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Sahlqvist Correspondence for Intuitionistic Modal Mu-Calculus

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SAHLQVIST CORRESPONDENCE FOR INTUITIONISTIC MODAL MU-CALCULUS

July 6, 2012

Dedication

This work is dedicated to my family.

Acknowledgments

This thesis was written under the supervision of Alessandra Palmigiano. I would like to thank her for putting a lot of time to supervise me. My gratitude also goes to my mentor Ulle Endriss who gave me a good advice that helped me to manage my time well. I also thank Sumit Sourabh for his help during the preparation of this thesis.

Abstract

The motivation of this thesis is the idea of extending the results in [13] to the case of logics with fixpoints and with a non-classical base. We focus in particular on the intuitionistic modal mu-calculus. We enhance the ALBA algorithm [26] for the elimination of monadic second order variables so that mu-formulas can be treated in which all the variables occur inside the scope of fixpoint binders. We prove that this enhancement is sound thanks to the order-theoretic properties of the interpretation of fixpoint binders in the algebraic semantics for intuitionistic mu-calculus. We define the class of recursive inequalities and informally justify that the enhanced ALBA is successful on this class.

Introduction

A glimpse at correspondence theory. Correspondence theory stems from the observation that Kripke frames can be used both as models for classical first-order logic and for modal logic. This implies that both modal formulas and first-order sentences can be used to define classes of Kripke frames (in the case of modal formulas this can be done though the notion of validity). So, we can say that a modal formula and a first-order sentence correspond to each other if they define the same class of Kripke frames. *Correspondence theory* is the part of the model theory of modal logic which systematically studies the phenomenon of correspondence. In particular, *Sahlqvist theory* (cf. [19], [8]) provides a syntactic approach to correspondence theory, which is aimed at characterizing classes of modal formulas which are guaranteed to have first order-correspondents by means of conditions on their syntactic shape. The most famous of these classes is the *Sahlqvist class* of modal formulas. Because they have a first-order correspondent, Sahlqvist formulas have additional good properties: for instance, their associated normal modal logics are canonical, and hence strongly complete with respect to the class of frames defined by their first-order correspondents (cf. thm 4.42 of [5]). *Sahlqvist theory* provides sufficient but not necessary conditions for the correspondence of modal formulas. Indeed it was shown in [2] (Chagrova's theorem), that it is undecidable whether an arbitrary basic modal formula has a first-order correspondent. Thus any decidable class of modal formulas each member of which is guaranteed to correspond to some first-order formula can only be an approximation of the class of modal formulas which admit a first-order correspondent.

Developments in Sahlqvist theory: syntactic characterizations. Researchers have been trying to improve the original results in [8], [19] . These improvements follow two separate directions, which occasionally come together:

1. In the original classical modal logic setting, developments have come in the form of finding proper extensions of the Sahlqvist class. This gave rise to the introduction of the classes of inductive and recursive formulas in [20], [21];

2. Developments have also come in the form of finding counterparts of the Sahlqvist class for other logics than classical modal logic. For, instance the Sahlqvist class has been defined for distributive modal logic [14], substructural logic [16] and modal mu calculus [13], [18].

These two separate directions have come together in [26], where the inductive formulas have been defined also for distributive and intuitionistic modal logics.

Developments in Sahlqvist theory: the algorithmic approach. Extensions of Sahlqvist formulas using syntactic characterizations are not always easy to use in practice, whence the need to look for easier ways to characterize formulas which correspond to first-order conditions on frames.
This motivates the introduction of algorithms for correspondence (e.g SQEMA [23], [24] and ALBA [26]), which, given an arbitrary modal formula in input, compute its first-order correspondent, or terminate with failure. The algorithmic approach is different from the syntactic approach in the sense that "it does not care " to which the syntactically defined class the input formula belongs to. The algorithm can either succeed or report failure. This is in fact very nice, because it is not always easy to use the syntactic characterizations in practice, but it is very easy to run the algorithm on a formula and see whether it succeeds. Moreover, one is typically not interested to know whether a formula is Sahlqvist or inductive; rather, one is interested in knowing to which condition a certain formula corresponds. This question is exactly what the algorithm can be used to give answer to. However, given an algorithm, one is also interested in knowing how powerful and effective it really is. To answer this question, it has been shown that:

1. SQEMA and ALBA are very effective and are respectively guaranteed to succeed on all the inductive formulas in the setting of classical modal logic (SQEMA [23], [24]), of distributive and intuitionistic modal logic (ALBA [26]).

2. The recursive extension of SQEMA [21] is guaranteed to succeed on the class of recursive formulas, which is larger than the class of all inductive formulas, which in turn properly extends the class of Sahlqvist formulas.

3. The ALBA algorithm is guaranteed to succeed on a class larger than the class of all inductive inequalities[1].

General methodology. This thesis focuses on the problem of extending correspondence theory to different logics. To be able to successfully extend Sahlqvist theory to different types of logics, we need to understand the mathematical principles which make Sahlqvist theory work independently of the specific logical signature. It turns out that the algebraic approach is extremely well suited for this kind of analysis; besides [4,13,15] and [16] it has been pursued earlier in Jónsson [6], Sambin and Vaccaro [17], Ghilardi and Meloni [15].

The methodological foundations of the algebraic approach, as well as how it connects to the classical model-theoretic approach has been extensively explained in [22]; in a nutshell, this perspective is based on the duality theory for modal logic, which makes it possible to systematically translate validity and satisfaction of modal formulas from Kripke frames to their *complex algebras*, and then explain the Sahlqvist mechanism in terms of the order-theoretic properties of the operations which interpret the logical connectives of modal formulas. Since these order-theoretic properties are not unique to modal logic but hold for many logics (for instance, for intuitionistic and substructural logics as well), this algebraic and order-theoretic account of Sahlqvist theory gives a way to define

[1] In [26], the class of inequalities on which ALBA is guaranteed to succeed is proven to be strictly larger than the inductive inequalities, which in turn properly extends the class of Sahlqvist inequalities of [14]. Inductive inequalities are the distributive counterpart of the inductive formulas of Goranko and Vakarelov in the classical setting.

Sahlqvist, inductive and recursive formulas for all the logics with the right order-theoretic properties.

Modularity and its advantages. The algebraic approach that we described above has another advantage: it is *modular* in nature. Indeed,

1. the order-theoretic machinery can be easily transferred to different logics, which provides a uniform way to identify syntactic classes and to define versions of existing algorithms just by introducing little changes; in this way, it can be seen that the order-theoretic properties that underly the characterization of the Sahlqvist class remain inherently unchanged when moving from logics based on classical propositional logic to weaker, non-classical settings, such as intuitionistic, distributive and non-distributive lattice-based logics;

2. moreover, certain logics, as for instance substructural logics and intuitionistic modal logics, admit more than one type of relational semantics, and therefore, it can be expected that the same formula admits different first-order correspondents in the different semantic structures. However, the different relational structures dually generate algebras which all enjoy certain crucial order-theoretic properties; the algebraic approach depends exclusively on these properties (more on this in Chapter 2). Therefore, in these cases, the quest for first-order correspondents can be neatly divided in two steps:

    - in the first step, the general algebraic setting is employed independently of the specification of the semantics for the given logic;
    - in the second step, the results are then translated taking into account the concrete specifications of each relational semantics. This is very nice because the first step can be used across all relational semantics for the same given logic.

The modal mu-calculus. In 1983, Dexter Kozen [7] introduced a logical framework that combined simple modalities with fixpoint operators to enrich the expressivity of modal logic so as to deal with infinite processes such as recursion. This logic, also known as the (modal) mu-calculus, has a simple syntax, an easily given semantics, and yet is decidable [10]. Modal mu-calculus has become a fundamental logical tool in theoretical computer science and has been extensively studied ([9],[10],[7],[11]) and applied, for instance in the context of temporal properties of systems (e.g., that a given property is verified infinitely often), or infinite properties of concurrent systems (e.g., when pieces of computations are performed by more than one computer in parallel, and so on). Besides, many expressive modal and temporal logics such as PDL, CTL, CTL^* can be seen as fragments of the modal mu-calculus ([10], section 4.1). Finally, modal mu-calculus provides a unifying framework connecting modal and temporal logics, automata theory and the theory of games.

Developing correspondence theory for modal mu-calculus can be useful because, besides helping to strengthen its general mathematical theory and facilitate the transfer of results from first-order logic with fixpoints, it can also help the understanding of the meaning of mu-formulas, which is often difficult to grasp.

To our knowledge, the only results about frame-correspondence theory for mu-calculus was developed in [13] by means of purely model-theoretic techniques. Given the existing algebraic approach to Sahlqvist theory, a natural question arising at this point is whether an algebraic and order-theoretic account can be given of the results in [13], which would hopefully help to extend them to fixpoint logics with a weaker-than-classical base. The answer turns out to be in the affirmative. Before turning to presenting the original contributions of this thesis, let us mention a few reasons why fixpoint logics with a weaker-than-classical bases might be interesting to study in their own right:

1. the phenomena captured by fixpoints are intrinsically independent of their being set in classical logic; hence we can gain a better insight on their nature;

2. we obtain a greater generality, and hence the possibility to apply our results also in contexts where classical axioms such as the excluded middle are not sound.

3. constructive modal logics and type theory are of increasing foundational and practical relevance in fields such as semantics of programming languages, so mu-calculus on intuitionistic base can be a valuable tool to these investigations.

Our contribution. In this thesis, we give an algorithmic, order-theoretic account of the correspondence theory for modal mu-calculus presented in [13]. We isolate the algebraic principles underlying these results, and, following the methodology developed in [22], we add an intermediate level of order-theoretic analysis to the model-theoretic analysis presented in [13]. This makes it possible to:

1. Recognize that the classical Boolean setting plays no essential role in the development of the theory;

2. Extend the theory to several different logics with fixpoints, of which we only present in detail the case study of intuitionistic modal mu calculus;

3. Recognize that the distributive setting plays essentially no role for the crucial order-theoretic preservation properties of fixpoints; accordingly, we develop the relevant theory of order-theoretic preservation properties of fixpoints in the vastly more general setting of complete semilattices of which no distributivity law is assumed. This paves the way to the development of correspondence theory for substructural logics expanded with fixpoints.

4. Observe that different relational semantics can be given for the same given (fixpoint) logic; accordingly, we develop the crucial part of correspondence theory independently of the specific way the relational semantics is specified.

In particular, in this thesis, we extend the algorithm ALBA of [26] to the language of intuitionistic modal logic with extremal fixpoints, so that the enhanced version of ALBA can be modularly extended in such a way as to manipulate (a wider class of) mu-inequalities (more of this in Chapter 2). Moreover, we define the class of recursive inequalities for intuitionistic modal mu-calculus. This class is the intuitionistic counterpart of the class of Sahlqvist mu-formulas defined in [13]. Then, we give an informal justification to the effect that the enhanced ALBA is always successful on all the recursive mu-inequalities.

It is worth to stress that all the results and in particular all the practical reductions which we develop for intuitionistic modal mu-calculus are immediately applicable to the classical case, as will be illustrated later on in some examples.

Structure of the thesis. This thesis is organized as follows: In Chapter 1, we give the needed preliminaries on intuitionistic modal logic, its algebraic and one of its relational semantics, as well as on the order-theoretic facts about fixpoints which will be useful for our work; in Chapter 2, we sketch the essential features of the algorithmic approach to correspondence and explain what our contribution to it consists of; in Chapter 3, we review some versions of the Ackermann's lemma, which is the core engine of the algorithmic monadic second order variable elimination process; the original contribution of this thesis starts in Chapter 4, where we develop the order-theoretic preservation properties of extremal fixpoints, in a general setting of complete (non-distributive) lattices of variable assignments; in Chapter 5, we introduce formally the language and semantics of intuitionistic modal mu-calculus, as well as the expanded language which facilitates the algebraic correspondence reductions; in Chapter 6, we introduce some of the formal tools for the enhanced version of ALBA, in the form of so-called *approximation rules* (see Chapter 2 for more details on approximation rules) for fixpoint binders; we also discuss a proposal for the so-called *residuation rules* (see Chapter 2) for fixpoint binders, which turns out not to be good for our purposes, and then a case study. In Chapter 7, the adjunction and residuation rules for (non-nested) fixpoints which we will be actually using are introduced, together with a method to recursively reduce nested fixpoints to the non-nested cases. In Chapter 8, we define the intuitionistic recursive mu-inequalities, and justify informally that ALBA, augmented with the rules defined in the previous chapters, is enough to cover them. In Chapter 9, we discuss the conclusions and what is left for future developments.

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PRELIMINARIES

In this Chapter, we introduce the preliminaries on intuitionistic modal logic, as well as the duality between its relational and algebraic semantics in the form of perfect distributive lattices with operators. Also, preliminaries on fixpoints will be given here, which will be useful later on. We will assume some familiarity with partial orders and lattices [4], and therefore we will not give preliminaries on this.

> *The material in Section 1.1 is essentially an adaptation of the distributive modal logic setting of [26] to the case of Intuitionistic modal logic. Different relational semantics are also adopted in the literature for the same language, see [3]. The material in Section 1.2 is taken from [1] and [28]. If the reader is familiar with these topics, this chapter can be skipped.*

1.1 Intuitionistic modal logic

1.1.1 Syntax

The language of Intuitionistic Modal Logic (or IML for short) is obtained by adding two unary modalities $\Diamond, \Box$ and one binary connective $\rightarrow$ to the propositional language of distributive lattices. The intuitive meanings of $\Diamond\varphi, \Box\varphi$, respectively are: '$\varphi$ is possible, φ is necessary'. Because of the absence of classical negation, the modal operators are not interdefinable and thus both of them have to be taken as primitive. For the sake of generality, they will be also semantically interpreted using two (different) accessibility relations. Let PROP be a denumerably infinite set of propositional variables. The elements of PROP will be denoted by p, q, r, possibly indexed. The well-formed formulas of intuitionistic modal logic are given by the following grammar:

$$\varphi ::= \bot \mid p \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \Diamond\varphi \mid \Box\varphi \mid \varphi \rightarrow \psi.$$

We will use $\neg\varphi$ and $\top$ as abbreviations for $\varphi \rightarrow \bot$ and $\bot \rightarrow \bot$ respectively.

Definition 1.1. *An Intuitionistic Modal Logic Λ is a subset of formulas which contains all theorems of intuitionistic propositional calculus (IPC) and closed under modus ponens, substitution and the rules:*

$\Box(p \wedge q) = \Box p \wedge \Box q \qquad \Diamond(p \vee q) = \Diamond p \vee \Diamond q$
$\Box\top = \top \qquad \Diamond\bot = \bot$
$\varphi \rightarrow \psi / \Diamond\varphi \rightarrow \Diamond\psi \qquad \varphi \rightarrow \psi / \Box\varphi \rightarrow \Box\psi.$

1.1.2 Relational semantics for intuitionistic modal logic

Given a poset $(W, \leq)$, we will use the symbol $\mathcal{P}^{\uparrow}(W)$ to indicate the set of all upward-closed subsets of $(W, \leq)$ (that is, all subsets $U \subseteq W$ such that if $u \in U$ and $u \leq x$ then $x \in U$). IML-frames are based on posets and not on sets.

Definition 1.2. *(IML-frame). An IML-frame is a structure $\mathcal{F} = (W, \leq, R_{\Diamond}, R_{\Box})$ such that $(W, \leq)$ is a non empty poset, $R_{\Diamond}, R_{\Box}$ are binary relations on W such that the following conditions hold:*

$$\geq \circ R_{\Diamond} \circ \geq \subseteq R_{\Diamond} \qquad \leq \circ R_{\Box} \circ \leq \subseteq R_{\Box}.$$

A model based on an IML-frame $\mathcal{F}$ is a pair $\mathcal{M} = (\mathcal{F}, V)$, where $V : PROP \to \mathcal{P}^{\uparrow}(W)$, assigns an upward-closed subset to every propositional variable. V is called a *persistent valuation* on $\mathcal{F}$.

Given a model $\mathcal{M} = (\mathcal{F}, V)$ based on an IML-frame $\mathcal{F} = (W, \leq, R_{\Diamond}, R_{\Box})$ and a state $w \in W$, the semantics of our language $\mathcal{L}_{term}$ is formally given by:

$\mathcal{M}, w \Vdash \bot$, never

$\mathcal{M}, w \Vdash \top$, always

$\mathcal{M}, w \Vdash p$ if and only if $w \in V(p)$

$\mathcal{M}, w \Vdash \varphi \vee \psi$ if and only if $\mathcal{M}, w \Vdash \varphi$ *or* $\mathcal{M}, w \Vdash \psi$

$\mathcal{M}, w \Vdash \varphi \wedge \psi$ if and only if $\mathcal{M}, w \Vdash \varphi$ and $\mathcal{M}, w \Vdash \psi$

$\mathcal{M}, w \Vdash \Diamond\varphi$ if and only if there exists $v \in W$ with ($wR_{\Diamond}v$ and $\mathcal{M}, v \Vdash \varphi$)

$\mathcal{M}, w \Vdash \Box\varphi$ if and only if for all $v \in W$ with $wR_{\Box}v$, we have $\mathcal{M}, v \Vdash \varphi$

$\mathcal{M}, w \Vdash \varphi \to \psi$ if and only if for all $v \in W$ with $w \leq v$ if $\mathcal{M}, v \Vdash \varphi$ then $\mathcal{M}, v \Vdash \psi$.

Observe that $\mathcal{M}, w \Vdash \neg\varphi$ if and only if for all $v \in W$ with $w \leq v$, $\mathcal{M}, v \nVdash \varphi$.

We will use $\mathcal{M} \Vdash \varphi$ if $\mathcal{M}, w \Vdash \varphi$ for every state $w \in W$.

As usual, the IML formula φ is valid at a state in $\mathcal{F}$ (denoted $\mathcal{F}, w \Vdash \varphi$), if $(\mathcal{F}, V), w \Vdash \varphi$ for every persistent valuation V on $\mathcal{F}$; the formula φ is valid on $\mathcal{F}$ if it is valid at every state in $\mathcal{F}$.

Definition 1.3. *(Complex algebra of a IML-frame). For every IML-frame*

$$\mathcal{F} = (W, \leq, R_{\Diamond}, R_{\Box}),$$

the complex algebra of $\mathcal{F}$ is

$$\mathcal{F}^{+} = (\mathcal{P}^{\uparrow}(W), \cup, \cap, \emptyset, \Rightarrow, \langle R_{\Diamond}\rangle, [R_{\Box}]),$$

where for each $X \subseteq W$ we have[1]*,*

[1] Let X^c be the relative complement of $X \subseteq W$, $X{\downarrow} = \{w \in W \mid w \leq x \text{ for some } x \in X\}$, and for every $R \subseteq W \times W$, let $R[w] = \{x \in W \mid wRx\}$.

$[R_\Box]X := \{w \in W \mid R_\Box[w] \subseteq X\}$
$\langle R_\Diamond \rangle X := \{w \in W \mid R_\Diamond[w] \cap X \neq \emptyset\}$
and, for $X, Y \in W$, $X \Rightarrow Y = (X \cap Y^c)\downarrow^c$.

1.1.3 Perfect intuitionistic algebras and their correspondence with frames

Definition 1.4. *An intuitionistic modal algebra (IMA) is an algebra*

$$\mathcal{A} = (A, \vee, \wedge, \rightarrow, \bot, \top, \Diamond, \Box)$$

such that $(A, \vee, \wedge, \rightarrow, \bot, \top)$ *is a Heyting algebra*[2] *and the additional operations satisfy the following identities:*

$$\Diamond(x \vee y) = \Diamond x \vee \Diamond y \qquad \Diamond\bot = \bot \qquad \Box x \wedge \Box y = \Box(x \wedge y) \qquad \top = \Box\top.$$

It is well known in the classical setting that the complex algebras of Kripke frames are characterized in purely algebraic terms as complete and atomic modal algebras. This can be extended to the IML setting. In fact, the complex algebras of IML-frames are abstractly characterized as perfect IMA's.
In what follows, C will be a complete lattice except otherwise stated. An element $j \in C$ is completely join prime if for every $S \subseteq C$, if $j \leq \bigvee S$ then $j \leq s$ for some $s \in S$. An element $m \in C$ is completely meet prime if for every $S \subseteq C$, if $\bigwedge S \leq m$ then $s \leq m$ for some $s \in S$.

We denote by $\mathcal{J}^\infty(C)$ the collection of *completely join-prime elements* of C and by $\mathcal{M}^\infty(C)$ the collection of *completely meet-prime elements* of C.

Definition 1.5. *(cf. [12], Def. 2.9) A perfect distributive lattice is a complete lattice* C *such that* $\mathcal{J}^\infty(C)$ *is join-dense*[3] *in* C *and* $\mathcal{M}^\infty(C)$ *is meet-dense in* C.

Perfect distributive lattices are also characterized as those lattices that are isomorphic to $\mathcal{P}^\uparrow(X)$ for some poset X [26]. Section 2.4 of [26] states that, if we consider $\mathcal{J}^\infty(C)$ and $\mathcal{M}^\infty(C)$ as subposets of any perfect distributive lattice C then we have the following proposition:

Proposition 1.1. *The map* $\kappa : \mathcal{J}^\infty(C) \to \mathcal{M}^\infty(C)$ *defined by* $j \mapsto \bigvee\{u \in C;\ j \nleq u\}$ *is an order isomorphism. The inverse of* κ *denoted* λ *is defined order dually.*

Definition 1.6. *(Perfect Heyting Algebra)*
A perfect Heyting Algebra is a perfect distributive lattice equipped with the Heyting implication.

Definition 1.7. *(Perfect IMA)*
An IMA $\mathcal{A} = (A, \vee, \wedge, \rightarrow, \bot, \top, \Diamond, \Box)$ *is perfect if* $(A, \vee, \wedge, \rightarrow, \bot, \top)$ *is a perfect Heyting Algebra and for every* $S \subseteq A$*:*

$$\Diamond(\bigvee S) = \bigvee\{\Diamond s :\ s \in S\} \qquad \Box(\bigwedge S) = \bigwedge\{\Box s :\ s \in S\}.$$

[2] A Heyting algebra is a bounded distributive lattice equipped with the Heyting implication (i.e., a binary operation $\rightarrow$ such that for every $a, b, c \in \mathcal{A}$, $a \wedge b \leq c$ iff $a \leq b \rightarrow c$.

[3] i.e., for all $u \in C$ $u = \bigvee\{r \in \mathcal{J}^\infty(C);\ r \leq u\}$. Meet-dense is defined order dually.

Proposition 1.2. *For every IML-frame $\mathcal{F}$, $\mathcal{F}^+$ is a perfect IMA.*

In the classical case, we can associate any complete atomic modal algebra with its *atom structure*, i.e., a Kripke frame based on the set of its atoms; likewise, we can associate any perfect Heyting algebra with its prime structure (cf. definition 2.12 of [26])

$$C_+ = (\mathcal{J}^\infty(C), \geq, R_\Diamond, R_\Box),$$

where $(\mathcal{J}^\infty(C), \geq)$ is the *dualized subposet* of the completely join-prime elements of C, and for every $\mathbf{i}, \mathbf{j} \in \mathcal{J}^\infty(C)$,

$$\mathbf{j} R_\Diamond \mathbf{i} \;\text{ iff }\; \mathbf{j} \leq \Diamond \mathbf{i} \qquad \mathbf{j} R_\Box \mathbf{i} \;\text{ iff }\; \Box\kappa(\mathbf{i}) \leq \kappa(\mathbf{j}).$$

The following proposition states the duality that exists between IML-frames and perfect IMA.

Proposition 1.3. *For every perfect IMA C and every IML-frame $\mathcal{F}$,*

$$(C_+)^+ \cong C \;\; \textit{and} \;\; (\mathcal{F}^+)_+ \cong \mathcal{F}.$$

1.1.4 The expanded language of perfect IMA's

A crucial role for our (algebraic) correspondence will be played by an expansion of the language of IML. Any perfect IMA is a complete lattice, and, by definition, the operations $\vee, \wedge, \Diamond, \Box$ of a perfect IMA are each either completely join- or meet-preserving in each coordinate. So each of them has a coordinatewise adjoint, or residual, as seen below. Each operation in the lower row is the (coordinatewise) adjoint of the corresponding operation in the upper row:

$\wedge$	$\vee$	$\Diamond$	$\Box$
$\rightarrow$	$-$	$\blacksquare$	$\blacklozenge$

where $-$ is the Heyting implication of the dual lattice. Indeed, the complex algebra of a frame $(W, \leq, R_\Diamond, R_\Box)$ (in the intuitionistic context) is a complete bi-Heyting[4] algebra with operators. For reasons which will be explained in Chapter 2, we find it useful to define an expanded language for perfect IMA's, which will include the connectives corresponding to all the adjoint operations, as well as a denumerably infinite set of sorted variables NOM called nominals, ranging over the completely join-prime elements of perfect IMA's, and a denumerably infinite set of sorted variables CNOM, called co-nominals, ranging over the completely meet-prime elements of perfect IMA's. The elements of NOM will be denoted with **i**; **j**, and those of CNOM with **m**; **n**.

Definition 1.8. *([26]) The formulas of the expanded language $\mathcal{L}^+_{term}$ are given by the following grammar:*

$$\varphi ::= \bot \mid p \mid \boldsymbol{j} \mid \boldsymbol{m} \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \varphi - \psi \mid \varphi \rightarrow \psi \mid \Diamond\varphi \mid \Box\varphi \mid \blacklozenge\varphi \mid \blacksquare\varphi$$

where $p \in PROP$, $\boldsymbol{j} \in NOM$, and $\boldsymbol{m} \in CNOM$.

[4] A bi-Heyting algebra is a Heyting algebra such that its dual is also a Heyting algebra.

A *valuation* for $\mathcal{L}^+_{term}$ on an IML frame $\mathcal{F}$ is any map: $V : PROP \cup NOM \cup CNOM \to \mathcal{P}^{\uparrow}(W)$ such that $V(p) \in \mathcal{P}^{\uparrow}(W)$. $V(\mathbf{i}) = x{\uparrow}$ for some $x \in W$, for each $\mathbf{i} \in$ NOM. For each $\mathbf{m} \in$ CO-NOM, $V(\mathbf{m}) = (x{\downarrow})^c = \kappa(x{\uparrow})$ for some $x \in W$. From now on till the end of this chapter, we will denote by $\mathcal{L}^+$ the set of inequalities between terms in $\mathcal{L}^+_{term}$. The satisfaction relation for formulas of $\mathcal{L}^+$ is defined recursively as can be seen in section 2.5 of [26]. The following proposition shows how the *local* satisfaction of modal formulas can be encoded as a special case of *global* satisfaction of inequalities.

Proposition 1.4. *For every IMA C, every IML frame $\mathcal{F}$, every $\mathcal{L}^+_{term}$-valuation V on $\mathcal{F}$ and $\mathcal{L}^+_{term}$-assignment v on C, and every $\varphi \in \mathcal{L}^+_{term}$,*

1. *the following are equivalent:*
   a) $\mathcal{F}, V, w \Vdash \varphi$;
   b) $\mathcal{F}^+, V'^+ \vDash \boldsymbol{j} \leq \varphi$, where $\boldsymbol{j}$ is a new nominal not occurring in φ and V' is the $\boldsymbol{j}$-variant of V such that $V'(\boldsymbol{j}) = \{w\}{\uparrow}$;
   c) $\mathcal{F}^+, V'^+ \nvDash \varphi \leq \boldsymbol{m}$, where $\boldsymbol{m}$ is a new co-nominal not occurring in φ and V' is the $\boldsymbol{m}$-variant of V such that $V'(\boldsymbol{m}) = (\{w\}{\downarrow})^c$.

2. *$C, v \vDash \boldsymbol{j} \leq \varphi$ if and only if $C_+, v_+, v(\boldsymbol{j}) \vDash \varphi$.*

3. *$C, v \vDash \varphi \leq \boldsymbol{m}$ if and only if $C_+, v_+, \lambda(v(\boldsymbol{m})) \nvDash \varphi$.*

For an account of the *standard translation* of $\mathcal{L}^+$ in the general setting of DML, see section 2.5.2 of [26].

1.2 Order theoretic facts about fixpoints

The logical operators of the modal μ-calculus can be interpreted using algebraic tools. Recall that given an ordered set S and a map $F : S \to S$, an element $x \in S$ is called a fixpoint of F if $F(x) = x$. The least and greatest fixpoints of F will be respectively denoted by $LFP.F$ and $GFP.F$.

Theorem 1.1. *(The Knaster-Tarski Fixpoint Theorem)*
Let $(L, \bigvee, \bigwedge)$ be a complete lattice and $F : L \to L$ be an order-preserving map. Then

1. *$LFP.F = \bigwedge\{x \in L \mid F(x) \leq x\}$ is the least fixpoint of F.*

2. *$GFP.F = \bigvee\{x \in L \mid x \leq F(x)\}$ is the greatest fixpoint of F.*

Proof. 1. We first have to show that $LFP.F$ is a fixpoint of F; i.e., $F(x) = x$. Let $G = \{x \in L \mid F(x) \leq x\}$. For all $x \in G$, $LFP.F \leq x$, so $F(LFP.F) \leq F(x)$ since F is order-preserving. Thus, by transitivity of $\leq$ and using the fact that $x \in G$, we get $F(LFP.F) \leq x$. Hence $F(LFP.F)$ is a lower bound of G, whence

$$F(LFP.F) \leq LFP.F. \tag{1.1}$$

Since F is order preserving, from (1.1) we get $F(F(LFP.F)) \leq F(LFP.F)$. This entails that $F(LFP.F) \in G$; now $LFP.F = \bigwedge G$, so

$$LFP.F \leq F(LFP.F). \tag{1.2}$$

Equalities (1.1) and (1.2) yield $F(LFP.F) = LFP.F$ as desired.
Next we show that $LFP.F$ is the least fixpoint of F. Let α be any fixpoint of F. Then $\alpha \in G$, and so $LFP.F \leq \alpha$; i.e., $LFP.F$ is the least fixpoint of F.

2. This follows from 1. by order duality.

□

Definition 1.9. *Let $\mathbb{C} = (C, \bigvee, \bigwedge)$ be a complete lattice, and let $f : C \to C$ be some map. Then by ordinal induction we define the following maps on C:*

$$f_\mu^0(c) \quad := \quad c$$

$$f_\mu^{\alpha+1}(c) := \ f(f_\mu^\alpha(c))$$

$$f_\mu^\lambda(c) \quad := \bigvee_{\alpha \leq \lambda} f_\mu^\alpha(c).$$

where λ denotes an arbitrary limit ordinal.
By replacing μ and $\bigvee$ respectively by ν and $\bigwedge$, we obtain an analogue (dual) definition.

Proposition 1.5. *Let $\mathbb{C} = (C, \bigvee, \bigwedge)$ be a complete lattice, and let $f : C \to C$ be a monotone map. Then f is inductive, that is,*

$$f_\mu^\alpha(\bot) \leq f_\mu^\beta(\bot).$$

for all ordinals α and β such that $\alpha < \beta$.

Corollary 1.1. *Let $\mathbb{C} = (C, \bigvee, \bigwedge)$ be a complete lattice, and let $f : C \to C$ be a monotone map. Then there is some α of size at most $\mid C \mid$ such that $LFP.f = f_\mu^\alpha(\bot)$.*

The following proposition will be useful later in this thesis.

Proposition 1.6. *(cf. proposition 1.2.23 of [1])*
Let E be a complete lattice and D be an ordered set. If $f : E \times D \to E$ is monotonic in its two arguments, then $\mu x.f(x, y)$ and $\nu x.f(x, y)$ are monotonic mappings from D to E.

In the following, we will write θ for either μ or ν.

Proposition 1.7. *(cf. proposition 1.3.2 of [1])*
Let E be a complete lattice, and $h : E \times E \to E$ be a map which is monotonic in its two arguments. Then,

$$\theta x.\theta y.h(x, y) = \theta x.h(x, x) = \theta y.\theta x.h(x, y).$$

The context of our work: order theoretic algorithmic correspondence

The material in this chapter is a re-elaboration of insights coming from [26] and [22]. If the reader is not familiar with these papers, this chapter cannot be skipped.

The original contribution of this thesis is rather technical and can only be understood in the context of the order theoretic algorithmic correspondence theory [26]. Rather than giving a full account of this theory, which will be very lengthy, in this chapter we are going to run the algorithm on some examples, while at the same time illustrate the order-theoretic principles on which it is based. Let us start with one of the best known examples in correspondence theory namely $\Diamond\Diamond p \rightarrow \Diamond p$. It is well known that for every kripke frame $\mathcal{F} = (W, R)$,

$$\mathcal{F} \Vdash \Diamond\Diamond p \rightarrow \Diamond p \text{ iff } \mathcal{F} \models \forall xyz\,(Rxy\,\&\,Ryz \ \rightarrow Rxz).$$

The first important insight is that, by duality, every piece of argument to prove this correspondence that we can perform on frames can be translated on their dual algebras, which are perfect distributive lattices with operators (actually, in the classical case, they will be in particular complete atomic boolean algebras with operators). More details on this can be found in subsection 1.1.3. Once we are in the setting of perfect distributive lattices, the algorithmic strategy will be implemented there. Let us show how this is done in the case of the example above. First of all, the validity condition on $\mathcal{F}$ which we have written above translates into the complex algebra $\mathbb{A}$ as $[\![\Diamond\Diamond p]\!] \subseteq [\![\Diamond p]\!]$ for every assignment of p into $\mathbb{A}$, so we can rephrase this validity clause as follows:

$$\mathbb{A} \models \forall p[\Diamond\Diamond p \leq \Diamond p].$$

Now observe that in every perfect distributive algebra, every element is both the join of the completely join prime elements below it and the meet of the completely meet prime elements above it. So the condition above can be equivalently rewritten as follows:

$$\mathbb{A} \models \forall p[\bigvee\{i \in \mathcal{J}^{\infty}(\mathbb{A}) \mid i \leq \Diamond\Diamond p\} \leq \bigwedge\{m \in \mathcal{M}^{\infty}(\mathbb{A}) \mid \Diamond p \leq m\}].$$

By elementary properties of least upper bounds and greatest lower bounds in posets (cf. [4]), this condition is true if and only if every element in the join is less than equal to every element in the meet; thus the condition above can be equivalently rewritten as:

$$\mathbb{A} \models \forall p\forall i\forall m[(i \leq \Diamond\Diamond p \ \ \& \ \ \Diamond p \leq m) \Rightarrow i \leq m],$$

where the variables i and m are special and range over $\mathcal{J}^\infty(\mathbb{A})$ and $\mathcal{M}^\infty(\mathbb{A})$ respectively. At this point we can proceed in two different ways:

The first option. The first way to proceed is based on the observation that the operation on $\mathbb{A}$ interpreting $\Diamond$ is in fact $\langle R\rangle$, defined as $\langle R\rangle(X) = \{w \in W \mid \exists x(Rwx \ \& \ x \in X)\}$ for every $X \subseteq W$. It is well known that this operation preserves arbitrary unions i.e., it preserves arbitrary joins in the complete lattice $\mathbb{A}$. By the general theory of adjunction in complete lattices, this is equivalent to $\langle R\rangle$ being a left adjoint (cf. [4, proposition 7.34]). It is also well known that the right adjoint of $\langle R\rangle$ is is the operation $[R^{-1}]$, defined as $[R^{-1}](X) = \{w \in W \mid \forall x(R^{-1}wx \ \Rightarrow \ x \in X)\}$ for every $X \subseteq W$. This implies that if we introduce the symbol $\blacksquare$ in our language which is interpreted by $[R^{-1}]$ on the algebra $\mathbb{A}$, then, for every interpretation of p and $m \in \mathcal{M}^\infty(\mathbb{A})$,

$$\mathbb{A} \models \Diamond p \leq m \ \text{ iff } \ \mathbb{A} \models p \leq \blacksquare m.$$

Hence the condition we had before can be equivalently rewritten as:

$$\mathbb{A} \models \forall p \forall i \forall m[(i \leq \Diamond\Diamond p \ \& \ p \leq \blacksquare m) \Rightarrow i \leq m],$$

and then as follows:

$$\mathbb{A} \models \forall i \forall m[\exists p(i \leq \Diamond\Diamond p \ \& \ p \leq \blacksquare m) \Rightarrow i \leq m].$$

At this point we are in a position to eliminate the variable p and equivalently rewrite the previous condition as follows:

$$\mathbb{A} \models \forall i \forall m[i \leq \Diamond\Diamond\blacksquare m \Rightarrow i \leq m].$$

Let us justify this equivalence: for the direction from top to bottom, fix an interpretation V of the variables $i \in \mathcal{J}^\infty(\mathbb{A})$ and $m \in \mathcal{M}^\infty(\mathbb{A})$, and assume that under this interpretation $i \leq \Diamond\Diamond\blacksquare m$. To prove that, under the interpretation V, the inequality $i \leq m$ holds, consider the variant V^* of V which maps $V^*(p) = \blacksquare V(m)$. Then under this interpretation, of course both the inequalities $i \leq \Diamond\Diamond p$ and $p \leq \blacksquare m$ hold; hence by the topmost clause $i \leq m$ holds under V^* and hence under V. Conversely, fix an interpretation V of the variables p, i and m, and assume that under this interpretation $i \leq \Diamond\Diamond p$ and $p \leq \blacksquare m$. Then by the monotonicity of the term function $\Diamond\Diamond p$, we have that under V, $i \leq \Diamond\Diamond p \leq \Diamond\Diamond\blacksquare m$. Hence by the lower clause, it follows that the inequality $i \leq m$ holds under V.

The argument above is nothing else than an instance of the (proof of) the following (left) *Ackermann's lemma*:

Lemma 2.1. *Let $\alpha, \beta(p), \gamma(p)$ be formulas of a (modal) language $\mathcal{L}^+$ over the set of variables* PROP*; let $p \in$* PROP *such that $p \notin FV(\alpha)$ and let V be any valuation on a frame $\mathcal{F}$; if β is negative in p and γ is positive in p, then the following are equivalent for every $\mathcal{L}^+$-Kripke frame $\mathcal{F}$:*

1. *$\mathcal{F}, V \Vdash \beta(\alpha/p) \leq \gamma(\alpha/p)$;*

2. *there exists a p-variant V^* of V such that $\mathcal{F}, V^* \Vdash p \leq \alpha$ and $\mathcal{F}, V^* \Vdash \beta(p) \leq \gamma(p)$.*

the proof of it similar to the proof of the right Ackermann's lemma which is stated and proved in Chapter 3 (cf. lemma 3.1) . The only thing we want to add at this point is that, when proving the direction from top to bottom, the variant V^* we considered was nothing else than the minimal valuation of the classical Sahlqvist correspondence argument.

Whenever, in a reduction process we reach a shape in which the lemma above (or its symmetric (right) version) can be applied, we say that the condition is in *Ackermann's shape*.

The second option. For this example, there is also another way to reach Ackermann's shape: starting again from

$$\mathbb{A} \models \forall p \forall i \forall m[(i \leq \Diamond\Diamond p \ \ \& \ \ \Diamond p \leq m) \Rightarrow i \leq m],$$

recall that $\mathbb{A}$ is a perfect distributive lattice and so the element of $\mathbb{A}$ interpreting p is the join of the completely join prime elements below it. Hence, if V is any interpretation such that the inequality $i \leq \Diamond\Diamond p$ holds, because the interpretation of $\Diamond\Diamond$ on $\mathbb{A}$ is completely join-preserving, we have that $V(i) \in \mathcal{J}^\infty(\mathbb{A})$ and

$$V(i) \leq \Diamond\Diamond(\bigvee\{j \in \mathcal{J}^\infty(\mathbb{A}) \mid j \leq V(p)\}) = \bigvee\{\Diamond\Diamond j \mid j \in \mathcal{J}^\infty(\mathbb{A}) \text{ and } j \leq V(p)\},$$

which implies that $V(i)$ will be below or equal to some element of the join in display. Hence, we can equivalently rewrite the validity clause above as follows:

$$\mathbb{A} \models \forall p \forall i \forall m[\exists j(i \leq \Diamond\Diamond j \ \ \& \ \ j \leq p \ \ \& \ \ \Diamond p \leq m) \Rightarrow i \leq m],$$

and then as follows:

$$\mathbb{A} \models \forall p \forall i \forall m \forall j[(i \leq \Diamond\Diamond j \ \ \& \ \ j \leq p \ \ \& \ \ \Diamond p \leq m) \Rightarrow i \leq m],$$

Here we are again in a position to eliminate the variable p with a similar argument as before (or applying the Ackermann's lemma), so as to obtain

$$\mathbb{A} \models \forall i \forall m \forall j[(i \leq \Diamond\Diamond j \ \ \& \ \ \Diamond j \leq m) \Rightarrow i \leq m].$$

At this point, using elementary properties of least upper bounds and greatest lower bounds plus the fact that $\mathcal{J}^\infty(\mathbb{A})$ is join-dense in $\mathbb{A}$ and $\mathcal{M}^\infty(\mathbb{A})$ is meet-dense in $\mathbb{A}$, we can further reduce the clause above first as follows:

$$\mathbb{A} \models \forall i \forall j[(i \leq \Diamond\Diamond j \ \ \Rightarrow \ \ \forall m[\Diamond j \leq m \Rightarrow i \leq m]].$$

then as follows:

$$\mathbb{A} \models \forall i \forall j[(i \leq \Diamond\Diamond j \ \ \Rightarrow \ \ i \leq \Diamond j].$$

then as follows:

$$\mathbb{A} \models \forall j[\Diamond\Diamond j \leq \Diamond j].$$

Again via duality, the clause above can be retranslated on frames as

$$\mathcal{F} \Vdash \forall w[\langle R\rangle\langle R\rangle\{w\} \subseteq \langle R\rangle\{w\}],$$

which by definition is

$$\mathcal{F} \Vdash \forall w[R^{-1}[R^{-1}[w]] \subseteq R^{-1}[w]],$$

which can be easily translated into the familiar transitivity condition of R^{-1} and hence of R.

A proof-theoretic presentation. The discussion so far illustrates the main strategy for the elimination of second order variables, which is the core of Sahlqvist theory, by means of the algorithmic approach: namely, we transform the initial validity condition in a shape in which the Ackermann's lemma is applicable (i.e., the Ackermann's shape). Clearly, the Ackermann's lemma is the endpoint of this strategy, but besides it, the two special properties which guaranteed the soundness of the rewriting process to be able to reach the Ackermann's shape were:

(a) The possibility of *approximating* elements of the algebra from above or from below using completely join prime and completely meet prime elements;

(b) the fact that $\langle R\rangle$ is a left *adjoint*. This is a general fact of all the operations which interpret the (non fixpoint) logical connectives in our language: indeed they are either residuals or adjoints.

Accordingly, we can repackage these observations in the form of *proof rules*. For instance, a mirror-image version of the Ackermann's lemma 2.1 (cf. lemma 3.1) implies that the following rules are sound and invertible w.r.t. the standard Kripke semantics (and hence also w.r.t. the algebraic semantics of complete atomic Boolean algebras):

$$\frac{\forall p[\alpha \leq p \Rightarrow \phi(p) \leq \psi(p)]}{\phi(\alpha/p) \leq \psi(\alpha/p)}\ (\mathrm{RA}) \qquad \frac{\forall p[\phi(p) \leq \psi(p)]}{\phi(\bot/p) \leq \psi(\bot/p)}\ (\bot)$$

subject to the restrictions that α be p-free, and that ϕ and ψ be respectively positive and negative in p. Notice that the rule $(\bot)$ can be regarded as the special case of (RA) in which $\alpha := \bot$. Likewise, lemma 2.1 implies that the following rules are sound and invertible w.r.t. the standard Kripke semantics (and hence also w.r.t. the algebraic semantics of complete atomic Boolean algebras):

$$\frac{\forall p[p \leq \alpha \Rightarrow \phi(p) \leq \psi(p)]}{\phi(\alpha/p) \leq \psi(\alpha/p)}\ (\mathrm{LA}) \qquad \frac{\forall p[\phi(p) \leq \psi(p)]}{\phi(\top/p) \leq \psi(\top/p)}\ (\top)$$

subject to the restrictions that α be p-free, and that ϕ and ψ be respectively negative and positive in p. Other rules which can be easily proved to be sound and invertible come in two types: the *approximation rules* and the *residuation/adjunction rules*, such as the following ones (below, we use the variables $\mathbf{i}, \mathbf{j}$ to be interpreted as completely join prime elements, and the variables $\mathbf{m}, \mathbf{n}$ to be interpreted as completely meet prime elements of the algebra):

$$\frac{\phi \leq \psi}{\forall \mathbf{j} \forall \mathbf{m}[(\mathbf{j} \leq \phi \ \&\ \psi \leq \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}]}\ (\mathrm{FA})$$

The rule above is called first approximation and its soundness and invertibility has been motivated in the discussion above. The following rules are sound and invertible thanks to the fact that $\vee$ is both a left adjoint and a right residual:

$$\frac{\phi \vee \chi \leq \psi}{\phi \leq \psi \quad \chi \leq \psi}\ (\vee\mathrm{LA}) \qquad \frac{\phi \leq \chi \vee \psi}{\phi - \chi \leq \psi}\ (\vee\mathrm{RR})$$

The following rules have been applied in the discussion above and their soundness and invertibility is motivated by the fact that $\Diamond$ is a left adjoint, and hence completely join preserving, plus the fact that $\mathbf{i}, \mathbf{j}$ are interpreted as completely join prime elements:

$$\frac{\Diamond\phi \leq \psi}{\phi \leq \blacksquare\psi}\ (\Diamond\text{LA}) \qquad \frac{\mathbf{j} \leq \Diamond\psi}{\exists \mathbf{i}(\mathbf{j} \leq \Diamond\mathbf{i}\ \&\ \mathbf{i} \leq \psi)}\ (\Diamond\text{Approx})$$

In a perfect distributive lattice, $\wedge$ is both a right adjoint and a left residual:

$$\frac{\phi \leq \chi \wedge \psi}{\phi \leq \chi \quad \phi \leq \psi}\ (\wedge\text{RA}) \qquad \frac{\phi \wedge \chi \leq \psi}{\phi \leq \chi \to \psi}\ (\wedge\text{LR})$$

The interpretation of $\Box$ is a right adjoint and and hence completely meet preserving, plus $\mathbf{m}, \mathbf{n}$ are interpreted as completely meet prime elements:

$$\frac{\phi \leq \Box\psi}{\blacklozenge\phi \leq \psi}\ (\Box\text{RA}) \qquad \frac{\Box\phi \leq \mathbf{m}}{\exists \mathbf{n}(\Box\mathbf{n} \leq \mathbf{m}\ \&\ \phi \leq \mathbf{n})}\ (\Box\text{Appr})$$

Applying the algorithmic approach to mu-calculus. Given the machinery which we have seen above, can we apply it successfully to mu-inequalities? Yes we can in some cases, as the following example will show:

The inequality $\nu X.\Box(p \wedge X) \leq p$ can be reduced as follows:

$$\begin{array}{lll} & & \forall p[\nu X.\Box(p \wedge X) \leq p] \\ & \text{iff} & \forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \nu X.\Box(p \wedge X)\ \&\ p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\ (*) & \text{iff} & \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \nu X.\Box(\mathbf{m} \wedge X) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\ & \text{iff} & \forall \mathbf{m}[\nu X.\Box(\mathbf{m} \wedge X) \leq \mathbf{m}]. \end{array}$$

The equivalence marked with $(*)$ is an application of the (left) Ackermann's lemma (or, equivalently of the rule (LA)), which is soundly applied because the term function $\gamma(p) = \nu X.\Box(p \wedge X)$ is monotone in p. So all the steps in the previous chain of equivalences can be justified in terms of the principles of order-theoretic algorithmic correspondence we have seen above. But two main questions now are:

(a) in which language should we interpret the clause $\forall \mathbf{m}[\nu X.\Box(\mathbf{m} \wedge X) \leq \mathbf{m}]$ on frames?

(b) Which classes of mu-formulas can be treated, with the tools we have so far?

The answer to (a) is: the clause can be interpreted in the correspondence language FO+LFP; this is done just in the same way as in the fixpoint-free case and as we have seen in the example above (for instance, by interpreting the variable $m \in \mathcal{M}^{\infty}(\mathbb{A})$ on classical kripke frames as the complement of a singleton subset; more of this in [26]).

The answer to (b) is: only very special formulas, namely those which can be transformed into Ackermann's shape; but to achieve this, we need to be able to display some occurrences of the variable p (i.e., we need to get them to the surface of an inequality, either to the left or to the right, by means of applications of the rules we have so far). Of course, if all the propositional variables occur inside the scope of fixpoint binders, there is no hope to achieve this with the tools we have so far. Therefore, all the work in the following chapters is aimed at defining rules (both of approximation-type and of residuation/adjunction-type) for fixpoint binders, which make it possible to reach Ackermann's shape also in cases in which the propositional variables occur in the scope of fixpoint binders.

THE ACKERMANN'S LEMMAS

In this chapter, we review the lemmas which enable us to eliminate the monadic second order variables. They come in two versions: the Ackermann's lemma which we have seen and applied in the previous chapter, and the recursive version of it.

> *The lemmas are not original contributions and can be found in [26], [23] [21], but the examples are.*

Lemma 3.1. *(Right Ackermann's lemma) Let $\alpha, \beta(p), \gamma(p)$ be formulas of a (modal) language $\mathcal{L}^+$ over the set of variables* PROP*; let $p \in$* PROP *such that $p \notin FV(\alpha)$ and let V be any valuation on a frame $\mathcal{F}$; if β is positive in p and γ is negative in p, then the following are equivalent for every $\mathcal{L}^+$-Kripke frame $\mathcal{F}$:*

1. *$\mathcal{F}, V \Vdash \beta(\alpha/p) \leq \gamma(\alpha/p)$;*
2. *there exists a p-variant V^* of V such that $\mathcal{F}, V^* \Vdash \alpha \leq p$ and $\mathcal{F}, V^* \Vdash \beta(p) \leq \gamma(p)$.*

Proof. As to the direction from (1) to (2): let V be a valuation on a frame $\mathcal{F}$, such that $[\![\beta(\alpha/p)]\!]_V \subseteq [\![\gamma(\alpha/p)]\!]_V$. Let V^* be the p-variant of V such that $V^*(p) := [\![\alpha]\!]_V$. Then, because the variable p does not occur in α, we have $[\![\alpha]\!]_{V^*} = [\![\alpha]\!]_V = V^*(p)$, which proves that $\mathcal{F}, V^* \Vdash \alpha \leq p$. However, for every formula ξ, the following chain of equalities holds: $[\![\xi(p)]\!]_{V^*} = [\![\xi]\!](V^*(p)) = [\![\xi]\!]([\![\alpha]\!]_V) = [\![\xi(\alpha/p)]\!]_V$. This and the assumption prove that $\mathcal{F}, V^* \Vdash \beta(p) \leq \gamma(p)$.
Conversely, let V be a valuation on a frame $\mathcal{F}$, assume 2., then we have $[\![\alpha]\!]_{V^*} \subseteq V^*(p)$. Now the variable p does not occur in α so $[\![\alpha]\!]_V = [\![\alpha]\!]_{V^*} \subseteq V^*(p)$. Thus, we have the following:

$$[\![\beta(\alpha/p)]\!]_V = [\![\beta]\!]([\![\alpha]\!]_V) \subseteq [\![\beta]\!](V^*(p)) \subseteq [\![\gamma]\!](V^*(p)) \subseteq [\![\gamma]\!]([\![\alpha]\!]_V).$$

Hence, 1. holds.
The second inclusion above is by the assumption that $\mathcal{F}, V^* \Vdash \beta(p) \leq \gamma(p)$. The first and third inclusions above are justified by the fact that β and γ are respectively positive and negative in p (i.e, $[\![\beta]\!]$ and $[\![\gamma]\!]$ are respectively monotone and antitone in p.) □

3.1 Recursive Ackermann lemma

For the purpose of this section, $\mathcal{F}$ is a DML/IML frame, and $\mathcal{L}^+$ is the expanded language appropriate to its corresponding complex algebra $\mathcal{F}^+$.

Lemma 3.2 (Recursive right Ackermann's Lemma)**.** *Let $\alpha(p)$ and $\beta(p) \in \mathcal{L}^+$ be positive in p, and let $\gamma(p) \in \mathcal{L}^+$ be negative in p. Let V be any valuation on a frame $\mathcal{F}$. Then, the following are equivalent:*

1. $\mathcal{F}, V \Vdash \beta(\mu p.\alpha(p)/p) \leq \gamma(\mu p.\alpha(p))/p)$;
2. *there exists some $V' \sim_p V$ such that $\mathcal{F}, V' \Vdash \alpha(p) \leq p$, and $\mathcal{F}, V' \Vdash \beta(p) \leq \gamma(p)$.*

Proof. As regards '1 ⇒ 2', letting $V'(p) := V(\mu p.\alpha(p))$, it is not difficult to see that 2. holds. Conversely, $\mathcal{F}, V' \Vdash \alpha(p) \leq p$ implies that $V'(p)$ is a pre-fixpoint of $[\![\alpha]\!]$, and hence $LFP([\![\alpha]\!]) \leq V'(p)$. Therefore using the fact that β and γ are respectively positive and negative in p we get:

$[\![\beta(\mu p.\alpha(p)/p)]\!]_V = [\![\beta]\!](LFP([\![\alpha]\!])) \leq [\![\beta]\!](V'(p)) \leq [\![\gamma]\!](V'(p)) \leq [\![\gamma]\!](LFP([\![\alpha]\!])) = [\![\gamma(\mu p.\alpha(p)/p)]\!]_V$. □

Example 3.1. Consider the Löb inequality $\Box(\Box p \to p) \leq \Box p$. In the classical Sahlqvist setting it is well known that this formula does not have a first order correspondent. Indeed, the minimal valuation exists, but it is not first-order definable (so the Löb example is much better behaved than formulas, like the McKinsey formulas, for which the minimal valuation does not exist altogether). Indeed, if we allow for the correspondence language to be more expressive than the simple frame correspondence language, as is done in [13], where the FO+LFP is taken as the correspondence language, then we can apply the recursive Ackermann's lemma, which again gives the minimal valuation, and find the correspondent of the Löb inequality, also in the intuitionistic case, as follows:

$$
\begin{array}{lll}
 & & \forall p[\Box(\Box p \to p) \leq \Box p] \\
 & \textit{iff} & \forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \Box(\Box p \to p)\ \&\ \Box p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
 & \textit{iff} & \forall p \forall \mathbf{i} \forall \mathbf{m}[(\blacklozenge \mathbf{i} \leq \Box p \to p\ \&\ \Box p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
 & \textit{iff} & \forall p \forall \mathbf{i} \forall \mathbf{m}[(\blacklozenge \mathbf{i} \wedge \Box p \leq p\ \&\ \Box p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
(*) & \textit{iff} & \forall \mathbf{i} \forall \mathbf{m}[\Box(\mu p.(\blacklozenge \mathbf{i} \wedge \Box p)) \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
 & \textit{iff} & \forall \mathbf{i}[\mathbf{i} \leq \Box(\mu p.(\blacklozenge \mathbf{i} \wedge \Box p))] \\
 & \textit{iff} & \forall \mathbf{i}[\blacklozenge \mathbf{i} \leq \mu p.(\blacklozenge \mathbf{i} \wedge \Box p)]
\end{array}
$$

In the equivalence marked with $(*)$, the Right Ackermann lemma has been applied with $\alpha(p) := \blacklozenge \mathbf{i} \wedge \Box p$ and $\beta(p) := \Box p$ being positive in p, and $\gamma(p) := \mathbf{m}$ being negative in p.

Example 3.2. Consider the van Benthem inequality $\Box\Diamond\top \leq \Box(\Box(\Box p \to p) \to p)$; as in the previous example, we have:

$$
\begin{array}{lll}
& & \forall p[\Box\Diamond\top \leq \Box(\Box(\Box p \to p) \to p)] \\
& \textit{iff} & \forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \Box\Diamond\top \ \&\ \Box(\Box(\Box p \to p) \to p) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
& \textit{iff} & \forall p \forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{n}[(\mathbf{i} \leq \Box\Diamond\top \ \&\ \Box\mathbf{n} \leq \mathbf{m} \ \&\ \Box(\Box p \to p) \to p \leq \mathbf{n}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
& \textit{iff} & \forall p \forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{m} \forall \mathbf{n}[(\mathbf{i} \leq \Box\Diamond\top \ \&\ \Box\mathbf{n} \leq \mathbf{m} \ \&\ \mathbf{j} \leq \Box(\Box p \to p) \ \&\ \mathbf{j} \to p \leq \mathbf{n}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
& \textit{iff} & \forall p \forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{m} \forall \mathbf{n}[(\mathbf{i} \leq \Box\Diamond\top \ \&\ \Box\mathbf{n} \leq \mathbf{m} \ \&\ \blacklozenge\mathbf{j} \leq \Box p \to p \ \&\ \mathbf{j} \to p \leq \mathbf{n}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
& \textit{iff} & \forall p \forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{m} \forall \mathbf{n}[(\mathbf{i} \leq \Box\Diamond\top \ \&\ \Box\mathbf{n} \leq \mathbf{m} \ \&\ \blacklozenge\mathbf{j} \wedge \Box p \leq p \ \&\ \mathbf{j} \to p \leq \mathbf{n}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
(*) & \textit{iff} & \forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{m} \forall \mathbf{n}[(\mathbf{i} \leq \Box\Diamond\top \ \&\ \Box\mathbf{n} \leq \mathbf{m} \ \&\ \mathbf{j} \to \mu p.(\blacklozenge\mathbf{j} \wedge \Box p) \leq \mathbf{n}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
& \textit{iff} & \forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{n}[(\mathbf{i} \leq \Box\Diamond\top \ \&\ \mathbf{j} \to \mu p.(\blacklozenge\mathbf{j} \wedge \Box p) \leq \mathbf{n}) \Rightarrow \forall \mathbf{m}[\Box\mathbf{n} \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}]] \\
& \textit{iff} & \forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{n}[(\mathbf{i} \leq \Box\Diamond\top \ \&\ \mathbf{j} \to \mu p.(\blacklozenge\mathbf{j} \wedge \Box p) \leq \mathbf{n}) \Rightarrow \blacklozenge\mathbf{i} \leq \mathbf{n}] \\
& \textit{iff} & \forall \mathbf{i} \forall \mathbf{j}[\mathbf{i} \leq \Box\Diamond\top \Rightarrow \forall \mathbf{n}[\mathbf{j} \to \mu p.(\blacklozenge\mathbf{j} \wedge \Box p) \leq \mathbf{n} \Rightarrow \blacklozenge\mathbf{i} \leq \mathbf{n}]] \\
& \textit{iff} & \forall \mathbf{i} \forall \mathbf{j}[\mathbf{i} \leq \Box\Diamond\top \Rightarrow \blacklozenge\mathbf{i} \leq \mathbf{j} \to \mu p.(\blacklozenge\mathbf{j} \wedge \Box p)] \\
& \textit{iff} & \forall \mathbf{i} \forall \mathbf{j}[\mathbf{i} \leq \Box\Diamond\top \Rightarrow \mathbf{i} \leq \Box(\mathbf{j} \to \mu p.(\blacklozenge\mathbf{j} \wedge \Box p))] \\
& \textit{iff} & \forall \mathbf{j}[\Box\Diamond\top \leq \Box(\mathbf{j} \to \mu p.(\blacklozenge\mathbf{j} \wedge \Box p))]
\end{array}
$$

In the equivalence marked with $(*)$, the Right Ackermann lemma has been applied with $\alpha(p) := \blacklozenge\mathbf{j} \wedge \Box p$ and $\beta(p) := \mathbf{j} \to p$ being positive in p, and $\gamma(p) := \mathbf{n}$ being negative in p.

Preservation properties of extremal fixpoints

In this chapter, we give the order-theoretic basis for the definition of sound approximation rules for fixpoint binders. For the sake of greater generality, the presentation is purely lattice-theoretic. In fact, we will be working in a setting of complete lattices which do not need to be distributive. This is interesting because it points at the possibility to extend the algorithmic theory to logics with fixpoints on a non distributive lattice base.

To our knowledge, the material in this chapter is entirely original.

Proposition 4.1. *Let L be a complete $\bigvee$-semilattice, and let $\mathcal{X}$ be a family of monotone maps $f : L \to L$. Then,*

1. $\bigvee\{LFP(f) \mid f \in \mathcal{X}\} \leq LFP(\bigvee \mathcal{X})$.
2. *If $\mathcal{X}$ is directed and every $f \in \mathcal{X}$ is completely join-preserving, then*

$$LFP(\bigvee \mathcal{X}) = \bigvee\{LFP(f) \mid f \in \mathcal{X}\}.$$

3. *If $\mathcal{X}$ is a collection of monotone maps $f : L \to L$, such that*

$$(\bigvee \mathcal{X})(\bigvee S) = \bigvee\{f(a_f) \mid f \in \mathcal{X}\} \tag{4.1}$$

for every $\mathcal{X}$-indexed subset $S = \{a_f \mid f \in \mathcal{X}\} \subseteq L$, then

$$LFP(\bigvee \mathcal{X}) = \bigvee\{LFP(f) \mid f \in \mathcal{X}\}.$$

Proof. 1. Let $x = LFP(\bigvee \mathcal{X})$; so in particular x is a pre-fixpoint of $\bigvee \mathcal{X}$, i.e., $(\bigvee \mathcal{X})(x) \leq x$. Hence, for every $f \in \mathcal{X}$,

$$f(x) \leq \bigvee\{f(x) \mid f \in \mathcal{X}\} = (\bigvee \mathcal{X})(x) \leq x.$$

This shows that any pre-fixpoint of $\bigvee \mathcal{X}$ (and x in particular) is a pre-fixpoint of every $f \in \mathcal{X}$. Therefore, since the least fixpoint is also the least pre-fixpoint, $LFP(f) \leq x$ for every $f \in \mathcal{X}$, which proves that $\bigvee\{LFP(f) \mid f \in \mathcal{X}\} \leq LFP(\bigvee \mathcal{X})$.
2. To prove that $LFP(\bigvee \mathcal{X}) \leq \bigvee\{LFP(f) \mid f \in \mathcal{X}\}$, observe that

$$\begin{aligned} LFP(\bigvee \mathcal{X}) &= \bigwedge\{a \mid (\bigvee \mathcal{X})(a) \leq a\} \\ &= \bigwedge\{a \mid f(a) \leq a \text{ for all } f \in \mathcal{X}\}; \end{aligned}$$

hence it is enough to show that, if $x = \bigvee\{LFP(g) \mid g \in \mathcal{X}\}$, then $f(x) \leq x$ for every $f \in \mathcal{X}$. Fix $f \in \mathcal{X}$; since by assumption f is completely join preserving, $f(x) \leq x$ is equivalent to

$$\bigvee\{f(LFP(g)) \mid g \in \mathcal{X}\} \leq \bigvee\{LFP(g) \mid g \in \mathcal{X}\}.$$

To prove this, it is enough to show that for every $g \in \mathcal{X}$ there exists some $h \in \mathcal{X}$ such that $f(LFP(g)) \leq LFP(h)$. Since by assumption $\mathcal{X}$ is directed and $f, g \in \mathcal{X}$, there exists some $h \in \mathcal{X}$ such that $f \leq h$ and $g \leq h$ (w.r.t. the pointwise order). We will prove the statement above for this choice of h.

Since $LFP(h)$ is a pre-fixpoint of h, we have that $f(LFP(h)) \leq h(LFP(h)) \leq LFP(h)$, and $g(LFP(h)) \leq h(LFP(h)) \leq LFP(h)$, i.e. $LFP(h)$ is a pre-fixpoint of both f and g; hence $LFP(g) \leq LFP(h)$, from which it follows by monotonicity of f that $f(LFP(g)) \leq f(LFP(h))$. Hence the following chain of inequalities holds:

$$f(LFP(g)) \leq f(LFP(h)) \leq LFP(h).$$

3. To prove that $LFP(\bigvee \mathcal{X}) \leq \bigvee\{LFP(f) \mid f \in \mathcal{X}\}$ it is enough to show that $b = \bigvee\{LFP(f) \mid f \in \mathcal{X}\}$ is a fixpoint of $\bigvee \mathcal{X}$. Let $a_f = LFP(f)$ and $S = \bigvee\{a_f \mid f \in \mathcal{X}\}$. We have: $(\bigvee \mathcal{X})(b) = (\bigvee \mathcal{X})(\bigvee S) = \bigvee\{f(a_f) \mid f \in \mathcal{X}\} = \bigvee\{a_f \mid f \in \mathcal{X}\} = b$. □

In what follows, we are going to introduce a setting in which the unlikely-looking assumption (4.1) of Proposition 4.1.3 will be naturally verified.

Let Var be a set, and L be a complete $\bigvee$-semilattice. Let $Val(L)$ be the set of maps from Var to L. The set $Val(L)$ inherits the structure of complete $\bigvee$-semilattice from L: indeed, for every $\mathcal{H} \subseteq Val(L)$, define $\bigvee \mathcal{H} : Var \to L$ by $x \mapsto \bigvee_L\{h(x) \mid h \in \mathcal{H}\}$. Consider an arbitrary map $\phi : Val(L) \to L$. For every $x \in Var$, the map ϕ induces a map $\Phi^x : Val(L) \to [L \to L]$ where, for every $h : Var \to L$, the map $\Phi^x_h : L \to L$ sends each $a \in L$ to $\phi(h^a_x)$, and $h^a_x : Var \to L$ is defined as follows[1]: for every $y \in Var$,

$$h^a_x(y) = \begin{cases} a & \text{if } y = x \\ h(y) & \text{otherwise.} \end{cases}$$

Fact 4.1. *For every $a \in L$, $S \subseteq L$, $h \in Val(L)$, $\mathcal{H} \subseteq Val(L)$,*

1. $h^{\bigvee S}_x = \bigvee_{Val(L)}\{h^a_x \mid a \in S\}$.
2. $(\bigvee \mathcal{H})^a_x = \bigvee\{h^a_x \mid h \in \mathcal{H}\}$.
3. $(\bigvee \mathcal{H})^{\bigvee S}_x = \bigvee\{h^a_x \mid a \in S \text{ and } h \in \mathcal{H}\}$.
4. *If* $S = \{a_h \mid h \in \mathcal{H}\}$, *then* $(\bigvee \mathcal{H})^{\bigvee S}_x = \bigvee\{h^{a_h}_x \mid h \in \mathcal{H}\}$.

Proof. Items 1 and 2 immediately follow from definitions; Item 3 follows from 1 and 2. As to 4,

$$\begin{aligned} \bigvee\{h^{a_h}_x(x) \mid h \in \mathcal{H}\} &= \bigvee\{a_h \mid h \in \mathcal{H}\} \\ &= (\bigvee \mathcal{H})^{\bigvee S}_x(x), \end{aligned}$$

[1] Sometimes we will also write e.g. $h^{(a,b,c)}_{(x,y,z)}$ in place of e.g. $((h^a_x)^b_y)^c_x$.

and for every $y \in Var \setminus \{x\}$,

$$\begin{array}{rcl} \bigvee\{h_x^{a_h}(y) \mid h \in \mathcal{H}\} & = & \bigvee\{h(y) \mid h \in \mathcal{H}\} \\ & = & (\bigvee \mathcal{H})(y) \\ & = & (\bigvee \mathcal{H})_x^{\bigvee S}(y). \end{array}$$

□

Remark 4.1. *It is interesting to contrast items 3 and 4 of the Fact above; indeed, applying item 3 to the special case in which* $S = \{a_h \mid h \in \mathcal{H}\}$ *yields* $(\bigvee \mathcal{H})_x^{\bigvee S} = \bigvee\{h_x^{a_g} \mid h, g \in \mathcal{H}\}$. *This join is in principle different from* $\bigvee\{h_x^{a_h} \mid h \in \mathcal{H}\}$, *because the former is indexed over* $(h, g) \in \mathcal{H} \times \mathcal{H}$ *and the latter is indexed over* $(h, h) \in \Delta_{\mathcal{H}} \cong \mathcal{H}$. *Therefore, it always holds that*

$$\bigvee\{h_x^{a_g} \mid h, g \in \mathcal{H}\} \geq \bigvee\{h_x^{a_h} \mid h \in \mathcal{H}\},$$

and proving the converse inequality typically requires being able to produce, for every $(h, g) \in \mathcal{H}^2$, *an index* $k \in \mathcal{H}$ *such that* $h_x^{a_g} \leq k_x^{a_k}$. *This requirement is essentially a form of the (upward) directedness which we need to assume in the statement of Proposition 4.1.2. The item 4 is then the crucial piece of information, the one which allows Proposition 4.2 below (where no assumption of directedness is made) to be proved as an instance of Proposition 4.1.3.*

Given a map $\phi : Val(L) \to L$ and $x \in Var$, every $\mathcal{H} \subseteq Val(L)$ induces a corresponding set $\mathcal{X} = \{\Phi_h^x : L \to L \mid h \in \mathcal{H}\}$, to which Proposition 4.1.3 will be applied in the proof of the following

Proposition 4.2. *Let* L *be a complete* $\bigvee$*-semilattice,* $x \in Var$, *and* $\mathcal{H} \subseteq Val(L)$; *let* $\phi : Val(L) \to L$ *be such that:*
(a) $\Phi_h^x : L \to L$ *is monotone for every* $h \in \mathcal{H}$, *and*
(b) $\Phi_{\bigvee \mathcal{H}}^x = \bigvee\{\Phi_h^x \mid h \in \mathcal{H}\}$.
Then

$$LFP(\Phi_{\bigvee \mathcal{H}}^x) = \bigvee\{LFP(\Phi_h^x) \mid h \in \mathcal{H}\}.$$

Proof. The inequality $\bigvee\{LFP(\Phi_h^x) \mid h \in \mathcal{H}\} \leq LFP(\Phi_{\bigvee \mathcal{H}}^x)$ immediately follows from Proposition 4.1.1 and assumptions (a) and (b). The converse inequality can be obtained as a direct consequence of Proposition 4.1.3, by observing that, thanks to Fact 4.1.4, the set $\mathcal{X} = \{\Phi_h^x : L \to L \mid h \in \mathcal{H}\}$ satisfies the assumption (4.1). Indeed, for every set $S = \{a_h \mid h \in \mathcal{H}\}$,

$$\begin{array}{rcll} (\bigvee \mathcal{X})(\bigvee S) & = & \Phi_{\bigvee \mathcal{H}}^x(\bigvee S) & \\ & = & \phi((\bigvee \mathcal{H})_x^{\bigvee S}) & \\ & = & \phi(\bigvee\{h_x^{a_h} \mid h \in \mathcal{H}\}) & \text{(Fact 4.1.4)} \\ & = & \bigvee\{\phi(h_x^{a_h}) \mid h \in \mathcal{H}\} & \\ & = & \bigvee\{\Phi_h^x(a_h) \mid h \in \mathcal{H}\}. & \end{array}$$

□

A sufficient condition for assumptions (a) and (b) of the above Proposition to be verified is that $\phi : Val(L) \to L$ be completely $\bigvee$-preserving, as is shown in the following

Fact 4.2. *If* $\phi : Val(L) \to L$ *is completely* $\bigvee$*-preserving, then for every* $x \in Var$,

1. $\Phi_h^x : L \to L$ *is completely* $\bigvee$*-preserving (hence monotone) for every* $h \in Val(L)$;

2. *Φ^x is completely $\bigvee$-preserving, i.e. for every $\mathcal{H} \subseteq Val(L)$,*

$$\Phi^x_{\bigvee \mathcal{H}} = \bigvee \{\Phi^x_h \mid h \in \mathcal{H}\}.$$

Proof. 1. For every $S \subseteq L$,

$$\begin{array}{rcll} \Phi^x_h(\bigvee S) & = & \phi(h_x^{\bigvee S}) & \\ & = & \phi(\bigvee\{h^a_x \mid a \in S\}) & \text{(Fact 4.1.1)} \\ & = & \bigvee\{\phi(h^a_x) \mid a \in S\} & (\phi \text{ is completely } \bigvee\text{-preserving}) \\ & = & \bigvee\{\Phi^x_h(a) \mid a \in S\}. & \end{array}$$

2. For every $a \in L$,

$$\begin{array}{rcll} \Phi^x_{\bigvee \mathcal{H}}(a) & = & \phi((\bigvee \mathcal{H})^a_x) & \\ & = & \phi(\bigvee\{h^a_x \mid h \in \mathcal{H}\}) & \text{(Fact 4.1.2)} \\ & = & \bigvee\{\phi(h^a_x) \mid h \in \mathcal{H}\} & (\phi \text{ is } \bigvee\text{-preserving}) \\ & = & \bigvee\{\Phi^x_h(a) \mid h \in \mathcal{H}\}. & \end{array}$$

□

As an immediate consequence of Proposition 4.2 and Fact 4.2 we get:

Proposition 4.3. *If L is a complete $\bigvee$-semilattice and $\phi : Val(L) \to L$ is completely $\bigvee$-preserving, then for every $\mathcal{H} \subseteq Val(L)$ and every $x \in Var$,*

$$LFP(\Phi^x_{\bigvee \mathcal{H}}) = \bigvee \{LFP(\Phi^x_h) \mid h \in \mathcal{H}\}.$$

Proof. The inequality $\bigvee\{LFP(\Phi^x_h) \mid h \in \mathcal{H}\} \leq LFP(\Phi^x_{\bigvee \mathcal{H}})$ immediately follows from Proposition 4.1.1 and Fact 4.2.2.
The converse inequality can be obtained as a direct consequence of Fact 4.2.1 and of Proposition 4.1.3, by observing that, thanks to Fact 4.1.4, the set $\mathcal{X} = \{\Phi^x_h : L \to L \mid h \in \mathcal{H}\}$ satisfies the assumption (4.1). Indeed, for every set $S = \{a_h \mid h \in \mathcal{H}\}$,

$$\begin{array}{rcll} (\bigvee \mathcal{X})(\bigvee S) & = & \Phi^x_{\bigvee \mathcal{H}}(\bigvee S) & \\ & = & \phi((\bigvee \mathcal{H})^{\bigvee S}_x) & \\ & = & \phi(\bigvee\{h^{a_h}_x \mid h \in \mathcal{H}\}) & \text{(Fact 4.1.4)} \\ & = & \bigvee\{\phi(h^{a_h}_x) \mid h \in \mathcal{H}\} & \\ & = & \bigvee\{\Phi^x_h(a_h) \mid h \in \mathcal{H}\}. & \end{array}$$

□

If L is a complete lattice and $\phi : Val(L) \to L$, then, for every $x \in Var$, let $\mu x.\phi : Val(L) \to L$ and $\nu x.\phi : Val(L) \to L$ be respectively defined by the assignments $h \mapsto LFP(\Phi^x_h)$ and $h \mapsto GFP(\Phi^x_h)$, provided these fixpoints exist for every $h \in Val(L)$. Then Propositions 4.2 and 4.3 (and their order-dual versions) immediately imply the following

Corollary 4.1. *1. Let L be a complete $\bigvee$-semilattice, $x \in Var$, and $\mathcal{H} \subseteq Val(L)$; let $\phi : Val(L) \to L$ be such that:*
(a) $\Phi^x_h : L \to L$ is monotone for every $h \in Val(L)$, and
(b) $\Phi^x_{\bigvee \mathcal{H}} = \bigvee\{\Phi^x_h \mid h \in \mathcal{H}\}$.
Then

$$(\mu x.\phi)(\bigvee \mathcal{H}) = \bigvee \{(\mu x.\phi)(h) \mid h \in \mathcal{H}\}.$$

2. *If L is a complete $\bigvee$-semilattice and $\phi : Val(L) \to L$ is completely $\bigvee$-preserving, then $\mu x.\phi : Val(L) \to L$ is completely $\bigvee$-preserving for every $x \in Var$.*

3. *Let L be a complete $\bigwedge$-semilattice, $x \in Var$, and $\mathcal{H} \subseteq Val(L)$; let $\phi : Val(L) \to L$ be such that:*
   (a) $\Phi^x_h : L \to L$ is monotone for every $h \in Val(L)$, and
   (b) $\Phi^x_{\bigwedge \mathcal{H}} = \bigwedge\{\Phi^x_h \mid h \in \mathcal{H}\}$.
   Then
$$(\nu x.\phi)(\bigwedge \mathcal{H}) = \bigwedge\{(\nu x.\phi)(h) \mid h \in \mathcal{H}\}.$$

4. *If L is a complete $\bigwedge$-semilattice and $\phi : Val(L) \to L$ is completely $\bigwedge$-preserving, then $\nu x.\phi : Val(L) \to L$ is completely $\bigwedge$-preserving for every $x \in Var$.*

Corollary 4.2. *Let L be a complete $\bigvee$-semilattice, and let $x, y \in Var$ with $x \neq y$. Let $h : Val(L) \to L$ and $\mathcal{H} = \{h^s_y \mid s \in S\}$ for some $S \subseteq L$. If $\phi : Val(L) \to L$ is such that*
(a) $\Phi^x_h : L \to L$ is monotone for every $h \in Val(L)$, and
(b) $\Phi^x_{\bigvee \mathcal{H}} = \bigvee\{\Phi^x_{h'} \mid h' \in \mathcal{H}\}$,
then
$$(\mu x.\phi)(\bigvee \mathcal{H}) = \bigvee\{(\mu x.\phi)(h^s_y) \mid s \in S\}.$$

Proof. By Corollary 4.1.1 it is enough to show that
(b) $\Phi^x_{\bigvee \mathcal{H}} = \bigvee\{\Phi^x_{h'} \mid h' \in \mathcal{H}\}$,
On the other hand, $\Phi^x_{\bigvee \mathcal{H}}(c) = \phi(\bigvee \mathcal{H}^c_x) = \phi(\bigvee\{(h^j_y)^c_x \mid j \in \mathcal{J}^\infty(a)\}) = \phi(\bigvee\{(h^c_x)^j_y \mid j \in \mathcal{J}^\infty(a)\}) = \bigvee\{\phi(h^c_x)^j_y \mid j \in \mathcal{J}^\infty(a)\}) = \bigvee\{\phi(h^j_y)^c_x \mid j \in \mathcal{J}^\infty(a)\})$ □

The theory developed so far is meant to be applied to the special setting in which L is the complex algebra of some relational structure, the maps $\phi : Val(L) \to L$ arise from formulas of a propositional language expanded with fixpoints, and their order-theoretic properties are induced by the interpretations of the logical connectives in the complex algebras (or equivalently in the relational structures). The following chapter is dedicated to presenting this setting in detail for the case study of intuitionistic modal mu-calculus.

INTUITIONISTIC MODAL MU-LANGUAGE AND ITS SEMANTICS

In this chapter, the language and algebraic interpretation of intuitionistic modal mu-calculus is introduced.

To our knowledge, the material in this chapter is entirely original.

Let $AtProp$ and $FVar$ be disjoint sets of proposition variables and of fixpoint variables (the elements of which are respectively denoted by p, q, r and by X, Y, Z). Let x, y, z denote variables in $AtProp \cup FVar$. Let us define
(a) the set $\mathcal{L}$ of modal mu-formulas over $AtProp$ and $FVar$,
(b) their signed (positive or negative) generation trees, and
(c) the set $FV(\varphi)$ of their free variables,
by simultaneous recursion, as follows: any $x \in AtProp \cup FVar$ is a modal mu-formula; its $*$-signed generation tree (for $* \in \{+, -\}$) consists of one node, labelled by $*x$, and $FV(x) = \{x\}$. If φ and ψ are modal mu-formulas, then so are $\varphi \wedge \psi, \varphi \vee \psi, \varphi \rightarrow \psi, \Box\varphi, \Diamond\varphi$; for $\odot \in \{\wedge, \vee, \Box, \Diamond\}$, their $*$-signed generation tree consists of a root node, labelled by $*\odot$, whose only child (children) is (are) the root(s) of the $*$-signed generation tree(s) of the immediate subformula(s); the $*$-signed generation tree of $\varphi \rightarrow \psi$ consists of a root node, labelled by $* \rightarrow$, whose only children are the roots of the $*^{\partial}$-signed generation tree of φ and of the $*$-signed generation tree of ψ (where $*^{\partial} = +$ if $* = -$, and $*^{\partial} = -$ if $* = +$); for $\odot \in \{\Box, \Diamond\}$, we let $FV(\odot\varphi) = FV(\varphi)$, and for $\odot \in \{\wedge, \vee \rightarrow\}$, we let $FV(\varphi \odot \psi) = FV(\varphi) \cup FV(\phi)$. If every free occurrence of X in the positive generation tree of φ is labelled positively, then $\mu X.\varphi$ and $\nu X.\varphi$ are modal mu-formulas; for $\odot \in \{\mu X, \nu X\}$, the $*$-signed generation tree of $\odot.\varphi$ consists of a root node, labelled by $*\odot$, whose only child is the root of the $*$-signed generation tree of φ; we let $FV(\odot.\varphi) = FV(\varphi) \setminus \{X\}$. An occurrence of X in φ is *bound* if $X \notin FV(\varphi)$. A *sentence* is a modal mu-formula with no free fixpoint variables. The symbol $\varphi(p_1, ..., p_n, X_1, ..., X_m)$ indicates that the proposition variables and free fixpoint variables in φ are among $p_1, ..., p_n$ and $X_1, ..., X_m$ respectively; in this symbol, the variables $p_1, ..., p_n, X_1, ..., X_m$ will be understood as pairwise distinct. For modal mu-formulas φ and ψ and $x \in AtProp \cup FVar$, the symbol $\varphi(\psi/x)$ denotes the mu-formula obtained by replacing all free occurrences of x in φ by ψ.

The non-fixpoint fragment of this language can be interpreted on several types of relational structures $\mathcal{F}$; each interpretation yields a different corresponding definition of complex algebra L. Irrespective of these differences, the complex algebras of these relational structures are always *perfect distributive lattices* (see Definition 1.5 of Section

1.1.3) endowed with additional operations interpreting the logical connectives, each of which is either a *residual* or an *adjoint* (see Subsection 1.1.4 for more details). The core of the theory presented in this thesis can (and will) be developed only on the basis of these properties, hence independently of the way the interpretation of the connectives is concretely defined. As is the case for the languages of distributive and intuitionistic modal logic [26], the special properties of perfect (distributive) lattices make it possible to define an interpretation for the following *expanded mu-language* $\mathcal{L}^+$, which is built over $AtProp \cup FVar \cup \mathsf{NOM} \cup \mathsf{CNOM}$, where the variables $\mathbf{i}, \mathbf{j} \in \mathsf{NOM}$ (called *nominals*) and $\mathbf{m}, \mathbf{n} \in \mathsf{CNOM}$ (called *conominals*) are interpreted in every complex algebra L as elements of $\mathcal{J}^\infty(L)$ and of $\mathcal{M}^\infty(L)$ respectively, additionally closing under the modal operators $\blacklozenge$ and $\blacksquare$ (respectively interpreted in every complex algebra L as the left adjoint of $\Box^L$ and as the right adjoint of $\Diamond^L$), and the subtraction operation $-$ (interpreted in every complex algebra L as the left residual of $\vee^L$). The $*$-signed generation tree and set $FV(\varphi)$ of any $\mathcal{L}^+$-formula φ are defined as in the case of $\mathcal{L}$-formulas (for instance, the $*$-signed generation tree of $\varphi - \psi$ consists of a root node, labelled by $*-$, whose only children are the roots of the $*$-signed generation tree of φ and of the $*^\partial$-signed generation tree of ψ).

Let $Var = AtProp \cup FVar$, and let L be a perfect distributive lattice arising as the complex algebra of some given relational structure $\mathcal{F}$. Every formula $\varphi \in \mathcal{L}$ induces a map $[\![\varphi]\!] : Val(L) \to L$, recursively defined as follows: for every assignment $h : Var \to L$,

$$
\begin{aligned}
[\![x]\!](h) &= h(x)\\
[\![\bot]\!](h) &= \bot^L\\
[\![\varphi \vee \psi]\!](h) &= [\![\varphi]\!](h) \vee^L [\![\psi]\!](h)\\
[\![\varphi \wedge \psi]\!](h) &= [\![\varphi]\!](h) \wedge^L [\![\psi]\!](h)\\
[\![\varphi \to \psi]\!](h) &= [\![\varphi]\!](h) \to^L [\![\psi]\!](h)\\
[\![\Diamond\varphi]\!](h) &= \Diamond^L[\![\varphi]\!](h)\\
[\![\Box\varphi]\!](h) &= \Box^L[\![\varphi]\!](h)\\
[\![\mu X.\varphi]\!](h) &= (\mu X.[\![\varphi]\!])(h)\\
[\![\nu X.\varphi]\!](h) &= (\nu X.[\![\varphi]\!])(h)
\end{aligned}
$$

where the maps $\mu X.[\![\varphi]\!]$ and $\nu X.[\![\varphi]\!]$ are defined by specializing to $[\![\varphi]\!]$ the definition given right before Corollary 4.1 for an arbitrary map $\phi : Val(L) \to L$.

Let $Var^+ = AtProp \cup FVar \cup \mathsf{NOM} \cup \mathsf{CNOM}$, and let $Val^+(L)$ be the set of $\mathcal{L}^+$-*assignments* into L, i.e., those maps $h : Var^+ \to L$ which map nominals into $\mathcal{J}^\infty(L)$ and conominals into $\mathcal{M}^\infty(L)$. Every formula $\varphi \in \mathcal{L}^+$ induces a map $[\![\varphi]\!] : Val^+(L) \to L$, recursively defined as follows: for every assignment $h : Var^+ \to L$ (here we only report the additional variable sorts and connectives),

$$
\begin{aligned}
[\![\mathbf{i}]\!](h) &= h(\mathbf{i})\\
[\![\mathbf{m}]\!](h) &= h(\mathbf{m})\\
[\![\varphi - \psi]\!](h) &= [\![\varphi]\!](h) -^L [\![\psi]\!](h)\\
[\![\blacklozenge\varphi]\!](h) &= \blacklozenge^L[\![\varphi]\!](h)\\
[\![\blacksquare\varphi]\!](h) &= \blacksquare^L[\![\varphi]\!](h).
\end{aligned}
$$

Let us stipulate that, unless specified otherwise, the variables x, y, z range in Var in the context of $\mathcal{L}$-formulas and range in Var^+ in the context of $\mathcal{L}^+$-formulas. In what follows, we find it useful to introduce the *term function* associated with any mu-formula $\varphi =$

$\varphi(x_1, \ldots, x_n)$ (either belonging to $\mathcal{L}$ or to $\mathcal{L}^+$) as the operation $[\varphi] : L^n \to L$ defined by the assignment $(a_1, \ldots, a_n) \mapsto [\![\varphi]\!](h)$ for any $h \in Val(L)$ (resp. any $h \in Val^+(L)$) such that $h(x_i) = a_i$ for $1 \leq i \leq n$. The following proposition guarantees that both definitions make sense.

Proposition 5.1. *If $\varphi(X, x_1, \ldots, x_n) \in \mathcal{L}^+$ and every free occurrence of $X \in FVar$ in the positive generation tree of φ is labelled positively, then for every $h \in Val^+(L)$, the map $\Phi_h^X : L \to L$ defined by the assignment $a \mapsto [\![\varphi]\!](h_X^a)$ is monotone. Hence, both the maps $\mu X.[\![\varphi]\!] : Val^+(L) \to L$ and $\nu X.[\![\varphi]\!] : Val^+(L) \to L$ and the term functions $[\mu X.\varphi] : L^n \to L$ and $[\nu X.\varphi] : L^n \to L$ are well defined.*

Proof. The proof follows from proposition 1.3.2 of [1]. □

For any model $M = (L, h)$ and all formulas φ, ψ, we write:

$M, w \Vdash \varphi$	if	$w \in [\![\varphi]\!](h)$;
$M \Vdash \varphi$	if	$[\![\varphi]\!](h) = \top^L$;
$M \Vdash \varphi \leq \psi$	if	$[\![\varphi]\!](h) \leq [\![\psi]\!](h)$;
$\mathcal{F} \Vdash \varphi$	if	$[\![\varphi]\!](h) = \top^L$ for any $h : Var \to L$;
$\mathcal{F} \Vdash \varphi \leq \psi$	if	$[\![\varphi]\!] \leq [\![\psi]\!]$ ordered pointwise.

A moment of reflection will convince the reader of the following

Fact 5.1. *For every mu-formula $\varphi = \varphi(x_1, \ldots, x_n) \in \mathcal{L}^+$ and every $h \in Val^+(L)$,*

$$[\![\varphi]\!](h) = [\varphi]([\![x_1]\!](h), \ldots, [\![x_n]\!](h)).$$

Hence, for all mu-formulas $\varphi(x_1, \ldots, x_n), \psi_1, \ldots, \psi_n \in \mathcal{L}^+$ and every $h \in Val^+(L)$,

$$[\![\varphi(\psi_1/x_1, \ldots, \psi_n/x_n)]\!](h) = [\varphi]([\![\psi_1]\!](h), \ldots, [\![\psi_n]\!](h)).$$

TOWARDS A CALCULUS FOR CORRESPONDENCE FOR FIXPOINTS

This chapter is aimed at introducing – and proving the soundness of – rewriting rules which make it possible to transform systems of inequalities in $\mathcal{L}^+$ into equivalent systems, hopefully to get to a (recursive) Ackermann's shape. The soundness of these rules is proven on suitably expanded perfect distributive lattices. The way these rules are given is very general, and in fact does not depend on the specific signature introduced in the previous section; therefore the same rules immediately apply to other distributive-lattice based mu-calculi. In Section 6.1 the approximation rules will be introduced and proven to be sound on suitably expanded perfect distributive lattices; in Section 6.2 residuation rules will be introduced and proven to be sound, which are unfortunately not good for our purposes: indeed, it will be illustrated that their application is guaranteed to *never* yield to clauses to which the Ackermann elimination rule is applicable. Finally, in Section 6.3, ideas for a proof strategy for adjunction and residuation rules will be illustrated, which will be generalized in the following section.

The material in this chapter is entirely original.

The symbol $\varphi(!x)$ indicates that the variable x occurs exactly once in φ. Throughout this section, we find it convenient to understand that, when writing e.g. $\varphi(X, !x)$, other variables might occur in φ as well.

6.1 Sound approximation rules for fixpoint binders

Let us consider the following rules:

$$\frac{\mathbf{i} \leq \mu X.\varphi(X, \psi/!x)}{\exists \mathbf{j}[\mathbf{i} \leq \mu X.\varphi(X, \mathbf{j}/!x) \ \&\ \mathbf{j} \leq \psi]}\ (\mu\text{-A}) \qquad \frac{\nu X.\varphi(X, \psi/!x) \leq \mathbf{m}}{\exists \mathbf{n}[\nu X.\varphi(X, \mathbf{n}/!x) \leq \mathbf{m} \ \&\ \psi \leq \mathbf{n}]}\ (\nu\text{-A})$$

where, in the rule on the left, φ is assumed to satisfy the additional assumption that its associated term function $[\varphi]$ be completely $\bigvee$-preserving in x (resp. completely $\bigwedge$-preserving in x in the rule on the right); moreover, in both rules the variable $x \in Var$ is assumed to not occur in ψ. The following proposition essentially says that (ν-A) is sound on perfect distributive lattices.

Proposition 6.1. *Let $\varphi(X, !x), \psi \in \mathcal{L}^+$ such that every occurrence of $X \in FVar$ in the positive generation tree of φ is positive, and $x \in Var$ does not occur in ψ. If the term function $[\varphi]$ is completely $\bigwedge$-preserving in the coordinate determined by x, then the following are equivalent on every perfect distributive lattice L and for every $h \in Val^+(L)$:*

1. $[\![\nu X.\varphi(X, \mathbf{n}/!x)]\!] \leq [\![\mathbf{m}]\!](h)$ *and* $[\![\psi]\!](h) \leq [\![\mathbf{n}]\!](h)$ *for some* $\mathbf{n} \in \mathsf{CNOM} \setminus \{\mathbf{m}\}$*, not occurring in φ, nor in ψ;*
2. $[\![\nu X.\varphi(X, \psi/!x)]\!](h) \leq [\![\mathbf{m}]\!](h)$.

Proof. (1. $\Rightarrow$ 2.) Assume that $[\![\nu X.\varphi(X, \mathbf{n}/!x)]\!] \leq [\![\mathbf{m}]\!](h)$ and $[\![\psi]\!](h) \leq [\![\mathbf{n}]\!](h)$ for some $\mathbf{n} \in \mathsf{CNOM}$. moreover, since $[\varphi(!x)]$ is order preserving in x, and $x \neq X$, we have that $[\nu X.\varphi(!x)]$ is also order preserving; hence we have:

$$\begin{array}{rcll} [\![\nu X.\varphi(\psi/!x)]\!](h) & = & [\nu X.\varphi]([\![\psi]\!](h)) & \text{(Fact 5.1)} \\ & \leq & [\nu X.\varphi]([\![\mathbf{n}]\!](h)) & \text{($[\nu X.\varphi]$ monot. in x.)} \\ & = & [\![\nu X.\varphi(\mathbf{n}/!x)]\!](h) & \text{(Fact 5.1)} \\ & \leq & [\![\mathbf{m}]\!](h). & \end{array}$$

(2. $\Rightarrow$ 1.) Assume that $[\![\nu X.\varphi(X, \psi/!x)]\!](h) \leq [\![\mathbf{m}]\!](h)$; since x does not occur in ψ, we can assume w.l.o.g. that $h(x) = [\![\psi]\!](h)$, so that our assumption becomes:

$$[\![\nu X.\varphi]\!](h) \leq [\![\mathbf{m}]\!](h). \tag{6.1}$$

Let $\mathcal{H} = \{h^n_x \mid n \in \mathcal{M}^\infty(h(x))\}$. Since L is perfect, $h(x) = \bigwedge \mathcal{M}^\infty(h(x))$, which implies that $\bigwedge \mathcal{H} = h$. Let us show that the map $[\![\varphi]\!]$ and the set $\mathcal{H}$ verify assumption (b) in Corollary 4.1.3; indeed, for any $h \in Var^+(L)$, let $\Phi^X_h : L \to L$ be defined by the assignment $a \mapsto [\![\varphi]\!](h^a_X)$, and let us show that:

$$\Phi^X_{\bigwedge \mathcal{H}} = \bigwedge \{\Phi^X_k \mid k \in \mathcal{H}\}.$$

Indeed, for every $b \in L$,

$$\begin{array}{cll} & \Phi^X_{\bigwedge \mathcal{H}}(b) & \\ = & [\![\varphi]\!]((\bigwedge \mathcal{H})^b_X) & \\ = & [\varphi](b, h(x)) & (\bigwedge \mathcal{H} = h \text{ and } x \neq X) \\ = & [\varphi](b, \bigwedge \mathcal{M}^\infty(h(x))) & (h(x) = \bigwedge \mathcal{M}^\infty(h(x))) \\ = & \bigwedge\{[\varphi](b, n) \mid n \in \mathcal{M}^\infty(h(x))\} & \text{($[\varphi]$ is c. $\bigwedge$-pr. in x.)} \\ = & \bigwedge\{[\![\varphi]\!]((h^b_X)^n_x \mid n \in \mathcal{M}^\infty(h(x))\} & \\ = & \bigwedge\{[\![\varphi]\!]((h^n_x)^b_X \mid n \in \mathcal{M}^\infty(h(x))\} & \\ = & \bigwedge\{\Phi^X(k)(b) \mid k \in \mathcal{H}\}. & \end{array}$$

The map $[\![\varphi]\!]$ verifies also assumption (a) of the same corollary, because of Proposition 5.1. Hence,

$$(\nu X[\![\varphi]\!])(h) = \bigwedge\{(\nu X[\![\varphi]\!])(h^n_x) \mid n \in \mathcal{M}^\infty(h(x))\},$$

which can be rewritten as

$$[\![\nu X.\varphi]\!](h) = \bigwedge\{[\![\nu X.\varphi]\!](h^n_x) \mid n \in \mathcal{M}^\infty(h(x))\}.$$

Hence our assumption (6.1) can be equivalently rewritten as

$$\bigwedge\{[\![\nu X.\varphi]\!](h^n_x) \mid n \in \mathcal{M}^\infty(h(x))\} \leq [\![\mathbf{m}]\!](h).$$

Since $[\![\mathbf{m}]\!](h) = h(\mathbf{m}) \in \mathcal{M}^\infty(L)$, this implies that $[\![\nu X.\varphi]\!](h^n_x) \leq [\![\mathbf{m}]\!](h)$ for some $n \in \mathcal{M}^\infty(h(x))$. Let $\mathbf{n} \in \mathsf{CNOM}$ be a fresh variable. We can assume w.l.o.g. that $h(\mathbf{n}) = n \geq h(x) = [\![\psi]\!](h)$. Hence we have $[\![\psi]\!](h) \leq [\![\mathbf{n}]\!](h)$, and $[\![\nu X.\varphi(X, \mathbf{n}/!x)]\!](h) = [\![\nu X.\varphi]\!](h^n_x) \leq [\![\mathbf{m}]\!](h)$, which finishes the proof. □

The following proposition can be proven similarly to the proposition above using Corollary 4.1.1, and takes care of the soundness of the rule (μ-A):

Proposition 6.2. *Let $\varphi(X, !x), \psi \in \mathcal{L}^+$ such that every occurrence of $X \in FVar$ in the positive generation tree of φ is positive, and $x \in Var$ does not occur in ψ. If the term function $[\varphi]$ is completely $\bigvee$-preserving in the coordinate determined by x, then the following are equivalent on every perfect distributive lattice L and for every $h \in Val^+(L)$:*

1. $[\![\mathbf{i}]\!](h) \leq [\![\mu X.\varphi(X, \mathbf{j}/!x)]\!]$ *and* $[\![\mathbf{j}]\!](h) \leq [\![\psi]\!](h)$ *for some* $\mathbf{j} \in \mathsf{NOM}\backslash\{\mathbf{i}\}$, *not occurring in φ, nor in ψ;*
2. $[\![\mathbf{i}]\!](h) \leq [\![\mu X.\varphi(X, \psi/!x)]\!](h)$.

Example 6.1. ***(Application of (ν-A))*** *The inequality $p \leq \nu X[\Box(X \wedge (q \to \bot)) \vee (\Diamond p \wedge \Diamond q)]$ can be reduced as follows:*

$$
\begin{array}{lll}
 & & \forall p \forall q[p \leq \nu X[\Box(X \wedge (q \to \bot)) \vee (\Diamond p \wedge \Diamond q)]] \\
 & \textit{iff} & \forall p \forall q \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq p \ \& \ \nu X[\Box(X \wedge (q \to \bot)) \vee (\Diamond p \wedge \Diamond q)] \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
 & \textit{iff} & \forall q \forall \mathbf{i} \forall \mathbf{m}[\nu X[\Box(X \wedge (q \to \bot)) \vee (\Diamond \mathbf{i} \wedge \Diamond q)] \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
(\nu\text{-}A) & \textit{iff} & \forall q \forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{n}[(\Diamond \mathbf{i} \wedge \Diamond q \leq \mathbf{n} \ \& \ \nu X[\Box(X \wedge (q \to \bot)) \vee \mathbf{n}] \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
 & \textit{iff} & \forall q \forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{n}[(q \leq \blacksquare(\Diamond \mathbf{i} \to \mathbf{n}) \ \& \ \nu X[\Box(X \wedge (q \to \bot)) \vee \mathbf{n}] \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
 & \textit{iff} & \forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{n}[\nu X[\Box(X \wedge (\blacksquare(\Diamond \mathbf{i} \to \mathbf{n}) \to \bot)) \vee \mathbf{n}] \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
 & \textit{iff} & \forall \mathbf{i} \forall \mathbf{n}[\mathbf{i} \leq \nu X[\Box(X \wedge (\blacksquare(\Diamond \mathbf{i} \to \mathbf{n}) \to \bot)) \vee \mathbf{n}].
\end{array}
$$

All the steps in the above chain of equivalences but the one marked with (ν-A) can be justified by the theory of algorithmic correspondence for distributive and intuitionistic modal logic [26]. The marked equivalence is an application of the approximation rule (ν-A), which is soundly applied, since all the assumptions of Proposition 6.1 are verified by $\varphi(X, !x) = \Box(X \wedge (q \to \bot)) \vee x$ and $\psi = \Diamond \mathbf{i} \wedge \Diamond q$.

6.2 Sound but useless residuation rules for fixpoint binders

$$
\frac{\mu X.\varphi(X, \xi/!x) \leq \chi}{\xi \leq \psi(\mu X.\varphi(X, \xi/!x)/X, \chi/!y)}\ (\mu\text{-Res}) \qquad \frac{\chi \leq \nu X.\psi(X, \xi/!y)}{\varphi(\nu X.\psi(X, \xi/!y)/X, \chi/!x) \leq \xi]}\ (\nu\text{-Res})
$$

where, in both rules, $\varphi(X, !x)$ and $\psi(X, !y)$ are assumed to satisfy the additional assumption that their associated term functions $[\varphi]$ and $[\psi]$ form a residuated pair in x and y: i.e. for every $h \in Val^+(L)$,

$$
[\![\varphi(X, !x)]\!](h) \leq [\![y]\!](h) \ \text{ iff } \ [\![x]\!](h) \leq [\![\psi(X, !y)]\!](h). \tag{6.2}
$$

Moreover, in both rules the variable $x \in Var$ is assumed to not occur in ψ, ξ or in χ, and the variable $y \in Var$ is assumed to not occur in φ, ξ or in χ. The following proposition essentially says that (μ-R) is sound on perfect distributive lattices.

Proposition 6.3. *Let $\varphi(X,!x), \psi(X,!y), \xi, \chi \in \mathcal{L}^+$ such that $X \in FVar$ does not occur free in ξ or χ, every free occurrence of X in the positive generation tree of φ is positive, $x \in Var$ does not occur in ψ, ξ or in χ, and $y \in Var$ does not occur in φ, ξ or in χ. Let L be a perfect distributive lattice, and assume that clause (6.2) holds for every $h \in Val^+(L)$. Then the following are equivalent:*

1. $[\![\mu X.\varphi(X, \xi/!x)]\!](h) \leq [\![\chi]\!](h)$;
2. $[\![\xi]\!](h) \leq [\![\psi(\mu X.\varphi(X, \xi/!x)/X, \chi/!y)]\!](h)$.

Proof. Assume that $[\![\mu X.\varphi(X, \xi/!x)]\!](h) \leq [\![\chi]\!](h)$. By definition, $[\![\mu X.\varphi(X, \xi/!x)]\!](h) = LFP(\Phi^X_h)$, where $\Phi^X_h : L \to L$ is defined by the assignment $a \mapsto [\![\varphi(X, \xi/!x)]\!](h^a_X)$. Let $b = LFP(\Phi^X_h)$; the following chain of identities holds:

$$\begin{array}{lll}[\![\mu X.\varphi(X, \xi/!x)]\!](h) & = LFP(\Phi^X_h) & \text{(definition of } [\![\cdot]\!])\\ & = \Phi^X_h(b) & (b = LFP(\Phi^X_h) \text{ is fixpt of } \Phi^X_h)\\ & = [\![\varphi(X, \xi/!x)]\!](h^b_X). & \text{(def. of } \Phi^X_h)\end{array}$$

Hence, the assumed inequality can be equivalently rewritten as $[\![\varphi(X, \xi/!x)]\!](h^b_X) \leq [\![\chi]\!](h)$. Let $c = [\![\xi]\!](h)$ and $d = [\![\chi]\!](h)$. By clause (6.2) applied to $h^{(b,c,d)}_{(X,x,y)} \in Val^+(L)$, this is equivalent to $[\![\xi]\!](h) \leq [\![\psi(\mu X.\varphi(X, \xi/!x)/X, \chi/!y)]\!](h)$. □

The following proposition can be proven similarly to the proposition above, and takes care of the soundness of the rule (ν-Res):

Proposition 6.4. *Let $\varphi(X,!x), \psi(X,!y), \xi, \chi \in \mathcal{L}^+$ such that $X \in FVar$ does not occur free in ξ or χ, every free occurrence of X in the positive generation tree of ψ is positive, $x \in Var$ does not occur in ψ, ξ or in χ, and $y \in Var$ does not occur in φ, ξ or in χ. Let L be a perfect distributive lattice, and assume that clause (6.2) holds for every $h \in Val^+(L)$. Then the following are equivalent:*

1. $[\![\chi]\!](h) \leq [\![\nu X.\psi(X, \xi/!y)]\!](h)$;
2. $[\![\varphi(\nu X.\psi(X, \xi/!y)/X, \chi/!x)]\!](h) \leq [\![\xi]\!](h)$.

Example 6.2. ***(Application of (μ-Res))*** Let $\neg q := q \to \bot$; consider the inequality $p \leq \nu X.[\Box(X \wedge \neg q) \vee \mu Y.[(\Diamond Y \wedge \Diamond p) \wedge \Diamond q]]$.

$$\begin{array}{ll} & \forall p \forall q[p \leq \nu X.[\Box(X \wedge \neg q) \vee \mu Y.[(\Diamond Y \wedge \Diamond p) \wedge \Diamond q]]]\\ \textit{iff} & \forall p \forall q \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq p \ \&\ \nu X[\Box(X \wedge \neg q) \vee \mu Y.[(\Diamond Y \wedge \Diamond p) \wedge \Diamond q]] \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]\\ \textit{iff} & \forall q \forall \mathbf{i} \forall \mathbf{m}[\nu X[\Box(X \wedge \neg q) \vee \mu Y.[(\Diamond Y \wedge \Diamond \mathbf{i}) \wedge \Diamond q]] \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}]\\ \textit{iff} & \forall q \forall \mathbf{i}[\mathbf{i} \leq \nu X[\Box(X \wedge \neg q) \vee \mu Y.[(\Diamond Y \wedge \Diamond \mathbf{i}) \wedge \Diamond q]]]\\ \textit{iff} & \forall q \forall \mathbf{i} \forall \mathbf{n}[\mu Y.[(\Diamond Y \wedge \Diamond \mathbf{i}) \wedge \Diamond q] \leq \mathbf{n} \ \&\ \nu X[\Box(X \wedge \neg q) \vee \mathbf{n}] \leq \mathbf{m}]\\ \textit{iff} & \forall q \forall \mathbf{i} \forall \mathbf{n}[q \leq \blacksquare((\Diamond \mu Y.[(\Diamond Y \wedge \Diamond \mathbf{i}) \wedge \Diamond q] \wedge \Diamond \mathbf{i}) \to \mathbf{n}) \ \&\ \nu X[\Box(X \wedge \neg q) \vee \mathbf{n}] \leq \mathbf{m}]\end{array}$$

The last 'iff' is due to an application of (μ-Res), where $\varphi(Y, x) := (\Diamond Y \wedge \Diamond \mathbf{i}) \wedge \Diamond x$, $\xi := q, \chi := \mathbf{n}$, and $\psi(Y, y) := \blacksquare((\Diamond Y \wedge \Diamond \mathbf{i}) \to y)$.

Notice that this last clause is *not* in recursive Ackermann's shape: indeed, the first inequality is of the form $q \leq \alpha(q)$ where $\alpha(q) := \blacksquare((\Diamond \mu Y.[(\Diamond Y \wedge \Diamond \mathbf{i}) \wedge \Diamond q] \wedge \Diamond \mathbf{i}) \to \mathbf{n}))$ is *negative* in q. This is a general problem of the -Res rules: indeed, if $\varphi(X, x)$ is positive in X and $\psi(X, y)$ is its right residual in x, y, then φ is positive in x (and hence so is $\mu X.\varphi$), and ψ is negative in X, hence $\alpha(x) := \psi(\mu X.\varphi(X, x)/X, y)$ is *negative* in x. The fact that the application of -Res rules does not lead to an Ackermann shape makes them unfortunately useless for our purposes.

6.3 A case study

Consider the inequality $\nu X.\Box(p \wedge X) \leq p$ which we have already reduced in Chapter 2. This inequality can also be alternatively reduced as follows:

$$\begin{array}{lll} & & \forall p[\nu X.\Box(p \wedge X) \leq p] \\ & \text{iff} & \forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \nu X.\Box(p \wedge X) \ \&\ p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\ (*) & \text{iff} & \forall p \forall \mathbf{i} \forall \mathbf{m}[(\mu X.\blacklozenge(X \vee \mathbf{i}) \leq p \ \&\ p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\ & \text{iff} & \forall \mathbf{i} \forall \mathbf{m}[\mu X.\blacklozenge(X \vee \mathbf{i}) \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] \\ & \text{iff} & \forall \mathbf{i}[\mathbf{i} \leq \mu X.\blacklozenge(X \vee \mathbf{i})]. \end{array}$$

The equivalence marked with $(*)$ is justified by the following steps (where $e_{\mathbf{i}}(*) = \blacklozenge(* \vee \mathbf{i})$):

$$\begin{array}{rcll} \mu X.\blacklozenge(X \vee \mathbf{i}) & = & \bigvee_{\kappa \geq 1} e_{\mathbf{i}}^{\kappa}(\bot) & \text{the justification of this equality comes from [28, section 3.1]} \\ & = & \bigvee_{\kappa \geq 1} \blacklozenge^{\kappa}\mathbf{i} & \text{(Claim 1)} \\ & & & \\ \bigvee_{\kappa \geq 1} \blacklozenge^{\kappa} & \dashv & \bigwedge_{\kappa \geq 1} \Box^{\kappa} & \text{(Claim 2)} \\ & & & \\ \nu X.\Box(X \wedge p) & = & \bigwedge_{\kappa \geq 1} h_p^{\kappa}(\top) & (\text{where } h_p(*) = \Box(* \wedge p)) \\ & = & \bigwedge_{\kappa \geq 1} \Box^{\kappa} p & \text{(Claim 3)} \end{array}$$

The proof of Claim 1 is analogous (in fact dual) to the proof of Claim 3, which immediately follows from the fact that for every κ,

$$h_p^{\kappa}(\top) = \bigwedge_{1 \leq i \leq \kappa} \Box^i p;$$

this can be shown by transfinite induction on κ. For instance, if κ is a limit ordinal, then

$$\begin{array}{rcll} h_p^{\kappa}(\top) & = & \bigwedge_{\lambda \leq \kappa} h_p^{\lambda}\top & (\text{def of } h_p^{\kappa} \text{ for limit ordinals}) \\ & = & \bigwedge_{\lambda \leq \kappa} \bigwedge_{1 \leq i \leq \lambda} \Box^i p & \text{(induction hypothesis)} \\ & = & \bigwedge_{1 \leq i \leq \kappa} \Box^i p. & \end{array}$$

Claim 2 is an instance of Fact 7.1.

Notice that, because of Claim 1, the correspondent $\forall \mathbf{i}[\mathbf{i} \leq \mu X.\blacklozenge(X \vee \mathbf{i})]$ can be rewritten as $\forall \mathbf{i}[\mathbf{i} \leq \bigvee_{\kappa \geq 1} \blacklozenge^{\kappa}\mathbf{i}]$, which immediately translates on Kripke frames into the condition expressing the reflexivity of the transitive closure of the relation interpreting $\Box$.

This case study suggests some ideas for a general strategy; firstly, the fixpoint formula $\nu X.\Box(p \wedge X)$, was unfolded and rewritten as a transfinite meet of terms (or term-functions), each of which is an adjoint; this guarantees that the rewritten unfolding itself gives rise to an adjoint function, which is given by the transfinite join of the adjoints of the members of the rewritten unfolding (see Fact 7.1). This is already excellent news, because this transfinite join is certainly expressible in the target FO+FP correspondence language. But there is even more! Indeed, the transfinite join of adjoints $\bigvee_{\kappa \geq 1} \blacklozenge^{\kappa}$ is itself a *finite term* (function) in the expanded language, which makes it possible to express the reduction using only *finite terms* of the expanded language. In the following subsections, this proof strategy will be developed for a hierarchy of more and more general types of formulas/inequalities, highlighting the cases in which finite reduction rules can be soundly defined.

DISPLAY METHODS FOR FIXPOINT BINDERS

This chapter is aimed at introducing – and proving the soundness of – methods which make it possible to equivalently rewrite inequalities of the form $\varphi \leq \psi(p)$ or $\varphi(p) \leq \psi$, where the proposition variable p occurs within the scope of some fixpoint binder(s), respectively as $\alpha \leq p$ or $p \leq \alpha$. As mentioned early on in Chapter 2, we refer to this rewriting as *displaying the variable p*; clearly, this step is crucial to be able to reach the Ackermann shape and then eliminate p. As mentioned early on, we will proceed analytically and introduce – whenever notationally convenient – rules applicable to progressively more general types of formulas/inequalities. In particular, first, the case of non-nested fixpoint binders will be treated in the following Section 7.1; then in Section 7.2, we will make use of the insights gained in the following subsection to treat the case of nested fixpoint binders. Before getting started, we need to warn the reader: while, up to the previous section, we were very explicit in keeping formulas, their associated extension maps and term functions notationally distinct (which was needed to be able to first justify and then apply Corollary 4.1), throughout the present section these distinctions are not needed anymore, and so, in particular, formulas and their associated term functions will be systematically identified.

The material in this chapter is entirely original.

7.1 Non-nested occurrences of fixpoint binders

7.1.1 The pure-adjunction case

Let us consider the following rules:

$$\frac{\mu X.(A(X) \vee B(p)) \leq \chi}{p \leq \nu X.(E(X) \wedge D(\chi/p))}\ (\mu\text{-Adj}) \qquad \frac{\chi \leq \nu X.(E(X) \wedge D(p))}{\mu X.(A(X) \vee B(\chi/p)) \leq p}\ (\nu\text{-Adj})$$

where, in each rule, $A(X) = \bigvee_{i\in I} \delta_i(X)$, $B(p) = \bigvee_{j\in J} \delta'_j(p)$, $E(X) = \bigwedge_{i\in I} \beta_i(X)$ and $D(p) = \bigwedge_{j\in J} \beta'_j(p)$ I and J are finite sets of indexes, each δ_i and δ'_j is a unary left adjoint (relativized to our signature, this means that δ_i and δ'_j are concatenations of $\Diamond$), and each β_i and β'_j is a unary right adjoint (relativized to our signature, this means that β_i and β'_j are concatenations of $\Box$). Finally, $\delta_i \dashv \beta_i$ and $\delta'_j \dashv \beta'_j$ for each i and j.

These rules cover the case study of the preceding subsection. The remainder of this subsection is aimed at showing the soundness of these rules. The following lemma holds more generally than under the assumptions above; in particular, notice that A and B do not need to be term functions (this will become important in Subsection 7.2).

Lemma 7.1. *Let $A(X) = A(X, \overrightarrow{z})$ be completely join preserving in X and let B be an arbitrary function which does not depend on X. Then,*

$$\mu X.(A(X) \vee B) = \bigvee_{\kappa \in Ord} A^{\kappa}(B).$$

Proof. Let $F(*) = A(*) \vee B$. It is well known that $\mu X.(A(X) \vee B) = \bigvee_{\kappa \geq 1} F^{\kappa}(\bot)$ (see [28, section 3.1]); then the equality is an immediate consequence of the following claim: for every ordinal κ,

$$F^{\kappa+1}(\bot) = \bigvee_{0 \leq i \leq \kappa} A^{i}(B). \tag{7.1}$$

By induction on κ. If $k = 0$, then $C_p(\bot) = A(\bot) \vee B = \bot \vee B = A^0(B)$ ($A(*)$ is c. join-pres., hence $A(\bot) = \bot$).

If κ is a successor ordinal, then

$$\begin{array}{rcll} F(F^{\kappa}(\bot)) & = & A(F^{\kappa}(\bot)) \vee B & \\ & = & A(\bigvee_{0 \leq i \leq \kappa - 1} A^{i}(B)) \vee B & \text{(induction hypothesis)} \\ & = & \bigvee_{1 \leq i \leq \kappa} A^{i}(B) \vee B & \text{(A is compl. join-pres.)} \\ & = & \bigvee_{0 \leq i \leq \kappa} A^{i}(B). & (B = A^0(B)) \end{array}$$

If κ is a limit ordinal, then

$$\begin{array}{rcll} F(F^{\kappa}(\bot)) & = & F(\bigvee_{\lambda \leq \kappa} F^{\lambda}(\bot)) & \\ & = & \bigvee_{\lambda \leq \kappa} F(F^{\lambda}(\bot)) & \text{(F is compl. join-pres.)} \\ & = & \bigvee_{\lambda \leq \kappa} F^{1+\lambda}(\bot) & \\ & = & \bigvee_{\lambda \leq \kappa} F^{\lambda}(\bot) & \text{(κ is limit ordinal)} \\ & = & \bigvee_{\lambda \leq \kappa} \bigvee_{0 \leq i \leq \lambda} A^{i}(B) & \text{(induction hypothesis)} \\ & = & \bigvee_{0 \leq i \leq \kappa} A^{i}(B). & \end{array}$$

□

By a straightforward dualization of the proof above we obtain the following

Lemma 7.2. *Let $E(X) = E(X, \overrightarrow{z})$ be completely meet preserving in X and let D be an arbitrary function which does not depend on X. Then,*

$$\nu X.(E(X) \wedge D) = \bigwedge_{\kappa \in Ord} E^{\kappa}(D).$$

Thanks to the very special shape of A and B in the rule (μ-Adj) above, the commutativity of their composition can be proved as a byproduct of associativity, which makes it possible for the following to hold:

Lemma 7.3. *If $A(X) = \bigvee_{i \in I} \delta_i(X)$ and $B(p) = \bigvee_{j \in J} \delta'_j(p)$, such that I and J are finite sets of indexes, and each δ_i and δ'_j is a unary left adjoint (relativized to our signature, this means that δ_i and δ'_j are concatenations of $\Diamond$). Then for every κ,*

$$A^{\kappa}(B(p)) = B(A^{\kappa}(p)).$$

Proof. by induction on κ. If $\kappa = 0$ it is immediate. If $\kappa = 1$, recall that δ_i and δ'_j are concatenations of $\Diamond$. In what follows, we will abuse notation and use the indexes i and j as standing for the number of concatenated diamonds that δ_i and δ'_j respectively consist of. With this stipulation, we have for every q:

$$
\begin{array}{rcll}
A(B(q)) & = & \bigvee_i \delta_i(\bigvee_j \delta'_j(q)) & \\
& = & \bigvee_i \bigvee_j \delta_i(\delta'_j(q)) & (\delta_i \text{ is c. j. pres.}) \\
& = & \bigvee_i \bigvee_j \delta_{i+j}(q)) & \text{(associativity of composition)} \\
& = & \bigvee_j \bigvee_i \delta_{j+i}(q)) & \\
& = & \bigvee_j \bigvee_i \delta'_j(\delta_i(q)) & \text{(associativity of composition)} \\
& = & \bigvee_j \delta'_j(\bigvee_i \delta_i(q)) & (\delta'_j \text{ is c. j. pres.}) \\
& = & B(A(q)). &
\end{array}
$$

Hence for successor ordinals, applying the induction hypothesis to A^κ and $B'(p) = A(B(p))$ and then the case $\kappa = 1$ to $q = A^\kappa(p)$, we get: $A^{\kappa+1}(B(p)) = A^\kappa(B'(p)) = B'(A^\kappa(p))) = A(B(A^\kappa(p))) = B(A(A^\kappa(p))) = B(A^{\kappa+1}(p))$.

For limit ordinals,

$$
\begin{array}{rcll}
A^\kappa(B(p)) & = & \bigvee_{\lambda\leq\kappa} A^\lambda(B(p)) & \\
& = & \bigvee_{\lambda\leq\kappa} B(A^\lambda(p)) & \text{(induction hypothesis)} \\
& = & B(\bigvee_{\lambda\leq\kappa} A^\lambda(p)) & (B \text{ is compl. join-pres.}) \\
& = & B(A^\kappa(p)). &
\end{array}
$$

□

By a straightforward dualization of the proof above we obtain the following

Lemma 7.4. *If $E(X) = \bigwedge_{i\in I} \beta_i(X)$ and $D(p) = \bigwedge_{j\in J} \beta'_j(p)$, such that I and J are finite sets of indexes, and each β_i and β'_j is a unary right adjoint (relativized to our signature, this means that β_i and β'_j are concatenations of $\Box$). Then for every κ,*

$$E^\kappa(D(p)) = D(E^\kappa(p)).$$

Fact 7.1. *If $f, g : L \to L$ and $f \dashv g$, then $f^\kappa \dashv g^\kappa$ for every κ, where for limit ordinals, $f^\kappa := \bigvee_{\lambda\leq\kappa} f^\lambda$, and $g^\kappa := \bigwedge_{\lambda\leq\kappa} g^\lambda$.*

Proof. By transfinite induction on κ. If κ is a successor, the statement follows immediately from the induction hypothesis and the adjoint of a composition being the composition of the adjoints. If κ is a limit ordinal, then we have $f^\kappa := \bigvee_{\lambda\leq\kappa} f^\lambda \dashv \bigwedge_{\lambda\leq\kappa} g^\lambda = g^\kappa$. □

Corollary 7.1. *Let $A(X) = \bigvee_{i\in I} \delta_i(X)$, $B(x) = \bigvee_{j\in J} \delta'_j(x)$, $E(X) = \bigwedge_{i\in I} \beta_i(X)$ and $D(y) = \bigwedge_{j\in J} \beta'_j(y)$, such that I and J are finite sets of indexes, and $\delta_i \dashv \beta_i$ and $\delta'_j \dashv \beta'_j$ for each i and j. Then, for every perfect distributive lattice L, every $h \in val^+(L)$, and $x, y \in Var^+$*

$$[\![\mu X.(A(X) \vee B(x))]\!](h) \leq [\![y]\!](h) \text{ iff } [\![x]\!](h) \leq [\![\nu X.(E(X) \wedge D(y))]\!](h).$$

Proof. By Lemma 7.1 and 7.2, it is enough to show that

$$\bigvee_{\kappa\in Ord} A^\kappa(B(*)) \dashv \bigwedge_{\kappa\in Ord} E^\kappa(D(*)).$$

For this, it is enough to show that $A^\kappa(B(*)) \dashv E^\kappa(D(*))$ for every κ. From the assumptions it readily follows that $A \dashv E$ and $B \dashv D$; hence By Fact 7.1, $A^\kappa \dashv E^\kappa$ for every κ, hence $A^\kappa B \dashv DE^\kappa = E^\kappa D$, the latter identity holding because of Lemma 7.4. □

Example 7.1. ***(Application of (ν-Adj))*** Consider the inequality $\nu X.\Box(p \wedge \Box X) \leq p$.

$$
\begin{array}{lll}
 & & \forall p[\nu X.\Box(p \wedge \Box X) \leq p] \\
 & \textit{iff} & \forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \nu X.\Box(p \wedge \Box X) \ \&\ p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
(*) & \textit{iff} & \forall p \forall \mathbf{i} \forall \mathbf{m}[\mu X.\blacklozenge(\blacklozenge X \vee \mathbf{i}) \leq p \ \&\ p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
 & \textit{iff} & \forall \mathbf{i} \forall \mathbf{m}[\mu X.\blacklozenge(\blacklozenge X \vee \mathbf{i}) \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
 & \textit{iff} & \forall \mathbf{i}[\mathbf{i} \leq \mu X.\blacklozenge(\blacklozenge X \vee \mathbf{i})].
\end{array}
$$

The equivalence marked with $(*)$ is an application of (ν-Adj), modulo distributing modal connectives.

7.1.2 Adding residuals

Let us consider the following rules:

$$
\frac{\mu X.(A'(X,\overrightarrow{z}) \vee B'(p,\overrightarrow{z})) \leq \chi}{p \leq \bigwedge_{\kappa \in Ord} D'(E'^{\kappa}(\chi/y,\overrightarrow{z}))}\ (\mu\text{-R}) \qquad \frac{\chi \leq \nu X.(A'(X,\overrightarrow{z}) \wedge B'(p,\overrightarrow{z}))}{\bigvee_{\kappa \in Ord} D'(E'^{\kappa}(\chi/y,\overrightarrow{z})) \leq p}\ (\nu\text{-R})
$$

where, in the rule on the left,

$$
\begin{array}{lllllll}
A(X) & = & \bigvee_{i \in I} \delta_i(X), & A'(X,\overrightarrow{z}) & = & \varphi(A(X)/!x,\overrightarrow{z}), \\
B(p) & = & \bigvee_{j \in J} \delta'_j(p), & B'(p,\overrightarrow{z}) & = & \varphi(B(p)/!x,\overrightarrow{z}), \\
E(x) & = & \bigwedge_{i \in I} \beta_i(x), & E'(y,\overrightarrow{z}) & = & E(\psi(y,\overrightarrow{z})/x), \\
D(x) & = & \bigwedge_{j \in J} \beta'_j(x), & D'(u,\overrightarrow{z}) & = & D(\psi(u,\overrightarrow{z})/x);
\end{array}
$$

(notice the inversion from A' and B' to E' and D'); moreover, I and J are finite sets of indexes, each δ_i and δ'_j is a unary left adjoint (relativized to our signature, this means that δ_i and δ'_j are concatenations of $\Diamond$), and each β_i and β'_j is a unary right adjoint. Finally, $\varphi(x,\overrightarrow{z}) \dashv \psi(y,\overrightarrow{z})$ in x and y, $\delta_i \dashv \beta_i$ and $\delta'_j \dashv \beta'_j$ for each i and j.

where, in the rule on the right,

$$
\begin{array}{lllllll}
A(X) & = & \bigwedge_{i \in I} \delta_i(X), & A'(X,\overrightarrow{z}) & = & \varphi(A(X)/!x,\overrightarrow{z}), \\
B(p) & = & \bigwedge_{j \in J} \delta'_j(p), & B'(p,\overrightarrow{z}) & = & \varphi(B(p)/!x,\overrightarrow{z}), \\
E(x) & = & \bigvee_{i \in I} \beta_i(x), & E'(y,\overrightarrow{z}) & = & E(\psi(y,\overrightarrow{z})/x), \\
D(x) & = & \bigvee_{j \in J} \beta'_j(x), & D'(u,\overrightarrow{z}) & = & D(\psi(u,\overrightarrow{z})/x);
\end{array}
$$

Here, I and J are finite sets of indexes, each δ_i and δ'_j is a unary right adjoint (relativized to our signature, this means that δ_i and δ'_j are concatenations of $\Box$), and each β_i and β'_j is a unary left adjoint. Finally, the delta's are the right adjoints of the beta's i.e., $\beta_i \dashv \delta_i$ and $\beta'_j \dashv \delta'_j$ for all i and j; and the $\varphi(x,\overrightarrow{z})$ is the right residual of the $\psi(y,\overrightarrow{z})$ in x and y.

The soundness of (μ-R) is a straightforward consequence of Lemma 7.1 and of the fact that for every $\overrightarrow{z}$,

$$
\bigvee_{\kappa \in Ord} A'^{\kappa}(B'(x,\overrightarrow{z})/X,\overrightarrow{z}) \dashv \bigwedge_{\kappa \in Ord} D'(E'^{\kappa}(y,\overrightarrow{z})/u,\overrightarrow{z})
$$

in x and y, which in its turn follows by the assumptions and Fact 7.1.

A straightforward dualization of this argument gives the soundness of (ν-R).

The counterparts of Lemmas 7.3 and 7.4 for the present set of assumptions do not hold. This implies that the strategy carried out in the preceding subsection cannot be implemented to equivalently reduce the adjoint function defined as a transfinite join (or meet) to some term function in $\mathcal{L}^+$. Whether this can be done using another strategy is left as an open problem. However, in the next subsection it is shown how this is still possible for a restricted subclass of formulas.

7.1.3 Finite residuation rule for the symmetrization of previous shape

Let us consider the special case of the assumptions above in which $B'(*) = A'(*)$; in this case, the application of Lemmas 7.1 and 7.2 yields

Lemma 7.5. *Let $\mu X.(A'(X) \vee A'(p))$ and $\nu X.(E'(X) \wedge E'(p))$ be as above. Then*

1. $\mu X.(A'(X) \vee A'(p)) = \bigvee_{\kappa \in Ord} A'^{\kappa}(A'(p))$.
2. $\nu X.(E'(X) \wedge E'(p)) = \bigwedge_{\kappa \in Ord} E'^{\kappa}(E'(p))$.

Moreover, the identities $A'^{\kappa}A' = A'A'^{\kappa}$ and $E'^{\kappa}E' = E'E'^{\kappa}$ clearly hold, which amount to the counterparts of Lemmas 7.3 and 7.4 for the present set of assumptions. So all the steps making it possible to implement the proof strategy of subsection 7.1.1 are in place, and indeed, by an analogous argument to the one given at the end of subsection 7.1.1 we get the following

Corollary 7.2. *Let $A'(*)$ and $E'(*)$ be as above. Then, for every perfect distributive lattice L, every $h \in val^+(L)$, and $x, y \in Var^+$,*

$$[\![\mu X.(A'(X) \vee A'(x))]\!](h) \leq [\![y]\!](h) \text{ iff } [\![x]\!](h) \leq [\![\nu X.(E'(X) \wedge E'(y))]\!](h).$$

This justifies the soundness of the following rules:

$$\frac{\mu X.(A'(X, \overrightarrow{z}) \vee A'(p, \overrightarrow{z})) \leq \chi}{p \leq \nu X.(E'(X, \overrightarrow{z}) \wedge E'(\chi/p, \overrightarrow{z}))}\ (\mu\text{-R}) \qquad \frac{\chi \leq \nu X.(A'(X, \overrightarrow{z}) \wedge A'(p, \overrightarrow{z}))}{\mu X.(E'(X, \overrightarrow{z}) \vee E'(\chi/p, \overrightarrow{z})) \leq p}\ (\nu\text{-R})$$

Example 7.2. ***(Application of (μ-R))*** Let $(q \to \bot) = \neg q$, and consider the inequality $q \leq \mu Y.[((\Diamond Y \vee \Diamond\Diamond q) \vee (\Diamond\Diamond Y \vee \Diamond q)) \wedge \neg\Box q]$. We can equivalently rewrite it as

$$q \leq \mu Y.[((\Diamond Y \vee \Diamond\Diamond Y) \wedge \neg\Box q) \vee ((\Diamond q \vee \Diamond\Diamond q) \wedge \neg\Box q)],$$

so $A(*) := \Diamond * \vee \Diamond\Diamond *$, and $\varphi(x, \overrightarrow{z}) = x \wedge \neg\Box q$. Solving for $\epsilon_q = \partial$, we get:

$\forall q[q \leq \mu Y.[((\Diamond Y \vee \Diamond\Diamond q) \vee (\Diamond\Diamond Y \vee \Diamond q)) \wedge \neg\Box q]]$

iff $\forall q \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq q \,\&\, \mu Y.[((\Diamond Y \vee \Diamond\Diamond q) \vee (\Diamond\Diamond Y \vee \Diamond q)) \wedge \neg\Box q] \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]$

(∗) iff $\forall q \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq q \,\&\, q \leq \nu Y.[(\blacksquare Y \wedge \blacksquare\blacksquare(\neg\Box q \to \mathbf{m})) \wedge (\blacksquare\blacksquare Y \wedge \blacksquare(\neg\Box q \to \mathbf{m}))]) \Rightarrow \mathbf{i} \leq \mathbf{m}]$

(∗∗) iff $\forall \mathbf{i} \forall \mathbf{m}[\mathbf{i} \leq \nu q.\nu Y.[(\blacksquare Y \wedge \blacksquare\blacksquare(\neg\Box q \to \mathbf{m})) \wedge (\blacksquare\blacksquare Y \wedge \blacksquare(\neg\Box q \to \mathbf{m}))] \Rightarrow \mathbf{i} \leq \mathbf{m}]$

iff $\forall \mathbf{m}[\nu q.\nu Y.[(\blacksquare Y \wedge \blacksquare\blacksquare(\neg\Box q \to \mathbf{m})) \wedge (\blacksquare\blacksquare Y \wedge \blacksquare(\neg\Box q \to \mathbf{m}))] \leq \mathbf{m}]$.

The equivalence marked with $(*)$ is an application of (μ-R); the equivalence marked with $(**)$ is an application of the recursive Ackermann, which is applicable, since $\nu Y.[(\blacksquare Y \wedge \blacksquare\blacksquare(\neg q \to \mathbf{m})) \wedge (\blacksquare\blacksquare Y \wedge (\neg q \to \mathbf{m}))]$ is positive in q.

7.2 Nested fixpoints

For reasons of notational convenience, in this section we will not describe the display methods in the form of rules to be applied globally on inequalities. Rather, we will show how to rewrite a certain type of fixpoint formula as an adjoint or residual, starting from the innermost occurrences of fixpoint binders. Notice preliminarily that from [1, Proposition 1.3.2], it follows immediately that the following pre-processing rules are sound:

$$\frac{\varphi(\mu X.\mu Y.\xi(X,Y)/!x)}{\varphi(\mu X.\xi(X,X)/!x)}\ (\mu\text{-Pre}) \qquad \frac{\varphi(\nu X.\nu Y.\xi(X,Y)/!x)}{\varphi(\nu X.\xi(X,X)/!x)}\ (\nu\text{-Pre})$$

where, in both rules, the formulas can occur either on the left or on the right of an inequality, and the "placeholder" variable x does not occur in ξ. Repeated applications of these rules will ensure that it can be assumed without loss of generality that fixpoint binders occur consecutively in the generation trees of formulas only if they are non-homogeneous (e.g., either $\mu X.\nu Y.\varphi$ or $\nu X.\mu Y.\varphi$).

For reasons which will become clear in the next section, we are mainly interested in the case of nested *homogeneous* fixpoints. Hence, because of the observation above, the main case we need to treat is the pattern represented by the formula

$$\mu X.\varphi(A_1(X) \vee B_1(\mu Y.[A_2(Y) \vee B_2(p) \vee B_3(X)])/!x, \overrightarrow{z}),$$

where,

$$\begin{array}{rclrcl}
A_2(X) & = & \bigvee_{i\in I_2}\delta_i(X), & A'_2(X,\overrightarrow{z}) & = & \varphi(A_2(X)/!x,\overrightarrow{z}),\\
A_1(Y) & = & \bigvee_{i\in I_1}\delta_i(X), & A'_1(Y,\overrightarrow{z}) & = & \varphi(A_1(X)/!x,\overrightarrow{z}),\\
B_3(X) & = & \bigvee_{j\in J_3}\delta'_j(X), & B'_3(X,\overrightarrow{z}) & = & \varphi(B_3(X)/!x,\overrightarrow{z}),\\
B_2(p) & = & \bigvee_{j\in J_2}\delta'_j(p), & B'_2(p,\overrightarrow{z}) & = & \varphi(B_2(p)/!x,\overrightarrow{z}),\\
B_1(v) & = & \delta'(v), & B'_1(v,\overrightarrow{z}) & = & \varphi(B_1(v)/!x,\overrightarrow{z}),
\end{array}$$

moreover, I_1, I_2 and J_1, J_2, J_3 are finite sets of indexes, for $i \in I_1 \cup I_2$ and $j \in J_1 \cup J_2, J_3$, each δ_i and δ'_j is a unary left adjoint (relativized to our signature, this means that δ_i and δ'_j are concatenations of $\Diamond$). In what follows, we are going to show that

Lemma 7.6. *Under the assumptions above,*

$$\mu X.\varphi(A_1(X) \vee B_1(\mu Y.[A_2(Y) \vee B_2(p) \vee B_3(X)])/!x, \overrightarrow{z}) = \bigvee_{\lambda\in Ord} A'^{\lambda}(B'(p))$$

for some left residuals $A'(x,\overrightarrow{z})$ and $B'(p,\overrightarrow{z})$.

Proof. The following chain of identities holds:

$$\begin{array}{cll}
 & B_1(\mu Y.[A_2(Y) \vee B_2(p) \vee B_3(X)]) & \\
= & B_1(\bigvee_{\kappa\in Ord} A_2^{\kappa}(B_2(p) \vee B_3(X))) & \text{(Lemma 7.1)}\\
= & B_1(\bigvee_{\kappa\in Ord} A_2^{\kappa}(B_2(p))) \vee B_1(\bigvee_{\kappa\in Ord} A_2^{\kappa}(B_3(X))) & \\
= & \bigvee_{\kappa\in Ord} B_1(A_2^{\kappa}(B_2(p))) \vee \bigvee_{\kappa\in Ord} B_1(A_2^{\kappa}(B_3(X))) & (B_1 \text{ c. join-pres.})\\
= & \bigvee_{\kappa\in Ord} A_2^{\kappa}(B_1(B_2(p))) \vee \bigvee_{\kappa\in Ord} A_2^{\kappa}(B_1(B_3(X))) & \text{(Lemma 7.3)}
\end{array}$$

Let $C_1(p) = \bigvee_{\kappa\in Ord} A_2^{\kappa}(B_1(B_2(p)))$ and $C_2(X) = \bigvee_{\kappa\in Ord} A_2^{\kappa}(B_1(B_3(X)))$. Then,

$$\begin{array}{ll} & \mu X.\varphi(A_1(X) \vee B_1(\mu Y.[A_2(Y) \vee B_2(p) \vee B_3(X)])/!x, \overrightarrow{z}) \\ = & \mu X.\varphi((A_1(X) \vee C_2(X)) \vee C_1(p)/!x, \overrightarrow{z}) \\ = & \mu X.[\varphi((A_1(X) \vee C_2(X))/!x, \overrightarrow{z}) \vee \varphi(C_1(p)/!x, \overrightarrow{z})] \\ = & \mu X.[A'(X, \overrightarrow{z}) \vee B'(p, \overrightarrow{z})] \\ = & \bigvee_{\lambda \in Ord} A'^{\lambda}(B'(p)), \qquad \text{(Lemma 7.1)} \end{array}$$

where $A'(X, \overrightarrow{z}) = \varphi((A_1(X) \vee C_2(X))/!x, \overrightarrow{z})$ and $B'(p, \overrightarrow{z}) = \varphi(C_1(p)/!x, \overrightarrow{z})$. □

The formula above can be easily made more general by assuming that $B_1(v) = \bigvee_{j \in J_1} \delta'_j(v)$ and by taking appropriately indexed copies of the innermost fixpoint formula $\mu Y.[A_2(Y) \vee B_2(p) \vee B_3(X)]$.

The dual version of the preceding lemma is proved by a straightforward dualization of the foregoing argument. Repeated applications of the lemmas above, starting from the innermost occurrences of fixpoint binders will make it possible to rewrite a fixpoint formula as a composition of right or left adjoints/residuals. This rewriting makes it possible to display critical occurrences of proposition variables, and hence hopefully to reach Ackermann shape.

Example 7.3. To illustrate the method described above, consider $p \leq \mu X.[\neg p \wedge (X \vee \mu Y.[(\Diamond Y \vee \Diamond p) \vee \Diamond\Diamond X])]$.

The nested fixpoint satisfies the assumptions of Lemma 7.1, for $A_1(Y) = \Diamond Y$ and $B_1 = \Diamond p \vee \Diamond\Diamond X$. Therefore,

$$\begin{array}{rcl} \mu Y.[(\Diamond Y \vee \Diamond p) \vee \Diamond\Diamond X] & = & \bigvee_{\kappa \in Ord} \Diamond^{\kappa}(\Diamond p \vee \Diamond\Diamond X) \\ & = & \bigvee_{\kappa \in Ord} \Diamond^{\kappa}\Diamond p \vee \bigvee_{\kappa \in Ord} \Diamond^{\kappa}\Diamond\Diamond X. \end{array}$$

Hence, we can rewrite $X \vee \mu Y.[(\Diamond Y \vee \Diamond p) \vee \Diamond\Diamond X]$ as

$$(X \vee \bigvee_{\kappa \in Ord} \Diamond^{\kappa}\Diamond\Diamond X) \vee \bigvee_{\kappa \in Ord} \Diamond^{\kappa}\Diamond p.$$

This is again an expression of type $A_2(X) \vee B_2$ satisfying the assumptions of Lemma 7.1. By distributing $\varphi(x, p) = x \wedge \neg p$ over $A_2(X) \vee B_2$, we obtain $(A_2(X) \wedge \neg p) \vee (B_2 \wedge \neg p)$, which still satisfies the assumptions of Lemma 7.1. So, if $A'_2(X) = \varphi(A_2(X)/!x, p) = A_2(X) \wedge \neg p$, and $B'_2 = \varphi(B_2/!x, p) = B_2 \wedge \neg p$, the nested fixpoint formula can be rewritten as

$$\bigvee_{\lambda \in Ord} A_2'^{\lambda}(B'_2) = \bigvee_{\lambda \in Ord} A_2'^{\lambda}(\neg p \wedge \bigvee_{\kappa \in Ord} \Diamond^{\kappa}\Diamond p),$$

which is an expression of the same type as we have seen in Subsection 7.1.2, and the residual of which can be calculated similarly.

Intuitionistic recursive mu-inequalities

In this section, the definition of recursive inequalities for the signature of intuitionistic modal mu-calculus is introduced. The style of this definition closely follows that of [26], in that it is grounded on a certain classification of the nodes in the signed generation trees of formulas (cf. Table 8.1). However, one major difference with [26] is that the classification of nodes adopted in this thesis is based on the *positive* properties of the operations interpreting the logical connectives. Since, however, the fixpoints escape to some extent this classification (because they do not quite enjoy the same order theoretic properties of the other connectives), we settled for denominations which do not explicitly refer to order-theoretic properties; instead, in a signed generation tree of an $\mathcal{L}$-sentence, we will regard certain nodes as *skeleton nodes* and others as *PIA nodes*, since these names help to establish connections with the model-theoretic analysis conducted in [13].

An *order type* over $n \in \mathbb{N}$ is an n-tuple $\epsilon \in \{1, \partial\}^n$. For every order type ϵ, let ϵ^∂ be its *opposite* order type, i.e., $\epsilon^\partial_i = 1$ iff $\epsilon_i = \partial$ for every $1 \leq i \leq n$.

For any $\mathcal{L}$-sentence $\varphi(p_1, \ldots p_n)$, any order type ϵ over n, and any $1 \leq i \leq n$, an *ϵ-critical node* in the signed generation tree of φ is a (leaf) node $+p_i$ with $\epsilon_i = 1$, or $-p_i$ with $\epsilon_i = \partial$. An *ϵ-critical branch* in the tree is a branch terminating in an ϵ-critical node. The intuition, which will be built upon later, is that variable occurrences corresponding to ϵ-critical nodes are *to be solved for, according to ϵ*.

For every $\mathcal{L}$-sentence $\varphi(p_1, \ldots p_n)$, and every order type ϵ, we say that $+\varphi$ (resp. $-\varphi$) *agrees with* ϵ, and write $\epsilon(+\varphi)$ (resp. $\epsilon(-\varphi)$), if every leaf node in the signed generation tree of $+\varphi$ (resp. $-\varphi$) which is labelled with a proposition variable is ϵ-critical. In other words, $\epsilon(+\varphi)$ (resp. $\epsilon(-\varphi)$) means that all proposition variable occurrences corresponding to leaves of $+\varphi$ (resp. $-\varphi$) are to be solved for according to ϵ. We will also make use of the *sub-tree relation* $\gamma \prec \varphi$, which extends to signed generation trees, and we will write $\epsilon(\gamma) \prec *\varphi$ to indicate that γ, regarded as a sub- (signed generation) tree of $*\varphi$, agrees with ϵ.

Definition 8.1. *Nodes in signed generation trees will be called* skeleton nodes *and* PIA nodes *according to the specification given in table 8.1. A branch in a signed generation tree $*\varphi$, for $* \in \{+, -\}$, ending in a proposition variable is an ϵ-*good branch *if it is the concatenation of two paths P_1 and P_2, one of which may possibly be of length* 0*, such that P_1 is a path from the leaf consisting (apart from the variable node) only of PIA-nodes, and P_2 consists (apart from the variable node) only of skeleton-nodes. Moreover, any PIA-node of an ϵ-good branch which is neither a leaf- nor an SRA$\cup\{+\nu X, -\mu X\}$-node is of the form $\gamma \star \beta$, where*

SLR	SRA	Skeleton	PIA
			$+ \quad \wedge$
$+ \quad \vee$		$+ \quad \vee$	$+ \quad \Box$
$+ \quad \wedge$	$+ \quad \wedge$	$+ \quad \wedge$	$+ \quad \vee$
$+ \quad \Diamond$	$+ \quad \Box$	$+ \quad \Diamond$	$+ \quad \rightarrow$
		$+ \quad \mu X$	$+ \quad \nu X$
$- \quad \vee$	$- \quad \vee$	$- \quad \vee$	$- \quad \vee$
$- \quad \wedge$	$- \quad \Diamond$	$- \quad \wedge$	$- \quad \Diamond$
$- \quad \Box$		$- \quad \Box$	$- \quad \wedge$
$- \quad \rightarrow$		$- \quad \rightarrow$	$- \quad \mu X$
		$- \quad \nu X$	

Table 8.1: SLR, SRA, Skeleton and PIA nodes.

(a) the side of $\star$ where the ϵ-good branch lies must be monotone[1]*, and*
(b) if β is the side where the branch lies, then γ is a mu-sentence[2] *and $\epsilon^{\partial}(\gamma) \prec *\varphi$.*
Unraveling the condition $\epsilon^{\partial}(\gamma) \prec *\varphi$ specifically to the $\mathcal{L}$-signature, we obtain:
b1) if $\gamma \star \beta = +(\gamma \vee \beta)$, then $\epsilon^{\partial}(+\gamma)$;
b2) if $\gamma \star \beta = +(\gamma \rightarrow \beta)$ or $\gamma \star \beta = -(\gamma \wedge \beta)$, then $\epsilon(+\gamma)$.
A good branch is skeleton *if P_1 has length* 0.

Definition 8.2. *Given an order type ϵ, the signed generation tree $*\varphi$, $* \in \{-, +\}$, of an $\mathcal{L}$-sentence $\varphi(p_1, \ldots p_n)$ is* ϵ-recursive *if every ϵ-critical branch is ϵ-good. An $\mathcal{L}$-inequality $\varphi \leq \psi$ is* ϵ-recursive *if the signed generation trees $+\varphi$ and $-\psi$ are both ϵ-recursive. An $\mathcal{L}$-inequality $\varphi \leq \psi$ is* recursive *if it is ϵ-recursive for some order type ϵ.*

8.1 Recursive inequalities and the enhanced ALBA

The intuitive idea of the ϵ-recursive shape $\varphi \leq \psi$ is that, on either side of the inequality, it consists of *three types of ingredients*. The first ingredient is an outer, approximation-friendly (exo)*skeleton* $\varphi'(!x_1, \ldots, !x_n) \leq \psi'(!y_1, \ldots, !y_m)$; in the skeleton, the variables x_i and y_j for $1 \leq i \leq n$ and $1 \leq j \leq m$ should be understood as *placeholders*. These placeholders mark the point where, traversing critical branches from leaf to root, PIA-nodes give way to skeleton-nodes. Indeed, the ϵ-recursive shape guarantees that, appended to the generation tree of the skeleton (i.e., replacing each placeholder variable), there are the two remaining types of ingredients, namely, either ϵ^{∂}*-formulas* γ (i.e. formulas such that either $\epsilon^{\partial}(\gamma) \prec +\varphi$ or $\epsilon^{\partial}(\gamma) \prec -\psi$), or *PIA-formulas* β. This guarantees that ϵ-critical branches must go through PIA-formulas. The analysis conducted in [13] about PIA-formulas can be summarized in the slogan "PIA formulas provide minimal valuations". This exactly translates in the Ackermann's strategy: indeed, the PIA shape guarantees

[1]This condition is needed because, in the case of a non-unary, residuated connective, its negative coordinates are the ones which switch sides in the residuation condition; therefore if an ϵ-critical branch passed through the negative coordinate of such a connective, its corresponding variable occurrence could never be 'displayed'.

[2]This condition excludes the occurrence of 'PIA'-only signed generation (sub)trees such as $-\mu X(\Diamond X \wedge \Diamond q)$ from recursive inequalities.

that ϵ-critical variables be *displayed* (more on this below), and, as discussed in Chapter 2, displaying an ϵ-critical variable exactly amounts to finding the value of the minimal valuation for that variable.

The ϵ-recursive shape, and the three types of ingredients it consists of, guarantee that the enhanced version of ALBA can be successfully applied on $\varphi \leq \psi$: indeed, as already remarked in Section 7.2, along to the preprocessing steps in ALBA (see [26] for more details on the preprocessing) we can add the preprocessing rules (μ-Pre) and (ν-Pre), which guarantee that we can assume wlog that occurrences of homogeneous nested fixpoints are always separated by the occurrence of some non-fixpoints node between them. After preprocessing, the first approximation takes place, yielding the initial clause

$$\forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \varphi'(\overrightarrow{\gamma}, \overrightarrow{\beta}) \ \&\ \psi'(\overrightarrow{\gamma}, \overrightarrow{\beta}) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}];$$

Claim 1: the skeleton shape guarantees that the approximation rules are applicable. Indeed, the skeleton shape guarantees that all the non-fixpoint nodes in φ' are completely join preserving (at least in the coordinates lying on critical branches) and all the non-fixpoint nodes in ψ' are completely meet preserving (at least in the coordinates lying on critical branches). Therefore, the approximation rules (both for the non-fixpoints connectives and for the fixpoint binders) appropriate to each side can be applied repeatedly so as to un-nest the skeleton fixpoint-nodes. In this way, either we reach a critical variable, which will be then put in display, or we have equivalently transformed the initial clause into the following clause

$$\forall \mathbf{i} \forall \mathbf{m} \forall \overrightarrow{\mathbf{j}}[(\overrightarrow{\mathbf{j}} \leq \overrightarrow{\beta} \ \&\ \mathbf{i} \leq \varphi'(\overrightarrow{\gamma}, \overrightarrow{\mathbf{j}}) \ \&\ \psi'(\ldots) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}];$$

where the generation trees of the β's (which we have extracted by means of repeated application of the approximation rules) only consists of PIA nodes.

Claim 2: the PIA shape guarantees that the adjunction/residuation rules and rewriting procedure are applicable. Indeed, PIA-nodes which are not fixpoint binders are either (syntactic) right adjoints or right residuals. While the occurrence of right adjoint-nodes is not subject to any condition, the occurrence of right residual-nodes is limited by condition (b) in Definition 8.1, which essentially says that, inside any (sub-) tree rooted on a fixpoint binder, the free fixpoint variables and ϵ-critical variables cannot be 'separated' by residual nodes (in the sense of being the leaves of branches which join up in a residual node). For instance, in $-\mu Y.((\Diamond Y \vee \Diamond q) \wedge \Diamond\Diamond p)$, condition (b) is violated if $\epsilon_p = \partial$. This implies that either one of the adjunction/residuation rules given in Section 7.1 can be applied to display the critical variable(s), or the rewriting procedure described in Section 7.2 can be applied starting from the innermost occurrences of fixpoint binders. In either case, we obtain each inequality $\mathbf{j} \leq \beta_i$ can be equivalently rewritten into (systems of) displayed inequalities $\alpha_i \leq p$; hence the clause above can be equivalently rewritten as

$$\forall \mathbf{i} \forall \mathbf{m} \forall \overrightarrow{\mathbf{j}}[(\bigvee_i \alpha_i \leq p \ \&\ \mathbf{i} \leq \varphi'(\overrightarrow{\gamma}, \overrightarrow{\mathbf{j}}) \ \&\ \psi'(\ldots) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}].$$

The fact that the γ's are ϵ^∂ guarantees that the Ackermann's rule can be applied, which eliminates p.

CONCLUSION AND FUTURE WORK

This thesis was motivated by the idea of extending the results in [13] to the case of logics with fixpoints and with a non-classical base. We focused in particular on intuitionistic modal mu-calculus. We enhanced the ALBA algorithm with rules for solving for variables which occur inside the scope of fixpoint binders, and proved their soundness thanks to the order-theoretic properties of the interpretation of the logical connectives in the signature of intuitionistic modal mu-calculus on perfect intuitionistic modal algebras. We defined the class of recursive inequalities and informally justified that the enhanced ALBA is guaranteed to eliminate the monadic second order propositional variables, yielding a correspondent in FO+EFP.

The work in this thesis covers the general principles of the order-theoretic algorithmic correspondence for intuitionistic modal mu-calculus; however, in order to complete the picture we have to a fully fledged theory, the following results need to be developed, which are left to future work:

- details of the *standard translation* need to be provided;
- evidence should be provided that these results are about *local correspondence* (which we believe is the case);
- the fact that the enhanced ALBA is successful on recursive inequalities, which we have justified informally, needs to be fully proved; in particular, the adjunction/residuation rules need to be extended so as to be able to solve for *all* the critical variables *at once*;
- in the algorithmic correspondence theory, we have the possibility of distinguishing between applications of recursive and non-recursive Ackermann's lemma; we would like to define syntactic shapes of recursive inequalities which guarantee the solvability through applications of the non-recursive Ackermann.

In proving the order-theoretic properties of fixpoints, which the soundness of the rules is based on, we realized that they do not rely on the lattice being distributive. This opens the opportunity to extend these results to fixpoint expansions of logics with a non-distributive lattice base; for instance the substructural logics. So a further direction is about extending the existing non-distributive version of ALBA [27] with rules for fixpoints.

Bibliography

[1] A. Arnold, D. Niwinski *Rudiments of μ-calculus*, Vol. 146, ©2001 Elsevier Science B.V.

[2] A. Chagrov and L. A. Chagrova, *The truth about algorithmic problems in correspondence theory*, In advances in modal logic, G. Governatori, Ian Hodkinson and Y. Venema, eds., vol.6, College publications, 2006, pp. 121-138.

[3] A. Simpson, *The Proof Theory and Semantics of Intuitionistic Modal Logic*, PhD dissertation, University of Edinburgh, 1994.

[4] B. A. Davey, H. A. Priestley, *Introduction to lattices and order*, Second edition, ©Cambridge University Press 1990, 2002

[5] Blackburn, P., M. de Rijke and Y. Venema, *Modal Logic*, Cambridge University Press, 2001.

[6] B. Jónsson *On the canonicity of Salhqvist identities*, Studia Logica, 53 (1994), pp. 473-491.

[7] D. Kozen, *Results on the propositional mu-calculus*, Theoretical computer science 27 (1983), 333-353

[8] H. Sahlqvist, *Correspondence and completeness in the first and second order semantics for modal logic*, in proceedings of the 3rd Scandinavian Logic Symposium, Uppsala 1973, S. Kanger, ed., Amsterdam 1975, Springer-Verlag, pp. 110-143.

[9] J. C. Bradfield, *On the expressivity of the modal mu-calculus*, Proc. 13th Annual Symposium on Theoretical Aspects of Computer Science (STACS '96) LNCS 1046 479-490 (1996).

[10] J. C. Bradfield, Julian and Stirling, Colin, *Modal mu-calculi* In: P. Blackburn, J. van Benthem and F. Wolter (eds.), The Handbook of Modal Logic pp. 721-756. Elsevier (2006).

[11] J. C. Bradfield, *Fixpoint Alternation: Arithmetic, transition systems, and the binary tree*, BRICS RS-98-53, ©1998

[12] J. M. Dunn, M. Gehrke and A. Palmigiano, *Canonical extensions and relational completeness of some substructural logics*, Journal of Symbolic Logic, 70(3), (2005), pp. 713-740.

[13] Johan van Benthem, Nick Bezhanishvili, Ian Hodkinson, *Sahlqvist correspondence for modal mu-calculus*, Studia Logica April 2012, Volume 100, Issue 1-2, pp 31-60

[14] M. Gehrke, H. Nagahashi, Y. Venema, *A Sahlqvist theorem for distributive modal logic*, Annals of pure and applied logic, 131 (2005), pp. 65-102.

[15] S. Ghilardi and G. Meloni, *Constructive canonicity in non classical logics*, Annals of Pure and Applied Logic, 86 (1997), pp. 1-32

[16] Tomoyuki Suzuki, *Canonicity results of substructural and lattice-based logic.*, The review of symbolic logic, 4(2001) pp. 1-42.

[17] Vaccaro, Sambin, *A new proof of Sahlqvist theorem on modal definability and completeness*, Journal of Symbolic Logic, 54 (1989) pp. 992-999.

[18] van Benthem, J.F.A.K., *Modal frame correspondence and fixed points* Studia Logica 83(2006) 133-155.

[19] van Benthem, J.F.A.K., *Modal logic and classical logic* Bibliopolis, 1983.

[20] V. Goranko, D. Vakarelov, *Elementary canonical formulae: Extending Sahlqvist theorem*, Annals of Pure and Applied Logic vol. 141 (1-2) (2006) 180-217.

[21] W. Conradie, V. Goranko, D. Vakarelov, *Algorithmic correspondence and completeness in modal logic, V. Recursive SQEMA.*,Journal of Applied Logic, Vol. 8, No. 4, 2010, p. 319-333.

[22] W. Conradie, A. Palmigiano, S. Sourabh, *Algebraic modal correspondence: Sahlqvist and beyond*, *to appear*

[23] W. Conradie, V. Goranko, D. Vakarelov, *Algorithmic correspondence and completeness in modal logic I, The core Algorithm SQEMA.*, Logical methods in Computer Science 2 (1:5) (2006).

[24] W. Conradie, V. Goranko, D. Vakarelov, *Algorithmic correspondence and completeness in modal logic II, Polyadic and hybrid extensions of the algorithm SQEMA.*, Journal of Logic and Computation 16 (2006), pp. 579-612.

[25] W. Conradie, V. Goranko, D. Vakarelov, *Elementary canonical formulae: A survey on syntactic, algorithmic and model theoretic aspects*,Advances in modal logic, R. Schmidt, I. Pratt-Hartmann, M. Reynolds, and H. Wansing, eds., vol. 5, Kings College, 2005, pp. 17-51.

[26] Willem Conradie, Alessandra Palmigiano, *Algorithmic correspondence and canonicity for distributive modal logic*, Annals of Pure and Applied Logic 163 (2012) 338-376.

[27] Willem Conradie, Alessandra Palmigiano *Algorithmic correspondence and canonicity for lattice based logic*, (manuscript).

[28] Yde Venema, *Lectures on the modal μ-calculus*, ©2008 (Institute for Logic, Language and Computation, University of Amsterdam).

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