle with a given line from a given point on a plane [...] a proposition of this sort cannot be proved, since the procedure that it demands is precisely that through which we first generate the concept of such a figure.

What I take Kant as saying here is that axioms are nothing else than rules that can be used in a constructive procedure, and thus they must be "made sensible", i.e., an object that instantiates their concept must be constructed by the *synthesis speciosa*. For instance, the axiom stating that given any point and line segment, a circle can be constructed having that point as a centre and that segment as a radius, is nothing else than a rule for the synthesis in intuition *a priori* by the *synthesis speciosa* of an object instantiating the concept of a circle. Given any point and line segment, the *synthesis speciosa* can produce a circular motion of the line segment around that point, thereby constructing a circle. Axioms, then, acquire their significance only if there is a procedure by the *synthesis speciosa* which provides them with a corresponding construction in pure intuition of an object.

[25] Consider the following passage from [10], *Notes on Logic*, R 2836, (16:539):

> *a. Conceptus est cogitatio. b. Exhibitio cogitati est relatio conceptus ad intuitum. c. Exhibitio a priori: constructio* (objective reality of the concept through presentation).

[26]Consider, e.g., the following quote from [11], A166/B207:

> Empirical intuition is possible only though the pure intuition (of space and time); what geometry says about the latter is therefore undeniably valid of the former, and evasions, as if objects of the senses did not have to be in agreement with the rules of construction in space (e.g., the rules of infinite divisibility of lines and angles), must cease [...] the synthesis of spaces and times, as the essential form of all intuition, is that which at the same time makes possible the apprehension of the appearance, thus [...] what mathematics in its pure use proves about the former is also necessarily valid of the latter.

The discussion in the previous chapter regarding the relation between the productive synthesis of the imagination *a priori* and the synthesis of apprehension is also relevant.

These same observations can also be used to clarify Kant's insistence on the notion of *potential* infinity. Consider the following important passage from the Metaphysical Foundations of Natural Science:

> [The philosopher] is thereby helped out of that difficulty due to the *infinite divisibility* of matter, whereby it still does *not* consist of *infinitely many parts*. Now this latter can perfectly well be thought through reason, even though it cannot be made intuitive and constructed. For what is only actual by being given in the representation also has no *more* given of it than what is met with in the representation - no more, that is, than the progress of representation reaches[27].

It is very well possible, according to Kant, to think through *reason* a totality of infinitely many parts, i.e., an actual infinity. It is, of course, what we do for instance with the existential axiom of density, by means of which we *think* an infinite series of points; or what we do with Peano's axioms, by means of which we *think* an actual infinity of natural numbers. It is also very well possible that an actual infinity exists in the *noumenon*. As far as human cognition is concerned, however, an object must be able to be constructed in intuition which corresponds to the concept. Since every process of construction in intuition is clearly finite (involving a finite number of steps), it follows that the only kind of infinity which can not simply be thought, but also be *cognized*, is that of the potentially infinite iterability of this construction. In other words, with potential infinity what we cognize is the possibility of an unbounded and progressive repetition of a constructive procedure, in which at every stage a finite number of new elements is introduced (constructed). The same remarks about infinity we find in the Antinomies of Pure Reason, for instance at *locus* A501/B529, which underpins the temporal interpretation I have given above with the adjective *progressive*: empirical synthesis is given only in time, one member after another, and thus one cannot presuppose an infinite totality of this synthesis to be given, but only a potential infinity is possible, given through successive iteration of this synthesis. Analogously, the process of construction of an object in pure spatial intuition by the activity of the *synthesis speciosa*, on which the empirical synthesis is founded, is a successive process which takes place in time; hence the infinite in it is never given as actual, but it can only consist in the "condition of the possibility of the *progressus in infinitum* or *indefinitum*"[28][29].

[27] [15], [507].

[28] [10], *Notes on Metaphysics*, R 5893, (18:377).

[29] Again, consider the following excerpt from Aristotle ([32], III, 7, 207 b 27):

> This account does not deprive the mathematicians of their study, though it does do away with anything's being infinite in such a way as to be actually untraversable in the direction

We are thus led to reject Friedman's claim that Kant's logic is essentially Aristotelian, along with the inference that his constructive approach was devised to supply to the deficiencies of his logic.

The next point to be addressed is that of continuity and of its relation to motion and the calculus. We have already remarked above how a process of iteration of finite constructions, carried out by the synthesis of the imagination *a priori*, would be able to produce only those points that we obtain by closing $\mathcal{Q}$ under the operation of taking square roots. The calculus, however, requires the construction of the full real numbers, i.e., of the full continuum. We have seen that Friedman claims that motion is responsible for constructing the missing points. I would like to challenge this view here. The *locus* where Kant is most explicit regarding continuity is the following important passage from the axioms of intuition[30]:

> The property of magnitudes on account of which no part of them is the smallest (no part is simple) is called their continuity. Space and time are *quanta continua*, because no part of them can be given except as enclosed between boundaries (points and instants), thus only in such a way that this part is again a space or a time. Space therefore consists only of spaces, time of times. Points and instants are only boundaries, i.e., mere places of their limitation; but places always presuppose those intuitions that limit or determine them, and from mere places, as components that could be given prior to space or time, neither space nor time can be composed. Magnitudes of this sort can also be called **flowing**, since the synthesis (of the productive imagination) in their generation is a process in time, the continuity of which is customarily designated by the expression "flowing" ("elapsing").

We can infer from this passage the conclusion that continuity of a magnitude for Kant means that this magnitude does not have a simple (smallest) part. In this notion, however, two concepts are contained. First, taking into account Kant's constructivism, this means that given any part x of a magnitude y one can construct a part z of y such that z is contained in x (z is a part of x). Second, this means that points, since they are simple, are not to be considered parts of space, but only *boundaries*. It would then be a

> of increase (ἀδιεξίτητον). For as it is, they have no need of the infinite (for they do not use it), but they need only that something finite can be as great as they want.

Heath translates ἀδιεξίτητον as "[it] cannot be gone through". In other words, it is impossible to give an actual infinity of objects in intuition. It is instructive to examine the entire discussion on the infinite contained in chapter 7 of the Physics; there, Aristotle also says that "the bisections of a magnitude are infinite" ([32], III, 7, 207b 10), which claim is also Kant's, and as we shall see it will prove important in characterising the action of the *synthesis speciosa*.

[30] [11], B211.

great mistake to equate this definition with infinite divisibility, as Friedman seems to do. Indeed, infinite divisibility of spaces holds even in systems of geometry, such as Tarski's, in which points are primitive, and thus certainly parts, of space; Kant, however, makes the strongest claim that no part exists which is simple. It follows that points can only be *boundaries* of spaces, which means that in themselves they have no reality, but are only defined (constructed) by means of the spaces which they delimit. That this is the case is confirmed by various *loci* in Kant's fragments. For instance, it is claimed that:

> Space and time do not consist of simple parts (their parts are themselves magnitudes), i.e., absolute unities: continuity[31].

It is interesting that Kant put this emphasis on the fact that continuity does not simply follow from infinite divisibility (every part of space is in turn a space), but from the stronger fact that points have a different epistemological status than spaces (since they are just boundaries of these). The acquainted reader will notice that Kant's notion of point is similar in spirit to what we find in the contemporary literature on mereotopology, a discipline originally initiated by Whitehead and De Laguna ([33–36]) which purports to develop a theory of space in which the primitive notion is not that, epistemologically problematic, of a dimensionless point, but of a spatial *region*. For instance, we can let the set of regions be the set $ROS(\mathbb{R})$ of the regular semi-algebraic[32] open subsets of $\mathbb{R}$, and define a point as a pair (r_1, r_2) of regions such that the following formula holds:

$$C(r_1, r_2) \wedge \forall y_1 \forall y_2 (y_1 \leq r_1 \wedge y_2 \leq r_2 \wedge C(y_1, r_2) \wedge C(y_2, r_1) \rightarrow C(y_1, y_2))$$

Where $C(x, y)$, "x is in contact with y", holds only if $x^- \cap y^- \neq \emptyset$, and $x \leq y$ holds only if x is a subset of y. In other words, a point is a pair of regions such that the intersection of the closures of these regions is a singleton set. This definition is, conceptually, very much akin to Kant's, in that a point is conceived here just as boundary between two regions, i.e., a limit which separates two spaces, without independent (primitive) reality

[31][10], *Notes on Metaphysics*, R 4756, (17:700). Consider also what is claimed at [24], (21:459):

> The combination of reality with the concept of magnitude (is intensive), namely absolute unity of reality can have magnitude. But what does not have reality and is absolute unity (the point) has no magnitude.

and again at [10], 4756 (17:699):

> All parts of space are in turn parts. The point is not a part, but a boundary. Continuity.

[32]A subset y of $\mathbb{R}^n$ is regular open if it is open and $y = y^{-^{\circ}}$. A subset y of $\mathbb{R}^n$ is semi-algebraic if it can be written as a finite boolean combination of sets of the form $\{\vec{x} : f(\vec{x} > 0\}$ and $\{\vec{x} : g(\vec{x}) = 0$, where f, g are polynomials in $x_1, ..., x_n$ over the reals. For some arguments on why this is a good choice as a definition of a region, see [37].

with respect to the intuitions that define it[33]. We can then interpret Kant as saying that continuity of a spatial magnitude follows from the (potential) infinite divisibility of any of its parts, *along with* the strong epistemological claim that points are not primitive cognitive entities, but are constructed at the boundary of spaces (here, Kant foresees contemporary mereotopological research[34], albeit in a constructive setting).

On this basis a difference is drawn between a continuous magnitude (a *quantum*, such as space and time) and a mere multitude. While the latter is given along with a predetermined unity, and is therefore made up of a collection of discrete elements, the former cannot be given as a collection of discrete elements (e.g., points or instants), but is instead such that every part of it can in turn be constructed as a sum of homogenous parts:

> The **geometrical** law of continuity: space and time, therefore spatial and temporal quantities are continuous, i.e., each of their parts in a homogeneous whole are themselves quantities. Any part of them is a sum of homogeneous parts: discrete quantities in them are contradictories, except in the sense that any space is a sum of homogenous parts. E.g., a vessel full of fruit is not a quantity of fruit, except in abstraction from the intervals between the materials of the fruit which fills the space. - A discrete quantity is a **multitude**[35].

These passages provide sufficient support for understanding why it would be a mistake to equate Kant's notion of continuity to the mere infinite divisibility of space expressed by the concept of density, which, indeed, was the customary interpretation of this notion before Dedekind. The distinction between multitudes and *quanta continua* illustrates that for Kant continuity of a magnitude (e.g., a line segment) means that the magnitude is not made up of discrete elements (points), but is such that any parts of it which can

[33]Of course, we are here in a non-constructive setting, as we are presupposing the existence of a well defined set of regions.

[34]It is interesting to notice that also the mereotopological notion of contact, the relation C interpreted as the set $\mathcal{C} \subset ROS(\mathbb{R})^2$ of the pairs of regions whose closures intersect, was somehow foreseen by Kant, although for him two spaces are in contact only if their frontiers intersect ([15], [512]):

> Contact in the mathematical sense is the common boundary of two spaces, which is therefore whithin neither the one nor the other space. Thus two straight lines cannot be in contact with one another; rather, if they have a point in common, it belongs as much to one of these lines as to the other when they are produced, that is, they intersect. But a circle and a straight line, or two circles, are in contact at a point, surfaces at a line, and bodies at surfaces[...]

Of course, neither lines in the plane nor circles are regular semi-algebraic open sets. However, the idea that contact between two spaces means that there is a common boundary which does not belong to either space is at the basis of contemporary mereotopology.

[35][10], *Notes on Metaphysics*, R 6338a, (18:664)

be constructed can in turn be represented as a sum of homogenous parts, and so on *ad infinitum*, in a process of iterated decomposition in which points are only created at the boundaries between the constructed parts (as limits of intuitions)[36].

In the passage quoted above from the axioms of intuition, Kant also states that continuous magnitudes are *flowing* magnitudes, since their production by the *synthesis speciosa* is a process in time which is continuous. Friedman takes this clear reference to the Newtonian theory of fluents (Kant uses in the passage the term *fliessende Grössen*, the standard german equivalent of the term "fluent" employed by Newton) as a proof that it is motion, and in particular the motion of a point, which allows for the construction of any real number - not simply of those that can be obtained by means of iterated division. The claim that for Kant it is through motion that continuous quantities such as spatial magnitudes can be constructed is one that I do not dispute. Indeed, Kant states this clearly in many *loci* during his critical period[37]. It is through motion that a space, be it a line or a circle, can be constructed, or described; the former by the motion of a point, the latter by the motion of a line segment around a centre. However, there is an important difference between the claim that any continuous magnitude can be constructed by means of motion, and the claim that the continuity of this magnitude consists in the fact that it has been constructed through motion. While the first claim only implies that a magnitude having the property of being continuous can be produced by means of a possibly iterated constructive procedure involving motion, the latter identifies continuity with motion itself; continuity, that is, becomes synonimous with construction by means of motion. It seems to me that Kant, in the above passage from the axioms of intuitions, subscribes only to the first of these claims. What he exactly states is that any continuous magnitude can be constructed (synthesized by the *synthesis speciosa*)

[36]It is instructive to compare Kant's definitions of points and continuity with what is said about these notions in the Elements and in Aristotle. Euclid defines the notion of point as "that which has no parts", but he also adds definition 3 stating that "the extremities of a line are points"; the latter was the customary definition of point before Euclid, and it conveys the idea that points are constructed as the boundaries of a line (and thus lines as the boundaries of surfaces, and so forth). This was most notably the position of Plato, who, in citicising the definition of point given by the pythagoreans, stated that a point was a mere geometrical fiction, and called it the "beginning of a line" (ἀρχέ γραμμῆς). Aristotle definied istead a point as that which is indivisible and has position, and in particular he identifies it with a *place* (τόπος), a term which also Kant employs (see [15], [482]: "for the place of a body is a point"). Most importantly, he claims that continuous magnitudes such as lines cannot be obtained by means of the accumulation of dimensionless points, and that a point is like the *now* in time, in that is indivisible and not a part of time, but only the beginning, the end, or the division of time; in other words, an instant is only a boundary of a time interval (See [25], p.156 for the references). The similarity between Kant's discussion and these observations is remarkable.

[37]Consider for instance the important footnote that Kant inserts at [11], B155:

> [...] But motion, as **description** of a space, is a pure act of the successive synthesis of the manifold in outer intuition in general through productive imagination, and belongs not only to geometry but even to transcendental philosophy[38].

Other similar passages are the handwritten note on Kant's copy of the A edition of the CPR ([11], note to passage A234/B287, p. 333) and [15], [489].

by means of a process in time (motion) which is continuous. The last occurrence of the term "continuous" in the passage is hence referred *not* to the magnitude itself, but to the temporal process by means of which the magnitude is constructed. Hence, what Kant is claiming is that a continuous magnitude can be constructed by means of continuous motion, but he does not make the stronger claim that these two notions of continuity (that attributed to the magnitude and that attributed to motion) are identical.

Indeed, Kant himself specifically distinguishes geometrical continuity from dynamical and mechanical continuity in the sequel of the last passage we quoted above[39]. Dynamical continuity consists in the continuity of the "momentum of accelerative forces", which just means that there is no smallest possible acceleration to which a body can be subjected. Mechanical continuity, in turn, consists in the fact that every change in motion, be it a change of state (from rest to movement and viceversa) or speed or direction, is possible only through infinitely smaller differences from the initial state, meaning that "in any change, no degree is the smallest, there is always another which preceeds or succeeds it". If we take this taxonomy to be at work in the Critique, then we must interpret Kant as saying that a magnitude having the property of being continuous can be produced by means of continuous motion, and not that the continuity of the magnitude consists in it having been produced through motion. As an example of this subtle distinction, consider for instance the following passage taken from the Metaphysical Foundations of Natural Science, in which Kant (*contra* Leibniz) is trying to disprove the existence of monads:

> [...] Let us assume that A is the place of a monad in space, and ab is the diameter of the sphere of its repulsive force, so that aA is the radius of this sphere. Then between a, where the penetration of an external monad into the space occupied by this sphere is resisted, and the center A, it is possible to specify a point c (*according to the infinite divisibility of space*) [...][40]

In this proof, Kant is choosing a point c along the radius of the sphere of the monad's repulsive force; in other words, he is choosing a point satisfying specific positional conditions, in the same fashion as Avigad *et al.* and Mumma do in their formalizations of Euclidean geometry. If the concept of the continuity of spatial magnitudes (in this case, of the radius aA) were simply the concept of their continuous production in time by means of the *synthesis speciosa* (as Friedman claims), then Kant would have had to justify the existence of point c by an appeal to motion. After all, for all we know the length of the segment ac could very well be π, as in Friedman's example. Kant, however,

[39] [10], *Notes on Metaphysics*, 6338a, (18:664).
[40] [15], [504], pp. 41-42. My emphasis.

appeals to the infinite divisibility of space in order to justify the existence of point c. In other words, the existence of point c is secured by the fact that the radius aA can be divided *ad infinitum*, along with the fact, which is left implicit in the passage, that points are only boundaries and not parts of space.

If we are to understand Kant's notion of geometrical continuity of a magnitude, then, we must avoid conflating it with the notion of dynamical or mechanical continuity of the process by means of which this magnitude is constructed. It could be of course argued that, for Kant, the latter is what brings about the former; that is, that it is the non-existence of a smallest change of degree or accelerative momentum in the production through motion of the magnitude that enforces its infinite divisibility and its not having a simple part. However, it is clear that Kant regarded the notion of geometrical continuity as conceptually distinct from that of mechanical and dynamical continuity. This implies that we must investigate whether Kant's definition of geometrical continuity, considered independently of motion and on its own terms, can provide us with a solution to the question of whether it is possible to give a constructive[41] account explaining why we come to think of geometrical space as a continuous and infinite medium. In the next section, we shall put forward a formal argument that purports to show that this is possible, and that indeed this is achieved thanks to the construction of pure sensible concepts by means of the pure *a priori* imagination.

§2.3 Kant's constructive continuum

We start this section with the definition of event orderings. These structures, consisting in a set of "events" satisfying a definite set of axioms, were introduced at the beginning of the past century in order to obtain a formalization of time in which instants were constructed, or derived, from events. Russell proposed a possible formalization of time using event orderings in [38, 39], while Walker developed an alternative approach in the context of the foundations of relativity theory [40, 41]. In what follows, we shall make use of some fundamental results by Thomason [42, 43]. Indeed, the purpose of this section will be to show that the theory of event orderings can be successfully applied to formalize the construction of space as a continuum in a Kantian fashion, thereby substantiating the claim that continuity (along with infinity) of space is a product[42] of the constructive activity of the pure productive synthesis of the imagination.

[41] In the sense of "constructive" emplyed by Kant, of course.

[42] Of course, "product" is here to be intended according to the analysis of the productive synthesis of the imagination *a priori* that was developed in the previous chapter.

Definition 2.5. An event ordering is a tuple $\mathcal{W} = (W, P, O)$, where W is a non-empty set, and (P, O) are binary relations over W such that the following holds[43]:

1. $P(a, b) \to \neg P(b, a)$
2. $P(a, b) \wedge P(b, c) \to P(a, c)$
3. $O(a, a)$
4. $O(a, b) \to O(b, a)$
5. $P(a, b) \to \neg O(a, b)$
6. $P(a, b) \wedge O(b, c) \wedge P(c, d) \to P(a, d)$
7. $P(a, b) \vee O(a, b) \vee P(b, a)$

We shall also make use of the following abbreviations:

- $B(a, b) \leftrightarrow \exists c(P(c, b) \wedge (P(a, c) \vee O(a, c)))$
- $E(a, b) \leftrightarrow \exists c(P(a, c) \wedge (P(c, b) \vee O(c, b))))$
- $A(a, b) \leftrightarrow P(a, b) \wedge \neg\exists c(P(c, b) \wedge E(a, c)) \wedge \neg\exists d(P(a, d) \wedge B(d, b))$

The interpretation of the above relations are as follows: $P(a, b)$ stands for "a wholly precedes b", $O(a, b)$ stands for "a overlaps with b", $B(a, b)$ stands for "a begins before b", $E(a, b)$ stands for "a ends before b", $A(a, b)$ stands for "a abuts b from the left". The interpretation of the axioms should be intuitive given the interpretation of the primitive relations; notice that axiom 2 enforces transitivity, while axiom 7 enforces linearity of events. We shall now define the notion of a Walker's (geometric) point.

Definition 2.6. Let $\mathcal{W} = (W, P, O)$ be an event ordering. A Walker's point is a triple (P, C, F) of subsets of W such that the following holds:

1. $C = W \setminus (P \cup F)$
2. $a \in P \wedge b \in F \Rightarrow P(a, b)$
3. $c \in C \Rightarrow (\exists a \in P)(\exists b \in F)(O(c, a) \wedge O(c, b))$

[43]The axiomatization of event orderings presented here, despite it differing from that of Thomason ([42], p. 87.), is to it essentially equivalent. In particular, one can easily see that all of Thomason's axioms can be derived as theorems from the present axioms, by stipulating $O(a, b)$ as an abbreviation for $\neg P(a, b) \wedge \neg P(b, a)$, and that viceversa, all of these axioms are theorems in Thomason's axiomatization.

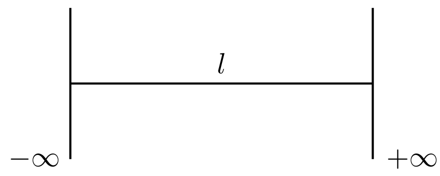

FIGURE 2.2: Event ordering $\mathcal{W}_0$

Notice that a point $p = (P, C, F)$ can be such that $P = \emptyset$ or $F = \emptyset$. Notice moreover that there can only be exactly one point with empty P. Indeed, if p has an empty P, then it also has an empty C (because of 3 above) and therefore $F = W$ (because of 1), hence p is unique and it is the point $p = (\emptyset, \emptyset, W)$, which always exists. The same observations hold in the case of a point with empty F[44]. Given an event ordering $\mathcal{W}$, we denote with the symbols $+\infty$ and $-\infty$ the two points with empty P and empty F, respectively. We also denote with $F(\mathcal{W})$ the set of all Walker's points of $\mathcal{W}$ such that $P \neq \emptyset \neq F$ (i.e., excluding $-\infty$ and $+\infty$), and with $F^*(\mathcal{W})$ we denote $F(\mathcal{W}) \cup \{-\infty, +\infty\}$. An order relation $<$ is defined over $F^*(\mathcal{W})$ by letting $(P, C, F) < (P', C', F')$ if and only if $P \subset P'$. The following lemma establishes that the points induced by an event ordering are linearly ordered. Recall that a linear ordering is complete if every non-empty subset having an upper (lower) bound has a least upper (greatest lower) bound. The proofs of the lemmata and of the theorems which are not proven here can be found in [42].

Lemma 2.7. *Let $\mathcal{W}$ be an event ordering. Then $F^*(\mathcal{W})$ and $F(\mathcal{W})$ are complete linear orders.*

Proof. Omitted. □

Consider now a line segment l joining two given points. We can represent this segment with the event ordering $\mathcal{W}_0$, presented in figure 2.2, where W is defined by letting $W = \{l\}$ and (P, O) are the empty relation. Point $-\infty$ is defined as $-\infty = (\emptyset, \emptyset, \{l\})$, and $+\infty$ is defined analogously. The purpose of points $(-\infty, +\infty)$ is not only to represent the two points which are joined by line l, but also to encode the fact that line l is, at the present stage of productive synthesis in pure intuition, the only space which has been constructed; i.e., line l is bounded by nothing, which amounts to say that l is considered as the *absolute space*.

If we take l as a unit, one can construct an extended line segment, represented by line n in the event ordering $\mathcal{W}_1$ (figure 2.3), which has $\mathcal{W}_0$ as a substructure. Notice that in $\mathcal{W}_1$ a new point p_2 is created at the boundary between l and h; moreover, one sees that

[44] More generally, it can be shown that every point is uniquely determined by its past: see [42], Proposition 2, p. 89.

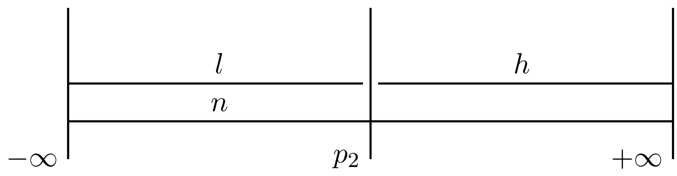

FIGURE 2.3: Event ordering $\mathcal{W}_1$

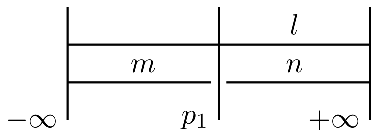

FIGURE 2.4: Event ordering $\mathcal{W}_3$

$B(l, h)$, $E(h, l)$, and $A(l, h)$ hold. The construction of line segment n would actually involve two steps. First, l is taken as a unit, and segment m is constructed. Second, segments l and m undergo a mathematical synthesis, i.e., they are combined together by the imagination according to the concept of extensive magnitude, which, as we have seen, is derived from the category of quantity. This second step produces the representation of n, which is then thought as the whole produced by the parts l and m (as one in many, or many that constitute one). l and m are thus synthesized, or combined, into n. As we shall remark later, this process of construction in pure intuition can be iterated *ad infinitum*, thereby giving rise to the original representation of an infinite line.

Given the event ordering $\mathcal{W}_0$ above, we can also represent the process of bisection of line segment l, by means of the event ordering $\mathcal{W}_3$ presented in figure 2.4. Events m and n represent the two half-spaces in which l is divided, while l constitutes in turn the whole in which m, n are "carved" or "limited" through bisection. A point p_1 is constructed at the boundary between m and n, as a "limit" which separates the two subspaces. Notice that the presence of event l in $\mathcal{W}_3$ is justified by the fact that the process of bisection of a line must first involve the construction of the line to be bisected (event ordering $\mathcal{W}_0$); hence, $\mathcal{W}_0$ has to be a substructure of $\mathcal{W}_3$, that is, $\mathcal{W}_3$ is obtained by extending $\mathcal{W}_0$ by means of an ulterior construction step (in this case, division by bisection). It is important to consider, though, that event ordering $\mathcal{W}_3$ is equivalent to event ordering $\mathcal{W}'$ (figure 2.5). In other words, event orderings themselves are just a means to represent the relations (P, B, O, E, A) which hold between events, and thus are of course blind to any possible metric relation between the line segments which constitute our set W. Still, event orderings allow us to model reliably the process of constructing points as boundaries of (linear) spaces. Furthermore, event orderings $\mathcal{W}_3$ and $\mathcal{W}'$ *are not* equal - elements m, n are not, respectively, equal to h, r, since they correspond to different spatial intuitions.

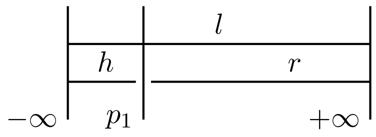

FIGURE 2.5: $\mathcal{W}'$

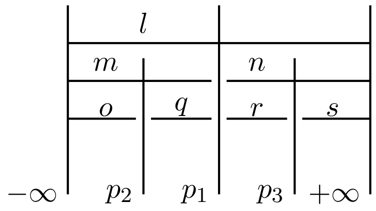

FIGURE 2.6: $\mathcal{W}_4$

In the same way in which we can iterate the process of extending a line indefinitely, one could now bisect both line m and line n, thus extending $\mathcal{W}_3$ to event ordering $\mathcal{W}_4$ in figure 2.6. Points p_2, p_3 are defined respectively as $p_2 = (\{o\}, \{m, l\}, \{q, n, r, s\})$ and $p_3 = (\{m, o, q, r\}, \{l, n\}, \{s\})$.

Let us now consider a problem that arises naturally when we consider the relation between event orderings and their associated linear orders. Given an event ordering $\mathcal{W}$, we would like to be able to map an event $e \in W$ to an interval of the corresponding linear order $F^*(\mathcal{W})$. The following definitions and lemmata are useful to this effect.

Definition 2.8. Let $(\mathbb{L}, <_{\mathbb{L}})$ and $(\mathbb{M}, <_{\mathbb{M}})$ be two linear orders. A function $f : \mathbb{L} \to \mathcal{P}(\mathbb{M})$ is an expansion if the following holds:

1. for all $x \in \mathbb{L}$, $f(x)$ is a non-empty subset of $\mathbb{M}$
2. for all $x, y \in \mathbb{L}$, if $x <_{\mathbb{L}} y$ then $(\forall v \in f(x))(\forall u \in f(y))(v <_{\mathbb{M}} y)$

An expansion is a multi-valued function which maps an element of a linear order $\mathbb{L}$ to its set of children in another linear order $\mathbb{M}$, and which satisfies the consistency requirement at point 1. In other words, an expansion function allows us to consider any element of $\mathbb{M}$ as a new point, or as a child of a point in $\mathbb{L}$.

Definition 2.9. Let $\mathbb{L}$ be a linear order. A formal open interval of $\mathbb{L}$ is an ordered pair (x, y) where $x \in \mathbb{L}$, $y \in \mathbb{L}$, and $x < y$. We denote with $\mathcal{I}(\mathbb{L})$ the set of all formal open intervals of $\mathbb{L}$.

In our present case, given an event ordering $\mathcal{W}$, we will consider the intervals defined over the linear order $\mathbb{L} = F^*(\mathcal{W})$, according to the above definition. Notice that the notion of a formal interval is radically different than that of the set of points comprised between two end-points. Indeed, in figure 2.4, $(-\infty, p_1)$ constitutes a formal open interval, even though the set of points between $-\infty$ and p_1 is empty.

Lemma 2.10. *Let $\mathbb{L}$ be a linear order. For all pairs (x, y), (u, v) in $\mathcal{I}(\mathbb{L})$, declare:*

- $P((x, y), (u, v))$ *iff* $y \leq u$
- $B((x, y), (u, v))$ *iff* $x < u$
- $E((x, y), (u, v))$ *iff* $y < v$

Then $\mathcal{I}(\mathbb{L})$ is an event ordering.

Proof. Omitted. □

Definition 2.11. Let $\mathcal{W}$ be an event ordering. Define the function $\eta_{\mathcal{W}} : W \to \mathcal{I}(F^*(\mathcal{W}))$ by letting $\eta_{\mathcal{W}}(a) = (x_a, y_a)$, where $x_a = (P, C, F)$ with:

- $P = \{c \in \mathcal{W} | P(c, a)\}$
- $F = \{c \in \mathcal{W} | \neg B(c, a)\}$
- $C = \mathcal{W} \setminus (P \cup F)$

Analogously, let $y_a = (P, C, F)$ with:

- $P = \{c \in \mathcal{W} | \neg E(a, c)\}$
- $F = \{c \in \mathcal{W} | P(a, c)\}$
- $C = \mathcal{W} \setminus (P \cup F)$

Definition 2.12. Let $\mathcal{W}, \mathcal{V}$ be two event orderings such that $f : \mathcal{W} \to \mathcal{V}$ is a homomorphism, i.e., a map which preserves and reflects (P, O). The Walker's expansion *exp* : $F^*(\mathcal{W}) \to \mathcal{P}(F^*(\mathcal{V}))$ is defined as follows: if $(P, C, F) \in F^*(\mathcal{W})$, then $exp((P, C, F))$ is the set of all $(P', C', F') \in F^*(\mathcal{V})$ such that $P' \cap f(\mathcal{W}) = P$, $C' \cap f(\mathcal{W}) = C$, $F' \cap f(\mathcal{W}) = F$, where $f(\mathcal{W}) = \{f(x) : x \in \mathcal{W}\}$

The above machinery is useful for the following purpose. Consider two finite event orderings $\mathcal{W}, \mathcal{V}$ and a homomorphism $f : \mathcal{W} \to \mathcal{V}$. Then not only relations (P, O) are preserved by the homomorphism, but also relations (B, E), due to the following basic lemma, and the fact that both B and E are defined by means of a positive primitive formula:

Lemma 2.13 (Lyndon)**.** *Let $h : \mathcal{A} \to \mathcal{B}$ be a homomorphism from a model $\mathcal{A}$ to a model $\mathcal{B}$. If $\psi(\vec{x})$ is positive primitive, and $\vec{a}$ is a tuple of elements from $\mathcal{A}$, then $\mathcal{A} \models \psi[\vec{a}]$ entails $\mathcal{B} \models \psi[h(\vec{a})]$.*

Proof. Omitted. □

If $\mathcal{W}$ is a substructure of $\mathcal{V}$, then there is an obvious homomorphism f from $\mathcal{W}$ to $\mathcal{V}$, namely, the identity $f(a) = a$, which then preserves and reflects not only (P, O) but also (B, E). Now, to $\mathcal{W}$ is associated a linear order $F^*(\mathcal{W})$ whose set of formal open intervals $\mathcal{I}(F^*(\mathcal{W}))$ (Definition 2.9) can be made into an event ordering in the natural way (Definition 2.10); and similarly for $\mathcal{V}$. Given $e \in \mathcal{W}$, and the same event $f(e) = e$ in $\mathcal{V}$, $\eta_{\mathcal{W}}$ maps e to its corresponding interval $\eta_{\mathcal{W}}(e)$ in $\mathcal{I}(F^*(\mathcal{W}))$, and $\eta_{\mathcal{V}}$ maps e to its corresponding interval $\eta_{\mathcal{V}}(e)$ in $\mathcal{I}(F^*(\mathcal{V}))$, allowing us to regard e, in each case, as a formal open interval of the linear order induced by the event ordering. The function exp maps every point in $F^*(\mathcal{W})$ to its set of children in $F^*(\mathcal{V})$, and hence allows us to regard every point of $F^*(\mathcal{V})$ as a child of a point in $F^*(\mathcal{W})$ or as a new point.

We can now define an homomorphism g between $\mathcal{I}(F^*(\mathcal{W}))$ and $\mathcal{I}(F^*(\mathcal{V}))$, which are event orderings, by letting $g((x, y)) = (max(exp(x)), min(exp(y)))$ (with $max(exp(-\infty)) = \infty$, and dually for $+\infty$). It is possible to check that with these definitions, we obtain the highly desirable property that if a point $y \in F^*(\mathcal{V})$ is a child of a point $x \in F^*(\mathcal{W})$, then y belongs to $\eta_{\mathcal{V}}(e)$ if and only if its parent belongs to $\eta_{\mathcal{W}}(e)$.

An illustrative example will clarify these concepts. Consider the two event orderings $\mathcal{V}, \mathcal{V}'$ depicted in figure 2.7. The set of intervals of $\mathcal{V}$ is:

$$\mathcal{I}(F^*(\mathcal{V})) = \{(-\infty, p_0), (-\infty, +\infty), (p_0, +\infty)\}$$

While the set of intervals of $\mathcal{V}'$ is:

$$\mathcal{I}(F^*(\mathcal{V}')) = \{(-\infty, p_2), (-\infty, p_1), (-\infty, p_3), (-\infty, +\infty), (p_2, p_1), (p_2, p_3), ...\}$$

If we consider $p_0 \in F^*(\mathcal{V})$, we have $exp(p_0) = \{p_1, p_2, p_3\}$, represented by the red arrows in the figure; in other words, point p_0 is split into points p_1, p_2, p_3 by the addition of events r, s to event ordering $\mathcal{V}$. Of course, points $-\infty, +\infty$ in $F^*(\mathcal{V})$ map to the same points in $F^*(\mathcal{V}')$. Moreover, event m corresponds to the formal open interval $\eta_{\mathcal{V}}(m) = (-\infty, p_0)$ with respect to $F^*(\mathcal{V})$, but to the interval $\eta_{\mathcal{V}'}(m) = (-\infty, p_2)$ with respect to $F^*(\mathcal{V}')$; and analogously for event n. Notice also that, e.g., interval $(-\infty, p_2)$ of $F^*(\mathcal{V})$, and the corresponding event m, are such that they contain no points.

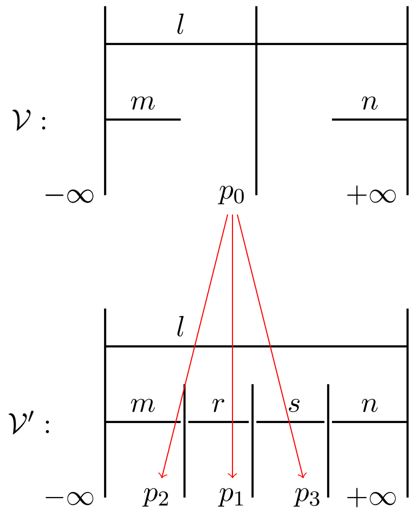

FIGURE 2.7: Example

We are now able, given an event ordering $\mathcal{W}$, to map an event $e \in W$ to the corresponding formal open interval of the associated linear order $F^*(\mathcal{W})$, in a fashion that is consistent with extending the event ordering $\mathcal{W}$ with new events. Since, as we have seen, an event ordering can be taken as the representation of a given step in the construction of, e.g., an infinite line, we might want to model this construction process by means of a (countably) infinite sequence of event orderings $\langle \mathcal{W}_0, \mathcal{W}_1, ...\rangle$, with the property that if $i < j$, then $\mathcal{W}_i$ be a substructure of $\mathcal{W}_j$.

Before we investigate this possibility, however, an observation must be made which is of conceptual importance. As we have seen, the extension of an event ordering preserves and reflects relations (P, O, B, E), but it does not necessarily preserve relation A. For instance, in the previous example (figure 2.7), we have that $\mathcal{V}$ is a substructure of $\mathcal{V}'$, but while $A(m, n)$ holds at $\mathcal{V}$, it is certainly not the case that $A(m, n)$ holds at $\mathcal{V}'$. The reason why abutness does not hold at $\mathcal{V}'$ is, of course, that events r and s have been constructed between m and n.

It is possible to notice, however, that in the previous examples of the extension or division of a line segment abutness was always preserved, in that no event could be given between two already constructed events. The interpretation I offer for this phenomenon is as follows. In the case presented in Figure 2.7, point p_0 in $\mathcal{V}$ is actually not a real point, but it is a boundary between the two spaces m and n which itself comprises a "hidden" space, that is, a space which the observer has not (yet) synthesized. Extending $\mathcal{V}$ into $\mathcal{V}'$ splits point p_0 into three points, and the "hidden" spaces r, s are revealed; the observer comes to recognize that point p_0 actually contains two different spaces. This amounts to say that at event ordering $\mathcal{V}$, the observer considers m, n as *fully* partitioning event l - the only intervals that the observer is aware of are $(-\infty, p_0)$, corrisponding to

m, $(p_0, +\infty)$, corresponding to n, and $(-\infty, +\infty)$, corresponding to the entire segment l. However, by extending $\mathcal{V}$ to $\mathcal{V}'$, this inference is retracted: m and n do not fully partition l, since it is now discovered that their boundary p_0 is not a real limit, but it instead contains spaces r and s.

Of course, in the empirical syntheses of spaces, we can never be sure of the fact that abutness will be preserved. We synthesize a space, and we synthesize two subspaces which seem to partition it, and which seem separated by a boundary; but we are not always in a position to be sure that this boundary itself is not a space, and that our partition, according to the category of community, is not mistaken[45]. Consider, for example, a situation in which a building in the distance, which is first thought as being composed of two parts, turns out upon closer inspection to be partitioned into three or more parts. On the contrary, in the process of construction in pure intuition by means of the *synthesis speciosa*, every boundary which arises at the boundary of two or more spaces is a *real* boundary (be it a point, a line, or surface), in the sense that within this boundary no space of higher dimensionality can be constructed. A point, which is constructed in pure intuition as the boundary between two line segments, cannot be found to hide another line segment any more than a square, conceived as the boundary between two cubes partitioning a parallelepiped, can be found to hide another cube inside it.

In particular, once an line segment l has been partitioned through bisection by construction of spaces m, n, the point arising at the boundary will never split through any further construction of events in pure intuition, as the relation $A(m, n)$ will be preserved in every extension of the event ordering. These considerations do no apply to points $-\infty$ and $+\infty$, which, since they are not *real* points, but simply indicate the place at which the synthesis of the pure productive imagination has stopped, not only can but must be conceived as containing ulterior spaces to be synthesized[46]. Since we are here interested in modelling the productive synthesis in pure intuition *a priori*, and not the empirical synthesis (which is exercised on an empirically given manifold), we shall reformulate our notion of an event ordering in such a way that abutness be preserved by the substructure relation.

Definition 2.14. An event ordering is now a tuple $\mathcal{W} = (W, P, O, A)$ such that W is a non-empty set, and (P, O, A) are binary relations over W satisfying axioms 1-8 in definition 2.5 along with the following three additional axioms:

1. $P(a, b) \rightarrow A(a, b) \vee \exists c(P(c, b) \wedge E(a, c)) \vee \exists d(P(a, d) \wedge B(d, b))$

[45] This corresponds to the case in which a concept is erroneously partitioned.

[46] The reader should notice that this is in thorough agreement with what Kant says, e.g., at [10], *Notes on Metaphysics*, R 5348, (18:158), and at R 4529, (17:584), where it is claimed that space can be perceived only if some appearance is placed in it.

2. $A(a, b) \to P(a, b)$

3. $A(a, b) \wedge [\exists c(P(c, b) \wedge E(a, c)) \vee \exists d(P(a, d) \wedge B(d, b))] \to \perp$

The three additional axioms govern the interaction between abutness (which is now primitive) and (P, O). With some efforts one can see that this axiom system for event orderings is equivalent to that given at the beginning of the present section, under the previous definition of abutness in terms of (P, B, E). However, since now abutness has been included as primitive in the signature, we have that if event ordering $\mathcal{V}$ is a substructure of event ordering $\mathcal{V}'$, then abutness must be preserved (and reflected). Thus situations like that in figure 2.7, which properly belong to the empirical synthesis, are ruled out. We can now proceed to expound some technical notions, which will allow us to show that from the Kantian notion of continuity one can recover the real continuum. Again, those proofs which are missing can be found in [42].

Lemma 2.15. *If $\mathcal{W}$ is an event ordering, $F(\mathcal{W})$ has a least element if and only if $(\exists a \in W)(\forall b \in W)(\neg E(b, a))$, and it has a greatest element if and only if $(\exists a \in W)(\forall b \in W)(\neg B(a, b))$*

Definition 2.16. If $\mathcal{W}$ is an event ordering with $a, c, d \in W$, then (c, d) splits a if $P(c, d) \wedge cOa \wedge dOa$. $\mathcal{W}$ is said to be dense if whenever $a, b \in W$ and (a, b) overlap, there are $c, d \in W$ such that (c, d) splits both a and b

It follows from the above lemma that if $\mathcal{W}$ is dense, $F(\mathcal{W})$ has no least or greatest element (recall that $(-\infty, +\infty) \notin F(\mathcal{W})$). For instance, suppose $F(\mathcal{W})$ has a least element, then there is $a \in W$ such that $(\forall b \in W)(\neg E(b, a))$, but $\mathcal{W}$ is dense, so a is split by some (c, d) (since $O(a, a)$), but then $E(c, a)$, a contradiction. The case of the greatest element is analogous.

Lemma 2.17. *If $\mathcal{W}$ is an event ordering, $\mathcal{W}$ is non-empty and dense if and only if the linear order $F(\mathcal{W})$ is non-empty and dense and has no least and greatest element.*

Definition 2.18. If $\mathcal{W}$ is a dense event ordering, then $\mathcal{S} \subset \mathcal{W}$ is dense in $\mathcal{W}$ if for any (a, b) in W, if aOb then there are (c, d) in S such that (c, d) splits both a and b.

Lemma 2.19. *Let $\mathcal{W}$ be a dense event ordering and k an infinite cardinal. $F(\mathcal{W})$ has a dense subset of cardinality at most k if and only if $\mathcal{W}$ has a dense subset of cardinality at most k.*

Definition 2.20. Let $(A, <)$ and $(B, <)$ be partially ordered sets. A function $f : A \to B$ is normal if for every $C \subset A$ which has a supremum in A the set $f \restriction C$ has a supremum in B and $f(supC) = sup(f \restriction C)$. f is called strictly normal if it is also one-to-one.

Lemma 2.21. *Let $(A,<)$ be a dense linear order that has no least and no greatest element, and let $B \subset A$ be dense in A. Then every normal function $f : B \to \mathbb{R}$ has only one normal extension $f^* : A \to \mathbb{R}$.*

Proof. If $f : B \to \mathbb{R}$ is a normal function, define $f^*(a) = sup_{x \in B, x<a} f(x)$ for all $a \in A$. Clearly, if $a \in B$, we have that $f^*(a) = sup_{x\in B,x<a} f(x) = f(sup\{x \in B : x < a\}) = f(a)$ (for normality of f). If $C \subset A$ such that $sup(C) = c$, $c \in A$, then $sup(f^* \restriction C) = sup_{a\in C} sup_{y\in B,y<a} f(y) = sup_{y\in B,y<c} f(y) = f^*(c)$, so f^* is normal. Uniqueness can be seen by the fact that if $a \in A$, then for f^* to be normal, one must have that $f^*(a) = sup_{x\in B,x<a} f^*(x) = sup_{x\in B,x<a} f(x)$; the first equality in reason of the fact that $a = sup\{x \in B, x < a\}$, the second equality because f^* is an extension of f. □

Lemma 2.22. *If $(A,<)$ is a dense linear order without end-points, and $B \subset A$ is dense in A and complete with respect to the ordering of A, then $B = A$*

Proof. Let $a \in A$ and $C = \{x \in B : x < a\}$. Then $sup(C) \in B$ by completeness and $sup(C) = a$ by denseness. Hence $a \in B$. □

Theorem 2.23. *if $\mathcal{W}$ is an event ordering which is not empty, dense, and which has a denumerable dense subset, then $F(\mathcal{W})$ is order-isomorphic to $\mathbb{R}$.*

Proof. Since $\mathcal{W}$ is non-empty and dense, then $F(\mathcal{W})$ is complete, non-empty, dense, and has no end points (Lemmata 2.7 and 2.17). Since $\mathcal{W}$ has a dense subset of cardinality $\aleph_0$, then by Lemma 2.19, $F(\mathcal{W})$ must have a dense subset of cardinality at most $\aleph_0$, which simply means that $F(\mathcal{W})$ has a denumerable dense subset (without end points). Consider now $(F(\mathcal{W}), >)$ and this denumerable dense subset A. Since Cantor proved that every denumerable dense linear order without end points is order-isomorphic to the rationals, there is an order-isomorphism f between A and the denumerable dense subset $\mathbb{Q}$ of $\mathbb{R}$. This isomorphism f is certainly normal (proof by contradiction), and hence there is a unique normal extension $f^* : F(\mathcal{W}) \to \mathbb{D}$, where $\mathbb{D} \subset \mathbb{R}$, of f (Lemma 2.21). However, $\mathbb{D}$ is dense in $\mathbb{R}$, and, being isomorphic to $F(\mathcal{W})$, it is complete (with respect to the ordering of the reals). By lemma 2.22, this means that $\mathbb{D} = \mathbb{R}$. □

We have shown that a non-empty and dense event ordering $\mathcal{W}$ which has a denumerable dense subset gives rise to a linear order of Walker's points $F(\mathcal{W})$ (distinct from $F^*(\mathcal{W})$) which is order-isomorphic to the real line. We can now use this result to show that one can construct the real continuum according to Kantian principles. Given an event ordering $\mathcal{W}$ and $a \in W$, let $O(a) = \{x \in W : O(a,x)\}$. Furthermore, let $min_P(T)$, where $T \subset W$, be the set $\{x \in T : (\forall y \in T)(\neg P(y,x))\}$. Define now an ω sequence $W_\xi = \langle \mathcal{W}_0, \mathcal{W}_1, ...\rangle$ of finite event orderings of line segments as follows: $\mathcal{W}_0$ consists of a single line segment l, and $\mathcal{W}_{n+1}$ is the extension obtained from $\mathcal{W}_n$ through bisection of the line segment (event) $min_P\{x \in W_n : \forall y \in W(|O(x)| \leq |O(y)|)\}$[47]. To put it in a simpler way, we keep on applying bisection iteratively from left to right, adding the partitioning line segments to the event orderings to obtain the next step in the construction process. Figure 2.8 shows the first three elements of the sequence, along with the expansion function indicated by the red arrows.

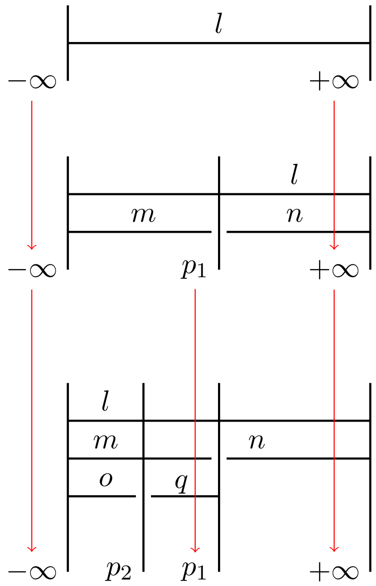

FIGURE 2.8: sequence of bisections

Every event ordering $\mathcal{W}_\alpha$ belonging to the sequence $\langle \mathcal{W}_\xi : \xi < \omega \rangle$ represents a step in the process of dividing a line segment *ad infinitum*. Considered from the standpoint of model theory, this sequence is a chain of structures ordered according to the substructure relation. We can then define a structure $\mathcal{M}$ by letting $dom(\mathcal{M}) = \bigcup_{\xi<\omega} dom(\mathcal{W}_\xi)$, and letting, for $R \in \{P, O, A\}$, $(a, b) \in R^{\mathcal{M}}$ iff there is $\alpha < \omega$ such that $(a, b) \in R^{\mathcal{W}_\alpha}$. Then $\mathcal{M}$ is the union of the chain $\langle \mathcal{W}_\xi : \xi < \omega \rangle$, and every $\mathcal{W}_\alpha$ embeds in $\mathcal{M}$; moreover, it follows from a well-known model-theoretic fact[48] (which can be easily verified directly) that the axioms 1-8 of event orderings hold at $\mathcal{M}$. Hence $\mathcal{M}$ is an event ordering, and it also holds that $|dom(\mathcal{M})| = \aleph_0$ and, by construction, that $\mathcal{M}$ is dense. Hence $\mathcal{M}$ has itself as a denumerable dense subset, and thus it follows from theorem 2.23 that $F(\mathcal{M})$ is order-isomorphic to the reals, and consequently that $F^*(\mathcal{M})$ is order-isomorphic to the 2-point compactification of the reals, i.e., it is homeomorphic to a closed interval of real numbers.

These facts show that it is possible to recover the spatial continuum by means of the iteration *ad infinitum* of the constructive procedure consisting in the bisection of a

[47] It is straightforward to see that this line segment (event) always exists and is unique.
[48] See [44], theorem 2.4.4, p. 50.

line segment, without recurring to motion, but relying only on the mereotopological principles that according to Kant constitute the notion of geometrical continuity. Of course, it follows from the previous philosophical analysis that the infinite construction represented by the sequence $\langle \mathcal{W}_\xi : \xi < \omega \rangle$ cannot be brought to completion by any observer, i.e., the event ordering $\mathcal{M}$ is itself only an idea of reason, which is never completely achieved, but only approximated:

> That this line can be infinitely divided is also not an idea, for it signifies only a continuation of the division unlimited by the size of the line. But to see this infinite division in its totality, and consequently as completed, is an idea of reason, the idea of an absolute totality of conditions (of synthesis) demanded of an object of sense, which is impossible since the unconditioned is not at all to be found among appearances[49].

Thus the complete totality of the division of the line segment (represented by the structure $\mathcal{M}$) is the idea of an object of sense in which the synthesis of the productive imagination *a priori* according to the category of community has been carried out to its utmost limit; still, the mind's consciousness of the *potential* infinite iterability of the construction, i.e., of the possibility of refining more and more, *ad libitum*, the linear order of points induced by a given stage in the division of the line segment, produces the consciousness of space as a continuous magnitude. Analogously, the consciousness of the infinite extendibility of a line segment first produces the consciousness of an infinite line, and with it of the infinity of space. The first steps of a sequence of event orderings corresponding to the extension of a line segment to infinity is given in figure 2.9.

We are now in a position to offer a constructive justification for the fact that we conceive of space as a continuous medium. Indeed, we can attribute this fact to the consciousness of the activity of the *synthesis speciosa* in the division of a line segment *ad infinitum*, whereby we become conscious of the possibility of infinitely refining the associated linear order of points. Thus, the action of the geometer, who freely chooses an arbitrary point on a line satisfying determinate positional conditions (as in the proof of the non-existence of monads above), can be philosophically grounded on his consciousness that this point can be constructed by means of a (possibly infinite) process of approximation, consisting in the construction by the *synthesis speciosa* of smaller and smaller spaces which contain that point, in a fashion which is not dissimilar to that of the "abstractive process" envisaged by Whitehead[50].

[49] [7], 96, [362], (11:53).
[50] See for instance [45], p. 392-395.

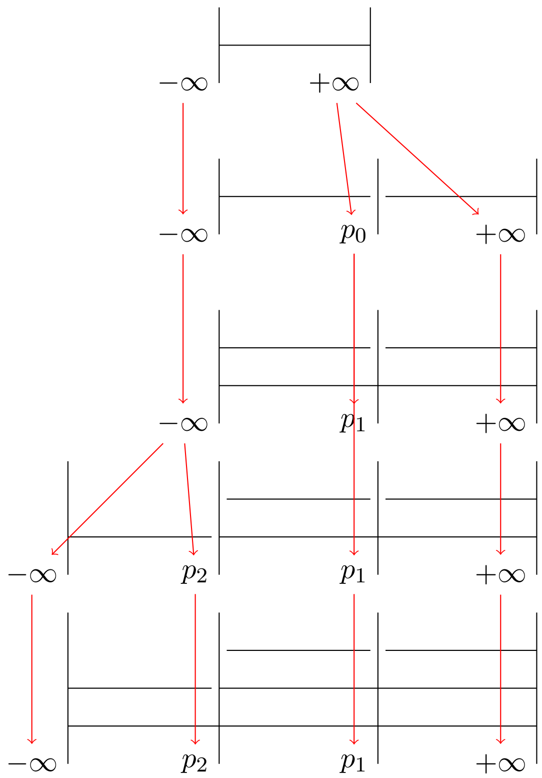

FIGURE 2.9: Construction of an infinite line

§2.4 Conclusions

In the present work we have tried to provide an accurate account of Kant's theory of space and geometry, starting from his general theory of human cognition and proceeding through the meanders of his characterization of the activity of the pure imagination with respect to geometrical reasoning. The main contributions which we hope to have made to the body of Kant's scholarship are:

1. the development of an exegesis of the Transcendental Deduction which, focusing on the notion of combination, clarifies the main argument regarding the applicability of the categories to objects of experience, and sheds light on the complex interaction which exists between the various syntheses of the imagination, geometrical reasoning, and the synthesis of the understanding.

2. the development of a solid argument purporting to show that Kant's constructive approach to geometrical reasoning was not merely motivated by matters related to the limits of the logical tools at his disposal, but was a conscious philosophical choice, deriving from his general theory of cognition.

3. the devlopment of a formal argument purporting to show that the notion of space as an infinite and continuous medium of points can be philosophically justified (from a constructive standpoint) starting from Kantian mereotopological principles.

4. the development of a formal argument which, along with other informal arguments, illustrates the process by means of which we become conscious of the properties of space and thereby form a concept of it, in order to substantiate the claim that the concept of space is a product of the activity of the pure imagination *a priori*.

Numerous interesting questions could be formulated in light of future research. The most pressing matter would be to investigate in detail the relationship between Kant's concept of geometrical continuity, which we have tried to analyse here, and his concepts of dynamical and mechanical continuity. In particular, if Kant's position is that a magnitude is geometrically continuous if and only if it has been constructed by means of a mechanically continuous motion, then it would follow that a magnitude is geometrically continuous if and only if it is differentiable. In this case, then, one could perhaps model the interaction between these two notions of continuity by means of smooth infinitesimal analysis. However, other interpretative options remain open. On the same line, it would be of remarkable interest to investigate how the considerations expounded here apply to the treatment of motion in the Metaphysical Foundations of Natural Science, in particular with respect to the construction of motion as a magnitude contained in the phoronomy, and to matter as a continuous quantity. Finally, the implications of the present analysis of Kant's notion of geometrical space could provide interesting hypotheses concerning human space perception.

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