# Dependence Logic in Algebra and Model Theory

**MSc Thesis** *(Afstudeerscriptie)*

written by

**Gianluca Paolini**
(born March 21st, 1988 in Ascoli Piceno, Italia)

under the supervision of **Prof Jouko Väänänen**, and submitted to the Board of Examiners in partial fulfillment of the requirements for the degree of

**MSc in Logic**

at the *Universiteit van Amsterdam.*

| Date of the public defense: | Members of the Thesis Committee: |
|---|---|
| *May 24, 2013* | Benedikt Loewe |
| | Juha Kontinen |
| | Alessandra Palmigiano |
| | Krzysztof Apt |

Institute for Logic, Language and Computation

*To Alberto Facchini, my first professor of algebra and the first to teach me the virtues of mathematical thinking.*

# Contents

# Chapter 1

# Introduction

## 1.1 Dependence Logic

Logics of dependence and independence (often called logics of imperfect information) go back to the work of Henkin [21], Enderton [11], Walkoe [27], Blass and Gurevich [8], and others on partially ordered (or Henkin-) quantifiers. A next step in this direction are the independence-friendly (IF) logics of Hintikka and Sandu [24], that incorporate explicit dependencies of quantifiers on each other. In all these cases the semantics are given in game-theoretic terms, using so-called games of imperfect information.

It had repeatedly been claimed that a compositional semantics, defined by induction on the construction of formulae, could not be given for IF-logic and similar formalisms. However, this claim had never been made precise, let alone proved. In fact the claim was later refuted by Hodges [25], who presented a compositional semantics for IF-logic in terms of what he called trumps, which are sets of assignments to a fixed finite set of variables.

In 2007, Väänänen [37], inspired by the above-mentioned work of Hodges, proposed a new approach to the subject, defining a logical formalism that goes under the name of dependence logic. The main idea of dependence logic is to treat dependence as a logical concept and to express it via logical constants. Following this idea, dependence logic is obtained by extending first-order logic with dependence atoms.

In order to characterize dependence in these terms, the semantics is formulated with respect to sets of assignments (also called *teams*) instead of single assignments, as it is the case for first-order logic and most of its extensions. This is the device that allows to characterize the intrinsic second-order concepts of dependence and independence without the introduction of variables of higher orders.

Väänänen's approach made the logical reasoning about dependence mathematically much more transparent, and led to a deeper understanding of the logical aspects of dependence as well as of the expressive power of IF-logic and dependence logic. This advance resulted in a series of studies by several authors of Väänänen's formalism, and brought new insights into the subject.

These works showed that dependence is just one among many different properties that give rise to interesting logics based on team semantics. Galliani [15] and Engström [12] have studied several logics with team properties based on notions originating in database dependency theory, and Väänänen and Grädel [19] analyzed the notion of independence (which is a much more delicate but also more powerful notion than that of dependence) introducing Independence Logic.

As a consequence of this flourishing situation, in more recent years the term 'dependence logic' extended its meaning from the study of a single formalism to a far-reaching research project. As Väänänen and Galliani claim in [16],

> The goal of dependence logic is to establish a basic theory of dependence and independence underlying such seemingly unrelated subjects as causality, random variables, bound variables in logic, database theory, the theory of social choice, and even quantum physics. There is an avalanche of new results in this field demonstrating remarkable convergence. The concepts of (in)dependence in the different fields of humanities and sciences have surprisingly much in common and a common logic is starting to emerge.

The present study is part of this research project, and intends to frame the algebraic and model-theoretic dependence and independence concepts in the more general theory of dependence logic.

## 1.2 Atomic Systems

As noticed, what distinguishes dependence logic from first-order logic is the presence of other logical atoms apart from the equational one. The richer structure of the atomic level of the language affects the semantic properties of all of the language. For example, the disjunction takes an intuitionistic behavior, and the model-theoretic consequence relation becomes too complex to be axiomatizable. Nevertheless, the study of the atomic fragment of the language remains the main tool in the analysis of the various forms of dependence and independence and their relations; this is the reason why we place our study at this level.

The first ones to study atomic dependence and independence systems were not logicians but experts in database theory and statistics, see for example [2], [7], [6], [17], [32] and [35]. It is to them that the definition of these atoms and the identification of axiomatic systems capable of characterizing them is due. A famous example is Armstrong's axiomatization of functional dependence in [2], where he isolates some deductive rules that completely describe the behavior of the functional dependence atom.

In all these systems the dependence and independence atoms are interpreted either as dependencies in a database, or as probabilistically independent variables. dependence logic incorporated and unified these approaches, under the abstract setting that is characteristic of its framework, giving to the known results a common ground.

The following work is placed at this crossroad, taking inspiration from the abstract theory of dependence logic for motivations and from the above-mentioned studies for proof techniques.

## 1.3 Dependence and Independence in Algebra and Model Theory

The aim of the present thesis is to analyze the dependence and independence concepts present in algebra and model theory from the point of view of dependence logic. How do we perform this analysis?

The idea is to use the atomic systems that we mentioned in the previous section. Specifically, for each case of dependence and independence that we want to analyze, the plan is the following.

1. Formulate interpretations of the dependence and independence atoms in the context under analysis.

2. Study the relationships between the model-theoretic consequence relations between atoms and the axiomatic systems that characterize them in abstract terms.

3. In case of mismatch, find axiomatic systems that characterize the behavior of the atoms with respect to this semantics.

4. Study the relations between the different kinds of atoms with respect to this semantics.

Let's take as an example the notions of linear dependence and independence. First, we take as models of our semantics only the vector spaces. Second, we say that an atom is satisfied under an assignment if the vectors that correspond to the variables occurring in the atom are dependent or independent. Then, we verify if the known deductive systems are sound and complete with respect to this semantics, and if not, we try to define others. Finally, we study the relations between the introduced atoms under this interpretation.

With this general plan in mind, we structure the work as follows: in Chapter 2 we define the abstract versions of the systems that we are going to deal with; in Chapter 3 we see how the semantics of these systems can be reinterpreted with vector space conditions; in Chapter 4 we analyze the linear dependence and independence of linear algebra and the algebraic dependence and independence of field theory; in Chapter 5 we deal with the dependence and independence notions definable in terms of the model-theoretic operator of algebraic closure; in Chapter 6 we generalize what we have done in Chapter 5 to any well-behaved closure operator; in Chapter 7 we develop the basics of stability theory; in Chapter 8 we study the forking independence relation in $\omega$-stable theories; and finally in Chapter 9 we draw some conclusions and explore the possibilities for future work.

## 1.4 Notation

Most of the notation is standard. We use $A \subseteq B$ to mean $A$ is subset of $B$ and $A \subset B$ or $A \subsetneq B$ to mean $A$ is a proper subset of $B$ (i.e. $A \subseteq B$ but $A \neq B$).

If $A$ is a set,

$$A^{<\omega} = \bigcup_{n=0}^{\infty} A^n$$

is the set of all finite sequences from $A$. We write $\overline{a}$ to indicate a finite sequence $(a_0, ..., a_{n-1})$. When we write $\overline{a} \in A$ we really mean $\overline{a} \in A^{<\omega}$. Given two sequences $\overline{a}, \overline{b} \in A^{<\omega}$ with $\overline{a} = (a_0, ..., a_{n-1})$ and $\overline{b} = (b_0, ..., b_{m-1})$ , we denote by $\overline{a}\,\overline{b}$ their concatenation. That is, the sequence $(a_0, ..., a_{n-1}, b_0, ..., b_{m-1})$.

In most of the cases we will consider sequences modulo repetitions and permutations and apply standard set theoretic operations to them. For example, given the sequences $\overline{a} = (a_0, ..., a_{n-1})$ and $\overline{b} = (b_0, ..., b_{m-1})$ we write $\overline{a} \subseteq \overline{b}$ to mean $\{a_0, ..., a_{n-1}\} \subseteq \{b_0, ..., b_{m-1}\}$.

If $A$ is a set, then $|A|$ is the cardinality of $A$. The *power set* of $A$ is $\mathcal{P}(A) = \{X : X \subseteq A\}$. By the expression $A_0 \subseteq_{\mathsf{fin}} A$ we mean that $A_0 \subseteq A$ and $A_0$ is finite.

In displays, we sometimes use both $\Leftarrow$ and $\Rightarrow$ as abbreviations for "implies", and $\Leftrightarrow$ as an abbreviation for "if and only if".

We reserve the style $\backslash mathbb$ for mathematical structures, $\backslash mathcal$ for model-theoretic structures, $\backslash mathbf$ for proper classes and $\backslash mathfrak$ for monster models. So, for example, we have respectively $\mathbb{M}$, $\mathcal{M}$, $\mathbf{M}$ and $\mathfrak{M}$. Given a mathematical structure $\mathbb{M}$ or a model-theoretic structure $\mathcal{M}$ we denote its domain with the symbol $M$.

For all the notations and conventions that we do not make explicit, we refer the reader to [37] for dependence logic and [29] for model theory.

# Chapter 2

# Abstract Systems

In this chapter we define the abstract versions of the systems that we are going to deal with for the rest of the thesis. They qualify as abstract because their semantics intend to give an abstract account of the concepts of dependence and independence, and do not refer to any particular version of these notions.

The variants of dependence and independence that are conceivable are several. In our analysis we will focus on the following: $\overline{y}$ is dependent on $\overline{x}$, $\overline{x}$ is independent, $\overline{x}$ is independent from $\overline{y}$, $\overline{x}$ is independent over $\overline{z}$, and $\overline{x}$ is independent from $\overline{y}$ over $\overline{z}$.

To these variants of the notions we associate respectively the following atoms:

$$=(\overline{x}, \overline{y}), \quad \perp(\overline{x}), \quad \overline{x} \perp \overline{y}, \quad \perp_{\overline{z}}(\overline{x}) \quad \text{and} \quad \overline{x} \perp_{\overline{z}} \overline{y}.$$

For completeness of the study that we develop in Chapter 3 we also consider the atom $\overline{x} = \overline{y}$.

All the atoms except for $\perp(\overline{x})$ and $\perp_{\overline{z}}(\overline{x})$ are present in the literature; these two have been introduced by the author.

Each kind of atom gives rise to an atomic language, and for each atomic language we define a team semantics and a deductive system.

The resulting systems are: Atomic Equational Logic, Atomic Dependence Logic, Atomic Absolute Independence Logic, Atomic Independence Logic, Atomic Absolute Conditional Independence Logic and Atomic Conditional Independence Logic.

The first four admit finite complete axiomatizations while the last one does not. Indeed, Parker and Parsaye-Ghomi [32] proved that it is not possible to find a *finite* complete axiomatization for the conditional independence atoms. Furthermore, in [23] and [22] Hermann proved that the consequence relation between these atoms is undecidable. However, in [31] Naumov and Nicholls developed a recursively enumerable axiomatization of them.

## 2.1 Atomic Equational Logic

In this section we define the system Atomic Equational Logic (AEL).

### 2.1.1 Syntax

The language of this logic is made only of equational atoms. That is, let $\overline{x}$ and $\overline{y}$ be finite sequences of variables of the same length, then the formula $\overline{x} = \overline{y}$ is a formula of the language of AEL.

### 2.1.2 Semantics

**Definition 2.1.1.** Let $\mathcal{M}$ be a FO structure. Let $X = \{s_i\}_{i \in I}$ with $s_i$ : $\mathrm{dom}(X) \to M$ and $\overline{x}\,\overline{y} \subseteq \mathrm{dom}(X) \subseteq \mathrm{Var}$. Let $\overline{x} = (x_{j_0}, ..., x_{j_{n-1}})$ and $\overline{y} = (y_{k_0}, ..., y_{k_{n-1}})$, we say that $\mathcal{M}$ satisfies $\overline{x} = \overline{y}$ under $X$, in symbols $\mathcal{M} \models_X \overline{x} = \overline{y}$, if

$$\forall s \in X \; \forall i \in \{0, ..., n-1\} \;\; (s(x_{n_i}) = s(y_{k_i})).$$

**Definition 2.1.2.** Let $\Sigma$ be a set of atoms and let $X$ be such that the set of variables occurring in $\Sigma$ is included in $\mathrm{dom}(X)$. We say that $\mathcal{M}$ satisfies $\Sigma$ under $X$, in symbols $\mathcal{M} \models_X \Sigma$, if $\mathcal{M}$ satisfies every atom in $\Sigma$ under $X$.

**Definition 2.1.3.** Let $\Sigma$ be a set of atoms. We say that $\overline{x} = \overline{y}$ is a logical consequence of $\Sigma$, in symbols $\Sigma \models \; \overline{x} = \overline{y}$, if for every $\mathcal{M}$ and $X$ such that the set of variables occurring in $\Sigma \cup \{\overline{x} = \overline{y}\}$ is included in $\mathrm{dom}(X)$ we have that

$$\text{if } \; \mathcal{M} \models_X \Sigma \; \text{ then } \; \mathcal{M} \models_X \overline{x} = \overline{y}.$$

### 2.1.3 Deductive system

The deductive system is characterized by the following rules:

($\mathrm{a}_0$.) $\overline{x} = \overline{x}$ [as a degenerate case of this rule we admit $\emptyset = \emptyset$];

($\mathrm{b}_0$.) If $\overline{x} = \overline{y}$, then $\overline{y} = \overline{x}$;

($\mathrm{c}_0$.) If $\overline{x} = \overline{y}$ and $\overline{y} = \overline{z}$, then $\overline{x} = \overline{z}$;

($\mathrm{d}_0$.) If $\overline{x} = \overline{y}$ and $\overline{z} = \overline{v}$, then $\overline{x}\,\overline{z} = \overline{y}\,\overline{v}$;

($\mathrm{e}_0$.) If $(x_{j_0}, ..., x_{j_{n-1}}) = (y_{k_0}, ..., y_{k_{n-1}})$, then $x_{j_i} = y_{k_i}$ [for any $i \in \{0, ..., n-1\}$].

### 2.1.4 Soundness and Completeness

**Theorem 2.1.4.** Let $\Sigma$ be a set of atoms, then

$$\Sigma \models \; \overline{x} = \overline{y} \text{ if and only if } \Sigma \vdash \; \overline{x} = \overline{y}.$$

□

## 2.2 Atomic Dependende Logic

In this section we define the system Atomic Dependende Logic (ADL).

### 2.2.1 Syntax

The language of this logic is made only of dependence atoms. That is, let $\overline{x}$ and $\overline{y}$ be finite sequences of variables, with $\overline{y} \neq \emptyset$ if $\overline{x} \neq \emptyset$, then the formula $=\!(\overline{x}, \overline{y})$ is a formula of the language of ADL.

### 2.2.2 Semantics

**Definition 2.2.1.** Let $\mathcal{M}$ be a FO structure. Let $X = \{s_i\}_{i \in I}$ with $s_i : \mathrm{dom}(X) \to M$ and $\overline{x}\,\overline{y} \subseteq \mathrm{dom}(X) \subseteq \mathrm{Var}$. We say that $\mathcal{M}$ satisfies $=(\overline{x}, \overline{y})$ under $X$, in symbols $\mathcal{M} \models_X =(\overline{x}, \overline{y})$, if

$$\forall s, s' \in X\ (s(\overline{x}) = s'(\overline{x}) \to s(\overline{y}) = s'(\overline{y})).$$

**Definition 2.2.2.** Let $\Sigma$ be a set of atoms and let $X$ be such that the set of variables occurring in $\Sigma$ is included in $\mathrm{dom}(X)$. We say that $\mathcal{M}$ satisfies $\Sigma$ under $X$, in symbols $\mathcal{M} \models_X \Sigma$, if $\mathcal{M}$ satisfies every atom in $\Sigma$ under $X$.

**Definition 2.2.3.** Let $\Sigma$ be a set of atoms. We say that $=(\overline{x}, \overline{y})$ is a logical consequence of $\Sigma$, in symbols $\Sigma \models =(\overline{x}, \overline{y})$, if for every $\mathcal{M}$ and $X$ such that the set of variables occurring in $\Sigma \cup \{=(\overline{x}, \overline{y})\}$ is included in $\mathrm{dom}(X)$ we have that

$$\text{if } \mathcal{M} \models_X \Sigma \text{ then } \mathcal{M} \models_X =(\overline{x}, \overline{y}).$$

### 2.2.3 Deductive system

The deductive system is characterized by the following rules:

(a$_1$.) $=(\overline{x}, \overline{x})$ [as a degenerate case of this rule we admit $=(\emptyset, \emptyset)$];

(b$_1$.) If $=(\overline{x}, \overline{y})$, $\overline{u} \subseteq \overline{y}$ and $\overline{x} \subseteq \overline{z}$, then $=(\overline{z}, \overline{u})$;

(c$_1$.) If $=(\overline{x}, \overline{y})$ and $=(\overline{y}, \overline{z})$, then $=(\overline{x}, \overline{z})$;

(d$_1$.) If $=(\overline{x}, \overline{y})$ and $=(\overline{z}, \overline{v})$, then $=(\overline{x}\,\overline{z}, \overline{y}\,\overline{v})$;

(e$_1$.) If $=(\overline{x}, \overline{y})$, then $=(\pi\overline{x}, \pi\overline{y})$ [where $\pi : \mathrm{Var}^{<\omega} \to \mathrm{Var}^{<\omega}$ is the function that eliminates repetitions in finite sequences of variables];

(f$_1$.) If $=(\overline{x}, \overline{y})$, then $=(\overline{x}', \overline{y}')$ [where $\overline{x}' \in R[\overline{x}]$, $\overline{y}' \in R[\overline{y}]$ and $R : \mathrm{Var}^{<\omega} \to \mathrm{Var}^{<\omega}$ is the relation that add repetitions to finite sequences of variables].

### 2.2.4 Soundness and Completeness

**Theorem 2.2.4.** Let $\Sigma$ be a set of atoms, then

$$\Sigma \models\ =(\overline{x}, \overline{y}) \text{ if and only if } \Sigma \vdash\ =(\overline{x}, \overline{y}).$$

*Proof.* See [2] and [13].

□

## 2.3 Atomic Absolute Independence Logic

In this section we define the system Atomic Absolute Independence Logic (AAIndL).

### 2.3.1 Syntax

The language of this logic is made only of absolute independence atoms. That is, let $\overline{x}$ be a finite sequence of variables, then $\perp(\overline{x})$ is a formula of the language of AAIndL.

### 2.3.2 Semantics

The intuition behind the atom $\perp(\overline{x})$ is that $\overline{x}$ is made of independent elements. That is, each element of $\overline{x}$ is independent of all the other elements of $\overline{x}$. In particular, we ask that each element of $\overline{x}$ does not depend on any other element, i.e. that it is not constant.

**Definition 2.3.1.** Let $\mathcal{M}$ be a FO structure. Let $X = \{s_i\}_{i \in I}$ with $s_i : \mathrm{dom}(X) \to M$ and $\overline{x} \subseteq \mathrm{dom}(X) \subseteq \mathrm{Var}$. Let $x \in \overline{x}$, we denote by $\overline{x} -_X x$ any enumeration of the set $\{x' \in \overline{x} \mid \mathcal{M} \not\models_X x' = x\}$. We say that $\mathcal{M}$ satisfies $\perp(\overline{x})$ under $X$, in symbols $\mathcal{M} \models_X \perp(\overline{x})$, if for all $x \in \overline{x}$

$$\forall s, s' \in X \; \exists s'' \in X \; (s''(x) = s(x) \wedge s''(\overline{x} -_X x) = s'(\overline{x} -_X x))$$

and

$$\exists s, s' \in X \; (s(x) \neq s'(x)).$$

**Definition 2.3.2.** Let $\Sigma$ be a set of atoms and let $X = \{s_i\}_{i \in I}$ be such that the set of variables occurring in $\Sigma$ is included in $\mathrm{dom}(X)$. We say that $\mathcal{M}$ satisfies $\Sigma$ under $X$, in symbols $\mathcal{M} \models_X \Sigma$, if $\mathcal{M}$ satisfies every atom in $\Sigma$ under $X$.

**Definition 2.3.3.** Let $\Sigma$ be a set of atoms. We say that $\perp(\overline{x})$ is a logical consequence of $\Sigma$, in symbols $\Sigma \models \perp(\overline{x})$, if for every $\mathcal{M}$ and $X$ such that the set of variables occurring in $\Sigma \cup \{\perp(\overline{x})\}$ is included in $\mathrm{dom}(X)$ we have that

$$\text{if } \; \mathcal{M} \models_X \Sigma \; \text{ then } \; \mathcal{M} \models_X \perp(\overline{x}).$$

### 2.3.3 Deductive system

The deductive system is characterized by the following rules:

($a_2$.) $\perp(\emptyset)$;

($b_2$.) If $\perp(\overline{x}\,\overline{y})$, then $\perp(\overline{x})$;

($c_2$.) If $\perp(\overline{x})$, then $\perp(\pi\overline{x})$ [where $\pi$ is a permutation of $\overline{x}$].

### 2.3.4 Soundness and Completeness

**Theorem 2.3.4.** Let $\Sigma$ be a set of atoms, then

$$\Sigma \models \perp(\overline{x}) \text{ if and only if } \Sigma \vdash \perp(\overline{x}).$$

*Proof.* ($\Leftarrow$) Obvious.

($\Rightarrow$) Suppose $\Sigma \nvdash \perp(\overline{x})$. Notice that if this is the case then $\overline{x} \neq \emptyset$. Indeed if $\overline{x} = \emptyset$ then $\Sigma \vdash \perp(\overline{x})$ because by rule ($a_2$.) $\vdash \perp(\emptyset)$.

We can assume that $\overline{x}$ is injective. This is without loss of generality because clearly $\mathcal{M} \models_X \perp(\overline{x})$ if and only if $\mathcal{M} \models_X \perp(\pi\overline{x})$, where $\pi : \mathrm{Var}^{<\omega} \to \mathrm{Var}^{<\omega}$ is the function that eliminates repetitions in finite sequences of variables. Let then $\overline{x} = (x_{j_0}, ..., x_{j_{n-1}}) \neq \emptyset$ be injective.

Let $\mathcal{M} = \{0, 1\}$. Define $X = \{s_t \mid t \in 2^\omega\}$ to be the set of assignments which give all the possible combinations of 0s and 1s to all the variables but $x_{j_0}$ and which at $x_{j_0}$ are such that

$$s_t(x_{j_0}) = 0 \qquad \text{if } \overline{x} = \{x_{j_0}\}$$

$$s_t(x_{j_0}) = p(s_t(x_{j_1}), ..., s_t(x_{j_{n-1}})) \qquad \text{if } \overline{x} \neq \{x_{j_0}\}$$

for all $t \in 2^\omega$, where $p : M^{<\omega} \to M$ is the function which assigns 1 to the sequences with an odd numbers of 1s and 0 to the sequences with an even numbers of 1s.

We claim that $\mathcal{M} \not\models_X \perp(\overline{x})$. There are two cases.

Case 1. For all $t \in 2^\omega$, $s_t(x_{j_0}) = 0$.

In this case there is not $s, s' \in X$ such that $s(x) \neq s'(x)$.

Case 2. For all $t \in 2^\omega$, $s_t(x_{j_0}) = p(s_t(x_{j_1}), ..., s_t(x_{j_{n-1}}))$.

Notice that if this is the case, then $n \geq 2$. Let $t, d \in 2^\omega$ be such that $s_t(x_{j_1}) = 0$, $s_t(x_{j_1}) = 1$ and $s_t(x_{j_i}) = s_d(x_{j_i})$ for every $i \in \{2, ..., n-1\}$. Clearly

$$p(s_t(x_{j_1}), ..., s_t(x_{j_{n-1}})) \neq p(s_d(x_{j_1}), ..., s_d(x_{j_{n-1}})).$$

Suppose that $\mathcal{M} \models_X \perp(\overline{x})$, then there exists $f \in 2^\omega$ such that

$$s_f(x_{j_0}) = s_t(x_{j_0}) \wedge s_f(\overline{x} -_X x_{j_0}) = s_d(\overline{x} -_X x_{j_0})).$$

Notice that under this $X$ we have that $\overline{x} -_X x_{j_0} = (x_{j_1}, ..., x_{j_n-1})$, thus

$$s_f(\overline{x} -_X x_{j_0}) = (s_f(x_{j_1}), ..., s_f(x_{j_{n-1}}))$$

and

$$s_d(\overline{x} -_X x_{j_0}) = (s_d(x_{j_1}), ..., s_d(x_{j_{n-1}})).$$

Hence

$$\begin{aligned} p(s_d(x_{j_1}), ..., s_d(x_{j_{n-1}})) &= p(s_f(x_{j_1}), ..., s_f(x_{j_{n-1}})) \\ &= s_f(x_{j_0}) \\ &= s_t(x_{j_0}) \\ &= p(s_t(x_{j_1}), ..., s_t(x_{j_{n-1}})), \end{aligned}$$

which is a contradiction.

Let now $\perp(\overline{v}) \in \Sigma$, we want to show that $\mathcal{M} \models_X \perp(\overline{v})$. As before, we assume, without loss of generality, that $\overline{v}$ is injective. Notice that if $\overline{v} = \emptyset$, then $\mathcal{M} \models_X \perp(\overline{v})$. Thus let $\overline{v} = (v_{h_0}, ..., v_{h_{c-1}}) \neq \emptyset$.

We make a case distinction on $\overline{v}$.

**Case 1.** $x_{j_0} \notin \overline{v}$.

Let $v \in \overline{v}$. Because of the assumption, $v \neq x_{j_0}$ and $x_{j_0} \notin \overline{v} -_X v$. Thus for every $t, d \in 2^\omega$ clearly there is $f \in 2^\omega$ such that

$$s_f(v) = s_t(v) \wedge s_f(\overline{v} -_X v) = s_d(\overline{x} -_X v).$$

**Case 2.** $x_{j_0} \in \overline{v}$.

**Subcase 1.** $\overline{x} \setminus \overline{v} \neq \emptyset$.

Notice that $\overline{x} \neq \{x_{j_0}\}$ because if not then $\overline{x} \setminus \overline{v} = \{x_{j_0}\}$ and so $x_{j_0} \notin \overline{v}$. Hence for every $t \in 2^\omega$ we have that

$$s_t(x_{j_0}) = p(s_t(x_{j_1}), ..., s_t(x_{j_{n-1}}).$$

Suppose, without loss of generality, that $\overline{v} = (x_{j_0}, v_{h_1}, ..., v_{h_{c-1}})$ and let $\overline{x}' = \overline{x} \cap \overline{v} = (u_{p_0}, ..., u_{p_{m-1}})$ and $z \in \overline{x} \setminus \overline{v}$. Let $v \in \overline{v}$.

Case 1. $v \neq x_{j_0}$.

Let $k \in \{1, ..., c-1\}$ and $v = v_{h_k}$. Let $t, d \in 2^\omega$ and let $f \in 2^\omega$ be such that:

i) $s_f(v_{h_k}) = s_t(v_{h_k})$;

ii) $s_f(v_{h_i}) = s_d(v_{h_i})$ for every $i \in \{1, ..., k-1, k+1, ..., c-1\}$;

iii) $s_f(u) = 0$ for every $u \in \overline{x} \setminus \overline{x}'z$;

iv) $s_f(z) = 0$, if $p(s_f(u_{p_0}), ..., s_f(u_{p_{m-1}})) = s_d(x_{j_0})$ and $s_f(z) = 1$ otherwise.

Then $f$ is such that

$$s_f(v_{h_k}) = s_t(v_{h_k})$$

and

$$(s_f(x_{j_0}), s_f(v_{h_1}), ..., s_f(v_{h_{k-1}}), s_f(v_{h_{k+1}}), ..., s_f(v_{h_{c-1}}))$$

$$\|$$

$$(s_d(x_{j_0}), s_d(v_{h_1}), ..., s_d(v_{h_{k-1}}), s_d(v_{h_{k+1}}), ..., s_d(v_{h_{c-1}})).$$

Case 2. $v = x_{j_0}$.
Let $t, d \in 2^\omega$ and let $f \in 2^\omega$ be such that:

i) $s_f(v_{h_i}) = s_d(v_{h_i})$ for every $i \in \{1, ..., c-1\}$;

ii) $s_f(u) = 0$ for every $u \in \overline{x} \setminus \overline{x}'\, z$;

iii) $s_f(z) = 0$, if $p(s_f(u_{p_0}), ..., s_f(u_{p_{m-1}})) = s_t(x_{j_0})$ and $s_f(z) = 1$ otherwise.

Then $f$ is such that

$$s_f(x_{j_0}) = s_t(x_{j_0})$$

and

$$(s_f(v_{h_1}), ..., s_f(v_{h_{c-1}})) = (s_d(v_{h_1}), ..., s_d(v_{h_{c-1}})).$$

**Subcase 2.** $\overline{x} \subseteq \overline{v}$.

This case is not possible. Suppose indeed it is, then by rule ($c_2$.) we can assume that $\overline{v} = \overline{x}\ \overline{v}'$ with $\overline{v}' \subseteq \text{Var} \setminus \overline{x}$. Thus by rule ($b_2$.) we have that $\Sigma \vdash \bot(\overline{x})$ which is absurd.

□

## 2.4 Atomic Independence Logic

In this section we define the system Atomic Independence Logic (AIndL).

### 2.4.1 Syntax

The language of this logic is made only of independence atoms. That is, let $\overline{x}$ and $\overline{y}$ be finite sequences of variables then the formula $\overline{x} \perp \overline{y}$ is a formula of the language of AIndL.

### 2.4.2 Semantics

The intuitive meaning of the atom $\overline{x} \perp \overline{y}$ in team semantics is that the values of the variables in $\overline{x}$ give no information about the values of the variables in $\overline{y}$ and vice versa. This is formally expressed via the following condition.

**Definition 2.4.1.** Let $\mathcal{M}$ be a FO structure. Let $X = \{s_i\}_{i \in I}$ with $s_i : \mathrm{dom}(X) \to M$ and $\overline{x}\,\overline{y} \subseteq \mathrm{dom}(X) \subseteq \mathrm{Var}$. We say that $\mathcal{M}$ satisfies $\overline{x} \perp \overline{y}$ under $X$, in symbols $\mathcal{M} \models_X \overline{x} \perp \overline{y}$, if

$$\forall s, s' \in X \; \exists s'' \in X \; (s''(\overline{x}) = s(\overline{x}) \wedge s''(\overline{y}) = s'(\overline{y})).$$

**Definition 2.4.2.** Let $\Sigma$ be a set of atoms and let $X$ be such that the set of variables occurring in $\Sigma$ is included in $\mathrm{dom}(X)$. We say that $\mathcal{M}$ satisfies $\Sigma$ under $X$, in symbols $\mathcal{M} \models_X \Sigma$, if $\mathcal{M}$ satisfies every atom in $\Sigma$ under $X$.

**Definition 2.4.3.** Let $\Sigma$ be a set of atoms. We say that $\overline{x} \perp \overline{y}$ is a logical consequence of $\Sigma$, in symbols $\Sigma \models \overline{x} \perp \overline{y}$, if for every $\mathcal{M}$ and $X$ such that the set of variables occurring in $\Sigma \cup \{\overline{x} \perp \overline{y}\}$ is included in $\mathrm{dom}(X)$ we have that

$$\text{if } \mathcal{M} \models_X \Sigma \text{ then } \mathcal{M} \models_X \overline{x} \perp \overline{y}.$$

### 2.4.3 Deductive system

The deductive system is characterized by the following rules:

($a_3$.) $\overline{x} \perp \emptyset$;

($b_3$.) If $\overline{x} \perp \overline{y}$, then $\overline{y} \perp \overline{x}$;

($c_3$.) If $\overline{x} \perp \overline{y}\,\overline{z}$, then $\overline{x} \perp \overline{y}$;

($d_3$.) If $\overline{x} \perp \overline{y}$ and $\overline{x}\,\overline{y} \perp \overline{z}$, then $\overline{x} \perp \overline{y}\,\overline{z}$;

($e_3$.) If $x \perp x$, then $x \perp \overline{y}$ [for arbitrary $\overline{y}$];

($f_3$.) If $\overline{x} \perp \overline{y}$, then $\pi\overline{x} \perp \sigma\overline{y}$ [where $\pi$ is a permutation of $\overline{x}$ and $\sigma$ is a permutation of $\overline{y}$].

### 2.4.4 Soundness and Completeness

**Theorem 2.4.4.** Let $\Sigma$ be a set of atoms, then

$$\Sigma \models \overline{x} \perp \overline{y} \text{ if and only if } \Sigma \vdash \overline{x} \perp \overline{y}.$$

*Proof.* See [17] and [16].

□

## 2.5 Atomic Absolute Conditional Independence Logic

In this section we define the system Atomic Absolute Conditional Independence Logic (AACIndL).

### 2.5.1 Syntax

The language of this logic is made only of absolute conditional independence atoms. That is, let $\overline{x}$ and $\overline{z}$ be finite sequences of variables, then $\perp_{\overline{z}}(\overline{x})$ is a formula of the language of AACIndL.

### 2.5.2 Semantics

**Definition 2.5.1.** Let $\mathcal{M}$ be a FO structure. Let $X = \{s_i\}_{i \in I}$ with $s_i$ : $\text{dom}(X) \to M$ and $\overline{x}\,\overline{z} \subseteq \text{dom}(X) \subseteq \text{Var}$. Let $x \in \overline{x}$, we denote by $\overline{x} -_X x$ any enumeration of the set $\{x' \in \overline{x} \mid \mathcal{M} \not\models_X x' = x\}$. We say that $\mathcal{M}$ satisfies $\perp_{\overline{z}}(\overline{x})$ under $X$, in symbols $\mathcal{M} \models_X \perp_{\overline{z}}(\overline{x})$, if for all $x \in \overline{x}$

$$\forall s, s' \in X\, (s(\overline{z}) = s'(\overline{z}) \to$$

$$\exists s'' \in X(s''(\overline{z}) = s(\overline{z}) \wedge (s''(x) = s(x) \wedge s''(\overline{x} -_X x) = s'(\overline{x} -_X x)))$$

and

$$\exists s, s' \in X\ (s(\overline{z}) = s'(\overline{z}) \wedge s(x) \neq s'(x)).$$

**Definition 2.5.2.** Let $\Sigma$ be a set of atoms and let $X = \{s_i\}_{i \in I}$ be such that the set of variables occurring in $\Sigma$ is included in $\text{dom}(X)$. We say that $\mathcal{M}$ satisfies $\Sigma$ under $X$, in symbols $\mathcal{M} \models_X \Sigma$, if $\mathcal{M}$ satisfies every atom in $\Sigma$ under $X$.

**Definition 2.5.3.** Let $\Sigma$ be a set of atoms. We say that $\perp_{\overline{z}}(\overline{x})$ is a logical consequence of $\Sigma$, in symbols $\Sigma \models \perp_{\overline{z}}(\overline{x})$, if for every $\mathcal{M}$ and $X$ such that the set of variables occurring in $\Sigma \cup \{\perp_{\overline{z}}(\overline{x})\}$ is included in $\text{dom}(X)$ we have that

$$\text{if } \ \mathcal{M} \models_X \Sigma \ \text{ then } \ \mathcal{M} \models_X \perp_{\overline{z}}(\overline{x}).$$

### 2.5.3 Deductive system

The deductive system is characterized by the following rules:

($a_4$.) $\perp_{\overline{z}}(\emptyset)$;

($b_4$.) If $\perp_{\overline{z}}(\overline{x}\,\overline{y})$, then $\perp_{\overline{z}}(\overline{x})$;

($c_4$.) If $\perp_{\overline{z},\overline{u}}(\overline{x})$, then $\perp_{\overline{z}}(\overline{x})$;

($d_4$.) If $\perp_{\overline{z}}(\overline{x})$ and $\perp_{\overline{x},\overline{z}}(\overline{u})$, then $\perp_{\overline{z},\overline{u}}(\overline{x})$;

($e_4$.) If $\perp_{\overline{x}}(\overline{x})$, then $\perp_{\overline{z}}(\overline{y})$ [for arbitrary $\overline{z}$ and $\overline{y}$];

($f_4$.) If $\perp_{\overline{z}}(\overline{x})$, then $\perp_{\pi\overline{z}}(\sigma\overline{x})$ [where $\pi$ is a permutation of $\overline{z}$ and $\sigma$ is a permutation of $\overline{x}$].

### 2.5.4 Soundness

**Theorem 2.5.4.** Let $\Sigma$ be a set of atoms, then

$$\Sigma \vdash \perp_{\overline{z}}(\overline{x}) \quad \Rightarrow \quad \Sigma \models \perp_{\overline{z}}(\overline{x}).$$

□

It is at present not known whether a completeness theorem holds for this system, but in the light of what is known about conditional independence in database theory it seems that this is not the case, at least with respect to a *finite* set of axioms.

Although a finite axiomatization is unlikely, it may still be possible that the system admits a recursively enumerable axiomatization, as it happens for the system Atomic Conditional Independence Logic.

## 2.6 Atomic Conditional Independence Logic

In this section we define the system Atomic Conditional Independence Logic (ACIndL).

### 2.6.1 Syntax

The language of this logic is made only of conditional independence atoms. That is, let $\overline{x}$, $\overline{y}$ and $\overline{z}$ be finite sequences of variables then the formula $\overline{x} \perp_{\overline{z}} \overline{y}$ is a formula of the language of ACIndL.

### 2.6.2 Semantics

**Definition 2.6.1.** Let $\mathcal{M}$ be a FO structure. Let $X = \{s_i\}_{i \in I}$ with $s_i$ : $\mathrm{dom}(X) \to M$ and $\overline{x}\,\overline{y}\,\overline{z} \subseteq \mathrm{dom}(X) \subseteq \mathrm{Var}$. We say that $\mathcal{M}$ satisfies $\overline{x} \perp_{\overline{z}} \overline{y}$ under $X$, in symbols $\mathcal{M} \models_X \overline{x} \perp_{\overline{z}} \overline{y}$, if

$$\forall s, s' \in X(s(\overline{z}) = s'(\overline{z}) \to \exists s'' \in X(s''(\overline{z}) = s(\overline{z}) \wedge s''(\overline{x}) = s(\overline{x}) \wedge s''(\overline{y}) = s'(\overline{y}))).$$

**Definition 2.6.2.** Let $\Sigma$ be a set of atoms and let $X$ be such that the set of variables occurring in $\Sigma$ is included in $\mathrm{dom}(X)$. We say that $\mathcal{M}$ satisfies $\Sigma$ under $X$, in symbols $\mathcal{M} \models_X \Sigma$, if $\mathcal{M}$ satisfies every atom in $\Sigma$ under $X$.

**Definition 2.6.3.** Let $\Sigma$ be a set of atoms. We say that $\overline{x} \perp_{\overline{z}} \overline{y}$ is a logical consequence of $\Sigma$, in symbols $\Sigma \models \overline{x} \perp_{\overline{z}} \overline{y}$, if for every $\mathcal{M}$ and $X$ such that the set of variables occurring in $\Sigma \cup \{\overline{x} \perp_{\overline{z}} \overline{y}\}$ is included in $\mathrm{dom}(X)$ we have that

$$\text{if } \mathcal{M} \models_X \Sigma \text{ then } \mathcal{M} \models_X \overline{x} \perp_{\overline{z}} \overline{y}.$$

### 2.6.3 Deductive system

The deductive system is characterized by the following rules:

($a_5$.) $\overline{x} \perp_{\overline{x}} \overline{y}$;

($b_5$.) If $\overline{x} \perp_{\overline{z}} \overline{y}$, then $\overline{y} \perp_{\overline{z}} \overline{x}$;

($c_5$.) If $\overline{x}\,\overline{x}' \perp_{\overline{z}} \overline{y}\,\overline{y}'$, then $\overline{x} \perp_{\overline{z}} \overline{y}$;

($d_5$.) If $\overline{x} \perp_{\overline{z}} \overline{y}$, then $\overline{x}\,\overline{z} \perp_{\overline{z}} \overline{y}\,\overline{z}$;

($e_5$.) If $\overline{x} \perp_{\overline{z}} \overline{y}$ and $\overline{u} \perp_{\overline{z},\overline{x}} \overline{y}$, then $\overline{u} \perp_{\overline{z}} \overline{y}$;

($f_5$.) If $\overline{y} \perp_{\overline{z}} \overline{y}$ and $\overline{z}\,\overline{x} \perp_{\overline{y}} \overline{u}$, then $\overline{x} \perp_{\overline{z}} \overline{u}$;

(g$_5$.) If $\overline{x} \perp_{\overline{z}} \overline{y}$ and $\overline{x}\,\overline{y} \perp_{\overline{z}} \overline{u}$, then $\overline{x} \perp_{\overline{z}} \overline{y}\,\overline{u}$;

(h$_5$.) If $\overline{x} \perp_{\overline{z}} \overline{y}$, then $\pi\overline{x} \perp_{\tau\overline{z}} \sigma\overline{y}$ [where $\pi$ is a permutation of $\overline{x}$, $\tau$ is a permutation of $\overline{z}$ and $\sigma$ is a permutation of $\overline{y}$].

### 2.6.4 Soundness

**Theorem 2.6.4.** Let $\Sigma$ be a set of atoms, then

$$\Sigma \vdash \ \overline{x} \perp_{\overline{z}} \overline{y} \quad \Rightarrow \quad \Sigma \models \ \overline{x} \perp_{\overline{z}} \overline{y}.$$

□

# Chapter 3

# Vector Space Interpretation of Atomic Dependence and Independence Logic

Let $\mathcal{M} = \{0, 1\}$ and consider the set of assignments $X = \{s_0, s_1\}$ where $\mathrm{dom}(X) = \{x\}$ and $s_0(x) = 0$ and $s_1(x) = 1$. The set $X$ can be naturally seen as made of one function with codomain $\{0, 1\}^2$ instead of two functions with codomain $\{0, 1\}$ by imposing $s(x)(i) = s_i(x)$ for $i \in \{0, 1\}$. If we now think of the set $\{0, 1\}$ as the domain of the two element field, then $X$ can be seen as the singleton that has as only element the vector $(0, 1)$.

This identification between sets of assignments and vectors can be made for any $X$ and $\mathcal{M}$ by taking a field with at least as many elements as the union of the images in $M$ of the assignments in $X$ and the power of this field of the same cardinality of $X$.

Linear algebra offers us a well established notion of dependence and independence, namely linear dependence and independence. We can then wonder what happens if we define a semantics in which the abstract conditions that we used in Chapter 2 are replaced by these notions. Are these systems meaningful? Are they sound? Are they complete? In this chapter we answer these questions.

The present chapter should be considered more as an algebraic analysis of the atomic fragment of Dependence and Independence Logic than as a logical analysis of the dependence and independence concepts arising from linear algebra. We include it here because it introduces ideas that are useful to our analysis and represents a link between the study of abstract atomic dependence and independence systems and the study of algebraic and model-theoretic ones.

The proofs of Theorem 3.2.5 and Theorem 3.4.4 are adaptations to our framework of the completeness proofs in [13] and [17] respectively.

The systems that we are going to study are: External Vector space Atomic Equational Logic, External Vector space Atomic Dependence Logic, External Vector Spaces Atomic Absolute Independence Logic and External Vector space Atomic Independence Logic.

## 3.1 External Vector Space Atomic Equational Logic

In this section we define the system External Vector Space Atomic Equational Logic (EVSAEL) and then prove its soundness and completeness. The syntax and deductive apparatus of this system are the same as those of AEL.

### 3.1.1 Semantics

**Definition 3.1.1.** Let $\mathbb{K}$ be a field. Let $X = \{s_i\}_{i \in I}$ with $s_i : \mathrm{dom}(X) \to K$ and $\overline{x}\,\overline{y} \subseteq \mathrm{dom}(X) \subseteq \mathrm{Var}$. For $z \in \mathrm{dom}(X)$ we let $s(z)$ be the element of $K^I$ such that $s(z)(i) = s_i(z)$ for every $i \in I$.

Let $\overline{x} = (x_{j_0}, ..., x_{j_{n-1}})$ and $\overline{y} = (y_{k_0}, ..., y_{k_{n-1}})$, we say that the vector space $\mathbb{K}^I$ over $\mathbb{K}$ satisfies $\overline{x} = \overline{y}$ under $X$, in symbols $\mathbb{K}^I \models_X \overline{x} = \overline{y}$, if for every $i \in \{0, ..., n-1\}$ we have that $s(x_{j_i}) = s(y_{k_i})$.

**Definition 3.1.2.** Let $\Sigma$ be a set of atoms and let $X = \{s_i\}_{i \in I}$ be such that the set of variables occurring in $\Sigma$ is included in $\mathrm{dom}(X)$. We say that $\mathbb{K}^I$ satisfies $\Sigma$ under $X$, in symbols $\mathbb{K}^I \models_X \Sigma$, if $\mathbb{K}^I$ satisfies every atom in $\Sigma$ under $X$.

**Definition 3.1.3.** Let $\Sigma$ be a set of atoms. We say that $\overline{x} = \overline{y}$ is a logical consequence of $\Sigma$, in symbols $\Sigma \models \overline{x} = \overline{y}$, if for every field $\mathbb{K}$ and $X = \{s_i\}_{i \in I}$ such that the set of variables occurring in $\Sigma \cup \{\overline{x} = \overline{y}\}$ is included in $\mathrm{dom}(X)$ we have that

$$\text{if } \mathbb{K}^I \models_X \Sigma \text{ then } \mathbb{K}^I \models_X \overline{x} = \overline{y}.$$

### 3.1.2 Soundness and Completeness

**Theorem 3.1.4.** Let $\Sigma$ be a set of atoms, then

$$\Sigma \models \overline{x} = \overline{y} \text{ if and only if } \Sigma \vdash \overline{x} = \overline{y}.$$

[The deductive system to which we refer has been defined in Section 2.1.3.]

*Proof.* ($\Leftarrow$) Obvious.

($\Rightarrow$) Suppose $\Sigma \nvdash \overline{x} = \overline{y}$. Notice that $\overline{x}, \overline{y} \neq \emptyset$, indeed if not so then, by the admitted degenerate case of rule ($a_0$.), we would have that $\Sigma \vdash \overline{x} = \overline{y}$. Let $\overline{x} = (x_{j_0}, ..., x_{j_{n-1}})$ and $\overline{y} = (y_{k_0}, ..., y_{k_{n-1}})$. Define $V_i = \{z \in \mathrm{Var} \mid \Sigma \vdash x_{j_i} = z\}$, $V = \bigcup_{i=0}^{n-1} V_i$ and $W = \mathrm{Var} \setminus V$.

Let $\mathbb{Q}$ be the field of rational numbers and $\mathbb{Q}^1$ the one dimensional vector space over it. Let $\{p_0, ..., p_{n-1}\}$ be an enumeration of the first $n$ prime numbers. Let $X = \{s_0\}$ where $s_0 : \mathrm{Var} \to Q$ is such that

$$s(v) = \begin{cases} 1 & \text{if } v \in W \\ p_{a_0} \cdots p_{a_{q-1}} & \text{if } v \in V \end{cases}$$

where $\{a_0, ..., a_{q-1}\} \subseteq \{0, ..., n-1\}$, $v \in V_{a_0}$, $v \in V_{a_1}$, $\ldots$ , $v \in V_{a_{q-1}}$ and $a_0 < ... < a_{q-1}$.

We claim that $\mathbb{Q}^1 \not\models_X \overline{x} = \overline{y}$. Suppose on the contrary that for every $i \in \{0, ..., n-1\}$ we have that $s(x_{j_i}) = s(y_{k_i})$. Let $i \in \{0, ..., n-1\}$. By rule ($a_0$.), $\Sigma \vdash x_{j_i} = x_{j_i}$ and so

$$s(x_{j_i}) = p_{a_0} \cdots p_i \cdots p_{a_{t-1}}.$$

Thus

$$s(y_{k_i}) = s(x_{j_i}) = p_{a_0} \cdots p_i \cdots p_{a_{t-1}}.$$

But if this is the case then for every $i \in \{0, ..., n-1\}$ we have that $\Sigma \vdash x_{j_i} = y_{k_i}$ and so, by an iterated application of rule (d$_0$.), $\Sigma \vdash \overline{x} = \overline{y}$, which is absurd.

Let now $\overline{x}' = \overline{y}' \in \Sigma$, with $\overline{x}' = x_{h_0}, ..., x_{h_{r-1}}$ and $\overline{y}' = y_{m_0}, ..., y_{m_{r-1}}$. We want to show that $\mathbb{Q}^1 \models_X \overline{x}' = \overline{y}'$.

If $\overline{y}' = \emptyset$ then $\overline{x}' = \emptyset$, so trivially $\mathbb{Q}^1 \models_X \overline{x}' = \overline{y}'$. Noticed this, for the rest of the proof we assume $\overline{y}' \neq \emptyset$. Let now $x_{h_p} \in \overline{x}'$

**Case 1.** $x_{h_p} \in V$.

Let $i \in \{0, ..., n-1\}$, we show that

$$x_{h_p} \in V_i \iff y_{m_p} \in V_i,$$

from this it follows that $s(x_{h_p}) = s(y_{m_p})$.

Let $x_{h_p} \in V_i$, then $\Sigma \vdash x_{j_i} = x_{h_p}$. Now by rule (e$_0$.) $\Sigma \vdash x_{h_p} = y_{m_p}$, so by rule (c$_0$.) $\Sigma \vdash x_{j_i} = y_{m_p}$ and hence $y_{m_p} \in V_i$.

Let $y_{m_p} \in V_i$, then $\Sigma \vdash x_{j_i} = y_{m_p}$. Now by rule (e$_0$.) $\Sigma \vdash x_{h_p} = y_{m_p}$, so by rule (b$_0$.) $\Sigma \vdash y_{m_p} = x_{h_p}$ and then by rule (c$_0$.) $\Sigma \vdash x_{j_i} = x_{h_p}$, thus $x_{h_p} \in V_i$.

**Case 2.** $x_{h_p} \in W$.

Suppose $s(y_{m_p}) \neq 1$, then there exists $i \in \{0, ..., n-1\}$ such that $y_{m_p} \in V_i$, that is $\Sigma \vdash x_{j_i} = y_{m_p}$. Now by rule (e$_0$.) $\Sigma \vdash x_{h_p} = y_{m_p}$, so by rule (b$_0$.) $\Sigma \vdash y_{m_p} = x_{h_p}$ and then by rule (c$_0$.) $\Sigma \vdash x_{j_i} = x_{h_p}$. Thus $x_{h_p} \in V$, which is absurd. So $s(x_{h_p}) = 1 = s(y_{m_p})$.

We showed that for every $p \in \{0, ..., r-1\}$ we have that $s(x_{h_p}) = s(y_{m_p})$, hence $\mathbb{Q}^1 \models_X \overline{x}' = \overline{y}'$. □

## 3.2 External Vector Space Atomic Dependence Logic

In this section we define the system External Vector Space Atomic Dependence Logic (EVSADL) and then prove its soundness and completeness. The syntax and deductive apparatus of this system are the same as those of ADL.

### 3.2.1 Semantics

The intuitive meaning of $=\!(\overline{x}, \overline{y})$ in the context of vector spaces is that each vector in $\overline{y}$ is a linear function of the vectors in $\overline{x}$. In order to formalize our intuitions we introduce the notion of span of a subset of a vector space.

**Proposition 3.2.1.** Let $\mathbb{K}$ be a field, $\mathbb{V}$ a vector space over $\mathbb{K}$ , $A \subseteq V$ and $\mathbb{W}$ a subspace of $\mathbb{V}$ containing $A$. The following are equivalent:

i) $\mathbb{W}$ is the smallest subspace of $\mathbb{V}$ containing $A$;

ii) $\mathbb{W}$ is the intersection of all the subspaces of $\mathbb{V}$ containing $A$;

iii) $w \in W$ if and only if there exists $\overline{a} \in A^n$ and $\overline{c} \in K^n$ such that

$$w = \sum_{i=0}^{n-1} c_i a_i.$$

*Proof.* See [9, Proposition 2.4].

□

Let $\mathbb{K}$ be a field, $\mathbb{V}$ a vector space over $\mathbb{K}$ and $A \subseteq V$. We denote by $\langle A \rangle$ the smallest subspace of $\mathbb{V}$ containing $A$ and refer to it as the subspace of $\mathbb{V}$ spanned by $A$.

**Definition 3.2.2.** Let $\mathbb{K}$ be a field. Let $X = \{s_i\}_{i \in I}$ with $s_i : \mathrm{dom}(X) \to K$ and $\overline{x}\,\overline{y} \subseteq \mathrm{dom}(X) \subseteq \mathrm{Var}$. For $z \in \mathrm{dom}(X)$ we let $s(z)$ be the element of $K^I$ such that $s(z)(i) = s_i(z)$ for every $i \in I$.

Let $\overline{x} = (x_{j_0}, ..., x_{j_{n-1}})$, we say that the vector space $\mathbb{K}^I$ over $\mathbb{K}$ satisfies $=\!(\overline{x}, \overline{y})$ under $X$, in symbols $\mathbb{K}^I \models_X =\!(\overline{x}, \overline{y})$, if for every $y \in \overline{y}$ we have that $s(y) \in \langle \{s(x) \mid x \in \overline{x}\} \rangle$, that is there exists $\overline{a} \in K^n$ such that

$$\sum_{i=0}^{n-1} a_i s(x_{j_i}) = s(y).$$

[Notice that under this formulation the case $\overline{x} = \emptyset$ and $\overline{x}, \overline{y} = \emptyset$ are taken care of, in the first case indeed we have that $\sum_{i=0}^{n-1} a_i s(x_{j_i}) = s(y)$ becomes just $s(y) = 0$, while in the second the condition is always trivially satisfied.]

**Definition 3.2.3.** Let $\Sigma$ be a set of atoms and let $X = \{s_i\}_{i \in I}$ be such that the set of variables occurring in $\Sigma$ is included in $\mathrm{dom}(X)$. We say that $\mathbb{K}^I$ satisfies $\Sigma$ under $X$, in symbols $\mathbb{K}^I \models_X \Sigma$, if $\mathbb{K}^I$ satisfies every atom in $\Sigma$ under $X$.

**Definition 3.2.4.** Let $\Sigma$ be a set of atoms. We say that $=\!(\overline{x}, \overline{y})$ is a logical consequence of $\Sigma$, in symbols $\Sigma \models =\!(\overline{x}, \overline{y})$, if for every field $\mathbb{K}$ and $X = \{s_i\}_{i \in I}$ such that the set of variables occurring in $\Sigma \cup \{=\!(\overline{x}, \overline{y})\}$ is included in $\mathrm{dom}(X)$ we have that

$$\text{if } \mathbb{K}^I \models_X \Sigma \text{ then } \mathbb{K}^I \models_X =\!(\overline{x}, \overline{y}).$$

### 3.2.2 Soundness and Completeness

**Theorem 3.2.5.** Let $\Sigma$ be a set of atoms, then

$$\Sigma \models =\!(\overline{x}, \overline{y}) \text{ if and only if } \Sigma \vdash =\!(\overline{x}, \overline{y}).$$

[The deductive system to which we refer has been defined in Section 2.2.3.]

*Proof.* ($\Leftarrow$) Easy exercise of linear algebra.

($\Rightarrow$) We adapt the proofs of [2] and [13] to our framework.

Suppose $\Sigma \nvdash =\!(\overline{x}, \overline{y})$. Let $V = \{z \in \mathrm{Var} \mid \Sigma \vdash =\!(\overline{x}, z)\}$ and $W = \mathrm{Var} \setminus V$. First notice that $\overline{y} \neq \emptyset$, indeed if not so, by the syntactic constraints that we put on the system, we have that $\overline{x}, \overline{y} = \emptyset$ and so by the admitted degenerate case of rule ($a_1$.) we have that $\Sigma \vdash =\!(\overline{x}, \overline{y})$. Furthermore $\overline{y} \cap W \neq \emptyset$, indeed if $\overline{y} \cap W = \emptyset$ then for every $y \in \overline{y}$ we have that $\Sigma \vdash =\!(\overline{x}, y)$ and so by rules ($d_1$.), ($e_1$.) and, if necessary, ($f_1$.)[1] we have that $\Sigma \vdash =\!(\overline{x}, \overline{y})$.

Let $\mathbb{K}$ be the two elements field and $\mathbb{K}^2$ the two dimensional vector space over it. Let $X = \{s_0, s_1\}$ where $s_0, s_1 : \mathrm{Var} \to K$, $s_0(v) = 0$ for all $v \in \mathrm{Var}$, $s_1(v) = 0$ if $v \in V$ and $s_1(v) = 1$ if $v \in W$.

[1]Notice that ($f_1$.) is necessary only if $\overline{x}$ or $\overline{y}$ contain repetitions.

We claim that $\mathbb{K}^2 \not\models_X =(\overline{x}, \overline{y})$. In accordance to the semantics we then have to show that there is $y \in \overline{y}$ such that $s(y) \notin \langle \{s(x) \mid x \in \overline{x}\} \rangle$. Let $y \in \overline{y} \cap W$ and $\overline{x} = (x_{j_0}, ..., x_{j_{n-1}}) \neq \emptyset$, we then have that $s(y) = (s_0(y), s_1(y)) = (0, 1)$ but for every $\overline{a} \in K^n$

$$\sum_{i=0}^{n-1} a_i s(x_{j_i}) = \sum_{i=0}^{n-1} a_i (s_0(x_{j_i}), s_1(x_{j_i})) = (0, 0)$$

because for $x \in \overline{x}$ we have that $\Sigma \vdash =(\overline{x}, x)$. Indeed by rule (a$_1$.) $\vdash =(\overline{x}, \overline{x})$ and so by rule (b$_1$.) $\vdash =(\overline{x}, x)$. Notice finally that if $\overline{x} = \emptyset$ then for every $\overline{a} \in K^n$ we have that

$$\sum_{i=0}^{n-1} a_i s(x_{j_i}) + s(y) = s(y) = (0, 1) \neq (0, 0).$$

Let now $=(\overline{x}', \overline{y}') \in \Sigma$, we want to show that $\mathbb{K}^2 \models_X =(\overline{x}', \overline{y}')$. If $\overline{y}' = \emptyset$ then also $\overline{x}' = \emptyset$ and so trivially $\mathbb{K}^2 \models_X =(\overline{x}', \overline{y}')$. Noticed this, for the rest of the proof we assume $\overline{y}' \neq \emptyset$.

**Case 1.** $\overline{x}' = \emptyset$.

Suppose that $\mathbb{K}^2 \not\models_X =(\emptyset, \overline{y}')$, then there exists $y' \in \overline{y}'$ such that $s(y') \neq (0, 0)$, so $s_1(y') = 1$ which means that $\Sigma \nvdash =(\overline{x}, y')$. Notice though that $\Sigma \vdash =(\emptyset, \overline{y}')$, so by rule (b$_1$.) $\Sigma \vdash =(\emptyset, y')$ and hence again by rule (b$_1$.) $\Sigma \vdash =(\overline{x}, y')$. Thus $\mathbb{K}^2 \models_X =(\overline{x}', \overline{y}')$.

**Case 2.** $\overline{x}' \neq \emptyset$ and $\overline{x}' \subseteq V$.

If this is the case, then

$$\begin{aligned} \forall x' \in \overline{x}' \;\; \Sigma \vdash =(\overline{x}, x') &\implies \Sigma \vdash =(\overline{x}, \overline{x}') \;\; \text{[by rules (d}_1\text{.), (e}_1\text{.) and (f}_1\text{.)]} \\ &\implies \Sigma \vdash =(\overline{x}, \overline{y}') \;\; \text{[by rule (c}_1\text{.)]} \\ &\implies \forall y' \in \overline{y}' \;\; \Sigma \vdash =(\overline{x}, y') \;\; \text{[by rule (b}_1\text{.)]} \\ &\implies \overline{y}' \subseteq V. \end{aligned}$$

Let $y' \in \overline{y}'$, then for any $x' \in \overline{x}'$ we have that $s(y') = (s_0(y'), s_1(y')) = (0, 0) = (s_0(x'), s_1(x')) = s(x')$. Hence $\mathbb{K}^2 \models_X =(\overline{x}', \overline{y}')$.

**Case 3.** $\overline{x}' \cap W \neq \emptyset$.

If this is the case, then there exists $x' \in \overline{x}'$ such that $\Sigma \nvdash =(\overline{x}, x')$, so we have $x' \in \overline{x}'$ such that $s(x') = (s_0(x'), s_1(x')) = (0, 1)$. Let now $y' \in \overline{y}'$.

**Subcase 1**. $y' \in W$.

In this case we have that $s_0(y') = 0$ and $s_1(y') = 1$, so $s(y') = (s_0(y'), s_1(y')) = (0, 1) = (s_0(x'), s_1(x')) = s(x')$. Hence $\mathbb{K}^2 \models_X =(\overline{x}', \overline{y}')$.

**Subcase 2**. $y' \in V$.

In this case we have that $s_0(y') = 0$ and $s_1(y') = 0$, but $0(s_0(x'), s_1(x')) = (0, 0) = (s_0(y'), s_1(y')) = s(y')$. Hence $\mathbb{K}^2 \models_X =(\overline{x}', \overline{y}')$.

□

## 3.3 External Vector Space Atomic Absolute Independence Logic

In this section we define the system External Vector Space Atomic Absolute Independence Logic (EVSAAIndL) and then prove its soundness and completeness. The syntax and deductive apparatus of this system are the same as those of AAIndL.

### 3.3.1 Semantics

The atom $\perp(\overline{x})$ has a natural interpretation in this context: the elements of $\overline{x}$ are linearly independent vectors. In the following proposition we see two equivalent formulations of linear independence.

**Proposition 3.3.1.** Let $\mathbb{K}$ be a field, $\mathbb{V}$ be a vector space over $\mathbb{K}$ and $W = \{w_0, ..., w_{n-1}\} \subseteq V$. The following are equivalent.

i) $W$ is linearly independent, that is for every $\overline{a} \in \mathbb{K}^n$ we have that

$$\sum_{i=0}^{n-1} a_i w_i = 0 \text{ if and only if } a_i = 0 \text{ for every } i \in \{0, ..., n-1\}.$$

ii) For every $w \in W$ we have that $w \notin \langle W \setminus \{w\} \rangle$.

□

**Definition 3.3.2.** Let $\mathbb{K}$ be a field. Let $X = \{s_i\}_{i \in I}$ with $s_i : \mathrm{dom}(X) \to K$ and $\overline{x} \subseteq \mathrm{dom}(X) \subseteq \mathrm{Var}$. For $z \in \mathrm{dom}(X)$ we let $s(z)$ be the element of $K^I$ such that $s(z)(i) = s_i(z)$ for every $i \in I$.

We say that the vector space $\mathbb{K}^I$ over $\mathbb{K}$ satisfies $\perp(\overline{x})$ under $X$, in symbols $\mathbb{K}^I \models_X \perp(\overline{x})$, if for every $x \in \overline{x}$ we have that

$$s(x) \notin \langle \{s(z) \mid z \in \overline{x}\} \setminus \{s(x)\} \rangle.$$

**Definition 3.3.3.** Let $\Sigma$ be a set of atoms and let $X = \{s_i\}_{i \in I}$ be such that the set of variables occurring in $\Sigma$ is included in $\mathrm{dom}(X)$. We say that $\mathbb{K}^I$ satisfies $\Sigma$ under $X$, in symbols $\mathbb{K}^I \models_X \Sigma$, if $\mathbb{K}^I$ satisfies every atom in $\Sigma$ under $X$.

**Definition 3.3.4.** Let $\Sigma$ be a set of atoms. We say that $\perp(\overline{x})$ is a logical consequence of $\Sigma$, in symbols $\Sigma \models \perp(\overline{x})$, if for every field $\mathbb{K}$ and $X = \{s_i\}_{i \in I}$ such that the set of variables occurring in $\Sigma \cup \{\perp(\overline{x})\}$ is included in $\mathrm{dom}(X)$ we have that

$$\text{if } \mathbb{K}^I \models_X \Sigma \text{ then } \mathbb{K}^I \models_X \perp(\overline{x}).$$

### 3.3.2 Soundness and Completeness

**Theorem 3.3.5.** Let $\Sigma$ be a set of atoms, then

$$\Sigma \models \perp(\overline{x}) \text{ if and only if } \Sigma \vdash \perp(\overline{x}).$$

[The deductive system to which we refer has been defined in Section 2.3.3.]

*Proof.* ($\Leftarrow$) Obvious.

($\Rightarrow$) Suppose $\Sigma \nvdash \perp(\overline{x})$. Notice that if this is the case then $\overline{x} \neq \emptyset$. Indeed if $\overline{x} = \emptyset$ then $\Sigma \vdash \perp(\overline{x})$ because by rule ($a_2$.) $\vdash \perp(\emptyset)$.

We can assume that $\overline{x}$ is injective. This is without loss of generality because clearly $\mathbb{K}^I \models_X \perp(\overline{x})$ if and only if $\mathbb{K}^I \models_X \perp(\pi\overline{x})$, where $\pi : \mathrm{Var}^{<\omega} \to \mathrm{Var}^{<\omega}$ is the function that eliminates repetitions in finite sequences of variables. Let then $\overline{x} = (x_{j_0}, ..., x_{j_{n-1}}) \neq \emptyset$ be injective.

Let $\mathbb{K}$ be the two elements field. We identify the domain of $\mathbb{K}$ with the set 2. Define $X = \{s_t \mid t \in 2^\omega\}$ to be the set of assignments which give all the possible combinations of 0s and 1s to all the variables but $x_{j_0}$ and which at $x_{j_0}$ are such that

$$s_t(x_{j_0}) = 0 \qquad \text{if } \overline{x} = \{x_{j_0}\}$$

$$s_t(x_{j_0}) = p(s_t(x_{j_1}), ..., s_t(x_{j_{n-1}})) \qquad \text{if } \overline{x} \neq \{x_{j_0}\}$$

for all $t \in 2^\omega$, where $p : K^{<\omega} \to K$ is the function which assigns 1 to the sequences with an odd numbers of 1s and 0 to the sequences with an even numbers of 1s.

We claim that $\mathbb{K}^{(2^\omega)} \not\models_X \perp(\overline{x})$. Indeed either

$$s(x_{j_0}) = 0 \in \langle \emptyset \rangle = \langle \{s(x) \mid x \in \overline{x}\} \setminus \{s(x_{j_0})\} \rangle$$

or

$$s(x_{j_0}) = \sum_{i=1}^{n-1} s(x_{j_i}) \in \langle \{s(x) \mid x \in \overline{x}\} \setminus \{s(x_{j_0})\} \rangle$$

because for every $t \in 2^\omega$

$$\begin{aligned} s(x_{j_0})(t) = 0 &\iff s_t(x_{j_0}) = 0 \\ &\iff p(s_t(x_{j_1}), ..., s_t(x_{j_{n-1}})) = 0 \\ &\iff p(s(x_{j_1})(t), ..., s(x_{j_{n-1}})(t)) = 0 \\ &\iff (\sum_{i=1}^{n-1} s(x_{j_i})(t)) = 0 \\ &\iff (\sum_{i=1}^{n-1} s(x_{j_i}))(t) = 0 \end{aligned}$$

and similarly for $s(x_{j_0})(t) = 1$. [To justify the second to last passage of the series of equivalences above just notice that in $\mathbb{K}$ we have that $1 + 1 = 0$].

Let now $\perp(\overline{v}) \in \Sigma$, we want to show that $\mathbb{K}^{(2^\omega)} \models_X \perp(\overline{v})$. As before, we assume, without loss of generality, that $\overline{v}$ is injective. Notice that if $\overline{v} = \emptyset$, then $\mathbb{K}^{(2^\omega)} \models_X \perp(\overline{v})$. Thus let $\overline{v} = (v_{h_0}, ..., v_{h_{c-1}}) \neq \emptyset$. We prove a lemma.

**Lemma 3.3.6.** If $v \in \mathrm{Var} \setminus \{x_{j_0}\}$ and $Z = \{z_{d_0}, ..., z_{d_{q-1}}\} \subseteq \mathrm{Var}$ with $v \notin Z$, then for every $\overline{a} \in K^q$ we have

$$s(v) \neq \sum_{i=0}^{q-1} a_i s(z_{d_i}).$$

*Proof.* If $Z = \emptyset$ then we are done because for every $v \in \mathrm{Var} \setminus \{x_{j_0}\}$ we clearly have that $s(v) \neq 0$. Let then $Z \neq \emptyset$. Consider first the case $x_{j_0} \notin Z$. Suppose that there exists $\overline{a} \in K^q$ such that

$$s(v) = \sum_{i=0}^{q-1} a_i s(z_{d_i}).$$

Let $P = \{i \in \{0, ..., q-1\} \mid a_i = 0\}$, then

$$s(v) = \sum_{\substack{i=0 \\ i \notin P}}^{q-1} s(z_{d_i}).$$

Let $t \in 2^\omega$ such that $s(v)(t) = 1$ and $s(z_{d_i})(t) = 0$ for every $i \in \{0, ..., q-1\}$, then

$$s(v)(t) = 1 = 0 = (\sum_{\substack{i=0 \\ i \notin P}}^{q-1} s(z_{d_i}))(t).$$

which is absurd.

Consider now the case $x_{j_0} \in Z$, then there exists $i^* \in \{0, ..., q-1\}$ such that $x_{j_0} = z_{d_{i^*}}$. Suppose that there exists $\overline{a} \in K^q$ such that

$$s(v) = \sum_{\substack{i=0 \\ i \neq i^*}}^{q-1} a_i s(z_{d_i}) + a_{i^*} s(x_{j_0}).$$

If $a_{i^*} = 0$ then we are in the same situation as the case above and so we are done. If $a_{i^*} \neq 0$ then

$$s(v) = \sum_{\substack{i=0 \\ i \neq i^* \\ i \notin P}}^{q-1} s(z_{d_i}) + s(x_{j_0}).$$

where $P$ is as above.

Let $t \in 2^\omega$ such that $s(v)(t) = 1$, $s(z_{d_i})(t) = 0$ for every $i \in \{0, ..., q-1\} \setminus \{i^*\}$ and $s(u)(t) = 0$ for every $u \in \overline{x} \setminus \{x_{j_0}\}$, then $s(x_{j_0}) = 0$ and so

$$s(v)(t) = 1 = 0 = (\sum_{\substack{i=0 \\ i \neq i^* \\ i \notin P}}^{q-1} s(z_{d_i}) + s(x_{j_0}))(t).$$

which is absurd.

□

We now make a case distinction on $\overline{v}$.

**Case 1.** $x_{j_0} \notin \overline{v}$.

The fact that $\mathbb{K}^{(2^\omega)} \models_X \perp(\overline{v})$ follows directly from the lemma. Indeed let $v \in \overline{v}$ and $Z = \{x \in \text{Var} \mid x \in \overline{v} \text{ and } x \neq v\}$, then by the lemma we have that

$$v \notin \langle \{s(z) \mid z \in Z\} \rangle \supseteq \langle \{s(z) \mid z \in \overline{v}\} \setminus \{s(v)\} \rangle.$$

**Case 2.** $x_{j_0} \in \overline{v}$.

**Subcase 1.** $\overline{x} \setminus \overline{v} \neq \emptyset$.

Notice that $\overline{x} \neq \{x_{j_0}\}$ because if not then $\overline{x} \setminus \overline{v} = \{x_{j_0}\}$ and so $x_{j_0} \notin \overline{v}$. Hence for every $d \in 2^\omega$

$$s_d(x_{j_0}) = p(s_d(x_{j_1}), ..., s_d(x_{j_{n-1}})).$$

If $v \in \overline{v}$ and $v \neq x_{j_0}$, then by the lemma we have what we want. We are then only left to consider the case $v = x_{j_0}$.

Let $\overline{x}' = \overline{x} \cap \overline{v}$, $\overline{u} = (u_{p_0}, ..., u_{p_{t-1}}) = \overline{v} \setminus \overline{x}$ and $V = \{i \in \{1, ..., n-1\} \mid x_{j_i} \notin \overline{x}'\}$. Suppose then that there exists $\overline{a}, \overline{c} \in K^{<\omega}$ such that

$$s(x_{j_0}) = \sum_{\substack{i=1 \\ i \notin V}}^{n-1} a_i s(x_{j_i}) \; + \sum_{i=0}^{t-1} c_i s(u_{p_i}).$$

Let now $V' = V \cup \{i \in \{1, ..., n-1\} \mid a_i = 0\}$ and $W = \{i \in \{0, ..., t-1\} \mid c_i = 0\}$, we then have that

$$s(x_{j_0}) = \sum_{\substack{i=1 \\ i \notin V'}}^{n-1} s(x_{j_i}) \;+\; \sum_{\substack{i=0 \\ i \notin W}}^{t-1} s(u_{p_i}).$$

As we noticed above though

$$s(x_{j_0}) = \sum_{i=1}^{n-1} s(x_{j_i}),$$

so

$$\sum_{i=1}^{n-1} s(x_{j_i}) \;=\; \sum_{\substack{i=1 \\ i \notin V'}}^{n-1} s(x_{j_i}) \;+\; \sum_{\substack{i=0 \\ i \notin W}}^{t-1} s(u_{p_i}).$$

Let $z^* \in \overline{x} \setminus \overline{v}$ with $z^* \neq x_{j_0}$, then there exists $i^* \in \{1, ..., n-1\}$ such that $z^* = x_{j_{i^*}}$. We then have that

$$s(x_{i^*}) = \sum_{\substack{i=1 \\ i \neq i^*}}^{n-1} s(x_{j_i}) \;+\; \sum_{\substack{i=1 \\ i \notin V'}}^{n-1} s(x_{j_i}) \;+\; \sum_{\substack{i=0 \\ i \notin W}}^{t-1} s(u_{p_i}).$$

But this contradicts the lemma, thus $\mathbb{K}^{(2^\omega)} \models_X \perp(\overline{v})$.

**Subcase 2.** $\overline{x} \subseteq \overline{v}$.

This case is not possible. Suppose indeed it is, then by rule (c$_2$.) we can assume that $\overline{v} = \overline{x}\ \overline{v}'$ with $\overline{v}' \subseteq \mathrm{Var} \setminus \overline{x}$. Thus by rule (b$_2$.) we have that $\Sigma \vdash \perp(\overline{x})$ which is absurd.

□

## 3.4 External Vector Space Atomic Independence Logic

In this section we define the system External Vector Space Atomic Independence Logic (EVSAIndL) and then prove its soundness and completeness. The syntax and deductive apparatus of this system are the same as those of AIndL.

### 3.4.1 Semantics

The intuitive meaning of the atom $\overline{x} \perp \overline{y}$ in the present framework is that the only common linear combination of the vectors in $\overline{x}$ and in $\overline{y}$ is the trivial one.

**Definition 3.4.1.** Let $\mathbb{K}$ be a field. Let $X = \{s_i\}_{i \in I}$ with $s_i : \mathrm{dom}(X) \to K$ and $\overline{x}\,\overline{y} \subseteq \mathrm{dom}(X) \subseteq \mathrm{Var}$. For $z \in \mathrm{dom}(X)$ we let $s(z)$ be the element of $K^I$ such that $s(z)(i) = s_i(z)$ for every $i \in I$.

We say that the vector space $\mathbb{K}^I$ over $\mathbb{K}$ satisfies $\overline{x} \perp \overline{y}$ under $X$, in symbols $\mathbb{K}^I \models_X \overline{x} \perp \overline{y}$, if $\langle \{s(x) \mid x \in \overline{x}\} \rangle \cap \langle \{s(y) \mid y \in \overline{y}\} \rangle = \{0\}$.

**Definition 3.4.2.** Let $\Sigma$ be a set of atoms and let $X = \{s_i\}_{i \in I}$ be such that the set of variables occurring in $\Sigma$ is included in $\mathrm{dom}(X)$. We say that $\mathbb{K}^I$ satisfies $\Sigma$ under $X$, in symbols $\mathbb{K}^I \models_X \Sigma$, if $\mathbb{K}^I$ satisfies every atom in $\Sigma$ under $X$.

**Definition 3.4.3.** Let $\Sigma$ be a set of atoms. We say that $\overline{x} \perp \overline{y}$ is a logical consequence of $\Sigma$, in symbols $\Sigma \models \overline{x} \perp \overline{y}$, if for every field $\mathbb{K}$ and $X = \{s_i\}_{i \in I}$ such that the set of variables occurring in $\Sigma \cup \{\overline{x} \perp \overline{y}\}$ is included in $\mathrm{dom}(X)$ we have that

$$\text{if } \mathbb{K}^I \models_X \Sigma \text{ then } \mathbb{K}^I \models_X \overline{x} \perp \overline{y}.$$

### 3.4.2 Soundness and Completeness

**Theorem 3.4.4.** Let $\Sigma$ be a set of atoms, then

$$\Sigma \models \overline{x} \perp \overline{y} \text{ if and only if } \Sigma \vdash \overline{x} \perp \overline{y}.$$

[The deductive system to which we refer has been defined in Section 2.4.3.]

*Proof.* ($\Leftarrow$) We only prove soundness of rule ($d_3$.) and ($e_3$).

($d_3$.) We want to show that if $\mathbb{K}^I \models_X \overline{x} \perp \overline{y}$ and $\mathbb{K}^I \models_X \overline{x}\,\overline{y} \perp \overline{z}$, then $\mathbb{K}^I \models_X \overline{x} \perp \overline{y}\,\overline{z}$. Let $\overline{x} = (x_{j_0}, ..., x_{j_{n-1}})$, $\overline{y} = (y_{k_0}, ..., y_{k_{m-1}})$ and $\overline{z} = (z_{p_0}, ..., z_{p_{q-1}})$. Suppose that $\langle\{s(x) \mid x \in \overline{x}\}\rangle \cap \langle\{s(y) \mid y \in \overline{y}\}\rangle = \{0\}$ and $\langle\{s(v) \mid v \in \overline{x}\,\overline{y}\}\rangle \cap \langle\{s(z) \mid z \in \overline{z}\}\rangle = \{0\}$. Let $a \in \langle\{s(x) \mid x \in \overline{x}\}\rangle \cap \langle\{s(v) \mid v \in \overline{y}\,\overline{z}\}\rangle$, then there exist $\overline{a} \in K^n$, $\overline{b} \in K^m$ and $\overline{c} \in K^q$ such that

$$\sum_{i=0}^{n-1} a_i s(x_{j_i}) = a = \sum_{i=0}^{m-1} b_i s(y_{k_i}) + \sum_{i=0}^{q-1} c_i s(z_{p_i}),$$

so we have that

$$\sum_{i=0}^{n-1} a_i s(x_{j_i}) - \sum_{i=0}^{m-1} b_i s(y_{k_i}) = \sum_{i=0}^{q-1} c_i s(z_{p_i}).$$

But then

$$\sum_{i=0}^{n-1} a_i s(x_{j_i}) - \sum_{i=0}^{m-1} b_i s(y_{k_i}) = 0$$

because by hypothesis $\langle\{s(v) \mid v \in \overline{x}\,\overline{y}\}\rangle \cap \langle\{s(z) \mid z \in \overline{z}\}\rangle = \{0\}$. Thus

$$\sum_{i=0}^{n-1} a_i s(x_{j_i}) = \sum_{i=0}^{m-1} b_i s(y_{k_i})$$

and hence

$$a = \sum_{i=0}^{n-1} a_i s(x_{j_i}) = 0$$

because by hypothesis $\langle\{s(x) \mid x \in \overline{x}\}\rangle \cap \langle\{s(y) \mid y \in \overline{y}\}\rangle = \{0\}$.

($e_3$.) If $\mathbb{K}^I \models_X x \perp x$, then $\langle s(x)\rangle \cap \langle s(x)\rangle = \{0\}$. So $s(x) = 0$ and hence

$$\langle s(x)\rangle \cap \langle\{s(y) \mid y \in \overline{y}\}\rangle = \langle 0\rangle \cap \langle\{s(y) \mid y \in \overline{y}\}\rangle = \langle 0\rangle = \{0\},$$

for any $\overline{y} \in \mathrm{Var}$.

($\Rightarrow$) We follow the line of argument of [17][2].

[2] In [17] the syntax of the system is such that for every atom $\overline{x} \perp \overline{y}$ we have that $\overline{x} \cap \overline{y} = \emptyset$, this syntactic restriction makes the proof much easier.

Suppose $\Sigma \nvdash \overline{x} \perp \overline{y}$. Notice that if this is the case then $\overline{x} \neq \emptyset$ and $\overline{y} \neq \emptyset$. Indeed if $\overline{y} = \emptyset$ then $\Sigma \vdash \overline{x} \perp \overline{y}$ because by rule (a$_3$.) $\vdash \overline{x} \perp \emptyset$. Analogously if $\overline{x} = \emptyset$ then $\Sigma \vdash \overline{x} \perp \overline{y}$ because by rule (a$_3$.) $\vdash \overline{y} \perp \emptyset$ and so by rule (b$_3$.) $\vdash \emptyset \perp \overline{y}$.

We can assume that $\overline{x}$ and $\overline{y}$ are injective. This is without loss of generality because clearly $\mathbb{K}^I \models_X \overline{x} \perp \overline{y}$ if and only if $\mathbb{K}^I \models_X \pi\overline{x} \perp \pi\overline{y}$, where $\pi : \mathrm{Var}^{<\omega} \to \mathrm{Var}^{<\omega}$ is the function that eliminates repetitions in finite sequences of variables.

Furthermore we can assume that $\overline{x} \perp \overline{y}$ is minimal, in the sense that if $\overline{x}' \subseteq \overline{x}$, $\overline{y}' \subseteq \overline{y}$ and $\overline{x}'\,\overline{y}' \neq \overline{x}\,\overline{y}$, then $\Sigma \vdash \overline{x}' \perp \overline{y}'$. This is for two reasons.

i) If $\overline{x} \perp \overline{y}$ is not minimal we can always find a minimal atom $\overline{x}^* \perp \overline{y}^*$ such that $\Sigma \nvdash \overline{x}^* \perp \overline{y}^*$, $\overline{x}^* \subseteq \overline{x}$ and $\overline{y}^* \subseteq \overline{y}$ $--$ just keep deleting elements of $\overline{x}$ and $\overline{y}$ until you obtain the desired property or until both $\overline{x}^*$ and $\overline{y}^*$ are singletons, in which case, due to the trivial independence rule (a$_3$.), $\overline{x}^* \perp \overline{y}^*$ is a minimal statement.

ii) For any $\overline{x}' \subseteq \overline{x}$ and $\overline{y}' \subseteq \overline{y}$ we have that if $\mathbb{K}^I \not\models_X \overline{x}' \perp \overline{y}'$ then $\mathbb{K}^I \not\models_X \overline{x} \perp \overline{y}$.

Let then $\overline{x} = (x_{j_0}, ..., x_{j_{n-1}})$ and $\overline{y} = (y_{k_0}, ..., y_{k_{m-1}})$ be injective and such that $\overline{x} \perp \overline{y}$ is minimal.

Let $V = \{v \in \mathrm{Var} \mid \Sigma \vdash v \perp v\}$ and $W = \mathrm{Var} \setminus V$. We claim that $\overline{x}, \overline{y} \not\subseteq V$. We prove it only for $\overline{x}$, the other case is symmetrical. Suppose that $\overline{x} \subseteq V$, then for every $x \in \overline{x}$ we have that $\Sigma \vdash x \perp x$ so by rule (e$_3$.), (b$_3$.) and (d$_3$.)

$$\Sigma \vdash \overline{y} \perp x_{j_0} \quad \text{and} \quad \Sigma \vdash \overline{y}\, x_{j_0} \perp x_{j_1} \Rightarrow \ \Sigma \vdash \overline{y} \perp x_{j_0}\, x_{j_1},$$

$$\Sigma \vdash \overline{y} \perp x_{j_0}\, x_{j_1} \quad \text{and} \quad \Sigma \vdash \overline{y}\, x_{j_0}\, x_{j_1} \perp x_{j_2} \Rightarrow \ \Sigma \vdash \overline{y} \perp x_{j_0}\, x_{j_1}\, x_{j_2},$$

$$\vdots$$

$$\Sigma \vdash \overline{y} \perp x_{j_0} \cdots x_{j_{n-2}} \quad \text{and} \quad \Sigma \vdash \overline{y}\, x_{j_0} \cdots x_{j_{n-2}} \perp x_{j_{n-1}} \Rightarrow \ \Sigma \vdash \overline{y} \perp \overline{x}.$$

Hence by rule (b$_3$.) $\Sigma \vdash \overline{x} \perp \overline{y}$.

Thus $\overline{x} \cap W \neq \emptyset$ and $\overline{y} \cap W \neq \emptyset$. Without loss of generality suppose that $x_{j_0} \in W$ and $y_{k_0} \in W$.

Let $\mathbb{K}$ be the two elements field and $\{w_i \mid i \in \kappa\}$ be an injective enumeration of $W \setminus \{x_{j_0}\}$. We identify the domain of $\mathbb{K}$ with the set 2. Let $X = \{s_t \mid t \in 2^\kappa\}$ be the set of assignment such that for every $t \in 2^\kappa$

i) $s_t(v) = 0$ for every $v \in V$,

ii) $s_t(w_i) = t(i)$ for every $i \in \kappa$,

iii) $s_t(x_{j_0}) = p(s_t(x_{j_1}), ..., s_t(x_{j_{n-1}}), s_t(y_{k_0}), ..., s_t(y_{k_{m-1}}))$,

where $p : K^{<\omega} \to K$ is the function which assigns 1 to the sequences with an odd numbers of 1s and 0 to the sequences with an even numbers of 1s.

We claim that $\mathbb{K}^{(2^\kappa)} \not\models_X \overline{x} \perp \overline{y}$. First notice that

$$s(x_{j_0}) = \sum_{i=1}^{n-1} s(x_{j_i}) \ + \sum_{i=0}^{m-1} s(y_{k_i}),$$

this is because for every $t \in 2^\omega$

$$\begin{aligned} s(x_{j_0})(t) = 0 &\iff s_t(x_{j_0}) = 0 \\ &\iff p(s_t(x_{j_1}), ..., s_t(x_{j_{n-1}}), s_t(y_{k_0}), ..., s_t(y_{k_{m-1}})) = 0 \\ &\iff p(s(x_{j_1})(t), ..., s(x_{j_{n-1}})(t), s(y_{k_0})(t), ..., s(y_{k_{m-1}})(t)) = 0 \\ &\iff (\sum_{i=1}^{n-1} s(x_{j_i})(t) \ + \sum_{i=0}^{m-1} s(y_{k_i})(t)) = 0 \\ &\iff (\sum_{i=1}^{n-1} s(x_{j_i}) \ + \sum_{i=0}^{m-1} s(y_{k_i}))(t) = 0 \end{aligned}$$

and similarly for $s(x_{j_0})(t) = 1$. [To justify the second to last passage of the series of equivalences above just notice that in $\mathbb{K}$ we have that $1 + 1 = 0$].

Thus

$$\sum_{i=0}^{n-1} s(x_{j_i}) = \sum_{i=0}^{m-1} s(y_{k_i})$$

because in $\mathbb{K}$ we have that $-1 = 1$. Notice finally that

$$\sum_{i=0}^{m-1} s(y_{k_i}) \neq 0.$$

Indeed let $\overline{y} \cap W = (y_{k'_0}, ..., y_{k'_{m'-1}})$, if $m'$ is odd consider $t \in 2^\kappa$ such that $s(y_{k'_i})(t) = 1$ for every $\{0, ..., m' - 1\}$ and if $m'$ is even consider $t \in 2^\kappa$ such that $s(y_{k'_0})(t) = 0$ and $s(y_{k'_i})(t) = 1$ for every $\{1, ..., m' - 1\}$. Then we have that

$$\sum_{i=0}^{m-1} s(y_{k_i})(t) = p(s_t(y_{k_0}), ..., s_t(y_{k_{m-1}})) = p(s_t(y_{k'_0}), ..., s_t(y_{k'_{m'-1}})) = 1.$$

Hence $\langle \{s(x) \mid x \in \overline{x}\} \rangle \cap \langle \{s(y) \mid y \in \overline{y}\} \rangle \neq \{0\}$.

Let now $\overline{v} \perp \overline{w} \in \Sigma$, we want to show that $\mathbb{K}^{(2^\omega)} \models_X \overline{v} \perp \overline{w}$. As before, we assume, without loss of generality, that $\overline{v}$ and $\overline{w}$ are injective. Notice also that if $\overline{v} = \emptyset$ or $\overline{w} = \emptyset$, then $\mathbb{K}^{(2^\omega)} \models_X \overline{v} \perp \overline{w}$. Thus let $\overline{v}, \overline{w} \neq \emptyset$.

**Case 1** $\overline{v} \subseteq V$ or $\overline{w} \subseteq V$.

Suppose that $\overline{v} \subseteq V$, the other case is symmetrical, then

$$\langle \{s(v) \mid v \in \overline{v}\} \rangle \cap \langle \{s(w) \mid w \in \overline{w}\} \rangle = \langle 0 \rangle \cap \langle \{s(w) \mid w \in \overline{w}\} \rangle = \langle 0 \rangle = \{0\}\,.$$

**Case 2** $\overline{v} \not\subseteq V$ and $\overline{w} \not\subseteq V$.

Let $\overline{v} \cap W = \overline{v}' = (v_{p_0}, ..., v_{p_{l-1}}) \neq \emptyset$ and $\overline{w} \cap W = \overline{w}' = (w'_{r_0}, ..., w'_{r_{q-1}}) \neq \emptyset$. Notice that

$$\langle \{s(v) \mid v \in \overline{v}\} \rangle \cap \langle \{s(w) \mid w \in \overline{w}\} \rangle = \langle \{s(v') \mid v' \in \overline{v}'\} \rangle \cap \langle \{s(w') \mid w' \in \overline{w}'\} \rangle$$

because if $u \in \overline{v}\,\overline{w} \setminus \overline{v}'\,\overline{w}'$, then $s(u) = 0$. Hence $\mathbb{K}^{(2^\omega)} \models_X \overline{v} \perp \overline{w}$ if and only if $\mathbb{K}^{(2^\omega)} \models_X \overline{v}' \perp \overline{w}'$.

We prove a lemma.

**Lemma 3.4.5.** If $v \in W \setminus \{x_{j_0}\}$ and $Z = \left\{z_{d_0}, ..., z_{d_{q-1}}\right\} \subseteq W$ with $v \notin Z$, then for every $\overline{a} \in K^q$ we have

$$s(v) \neq \sum_{i=0}^{q-1} a_i s(z_{d_i}).$$

*Proof.* Straightforward adaptation of Lemma 3.3.6.

□

We now go back to our main proof making a case distinction on $\overline{v}'\,\overline{w}'$.

**Subcase 2.1** $x_{j_0} \notin \overline{v}'\,\overline{w}'$.

Suppose that

$$\langle\{s(v') \mid v' \in \overline{v}'\}\rangle \cap \langle\{s(w') \mid w' \in \overline{w}'\}\rangle \neq \{0\}\,,$$

then there exist $\overline{b} \in K^l$ and $\overline{c} \in K^q$ such that

$$0 \neq \sum_{i=0}^{l-1} b_i s(v_{p_i}) = \sum_{i=0}^{q-1} c_i s(w'_{r_i}) \neq 0.$$

Let $i^*$ be such that $b_{i^*} \neq 0$, then we have that

$$s(v_{p_{i^*}}) = \sum_{\substack{i=0\\ i\neq i^*}}^{l-1} b_i s(v_{p_i}) + \sum_{i=0}^{q-1} c_i s(w'_{r_i})$$

which contradicts the lemma.

**Subcase 2.2** $x_{j_0} \in \overline{v}'\,\overline{w}'$.

Let $\overline{x} \cap W = \overline{x}' = (x_{j'_0}, ..., x_{j'_{n'-1}}) = (x_{j_0}, ..., x_{j'_{n'-1}}) \neq \emptyset$ and $\overline{y} \cap W = \overline{y}' = (y_{k'_0}, ..., y_{k'_{m'-1}})$. Notice that $\overline{x}' \cap \overline{y}' = \emptyset$. Indeed let $z \in \overline{x}' \cap \overline{y}'$, then by rules ($b_3$.) and ($c_3$.) we have that $\Sigma \vdash z \perp z$. Thus $z \in V$, a contradiction.

**Subsubcase 2.2.1** $\overline{x}'\,\overline{y}' \setminus \overline{v}'\,\overline{w}' \neq \emptyset$.

Suppose that

$$\langle\{s(v') \mid v' \in \overline{v}'\}\rangle \cap \langle\{s(w') \mid w' \in \overline{w}'\}\rangle \neq \{0\}\,,$$

then there exist $\overline{d} \in K^l$ and $\overline{f} \in K^q$ such that

$$0 \neq \sum_{i=0}^{l-1} d_i s(v_{p_i}) = \sum_{i=0}^{q-1} f_i s(w'_{r_i}) \neq 0.$$

Suppose that $x_{j_0} \in \overline{v}'$, the case $x_{j_0} \in \overline{w}'$ is symmetrical, then there exists $\hat{i}$ such that $x_{j_0} = v_{p_{\hat{i}}}$. Suppose that $d_{\hat{i}} = 0$, then

$$0 \neq \sum_{\substack{i=0\\ i\neq\hat{i}}}^{l-1} d_i s(v_{p_i}) = \sum_{i=0}^{q-1} f_i s(w'_{r_i}) \neq 0$$

so we are in the same situation as the case above and hence we have a contradiction.

Suppose that $d_{\hat{i}} = 1$, then we have that

$$s(x_{j_0}) = \sum_{\substack{i=0\\ i\neq\hat{i}}}^{l-1} d_i s(v_{p_i}) + \sum_{i=0}^{q-1} f_i s(w'_{r_i}).$$

Let now $\overline{u} = (u_{p_0}, ..., u_{p_{t-1}}) = \overline{v}'\,\overline{w}' \setminus \overline{x}'\,\overline{y}'$, $\overline{x}'' = \overline{x}' \cap \overline{v}'\,\overline{w}'$ and $\overline{y}'' = \overline{y}' \cap \overline{v}'\,\overline{w}'$. Let also $V = \{i \in \{1, ..., n'-1\} \mid x_{j'_i} \notin \overline{x}''\}$ and $U = \{i \in \{0, ..., m'-1\} \mid y_{k'_i} \notin \overline{y}''\}$. Then there exists $\overline{a}, \overline{b}, \overline{c} \in K^{<\omega}$ such that

$$s(x_{j_0}) = \sum_{\substack{i=1 \\ i \notin V}}^{n'-1} a_i s(x_{j'_i}) \;+ \sum_{\substack{i=0 \\ i \notin U}}^{m'-1} b_i s(y_{k'_i}) \;+ \sum_{i=0}^{t-1} c_i s(u_{p_i}).$$

Let $V' = V \cup \{i \in \{1, ..., n'-1\} \mid a_i = 0\}$, $U' = U \cup \{i \in \{0, ..., m'-1\} \mid b_i = 0\}$ and $W = \{i \in \{0, ..., t-1\} \mid c_i = 0\}$, we then have that

$$s(x_{j_0}) = \sum_{\substack{i=1 \\ i \notin V'}}^{n'-1} s(x_{j'_i}) \;+ \sum_{\substack{i=0 \\ i \notin U'}}^{m'-1} s(y_{k'_i}) \;+ \sum_{\substack{i=0 \\ i \notin W}}^{t-1} s(u_{p_i}).$$

As we noticed above though

$$s(x_{j_0}) = \sum_{i=1}^{n-1} s(x_{j_i}) \;+ \sum_{i=0}^{m-1} s(y_{k_i}) = \sum_{i=1}^{n'-1} s(x_{j'_i}) \;+ \sum_{i=0}^{m'-1} s(y_{k'_i}),$$

so

$$\sum_{i=1}^{n'-1} s(x_{j'_i}) \;+ \sum_{i=0}^{m'-1} s(y_{k'_i}) = \sum_{\substack{i=1 \\ i \notin V'}}^{n'-1} s(x_{j'_i}) \;+ \sum_{\substack{i=0 \\ i \notin U'}}^{m'-1} s(y_{k'_i}) \;+ \sum_{\substack{i=0 \\ i \notin W}}^{t-1} s(u_{p_i}).$$

Let $z^* \in \overline{x}'\,\overline{y}' \setminus \overline{v}'\,\overline{w}'$ and suppose that $z^* \in \overline{x}''$, the other case is symmetrical, then there exists $i^* \in \{1, ..., n'-1\}$ such that $z^* = x_{j'_{i^*}}$. We then have that

$$s(x_{j'_{i^*}}) = \sum_{\substack{i=1 \\ i \neq i^*}}^{n'-1} s(x_{j'_i}) \;+ \sum_{i=0}^{m'-1} s(y_{k'_i}) \;+ \sum_{\substack{i=1 \\ i \notin V'}}^{n'-1} s(x_{j'_i}) \;+ \sum_{\substack{i=0 \\ i \notin U'}}^{m'-1} s(y_{k'_i}) \;+ \sum_{\substack{i=0 \\ i \notin W}}^{t-1} s(u_{p_i}).$$

But this contradicts the lemma, thus $\mathbb{K}^{(2^\omega)} \models_X \overline{v} \perp \overline{w}$.

**Subsubcase 2.2.2** $\overline{x}'\,\overline{y}' \subseteq \overline{v}'\,\overline{w}'$.

This case is not possible. First notice that if $\Sigma \vdash \overline{x}' \perp \overline{y}'$, then $\Sigma \vdash \overline{x} \perp \overline{y}$. Let $\overline{x} \setminus \overline{x}' = (x_{s_0}, ..., x_{s_{b-1}})$ and $\overline{y} \setminus \overline{y}' = (y_{g_0}, ..., y_{g_{c-1}})$, then by rule (e$_3$.), (b$_3$.) and (d$_3$.) we have that

$$\Sigma \vdash \overline{x}' \perp \overline{y}' \quad \text{and} \quad \Sigma \vdash \overline{x}'\,\overline{y}' \perp y_{g_0}$$

$$\Downarrow$$

$$\Sigma \vdash \overline{x}' \perp \overline{y}'\, y_{g_0}$$

$$\vdots$$

$$\Sigma \vdash \overline{x}' \perp \overline{y}'\, y_{g_0} \cdots y_{g_{c-2}} \quad \text{and} \quad \Sigma \vdash \overline{x}'\,\overline{y}'\, y_{g_0} \cdots y_{g_{c-2}} \perp y_{g_{c-1}}$$

$$\Downarrow$$

$$\Sigma \vdash \overline{x}' \perp \overline{y}'\, y_{g_0} \cdots y_{g_{c-1}}$$

and hence by rule (f$_3$.) and (b$_3$.) we have that $\Sigma \vdash \overline{y} \perp \overline{x}'$. Thus

$$\Sigma \vdash \overline{y} \perp \overline{x}' \quad \text{and} \quad \Sigma \vdash \overline{y}\,\overline{x}' \perp x_{s_0}$$
$$\Downarrow$$
$$\Sigma \vdash \overline{y} \perp \overline{x}'\, x_{s_0}$$
$$\vdots$$
$$\Sigma \vdash \overline{y} \perp \overline{x}'\, x_{s_0} \cdots x_{s_{b-2}} \quad \text{and} \quad \Sigma \vdash \overline{y}\,\overline{x}'\, x_{s_0} \cdots x_{s_{b-2}} \perp x_{s_{b-1}}$$
$$\Downarrow$$
$$\Sigma \vdash \overline{y} \perp \overline{x}'\, x_{s_0} \cdots x_{s_{b-1}}$$

and hence by rule (f$_3$.) and (b$_3$.) we have that $\Sigma \vdash \overline{x} \perp \overline{y}$.

By rule (f$_3$.) we can assume that $\overline{v} = \overline{v}'\,\overline{u}$ and $\overline{w} = \overline{w}'\,\overline{u}'$ with $\overline{u}\,\overline{u}' \subseteq \mathrm{Var} \setminus \overline{v}'\,\overline{w}'$. Furthermore because $\overline{x}'\,\overline{y}' \subseteq \overline{v}'\,\overline{w}'$ again by rule (f$_3$.) we can assume that $\overline{v}' = \overline{x}''\,\overline{y}''\,\overline{z}'$ and $\overline{w}' = \overline{x}'''\,\overline{y}'''\,\overline{z}''$ with $\overline{x}''\,\overline{x}''' = \overline{x}'$, $\overline{y}''\,\overline{y}''' = \overline{y}'$ and $\overline{z}'\,\overline{z}'' \subseteq \mathrm{Var} \setminus \overline{x}'\,\overline{y}'$. Hence $\overline{v} = \overline{x}''\,\overline{y}''\,\overline{z}'\,\overline{u}$ and $\overline{w} = \overline{x}'''\,\overline{y}'''\,\overline{z}''\,\overline{u}'$.

By hypothesis we have that $\overline{v} \perp \overline{w} \in \Sigma$ so by rules (c$_3$.) and (b$_3$.) we can conclude that $\Sigma \vdash \overline{x}''\,\overline{y}'' \perp \overline{x}'''\,\overline{y}'''$.

If $\overline{x}'' = \overline{x}'$ and $\overline{y}''' = \overline{y}'$, then $\Sigma \vdash \overline{x}' \perp \overline{y}'$ because as we noticed $\overline{v}' \cap \overline{w}' = \emptyset$. Thus $\Sigma \vdash \overline{x} \perp \overline{y}$, a contradiction. Analogously if $\overline{x}''' = \overline{x}'$ and $\overline{y}'' = \overline{y}'$, then $\Sigma \vdash \overline{y}' \perp \overline{x}'$. Thus by rule (b$_3$.) $\Sigma \vdash \overline{x}' \perp \overline{y}'$ and hence $\Sigma \vdash \overline{x} \perp \overline{y}$, a contradiction. There are then four cases

i) $\overline{x}'' \neq \overline{x}'$ and $\overline{x}''' \neq \overline{x}'$

ii) $\overline{y}'' \neq \overline{y}'$ and $\overline{x}''' \neq \overline{x}'$

iii) $\overline{y}'' \neq \overline{y}'$ and $\overline{y}''' \neq \overline{y}'$

iv) $\overline{x}'' \neq \overline{x}'$ and $\overline{y}''' \neq \overline{y}'$

Suppose that either i) or ii) holds. If this is the case, then $\Sigma \vdash \overline{x}'' \perp \overline{y}''$ because by hypothesis $\overline{x} \perp \overline{y}$ is minimal. So $\Sigma \vdash \overline{x}'' \perp \overline{y}''\,\overline{x}'''\,\overline{y}'''$, because by rule (d$_3$.)

$$\Sigma \vdash \overline{x}'' \perp \overline{y}'' \text{ and } \Sigma \vdash \overline{x}''\,\overline{y}'' \perp \overline{x}'''\,\overline{y}''' \Rightarrow \Sigma \vdash \overline{x}'' \perp \overline{y}''\,\overline{x}'''\,\overline{y}'''.$$

Hence by rule (e$_3$.) $\Sigma \vdash \overline{x}'' \perp \overline{x}'''\,\overline{y}'$ and then by rule (b$_3$.) $\Sigma \vdash \overline{x}'''\,\overline{y}' \perp \overline{x}''$. So by rule (e$_3$.) $\Sigma \vdash \overline{y}'\,\overline{x}''' \perp \overline{x}''$.

We are under the assumption that $\overline{x}''' \neq \overline{x}'$ thus again by minimality of $\overline{x} \perp \overline{y}$ we have that $\Sigma \vdash \overline{x}''' \perp \overline{y}'$ and so by rule (b$_3$.) we conclude that $\Sigma \vdash \overline{y}' \perp \overline{x}'''$. Hence $\Sigma \vdash \overline{y}' \perp \overline{x}'''\,\overline{x}''$, because by rule (d$_3$.)

$$\Sigma \vdash \overline{y}' \perp \overline{x}''' \text{ and } \Sigma \vdash \overline{y}'\,\overline{x}''' \perp \overline{x}'' \Rightarrow \Sigma \vdash \overline{y}' \perp \overline{x}'''\,\overline{x}''.$$

Then finally by rule (e$_3$.) and (b$_3$.) we can conclude that $\Sigma \vdash \overline{x}' \perp \overline{y}'$ and so $\Sigma \vdash \overline{x} \perp \overline{y}$, a contradiction.

The case in which either iii) or iv) holds is symmetrical.

□

## 3.5 Relations between EVSAIndL and EVSAAIndL

In this section we study the relations, under the given semantics, between the independence atom and the absolute independence one.

**Lemma 3.5.1.** Let $\mathbb{K}$ be a field, $\mathbb{V}$ a vector space over $\mathbb{K}$ and $A \subseteq V$. The following are equivalent:

i) $a \notin \langle A \setminus \{a\} \rangle$ for every $a \in A$;

ii) $0 \notin A$ and $\langle a \rangle \cap \langle A \setminus \{a\} \rangle = \{0\}$ for every $a \in A$.

*Proof.* i) $\Rightarrow$ ii) If $0 \in A$ then $0 \in \langle \emptyset \rangle \subseteq \langle A \setminus \{0\} \rangle$. Let $a \in A$ and $b \in \langle a \rangle \cap \langle A \setminus \{a\} \rangle$ with $b \neq 0$, then there exists $\overline{a} \in (A \setminus \{a\})^n$, $c \in K$ and $\overline{c} \in K^n$ such that

$$ca = b = \sum_{i=0}^{n-1} c_i a_i.$$

Now $b \neq 0$, so $c \neq 0$ and thus

$$a = \sum_{i=0}^{n-1} \frac{c_i}{c} a_i.$$

Hence $a \in \langle A \setminus \{a\} \rangle$.

ii) $\Rightarrow$ i) Let $a \in \langle A \setminus \{a\} \rangle$, then there exists $\overline{a} \in (A \setminus \{a\})^n$ and $\overline{c} \in K^n$ such that

$$a = \sum_{i=0}^{n-1} c_i a_i.$$

Either $a = 0$, in which case $0 \in A$, or $a \neq 0$, in which case $\langle a \rangle \cap \langle A \setminus \{a\} \rangle \neq \{0\}$. □

From the above lemma it follows directly the following characterization of EVSAAIndL in terms of EVSAIndL.

**Lemma 3.5.2.** Let $\mathbb{K}$ be a field and $X = \{s_i\}_{i \in I}$ be such that the set of variables occurring in $\Sigma \cup \{\overline{x}\}$ is included in $\mathrm{dom}(X)$. Let $x \in \overline{x}$, we denote by $\overline{x} -_X x$ any enumeration of the set $\{x' \in \overline{x} \mid \mathbb{K}^I \not\models_X x' = x\}$. Then

$$\mathbb{K}^I \models_X \perp(\overline{x}) \Longleftrightarrow \mathbb{K}^I \models_X x \perp \overline{x} -_X x \text{ and } \mathbb{K}^I \not\models_X x \perp x, \text{ for all } x \in \overline{x}.$$

□

## 3.6 Remarks

The completeness of EVSAEL, EVSADL, EVSAAIndL and EVSAIndL answer positively to the questions we stated at the beginning of the chapter and show that the idea that a set of assignments on a variable determines a vector is a fruitful one. In the light of these results we can bring ideas from linear algebra into the general theory of dependence logic and the other way around.

For example the fact that in the proof of the completeness of EVSAEL we used a unidimensional vector space reflects the genuine first-order nature of this concept. Indeed the fact that a single dimension suffices for the proof is indicative of the fact that all the various forms of dependence logic are conservative extensions of FO logic and so the team semantics on FO formulas collapses on the standard one.

On the other side the fact that in the proof of the completeness of EVSADL the use of a bidimensional vector space sufficed while in the case of EVSAIndL and EVSAAIndL we had to use a space with possibly infinitely many dimensions is indicative of the fact that the concept of independence is much more complex than that of dependence, as indeed the independence atom is much more expressive than the dependence one.

Finally the completeness results for EVSADL, EVSAAIndL and EVSAIndL show that the dependence and independence concepts arising from linear algebra not only fit with the abstract ones but they also have a certain absoluteness.

Indeed if we combine the completeness of ADL, AAIndL and AIndL with that of EVSADL, EVSAAIndL and EVSAIndL respectively we have that the abstract semantics and the algebraic ones are deductively equivalent.

# Chapter 4

# Dependence and Independence in Vector Spaces and Algebraically Closed Fields

In this chapter we start the analysis that we announced in the introductory chapter, treating the linear and algebraic dependence and independence notions of linear algebra and field theory. Among the various notions of dependence and independence that we encounter in algebra, these are by far the most well-established and studied and can be considered, in a sense, as canonical.

The difference between the analysis of linear algebra that we develop here and what we have done in the previous chapter is that here we interpret the variables directly in the vector space, instead of seeing a set of assignments into a field as a set of vectors on that field. We then go back to the use of standard first-order semantics and do not refer anymore to team semantics, as was the case in chapters 2 and 3.

This will be our strategy for the rest of the work. We will define structures in which interesting dependence and independence notions have been formulated, interpret our atoms in these structures via first-order assignments, and then study the properties of the resulting systems.

An interesting feature of the systems studied here is that the algebraic structure that their models carry allow us to define dependence and independence as being negations of each other. This is not the case in the abstract setting, where independence is a very strong denial of dependence rather than simply being its negation.

The systems that we are going to study are: Vector Space Atomic Dependence Logic, Vector Space Atomic Absolute Independence Logic, Vector Space Atomic Independence Logic, Algebraically Closed Fields Atomic Dependence Logic, Algebraically Closed Fields Atomic Absolute Independence Logic and Algebraically Closed Fields Atomic Independence Logic.

## 4.1 Vector Space Atomic Dependence Logic

In this section we define the system Vector Space Atomic Dependence Logic (VSADL) and then prove its soundness and completeness. The syntax and deductive apparatus of this system are the same as those of ADL.

### 4.1.1 Semantics

As we saw in Section 3.2.1, the intuitive meaning of $=(\overline{x}, \overline{y})$ in the context of vector spaces is that each vector in $\overline{y}$ is a linear function of the vectors in $\overline{x}$.

**Definition 4.1.1.** Let $\mathbb{K}$ be a field and $\mathbb{V}$ a vector space over $\mathbb{K}$. Let $s : \mathrm{dom}(s) \to V$ with $\overline{x}\,\overline{y} \subseteq \mathrm{dom}(s) \subseteq \mathrm{Var}$. Let $\overline{x} = (x_{j_0}, ..., x_{j_{n-1}})$, we say that $\mathbb{V}$ satisfies $=(\overline{x}, \overline{y})$ under $s$, in symbols $\mathbb{V} \models_s =(\overline{x}, \overline{y})$, if for every $y \in \overline{y}$ we have that $s(y) \in \langle \{s(x) \mid x \in \overline{x}\} \rangle$, that is there exists $\overline{a} \in K^n$ such that

$$\sum_{i=0}^{n-1} a_i s(x_{j_i}) = s(y).$$

[Notice that under this formulation the case $\overline{x} = \emptyset$ and $\overline{x}, \overline{y} = \emptyset$ are taken care of, in the first case indeed we have that $\sum_{i=0}^{n-1} a_i s(x_{j_i}) = s(y)$ becomes just $s(y) = 0$, while in the second the condition is always trivially satisfied.]

**Definition 4.1.2.** Let $\Sigma$ be a set of atoms and let $s$ be such that the set of variables occurring in $\Sigma$ is included in $\mathrm{dom}(s)$. We say that $\mathbb{V}$ satisfies $\Sigma$ under $s$, in symbols $\mathbb{V} \models_s \Sigma$, if $\mathbb{V}$ satisfies every atom in $\Sigma$ under $s$.

**Definition 4.1.3.** Let $\Sigma$ be a set of atoms. We say that $=(\overline{x}, \overline{y})$ is a logical consequence of $\Sigma$, in symbols $\Sigma \models =(\overline{x}, \overline{y})$, if for every field $\mathbb{K}$, vector space $\mathbb{V}$ over $\mathbb{K}$ and $s$ such that the set of variables occurring in $\Sigma \cup \{=(\overline{x}, \overline{y})\}$ is included in $\mathrm{dom}(s)$ we have that

$$\text{if } \mathbb{V} \models_s \Sigma \text{ then } \mathbb{V} \models_s =(\overline{x}, \overline{y}).$$

### 4.1.2 Soundness and Completeness

**Theorem 4.1.4.** Let $\Sigma$ be a set of atoms, then

$$\Sigma \models =(\overline{x}, \overline{y}) \text{ if and only if } \Sigma \vdash =(\overline{x}, \overline{y}).$$

[The deductive system to which we refer has been defined in Section 2.2.3.]

*Proof.* ($\Leftarrow$) Easy exercise of linear algebra.

($\Rightarrow$) We adapt the proofs of [2] and [13] to our framework.

Suppose $\Sigma \nvdash =(\overline{x}, \overline{y})$. Let $V = \{z \in \mathrm{Var} \mid \Sigma \vdash =(\overline{x}, z)\}$ and $W = \mathrm{Var} \setminus V$. First notice that $\overline{y} \neq \emptyset$, indeed if not so, by the syntactic constraints that we put on the system, we have that $\overline{x}, \overline{y} = \emptyset$ and so by the admitted degenerate case of rule ($a_1$.) we have that $\Sigma \vdash =(\overline{x}, \overline{y})$. Furthermore $\overline{y} \cap W \neq \emptyset$, indeed if $\overline{y} \cap W = \emptyset$ then for every $y \in \overline{y}$ we have that $\Sigma \vdash =(\overline{x}, y)$ and so by rules ($d_1$.), ($e_1$.) and, if necessary, ($f_1$.)[1] we have that $\Sigma \vdash =(\overline{x}, \overline{y})$.

[1]Notice that ($f_1$.) is necessary only if $\overline{x}$ or $\overline{y}$ contain repetitions.

Let $\mathbb{K}$ be the two elements field and $\mathbb{K}^2$ the two dimensional vector space over it. Let $s : \text{Var} \to K^2$ be the following assignment

$$s(v) = \begin{cases} (0,0) & \text{if } v \in V \\ (0,1) & \text{if } v \in W. \end{cases}$$

We claim that $\mathbb{K}^2 \not\models_s =(\overline{x}, \overline{y})$. In accordance to the semantics we then have to show that there is $y \in \overline{y}$ such that $s(y) \notin \langle\{s(x) \mid x \in \overline{x}\}\rangle$. Let $y \in \overline{y} \cap W$ and $\overline{x} = (x_{j_0}, ..., x_{j_{n-1}}) \neq \emptyset$, we then have that $s(y) = (0,1)$ but for every $\overline{a} \in K^n$

$$\sum_{i=0}^{n-1} a_i s(x_{j_i}) = (0,0)$$

because for $x \in \overline{x}$ we have that $\Sigma \vdash =(\overline{x}, x)$. Indeed by rule (a$_1$.) $\vdash =(\overline{x}, \overline{x})$ and so by rule (b$_1$.) $\vdash =(\overline{x}, x)$. Notice finally that if $\overline{x} = \emptyset$ then for every $\overline{a} \in K^n$ we have that

$$\sum_{i=0}^{n-1} a_i s(x_{j_i}) + s(y) = s(y) = (0,1) \neq (0,0).$$

Let now $=(\overline{x}', \overline{y}') \in \Sigma$, we want to show that $\mathbb{K}^2 \models_s =(\overline{x}', \overline{y}')$. If $\overline{y}' = \emptyset$ then also $\overline{x}' = \emptyset$ and so trivially $\mathbb{K}^2 \models_s =(\overline{x}', \overline{y}')$. Noticed this, for the rest of the proof we assume $\overline{y}' \neq \emptyset$.

**Case 1.** $\overline{x}' = \emptyset$.

Suppose that $\mathbb{K}^2 \not\models_s =(\emptyset, \overline{y}')$, then there exists $y' \in \overline{y}'$ such that $s(y') \neq (0,0)$, so $s(y') = (0,1)$ which means that $\Sigma \nvdash =(\overline{x}, y')$. Notice though that $\Sigma \vdash =(\emptyset, \overline{y}')$, so by rule (b$_1$.) $\Sigma \vdash =(\emptyset, y')$ and hence again by rule (b$_1$.) $\Sigma \vdash =(\overline{x}, y')$. Thus $\mathbb{K}^2 \models_s =(\overline{x}', \overline{y}')$.

**Case 2.** $\overline{x}' \neq \emptyset$ and $\overline{x}' \subseteq V$.

If this is the case, then

$$\begin{aligned} \forall x' \in \overline{x}' \ \ \Sigma \vdash =(\overline{x}, x') &\implies \Sigma \vdash =(\overline{x}, \overline{x}') \ \ \text{[by rules (d}_1\text{.), (e}_1\text{.) and (f}_1\text{.)]} \\ &\implies \Sigma \vdash =(\overline{x}, \overline{y}') \ \ \text{[by rule (c}_1\text{.)]} \\ &\implies \forall y' \in \overline{y}' \ \ \Sigma \vdash =(\overline{x}, y') \ \ \text{[by rule (b}_1\text{.)]} \\ &\implies \overline{y}' \subseteq V. \end{aligned}$$

Let $y' \in \overline{y}'$, then for any $x' \in \overline{x}'$ we have that $s(y') = (0,0) = s(x')$. Hence $\mathbb{K}^2 \models_s =(\overline{x}', \overline{y}')$.

**Case 3.** $\overline{x}' \cap W \neq \emptyset$.

If this is the case, then there exists $x' \in \overline{x}'$ such that $\Sigma \nvdash =(\overline{x}, x')$, so we have $x' \in \overline{x}'$ such that $s(x') = (0,1)$. Let now $y' \in \overline{y}'$.

**Subcase 1**. $y' \in W$.

In this case we have that $s(y') = (0,1) = s(x')$. Hence $\mathbb{K}^2 \models_s =(\overline{x}', \overline{y}')$.

**Subcase 2**. $y' \in V$.

In this case we have that $s(y') = (0,0) = 0s(x')$. Hence $\mathbb{K}^2 \models_s =(\overline{x}', \overline{y}')$.

□

## 4.2 Vector Space Atomic Absolute Independence Logic

In this section we define the system Vector Space Atomic Absolute Independence Logic (VSAAIndL) and then prove its soundness and completeness. The syntax and deductive apparatus of this system are the same as those of AAIndL.

### 4.2.1 Semantics

As we noticed in Section 3.3.1, the atom $\perp(\overline{x})$ has a natural interpretation in the context of vector spaces: the elements of $\overline{x}$ are linearly independent vectors.

**Definition 4.2.1.** Let $\mathbb{K}$ be a field and $\mathbb{V}$ a vector space over $\mathbb{K}$. Let $s : \mathrm{dom}(s) \to V$ with $\overline{x} \subseteq \mathrm{dom}(s) \subseteq \mathrm{Var}$. We say that $\mathbb{V}$ satisfies $\perp(\overline{x})$ under $s$, in symbols $\mathbb{V} \models_s \perp(\overline{x})$, if for every $x \in \overline{x}$ we have that

$$s(x) \notin \langle \{s(z) \mid z \in \overline{x}\} \setminus \{s(x)\} \rangle.$$

Notice that, because of Proposition 3.3.1, the above condition is equivalent to the classical characterization of linear independence, according to which a set of vectors is independent if and only if no non-trivial linear combination has value 0.

**Definition 4.2.2.** Let $\Sigma$ be a set of atoms and let $s$ be such that the set of variables occurring in $\Sigma$ is included in $\mathrm{dom}(s)$. We say that $\mathbb{V}$ satisfies $\Sigma$ under $s$, in symbols $\mathbb{V} \models_s \Sigma$, if $\mathbb{V}$ satisfies every atom in $\Sigma$ under $s$.

**Definition 4.2.3.** Let $\Sigma$ be a set of atoms. We say that $\perp(\overline{x})$ is a logical consequence of $\Sigma$, in symbols $\Sigma \models \perp(\overline{x})$, if for every field $\mathbb{K}$, vector space $\mathbb{V}$ over $\mathbb{K}$ and $s$ such that the set of variables occurring in $\Sigma \cup \{\perp(\overline{x})\}$ is included in $\mathrm{dom}(s)$ we have that

$$\text{if } \ \mathbb{V} \models_s \Sigma \ \text{ then } \ \mathbb{V} \models_s \perp(\overline{x}).$$

### 4.2.2 Soundness and Completeness

**Theorem 4.2.4.** Let $\Sigma$ be a set of atoms, then

$$\Sigma \models \perp(\overline{x}) \text{ if and only if } \Sigma \vdash \perp(\overline{x}).$$

[The deductive system to which we refer has been defined in Section 2.3.3.]

*Proof.* ($\Leftarrow$) Obvious.

($\Rightarrow$) Suppose $\Sigma \nvdash \perp(\overline{x})$. Notice that if this is the case then $\overline{x} \neq \emptyset$. Indeed if $\overline{x} = \emptyset$ then $\Sigma \vdash \perp(\overline{x})$ because by rule ($a_2$.) $\vdash \perp(\emptyset)$.

We can assume that $\overline{x}$ is injective. This is without loss of generality because clearly $\mathbb{V} \models_s \perp(\overline{x})$ if and only if $\mathbb{V} \models_s \perp(\pi\overline{x})$, where $\pi : \mathrm{Var}^{<\omega} \to \mathrm{Var}^{<\omega}$ is the function that eliminates repetitions in finite sequences of variables. Let then $\overline{x} = (x_{j_0}, ..., x_{j_{n-1}}) \neq \emptyset$ be injective.

Let $\mathbb{K}$ be the two elements field and $\mathbb{V}$ the $\aleph_0$-infinite dimensional vector space over $\mathbb{K}$.

Let then $\{a_i \mid i \in \omega\}$ be an injective enumeration of a basis $A$ of $\mathbb{V}$, $\{w_i \mid i \in \omega\}$ an injective enumeration of $\mathrm{Var} \setminus \{x_{j_0}\}$ and let $s$ be the following assignment:

$$s(w_i) = a_i$$

and

$$s(x_{j_0}) = 0 \qquad\qquad \text{if } \overline{x} = \{x_{j_0}\}$$

$$s(x_{j_0}) = \sum_{i=1}^{n-1} a_{p_i} \qquad \text{if } \overline{x} \neq \{x_{j_0}\},$$

where $w_{p_i} = x_{j_i}$ for every $i \in \{1, ..., n-1\}$.

We claim that $\mathbb{V} \not\models_s \bot(\overline{x})$. In accordance to the semantic we then have to show that there is $x \in \overline{x}$ such that $s(x) \in \langle \{s(z) \mid z \in \overline{x}\} \setminus \{s(x)\}\rangle$. But $x_{j_0}$ satisfies this condition, indeed either

$$s(x_{j_0}) = 0 \quad \text{or} \quad s(x_{j_0}) = \sum_{i=1}^{n-1} s(x_{j_i})$$

and clearly in both cases $s(x_{j_0}) \in \langle \{s(x) \mid x \in \overline{x}\} \setminus \{s(x_{j_0})\}\rangle$.

Let now $\bot(\overline{v}) \in \Sigma$, we want to show that $\mathbb{V} \models_s \bot(\overline{v})$. As before, we assume, without loss of generality, that $\overline{v}$ is injective. Notice that if $\overline{v} = \emptyset$, then $\mathbb{V} \models_s \bot(\overline{v})$. Thus let $\overline{v} = (v_{h_0}, ..., v_{h_{c-1}}) \neq \emptyset$.

**Case 1.** $x_{j_0} \notin \overline{v}$.

Let $w_{r_i} = v_{h_i}$ for every $i \in \{0, ..., c-1\}$, then

$$\left\{s(v_{h_0}) = s(w_{r_0}) = a_{r_0}, ..., s(v_{h_{c-1}}) = s(w_{r_{c-1}}) = a_{r_{c-1}}\right\}$$

is linearly independent.

**Case 2.** $x_{j_0} \in \overline{v}$.

**Subcase 1.** $\overline{x} \setminus \overline{v} \neq \emptyset$.

Notice that $\overline{x} \neq \{x_{j_0}\}$ because if not then $\overline{x} \setminus \overline{v} = \{x_{j_0}\}$ and so $x_{j_0} \notin \overline{v}$. Hence

$$s(x_{j_0}) = \sum_{i=1}^{n-1} a_{p_i}.$$

Let $(\overline{v} \setminus \{x_{j_0}\}) \cap \overline{x} = \left\{v_{h'_0}, ..., v_{h'_{d-1}}\right\}$, $\overline{v} \setminus \overline{x} = \left\{v_{h''_0}, ..., v_{h''_{t-1}}\right\}$, $w_{r'_i} = v_{h'_i}$ for every $i \in \{0, ..., d-1\}$ and $w_{r''_i} = v_{h''_i}$ for every $i \in \{0, ..., t-1\}$

Suppose now that the set

$$\left\{a_{r'_0}, ..., a_{r'_{d-1}}, \sum_{i=1}^{n-1} a_{p_i}, a_{r''_0}, ..., a_{r''_{t-1}}\right\}$$

is linearly dependent, then there exists $\overline{f} \in \mathbb{K}^d$, $l \in \mathbb{K}$ and $\overline{g} \in \mathbb{K}^t$ such that

$$\sum_{i=0}^{d-1} f_i(a_{r'_i}) + l(\sum_{i=1}^{n-1} a_{p_i}) + \sum_{i=0}^{t-1} g_i(a_{r''_i}) = 0$$

with $\overline{f} \neq (0_0, ..., 0_{d-1})$ or $l \neq 0$ or $\overline{g} \neq (0_0, ..., 0_{t-1})$.

Let $V = \{i \in \{1, ..., n-1\} \mid x_i \in (\overline{v} \setminus \{x_{j_0}\}) \cap \overline{x}\}$. In each three of the cases the linear combination

$$\sum_{i=0}^{d-1} (f_i + l)(a_{r'_i}) + l(\sum_{\substack{i=1 \\ i \notin V}}^{n-1} a_{p_i}) + \sum_{i=0}^{t-1} g_i(a_{r''_i}) = 0$$

is non trivial. Thus the set

$$\left\{s(x_{j_1}), ..., s(x_{j_{n-1}}), a_{r''_0}, ..., a_{r''_{t-1}}\right\}$$

is linearly dependent, which is absurd.

**Subcase 2.** $\overline{x} \subseteq \overline{v}$.

This case is not possible. Suppose indeed it is, then by rule ($c_2$.) we can assume that $\overline{v} = \overline{x}\,\overline{v}'$ with $\overline{v}' \subseteq \mathrm{Var} \setminus \overline{x}$. Thus by rule ($b_2$.) we have that $\Sigma \vdash \perp(\overline{x})$ which is absurd.

□

## 4.3 Vector Space Atomic Independence Logic

In this section we define the system Vector Space Atomic Independence Logic (VSAIndL) and then prove its soundness and completeness. The syntax and deductive apparatus of this system are the same as those of AIndL.

### 4.3.1 Semantics

As in Section 3.4.1, the intuitive meaning of the atom $\overline{x} \perp \overline{y}$ in the context of vector spaces is that the only common linear combination of the vectors in $\overline{x}$ and in $\overline{y}$ is the trivial one.

**Definition 4.3.1.** Let $\mathbb{K}$ be a field and $\mathbb{V}$ a vector space over $\mathbb{K}$. Let $s : \mathrm{dom}(s) \to V$ with $\overline{x}\,\overline{y} \subseteq \mathrm{dom}(s) \subseteq \mathrm{Var}$. We say that $\mathbb{V}$ satisfies $\overline{x} \perp \overline{y}$ under $s$, in symbols $\mathbb{V} \models_s \overline{x} \perp \overline{y}$, if $\langle \{s(x) \mid x \in \overline{x}\} \rangle \cap \langle \{s(y) \mid y \in \overline{y}\} \rangle = \{0\}$.

**Definition 4.3.2.** Let $\Sigma$ be a set of atoms and let $s$ be such that the set of variables occurring in $\Sigma$ is included in $\mathrm{dom}(s)$. We say that $\mathbb{V}$ satisfies $\Sigma$ under $s$, in symbols $\mathbb{V} \models_s \Sigma$, if $\mathbb{V}$ satisfies every atom in $\Sigma$ under $s$.

**Definition 4.3.3.** Let $\Sigma$ be a set of atoms. We say that $\overline{x} \perp \overline{y}$ is a logical consequence of $\Sigma$, in symbols $\Sigma \models \overline{x} \perp \overline{y}$, if for every field $\mathbb{K}$, vector space $\mathbb{V}$ over $\mathbb{K}$ and $s$ such that the set of variables occurring in $\Sigma \cup \{\overline{x} \perp \overline{y}\}$ is included in $\mathrm{dom}(s)$ we have that

$$\text{if } \mathbb{V} \models_s \Sigma \text{ then } \mathbb{V} \models_s \overline{x} \perp \overline{y}.$$

### 4.3.2 Soundness and Completeness

**Theorem 4.3.4.** Let $\Sigma$ be a set of atoms, then

$$\Sigma \models \overline{x} \perp \overline{y} \text{ if and only if } \Sigma \vdash \overline{x} \perp \overline{y}.$$

[The deductive system to which we refer has been defined in Section 2.4.3.]

*Proof.* ($\Leftarrow$) As in Theorem 3.4.4.

($\Rightarrow$) We follow the line of argument of [17][2].

Suppose $\Sigma \nvdash \overline{x} \perp \overline{y}$. Notice that if this is the case then $\overline{x} \neq \emptyset$ and $\overline{y} \neq \emptyset$. Indeed if $\overline{y} = \emptyset$ then $\Sigma \vdash \overline{x} \perp \overline{y}$ because by rule ($a_3$.) $\vdash \overline{x} \perp \emptyset$. Analogously if $\overline{x} = \emptyset$ then $\Sigma \vdash \overline{x} \perp \overline{y}$ because by rule ($a_3$.) $\vdash \overline{y} \perp \emptyset$ and so by rule ($b_3$.) $\vdash \emptyset \perp \overline{y}$.

[2] In [17] the syntax of the system is such that for every atom $\overline{x} \perp \overline{y}$ we have that $\overline{x} \cap \overline{y} = \emptyset$, this syntactic restriction makes the proof much easier.

We can assume that $\overline{x}$ and $\overline{y}$ are injective. This is without loss of generality because clearly $\mathbb{V} \models_s \overline{x} \perp \overline{y}$ if and only if $\mathbb{V} \models_s \pi\overline{x} \perp \pi\overline{y}$, where $\pi : \text{Var}^{<\omega} \to \text{Var}^{<\omega}$ is the function that eliminates repetitions in finite sequences of variables.

Furthermore we can assume that $\overline{x} \perp \overline{y}$ is minimal, in the sense that if $\overline{x}' \subseteq \overline{x}$, $\overline{y}' \subseteq \overline{y}$ and $\overline{x}'\,\overline{y}' \neq \overline{x}\,\overline{y}$, then $\Sigma \vdash \overline{x}' \perp \overline{y}'$. This is for two reasons.

i) If $\overline{x} \perp \overline{y}$ is not minimal we can always find a minimal atom $\overline{x}^* \perp \overline{y}^*$ such that $\Sigma \nvdash \overline{x}^* \perp \overline{y}^*$, $\overline{x}^* \subseteq \overline{x}$ and $\overline{y}^* \subseteq \overline{y}$ $--$ just keep deleting elements of $\overline{x}$ and $\overline{y}$ until you obtain the desired property or until both $\overline{x}^*$ and $\overline{y}^*$ are singletons, in which case, due to the trivial independence rule (a$_3$.), $\overline{x}^* \perp \overline{y}^*$ is a minimal statement.

ii) For any $\overline{x}' \subseteq \overline{x}$ and $\overline{y}' \subseteq \overline{y}$ we have that if $\mathbb{V} \not\models_s \overline{x}' \perp \overline{y}'$ then $\mathbb{V} \not\models_s \overline{x} \perp \overline{y}$.

Let then $\overline{x} = (x_{j_0}, ..., x_{j_{n-1}})$ and $\overline{y} = (y_{k_0}, ..., y_{k_{m-1}})$ be injective and such that $\overline{x} \perp \overline{y}$ is minimal.

Let $V = \{v \in \text{Var} \mid \Sigma \vdash v \perp v\}$ and $W = \text{Var} \setminus V$. We claim that $\overline{x}, \overline{y} \not\subseteq V$. We prove it only for $\overline{x}$, the other case is symmetrical. Suppose that $\overline{x} \subseteq V$, then for every $x \in \overline{x}$ we have that $\Sigma \vdash x \perp x$ so by rule (e$_3$.), (b$_3$.) and (d$_3$.)

$$\Sigma \vdash \overline{y} \perp x_{j_0} \quad \text{and} \quad \Sigma \vdash \overline{y}\, x_{j_0} \perp x_{j_1} \Rightarrow \quad \Sigma \vdash \overline{y} \perp x_{j_0}\, x_{j_1},$$

$$\Sigma \vdash \overline{y} \perp x_{j_0}\, x_{j_1} \quad \text{and} \quad \Sigma \vdash \overline{y}\, x_{j_0}\, x_{j_1} \perp x_{j_2} \Rightarrow \quad \Sigma \vdash \overline{y} \perp x_{j_0}\, x_{j_1}\, x_{j_2},$$

$$\vdots$$

$$\Sigma \vdash \overline{y} \perp x_{j_0} \cdots x_{j_{n-2}} \quad \text{and} \quad \Sigma \vdash \overline{y}\, x_{j_0} \cdots x_{j_{n-2}} \perp x_{j_{n-1}} \Rightarrow \quad \Sigma \vdash \overline{y} \perp \overline{x}.$$

Hence by rule (b$_3$.) $\Sigma \vdash \overline{x} \perp \overline{y}$.

Thus $\overline{x} \cap W \neq \emptyset$ and $\overline{y} \cap W \neq \emptyset$. Without loss of generality suppose that $x_{j_0} \in W$ and $y_{k_0} \in W$. Let $\overline{x} \cap W = \overline{x}' = (x_{j'_0}, ..., x_{j'_{n'-1}}) = (x_{j_0}, ..., x_{j'_{n'-1}}) \neq \emptyset$ and $\overline{y} \cap W = \overline{y}' = (y_{k'_0}, ..., y_{k'_{m'-1}})$.

Let $\{w_i \mid i \in \kappa\}$ be an injective enumeration of $W \setminus \{x_{j_0}\}$. Let $\mathbb{K}$ be the two elements field, $\mathbb{V}$ the $\kappa$-dimensional vector space over it and $\{a_i \mid i \in \kappa\}$ an injective enumeration of a basis $B$ for $\mathbb{V}$. Let $s$ be the following assignment:

i) $s(v) = 0$ for every $v \in V$,

ii) $s(w_i) = a_i$ for every $i \in \kappa$,

iii) $s(x_{j_0}) = \sum_{i=1}^{n-1} s(x_{j_i}) \ + \sum_{i=0}^{m-1} s(y_{k_i})$.

We claim that $\mathbb{V} \not\models_s \overline{x} \perp \overline{y}$. By construction

$$s(x_{j_0}) = \sum_{i=1}^{n-1} s(x_{j_i}) \ + \sum_{i=0}^{m-1} s(y_{k_i}),$$

thus

$$\sum_{i=0}^{n-1} s(x_{j_i}) = \sum_{i=0}^{m-1} s(y_{k_i}).$$

Furthermore

$$\sum_{i=0}^{m-1} s(y_{k_i}) \neq 0,$$

because otherwise the set $\left\{s(y_{k_0}), ..., s(y_{k_{m-1}})\right\}$ would be linearly dependent.

Let now $\overline{v} \perp \overline{w} \in \Sigma$, we want to show that $\mathbb{V} \models_s \overline{v} \perp \overline{w}$. As before, we assume, without loss of generality, that $\overline{v}$ and $\overline{w}$ are injective. Notice also that if $\overline{v} = \emptyset$ or $\overline{w} = \emptyset$, then $\mathbb{V} \models_s \overline{v} \perp \overline{w}$. Thus let $\overline{v}, \overline{w} \neq \emptyset$.

**Case 1** $\overline{v} \subseteq V$ or $\overline{w} \subseteq V$.

Suppose that $\overline{v} \subseteq V$, the other case is symmetrical, then

$$\langle\{s(v) \mid v \in \overline{v}\}\rangle \cap \langle\{s(w) \mid w \in \overline{w}\}\rangle = \langle 0\rangle \cap \langle\{s(w) \mid w \in \overline{w}\}\rangle = \langle 0\rangle = \{0\}\,.$$

**Case 2** $\overline{v} \not\subseteq V$ and $\overline{w} \not\subseteq V$.

Let $\overline{v} \cap W = \overline{v}' = (v_{p_0}, ..., v_{p_{l-1}}) \neq \emptyset$ and $\overline{w} \cap W = \overline{w}' = (w'_{r_0}, ..., w'_{r_{q-1}}) \neq \emptyset$. Notice that

$$\langle\{s(v) \mid v \in \overline{v}\}\rangle \cap \langle\{s(w) \mid w \in \overline{w}\}\rangle = \langle\{s(v') \mid v' \in \overline{v}'\}\rangle \cap \langle\{s(w') \mid w' \in \overline{w}'\}\rangle$$

because if $u \in \overline{v}\,\overline{w} \setminus \overline{v}'\,\overline{w}'$, then $s(u) = 0$. Hence $\mathbb{V} \models_s \overline{v} \perp \overline{w}$ if and only if $\mathbb{V} \models_s \overline{v}' \perp \overline{w}'$.

**Subcase 2.1** $x_{j_0} \notin \overline{v}'\,\overline{w}'$.

Suppose that

$$\langle\{s(v') \mid v' \in \overline{v}'\}\rangle \cap \langle\{s(w') \mid w' \in \overline{w}'\}\rangle \neq \{0\}\,,$$

then there exist $\overline{b} \in K^l$ and $\overline{c} \in K^q$ such that

$$0 \neq \sum_{i=0}^{l-1} b_i s(v_{p_i}) = \sum_{i=0}^{q-1} c_i s(w'_{r_i}) \neq 0.$$

Let $i^*$ be such that $b_{i^*} \neq 0$, then we have that

$$s(v_{p_{i^*}}) = \sum_{\substack{i=0\\ i\neq i^*}}^{l-1} b_i s(v_{p_i}) + \sum_{i=0}^{q-1} c_i s(w'_{r_i}).$$

Hence the set

$$\left\{s(v_{p_0}), ..., s(v_{p_{l-1}}), s(w'_{r_0}), ..., s(w'_{r_{q-1}})\right\}$$

is not linearly independent, which is absurd.

**Subcase 2.2** $x_{j_0} \in \overline{v}'\,\overline{w}'$.

Let $\overline{x} \cap W = \overline{x}' = (x_{j'_0}, ..., x_{j'_{n'-1}}) = (x_{j_0}, ..., x_{j'_{n'-1}}) \neq \emptyset$ and $\overline{y} \cap W = \overline{y}' = (y_{k'_0}, ..., y_{k'_{m'-1}})$. Notice that $\overline{x}' \cap \overline{y}' = \emptyset$. Indeed let $z \in \overline{x}' \cap \overline{y}'$, then by rules ($b_3$.) and ($c_3$.) we have that $\Sigma \vdash z \perp z$. Thus $z \in V$, a contradiction.

**Subsubcase 2.2.1** $\overline{x}'\,\overline{y}' \setminus \overline{v}'\,\overline{w}' \neq \emptyset$.

Suppose that

$$\langle\{s(v') \mid v' \in \overline{v}'\}\rangle \cap \langle\{s(w') \mid w' \in \overline{w}'\}\rangle \neq \{0\}\,,$$

then there exist $\overline{d} \in K^l$ and $\overline{f} \in K^q$ such that

$$0 \neq \sum_{i=0}^{l-1} d_i s(v_{p_i}) = \sum_{i=0}^{q-1} f_i s(w'_{r_i}) \neq 0.$$

Suppose that $x_{j_0} \in \overline{v}'$, the case $x_{j_0} \in \overline{w}'$ is symmetrical, then there exists $\hat{i}$ such that $x_{j_0} = v_{p_{\hat{i}}}$. Suppose that $d_{\hat{i}} = 0$, then

$$0 \neq \sum_{\substack{i=0 \\ i \neq \hat{i}}}^{l-1} d_i s(v_{p_i}) = \sum_{i=0}^{q-1} f_i s(w'_{r_i}) \neq 0$$

so we are in the same situation as the case above and hence we have a contradiction.

Suppose that $d_{\hat{i}} = 1$, then we have that

$$s(x_{j_0}) = \sum_{\substack{i=0 \\ i \neq \hat{i}}}^{l-1} d_i s(v_{p_i}) + \sum_{i=0}^{q-1} f_i s(w'_{r_i}).$$

Let now $\overline{u} = (u_{p_0}, ..., u_{p_{t-1}}) = \overline{v}' \, \overline{w}' \setminus \overline{x}' \, \overline{y}'$, $\overline{x}'' = \overline{x}' \cap (\overline{v}' \, \overline{w}')$ and $\overline{y}'' = \overline{y}' \cap (\overline{v}' \, \overline{w}')$. Let also $V = \{ i \in \{1, ..., n'-1\} \mid x_{j'_i} \notin \overline{x}'' \}$ and $U = \{ i \in \{0, ..., m'-1\} \mid y_{k'_i} \notin \overline{y}'' \}$. Then there exists $\overline{a}, \overline{b}, \overline{c} \in K^{<\omega}$ such that

$$s(x_{j_0}) = \sum_{\substack{i=1 \\ i \notin V}}^{n'-1} a_i s(x_{j'_i}) \; + \sum_{\substack{i=0 \\ i \notin U}}^{m'-1} b_i s(y_{k'_i}) \; + \sum_{i=0}^{t-1} c_i s(u_{p_i}).$$

Let $V' = V \cup \{i \in \{1, ..., n'-1\} \mid a_i = 0\}$, $U' = U \cup \{i \in \{0, ..., m'-1\} \mid b_i = 0\}$ and $W = \{i \in \{0, ..., t-1\} \mid c_i = 0\}$, we then have that

$$s(x_{j_0}) = \sum_{\substack{i=1 \\ i \notin V'}}^{n'-1} s(x_{j'_i}) \; + \sum_{\substack{i=0 \\ i \notin U'}}^{m'-1} s(y_{k'_i}) \; + \sum_{\substack{i=0 \\ i \notin W}}^{t-1} s(u_{p_i}).$$

As we noticed above though

$$s(x_{j_0}) = \sum_{i=1}^{n-1} s(x_{j_i}) \; + \sum_{i=0}^{m-1} s(y_{k_i}) = \sum_{i=1}^{n'-1} s(x_{j'_i}) \; + \sum_{i=0}^{m'-1} s(y_{k'_i}),$$

so

$$\sum_{i=1}^{n'-1} s(x_{j'_i}) \; + \sum_{i=0}^{m'-1} s(y_{k'_i}) = \sum_{\substack{i=1 \\ i \notin V'}}^{n'-1} s(x_{j'_i}) \; + \sum_{\substack{i=0 \\ i \notin U'}}^{m'-1} s(y_{k'_i}) \; + \sum_{\substack{i=0 \\ i \notin W}}^{t-1} s(u_{p_i}).$$

Let $z^* \in \overline{x}' \, \overline{y}' \setminus \overline{v}' \, \overline{w}'$ and suppose that $z^* \in \overline{x}''$, the other case is symmetrical, then there exists $i^* \in \{1, ..., n'-1\}$ such that $z^* = x_{j'_{i^*}}$. We then have that

$$s(x_{j'_{i^*}}) = \sum_{\substack{i=1 \\ i \neq i^*}}^{n'-1} s(x_{j'_i}) \; + \sum_{i=0}^{m'-1} s(y_{k'_i}) \; + \sum_{\substack{i=1 \\ i \notin V'}}^{n'-1} s(x_{j'_i}) \; + \sum_{\substack{i=0 \\ i \notin U'}}^{m'-1} s(y_{k'_i}) \; + \sum_{\substack{i=0 \\ i \notin W}}^{t-1} s(u_{p_i}).$$

Hence the set

$$\left\{ s(x_{j'_1}), ..., s(x_{j'_{n'-1}}), s(y_{k'_0}), ..., s(y_{k'_{m'-1}}), s(u_{p_0}), ..., s(u_{p_{t-1}}) \right\}$$

is not linearly independent, which is absurd.

**Subsubcase 2.2.2** $\overline{x}' \, \overline{y}' \subseteq \overline{v}' \, \overline{w}'$.

As shown in Theorem 3.4.4, this case is not possible.

□

## 4.4 Algebraically Closed Field Atomic Dependence Logic

From this section on we pass from the analysis of linear dependence and independence to the analysis of algebraic dependence and independence.

In this section in particular we define the system Algebraically Closed Field Atomic Dependence Logic (ACFADL) and then prove its soundness and completeness. The syntax and deductive apparatus of this system are the same as those of ADL.

### 4.4.1 Semantics

In the context of algebraically closed fields we think of the atom $=(\overline{x}, \overline{y})$ as expressing that each element in $\overline{y}$ is bound to the elements in $\overline{x}$ via the existence of a polynomial with coefficients from the subfield generated by the elements in $\overline{x}$. We do not define $=(\overline{x}, \overline{y})$ by saying that each variable in $\overline{y}$ is a polynomial or an algebraic expression of the elements in $\overline{x}$, although we could and that would perhaps be worth studying. The reason for the adopted concept, which is also the concept of dependence used in algebra, is that there famously are polynomial equations of even as low degree as five, which cannot be solved in terms of radicals.

**Definition 4.4.1.** Let $\mathbb{K}$ be an algebraically closed field and $s : \mathrm{dom}(s) \to K$ with $\overline{x}\,\overline{y} \subseteq \mathrm{dom}(s) \subseteq \mathrm{Var}$. We say that $\mathbb{K}$ satisfies $=(\overline{x}, \overline{y})$ under $s$, in symbols $\mathbb{K} \models_s =(\overline{x}, \overline{y})$, if for every $y \in \overline{y}$ we have that $s(y)$ is algebraic over the subfield $\mathbb{F}$ of $\mathbb{K}$ generated by $\{s(x) \mid x \in \overline{x}\}$, that is there exists a non-trivial polynomial

$$P(X) = X^n + \sum_{i=0}^{n-1} a_i X^i$$

with coefficients in $\mathbb{F}$ such that $P(s(y)) = 0$.

Let $\mathbb{K}$ be an algebraically closed field, $\mathbb{F}$ a subfield of $\mathbb{K}$ and $a \in K$. If $a$ is not algebraic over $\mathbb{F}$, then we say that $a$ is transcendental over $\mathbb{F}$.

**Definition 4.4.2.** Let $\Sigma$ be a set of atoms and let $s$ be such that the set of variables occurring in $\Sigma$ is included in $\mathrm{dom}(s)$. We say that $\mathbb{K}$ satisfies $\Sigma$ under $s$, in symbols $\mathbb{K} \models_s \Sigma$, if $\mathbb{K}$ satisfies every atom in $\Sigma$ under $s$.

**Definition 4.4.3.** Let $\Sigma$ be a set of atoms. We say that $=(\overline{x}, \overline{y})$ is a logical consequence of $\Sigma$, in symbols $\Sigma \models =(\overline{x}, \overline{y})$, if for every algebraically closed field $\mathbb{K}$ and $s$ such that the set of variables occurring in $\Sigma \cup \{=(\overline{x}, \overline{y})\}$ is included in $\mathrm{dom}(s)$ we have that

$$\text{if } \mathbb{K} \models_s \Sigma \text{ then } \mathbb{K} \models_s =(\overline{x}, \overline{y}).$$

### 4.4.2 Soundness and Completeness

**Theorem 4.4.4.** Let $\Sigma$ be a set of atoms, then

$$\Sigma \models =(\overline{x}, \overline{y}) \text{ if and only if } \Sigma \vdash =(\overline{x}, \overline{y}).$$

[The deductive system to which we refer has been defined in Section 2.2.3.]

*Proof.* ($\Leftarrow$) We prove only the soundness of rules ($a_1$.) and ($c_1$.) . Let $\mathbb{K}$ be an algebraically closed field and $s$ an appropriate assignment.

($a_1$.) We want to show that $\mathbb{K} \models_s =(\overline{x}, \overline{x})$. If $\overline{x} = \emptyset$ this is trivially true, suppose then that $\overline{x} \neq \emptyset$ and let $x \in \overline{x}$ and $s(x) = a$. Let $P(X) = X - a$, then clearly $P(a) = 0$. Notice also that $P$ has coefficients in the subfield $\mathbb{F}$ of $\mathbb{K}$ generated by $\{s(x) \mid x \in \overline{x}\}$ because $a \in F$.

($c_1$.) First we state a useful characterization of algebraic dependence and a lemma about field extensions, for the proofs of these results see [30, Proposition 1.30 and Proposition 1.20].

**Proposition 4.4.5.** Let $\mathbb{F}$ be a subfield of a field $\mathbb{K}$ and $\mathbb{F}(a)$ be the subfield of $\mathbb{K}$ generated over $\mathbb{F}$ by $\{a\}$. Then $a$ is algebraic over $\mathbb{F}$ if and only if $[\mathbb{F}(a) : \mathbb{F}]$ is finite, where $[\mathbb{F}(a) : \mathbb{F}]$ denotes the degree of $\mathbb{F}(a)$ over $\mathbb{F}$, that is the dimension of $\mathbb{F}(a)$ considered as a vector space over $\mathbb{F}$. □

**Lemma 4.4.6.** Let $\mathbb{K}$, $\mathbb{F}$ and $\mathbb{E}$ be fields with $\mathbb{K}$ extending $\mathbb{F}$ and $\mathbb{F}$ extending $\mathbb{E}$. Then $\mathbb{K}/\mathbb{E}$ is of finite degree if and only if $\mathbb{K}/\mathbb{F}$ and $\mathbb{F}/\mathbb{E}$ are both of finite degree. □

Suppose now that $\mathbb{K} \models_s =(\overline{x}, \overline{y})$ and $\mathbb{K} \models_s =(\overline{y}, \overline{z})$, we want to show that $\mathbb{K} \models_s =(\overline{x}, \overline{z})$. Let $z \in \overline{z}$, $s(z) = c$, $s(\overline{x}) = \overline{a}$ and $s(\overline{y}) = \overline{b}$. Let then $\mathbb{F}$ be the subfield of $\mathbb{K}$ generated by $\overline{a}$ and $\mathbb{F}'$ be the subfield of $\mathbb{K}$ generated by $\overline{b}$.

By assumption every $b \in \overline{b}$ is algebraic over $\mathbb{F}$, so by Proposition 4.4.5 for every $b \in \overline{b}$ we have that $[\mathbb{F}(b) : \mathbb{F}]$ is finite and hence by Lemma 4.4.6 we have that $[\mathbb{F}(\overline{b}) : \mathbb{F}]$ is finite. Furthermore $c$ is algebraic over $\mathbb{F}'$ so clearly it is algebraic over $\mathbb{F}'(\overline{a})$ and hence by Proposition 4.4.5 $[\mathbb{F}'(\overline{a}c) : \mathbb{F}'(\overline{a})]$ is finite. Notice now that $\mathbb{F}'(\overline{a}) = \mathbb{F}(\overline{b})$ because this field is nothing but the subfield of $\mathbb{K}$ generated by $\overline{a}\,\overline{b}$. Thus we conclude that both $[\mathbb{F}(\overline{b}) : \mathbb{F}]$ and $[\mathbb{F}(\overline{b}c) : \mathbb{F}(\overline{b})]$ are finite and hence by Lemma 4.4.6 we have that $[\mathbb{F}(\overline{b}c) : \mathbb{F}]$ is finite.

Suppose now that $c$ is transcendental over $\mathbb{F}$, then by Proposition 4.4.5 $[\mathbb{F}(c) : \mathbb{F}]$ is infinite and hence by Lemma 4.4.6 $[\mathbb{F}(\overline{b}c) : \mathbb{F}]$ is also infinite, which is a contradiction. Thus $c$ is algebraic over $\mathbb{F}$ and hence $\mathbb{K} \models_s =(\overline{x}, \overline{y})$.

($\Rightarrow$) Suppose $\Sigma \nvdash =(\overline{x}, \overline{y})$. Let $V = \{z \in \mathrm{Var} \mid \Sigma \vdash =(\overline{x}, z)\}$ and $W = \mathrm{Var} \backslash V$. First notice that $\overline{y} \neq \emptyset$, indeed if not so, by the syntactic constraints that we put on the system, we have that $\overline{x}, \overline{y} = \emptyset$ and so by the admitted degenerate case of rule ($a_1$.) we have that $\Sigma \vdash =(\overline{x}, \overline{y})$. Furthermore $\overline{y} \cap W \neq \emptyset$, indeed if $\overline{y} \cap W = \emptyset$ then for every $y \in \overline{y}$ we have that $\Sigma \vdash =(\overline{x}, y)$ and so by rules ($d_1$.), ($e_1$.) and, if necessary, ($f_1$.)[3] we have that $\Sigma \vdash =(\overline{x}, \overline{y})$.

Let $\mathbb{C}$ be the field of complex numbers and let $s : \mathrm{Var} \to C$ be the following assignment

$$s(v) = \begin{cases} 0 & \text{if } v \in V \\ \pi & \text{if } v \in W. \end{cases}$$

We claim that $\mathbb{K} \not\models_s =(\overline{x}, \overline{y})$. In accordance to the semantic we then have to show that there is $y \in \overline{y}$ such that $s(y)$ is transcendental over the subfield of $\mathbb{C}$ generated by $\{s(x) \mid x \in \overline{x}\}$. Let $y \in \overline{y} \cap W$ and $\overline{x} \neq \emptyset$, we then have that $s(y) = \pi$ and $s(x) = 0$ for every $x \in \overline{x}$, because for $x \in \overline{x}$ we have $\Sigma \vdash =(\overline{x}, x)$. Indeed by rule ($a_1$.) $\vdash =(\overline{x}, \overline{x})$ and so by rule ($b_1$.) $\vdash =(\overline{x}, x)$.

[3]Notice that ($f_1$.) is necessary only if $\overline{x}$ or $\overline{y}$ contain repetitions.

Notice that the subfield of $\mathbb{C}$ generated by 0 is the field $\mathbb{Q}$ of rational numbers and clearly $\pi$ is transcendental over it. Finally if $\overline{x} = \emptyset$ then we are also done because the subfield of $\mathbb{C}$ generated by the empty set is again the field $\mathbb{Q}$.

Let now $=(\overline{x}', \overline{y}') \in \Sigma$, we want to show that $\mathbb{K} \models_s =(\overline{x}', \overline{y}')$. If $\overline{y}' = \emptyset$ then also $\overline{x}' = \emptyset$ and so trivially $\mathbb{K} \models_s =(\overline{x}', \overline{y}')$. Noticed this, for the rest of the proof we assume $\overline{y}' \neq \emptyset$.

**Case 1.** $\overline{x}' = \emptyset$.

Suppose that $\mathbb{K} \not\models_s =(\emptyset, \overline{y}')$, then there exists $y' \in \overline{y}'$ such that $s(y') = \pi$, so $\Sigma \nvdash =(\overline{x}, y')$. Notice though that $\Sigma \vdash =(\emptyset, \overline{y}')$, so by rule ($b_1$.) $\Sigma \vdash =(\emptyset, y')$ and hence again by rule ($b_1$.) $\Sigma \vdash =(\overline{x}, y')$.

**Case 2.** $\overline{x}' \neq \emptyset$ and $\overline{x}' \subseteq V$.

If this is the case, then

$$\begin{aligned}
\forall x' \in \overline{x}' \ \ \Sigma \vdash =(\overline{x}, x') &\implies \Sigma \vdash =(\overline{x}, \overline{x}') \ \ \text{[by rules (d}_1\text{.), (e}_1\text{.) and (f}_1\text{.)]} \\
&\implies \Sigma \vdash =(\overline{x}, \overline{y}') \ \ \text{[by rule (c}_1\text{.)]} \\
&\implies \forall y' \in \overline{y}' \ \ \Sigma \vdash =(\overline{x}', y') \ \ \text{[by rule (b}_1\text{.)]} \\
&\implies \overline{y}' \subseteq V.
\end{aligned}$$

If $\overline{x}' \subseteq V$ then for every $x' \in \overline{x}'$ we have that $s(x') = 0$ so again the subfield of $\mathbb{C}$ generated by $\{s(x') \mid x' \in \overline{x}'\}$ is $\mathbb{Q}$. Let now $y' \in \overline{y}'$, then we have that $s(y') = 0$ and clearly 0 is algebraic over $\mathbb{Q}$.

**Case 3.** $\overline{x}' \cap W \neq \emptyset$.

If this is the case, then there exists $x' \in \overline{x}'$ such that $\Sigma \nvdash =(\overline{x}, x')$, so we have $x' \in \overline{x}'$ such that $s(x') = \pi$. Hence the subfield of $\mathbb{K}$ generated by $\{s(x) \mid x \in \overline{x}\}$ is $\mathbb{Q}(\pi)$ and then in both cases $s(y') = \pi$ and $s(y') = 0$ we have algebraic dependence.

□

## 4.5 Independence in Algebraically Closed Fields

In this section we develop some of the theory of independent sets in algebraically closed fields and then define a ternary independence relation between a tuple of elements, a subset and a subfield of an algebraically closed field $\mathbb{K}$. In Section 4.7 we will use this relation to give an algebraically closed field interpretation of the independence atom $\overline{x} \perp \overline{y}$.

Let $\mathbb{K}$ be an algebraically closed field and $\mathbb{D}, \mathbb{E}, \mathbb{F}$ subfields of $\mathbb{K}$.

**Lemma 4.5.1.** Let $A = \{a_0, ..., a_{n-1}\} \subseteq K$. The following are equivalent.

i) For every $a \in A$ we have that $a$ is transcendental over the subfield of $\mathbb{K}$ generated by $A \setminus \{a\}$ over $\mathbb{F}$.

ii) A is algebraically independent over $\mathbb{F}$, that is for every non-trivial polynomial $P(X_0, ..., X_{n-1}) \in \mathbb{F}[X_0, ..., X_{n-1}]$ we have that

$$P(a_0, ..., a_{n-1}) \neq 0.$$

*Proof.* i) $\Rightarrow$ ii) Let $P(X_0, ..., X_{n-1}) \in \mathbb{F}[X_0, ..., X_{n-1}]$ be a non-trivial polynomial such that $P(a_0, ..., a_{n-1}) = 0$. Suppose that $P(X_0, ..., X_{n-1})$ is a constant polynomial, then there are two cases, either $P(X_0, ..., X_{n-1}) = 0$ or

$P(X_0, ..., X_{n-1}) \neq 0$, in the first case $P(X_0, ..., X_{n-1})$ is trivial and in the second $0 \neq P(X_0, ..., X_{n-1}) = P(a_0, ..., a_{n-1}) = 0$. Let then $i^*$ be such that $X_{i^*}$ occurs in $P(X_0, ..., X_{n-1})$ and let $A \setminus \{a_{i^*}\} = \{a_{k_1}, ..., a_{k_{n-1}}\}$. Then we have that $P(X_{i^*}, a_{k_1}, ..., a_{k_{n-1}}) := Q(X_{i^*}) \in \mathbb{F}(A \setminus \{a_{i^*}\})[X_{i^*}]$ and $Q(a_{i^*}) = 0$ so $a_{i^*}$ is algebraic over $\mathbb{F}(A \setminus \{a_{i^*}\})$.

ii) $\Rightarrow$ i) Let $a \in A$ and $A \setminus \{a\} = \{a_{k_1}, ..., a_{k_{n-1}}\}$. Suppose that $a$ is not transcendental over $\mathbb{F}(A \setminus \{a\})$, then there exists a non-trivial $Q(X_0) \in \mathbb{F}(A \setminus \{a\})[X_0]$ such that $Q(a) = 0$.

Let $Q(X_0) = X_0^p + \sum_{i=0}^{p-1} c_i X_0^i$. Now $Q(X_0) \in \mathbb{F}(A \setminus \{a\})[X_0]$ so each $c_i \in F(A \setminus \{a\})$, which means that

$$c_i = \sum_{j=0}^{m_i - 1} q_{j_{1,i} \cdots j_{n-1,i}} a_{k_1}^{j_{1,i}} \cdots a_{k_{n-1}}^{j_{n-1,i}}$$

where $q_{j_{1,i} \cdots j_{n-1,i}} \in F$ for every $j \in \{0, ..., m_i - 1\}$, for this see [30, Lemma 1.24]. So

$$Q(X_0) = X_0^p + \sum_{i=0}^{p-1} \sum_{j=0}^{m_i - 1} q_{j_{1,i} \cdots j_{n-1,i}} a_{k_1}^{j_{1,i}} \cdots a_{k_{n-1}}^{j_{n-1,i}} X_0^i.$$

Consider now the following polynomial

$$P(X_0, X_{k_1}, ..., X_{k_{n-1}}) = X_0^p + \sum_{i=0}^{p-1} \sum_{j=0}^{m_i - 1} q_{j_{1,i} \cdots j_{n-1,i}} X_{k_1}^{j_{1,i}} \cdots X_{k_{n-1}}^{j_{n-1,i}} X_0^i,$$

$P(X_0, X_{k_1}, ..., X_{k_{n-1}}) \in \mathbb{F}[X_0, ..., X_{n-1}]$, $P(X_0, X_{k_1}, ..., X_{k_{n-1}})$ is non-trivial and $P(a, a_{k_1}, ..., a_{k_{n-1}}) = 0$. Hence A is not algebraically independent over $\mathbb{F}$.

□

If $A \subseteq K$ is infinite, then we say that $A$ is independent over $\mathbb{F}$ if every finite subset of $A$ is independent over $\mathbb{F}$. Using the above characterization of algebraic independence and the fact that if $a \in K$ is algebraic over $\mathbb{F}(A)$ then $a$ is algebraic over $\mathbb{F}(A_0)$ with $A_0 \subseteq_{\text{fin}} A$, it is possible to prove the following proposition.

**Proposition 4.5.2.** Let $A \subseteq K$, then $A$ is independent over $\mathbb{F}$ if and only if for every $a \in A$ we have that $a$ is transcendental over the subfield of $\mathbb{K}$ generated by $A \setminus \{a\}$ over $\mathbb{F}$. □

We denote by $\mathbb{P}$ the subfield of $\mathbb{K}$ generated by the empty set, we call this field the prime field of $\mathbb{K}$. We say that a set $A \subseteq K$ is independent if it is independent over $\mathbb{P}$.

**Lemma 4.5.3** (Exchange Principle)**.** Let $A \subseteq K$ and $b \in K$. If $a$ is algebraic over $\mathbb{F}(A \cup \{b\})$ but $a$ is not algebraic over $\mathbb{F}(A)$, then $b$ is algebraic over $\mathbb{F}(A \cup \{a\})$.

*Proof.* See [30, Lemma 8.6].

□

Let $\mathbb{F}$ be a subfield of $\mathbb{E}$. We say that $\mathbb{E}$ is algebraic over $\mathbb{F}$ or that $\mathbb{E}$ is an algebraic extension of $\mathbb{F}$ if every element of $\mathbb{E}$ is algebraic over $\mathbb{F}$.

**Lemma 4.5.4** (Transitivity of Algebraic Dependence)**.** If $\mathbb{D}$ is algebraic over $\mathbb{E}$, and $\mathbb{E}$ is algebraic over $\mathbb{F}$, then $\mathbb{D}$ is algebraic over $\mathbb{F}$.

*Proof.* See [30, Lemma 8.7]. □

**Definition 4.5.5.** We say that $B \subseteq E$ is a *transcendence basis* for $\mathbb{E}$ over $\mathbb{F}$ if $B$ is algebraically independent over $\mathbb{F}$ and $\mathbb{E}$ is algebraic over $\mathbb{F}(B)$.

**Proposition 4.5.6.** Let $B \subseteq E$, the following are equivalent:

i) $B$ is a maximally independent over $\mathbb{F}$ subset of $E$;

ii) $B$ is a basis for $\mathbb{E}$ over $\mathbb{F}$;

iii) $B$ is a minimal subset of $E$ such that $\mathbb{E}$ is algebraic over $\mathbb{F}(B)$.

*Proof.* i) $\Rightarrow$ ii). Suppose that there exists $a \in E$ such that $a$ is not algebraic over $\mathbb{F}(B)$. We claim that $B \cup \{a\}$ is independent over $\mathbb{F}$. Suppose not, then there exists $b \in B \cup \{a\}$ such that $b$ is algebraic over $\mathbb{F}((B \cup \{a\}) \setminus \{b\})$. By hypothesis $a$ is not algebraic over $\mathbb{F}(B)$ so $b \neq a$ and hence $b$ is algebraic over $\mathbb{F}((B \setminus \{b\}) \cup \{a\})$. Notice now that $B$ is independent over $\mathbb{F}$, so $b$ is not algebraic over $\mathbb{F}(B \setminus \{b\})$. Hence by the Exchange Principle we have that $a$ is algebraic over $\mathbb{F}(B)$, a contradiction.

ii) $\Rightarrow$ iii). Suppose there exists $B' \subsetneq B$ such that $\mathbb{E}$ is algebraic over $\mathbb{F}(B')$. Let $b \in B \setminus B'$, then $b$ is algebraic over $\mathbb{F}(B')$ and so $b$ is algebraic over $\mathbb{F}(B \setminus \{b\})$ since $B' \subseteq B \setminus \{b\}$. Hence $B$ is not independent over $\mathbb{F}$.

iii) $\Rightarrow$ i). Suppose there exists $b \in B$ such that $b$ is algebraic over $\mathbb{F}(B \setminus \{b\})$, then $\mathbb{F}(B)$ is algebraic over $\mathbb{F}(B \setminus \{b\})$. By hypothesis $\mathbb{E}$ is algebraic over $\mathbb{F}(B)$ so by Transitivity of Algebraic Dependence we have that $\mathbb{E}$ is algebraic over $\mathbb{F}(B \setminus \{b\})$ which contradicts the minimality of $B$. Thus $B$ is algebraically independent over $\mathbb{F}$. Let now $a \in E \setminus B$ and suppose that $B \cup \{a\}$ is independent over $\mathbb{F}$, then $a$ is not algebraic over $\mathbb{F}(B)$ and so $\mathbb{E}$ is not algebraic over $\mathbb{F}(B)$. □

**Proposition 4.5.7.** Let $A_1 \subseteq K$ and $A_0 \subseteq A_1$ independent over $\mathbb{F}$. Then $A_0$ can be extended to a maximally independent over $\mathbb{F}$ subset of $A_1$. In particular for every subfield $\mathbb{F}$ and $\mathbb{E}$ of $\mathbb{K}$, there exists $A \subseteq E$ such that $A$ is a basis for $\mathbb{E}$ over $\mathbb{F}$.

*Proof.* See [30, Theorem 8.13]. Notice that the proof of this theorem requires Zorn's Lemma. □

It is possible to show that any two (possibly infinite) transcendence bases for $\mathbb{E}$ over $\mathbb{F}$ have the same cardinality. The cardinality of a transcendence basis for $\mathbb{E}$ over $\mathbb{F}$ is called the *transcendence degree* of $\mathbb{E}$ over $\mathbb{F}$ and denoted by $\mathrm{trdg}(\mathbb{E}/\mathbb{F})$.

The following two lemmas are not of particular interest but they will be relevant in the proof of Theorem 4.7.4, this is the reason for which we state them here.

**Lemma 4.5.8.** If $A$ is independent over $\mathbb{F}$, then $\text{trdg}(\mathbb{F}(A)/\mathbb{F}) = |A|$.

*Proof.* We show that $A$ is a minimal subset of $F(A)$ such that $\mathbb{F}(A)$ is algebraic over $\mathbb{F}(A)$, by Proposition 4.5.6 this suffices. Suppose that there is $B \subsetneq A$ such that $\mathbb{F}(A)$ is algebraic over $\mathbb{F}(B)$. Let $b \in A \setminus B$, then $b$ is algebraic over $\mathbb{F}(B)$ so $b$ is algebraic over $\mathbb{F}(A \setminus \{b\}))$, since $B \subseteq A \setminus \{b\}$. Thus $A$ is not independent over $\mathbb{F}$.

□

**Lemma 4.5.9.** Let $A \subseteq K$ be independent over $\mathbb{F}$. Let $D_0, D_1 \subseteq A$ and $D_0 \cap D_1 = \emptyset$, then

i) $D_0$ is independent over $\mathbb{F}(D_1)$;

ii) $\text{trdg}(\mathbb{F}(D_0)/\mathbb{F}(D_1)) = \text{trdg}(\mathbb{F}(D_0)/\mathbb{F})$.

*Proof.* i) Suppose that $D_0$ is not independent over $\mathbb{F}(D_1)$, then there exists $d \in D_0$ such that $d$ is algebraic over $\mathbb{F}(D_1 \cup (D_0 \setminus \{d\}))$. By hypothesis $D_0 \cap D_1 = \emptyset$, so $d$ is algebraic over $\mathbb{F}((D_1 \cup D_0) \setminus \{d\})$. Thus $D_0 \cup D_1$ is dependent over $\mathbb{F}$, a contradiction.

ii) By i) $D_0$ is independent over $\mathbb{F}(D_1)$ and thus independent over $\mathbb{F}$, hence by Lemma 4.5.8 we have that $\text{trdg}(\mathbb{F}(D_0)/\mathbb{F}(D_1)) = |D_0| = \text{trdg}(\mathbb{F}(D_0)/\mathbb{F})$.

□

**Proposition 4.5.10.** $\text{trdg}(\mathbb{F}(\overline{a})/\mathbb{F}) \leq |\overline{a}|$.

*Proof.* Let $\overline{a}' \subseteq \overline{a}$ be independent over $\mathbb{F}$ and such that $\mathbb{F}(\overline{a})$ is algebraic over $\mathbb{F}(\overline{a}')$. Then $\overline{a}'$ is a basis for $\mathbb{F}(\overline{a})$ over $\mathbb{F}$.

□

**Proposition 4.5.11.** If $\mathbb{F}$ is a subfield of $\mathbb{E}$, then $\text{trdg}(\mathbb{D}/\mathbb{E}) \leq \text{trdg}(\mathbb{D}/\mathbb{F})$.

*Proof.* Let $B$ be a basis for $\mathbb{D}$ over $\mathbb{F}$, then $B$ is independent over $\mathbb{F}$ and $\mathbb{D}$ is algebraic over $\mathbb{F}(B)$. Let $B' \subseteq B$ be such that $B'$ is independent over $\mathbb{E}$ and $\mathbb{F}(B)$ is algebraic over $\mathbb{E}(B')$. By choice $B'$ is independent over $\mathbb{E}$, furthermore $\mathbb{D}$ is algebraic over $\mathbb{E}(B')$ because $\mathbb{D}$ is algebraic over $\mathbb{F}(B)$ and $\mathbb{F}(B)$ is algebraic over $\mathbb{E}(B')$. Hence $B'$ is a basis for $\mathbb{D}$ over $\mathbb{E}$.

□

The notion of transcendence degree allow us to define a notion of independence with many desirable properties.

**Definition 4.5.12.** We say that $\overline{a}$ is independent from $B$ over $\mathbb{F}$ if

$$\text{trdg}(\mathbb{F}(\overline{a})/\mathbb{F}(B)) = \text{trdg}(\mathbb{F}(\overline{a})/\mathbb{F}).$$

We write $\overline{a} \mathop{\smile\!\!\!\!\!\mid}^{\text{tr}}_{\mathbb{F}} B$.

**Proposition 4.5.13.** The following are equivalent:

i) $\overline{a} \mathop{\smile\!\!\!\!\!\mid}^{\text{tr}}_{\mathbb{F}} B$;

ii) every basis for $\mathbb{F}(\overline{a})$ over $\mathbb{F}$ is a basis for $\mathbb{F}(\overline{a})$ over $\mathbb{F}(B)$;

iii) every maximally independent over $\mathbb{F}$ subset of $F(\overline{a})$ is independent over $\mathbb{F}(B)$;

iv) if $A \subseteq \mathbb{F}(\overline{a})$ is independent over $\mathbb{F}$, then $A$ is independent over $\mathbb{F}(B)$.

*Proof.* i) $\Rightarrow$ ii) Let $C$ be a basis for $\mathbb{F}(\overline{a})$ over $\mathbb{F}$, then $C$ is independent over $\mathbb{F}$ and $\mathbb{F}(\overline{a})$ is algebraic over $\mathbb{F}(C)$. Let $C' \subseteq C$ be such that $C'$ is independent over $\mathbb{F}(B)$ and $\mathbb{F}(C)$ is algebraic over $\mathbb{F}(B \cup C')$. By choice $C'$ is independent over $\mathbb{F}(B)$, furthermore $\mathbb{F}(\overline{a})$ is algebraic over $\mathbb{F}(B \cup C')$ because $\mathbb{F}(\overline{a})$ is algebraic over $\mathbb{F}(B)$ and $\mathbb{F}(B)$ is algebraic over $\mathbb{F}(B \cup C')$. Hence $C'$ is a basis for $\mathbb{F}(\overline{a})$ over $\mathbb{F}(B)$. Now, by hypothesis we have that $\operatorname{trdg}(\mathbb{F}(\overline{a})/\mathbb{F}) = \operatorname{trdg}(\mathbb{F}(\overline{a})/\mathbb{F}(B))$ and by Proposition 4.5.10 we have that $C$ is finite. Hence $C = C'$.

ii) $\Rightarrow$ iii) Immediate from Proposition 4.5.6.

iii) $\Rightarrow$ iv) Suppose that there exists $D$ independent over $\mathbb{F}$ but not over $\mathbb{F}(B)$. By Proposition 4.5.7, $D$ can be extended to a $D'$ maximally independent over $\mathbb{F}$ so there exists a maximally independent over $\mathbb{F}$ subset of $\mathbb{F}(\overline{a})$ that is dependent over $\mathbb{F}(B)$.

iv) $\Rightarrow$ i) Let $D$ be a basis for $\mathbb{F}(\overline{a})$ over $\mathbb{F}$, then $D$ is independent over $\mathbb{F}$ and so by the hypothesis it is independent over $\mathbb{F}(B)$. Furthermore $\mathbb{F}(\overline{a})$ is algebraic over $\mathbb{F}(B \cup D)$ because $\mathbb{F}(\overline{a})$ is algebraic over $\mathbb{F}(D)$. Thus $D$ is a basis for $\mathbb{F}(\overline{a})$ over $\mathbb{F}(B)$ and hence $\operatorname{trdg}(\mathbb{F}(\overline{a})/\mathbb{F}(B)) = \operatorname{trdg}(\mathbb{F}(\overline{a})/\mathbb{F})$.

□

**Lemma 4.5.14** (Monotonicity)**.** If $\overline{a} \mathop{\smile\!\!\!\!|}^{\mathrm{tr}}_{\mathbb{F}} B$ and $C \subseteq B$, then $\overline{a} \mathop{\smile\!\!\!\!|}^{\mathrm{tr}}_{\mathbb{F}} C$.

*Proof.* By Proposition 4.5.11,

$$\operatorname{trdg}(\mathbb{F}(\overline{a})/\mathbb{F}(B)) \leq \operatorname{trdg}(\mathbb{F}(\overline{a})/\mathbb{F}(C)) \leq \operatorname{trdg}(\mathbb{F}(\overline{a})/\mathbb{F}).$$

So if $\operatorname{trdg}(\mathbb{F}(\overline{a})/\mathbb{F}(B)) = \operatorname{trdg}(\mathbb{F}(\overline{a})/\mathbb{F})$, then $\operatorname{trdg}(\mathbb{F}(\overline{a})/\mathbb{F}(C)) = \operatorname{trdg}(\mathbb{F}(\overline{a})/\mathbb{F})$.

□

**Lemma 4.5.15** (Transitivity)**.** $\overline{a} \mathop{\smile\!\!\!\!|}^{\mathrm{tr}}_{\mathbb{F}} \overline{b}\,\overline{c}$ if and only if $\overline{a} \mathop{\smile\!\!\!\!|}^{\mathrm{tr}}_{\mathbb{F}} \overline{b}$ and $\overline{a} \mathop{\smile\!\!\!\!|}^{\mathrm{tr}}_{\mathbb{F}(\overline{b})} \overline{c}$.

*Proof.* By Proposition 4.5.11,

$$\operatorname{trdg}(\mathbb{F}(\overline{a})/\mathbb{F}(\overline{b}\,\overline{c})) \leq \operatorname{trdg}(\mathbb{F}(\overline{a})/\mathbb{F}(\overline{b})) \leq \operatorname{trdg}(\mathbb{F}(\overline{a})/\mathbb{F}).$$

Thus

$$\operatorname{trdg}(\mathbb{F}(\overline{a})/\mathbb{F}(\overline{b}\,\overline{c})) = \operatorname{trdg}(\mathbb{F}(\overline{a})/\mathbb{F})$$
$$\Updownarrow$$
$$\operatorname{trdg}(\mathbb{F}(\overline{a})/\mathbb{F}(\overline{b})) = \operatorname{trdg}(\mathbb{F}(\overline{a})/\mathbb{F}) \text{ and } \operatorname{trdg}(\mathbb{F}(\overline{a})/\mathbb{F}(\overline{b}\,\overline{c})) = \operatorname{trdg}(\mathbb{F}(\overline{a})/\mathbb{F}(\overline{b})).$$

□

**Lemma 4.5.16** (Symmetry)**.** If $\overline{c} \mathop{\smile\!\!\!\!|}^{\mathrm{tr}}_{\mathbb{F}} \overline{b}$, then $\overline{b} \mathop{\smile\!\!\!\!|}^{\mathrm{tr}}_{\mathbb{F}} \overline{c}$.

*Proof.* Let $D$ be a basis for $\mathbb{F}(\overline{b})$ over $\mathbb{F}$ and $C = \{c_0, ..., c_{m-1}\}$ be a basis for $\mathbb{F}(\overline{c})$ over $\mathbb{F}$. Notice that if $D = \emptyset$ or $C = \emptyset$, then $\overline{b} \mathop{\smile\!\!\!\!|}^{\mathrm{tr}}_{A} \overline{c}$. In the first case we have that $\operatorname{trdg}(\mathbb{F}(\overline{b})/\mathbb{F}) = 0 = \operatorname{trdg}(\mathbb{F}(\overline{b})/\mathbb{F}(\overline{c}))$. In the second we have that $\operatorname{trdg}(\mathbb{F}(\overline{c})/\mathbb{F}) = 0$, which implies that $\mathbb{F}(\overline{c})$ is algebraic over $\mathbb{F}$ and hence that $\operatorname{trdg}(\mathbb{F}(\overline{b})/\mathbb{F}) = \operatorname{trdg}(\mathbb{F}(\overline{b})/\mathbb{F}(\overline{c}))$ because if an element is transcendental over a field it is also transcendental over an algebraic extension of it.

Suppose that $\overline{b} \not\mathop{\smile\!\!\!\!|}^{\mathrm{tr}}_{\mathbb{F}} \overline{c}$, then there exists $d \in D$ such that $d$ is algebraic over $\mathbb{F}(\overline{c} \cup (D \setminus \{d\}))$. By hypothesis $\mathbb{F}(\overline{c})$ is algebraic over $\mathbb{F}(C)$ so we can conclude that $d$ is algebraic over $\mathbb{F}(C \cup (D \setminus \{d\}))$.

Now $D$ is independent over $\mathbb{F}$ so there exists $p \in \{0, ..., m-1\}$ such that $d$ is not algebraic over $\mathbb{F}(\{c_0, ..., c_{p-1}\} \cup (D \setminus \{d\}))$ but $d$ is algebraic over $\mathbb{F}(\{c_0, ..., c_p\} \cup (D \setminus \{d\}))$.

But then by the Exchange Principle we have that $c_p$ is algebraic over $\mathbb{F}(D \cup \{c_0, ..., c_{p-1}\})$ and hence algebraic over $\mathbb{F}(\bar{b} \cup \{c_0, ..., c_{p-1}\})$. Thus $C$ is not independent over $\mathbb{F}(\bar{b})$ and hence $\bar{c} \not\downarrow^{\mathrm{tr}}_{\mathbb{F}} \bar{b}$.

□

**Corollary 4.5.17.** $\bar{a}\,\bar{b} \downarrow^{\mathrm{tr}}_{\mathbb{F}} \bar{c}$ if and only if $\bar{a} \downarrow^{\mathrm{tr}}_{\mathbb{F}} \bar{c}$ and $\bar{b} \downarrow^{\mathrm{tr}}_{\mathbb{F}(\bar{a})} \bar{c}$.

*Proof.*

$$
\begin{array}{rcll}
\bar{a}\,\bar{b} \downarrow^{\mathrm{tr}}_{\mathbb{F}} \bar{c} & \Leftrightarrow & \bar{c} \downarrow^{\mathrm{tr}}_{\mathbb{F}} \bar{a}\,\bar{b} & \text{[by symmetry]} \\
& \Leftrightarrow & \bar{c} \downarrow^{\mathrm{tr}}_{\mathbb{F}} \bar{a} \text{ and } \bar{c} \downarrow^{\mathrm{tr}}_{\mathbb{F}(\bar{a})} \bar{b} & \text{[by transitivity]} \\
& \Leftrightarrow & \bar{a} \downarrow^{\mathrm{tr}}_{\mathbb{F}} \bar{c} \text{ and } \bar{b} \downarrow^{\mathrm{tr}}_{\mathbb{F}(\bar{a})} \bar{c} & \text{[by symmetry].}
\end{array}
$$

□

**Corollary 4.5.18.** If $\bar{a} \downarrow^{\mathrm{tr}}_{\mathbb{F}} \bar{b}$ and $\bar{a}\,\bar{b} \downarrow^{\mathrm{tr}}_{\mathbb{F}} \bar{c}$, then $\bar{a} \downarrow^{\mathrm{tr}}_{\mathbb{F}} \bar{b}\,\bar{c}$.

*Proof.*

$$
\begin{array}{cl}
\bar{a} \downarrow^{\mathrm{tr}}_{\mathbb{F}} \bar{b} \text{ and } \bar{a}\,\bar{b} \downarrow^{\mathrm{tr}}_{\mathbb{F}} \bar{c} & \\
\Downarrow & \\
\bar{a} \downarrow^{\mathrm{tr}}_{\mathbb{F}} \bar{b} \text{ and } \bar{b}\,\bar{a} \downarrow^{\mathrm{tr}}_{\mathbb{F}} \bar{c} & \\
\Downarrow & \\
\bar{a} \downarrow^{\mathrm{tr}}_{\mathbb{F}} \bar{b} \text{ and } \bar{a} \downarrow^{\mathrm{tr}}_{\mathbb{F}(\bar{b})} \bar{c} & \text{[by Corollary 4.5.17]} \\
\Downarrow & \\
\bar{a} \downarrow^{\mathrm{tr}}_{\mathbb{F}} \bar{b}\,\bar{c} & \text{[by Transitivity].}
\end{array}
$$

□

## 4.6 Algebraically Closed Field Atomic Absolute Independence Logic

In this section we define the system Algebraically Closed Field Atomic Absolute Independence Logic (ACFAAIndL) and then prove its soundness and completeness. The syntax and deductive apparatus of this system are the same as those of AAIndL.

### 4.6.1 Semantics

As for the case of vector spaces, the atom $\perp(\bar{x})$ has a natural interpretation in this context: the elements in $\bar{x}$ are algebraically independent elements.

**Definition 4.6.1.** Let $\mathbb{K}$ be an algebraically closed field and $s : \mathrm{dom}(s) \to K$ with $\bar{x} \subseteq \mathrm{dom}(s) \subseteq \mathrm{Var}$. We say that $\mathbb{K}$ satisfies $\perp(\bar{x})$ under $s$, in symbols $\mathbb{K} \models_s \perp(\bar{x})$, if for every $x \in \bar{x}$ we have that $s(x)$ is transcendental over the subfield $\mathbb{F}$ of $\mathbb{K}$ generated by $\{s(z) \mid z \in \bar{x}\} \setminus \{s(x)\}$, that is for every non-trivial polynomial

$$P(X) = X^n + \sum_{i=0}^{n-1} a_i X^i$$

with coefficients in $\mathbb{F}$ we have that $P(s(z)) \neq 0$.

Notice that, because of Lemma 4.5.1, the above condition is equivalent to the classical characterization of algebraic independence, according to which a set of elements is independent if and only if the elements in the set do not satisfy any non-trivial polynomial equation with coefficients in the prime field of $\mathbb{K}$.

**Definition 4.6.2.** Let $\Sigma$ be a set of atoms and let $s$ be such that the set of variables occurring in $\Sigma$ is included in $\mathrm{dom}(s)$. We say that $\mathbb{K}$ satisfies $\Sigma$ under $s$, in symbols $\mathbb{K} \models_s \Sigma$, if $\mathbb{K}$ satisfies every atom in $\Sigma$ under $s$.

**Definition 4.6.3.** Let $\Sigma$ be a set of atoms. We say that $\perp(\overline{x})$ is a logical consequence of $\Sigma$, in symbols $\Sigma \models \perp(\overline{x})$, if for every algebraically closed field $\mathbb{K}$ and $s$ such that the set of variables occurring in $\Sigma \cup \{\perp(\overline{x})\}$ is included in $\mathrm{dom}(s)$ we have that

$$\text{if } \mathbb{K} \models_s \Sigma \text{ then } \mathbb{K} \models_s \perp(\overline{x}).$$

### 4.6.2 Soundness and Completeness

**Theorem 4.6.4.** Let $\Sigma$ be a set of atoms, then

$$\Sigma \models \perp(\overline{x}) \text{ if and only if } \Sigma \vdash \perp(\overline{x}).$$

[The deductive system to which we refer has been defined in Section 2.3.3.]

*Proof.* ($\Leftarrow$) Obvious.

($\Rightarrow$) Suppose $\Sigma \nvdash \perp(\overline{x})$. Notice that if this is the case then $\overline{x} \neq \emptyset$. Indeed if $\overline{x} = \emptyset$ then $\Sigma \vdash \perp(\overline{x})$ because by rule (a$_2$.) $\vdash \perp(\emptyset)$.

We can assume that $\overline{x}$ is injective. This is without loss of generality because clearly $\mathbb{K} \models_s \perp(\overline{x})$ if and only if $\mathbb{K} \models_s \perp(\pi\overline{x})$, where $\pi : \mathrm{Var}^{<\omega} \to \mathrm{Var}^{<\omega}$ is the function that eliminates repetitions in finite sequences of variables. Let then $\overline{x} = (x_{j_0}, ..., x_{j_{n-1}}) \neq \emptyset$ be injective.

Let $\mathbb{Q}$ be the field of rational numbers and $\mathbb{E}$ be the field of algebraic numbers. As known, $\mathbb{E}$ is infinite-dimensional over $\mathbb{Q}$. Indeed suppose not, say $[\mathbb{E} : \mathbb{Q}] = n$, let $q$ be a root of an irreducible polynomial of degree $n+1$, then $[\mathbb{Q}(q) : \mathbb{Q}] = n+1$ but a subspace of a vector space can not be of dimension grater than it.

Let then $\{a_i \mid i \in \omega\}$ be an injective enumeration of a basis $A$ of $\mathbb{E}$ over $\mathbb{Q}$ and $\{w_i \mid i \in \omega\}$ an injective enumeration of $\mathrm{Var} \setminus \{x_{j_0}\}$. Let $\mathbb{C}$ be the field of complex numbers and let $s$ be the following assignment:

$$s(w_i) = e^{a_i}$$

and

$$s(x_{j_0}) = 0 \qquad \text{if } \overline{x} = \{x_{j_0}\}$$

$$s(x_{j_0}) = e^{\sum_{i=1}^{n-1} a_{p_i}} \qquad \text{if } \overline{x} \neq \{x_{j_0}\},$$

where $e$ is the Euler number and $w_{p_i} = x_{j_i}$ for every $i \in \{1, ..., n-1\}$.

We claim that $\mathbb{C} \not\models_s \perp(\overline{x})$. In accordance to the semantic we then have to show that there is $x \in \overline{x}$ such that $s(x)$ is algebraic over the subfield of $\mathbb{C}$ generated by $\{s(z) \mid z \in \overline{x}\} \setminus \{s(x)\}$. But $x_{j_0}$ satisfies this condition, indeed either

$$s(x_{j_0}) = 0 \quad \text{or} \quad s(x_{j_0}) = \prod_{i=1}^{n-1} s(x_{j_i})$$

and clearly in both cases $s(x_{j_0})$ is algebraic over the subfield of $\mathbb{C}$ generated by $\{s(v) \mid v \in \overline{x}\} \setminus \{s(x_{j_0})\}$.

Let now $\perp(\overline{v}) \in \Sigma$, we want to show that $\mathbb{C} \models_s \perp(\overline{v})$. As before, we assume, without loss of generality, that $\overline{v}$ is injective. Notice that if $\overline{v} = \emptyset$, then $\mathbb{C} \models_s \perp(\overline{v})$. Thus let $\overline{v} = (v_{h_0}, ..., v_{h_{c-1}}) \neq \emptyset$. We first state the following deep theorem.

**Theorem 4.6.5** (Lindemann–Weierstrass Theorem)**.** If $\{b_0, ..., b_{d-1}\}$ is a set of algebraic numbers which are linearly independent over $\mathbb{Q}$, then the set $\left\{e^{b_0}, ..., e^{b_{d-1}}\right\}$ is algebraically independent over $\mathbb{Q}$.

*Proof.* See [3, Theorem 1.4]. ☐

**Case 1.** $x_{j_0} \notin \overline{v}$.

Let $w_{r_i} = v_{h_i}$ for every $i \in \{0, ..., c-1\}$, we then have that $\left\{a_{r_0}, ..., a_{r_{c-1}}\right\}$ is linearly independent over $\mathbb{Q}$ and so by the theorem

$$\left\{s(v_{h_0}) = s(w_{r_0}) = e^{a_{r_0}}, ..., s(v_{h_{c-1}}) = s(w_{r_{c-1}}) = e^{a_{r_{c-1}}}\right\}$$

is algebraically independent over $\mathbb{Q}$.

**Case 2.** $x_{j_0} \in \overline{v}$.

**Subcase 1.** $\overline{x} \setminus \overline{v} \neq \emptyset$.

Notice that $\overline{x} \neq \{x_{j_0}\}$ because if not then $\overline{x} \setminus \overline{v} = \{x_{j_0}\}$ and so $x_{j_0} \notin \overline{v}$. Hence

$$s(x_{j_0}) = e^{\sum_{i=1}^{n-1} a_{p_i}}.$$

Let $(\overline{v} \setminus \{x_{j_0}\}) \cap \overline{x} = \left\{v_{h'_0}, ..., v_{h'_{d-1}}\right\}$, $\overline{v} \setminus \overline{x} = \left\{v_{h''_0}, ..., v_{h''_{t-1}}\right\}$, $w_{r'_i} = v_{h'_i}$ for every $i \in \{0, ..., d-1\}$ and $w_{r''_i} = v_{h''_i}$ for every $i \in \{0, ..., t-1\}$

Suppose now that the set

$$\left\{a_{r'_0}, ..., a_{r'_{d-1}}, \sum_{i=1}^{n-1} a_{p_i}, a_{r''_0}, ..., a_{r''_{t-1}}\right\}$$

is linearly dependent, then there exists $\overline{f} \in \mathbb{K}^d$, $l \in \mathbb{K}$ and $\overline{g} \in \mathbb{K}^t$ such that

$$\sum_{i=0}^{d-1} f_i(a_{r'_i}) + l(\sum_{i=1}^{n-1} a_{p_i}) + \sum_{i=0}^{t-1} g_i(a_{r''_i}) = 0$$

with $\overline{f} \neq (0_0, ..., 0_{d-1})$ or $l \neq 0$ or $\overline{g} \neq (0_0, ..., 0_{t-1})$.

Let $V = \{i \in \{1, ..., n-1\} \mid x_i \in (\overline{v} \setminus \{x_{j_0}\}) \cap \overline{x}\}$. In each three of the cases the linear combination

$$\sum_{i=0}^{d-1} (f_i + l)(a_{r'_i}) + l(\sum_{\substack{i=1 \\ i \notin V}}^{n-1} a_{p_i}) + \sum_{i=0}^{t-1} g_i(a_{r''_i}) = 0$$

is non trivial. Thus the set

$$\left\{s(x_{j_1}), ..., s(x_{j_{n-1}}), a_{r''_0}, ..., a_{r''_{t-1}}\right\}$$

is linearly dependent, which is absurd.

Hence again by the theorem, the set

$$\left\{e^{a_{r'_0}}, ..., e^{a_{r'_{d-1}}}, e^{\sum_{i=1}^{n-1} a_{p_i}}, e^{a_{r''_0}}, ..., e^{a_{r''_{t-1}}}\right\}$$

is algebraically independent over $\mathbb{Q}$.

**Subcase 2.** $\overline{x} \subseteq \overline{v}$.

This case is not possible. Suppose indeed it is, then by rule ($c_2$.) we can assume that $\overline{v} = \overline{x}\ \overline{v}'$ with $\overline{v}' \subseteq \mathrm{Var} \setminus \overline{x}$. Thus by rule ($b_2$.) we have that $\Sigma \vdash \perp(\overline{x})$ which is absurd.

□

## 4.7 Algebraically Closed Field Atomic Independence Logic

In this section we define the system Algebraically Closed Field Atomic Independence Logic (ACFAIndL) and then prove its soundness and completeness. The syntax and deductive apparatus of this system are the same as those of AIndL.

### 4.7.1 Semantics

The intuition behind the atom $\overline{x} \perp \overline{y}$ in this context is that if some elements from $\overline{x}$ are algebraically independent, then they are also algebraically independent over the subfield generated by $\overline{y}$. We do not define $\overline{x} \perp \overline{y}$ by saying that no variable in $\overline{x}$ is an algebraic expression of the elements in $\overline{y}$, although we could and that would perhaps be worth studying. The reasons for our choice are related to the unsolvability of the quintic by radicals. In the case of the unsolvable quintic there is a clear dependence even if it cannot be expressed in terms of an algebraic expression, even by means of radicals.

**Definition 4.7.1.** Let $\mathbb{K}$ be an algebraically closed field, $\mathbb{P}$ its prime field and $s : \mathrm{dom}(s) \to K$ with $\overline{x}\,\overline{y} \subseteq \mathrm{dom}(s) \subseteq \mathrm{Var}$. We say that $\mathbb{K}$ satisfies $\overline{x} \perp \overline{y}$ under $s$, in symbols $\mathbb{K} \models_s \overline{x} \perp \overline{y}$, if $\mathrm{trdg}(\mathbb{P}(s(\overline{x}))/\mathbb{P}(s(\overline{y}))) = \mathrm{trdg}(\mathbb{P}(s(\overline{x}))/\mathbb{P})$.

Notice that, because of Proposition 4.5.13, the condition that we used in the above definition is equivalent to the intuitive condition that we mentioned at the beginning of the section.

**Definition 4.7.2.** Let $\Sigma$ be a set of atoms and let $s$ be such that the set of variables occurring in $\Sigma$ is included in $\mathrm{dom}(s)$. We say that $\mathbb{K}$ satisfies $\Sigma$ under $s$, in symbols $\mathbb{K} \models_s \Sigma$, if $\mathbb{K}$ satisfies every atom in $\Sigma$ under $s$.

**Definition 4.7.3.** Let $\Sigma$ be a set of atoms. We say that $\overline{x} \perp \overline{y}$ is a logical consequence of $\Sigma$, in symbols $\Sigma \models \overline{x} \perp \overline{y}$, if for every algebraically closed field $\mathbb{K}$ and $s$ such that the set of variables occurring in $\Sigma \cup \{\overline{x} \perp \overline{y}\}$ is included in $\mathrm{dom}(s)$ we have that

$$\text{if } \mathbb{K} \models_s \Sigma \text{ then } \mathbb{K} \models_s \overline{x} \perp \overline{y}.$$

### 4.7.2 Soundness and Completeness

**Theorem 4.7.4.** Let $\Sigma$ be a set of atoms, then

$$\Sigma \models \overline{x} \perp \overline{y} \text{ if and only if } \Sigma \vdash \overline{x} \perp \overline{y}.$$

[The deductive system to which we refer has been defined in Section 2.4.3.]

*Proof.* ($\Leftarrow$) ($a_3$.) Obvious

($b_3$.)

$$\begin{array}{rcll} \mathbb{K} \models_s \overline{x} \perp \overline{y} & \Longrightarrow & s(\overline{x}) \downarrow^{\text{tr}}_{\mathbb{P}} s(\overline{y}) & \\ & \Longrightarrow & s(\overline{y}) \downarrow^{\text{tr}}_{\mathbb{P}} s(\overline{x}) & \text{[By Lemma 4.5.16]} \\ & \Longrightarrow & \mathbb{K} \models_s \overline{y} \perp \overline{x}. & \end{array}$$

($c_3$.)

$$\begin{array}{rcll} \mathbb{K} \models_s \overline{x} \perp \overline{y}\,\overline{z} & \Longrightarrow & s(\overline{x}) \downarrow^{\text{tr}}_{\mathbb{P}} s(\overline{y}\,\overline{z}) & \\ & \Longrightarrow & s(\overline{x}) \downarrow^{\text{tr}}_{\mathbb{P}} s(\overline{y}) & \text{[By Lemma 4.5.14]} \\ & \Longrightarrow & \mathbb{K} \models_s \overline{x} \perp \overline{y}. & \end{array}$$

($d_3$.)

$$\begin{array}{c} \mathbb{K} \models_s \overline{x} \perp \overline{y} \text{ and } \mathbb{K} \models_s \overline{x}\,\overline{y} \perp \overline{z} \\ \Downarrow \\ s(\overline{x}) \downarrow^{\text{tr}}_{\mathbb{P}} s(\overline{y}) \text{ and } s(\overline{x})s(\overline{y}) \downarrow^{\text{tr}}_{\mathbb{P}} s(\overline{z}) \\ \Downarrow \\ s(\overline{x}) \downarrow^{\text{tr}}_{\mathbb{P}} s(\overline{y})s(\overline{z}) \qquad \text{[By Corollary 4.5.18]} \\ \Downarrow \\ \mathbb{K} \models_s \overline{x} \perp \overline{y}\,\overline{z} \end{array}$$

($e_3$.) Suppose that $\mathbb{K} \models_s x \perp x$, then $\text{trdg}(\mathbb{P}(s(x))/\mathbb{P}(s(x))) = \text{trdg}(\mathbb{P}(s(x))/\mathbb{P})$. Thus $\text{trdg}(\mathbb{P}(s(x))/\mathbb{P}) = 0$ and so for any $\overline{y} \in \text{Var}$ we have that

$$\text{trdg}(\mathbb{P}(s(x))/\mathbb{P}(s(\overline{y}))) = 0 = \text{trdg}(\mathbb{P}(s(x))/\mathbb{P}).$$

($f_3$.) Obvious.

($\Rightarrow$) Suppose $\Sigma \nvdash \overline{x} \perp \overline{y}$. Notice that if this is the case then $\overline{x} \neq \emptyset$ and $\overline{y} \neq \emptyset$. Indeed if $\overline{y} = \emptyset$ then $\Sigma \vdash \overline{x} \perp \overline{y}$ because by rule ($a_3$.) $\vdash \overline{x} \perp \emptyset$. Analogously if $\overline{x} = \emptyset$ then $\Sigma \vdash \overline{x} \perp \overline{y}$ because by rule ($a_3$.) $\vdash \overline{y} \perp \emptyset$ and so by rule ($b_3$.) $\vdash \emptyset \perp \overline{y}$.

We can assume that $\overline{x}$ and $\overline{y}$ are injective. This is without loss of generality because clearly $\mathbb{K} \models_s \overline{x} \perp \overline{y}$ if and only if $\mathbb{K} \models_s \pi\overline{x} \perp \pi\overline{y}$, where $\pi : \text{Var}^{<\omega} \to \text{Var}^{<\omega}$ is the function that eliminates repetitions in finite sequences of variables.

Furthermore we can assume that $\overline{x} \perp \overline{y}$ is minimal, in the sense that if $\overline{x}' \subseteq \overline{x}$, $\overline{y}' \subseteq \overline{y}$ and $\overline{x}'\,\overline{y}' \neq \overline{x}\,\overline{y}$, then $\Sigma \vdash \overline{x}' \perp \overline{y}'$. This is for two reasons.

i) If $\overline{x} \perp \overline{y}$ is not minimal we can always find a minimal atom $\overline{x}^* \perp \overline{y}^*$ such that $\Sigma \nvdash \overline{x}^* \perp \overline{y}^*$, $\overline{x}^* \subseteq \overline{x}$ and $\overline{y}^* \subseteq \overline{y}$ $--$ just keep deleting elements of $\overline{x}$ and $\overline{y}$ until you obtain the desired property or until both $\overline{x}^*$ and $\overline{y}^*$ are singletons, in which case, due to the trivial independence rule ($a_3$.), $\overline{x}^* \perp \overline{y}^*$ is a minimal statement.

ii) For any $\overline{x}' \subseteq \overline{x}$ and $\overline{y}' \subseteq \overline{y}$ we have that if $\mathbb{K} \not\models_s \overline{x}' \perp \overline{y}'$ then $\mathbb{K} \not\models_s \overline{x} \perp \overline{y}$.

Let indeed $\overline{x} = \overline{x}'\overline{x}''$ and $\overline{y} = \overline{y}'\overline{y}''$, then

$$\begin{array}{lll} \mathbb{K} \models_s \overline{x}'\,\overline{x}'' \perp \overline{y}'\overline{y}'' & \Longrightarrow & s(\overline{x}')s(\overline{x}'') \mathop{\downarrow}^{\mathrm{tr}}_{\mathbb{P}} s(\overline{y}')s(\overline{y}'') \\ & \Longrightarrow & s(\overline{x}')s(\overline{x}'') \mathop{\downarrow}^{\mathrm{tr}}_{\mathbb{P}} s(\overline{y}') \qquad \text{[By Lemma 4.5.14]} \\ & \Longrightarrow & s(\overline{x}') \mathop{\downarrow}^{\mathrm{tr}}_{\mathbb{P}} s(\overline{y}') \qquad \text{[By Corollary 4.5.17]} \\ & \Longrightarrow & \mathbb{K} \models_s \overline{x}' \perp \overline{y}'. \end{array}$$

Let then $\overline{x} = (x_{j_0}, ..., x_{j_{n-1}})$ and $\overline{y} = (y_{k_0}, ..., y_{k_{m-1}})$ be injective and such that $\overline{x} \perp \overline{y}$ is minimal.

Let $V = \{v \in \mathrm{Var} \mid \Sigma \vdash v \perp v\}$ and $W = \mathrm{Var} \setminus V$. We claim that $\overline{x}, \overline{y} \not\subseteq V$. We prove it only for $\overline{x}$, the other case is symmetrical. Suppose that $\overline{x} \subseteq V$, then for every $x \in \overline{x}$ we have that $\Sigma \vdash x \perp x$ so by rule (e$_3$.), (b$_3$.) and (d$_3$.)

$$\Sigma \vdash \overline{y} \perp x_{j_0} \quad \text{and} \quad \Sigma \vdash \overline{y}\,x_{j_0} \perp x_{j_1} \Rightarrow \ \Sigma \vdash \overline{y} \perp x_{j_0}\,x_{j_1},$$

$$\Sigma \vdash \overline{y} \perp x_{j_0}\,x_{j_1} \quad \text{and} \quad \Sigma \vdash \overline{y}\,x_{j_0}\,x_{j_1} \perp x_{j_2} \Rightarrow \ \Sigma \vdash \overline{y} \perp x_{j_0}\,x_{j_1}\,x_{j_2},$$

$$\vdots$$

$$\Sigma \vdash \overline{y} \perp x_{j_0} \cdots x_{j_{n-2}} \quad \text{and} \quad \Sigma \vdash \overline{y}\,x_{j_0} \cdots x_{j_{n-2}} \perp x_{j_{n-1}} \Rightarrow \ \Sigma \vdash \overline{y} \perp \overline{x}.$$

Hence by rule (b$_3$.) $\Sigma \vdash \overline{x} \perp \overline{y}$.

Thus $\overline{x} \cap W \neq \emptyset$ and $\overline{y} \cap W \neq \emptyset$. Without loss of generality suppose that $x_{j_0} \in W$ and $y_{k_0} \in W$. Let $\overline{x} \cap W = \overline{x}' = (x_{j'_0}, ..., x_{j'_{n'-1}}) = (x_{j_0}, ..., x_{j'_{n'-1}}) \neq \emptyset$ and $\overline{y} \cap W = \overline{y}' = (y_{k'_0}, ..., y_{k'_{m'-1}})$. Notice that $\overline{x}' \cap \overline{y}' = \emptyset$. Indeed let $z \in \overline{x}' \cap \overline{y}'$, then by rules (b$_3$.) and (c$_3$.) we have that $\Sigma \vdash z \perp z$. Thus $z \in V$, a contradiction.

Let $\{w_i \mid i \in \kappa\}$ be an injective enumeration of $W \setminus \{x_{j_0}\}$. Let $\mathbb{Q}$ be the field of rational numbers, $\mathbb{E}$ the field of algebraic numbers and $\{a_i \mid i \in \omega\}$ an injective enumeration of a basis $A$ of $\mathbb{E}$ over $\mathbb{Q}$. Let $\mathbb{C}$ be the field of complex numbers and let $s$ be the following assignment:

i) $s(v) = 1$ for every $v \in V$,

ii) $s(w_i) = e^{a_i}$ for every $i \in \kappa$,

iii) $s(x_{j_0}) = \prod_{i=1}^{n-1} s(x_{j_i}) \cdot \prod_{i=0}^{m-1} s(y_{k_i})$,

where $e$ is the Euler number.

We claim that $\mathbb{C} \not\models_s \overline{x}' \perp \overline{y}'$, as noticed this implies that $\mathbb{C} \not\models_s \overline{x} \perp \overline{y}$. First we show that the set $\{s(x') \mid x' \in \overline{x}'\}$ is independent. As we will see in Chapter 5, Theorem 5.6.2, $s(x_{j_0})$ is not algebraic over $\mathbb{Q}(\{s(x') \mid x' \in \overline{x}'\} \setminus \{s(x_{j_0})\})$. Let then $i \in \{1, ..., n'-1\}$ and suppose that $s(x_{j'_i})$ is algebraic over $\mathbb{Q}(\left\{s(x_{j_0}), ..., s(x_{j'_{i-1}}), s(x_{j'_{i+1}}), ..., s(x_{j'_{n-1}})\right\})$.

By Theorem 4.6.5 the set $\left\{s(x_{j'_1}), ..., s(x_{j'_{n-1}})\right\}$ is independent, so $s(x_{j'_i})$ is not algebraic over $\mathbb{Q}(\left\{s(x_{j'_1}), ..., s(x_{j'_{i-1}}), s(x_{j'_{i+1}}), ..., s(x_{j'_{n-1}})\right\})$. Hence by the Exchange Principle $s(x_{j_0})$ is algebraic over $\mathbb{Q}(\left\{s(x_{j'_1}), ..., s(x_{j'_{n-1}})\right\})$, a contradiction. We then conclude that $\mathrm{trdg}(\mathbb{Q}(s(\overline{x}'))/\mathbb{Q}) = |\{s(x') \mid x' \in \overline{x}'\}|$.

We now show that $\left\{s(x_{j'_1}), ..., s(x_{j'_{n-1}})\right\}$ is a basis for $\mathbb{Q}(\{s(x') \mid x' \in \overline{x}'\})$ over $\mathbb{Q}(\{s(y') \mid y' \in \overline{y}'\})$.

As we noticed above $\overline{x}' \cap \overline{y}' = \emptyset$, so by properties of our assignment $s(\overline{x}') \cap s(\overline{y}') = \emptyset$. Thus by Lemma 4.5.9 $\left\{s(x_{j'_1}), ..., s(x_{j'_{n-1}})\right\}$ is independent over $\mathbb{Q}(\{s(y') \mid y' \in \overline{y}'\})$, also $\mathbb{Q}(\left\{s(x_{j_0}), ..., s(x_{j'_{n-1}})\right\})$ is algebraic over $\mathbb{Q}(s(\overline{y}') \cup \left\{s(x_{j'_1}), ..., s(x_{j'_{n-1}})\right\})$ because by construction

$$s(x_{j_0}) = \prod_{i=1}^{n-1} s(x_{j_i}) \cdot \prod_{i=0}^{m-1} s(y_{k_i}) = \prod_{i=1}^{n'-1} s(x_{j'_i}) \cdot \prod_{i=0}^{m'-1} s(y_{k'_i})$$

and so $s(x_{j_0})$ is algebraic over $\mathbb{Q}(\left\{s(x_{j'_1}), ..., s(x_{j'_{n'-1}}), s(y_{k'_0}), ..., s(y_{k'_{m'-1}})\right\})$.

Hence

$$\operatorname{trdg}(\mathbb{Q}(s(\overline{x}'))/\mathbb{Q}(s(\overline{y}'))) = |\left\{s(x_{j'_1}), ..., s(x_{j'_{n-1}})\right\}| = \operatorname{trdg}(\mathbb{Q}(s(\overline{x}'))/\mathbb{Q}) - 1.$$

Let now $\overline{v} \perp \overline{w} \in \Sigma$, we want to show that $\mathbb{C} \models_s \overline{v} \perp \overline{w}$. As before, we assume, without loss of generality, that $\overline{v}$ and $\overline{w}$ are injective. Notice also that if $\overline{v} = \emptyset$ or $\overline{w} = \emptyset$, then $\mathbb{C} \models_s \overline{v} \perp \overline{w}$. Thus let $\overline{v}, \overline{w} \neq \emptyset$.

**Case 1** $\overline{v} \subseteq V$ or $\overline{w} \subseteq V$.

Suppose that $\overline{v} \subseteq V$, the other case is symmetrical, then $\mathbb{Q}(s(\overline{v})) = \mathbb{Q}$ and so $\operatorname{trdg}(\mathbb{Q}(s(\overline{v}))/\mathbb{Q}(s(\overline{w}))) = 0 = \operatorname{trdg}(\mathbb{Q}(s(\overline{v}))/\mathbb{Q})$.

**Case 2** $\overline{v} \not\subseteq V$ and $\overline{w} \not\subseteq V$.

Let $\overline{v} \cap W = \overline{v}' \neq \emptyset$ and $\overline{w} \cap W = \overline{w}' \neq \emptyset$.

Notice that

$$s(\overline{v}) \underset{\mathbb{P}}{\overset{\text{tr}}{\downarrow}} s(\overline{w}) \text{ if and only if } s(\overline{v'}) \underset{\mathbb{P}}{\overset{\text{tr}}{\downarrow}} s(\overline{w'})$$

Left to right holds in general. As for the other direction it follows from the fact that $\mathbb{Q}(s(\overline{v})) = \mathbb{Q}(s(\overline{v}'))$ and $\mathbb{Q}(s(\overline{w})) = \mathbb{Q}(s(\overline{w}'))$ because if $u \in \overline{v}\,\overline{w} \setminus \overline{v}'\,\overline{w}'$, then $s(u) = 1$.

**Subcase 2.1** $x_{j_0} \notin \overline{v}'\,\overline{w}'$.

By Theorem 4.6.5 we have that $s(\overline{v}') \cup s(\overline{w}')$ is algebraically independent. Furthermore $\overline{v}' \cap \overline{w}' = \emptyset$ and so by properties of our assignment $s(\overline{v}') \cap s(\overline{w}') = \emptyset$. Thus by Lemma 4.5.9 it follows that $\operatorname{trdg}(\mathbb{Q}(s(\overline{v}'))/\mathbb{Q}(s(\overline{w}'))) = \operatorname{trdg}(\mathbb{Q}(s(\overline{v}'))/\mathbb{Q})$.

**Subcase 2.2** $x_{j_0} \in \overline{v}'\,\overline{w}'$.

**Subsubcase 2.2.1** $\overline{x}'\,\overline{y}' \setminus \overline{v}'\,\overline{w}' \neq \emptyset$.

Let $w_{p_i} = x_{j'_i}$ for every $i \in \{1, ..., n'-1\}$ and $w_{q_i} = y_{k'_i}$ for every $i \in \{0, ..., m'-1\}$. Notice that

$$s(x_{j_0}) = e^{\sum_{i=1}^{n'-1} a_{p_i} + \sum_{i=0}^{m'-1} a_{q_i}}.$$

Let $\overline{v}'\,\overline{w}' \setminus \{x_{j_0}\} \cap \overline{x}'\,\overline{y}' = \left\{u_{h'_0}, ..., u_{h'_{d-1}}\right\}$, $\overline{v}'\,\overline{w}' \setminus \overline{x}'\,\overline{y}' = \left\{u_{h''_0}, ..., u_{h''_{t-1}}\right\}$, $w_{r'_i} = u_{h'_i}$ for every $i \in \{0, ..., d-1\}$ and $w_{r''_i} = u_{h''_i}$ for every $i \in \{0, ..., t-1\}$.

Suppose now that the set

$$\left\{a_{r'_0}, ..., a_{r'_{d-1}}, \sum_{i=1}^{n'-1} a_{p_i} + \sum_{i=0}^{m'-1} a_{q_i}, a_{r''_0}, ..., a_{r''_{t-1}}\right\}$$

is linearly dependent over $\mathbb{Q}$, then there exists $\overline{f}$, $l$ and $\overline{g} \in Q^{<\omega}$ such that

$$\sum_{i=0}^{d-1} f_i(a_{r'_i}) + l(\sum_{i=1}^{n'-1} a_{p_i} + \sum_{i=0}^{m'-1} a_{q_i}) + \sum_{i=0}^{t-1} g_i(a_{r''_i}) = 0$$

with $\overline{f} \neq (0_0, ..., 0_{d-1})$ or $l \neq 0$ or $\overline{g} \neq (0_0, ..., 0_{t-1})$.

Let $V = \{i \in \{1, ..., n'-1\} \mid x_i \in (\overline{v}' \setminus \{x_{j_0}\}) \cap \overline{x}'\}$ and let $U = \{i \in \{0, ..., m'-1\} \mid y_i \in (\overline{w}' \setminus \{x_{j_0}\}) \cap \overline{y}'\}$. In each three of the cases the linear combination

$$\sum_{i=0}^{d-1}(f_i + l)(a_{r'_i}) + l(\sum_{\substack{i=1 \\ i \notin V}}^{n'-1} a_{p_i} + \sum_{\substack{i=0 \\ i \notin U}}^{m'-1} a_{q_i}) + \sum_{i=0}^{t-1} g_i(a_{r''_i}) = 0$$

is non trivial. Thus the set

$$\left\{ s(x_{j_1}), ..., s(x_{j_{n-1}}), a_{r''_0}, ..., a_{r''_{t-1}} \right\}$$

is linearly dependent over $\mathbb{Q}$, which is absurd.

Hence by Theorem 4.6.5, the set

$$\left\{ e^{a_{r'_0}}, ..., e^{a_{r'_{d-1}}}, e^{\sum_{i=1}^{n'-1} a_{p_i} + \sum_{i=0}^{m'-1} a_{q_i}}, ..., e^{a_{r''_{t-1}}} \right\}$$

is algebraically independent.

Clearly $\{s(v') \mid v' \in \overline{v}'\}$ and $\{s(w') \mid w' \in \overline{w}'\}$ are two subsets of the set above, indeed their union is that set. Also, as we noticed, $s(\overline{v}') \cap s(\overline{w}') = \emptyset$. Hence by Lemma 4.5.9 we have that $\text{trdg}(\mathbb{Q}(s(\overline{v}'))/\mathbb{Q}(s(\overline{w}'))) = \text{trdg}(\mathbb{Q}(s(\overline{v}'))/\mathbb{Q})$.

**Subsubcase 2.2.2** $\overline{x}'\,\overline{y}' \subseteq \overline{v}'\,\overline{w}'$.

As shown in Theorem 3.4.4, this case is not possible.

□

# Chapter 5

# Dependence and Independence of the Algebraic Closure Operator

From this chapter on, we pass from the analysis of the dependence and independence concepts occurring in algebra to the ones occurring in model theory. In particular, we will study the dependence and independence notions that have been formulated in the context of geometric model theory.

As typical of model theory, we will not study particular theories but classes of theories satisfying certain properties. This step will allow us to analyze several cases at once and subsume both the analyses of linear algebra and field theory of Chapter 4 into one general case, as it will be clear from combining Section 5.6 and Section 6.6.

In this chapter in particular, our topic of study will be theories in whose models the algebraic closure operator is well-behaved. This operator is a natural generalization of the operator of algebraic closure in algebraically closed fields. As we saw in the previous chapter, in algebraically closed fields the notion of dependence is defined via the existence of a particular kind of formal expression, the well-known polynomials. In the abstract setting of model theory, this notion is substituted by the more general notion of algebraic formula. This allows for a generalization of this form of dependence to any signature, reaching a great deal of abstraction.

The systems that we are going to study are: Algebraic Closure Atomic Dependence Logic, Algebraic Closure Atomic Absolute Independence Logic and Algebraic Closure Independence Logic.

## 5.1 Algebraic Closure

In this section we define the model-theoretic operator of algebraic closure and prove some fundamental results about it. We fix a signature $\mathcal{L}$.

**Definition 5.1.1.** Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $A \subseteq M$. We say that $b$ is *algebraic* over $A$ if there is an $\mathcal{L}$-formula $\phi(v, \overline{w})$ and $\overline{a} \in A$ such that

$\mathcal{M} \models \phi(b, \overline{a})$ and $\phi(\mathcal{M}, \overline{a}) = \{m \in M \mid \mathcal{M} \models \phi(m, \overline{a})\}$ is finite. We then let

$$\operatorname{acl}_{\mathcal{M}}(A) = \{m \in M \mid m \text{ is algebraic over } A\}.$$

If $\mathcal{M}$ is clear from the context, we write $\operatorname{acl}(A)$.

**Lemma 5.1.2.** Let $\mathcal{M}$ be a model and $A, B \subseteq M$. Then

i) $A \subseteq \operatorname{acl}(A)$;

ii) If $A \subseteq B$ then $\operatorname{acl}(A) \subseteq \operatorname{acl}(B)$;

iii) $\operatorname{acl}(A) = \operatorname{acl}(\operatorname{acl}(A))$;

iv) If $A \subseteq \operatorname{acl}(B)$ then $\operatorname{acl}(A) \subseteq \operatorname{acl}(B)$;

v) If $c \in \operatorname{acl}(A)$, then $c \in \operatorname{acl}(A_0)$ for $A_0 \subseteq_{\text{fin}} A$.

*Proof.* i) Let $a \in A$ and $\phi(x)$ be the formula $x = y$ then clearly

$$\phi(\mathcal{M}, a) = \{a\}.$$

ii) Obvious.
iii) See [34, Proposition 11.5.3].
iv) Suppose that $A \subseteq \operatorname{acl}(B)$, then by property ii) and iii)

$$\operatorname{acl}(A) \subseteq \operatorname{acl}(\operatorname{acl}(B)) = \operatorname{acl}(B).$$

(v) Let $c \in \operatorname{acl}(A)$, then there exists $\phi(v, \overline{w}) \in \mathcal{L}$ and $\overline{a} \in A$ such that $\mathcal{M} \models \phi(c, \overline{a})$ and $|\phi(\mathcal{M}, \overline{a})| < \infty$. Thus $c \in \operatorname{acl}(\overline{a})$.

□

**Lemma 5.1.3.** Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $A \subseteq M$, the following are equivalent:

i) $\overline{a} = (a_0, ..., a_{n-1}) \in \operatorname{acl}(A)$;

ii) there exists $\phi(v_0, ..., v_{n-1}) \in \mathcal{L}_A$ such that $\mathcal{M} \models \phi(\overline{a})$ and $|\phi(\mathcal{M})| < \infty$.

*Proof.* i) $\Rightarrow$ ii). If $\overline{a} \in \operatorname{acl}(A)$, then for every $i \in \{0, ..., n-1\}$ there exists $\psi_i(v_i) \in \mathcal{L}_A$ such that $\mathcal{M} \models \psi_i(a_i)$ and $|\psi_i(\mathcal{M})| < \infty$. Let $\phi(v_0, ..., v_{n-1})$ be the formula $\bigwedge_{i=0}^{n-1} \psi_i(v_i)$, then $\mathcal{M} \models \phi(\overline{a})$ and $|\phi(\mathcal{M})| \leq |\psi_i(\mathcal{M})| < \infty$.

ii) $\Rightarrow$ i). Let $a_i \in \overline{a}$ and let $\psi_i(v_i)$ be the formula

$$\exists v_0 \cdots \exists v_{i-1} \exists v_{i+1} \cdots \exists v_{n-1} \phi(v_0, ..., v_{n-1}).$$

Clearly $\mathcal{M} \models \psi_i(a_i)$, we want to show that $|\psi_i(\mathcal{M})| < \infty$.

Let $b \in \psi_i(\mathcal{M})$, then $\mathcal{M} \models \psi_i(b)$ so there exist $(a_0, ..., a_{i-1}, a_{i+1}, ..., a_{n-1})$ such that

$$\mathcal{M} \models \phi(a_0, ..., a_{i-1}, b, a_{i+1}, ..., a_{n-1}),$$

hence $\overline{a}_b := (a_0, ..., a_{i-1}, b, a_{i+1}, ..., a_{n-1}) \in \phi(\mathcal{M})$.

The attribution $b \mapsto \overline{a}_b$ define an injective function

$$f : \psi_i(\mathcal{M}) \to \phi(\mathcal{M})$$

so $|\psi_i(\mathcal{M})| \leq |\phi(\mathcal{M})| < \infty$.

□

## 5.2 Algebraic Closure Atomic Dependence Logic

In this section we define the system Algebraic Closure Atomic Dependence Logic (AclADL) and then prove its soundness and completeness. The syntax and deductive apparatus of this system are the same as those of ADL.

### 5.2.1 Semantics

Generalizing the interpretation that we gave of the dependence atom in the context of algebraically closed fields, we think of the atom $=(\overline{x}, \overline{y})$ as expressing the following: each element in $\overline{y}$ is bound to the elements in $\overline{x}$ via the existence of an algebraic formula with parameters from $\overline{x}$.

**Definition 5.2.1.** Let $T$ be a first-order theory with infinite models. We say that $T$ has the ACL-dependence property if for every $\mathcal{M} \models T$ we have $\mathrm{acl}(\emptyset) \neq \emptyset$.

Notice that if $T$ is a theory in a signature with at least one constant symbol this requirement is satisfied, indeed let $c$ be in the signature and $\mathcal{M} \models T$ then $c^{\mathcal{M}} \in \mathrm{acl}(\emptyset)$ because the formula $x = c$ is such that $\{m \in M \mid \mathcal{M} \models m = c\} = \{c^{\mathcal{M}}\}$.

Let $T$ be a theory with the ACL-dependence property.

**Definition 5.2.2.** Let $\mathcal{M} \models T$ and $s : \mathrm{dom}(s) \to M$ with $\overline{x}\,\overline{y} \subseteq \mathrm{dom}(s) \subseteq \mathrm{Var}$. We say that $\mathcal{M}$ satisfies $=(\overline{x}, \overline{y})$ under $s$, in symbols $\mathcal{M} \models_s =(\overline{x}, \overline{y})$, if for every $y \in \overline{y}$ we have that $s(y) \in \mathrm{acl}(\{s(x) \mid x \in \overline{x}\})$.

**Definition 5.2.3.** Let $\Sigma$ be a set of atoms and let $s$ be such that the set of variables occurring in $\Sigma$ is included in $\mathrm{dom}(s)$. We say that $\mathcal{M}$ satisfies $\Sigma$ under $s$, in symbols $\mathcal{M} \models_s \Sigma$, if $\mathcal{M}$ satisfies every atom in $\Sigma$ under $s$.

**Definition 5.2.4.** Let $\Sigma$ be a set of atoms. We say that $=(\overline{x}, \overline{y})$ is a logical consequence of $\Sigma$, in symbols $\Sigma \models =(\overline{x}, \overline{y})$, if for every $\mathcal{M} \models T$ and $s$ such that the set of variables occurring in $\Sigma \cup \{=(\overline{x}, \overline{y})\}$ is included in $\mathrm{dom}(s)$ we have that

$$\text{if } \mathcal{M} \models_s \Sigma \text{ then } \mathcal{M} \models_s =(\overline{x}, \overline{y}).$$

### 5.2.2 Soundness and Completeness

**Theorem 5.2.5.** Let $\Sigma$ be a set of atoms, then

$$\Sigma \models =(\overline{x}, \overline{y}) \text{ if and only if } \Sigma \vdash =(\overline{x}, \overline{y}).$$

[The deductive system to which we refer has been defined in Section 2.2.3.]

*Proof.* ($\Leftarrow$) We only prove the soundness of rules ($a_1$.), ($b_1$.) and ($c_1$.). Let $\mathcal{M} \models T$ and $s$ an appropriate assignment.

($a_1$.) By Lemma 5.1.2 i) we have that $\{s(x) \mid x \in \overline{x}\} \subseteq \mathrm{acl}(\{s(x) \mid x \in \overline{x}\})$, so $\mathcal{M} \models_s =(\overline{x}, \overline{x})$.

($b_1$.) Suppose that $\mathcal{M} \models_s =(\overline{x}, \overline{y})$ and let $\overline{u} \subseteq \overline{y}$ and $\overline{x} \subseteq \overline{z}$. Let now $u \in \overline{u}$, then by the fact that $\mathcal{M} \models_s =(\overline{x}, \overline{y})$ we have that $s(u) \in \mathrm{acl}(\{s(x) \mid x \in \overline{x}\})$ and so by Lemma 5.1.2 ii) we have $s(u) \in \mathrm{acl}(\{s(z) \mid z \in \overline{z}\})$.

($c_1$.) Suppose that $\mathcal{M} \models_s =(\overline{x}, \overline{y})$ and $\mathcal{M} \models_s =(\overline{y}, \overline{z})$. Let $z \in \overline{z}$, then $s(z) \in \mathrm{acl}(\{s(y) \mid y \in \overline{y}\})$ because $\mathcal{M} \models_s =(\overline{y}, \overline{z})$. Furthermore we have that

$\{s(y) \mid y \in \overline{y}\} \subseteq \text{acl}(\{s(x) \mid x \in \overline{x}\})$ because $\mathcal{M} \models_s =(\overline{x}, \overline{y})$. Thus, by Lemma 5.1.2 iv), we have that $\text{acl}(\{s(y) \mid y \in \overline{y}\}) \subseteq \text{acl}(\{s(x) \mid x \in \overline{x}\})$ and hence that $s(z) \in \text{acl}(\{s(x) \mid x \in \overline{x}\})$.

($\Rightarrow$) Suppose $\Sigma \nvdash =(\overline{x}, \overline{y})$. Let $V = \{z \in \text{Var} \mid \Sigma \vdash =(\overline{x}, z)\}$ and $W = \text{Var} \backslash V$. First notice that $\overline{y} \neq \emptyset$, indeed if not so, by the syntactic constraints that we put on the system, we have that $\overline{x}, \overline{y} = \emptyset$ and so by the admitted degenerate case of rule (a$_1$.) we have that $\Sigma \vdash =(\overline{x}, \overline{y})$. Furthermore $\overline{y} \cap W \neq \emptyset$, indeed if $\overline{y} \cap W = \emptyset$ then for every $y \in \overline{y}$ we have that $\Sigma \vdash =(\overline{x}, y)$ and so by rules (d$_1$.), (e$_1$.) and, if necessary, (f$_1$.)[1] we have that $\Sigma \vdash =(\overline{x}, \overline{y})$.

Let $\kappa > |\mathcal{L}| + \aleph_0$ where $\mathcal{L}$ is the signature of $T$. The theory $T$ has infinite models so by the Löwenheim-Skolem Theorem there is a structure $\mathcal{M}$ such that $\mathcal{M} \models T$ and $|M| = \kappa$. Notice now that if $|M| = \kappa$ then for every $m \in M$ we have that $\text{acl}(\{m\}) \neq M$ because $|\text{acl}(\{m\})| \leq |\mathcal{L}| + \aleph_0$.

Let then $a \in \text{acl}(\emptyset)$, $b \in M \setminus \text{acl}(\{a\})$ and let $s$ be the following assignment:

$$s(v) = \begin{cases} a & \text{if } v \in V \\ b & \text{if } v \in W. \end{cases}$$

We claim that $\mathcal{M} \not\models_s =(\overline{x}, \overline{y})$. In accordance to the semantic we then have to show that there is $y \in \overline{y}$ such that $s(y) \notin \text{acl}(\{s(x) \mid x \in \overline{x}\})$. Let $y \in \overline{y} \cap W$, then

$$s(y) = b \notin \text{acl}(\{a\}) = \text{acl}(\{s(x) \mid x \in \overline{x}\})$$

because for $x \in \overline{x}$ we have that $\Sigma \vdash =(\overline{x}, x)$. Indeed by rule (a$_1$.) $\vdash =(\overline{x}, \overline{x})$ and so by rule (b$_1$.) $\vdash =(\overline{x}, x)$. Notice that in the case $\overline{x} = \emptyset$, we have that

$$s(y) = b \notin \text{acl}(\{a\}) \supseteq \text{acl}(\emptyset) = \text{acl}(\{s(x) \mid x \in \overline{x}\}).$$

Let now $=(\overline{x}', \overline{y}') \in \Sigma$, we want to show that $\mathcal{M} \models_s =(\overline{x}', \overline{y}')$. If $\overline{y}' = \emptyset$ then also $\overline{x}' = \emptyset$ and so trivially $\mathcal{M} \models_s =(\overline{x}', \overline{y}')$. Noticed this, for the rest of the proof we assume $\overline{y}' \neq \emptyset$.

**Case 1.** $\overline{x}' = \emptyset$.

Suppose that $\mathcal{M} \not\models_s =(\emptyset, \overline{y}')$, then there exists $y' \in \overline{y}'$ such that $s(y') = b$, so $\Sigma \nvdash =(\overline{x}, y')$. Notice though that $\Sigma \vdash =(\emptyset, \overline{y}')$, so by rule (b$_1$.) $\Sigma \vdash =(\emptyset, y')$ and hence again by rule (b$_1$.) $\Sigma \vdash =(\overline{x}, y')$.

**Case 2.** $\overline{x}' \neq \emptyset$ and $\overline{x}' \subseteq V$.

If this is the case, then

$$\begin{aligned} \forall x' \in \overline{x}' \;\; \Sigma \vdash =(\overline{x}, x') &\implies \Sigma \vdash =(\overline{x}, \overline{x}') \;\; \text{[by rules (d}_1.\text{), (e}_1.\text{) and (f}_1.\text{)]} \\ &\implies \Sigma \vdash =(\overline{x}, \overline{y}') \;\; \text{[by rule (c}_1.\text{)]} \\ &\implies \forall y' \in \overline{y}' \;\; \Sigma \vdash =(\overline{x}', y') \;\; \text{[by rule (b}_1.\text{)]} \\ &\implies \overline{y}' \subseteq V. \end{aligned}$$

If $\overline{x}' \subseteq V$ then for every $x' \in \overline{x}'$ we have that $s(x') = a$ so

$$\text{acl}(\{s(x') \mid x' \in \overline{x}'\}) = \text{acl}(\{a\}).$$

Let $y' \in \overline{y}'$, then we have that $s(y') = a$ and clearly $a \in \text{acl}(\{a\})$. Hence $\mathcal{M} \models_s =(\overline{x}', \overline{y}')$

**Case 3.** $\overline{x}' \cap W \neq \emptyset$.

[1] Notice that (f$_1$.) is necessary only if $\overline{x}$ or $\overline{y}$ contain repetitions.

If this is the case, then there exists $w \in \overline{x}'$ such that $\Sigma \not\vdash =(\overline{x}, w)$, so we have $w \in \overline{x}'$ such that $s(w) = b$ and hence $\mathrm{acl}(\{s(x') \mid x' \in \overline{x}'\}) \supseteq \mathrm{acl}(\{b\})$. Let now $y' \in \overline{y}'$.

**Subcase 1**. $y' \in W$.

In this case we have that $s(y') = b$. Clearly

$$b \in \mathrm{acl}(\{b\}) \subseteq \mathrm{acl}(\{s(x') \mid x' \in \overline{x}'\}).$$

Hence $\mathcal{M} \models_s =(\overline{x}', \overline{y}')$.

**Subcase 2**. $y' \in V$.

In this case we have that $s(y') = a$. By choice of $a$

$$a \in \mathrm{acl}(\{b\}) \subseteq \mathrm{acl}(\{s(x') \mid x' \in \overline{x}'\}).$$

Hence $\mathcal{M} \models_s =(\overline{x}', \overline{y}')$.

□

## 5.3 Theories with the ACL-Independence Property

In this section we see how the imposition on the algebraic closure operator of a single simple condition, namely the Exchange Principle, allows us to develop a rich theory of independent sets and to consequently define several geometric concepts in an abstract setting. We will use this absolute notion of independence to define a ternary independence relation between a tuple of elements and two subset of a model satisfying the Exchange Principle. This relation will be used in Section 5.5 to give an algebraic closure interpretation of the independence atom $\overline{x} \perp \overline{y}$. We conclude this section by defining the notion of a theory with the ACL-independence property, the conditions that we impose in this definition will play a key role in the proofs of Theorems 5.4.4 and 5.5.4.

Let $\mathcal{M}$ be a model in the signature $\mathcal{L}$.

**Definition 5.3.1.** Let $A \subseteq M$, we say that $A$ is independent if for all $a \in A$ we have $a \notin \mathrm{acl}(A \setminus \{a\})$. If $C \subseteq M$, we say that $A$ is *independent over* $C$ if for every $a \in A$ we have $a \notin \mathrm{acl}(C \cup (A \setminus \{a\}))$.

We now define the promised Exchange Principle. In Lemma 7.6.2 we will see how, in any model of a so-called strongly minimal theory, the algebraic closure operator satisfies the Exchange Principle. Furthermore, this principle can be generalized to other kinds of operators, giving rise to the notion of a pregeometry, which will be object of study in the next chapter.

**Definition 5.3.2** (Exchange Principle)**.** Let $A \subseteq M$ and $b \in M$. If $a \in \mathrm{acl}(A \cup \{b\}) \setminus \mathrm{acl}(A)$, then $b \in \mathrm{acl}(A \cup \{a\})$.

For the rest of the section, let $\mathcal{M}$ be a model in the signature $\mathcal{L}$ satisfying the Exchange Principle.

**Definition 5.3.3.** Let $B \subseteq A \subseteq M$ and $C \subseteq M$. We say that $B$ is a *basis* for $A$ if $B$ is independent and $A \subseteq \mathrm{acl}(B)$. We say that $B$ is a basis for $A$ over $C$ if $B$ is independent over $C$ and $A \subseteq \mathrm{acl}(C \cup B)$.

**Proposition 5.3.4.** Let $B \subseteq A \subseteq M$ and $C \subseteq M$, the following are equivalent:

i) $B$ is a maximally independent over $C$ subset of $A$;

ii) $B$ is a basis for $A$ over $C$;

iii) $B$ is a minimal subset of $A$ such that $A \subseteq \mathrm{acl}(C \cup B)$.

*Proof.* i) $\Rightarrow$ ii). Suppose that there exists $a \in A$ such that $a \notin \mathrm{acl}(C \cup B)$. We claim that $B \cup \{a\}$ is independent over $C$. Suppose not, then there exists $b \in B \cup \{a\}$ such that $b \in \mathrm{acl}(C \cup ((B \cup \{a\}) \setminus \{b\}))$. By hypothesis $a \notin \mathrm{acl}(C \cup B)$ so $b \neq a$ and hence $b \in \mathrm{acl}(C \cup ((B \setminus \{b\}) \cup \{a\}))$. Notice now that $B$ is independent over $C$ so $b \notin \mathrm{acl}(C \cup (B \setminus \{b\}))$. Hence by the Exchange Principle we have that $a \in \mathrm{acl}(C \cup B)$, a contradiction.

ii) $\Rightarrow$ iii). Suppose there exists $B' \subsetneq B$ such that $A \subseteq \mathrm{acl}(C \cup B')$. Let $b \in B \setminus B'$, then $b \in \mathrm{acl}(C \cup B')$ because $A \subseteq \mathrm{acl}(C \cup B')$ and so $b \in \mathrm{acl}(C \cup (B \setminus \{b\}))$ since $B' \subseteq B \setminus \{b\}$. Hence $B$ is not independent over $C$.

iii) $\Rightarrow$ i). Suppose there exists $b \in B$ such that $b \in \mathrm{acl}(C \cup (B \setminus \{b\}))$, then $A \subseteq \mathrm{acl}(C \cup B) \subseteq \mathrm{acl}(C \cup (B \setminus \{b\}))$ so $B$ is not minimal. Thus $B$ is independent over $C$. Let now $a \in A \setminus B$ and suppose that $B \cup \{a\}$ is independent over $C$, then $a \notin \mathrm{acl}(C \cup B)$ and so $A \not\subseteq \mathrm{acl}(C \cup B)$.

□

**Proposition 5.3.5.** Let $A_1 \subseteq M$, $C \subseteq M$ and $A_0 \subseteq A_1$ independent over $C$. Then $A_0$ can be extended to a maximally independent over $C$ subset of $A_1$. In particular for every $A \subseteq M$, there exists $B \subseteq A$ such that $B$ is a basis for $A$ over $C$.

*Proof.* See [10, Proposition 1.7]. Notice that the proof of this theorem requires Zorn's Lemma.

□

**Lemma 5.3.6.** Let $A, B, C, D \subseteq M$ with $A \subseteq D$ and $B \subseteq D$. If $A$ and $B$ are bases for $D$ over $C$, then $|A| = |B|$.

*Proof.* See [29, Lemma 6.1.9].

□

**Definition 5.3.7.** Let $A, C \subseteq M$. The *dimension* of $A$ is the cardinality of a basis for $A$. We let $\dim(A)$ denote the dimension of $A$. The dimension of $A$ over $C$ is the cardinality of a basis for $A$ over $C$. We let $\dim(A/C)$ denote the dimension of $A$ over $C$.

The following two lemmas are not of particular interest but they will be relevant in the proof of Theorem 5.5.4, this is the reason for which we state them here.

**Lemma 5.3.8.** Let $A, C \subseteq M$. If $A$ is independent over $C$ and $B$ is a basis for $A$ over $C$, then $A = B$. In particular if $A$ is independent over $C$, then $\dim(A/C) = |A|$.

*Proof.* Suppose that there is $B \subsetneq A$ such that $B$ is a basis for $A$ over $C$. Let $b \in A \setminus B$, then $b \in \mathrm{acl}(C \cup B)$ because $A \subseteq \mathrm{acl}(C \cup B)$. So $b \in \mathrm{acl}(C \cup (A \setminus \{b\}))$, since $B \subseteq A \setminus \{b\}$. Thus $A$ is not independent over $C$.

□

**Lemma 5.3.9.** Let $A \subseteq M$ be an independent set. Let $D_0, D_1 \subseteq A$ and $D_0 \cap D_1 = \emptyset$, then

i) $D_0$ is independent over $D_1$;

ii) $\dim(D_0/D_1) = \dim(D_0)$.

*Proof.* i) Suppose that $D_0$ is not independent over $D_1$, then there exists $d \in D_0$ such that $d \in \operatorname{acl}(D_1 \cup (D_0 \setminus \{d\}))$. By hypothesis $D_0 \cap D_1 = \emptyset$, so $d \in \operatorname{acl}((D_1 \cup D_0) \setminus \{d\})$. Thus $D_0 \cup D_1$ is dependent, a contradiction.

ii) By i) $D_0$ is independent over $D_1$ and thus independent, hence by Lemma 5.3.8 we have that $\dim(D_0/D_1) = |D_0| = \dim(D_0)$.

□

**Proposition 5.3.10.** Let $A, C, D \subseteq M$. If $C \subseteq D$, then $\dim(A/D) \leq \dim(A/C)$.

*Proof.* Let $B$ be a basis for $A$ over $C$, then $B$ is independent over $C$ and $A \subseteq \operatorname{acl}(C \cup B)$. Let $B' \subseteq B$ be such that $B'$ is independent over $D$ and $B \subseteq \operatorname{acl}(D \cup B')$. By choice $B'$ is independent over $D$, furthermore $A \subseteq \operatorname{acl}(C \cup B) \subseteq \operatorname{acl}(D \cup B')$. Hence $B'$ is a basis for $A$ over $D$.

□

The notion of dimension that we have been dealing with allow us to define a notion of independence with many desirable properties.

**Definition 5.3.11.** Let $B, C \subseteq M$ and $\overline{a} \in M^{<\omega}$. We say that $\overline{a}$ is independent from $C$ over $B$ if $\dim(\overline{a}/B \cup C) = \dim(\overline{a}/B)$. We write $\overline{a} \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\mathrm{acl}}_{B} C$.

**Proposition 5.3.12.** The following are equivalent:

i) $\overline{a} \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\mathrm{acl}}_{B} C$;

ii) every basis for $\overline{a}$ over $B$ is a basis for $\overline{a}$ over $B \cup C$;

iii) every maximally independent over $B$ subset of $\overline{a}$ is independent over $B \cup C$;

iv) if $\overline{a}' \subseteq \overline{a}$ is independent over $B$, then $\overline{a}'$ is independent over $B \cup C$.

*Proof.* i) $\Rightarrow$ ii) Let $\overline{b}$ be a basis for $\overline{a}$ over $B$, then $\overline{b}$ is independent over $B$ and $\overline{a} \subseteq \operatorname{acl}(B \cup \overline{b})$. Let $\overline{b}'$ be such that $\overline{b}'$ is independent over $B \cup C$ and $\overline{b} \subseteq \operatorname{acl}(C \cup \overline{b}')$. By choice $\overline{b}'$ is independent over $B \cup C$, furthermore $\overline{a} \subseteq \operatorname{acl}(B \cup \overline{b}) \subseteq \operatorname{acl}((B \cup C) \cup \overline{b}')$, so $\overline{b}'$ is a basis for $\overline{a}$ over $B \cup C$. Now if $\overline{b}' \subsetneq \overline{b}$, then $\dim(\overline{a}/B \cup C) < \dim(\overline{a}/B)$. Thus $\overline{b}' = \overline{b}$.

ii) $\Rightarrow$ iii) Immediate from Proposition 5.3.4.

iii) $\Rightarrow$ iv) Suppose that there exists $\overline{a}' \subseteq \overline{a}$ independent over $B$ but not over $B \cup C$. By Proposition 5.3.5, $\overline{a}'$ can be extended to a $\overline{b} \subseteq \overline{a}$ maximally independent over $B$, so there exists a maximally independent over $B$ subset of $\overline{a}$ that is dependent over $B \cup C$.

iv) $\Rightarrow$ i) Let $\overline{b}$ be a basis for $\overline{a}$ over $B$, then $\overline{b}$ is independent over $B$ and so by the hypothesis it is independent over $B \cup C$. Furthermore $\overline{a} \subseteq \operatorname{acl}(B \cup \overline{b}) \subseteq \operatorname{acl}((B \cup C) \cup \overline{b})$. Thus $\overline{b}$ is a basis for $\overline{a}$ over $B \cup C$ and hence $\dim(\overline{a}/B) = |\overline{b}| = \dim(\overline{a}/B \cup C)$.

□

**Lemma 5.3.13** (Monotonicity)**.** If $\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_A B$ and $C \subseteq B$, then $\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_A C$.

*Proof.* By Proposition 5.3.10, $\dim(\overline{a}/A \cup B) \leq \dim(\overline{a}/A \cup C) \leq \dim(\overline{a}/A)$. Thus if $\dim(\overline{a}/A \cup B) = \dim(\overline{a}/A)$, then $\dim(\overline{a}/A \cup C) = \dim(\overline{a}/A)$.

□

**Lemma 5.3.14** (Transitivity)**.** $\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_A \overline{b}\,\overline{c}$ if and only if $\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_A \overline{b}$ and $\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_{A \cup \overline{b}} \overline{c}$.

*Proof.* By Proposition 5.3.10, $\dim(\overline{a}/A \cup \overline{b}\,\overline{c}) \leq \dim(\overline{a}/A \cup \overline{b}) \leq \dim(\overline{a}/A)$. Thus $\dim(\overline{a}/A \cup \overline{b}\,\overline{c}) = \dim(\overline{a}/A)$ if and only if $\dim(\overline{a}/A \cup \overline{b}) = \dim(\overline{a}/A)$ and $\dim(\overline{a}/(A \cup \overline{b}) \cup \overline{c}) = \dim(\overline{a}/A \cup \overline{b})$.

□

**Lemma 5.3.15** (Finite Basis)**.** $\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_A B$ if and only if $\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_A B_0$ for all finite $B_0 \subseteq B$.

*Proof.* ($\Rightarrow$) Follows from Monotonicity.

($\Leftarrow$) Suppose that $\overline{a} \not\mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_A B$, then there exists $\overline{a}' \subseteq \overline{a}$ such that $\overline{a}'$ is independent over $A$ but not over $A \cup B$. Thus there exists $a' \in \overline{a}'$ such that $a' \in \text{acl}((A \cup B) \cup (\overline{a}' \setminus \{a'\}))$. By Lemma 5.1.2 vi) there exists $A_0 \subseteq_{\text{fin}} A$ and $B_0 \subseteq_{\text{fin}} B$ such that $a' \in \text{acl}(A_0 \cup B_0) \cup (\overline{a}' \setminus \{a\}))$, thus $\overline{a}'$ is independent over $A$ but not over $A \cup B_0$. Hence $\overline{a} \not\mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_A B_0$.

□

**Lemma 5.3.16.** For any $\overline{a}$, $\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_A \text{acl}(A)$.

*Proof.* Let $\overline{a}' \subseteq \overline{a}$ be independent over $A$, then for every $a' \in \overline{a}'$ we have $a' \notin \text{acl}(A \cup (\overline{a}' \setminus \{a'\}))$. By Lemma 5.1.2 we have that $\text{acl}(A \cup (\overline{a}' \setminus \{a'\})) = \text{acl}(A \cup \text{acl}(A) \cup (\overline{a}' \setminus \{a'\}))$, thus $\overline{a}'$ is also independent over $A \cup \text{acl}(A)$.

□

**Lemma 5.3.17** (Symmetry)**.** If $\overline{c} \mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_A \overline{b}$, then $\overline{b} \mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_A \overline{c}$.

*Proof.* Let $\overline{b}'$ be a basis for $\overline{b}$ over $A$ and $\overline{c}' = \{c_0, ..., c_{m-1}\}$ be a basis for $\overline{c}$ over $A$. Notice that if $\overline{b}' = \emptyset$ or $\overline{c}' = \emptyset$, then $\overline{b} \mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_A \overline{c}$. In the first case we have that $\dim(\overline{b}/A) = 0 = \dim(\overline{b}/A \cup \overline{c})$. In the second we have that $\dim(\overline{c}/A) = 0$, which implies that $\overline{c} \subseteq \text{acl}(A)$, thus, by Lemma 5.3.13, we can conclude that $\overline{b} \mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_A \overline{c}$ because by Lemma 5.3.16 we have that $\overline{b} \mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_A \text{acl}(A)$.

Suppose that $\overline{b} \not\mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_A \overline{c}$, then there exists $b \in \overline{b}'$ such that $b \in \text{acl}((A \cup \overline{c}) \cup (\overline{b}' \setminus \{b\}))$. By hypothesis $\overline{c} \subseteq \text{acl}(A \cup \overline{c}')$ so we can conclude that $b \in \text{acl}((A \cup \overline{c}') \cup (\overline{b}' \setminus \{b\}))$.

Now $\overline{b}'$ is independent over $A$ so there exists $p \in \{0, ..., m-1\}$ such that $b \notin \text{acl}((A \cup \{c_0, ..., c_{p-1}\}) \cup (\overline{b}' \setminus \{b\}))$ but $b \in \text{acl}((A \cup \{c_0, ..., c_p\}) \cup (\overline{b}' \setminus \{b\}))$.

But then by the Exchange Principle we have that

$$c_p \in \text{acl}((A \cup \overline{b}') \cup \{c_0, ..., c_{p-1}\}) = \text{acl}((A \cup \overline{b}') \cup \{c_0, ..., c_{p-1}\}).$$

Thus $\overline{c}'$ is not independent over $A \cup \overline{b}$ and hence $\overline{c} \not\mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_A \overline{b}$.

□

**Corollary 5.3.18.** $\overline{a}\,\overline{b} \mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_A \overline{c}$ if and only if $\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_A \overline{c}$ and $\overline{b} \mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_{A \cup \overline{a}} \overline{c}$.

*Proof.*

$$\begin{aligned} \overline{a}\,\overline{b} \mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_{A} \overline{c} &\Leftrightarrow \overline{c} \mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_{A} \overline{a}\,\overline{b} && \text{[by Symmetry]} \\ &\Leftrightarrow \overline{c} \mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_{A} \overline{a} \text{ and } \overline{c} \mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_{A\cup\overline{a}} \overline{b} && \text{[by Transitivity]} \\ &\Leftrightarrow \overline{a} \mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_{A} \overline{c} \text{ and } \overline{b} \mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_{A\cup\overline{a}} \overline{c} && \text{[by Symmetry].} \end{aligned}$$

□

**Corollary 5.3.19.** If $\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_{A} \overline{b}$ and $\overline{a}\,\overline{b} \mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_{A} \overline{c}$, then $\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_{A} \overline{b}\,\overline{c}$.

*Proof.*

$$\begin{aligned} &\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_{A} \overline{b} \text{ and } \overline{a}\,\overline{b} \mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_{A} \overline{c} \\ &\Downarrow \\ &\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_{A} \overline{b} \text{ and } \overline{b}\,\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_{A} \overline{c} \\ &\Downarrow \\ &\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_{A} \overline{b} \text{ and } \overline{a} \mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_{A\cup\overline{b}} \overline{c} && \text{[by Corollary 5.3.18]} \\ &\Downarrow \\ &\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_{A} \overline{b}\,\overline{c} && \text{[by Transitivity].} \end{aligned}$$

□

**Proposition 5.3.20.** If $\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_{B} \overline{a}$, then $\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_{B} \overline{b}$ for any $\overline{b} \in M$.

*Proof.* If $\dim(\overline{a}/B\cup\overline{a}) = \dim(\overline{a}/B)$, then $\dim(\overline{a}/B) = 0$ because $\dim(\overline{a}/B\cup\overline{a}) = 0$. So $\emptyset$ is basis for $\overline{a}$ over $B$ and hence $\overline{a} \subseteq \text{acl}(B\cup\emptyset) = \text{acl}(B)$. Let now $\overline{b} \in M$, by Lemma 5.3.16 we have that $\overline{b} \mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_{B} \text{acl}(B)$ and hence by Lemma 5.3.13 and Lemma 5.3.17 we can conclude that $\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{\text{acl}}_{B} \overline{b}$.

□

We conclude this section with the definition of a theory with the ACL-independence property. Among the others conditions, the following definition impose to the models of the theory to satisfy the Exchange Principle and so all the theory developed in this section applies to models of such a theory.

**Definition 5.3.21.** Let $T$ be a first-order theory with infinite models. We say that $T$ has the ACL-independence property if for every $\mathcal{M} \models T$, $A, B, C \subseteq M$, $D_0 \subseteq_{\text{fin}} M$ independent and $a, b, \in M$, the following conditions hold:

i) $\text{acl}(\emptyset) \neq \emptyset$;

ii) $\text{acl}(D_0) \neq \bigcup_{D \subsetneq D_0} \text{acl}(D)$;

iii) If $a \in \text{acl}(A \cup \{b\}) \setminus \text{acl}(A)$, then $b \in \text{acl}(A \cup \{a\})$.

In Section 5.6 we will see that the formal theory of algebraically closed fields has the ACL-independence property.

## 5.4 Algebraic Closure Atomic Absolute Independence Logic

In this section we define the system Algebraic Closure Atomic Absolute Independence Logic (AclAAIndL) and then prove its soundness and completeness. The syntax and deductive apparatus of this system are the same as those of AAIndL.

### 5.4.1 Semantics

Let $T$ be a theory with the ACL-independence property.

**Definition 5.4.1.** Let $\mathcal{M} \models T$ and $s : \mathrm{dom}(s) \to M$ with $\overline{x} \subseteq \mathrm{dom}(s) \subseteq \mathrm{Var}$. We say that $\mathcal{M}$ satisfies $\bot(\overline{x})$ under $s$, in symbols $\mathcal{M} \models_s \bot(\overline{x})$, if for every $x \in \overline{x}$ we have that $s(x) \notin \mathrm{acl}(\{s(z) \mid z \in \overline{x}\} \setminus \{s(x)\})$.

**Definition 5.4.2.** Let $\Sigma$ be a set of atoms and let $s$ be such that the set of variables occurring in $\Sigma$ is included in $\mathrm{dom}(s)$. We say that $\mathcal{M}$ satisfies $\Sigma$ under $s$, in symbols $\mathcal{M} \models_s \Sigma$, if $\mathcal{M}$ satisfies every atom in $\Sigma$ under $s$.

**Definition 5.4.3.** Let $\Sigma$ be a set of atoms. We say that $\bot(\overline{x})$ is a logical consequence of $\Sigma$, in symbols $\Sigma \models \bot(\overline{x})$, if for every $\mathcal{M} \models T$ and $s$ such that the set of variables occurring in $\Sigma \cup \{\bot(\overline{x})\}$ is included in $\mathrm{dom}(s)$ we have that

$$\text{if } \ \mathcal{M} \models_s \Sigma \ \text{ then } \ \mathcal{M} \models_s \bot(\overline{x}).$$

### 5.4.2 Soundness and Completeness

**Theorem 5.4.4.** Let $\Sigma$ be a set of atoms, then

$$\Sigma \models \bot(\overline{x}) \text{ if and only if } \Sigma \vdash \bot(\overline{x}).$$

[The deductive system to which we refer has been defined in Section 2.3.3.]

*Proof.* ($\Leftarrow$) Obvious.

($\Rightarrow$) Suppose $\Sigma \nvdash \bot(\overline{x})$. Notice that if this is the case then $\overline{x} \neq \emptyset$. Indeed if $\overline{x} = \emptyset$ then $\Sigma \vdash \bot(\overline{x})$ because by rule ($a_2$.) $\vdash \bot(\emptyset)$.

We can assume that $\overline{x}$ is injective. This is without loss of generality because clearly $\mathcal{M} \models_s \bot(\overline{x})$ if and only if $\mathcal{M} \models_s \bot(\pi\overline{x})$, where $\pi : \mathrm{Var}^{<\omega} \to \mathrm{Var}^{<\omega}$ is the function that eliminates repetitions in finite sequences of variables. Let then $\overline{x} = (x_{j_0}, ..., x_{j_{n-1}}) \neq \emptyset$ be injective.

Let $\kappa > |\mathcal{L}| + \aleph_0$ where $\mathcal{L}$ is the signature of $T$. The theory $T$ has infinite models so by the Löwenheim-Skolem theorem there is a structure $\mathcal{M}$ such that $\mathcal{M} \models T$ and $|M| = \kappa$. By property iii) of Definition 5.3.21 we can assign dimensions to subset of $M$. Now, $|M| = \kappa$ and so we have that $\dim(M) = \kappa$ because for every $A \subseteq M$ such that $|A| < \kappa$ we have that $\mathrm{acl}(A) \leq \aleph_0 + |A| + |\mathcal{L}| < \kappa$.

Let then $\{a_i \mid i \in \kappa\}$ be an injective enumeration of a basis $B$ for $\mathcal{M}$ and $\{w_i \mid i \in \omega\}$ an injective enumeration of $\mathrm{Var} \setminus \{x_{j_0}\}$. Let $s$ be the following assignment:

$$s(w_i) = a_i$$

and

$$\begin{aligned} s(x_{j_0}) &= e \qquad \text{if } \overline{x} = \{x_{j_0}\} \\ s(x_{j_0}) &= d \qquad \text{if } \overline{x} \neq \{x_{j_0}\}, \end{aligned}$$

where $e \in \mathrm{acl}(\emptyset)$ and $d$ is such that $d \in \mathrm{acl}(\{s(x_{j_1}), ..., s(x_{j_{n-1}})\})$ but $d \notin \mathrm{acl}(D)$ for every $D \subsetneq \{s(x_{j_1}), ..., s(x_{j_{n-1}})\}$. Notice that $e$ and $d$ do exist because of properties i) and ii) of Definition 5.3.21.

We claim that $\mathcal{M} \not\models_s \bot(\overline{x})$. This is immediate because either

$$s(x_{j_0}) = e \quad \text{ or } \quad s(x_{j_0}) = d,$$

and

$$e \in \mathrm{acl}(\emptyset) \subseteq \mathrm{acl}(\{s(x) \mid x \in \overline{x}\} \setminus \{s(x_{j_0})\})$$

$$d \in \mathrm{acl}(\{s(x_{j_1}), ..., s(x_{j_{n-1}})\}) = \mathrm{acl}(\{s(x) \mid x \in \overline{x}\} \setminus \{s(x_{j_0})\}).$$

Let now $\perp(\overline{v}) \in \Sigma$, we want to show that $\mathcal{M} \models_s \perp(\overline{v})$. As before, we assume, without loss of generality, that $\overline{v}$ is injective. Notice that if $\overline{v} = \emptyset$, then $\mathcal{M} \models_s \perp(\overline{v})$. Thus let $\overline{v} = (v_{h_0}, ..., v_{h_{c-1}}) \neq \emptyset$.

**Case 1.** $x_{j_0} \notin \overline{v}$.

Let $w_{r_i} = v_{h_i}$ for every $i \in \{0, ..., c-1\}$, we then have that

$$\{s(v_{h_0}) = s(w_{r_0}) = a_{r_0}, ..., s(v_{h_{c-1}}) = s(w_{r_{c-1}}) = a_{r_{c-1}}\}$$

is independent.

**Case 2.** $x_{j_0} \in \overline{v}$.

**Subcase 1.** $\overline{x} \setminus \overline{v} \neq \emptyset$.

Notice that $\overline{x} \neq \{x_{j_0}\}$ because if not then $\overline{x} \setminus \overline{v} = \{x_{j_0}\}$ and so $x_{j_0} \notin \overline{v}$. Hence $s(x_{j_0}) = d$.

Let $(\overline{v} \setminus \{x_{j_0}\}) \cap \overline{x} = \left\{v_{h'_0}, ..., v_{h'_{b-1}}\right\}$, $\overline{v} \setminus \overline{x} = \left\{v_{h''_0}, ..., v_{h''_{t-1}}\right\}$, $w_{r'_i} = v_{h'_i}$ for every $i \in \{0, ..., b-1\}$ and $w_{r''_i} = v_{h''_i}$ for every $i \in \{0, ..., t-1\}$.

Suppose now that the set $\left\{a_{r'_0}, ..., a_{r'_{b-1}}, d, a_{r''_0}, ..., a_{r''_{t-1}}\right\}$ is dependent. The set $\left\{a_{r'_0}, ..., a_{r'_{b-1}}, a_{r''_0}, ..., a_{r''_{t-1}}\right\}$ is independent, so there are three cases.

Case 1. $a_{r''_l} \in \mathrm{acl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}, d, a_{r''_0}, ..., a_{r''_{l-1}}, a_{r''_{l+1}}, ..., a_{r''_{t-1}}\right\})$.

If this is the case, then

$$a_{r''_l} \in \mathrm{acl}(\left\{s(x_{j_1}), ..., s(x_{j_{n-1}}), a_{r''_0}, ..., a_{r''_{l-1}}, a_{r''_{l+1}}, ..., a_{r''_{t-1}}\right\})$$

because $d \in \mathrm{acl}(\{s(x_{j_1}), ..., s(x_{j_{n-1}})\})$. This is absurd though because the set $\left\{s(x_{j_1}), ..., s(x_{j_{n-1}}), a_{r''_0}, ..., a_{r''_{t-1}}\right\}$ is made of distinct elements of the basis $B$ and so it is independent.

Case 2. $d \in \mathrm{acl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}, a_{r''_0}, ..., a_{r''_{t-1}}\right\})$.

Notice that

$$d \notin \mathrm{acl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}\right\})$$

because $\left\{a_{r'_0}, ..., a_{r'_{b-1}}\right\} \subsetneq \{s(x_{j_1}), ..., s(x_{j_{n-1}})\}$ and $d$ has been chosen such that $d \in \mathrm{acl}(\{s(x_{j_1}), ..., s(x_{j_{n-1}})\})$ but $d \notin \mathrm{acl}(D)$ for every $D \subsetneq \{s(x_{j_1}), ..., s(x_{j_{n-1}})\}$.

Thus there is $l \leq t-1$ such that

$$d \in \mathrm{acl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}, a_{r''_0}, ..., a_{r''_{l-1}}\right\} \cup \left\{a_{r''_l}\right\}) \setminus \mathrm{acl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}, a_{r''_0}, ..., a_{r''_{l-1}}\right\})$$

and then by the Exchange Principle we have that

$$a_{r''_l} \in \mathrm{acl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}, a_{r''_0}, ..., a_{r''_{l-1}}\right\} \cup \{d\}).$$

Thus we have that $a_{r''_l} \in \mathrm{acl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}, d, a_{r''_0}, ..., a_{r''_{l-1}}, a_{r''_{l+1}}, ..., a_{r''_{t-1}}\right\})$, which is impossible as we saw in Case 1.

Case 3. $a_{r'_c} \in \mathrm{acl}(\left\{a_{r'_0}, ..., a_{r'_{c-1}}, a_{r'_{c+1}}, ..., a_{r'_{b-1}}, d, a_{r''_0}, ..., a_{r''_{t-1}}\right\})$.

Notice that

$$a_{r'_c} \notin \mathrm{acl}(\Big\{a_{r'_0}, ..., a_{r'_{c-1}}, a_{r'_{c+1}}, ..., a_{r'_{b-1}}, a_{r''_0}, ..., a_{r''_{t-1}}\Big\}).$$

Thus by the Exchange Principle we have that $d \in \mathrm{acl}(\Big\{a_{r'_0}, ..., a_{r'_{b-1}}, a_{r''_0}, ..., a_{r''_{t-1}}\Big\})$, which is impossible as we saw in Case 2.

**Subcase 2.** $\overline{x} \subseteq \overline{v}$.

This case is not possible. Suppose indeed it is, then by rule (c$_2$.) we can assume that $\overline{v} = \overline{x}\ \overline{v}'$ with $\overline{v}' \subseteq \mathrm{Var} \setminus \overline{x}$. Thus by rule (b$_2$.) we have that $\Sigma \vdash \perp(\overline{x})$ which is absurd.

□

## 5.5 Algebraic Closure Atomic Independence Logic

In this section we define the system Algebraic Closure Atomic Independence Logic (AclAIndL) and then prove its soundness and completeness. The syntax and deductive apparatus of this system are the same as those of AIndL.

### 5.5.1 Semantics

Let $T$ be a first-order theory with the ACL-independence property.

**Definition 5.5.1.** Let $\mathcal{M} \models T$ and $s : \mathrm{dom}(s) \to M$ with $\overline{x}\,\overline{y} \subseteq \mathrm{dom}(s) \subseteq \mathrm{Var}$. We say that $\mathcal{M}$ satisfies $\overline{x} \perp \overline{y}$ under $s$, in symbols $\mathcal{M} \models_s \overline{x} \perp \overline{y}$, if $\dim(s(\overline{x})/s(\overline{y})) = \dim(s(\overline{x}))$.

**Definition 5.5.2.** Let $\Sigma$ be a set of atoms and let $s$ be such that the set of variables occurring in $\Sigma$ is included in $\mathrm{dom}(s)$. We say that $\mathcal{M}$ satisfies $\Sigma$ under $s$, in symbols $\mathcal{M} \models_s \Sigma$, if $\mathcal{M}$ satisfies every atom in $\Sigma$ under $s$.

**Definition 5.5.3.** Let $\Sigma$ be a set of atoms. We say that $\overline{x} \perp \overline{y}$ is a logical consequence of $\Sigma$, in symbols $\Sigma \models \overline{x} \perp \overline{y}$, if for every $\mathcal{M} \models T$ and $s$ such that the set of variables occurring in $\Sigma \cup \{\overline{x} \perp \overline{y}\}$ is included in $\mathrm{dom}(s)$ we have that

$$\text{if } \mathcal{M} \models_s \Sigma \text{ then } \mathcal{M} \models_s \overline{x} \perp \overline{y}.$$

### 5.5.2 Soundness and Completeness

**Theorem 5.5.4.** Let $\Sigma$ be a set of atoms, then

$$\Sigma \models \overline{x} \perp \overline{y} \text{ if and only if } \Sigma \vdash \overline{x} \perp \overline{y}.$$

[The deductive system to which we refer has been defined in Section 2.4.3.]

*Proof.* ($\Leftarrow$) (a$_3$.) Obvious.

(b$_3$.)

$$\begin{aligned} \mathcal{M} \models_s \overline{x} \perp \overline{y} \quad &\Longrightarrow \quad s(\overline{x}) \mathop{\downarrow}^{\mathrm{acl}}_{\emptyset} s(\overline{y}) \\ &\Longrightarrow \quad s(\overline{y}) \mathop{\downarrow}^{\mathrm{acl}}_{\emptyset} s(\overline{x}) \quad \text{[By Lemma 5.3.17]} \\ &\Longrightarrow \quad \mathcal{M} \models_s \overline{y} \perp \overline{x}. \end{aligned}$$

(c$_3$.)

$$\begin{array}{rcll} \mathcal{M} \models_s \overline{x} \perp \overline{y}\,\overline{z} & \Longrightarrow & s(\overline{x}) \mathop{\smile\!\!\!\!\!\!\downarrow}^{\mathrm{acl}}_{\emptyset} s(\overline{y}\,\overline{z}) & \\ & \Longrightarrow & s(\overline{x}) \mathop{\smile\!\!\!\!\!\!\downarrow}^{\mathrm{acl}}_{\emptyset} s(\overline{y}) & \text{[By Lemma 5.3.13]} \\ & \Longrightarrow & \mathcal{M} \models_s \overline{x} \perp \overline{y}. & \end{array}$$

(d$_3$.)

$$\begin{array}{cl} \mathcal{M} \models_s \overline{x} \perp \overline{y} \text{ and } \mathcal{M} \models_s \overline{x}\,\overline{y} \perp \overline{z} & \\ \Downarrow & \\ s(\overline{x}) \mathop{\smile\!\!\!\!\!\!\downarrow}^{\mathrm{acl}}_{\emptyset} s(\overline{y}) \text{ and } s(\overline{x})s(\overline{y}) \mathop{\smile\!\!\!\!\!\!\downarrow}^{\mathrm{acl}}_{\emptyset} s(\overline{z}) & \\ \Downarrow & \\ s(\overline{x}) \mathop{\smile\!\!\!\!\!\!\downarrow}^{\mathrm{acl}}_{\emptyset} s(\overline{y})s(\overline{z}) & \text{[By Corollary 5.3.19]} \\ \Downarrow & \\ \mathcal{M} \models_s \overline{x} \perp \overline{y}\,\overline{z}. & \end{array}$$

(e$_3$.) Suppose that $\mathcal{M} \models_s x \perp x$, then $s(x) \mathop{\smile\!\!\!\!\!\!\downarrow}^{\mathrm{acl}}_{\emptyset} s(x)$ and so by Proposition 5.3.20 we have that $s(x) \mathop{\smile\!\!\!\!\!\!\downarrow}^{\mathrm{acl}}_{\emptyset} s(\overline{y})$ for any $\overline{y} \in \mathrm{Var}$.

(f$_3$.) Obvious.

($\Rightarrow$) Suppose $\Sigma \nvdash \overline{x} \perp \overline{y}$. Notice that if this is the case then $\overline{x} \neq \emptyset$ and $\overline{y} \neq \emptyset$. Indeed if $\overline{y} = \emptyset$ then $\Sigma \vdash \overline{x} \perp \overline{y}$ because by rule (a$_3$.) $\vdash \overline{x} \perp \emptyset$. Analogously if $\overline{x} = \emptyset$ then $\Sigma \vdash \overline{x} \perp \overline{y}$ because by rule (a$_3$.) $\vdash \overline{y} \perp \emptyset$ and so by rule (b$_3$.) $\vdash \emptyset \perp \overline{y}$.

We can assume that $\overline{x}$ and $\overline{y}$ are injective. This is without loss of generality because clearly $\mathcal{M} \models_s \overline{x} \perp \overline{y}$ if and only if $\mathcal{M} \models_s \pi\overline{x} \perp \pi\overline{y}$, where $\pi : \mathrm{Var}^{<\omega} \to \mathrm{Var}^{<\omega}$ is the function that eliminates repetitions in finite sequences of variables.

Furthermore we can assume that $\overline{x} \perp \overline{y}$ is minimal, in the sense that if $\overline{x}' \subseteq \overline{x}$, $\overline{y}' \subseteq \overline{y}$ and $\overline{x}'\,\overline{y}' \neq \overline{x}\,\overline{y}$, then $\Sigma \vdash \overline{x}' \perp \overline{y}'$. This is for two reasons.

i) If $\overline{x} \perp \overline{y}$ is not minimal we can always find a minimal atom $\overline{x}^* \perp \overline{y}^*$ such that $\Sigma \nvdash \overline{x}^* \perp \overline{y}^*$, $\overline{x}^* \subseteq \overline{x}$ and $\overline{y}^* \subseteq \overline{y}$ $--$ just keep deleting elements of $\overline{x}$ and $\overline{y}$ until you obtain the desired property or until both $\overline{x}^*$ and $\overline{y}^*$ are singletons, in which case, due to the trivial independence rule (a$_3$.), $\overline{x}^* \perp \overline{y}^*$ is a minimal statement.

ii) For any $\overline{x}' \subseteq \overline{x}$ and $\overline{y}' \subseteq \overline{y}$ we have that if $\mathcal{M} \not\models_s \overline{x}' \perp \overline{y}'$ then $\mathcal{M} \not\models_s \overline{x} \perp \overline{y}$, for every $\mathcal{M}$ and $s$.

Let indeed $\overline{x} = \overline{x}'\overline{x}''$ and $\overline{y} = \overline{y}'\overline{y}''$, then

$$\begin{array}{rcll} \mathcal{M} \models_s \overline{x}'\,\overline{x}'' \perp \overline{y}'\overline{y}'' & \Longrightarrow & s(\overline{x}')s(\overline{x}'') \mathop{\smile\!\!\!\!\!\!\downarrow}^{\mathrm{acl}}_{\emptyset} s(\overline{y}')s(\overline{y}'') & \\ & \Longrightarrow & s(\overline{x}')s(\overline{x}'') \mathop{\smile\!\!\!\!\!\!\downarrow}^{\mathrm{acl}}_{\emptyset} s(\overline{y}') & \text{[By Lemma 5.3.13]} \\ & \Longrightarrow & s(\overline{x}') \mathop{\smile\!\!\!\!\!\!\downarrow}^{\mathrm{acl}}_{\emptyset} s(\overline{y}') & \text{[By Corollary 5.3.18]} \\ & \Longrightarrow & \mathcal{M} \models_s \overline{x}' \perp \overline{y}'. & \end{array}$$

Let then $\overline{x} = (x_{j_0}, ..., x_{j_{n-1}})$ and $\overline{y} = (y_{k_0}, ..., y_{k_{m-1}})$ be injective and such that $\overline{x} \perp \overline{y}$ is minimal.

Let $V = \{v \in \mathrm{Var} \mid \Sigma \vdash v \perp v\}$ and $W = \mathrm{Var} \setminus V$. We claim that $\overline{x}, \overline{y} \not\subseteq V$. We prove it only for $\overline{x}$, the other case is symmetrical. Suppose that $\overline{x} \subseteq V$, then for every $x \in \overline{x}$ we have that $\Sigma \vdash x \perp x$ so by rule (e$_3$.), (b$_3$.) and (d$_3$.)

$$\Sigma \vdash \overline{y} \perp x_{j_0} \quad \text{and} \quad \Sigma \vdash \overline{y}\,x_{j_0} \perp x_{j_1} \Rightarrow \quad \Sigma \vdash \overline{y} \perp x_{j_0}\,x_{j_1},$$

$$\Sigma \vdash \overline{y} \perp x_{j_0} x_{j_1} \quad \text{and} \quad \Sigma \vdash \overline{y} x_{j_0} x_{j_1} \perp x_{j_2} \Rightarrow \quad \Sigma \vdash \overline{y} \perp x_{j_0} x_{j_1} x_{j_2},$$

$$\vdots$$

$$\Sigma \vdash \overline{y} \perp x_{j_0} \cdots x_{j_{n-2}} \quad \text{and} \quad \Sigma \vdash \overline{y} x_{j_0} \cdots x_{j_{n-2}} \perp x_{j_{n-1}} \Rightarrow \quad \Sigma \vdash \overline{y} \perp \overline{x}.$$

Hence by rule (b$_3$.) $\Sigma \vdash \overline{x} \perp \overline{y}$.

Thus $\overline{x} \cap W \neq \emptyset$ and $\overline{y} \cap W \neq \emptyset$. Without loss of generality suppose that $x_{j_0} \in W$ and $y_{k_0} \in W$. Let $\overline{x} \cap W = \overline{x}' = (x_{j'_0}, ..., x_{j'_{n'-1}}) = (x_{j_0}, ..., x_{j'_{n'-1}}) \neq \emptyset$ and $\overline{y} \cap W = \overline{y}' = (y_{k'_0}, ..., y_{k'_{m'-1}})$. Notice that $\overline{x}' \cap \overline{y}' = \emptyset$. Indeed let $z \in \overline{x}' \cap \overline{y}'$, then by rules (b$_3$.) and (c$_3$.) we have that $\Sigma \vdash z \perp z$. Thus $z \in V$, a contradiction.

Let $\kappa > |\mathcal{L}| + \aleph_0$ where $\mathcal{L}$ is the signature of $T$. The theory $T$ has infinite models so by the Löwenheim-Skolem theorem there is a structure $\mathcal{M}$ such that $\mathcal{M} \models T$ and $|M| = \kappa$. By property iii) of Definition 5.3.21 we can assign dimensions to subset of $M$. Now, $|M| = \kappa$ and so we have that $\dim(M) = \kappa$ because for every $A \subseteq M$ such that $|A| < \kappa$ we have that $\mathrm{acl}(A) \leq \aleph_0 + |A| + |\mathcal{L}| < \kappa$.

Let then $\{a_i \mid i \in \kappa\}$ be an injective enumeration of a basis $B$ for $\mathcal{M}$ and $\{w_i \mid i \in \lambda\}$ be an injective enumeration of $W \setminus \{x_{j_0}\}$. Let $s$ be the following assignment:

i) $s(v) = e$ for every $v \in V$,

ii) $s(w_i) = a_i$ for every $i \in \lambda$,

iii) $s(x_{j_0}) = d$,

where $e \in \mathrm{acl}(\emptyset)$ and $d$ is such that

$$d \in \mathrm{acl}(\Big\{ s(x_{j'_1}), ..., s(x_{j'_{n'-1}}), s(y_{k'_0}), ..., s(y_{k'_{m'-1}}) \Big\})$$

but $d \notin \mathrm{acl}(D)$ for every $D \subsetneq \Big\{ s(x_{j'_1}), ..., s(x_{j'_{n'-1}}), s(y_{k'_0}), ..., s(y_{k'_{m'-1}}) \Big\}$. Notice that $e$ and $d$ do exist because of properties i) and ii) of Definition 5.3.21.

We claim that $\mathcal{M} \not\models_s \overline{x}' \perp \overline{y}'$, as noticed this implies that $\mathcal{M} \not\models_s \overline{x} \perp \overline{y}$.

First we show that the set $\{s(x') \mid x' \in \overline{x}'\}$ is independent. By construction $s(x_{j_0}) \notin \mathrm{acl}(\{s(x') \mid x' \in \overline{x}'\} \setminus \{s(x_{j_0})\})$. Let then $i \in \{1, ..., n' - 1\}$ and suppose that $s(x_{j'_i}) \in \mathrm{acl}(\Big\{ s(x_{j_0}), ..., s(x_{j'_{i-1}}), s(x_{j'_{i+1}}), ..., s(x_{j'_{n-1}}) \Big\})$.

The set $\Big\{ s(x_{j'_1}), ..., s(x_{j'_{i-1}}), s(x_{j'_{i+1}}), ..., s(x_{j'_{n-1}}) \Big\}$ is independent, so

$$s(x_{j'_i}) \in \mathrm{acl}(\Big\{ s(x_{j'_1}), ..., s(x_{j'_{i-1}}), s(x_{j'_{i+1}}), ..., s(x_{j'_{n-1}}) \Big\} \cup \{s(x_{j_0})\})$$

but

$$s(x_{j'_i}) \notin \mathrm{acl}(\Big\{ s(x_{j'_1}), ..., s(x_{j'_{i-1}}), s(x_{j'_{i+1}}), ..., s(x_{j'_{n-1}}) \Big\}).$$

Hence by the Exchange Principle

$$s(x_{j_0}) \in \mathrm{acl}(\Big\{ s(x_{j'_1}), ..., s(x_{j'_{n-1}}) \Big\}),$$

a contradiction. Thus $\dim(s(\overline{x}')) = |\{s(x') \mid x' \in \overline{x}'\}|$.

We now show that $\left\{s(x_{j'_1}),...,s(x_{j'_{n-1}})\right\}$ is a basis for $\{s(x') \mid x' \in \overline{x}'\}$ over $\{s(y') \mid y' \in \overline{y}'\}$.

As we noticed above $\overline{x}' \cap \overline{y}' = \emptyset$, so by properties of our assignment $s(\overline{x}') \cap s(\overline{y}') = \emptyset$. Thus, by Lemma 5.3.9, $\left\{s(x_{j'_1}),...,s(x_{j'_{n-1}})\right\}$ is independent over $\{s(y') \mid y' \in \overline{y}'\}$, also $\left\{s(x_{j_0}),...,s(x_{j'_{n-1}})\right\} \subseteq \mathrm{acl}(s(\overline{y}') \cup \left\{s(x_{j'_1}),...,s(x_{j'_{n-1}})\right\})$ because $s(x_{j_0}) \in \mathrm{acl}(\left\{s(x_{j'_1}),...,s(x_{j'_{n'-1}}),s(y_{k'_0}),...,s(y_{k'_{m'-1}})\right\})$.

Hence

$$\dim(s(\overline{x}')/s(\overline{y}')) = |\left\{s(x_{j'_1}),...,s(x_{j'_{n-1}})\right\}| = \dim(s(\overline{x}')) - 1.$$

Let now $\overline{v} \perp \overline{w} \in \Sigma$, we want to show that $\mathcal{M} \models_s \overline{v} \perp \overline{w}$. As before, we assume, without loss of generality, that $\overline{v}$ and $\overline{w}$ are injective. Notice also that if $\overline{v} = \emptyset$ or $\overline{w} = \emptyset$, then $\mathcal{M} \models_s \overline{v} \perp \overline{w}$. Thus let $\overline{v}, \overline{w} \neq \emptyset$.

**Case 1.** $\overline{v} \subseteq V$ or $\overline{w} \subseteq V$.

Suppose that $\overline{v} \subseteq V$, the other case is symmetrical, then $s(\overline{v}) \subseteq \mathrm{acl}(\emptyset)$. Thus $\dim(s(\overline{v})/s(\overline{w})) = 0 = \dim(s(\overline{v}))$.

**Case 2.** $\overline{v} \not\subseteq V$ and $\overline{w} \not\subseteq V$.

Let $\overline{v} \cap W = \overline{v}' \neq \emptyset$ and $\overline{w} \cap W = \overline{w}' \neq \emptyset$.

Notice that

$$s(\overline{v}) \underset{\emptyset}{\overset{\mathrm{acl}}{\perp}} s(\overline{w}) \text{ if and only if } s(\overline{v'}) \underset{\emptyset}{\overset{\mathrm{acl}}{\perp}} s(\overline{w'})$$

Left to right holds in general. As for the other direction, suppose that $s(\overline{v}') \perp^{\mathrm{acl}}_{\emptyset} s(\overline{w}')$. If $u \in \overline{v}\,\overline{w} \setminus \overline{v'}\,\overline{w'}$, then $s(u) = e \in \mathrm{acl}(\emptyset)$. Thus

$$\begin{gathered} s(\overline{v}') \perp^{\mathrm{acl}}_{\emptyset} s(\overline{w}') \text{ and } s(\overline{v}')\,s(\overline{w}') \perp^{\mathrm{acl}}_{\emptyset} \mathrm{acl}(\emptyset) \quad \text{[By Lemma 5.3.16 ]} \\ \Downarrow \\ s(\overline{v}') \perp^{\mathrm{acl}}_{\emptyset} s(\overline{w}') \cup \mathrm{acl}(\emptyset) \\ \Downarrow \\ s(\overline{v}') \perp^{\mathrm{acl}}_{\emptyset} s(\overline{w}') \cup (\mathrm{acl}(\emptyset) \cap s(\overline{w})) \\ \Downarrow \\ s(\overline{v}') \perp^{\mathrm{acl}}_{\emptyset} s(\overline{w}). \end{gathered}$$

So

$$\begin{gathered} s(\overline{w}) \perp^{\mathrm{acl}}_{\emptyset} s(\overline{v}') \text{ and } s(\overline{w})\,s(\overline{v}') \perp^{\mathrm{acl}}_{\emptyset} \mathrm{acl}(\emptyset) \quad \text{[By Lemma 5.3.16 ]} \\ \Downarrow \\ s(\overline{w}) \perp^{\mathrm{acl}}_{\emptyset} s(\overline{v}') \cup \mathrm{acl}(\emptyset) \\ \Downarrow \\ s(\overline{w}) \perp^{\mathrm{acl}}_{\emptyset} s(\overline{v}') \cup (\mathrm{acl}(\emptyset) \cap s(\overline{v})) \\ \Downarrow \\ s(\overline{w}) \perp^{\mathrm{acl}}_{\emptyset} s(\overline{v}). \end{gathered}$$

**Subcase 2.1.** $x_{j_0} \notin \overline{v}'\,\overline{w}'$.

Notice that $\overline{v}' \cap \overline{w}' = \emptyset$, so by properties of our assignment $s(\overline{v}') \cap s(\overline{w}') = \emptyset$. Thus by Lemma 5.3.9 it follows that $\dim(s(\overline{v}')/s(\overline{w}')) = \dim(s(\overline{v}'))$.

**Subcase 2.2.** $x_{j_0} \in \overline{v}'\,\overline{w}'$.

**Subsubcase 2.2.1.** $\overline{x}'\,\overline{y}' \setminus \overline{v}'\,\overline{w}' \neq \emptyset$.

Let $\overline{v}'\,\overline{w}' \setminus \{x_{j_0}\} \cap (\overline{x}'\,\overline{y}') = \left\{u_{h'_0}, ..., u_{h'_{b-1}}\right\}$, $\overline{v}'\,\overline{w}' \setminus \overline{x}'\,\overline{y}' = \left\{u_{h''_0}, ..., u_{h''_{t-1}}\right\}$, $w_{r'_i} = u_{h'_i}$ for every $i \in \{0, ..., b-1\}$ and $w_{r''_i} = u_{h''_i}$ for every $i \in \{0, ..., t-1\}$

Suppose now that the set $\left\{a_{r'_0}, ..., a_{r'_{b-1}}, d, a_{r''_0}, ..., a_{r''_{t-1}}\right\}$ is dependent. There are three cases.

Case 1. $a_{r''_l} \in \text{acl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}, d, a_{r''_0}, ..., a_{r''_{l-1}}, a_{r''_{l+1}}, ..., a_{r''_{t-1}}\right\})$.

If this is the case, then

$$a_{r''_l} \in \text{acl}(\left\{s(x_{j'_1}), ..., s(x_{j'_{n'-1}}), s(y_{k'_0}), ..., s(y_{k'_{m'-1}}), a_{r''_0}, ..., a_{r''_{l-1}}, a_{r''_{l+1}}, ..., a_{r''_{t-1}}\right\})$$

because $d \in \text{acl}(\left\{s(x_{j'_1}), ..., s(x_{j'_{n'-1}}), s(y_{k'_0}), ..., s(y_{k'_{m'-1}})\right\})$. This is absurd though because the set $\left\{s(x_{j'_1}), ..., s(x_{j'_{n'-1}}), s(y_{k'_0}), ..., s(y_{k'_{m'-1}}), a_{r''_0}, ..., a_{r''_{t-1}}\right\}$ is made of distinct elements of the basis $B$ and so it is independent.

Case 2. $d \in \text{acl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}, a_{r''_0}, ..., a_{r''_{t-1}}\right\})$.

Notice that

$$d \notin \text{acl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}\right\})$$

because $\left\{a_{r'_0}, ..., a_{r'_{b-1}}\right\} \subsetneq \left\{s(x_{j'_1}), ..., s(x_{j'_{n'-1}}), s(y_{k'_0}), ..., s(y_{k'_{m'-1}})\right\}$ and $d$ has been chosen such that $d \in \text{acl}(\left\{s(x_{j'_1}), ..., s(x_{j'_{n'-1}}), s(y_{k'_0}), ..., s(y_{k'_{m'-1}})\right\})$ but $d \notin \text{acl}(D)$ for every $D \subsetneq \left\{s(x_{j'_1}), ..., s(x_{j'_{n'-1}}), s(y_{k'_0}), ..., s(y_{k'_{m'-1}})\right\}$.

Thus there is $l \leq t-1$ such that

$$d \in \text{acl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}, a_{r''_0}, ..., a_{r''_{l-1}}\right\} \cup \left\{a_{r''_l}\right\}) \setminus \text{acl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}, a_{r''_0}, ..., a_{r''_{l-1}}\right\})$$

and then by the Exchange Property we have that

$$a_{r''_l} \in \text{acl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}, a_{r''_0}, ..., a_{r''_{l-1}}\right\} \cup \{d\}).$$

Thus we have that $a_{r''_l} \in \text{acl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}, d, a_{r''_0}, ..., a_{r''_{l-1}}, a_{r''_{l+1}}, ..., a_{r''_{t-1}}\right\})$, which is impossible as we saw in Case 1.

Case 3. $a_{r'_c} \in \text{acl}(\left\{a_{r'_0}, ..., a_{r'_{c-1}}, a_{r'_{c+1}}, ..., a_{r'_{b-1}}, d, a_{r''_0}, ..., a_{r''_{t-1}}\right\})$.

Notice that

$$a_{r'_c} \notin \text{acl}(\left\{a_{r'_0}, ..., a_{r'_{c-1}}, a_{r'_{c+1}}, ..., a_{r'_{b-1}}, a_{r''_0}, ..., a_{r''_{t-1}}\right\}).$$

Thus by the Exchange Principle we have that $d \in \text{acl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}, a_{r''_0}, ..., a_{r''_{t-1}}\right\})$, which is impossible as we saw in Case 2.

We can then conclude that the set $\left\{a_{r'_0}, ..., a_{r'_{b-1}}, d, a_{r''_0}, ..., a_{r''_{t-1}}\right\}$ is independent. Clearly $\{s(v') \mid v' \in \overline{v}'\} \cup \{s(w') \mid w' \in \overline{w}'\} = \left\{a_{r'_0}, ..., a_{r'_{b-1}}, d, a_{r''_0}, ..., a_{r''_{t-1}}\right\}$. Furthermore, as we noticed above, $s(\overline{v}') \cap s(\overline{w}') = \emptyset$. Hence by Lemma 5.3.9 we have that $\dim(s(\overline{v}')/s(\overline{w}')) = \dim(s(\overline{v}'))$.

**Subsubcase 2.2.2.** $\overline{x}'\,\overline{y}' \subseteq \overline{v}'\,\overline{w}'$.

As shown in Theorem 3.4.4, this case is not possible.

□

## 5.6 Algebraic Closure in Algebraically Closed Fields

In this section we begin by defining the formal theory of algebraically closed fields and see how the operator of algebraic closure in algebraically closed field relates with its formal counterpart. Then we show that the formal theory of algebraically closed fields has the ACL-independence property and see what are the relations between the notion of transcendence degree defined in Chapter 4 and the formal notion of dimension defined in this one. Finally we deduce from these results that the algebraically closed fields systems that we studied in the previous chapter are particular cases of the systems treated in this one.

Let $\mathcal{L}_r$ be the language or rings $\{+, -, \cdot, 0, 1\}$, where $+, -$ and $\cdot$ are binary function symbols and 0 and 1 are constants. We axiomatize the class of algebraically closed fields by adding to the field axioms the sentences

$$\forall a_0 \cdots \forall a_{n-1} \exists x \; (x^n + \sum_{i=0}^{n-1} a_i \, x^i = 0)$$

for $n = 1, 2, ...$ . We denote by ACF the axioms for algebraically closed fields.

**Proposition 5.6.1.** Let $\mathcal{K} \models$ ACF and $A \subseteq K$. Then $a \in \mathrm{acl}(A)$ if and only if $a$ is algebraic over the subfield of $\mathcal{K}$ generated by $A$.

*Proof.* See [29, Proposition 3.2.15]. □

**Theorem 5.6.2.** ACF has the ACL-independence property.

*Proof.* Let $\mathcal{K} \models$ ACF and $\mathbb{K}$ be the corresponding field. Let $\mathbb{P}$ be the prime field of $\mathbb{K}$, $D_0 \subseteq K$ finite and independent in $\mathcal{K}$ and $a, b \in K$.

i) By Proposition 5.6.1, $\mathrm{acl}(\emptyset) = \{a \in K \mid a \text{ is algebraic over } \mathbb{P}\} \neq \emptyset$.

ii) Let $D_0 = \{d_0, .., d_{n-1}\}$. Clearly $\prod_{i=0}^{n-1} d_i \in \mathrm{acl}(D_0)$, indeed let $\mathbb{F}$ be the subfield of $\mathbb{K}$ generated by $D_0$, then $\prod_{i=0}^{n-1} d_i \in \mathbb{F}$ and so clearly it is algebraic over $\mathbb{F}$.

We claim that $\prod_{i=0}^{n-1} d_i \notin \bigcup_{D \subsetneq D_0} \mathrm{acl}(D)$. It suffices to show that $\prod_{i=0}^{n-1} d_i \notin \mathrm{acl}(T)$ for $T \subseteq D_0$ and $|T| = n - 1$. Let then $T = \{d_0, ..., d_{l-1}, d_{l+1}, ..., d_{n-1}\}$ and suppose that $\prod_{i=0}^{n-1} d_i \in \mathrm{acl}(T)$. By Proposition 5.6.1 $\mathrm{acl}(T)$ is the set of algebraic elements over the subfield $\mathbb{F}'$ of $\mathbb{K}$ generated by $T$. This set is actually a field, so for every $i \in \{0, ..., l-1, l+1, ..., n-1\}$ we have that $d_i^{-1} \in \mathrm{acl}(T)$ and thus that $\prod_{\substack{i=0 \\ i \neq l}}^{n-1} d_i^{-1} \in \mathrm{acl}(T)$.

Hence also

$$\prod_{i=0}^{n-1} d_i \cdot \prod_{\substack{i=0 \\ i \neq l}}^{n-1} d_i^{-1} = d_l \in \mathrm{acl}(T).$$

Thus $D_0$ is not independent in $\mathcal{K}$, a contradiction.

iii) Suppose that $a \in \mathrm{acl}(A \cup \{b\}) \setminus \mathrm{acl}(A)$, then $a$ is algebraic over $\mathbb{P}(A \cup \{b\})$ but $a$ is not algebraic over $\mathbb{P}(A)$ so, by [30, Lemma 8.6], we have that $b$ is algebraic over $\mathbb{P}(A \cup \{a\})$ and hence, by Proposition 5.6.1, $b \in \mathrm{acl}(A \cup \{a\})$.

□

**Lemma 5.6.3.** Let $\mathbb{K}$ be an algebraically closed field and $\mathcal{K}$ the corresponding model of ACF. Let $A, B, C \subseteq K$ and $\mathbb{F}$ the subfield of $\mathbb{K}$ generated by $C$, then

$$\mathrm{trdg}(\mathbb{F}(A)/\mathbb{F}(B)) = \dim(A/C \cup B),$$

where $\dim(A/C \cup B)$ is computed in $\mathcal{K}$.

*Proof.* Let $A'$ be a basis for $A$ over $C \cup B$, we want to show that $A'$ is a transcendence basis for $\mathbb{F}(A)$ over $\mathbb{F}(B)$. Let $a' \in A'$, then $a' \notin \mathrm{acl}(C \cup B \cup (A' \setminus \{a'\}))$. By Proposition 5.6.1, $\mathrm{acl}(C \cup B \cup (A' \setminus \{a'\}))$ is the set of algebraic elements over the subfield of $\mathbb{K}$ generated by $C \cup B \cup (A' \setminus \{a'\})$, that is $\mathbb{F}(B \cup (A' \setminus \{a'\})$, thus $a'$ is transcendental over $\mathbb{F}(B \cup (A' \setminus \{a'\})$. Hence $A'$ is algebraically independent over $\mathbb{F}(B)$.

Let $a$ be an element of $\mathbb{F}(A)$. By Proposition 5.6.1, $a \in \mathrm{acl}(C \cup A)$. By hypothesis $A \subseteq \mathrm{acl}(C \cup B \cup A')$ and so $\mathrm{acl}(C \cup A) \subseteq \mathrm{acl}(C \cup B \cup A')$. Thus $a \in \mathrm{acl}(C \cup B \cup A')$ and then, again by Proposition 5.6.1, we have that $a$ is algebraic over $\mathbb{F}(B \cup A')$. Hence $\mathbb{F}(A)$ is algebraic over $\mathbb{F}(B \cup A')$.

□

**Corollary 5.6.4.** Let $\mathbb{K}$ be an algebraically closed field and $\mathcal{K}$ the corresponding model of ACF. Let $B, C \subseteq K$, $\mathbb{F}$ the subfield of $\mathbb{K}$ generated by $C$ and $\overline{a} \in K$, then

$$\overline{a} \overset{\mathrm{acl}}{\underset{C}{\perp}} B \iff \overline{a} \overset{\mathrm{tr}}{\underset{\mathbb{F}}{\perp}} B.$$

□

We denote by $\models^{\mathrm{ACF}}$ the satisfaction relation of the systems ACFADL, ACFAAIndL and ACFAIndL and with ${}^{\mathrm{acl}}\!\models_s^{\mathrm{ACF}}$ the satisfaction relation of the systems AclADL, AclAAIndL and AclAIndL relative to the theory ACF. From what we showed in this section it follows directly the following theorem.

**Theorem 5.6.5.** Let $\mathbb{K}$ be an algebraically closed field and $\mathcal{K}$ the corresponding model of ACF. Let $s : \mathrm{dom}(s) \to K$ with $\overline{x}\,\overline{y} \subseteq \mathrm{dom}(s) \subseteq \mathrm{Var}$. Then the following hold:

i) $\mathbb{K} \models_s^{\mathrm{ACF}} =\!(\overline{x}, \overline{y})$ if and only if $\mathcal{K}\ {}^{\mathrm{acl}}\!\models_s^{\mathrm{ACF}} =\!(\overline{x}, \overline{y})$;

ii) $\mathbb{K} \models_s^{\mathrm{ACF}} \perp(\overline{x})$ if and only if $\mathcal{K}\ {}^{\mathrm{acl}}\!\models_s^{\mathrm{ACF}} \perp(\overline{x})$;

iii) $\mathbb{K} \models_s^{\mathrm{ACF}} \overline{x} \perp \overline{y}$ if and only if $\mathcal{K}\ {}^{\mathrm{acl}}\!\models_s^{\mathrm{ACF}} \overline{x} \perp \overline{y}$. □

## 5.7 Relations between AclAIndL and AclAAIndL

In this section, which is an analog of Section 3.5, we study the relations, under the given semantics, between the independence atom and the absolute independence one.

**Lemma 5.7.1.** Let $T$ be a theory with the ACL-independence property, $\mathcal{M} \models T$ and $A \subseteq M$. The following are equivalent:

i) $a \notin \mathrm{acl}(A \setminus \{a\})$ for every $a \in A$;

ii) $A \cap \mathrm{acl}(\emptyset) = \emptyset$ and $\dim(a/A \setminus \{a\}) = \dim(a)$ for every $a \in A$.

*Proof.* ii) $\Rightarrow$ i) Suppose that there exists $a \in A$ such that $a \in \mathrm{acl}(A \setminus \{a\})$ and $a \notin \mathrm{acl}(\emptyset)$, then $\dim(a) = 1$ and $\dim(a/A\setminus\{a\}) = 0$ because $a \in \mathrm{acl}((A\setminus\{a\})\cup\emptyset)$.

i) $\Rightarrow$ ii) If there exists $a \in A$ such that $a \in \mathrm{acl}(\emptyset)$, then $a \in \mathrm{acl}(A \setminus \{a\})$. If there exists $a \in A$ such that $\dim(a/A \setminus \{a\}) \neq \dim(a)$, then $\dim(a/A \setminus \{a\}) = 0$ and $\dim(a) = 1$, so $a \in \mathrm{acl}((A \setminus \{a\}) \cup \emptyset) = \mathrm{acl}(A \setminus \{a\})$.

□

From the above Lemma it follows directly the following characterization of AclAAIndL in terms of AclAIndL.

**Lemma 5.7.2.** Let $s$ be an assignment such that the set of variables occurring in $\Sigma \cup \{\perp(\overline{x})\}$ is included in $\mathrm{dom}(s)$. Let $x \in \overline{x}$, we denote by $\overline{x} -_X x$ any enumeration of the set $\{x' \in \overline{x} \mid \mathcal{M} \not\models_s x' = x\}$. Then

$$\mathcal{M} \models_s \perp(\overline{x}) \Longleftrightarrow \mathcal{M} \models_s x \perp \overline{x} -_X x \text{ and } \mathcal{M} \not\models_s x \perp x, \text{ for all } x \in \overline{x}.$$

□

# Chapter 6

# Dependence and Independence in Pregeometries

In this chapter we develop a straightforward generalization of what we have done in Chapter 5. As one can easily notice through a thoughtful analysis of the proofs of the results of the previous chapter, all the theory that we developed with respect to the algebraic closure operator does not depend intrinsically on this operator but only on a few structural properties that it satisfies. We can then develop the same theory working just with a closure operator that satisfies these essential conditions, leading to the very general notion of pregeometry.

The notion of pregeometry is a key notion of geometric model theory because it allows to define fundamental geometric concepts in a completely abstract setting. Pregeometries generalize the well-behavedness of concrete operators of wholly different sorts. Under the scope of this notion fall some cases of operators of substructure generation, operators of algebraic closure, operators of definable closure, and several others.

Although the generalization step between this and the previous chapter is quite wide, the definitions and proofs that we present here are very similar to the ones presented in Chapter 5. We could have avoided these similarities by starting to work directly with an arbitrary closure operator, and noticed that the operator of algebraic closure is such an operator. We preferred to maintain this organization of the matter as an expository device in order to show the generalization path that brought us from the analysis of the concrete notion of linear dependence to the very abstract setting in which the subject is developed here.

In the present chapter we will use for the first time the conditional atoms, giving pregeometric semantics for them and proving soundness results. As it can be noticed, these atoms have been neglected so far. The reason for this neglect is that in light of the already mentioned results of [32], we should probably refer to recursively enumerable axiomatizations of these atoms and this is outside of the scope of the present work.

Nonetheless, the case that we treat here subsumes all the previous ones and so, via instantiation, we receive formulations of the semantics for conditional atoms for all the contexts that we dealt with in the previous chapters.

The systems that we are going to study are: Pregeometry Atomic Dependence Logic, Pregeometry Atomic Absolute Independence Logic, Pregeometry Independence Logic, Pregeometry Atomic Absolute Conditional Independence Logic and Pregeometry Atomic Conditional Independence Logic.

## 6.1 Pregeometries

In this section we define the notions of pregeometry and pregeometry with the dependence property.

**Definition 6.1.1.** Let $\mathcal{M}$ be a structure in the signature $\mathcal{L}$ and $\mathrm{cl} : \mathcal{P}(M) \to \mathcal{P}(M)$ an operator on the power set of $M$. We say that $(\mathcal{M}, \mathrm{cl})$ is a *pregeometry* in the signature $\mathcal{L}$ if for every $A, B \subseteq M$ and $a, b \in M$ the following conditions are satisfied:

i) $A \subseteq \mathrm{cl}(A)$;

ii) If $A \subseteq B$ then $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$;

iii) $\mathrm{cl}(A) = \mathrm{cl}(\mathrm{cl}(A))$;

iv) If $a \in \mathrm{cl}(A \cup \{b\}) \setminus \mathrm{cl}(A)$, then $b \in \mathrm{cl}(A \cup \{a\})$;

v) If $a \in \mathrm{cl}(A)$, then $a \in \mathrm{cl}(A_0)$ for some $A_0 \subseteq_{\mathrm{fin}} A$.

When it is clear to which signature we refer we do not specify the signature explicitly.

Condition iv) of the above definition is called the Exchange Principle. This is clearly a generalization of the Exchange Principle for algebraically closed fields, and the one for the operator of algebraic closure that we met in Chapters 4 and 5 respectively.

**Proposition 6.1.2.** Let $(\mathcal{M}, \mathrm{cl})$ be a pregeometry and $A, B \subseteq X$. If $A \subseteq \mathrm{cl}(B)$ then $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$.

*Proof.* Suppose that $A \subseteq \mathrm{cl}(B)$, then by property ii) and iii) of the definition $\mathrm{cl}(A) \subseteq \mathrm{cl}(\mathrm{cl}(B)) = \mathrm{cl}(B)$.

□

**Definition 6.1.3.** Let $(\mathcal{M}, \mathrm{cl})$ be a pregeometry. We say that $(\mathcal{M}, \mathrm{cl})$ has the dependence property if $\mathrm{cl}(\emptyset) \neq \emptyset$ and for every set $A \subseteq M$ we have that $|\mathrm{cl}(A)| \leq \aleph_0 + |A| + |\mathcal{L}|$.

From what we showed in Section 5.6, it follows that any model of the formal theory of algebraically closed fields is a pregeometry with the dependence property with respect to the algebraic closure operator. In Section 6.6 we will see that the same holds for the formal theory of vector spaces over a fixed field $\mathbb{K}$ with respect to the span pregeometry.

## 6.2 Pregeometry Atomic Dependence Logic

In this section we define the system Pregeometry Atomic Dependence Logic (PGADL) and then prove its soundness and completeness. The syntax and deductive apparatus of this system are the same as those of ADL.

Let $T$ be a first-order theory, we denote by $\mathrm{Mod}(T)$ the class of its models, by $U(\mathrm{Mod}(T))$ the class of its domains and by $\mathbf{V}$ the set theoretical universe. A second order $n$-ary operator op on models of $T$ is a class function $\mathrm{op} : U(\mathrm{Mod}(T)) \to \mathbf{V}$ such that

$$\mathrm{op}(M) : \mathcal{P}^n(M) \to \mathcal{P}^n(M).$$

Given such an operator, for every $M \models T$ we can consider the second order structure $(\mathcal{M}, \mathrm{op}(M))$. For ease of notation we denote the structure $(\mathcal{M}, \mathrm{op}(M))$ simply as $(\mathcal{M}, \mathrm{op})$.

### 6.2.1 Semantics

Let $T$ be a first-order theory with infinite models and cl a unary second order operator on models of $T$ such that for every $\mathcal{M} \models T$ we have that $(\mathcal{M}, \mathrm{cl})$ is a pregeometry with the dependence property.

**Definition 6.2.1.** Let $\mathcal{M} \models T$ and $s : \mathrm{dom}(s) \to M$ with $\overline{x}\,\overline{y} \subseteq \mathrm{dom}(s) \subseteq \mathrm{Var}$. We say that $\mathcal{M}$ satisfies $=(\overline{x}, \overline{y})$ under $s$, in symbols $\mathcal{M} \models_s =(\overline{x}, \overline{y})$, if for every $y \in \overline{y}$ we have that $s(y) \in \mathrm{cl}(\{s(x) \mid x \in \overline{x}\})$.

**Definition 6.2.2.** Let $\Sigma$ be a set of atoms and let $s$ be such that the set of variables occurring in $\Sigma$ is included in $\mathrm{dom}(s)$. We say that $\mathcal{M}$ satisfies $\Sigma$ under $s$, in symbols $\mathcal{M} \models_s \Sigma$, if $\mathcal{M}$ satisfies every atom in $\Sigma$ under $s$.

**Definition 6.2.3.** Let $\Sigma$ be a set of atoms. We say that $=(\overline{x}, \overline{y})$ is a logical consequence of $\Sigma$, in symbols $\Sigma \models =(\overline{x}, \overline{y})$, if for every $\mathcal{M} \models T$ and $s$ such that the set of variables occurring in $\Sigma \cup \{=(\overline{x}, \overline{y})\}$ is included in $\mathrm{dom}(s)$ we have that

$$\text{if } \mathcal{M} \models_s \Sigma \text{ then } \mathcal{M} \models_s =(\overline{x}, \overline{y}).$$

### 6.2.2 Soundness and Completeness

**Theorem 6.2.4.** Let $\Sigma$ be a set of atoms, then

$$\Sigma \models =(\overline{x}, \overline{y}) \text{ if and only if } \Sigma \vdash =(\overline{x}, \overline{y}).$$

[The deductive system to which we refer has been defined in Section 2.2.3.]

*Proof.* ($\Leftarrow$) We only prove the soundness of rules ($a_1$.), ($b_1$.) and ($c_1$.). Let $\mathcal{M} \models T$ and $s$ an appropriate assignment.

($a_1$.) By Definition 6.1.1 i) we have that $\{s(x) \mid x \in \overline{x}\} \subseteq \mathrm{cl}(\{s(x) \mid x \in \overline{x}\})$ so clearly $\mathcal{M} \models_s =(\overline{x}, \overline{x})$.

($b_1$.) Suppose that $\mathcal{M} \models_s =(\overline{x}, \overline{y})$ and let $\overline{u} \subseteq \overline{x} \cup \overline{y}$ and $\overline{x} \subseteq \overline{z}$. Let now $u \in \overline{u}$, then by the fact that $\mathcal{M} \models_s =(\overline{x}, \overline{y})$ we have that $s(u) \in \mathrm{cl}(\{s(x) \mid x \in \overline{x}\})$ and so by Definition 6.1.1 ii) we have $s(u) \in \mathrm{cl}(\{s(z) \mid z \in \overline{z}\})$.

($c_1$.) Suppose that $\mathcal{M} \models_s =(\overline{x}, \overline{y})$ and $\mathcal{M} \models_s =(\overline{y}, \overline{z})$. Let $z \in \overline{z}$, then $s(z) \in \mathrm{cl}(\{s(y) \mid y \in \overline{y}\})$ because $\mathcal{M} \models_s =(\overline{y}, \overline{z})$. Furthermore we have that

$\{s(y) \mid y \in \overline{y}\} \subseteq \mathrm{cl}(\{s(x) \mid x \in \overline{x}\})$ because $\mathcal{M} \models_s =(\overline{x}, \overline{y})$. Thus, by Proposition 6.1.2, we have that $\mathrm{cl}(\{s(y) \mid y \in \overline{y}\}) \subseteq \mathrm{cl}(\{s(x) \mid x \in \overline{x}\})$ and hence that $s(z) \in \mathrm{cl}(\{s(x) \mid x \in \overline{x}\})$.

($\Rightarrow$) Suppose $\Sigma \nvdash =(\overline{x}, \overline{y})$. Let $V = \{z \in \mathrm{Var} \mid \Sigma \vdash =(\overline{x}, z)\}$ and $W = \mathrm{Var} \backslash V$. First notice that $\overline{y} \neq \emptyset$, indeed if not so, by the syntactic constraints that we put on the system, we have that $\overline{x}, \overline{y} = \emptyset$ and so by the admitted degenerate case of rule ($a_1$.) we have that $\Sigma \vdash =(\overline{x}, \overline{y})$. Furthermore $\overline{y} \cap W \neq \emptyset$, indeed if $\overline{y} \cap W = \emptyset$ then for every $y \in \overline{y}$ we have that $\Sigma \vdash =(\overline{x}, y)$ and so by rules ($d_1$.), ($e_1$.) and, if necessary, ($f_1$.)[1] we have that $\Sigma \vdash =(\overline{x}, \overline{y})$.

Let $\kappa > \aleph_0 + |\mathcal{L}|$ where $\mathcal{L}$ is the signature of $T$. The theory $T$ has infinite models so by the Löwenheim-Skolem theorem there is a structure $\mathcal{M}$ such that $\mathcal{M} \models T$ and $|M| = \kappa$. Notice now that if $|M| = \kappa$ then for every $m \in M$ we have that $\mathrm{cl}(\{m\}) \neq M$ because one of the conditions of Definition 6.1.3 ensures that $|\mathrm{cl}(\{m\})| \leq |\mathcal{L}| + \aleph_0$.

Let then $a \in \mathrm{cl}(\emptyset)$, $b \in M \setminus \mathrm{cl}(\{a\})$ and let $s$ be the following assignment:

$$s(v) = \begin{cases} a & \text{if } v \in V \\ b & \text{if } v \in W. \end{cases}$$

We claim that $\mathcal{M} \not\models_s =(\overline{x}, \overline{y})$. In accordance to the semantic we then have to show that there is $y \in \overline{y}$ such that $s(y) \notin \mathrm{cl}(\{s(x) \mid x \in \overline{x}\})$. Let $y \in \overline{y} \cap W$, then

$$s(y) = b \notin \mathrm{cl}(\{a\}) = \mathrm{cl}(\{s(x) \mid x \in \overline{x}\})$$

because for $x \in \overline{x}$ we have that $\Sigma \vdash =(\overline{x}, x)$. Indeed by rule ($a_1$.) $\vdash =(\overline{x}, \overline{x})$ and so by rule ($b_1$.) $\vdash =(\overline{x}, x)$. Notice that in the case $\overline{x} = \emptyset$, we have that

$$s(y) = b \notin \mathrm{cl}(\{a\}) \supseteq \mathrm{cl}(\emptyset) = \mathrm{cl}(\{s(x) \mid x \in \overline{x}\}).$$

Let now $=(\overline{x}', \overline{y}') \in \Sigma$, we want to show that $\mathcal{M} \models_s =(\overline{x}', \overline{y}')$. If $\overline{y}' = \emptyset$ then also $\overline{x}' = \emptyset$ and so trivially $\mathcal{M} \models_s =(\overline{x}', \overline{y}')$. Noticed this, for the rest of the proof we assume $\overline{y}' \neq \emptyset$.

**Case 1.** $\overline{x}' = \emptyset$.

Suppose that $\mathcal{M} \not\models_s =(\emptyset, \overline{y}')$, then there exists $y' \in \overline{y}'$ such that $s(y') = b$, so $\Sigma \nvdash =(\overline{x}, y')$. Notice though that $\Sigma \vdash =(\emptyset, \overline{y}')$, so by rule ($b_1$.) $\Sigma \vdash =(\emptyset, y')$ and hence again by rule ($b_1$.) $\Sigma \vdash =(\overline{x}, y')$.

**Case 2.** $\overline{x}' \neq \emptyset$ and $\overline{x}' \subseteq V$.

If this is the case, then

$$\begin{aligned} \forall x' \in \overline{x}' \;\; \Sigma \vdash =(\overline{x}, x') &\implies \Sigma \vdash =(\overline{x}, \overline{x}') \;\; \text{[by rules (d}_1\text{.), (e}_1\text{.) and (f}_1\text{.)]} \\ &\implies \Sigma \vdash =(\overline{x}, \overline{y}') \;\; \text{[by rule (c}_1\text{.)]} \\ &\implies \forall y' \in \overline{y}' \;\; \Sigma \vdash =(\overline{x}', y') \;\; \text{[by rule (b}_1\text{.)]} \\ &\implies \overline{y}' \subseteq V. \end{aligned}$$

If $\overline{x}' \subseteq V$ then for every $x' \in \overline{x}'$ we have that $s(x') = a$ so

$$\mathrm{cl}(\{s(x') \mid x' \in \overline{x}'\}) = \mathrm{cl}(\{a\}).$$

Let $y' \in \overline{y}'$, then we have that $s(y') = a$ and clearly $a \in \mathrm{cl}(\{a\})$. Hence $\mathcal{M} \models_s =(\overline{x}', \overline{y}')$

[1]Notice that ($f_1$.) is necessary only if $\overline{x}$ or $\overline{y}$ contain repetitions.

**Case 3.** $\overline{x}' \cap W \neq \emptyset$.

If this is the case, then there exists $w \in \overline{x}'$ such that $\Sigma \nvDash =(\overline{x}, w)$, so we have $w \in \overline{x}'$ such that $s(w) = b$ and hence $\mathrm{cl}(\{s(x') \mid x' \in \overline{x}'\}) \supseteq \mathrm{cl}(\{b\})$. Let now $y' \in \overline{y}'$.

**Subcase 1**. $y' \in W$.

In this case we have that $s(y') = b$. Clearly

$$b \in \mathrm{cl}(\{b\}) \subseteq \mathrm{cl}(\{s(x') \mid x' \in \overline{x}'\}).$$

Hence $\mathcal{M} \models_s =(\overline{x}', \overline{y}')$.

**Subcase 2**. $y' \in V$.

In this case we have that $s(y') = a$. By choice of $a$

$$a \in \mathrm{cl}(\{b\}) \subseteq \mathrm{cl}(\{s(x') \mid x' \in \overline{x}'\}).$$

Hence $\mathcal{M} \models_s =(\overline{x}', \overline{y}')$.

□

## 6.3 Indepedence in Pregeometries

In this section we see how all the definitions and results of Section 5.3 can be generalized to an arbitrary pregeometry. In particular we will see how also in this more general context we can define a notion of dimension and with that define a ternary independence relation that satisfies all the nice properties that we saw in that section. This allow us, in Section 6.5 and 6.8, to give a pregeometric interpretation of the independence and conditional independence atoms $\overline{x} \perp \overline{y}$ and $\overline{x} \perp_{\overline{z}} \overline{y}$. The notion of a pregeometry with the independence property, which we define at the end of the section, plays a role that is the analog of the one played by the theories with the ACL-independence property in the more restrictive context of Chapter 5.

**Definition 6.3.1.** Let $(\mathcal{M}, \mathrm{cl})$ be a pregeometry and $A \subseteq M$. We say that $A$ is independent if for all $a \in A$ we have $a \notin \mathrm{cl}(A \setminus \{a\})$.

**Definition 6.3.2.** Let $(\mathcal{M}, \mathrm{cl})$ be a pregeometry and $B \subseteq A \subseteq M$. We say that $B$ is a *basis* for $A$ if $B$ is independent and $A \subseteq \mathrm{cl}(B)$.

**Proposition 6.3.3.** Let $(\mathcal{M}, \mathrm{cl})$ be a pregeometry and $B \subseteq A \subseteq M$. The following are equivalent:

i) $B$ is a maximally independent subset of $A$;

ii) $B$ is a basis for $A$;

iii) $B$ is a minimal subset of $A$ such that $A \subseteq \mathrm{cl}(B)$.

*Proof.* i) $\Rightarrow$ ii). Suppose that there exists $a \in A$ such that $a \notin \mathrm{cl}(B)$. We claim that $B \cup \{a\}$ is independent. Suppose not, then there exists $b \in B \cup \{a\}$ such that $b \in \mathrm{cl}((B \cup \{a\}) \setminus \{b\})$. By hypothesis $a \notin \mathrm{cl}(B)$ so $b \neq a$ and hence $b \in \mathrm{cl}((B \setminus \{b\}) \cup \{a\})$. Notice now that $B$ is independent so $b \notin \mathrm{cl}(B \setminus \{b\})$. Hence by the Exchange Principle we have that $a \in \mathrm{cl}(B)$, a contradiction.

ii) $\Rightarrow$ iii). Suppose there exists $B' \subsetneq B$ such that $A \subseteq \text{cl}(B')$. Let $b \in B \setminus B'$, then $b \in \text{cl}(B')$ because $A \subseteq \text{cl}(B')$ and so $b \in \text{cl}(B \setminus \{b\})$ since $B' \subseteq B \setminus \{b\}$. Hence $B$ is not independent.

iii) $\Rightarrow$ i). Suppose there exists $b \in B$ such that $b \in \text{cl}(B \setminus \{b\})$, then $A \subseteq \text{cl}(B) \subseteq \text{cl}(B \setminus \{b\})$ so $B$ is not minimal. Thus $B$ is independent. Let now $a \in A \setminus B$ and suppose that $B \cup \{a\}$ is independent, then $a \notin \text{cl}(B)$ and so $A \not\subseteq \text{cl}(B)$.

□

**Proposition 6.3.4.** Let $(\mathcal{M}, \text{cl})$ be a pregeometry, $A_1 \subseteq M$ and $A_0 \subseteq A_1$ independent. Then $A_0$ can be extended to a maximally independent subset of $A_1$.

*Proof.* See [10, Proposition 1.7]. Notice that the proof of this theorem requires Zorn's Lemma.

□

**Lemma 6.3.5.** Let $(\mathcal{M}, \text{cl})$ be a pregeometry and $A, B, C \subseteq M$ with $A \subseteq C$ and $B \subseteq C$. If $A$ and $B$ are bases for $C$, then $|A| = |B|$.

*Proof.* See [29, Lemma 8.1.3].

□

**Definition 6.3.6.** Let $(\mathcal{M}, \text{cl})$ be a pregeometry and $A \subseteq M$. The *dimension* of $A$ is the cardinality of a basis for $A$. We let $\dim(A)$ denote the dimension of $A$.

**Lemma 6.3.7.** Let $(\mathcal{M}, \text{cl})$ be a pregeometry and $A \subseteq M$. If $A$ be independent and $B$ is a basis for $A$, then $A = B$. In particular if $A$ is independent, then $\dim(A) = |A|$.

*Proof.* Suppose that there is $B \subsetneq A$ such that $B$ is a basis for $A$. Let $b \in A \setminus B$, then $b \in \text{cl}(B)$ because $A \subseteq \text{cl}(B)$. So $b \in \text{cl}(A \setminus \{b\})$, since $B \subseteq A \setminus \{b\}$. Thus $A$ is not independent.

□

If $(\mathcal{M}, \text{cl})$ is a pregeometry and $A, C \subseteq M$, we also consider the *localization* $\text{cl}_C(A) = \text{cl}(C \cup A)$.

**Lemma 6.3.8.** If $(\mathcal{M}, \text{cl})$ is a pregeometry and $C \subseteq M$, then $(\mathcal{M}, \text{cl}_C)$ is a pregeometry. □

**Definition 6.3.9.** Let $(\mathcal{M}, \text{cl})$ be a pregeometry and $A, C \subseteq M$. We say that $A$ is independent over $C$ if $A$ is independent in $(\mathcal{M}, \text{cl}_C)$ and that $B \subseteq A$ is basis for $A$ over $C$ if $B$ is a basis for $A$ in $(\mathcal{M}, \text{cl}_C)$. We let $\dim(A/C)$ be the dimension of $A$ in $(\mathcal{M}, \text{cl}_C)$ and call $\dim(A/C)$ the dimension of $A$ over $C$.

**Corollary 6.3.10.** Let $(\mathcal{M}, \text{cl})$ be a pregeometry and $C \subseteq M$. For every $A \subseteq M$, there exists $B \subseteq A$ such that $B$ is a basis for $A$ over $C$.

*Proof.* Immediate from Proposition 6.3.4.

□

The following lemma is not of particular interest but it will be relevant in the proof of Theorem 6.5.4, this is the reason for which we state it here.

**Lemma 6.3.11.** Let $(\mathcal{M}, \mathrm{cl})$ be a pregeometry and $A \subseteq M$ be an independent set. Let $D_0, D_1 \subseteq A$ and $D_0 \cap D_1 = \emptyset$, then

i) $D_0$ is independent over $D_1$;

ii) $\dim(D_0/D_1) = \dim(D_0)$.

*Proof.* i) Suppose that $D_0$ is not independent over $D_1$, then there exists $d \in D_0$ such that $d \in \mathrm{cl}_{D_1}(D_0 \setminus \{d\}) = \mathrm{cl}(D_1 \cup (D_0 \setminus \{d\}))$. By hypothesis $D_0 \cap D_1 = \emptyset$, so $d \in \mathrm{cl}((D_1 \cup D_0) \setminus \{d\})$. Thus $D_0 \cup D_1$ is dependent, a contradiction.

ii) By i) $D_0$ is independent over $D_1$ and thus it is independent in both the pregeometries $(\mathcal{M}, \mathrm{cl})$ and $(\mathcal{M}, \mathrm{cl}_{D_1})$, hence by Lemma 6.3.7 we have that $\dim(D_0/D_1) = |D_0| = \dim(D_0)$.

□

**Proposition 6.3.12.** Let $(\mathcal{M}, \mathrm{cl})$ be a pregeometry and $A, C, D \subseteq M$. If $C \subseteq D$, then $\dim(A/D) \leq \dim(A/C)$.

*Proof.* Let $B$ be a basis for $A$ over $C$, then $B$ is independent over $C$ and $A \subseteq \mathrm{cl}(C \cup B)$. Let $B' \subseteq B$ be such that $B'$ is independent over $D$ and $B \subseteq \mathrm{cl}(D \cup B')$. By choice $B'$ is independent over $D$, furthermore $A \subseteq \mathrm{cl}(C \cup B) \subseteq \mathrm{cl}(D \cup B')$. Hence $B'$ is a basis for $A$ over $D$.

□

The notion of dimension that we have been dealing with allow us to define a notion of independence with many desirable properties.

**Definition 6.3.13.** Let $(\mathcal{M}, \mathrm{cl})$ be a pregeometry, $B, C \subseteq M$ and $\overline{a} \in M^{<\omega}$. We say that $\overline{a}$ is independent from $C$ over $B$ if $\dim(\overline{a}/B \cup C) = \dim(\overline{a}/B)$. We write $\overline{a} \mathop{\smile\!\!\!\!\!|}^{\mathrm{cl}}_{B} C$.

**Proposition 6.3.14.** Let $(\mathcal{M}, \mathrm{cl})$ be a pregeometry. The following are equivalent:

i) $\overline{a} \mathop{\smile\!\!\!\!\!|}^{\mathrm{cl}}_{B} C$;

ii) every basis for $\overline{a}$ over $B$ is a basis for $\overline{a}$ over $B \cup C$;

iii) every maximally independent over $B$ subset of $\overline{a}$ is independent over $B \cup C$;

iv) if $\overline{a}' \subseteq \overline{a}$ is independent over $B$, then $\overline{a}'$ is independent over $B \cup C$.

*Proof.* i) $\Rightarrow$ ii) Let $\overline{b}$ be a basis for $\overline{a}$ over $B$, then $\overline{b}$ is independent over $B$ and $\overline{a} \subseteq \mathrm{cl}(B \cup \overline{b})$. Let $\overline{b}'$ be such that $\overline{b}'$ is independent over $B \cup C$ and $\overline{b} \subseteq \mathrm{cl}((B \cup C) \cup \overline{b}')$. By choice $\overline{b}'$ is independent over $B \cup C$, furthermore $\overline{a} \subseteq \mathrm{cl}(B \cup \overline{b}) \subseteq \mathrm{cl}((B \cup C) \cup \overline{b}')$, so $\overline{b}'$ is a basis for $\overline{a}$ over $B \cup C$. Now if $\overline{b}' \subsetneq \overline{b}$, then $\dim(\overline{a}/B \cup C) < \dim(\overline{a}/B)$. Thus $\overline{b}' = \overline{b}$.

ii) $\Rightarrow$ iii) Immediate from Proposition 6.3.3.

iii) $\Rightarrow$ iv) Suppose that there exists $\overline{a}' \subseteq \overline{a}$ independent over $B$ but not over $B \cup C$. By Proposition 6.3.4, $\overline{a}'$ can be extended to a $\overline{b} \subseteq \overline{a}$ maximally independent over $B$, so there exists a maximally independent over $B$ subset of $\overline{a}$ that is dependent over $B \cup C$.

iv) $\Rightarrow$ i) Let $\overline{b}$ be a basis for $\overline{a}$ over $B$, then $\overline{b}$ is independent over $B$ and so by the hypothesis it is independent over $B \cup C$. Furthermore $\overline{a} \subseteq \mathrm{cl}(B \cup \overline{b}) \subseteq \mathrm{cl}((B \cup C) \cup \overline{b})$. Thus $\overline{b}$ is a basis for $\overline{a}$ over $B \cup C$ and hence $\dim(\overline{a}/B) = |\overline{b}| = \dim(\overline{a}/B \cup C)$.

□

**Lemma 6.3.15** (Monotonicity)**.** If $\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_A B$ and $C \subseteq B$, then $\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_A C$.

*Proof.* By Proposition 6.3.12, $\dim(\overline{a}/A \cup B) \leq \dim(\overline{a}/A \cup C) \leq \dim(\overline{a}/A)$. Thus if $\dim(\overline{a}/A \cup B) = \dim(\overline{a}/A)$, then $\dim(\overline{a}/A \cup C) = \dim(\overline{a}/A)$.

□

**Lemma 6.3.16** (Transitivity)**.** $\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_A \overline{b}\,\overline{c}$ if and only if $\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_A \overline{b}$ and $\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_{A \cup \overline{b}} \overline{c}$.

*Proof.* By Proposition 6.3.12, $\dim(\overline{a}/A \cup \overline{b}\,\overline{c}) \leq \dim(\overline{a}/A \cup \overline{b}) \leq \dim(\overline{a}/A)$. Thus $\dim(\overline{a}/A \cup \overline{b}\,\overline{c}) = \dim(\overline{a}/A)$ if and only if $\dim(\overline{a}/A \cup \overline{b}) = \dim(\overline{a}/A)$ and $\dim(\overline{a}/(A \cup \overline{b}) \cup \overline{c}) = \dim(\overline{a}/A \cup \overline{b})$.

□

**Lemma 6.3.17** (Finite Basis)**.** $\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_A B$ if and only if $\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_A B_0$ for all finite $B_0 \subseteq B$.

*Proof.* ($\Rightarrow$) Follows from Monotonicity.

($\Leftarrow$) Suppose that $\overline{a} \not\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_A B$, then there exists $\overline{a}' \subseteq \overline{a}$ such that $\overline{a}'$ is independent over $A$ but not over $A \cup B$. Thus there exists $a' \in \overline{a}'$ such that $a' \in \mathrm{cl}((A \cup B) \cup (\overline{a}' \setminus \{a'\}))$. By Property v) of Definition 6.1.1 there exists $A_0 \subseteq_{\mathrm{fin}} A$ and $B_0 \subseteq_{\mathrm{fin}} B$ such that $a' \in \mathrm{cl}(A_0 \cup B_0) \cup (\overline{a}' \setminus \{a\}))$, thus $\overline{a}'$ is independent over $A$ but not over $A \cup B_0$. Hence $\overline{a} \not\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_A B_0$.

□

**Lemma 6.3.18.** For any $\overline{a}$, $\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_A \mathrm{cl}(A)$.

*Proof.* Let $\overline{a}' \subseteq \overline{a}$ be independent over $A$, then for every $a' \in \overline{a}'$ we have $a' \notin \mathrm{cl}(A \cup (\overline{a}' \setminus \{a'\}))$. By Lemma 6.1.2 we have that $\mathrm{cl}(A \cup (\overline{a}' \setminus \{a'\})) = \mathrm{cl}(A \cup \mathrm{cl}(A) \cup (\overline{a}' \setminus \{a'\}))$, thus $\overline{a}'$ is also independent over $A \cup \mathrm{cl}(A)$.

□

**Lemma 6.3.19** (Symmetry)**.** If $\overline{c} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_A \overline{b}$, then $\overline{b} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_A \overline{c}$.

*Proof.* Let $\overline{b}'$ be a basis for $\overline{b}$ over $A$ and $\overline{c}' = \{c_0, ..., c_{m-1}\}$ be a basis for $\overline{c}$ over $A$. Notice that if $\overline{b}' = \emptyset$ or $\overline{c}' = \emptyset$, then $\overline{b} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_A \overline{c}$. In the first case we have that $\dim(\overline{b}/A) = 0 = \dim(\overline{b}/A \cup \overline{c})$. In the second we have that $\dim(\overline{c}/A) = 0$, which implies that $\overline{c} \subseteq \mathrm{cl}(A)$, thus by Lemma 6.3.15 we can conclude that $\overline{b} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_A \overline{c}$ because by Lemma 6.3.18 we have that $\overline{b} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_A \mathrm{cl}(A)$.

Suppose that $\overline{b} \not\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_A \overline{c}$, then there exists $b \in \overline{b}'$ such that $b \in \mathrm{cl}((A \cup \overline{c}) \cup (\overline{b}' \setminus \{b\}))$. By hypothesis $\overline{c} \subseteq \mathrm{cl}(A \cup \overline{c}')$ so we can conclude that $b \in \mathrm{cl}((A \cup \overline{c}') \cup (\overline{b}' \setminus \{b\}))$.

Now $\overline{b}'$ is independent over $A$ so there exists $p \in \{0, ..., m-1\}$ such that $b \notin \mathrm{cl}((A \cup \{c_0, ..., c_{p-1}\}) \cup (\overline{b}' \setminus \{b\}))$ but $b \in \mathrm{cl}((A \cup \{c_0, ..., c_p\}) \cup (\overline{b}' \setminus \{b\}))$.

But then by the Exchange Principle we have that

$$c_p \in \mathrm{cl}((A \cup \overline{b}') \cup \{c_0, ..., c_{p-1}\}) = \mathrm{cl}((A \cup \overline{b}) \cup \{c_0, ..., c_{p-1}\}).$$

Thus $\overline{c}'$ is not independent over $A \cup \overline{b}$ and hence $\overline{c} \not\perp\!\!\!\perp^{\mathrm{cl}}_A \overline{b}$.

□

**Corollary 6.3.20.** $\overline{a}\,\overline{b} \perp\!\!\!\perp^{\mathrm{cl}}_A \overline{c}$ if and only if $\overline{a} \perp\!\!\!\perp^{\mathrm{cl}}_A \overline{c}$ and $\overline{b} \perp\!\!\!\perp^{\mathrm{cl}}_{A\cup\overline{a}} \overline{c}$.

*Proof.*

$$\begin{array}{rcll} \overline{a}\,\overline{b} \perp\!\!\!\perp^{\mathrm{cl}}_A \overline{c} & \Leftrightarrow & \overline{c} \perp\!\!\!\perp^{\mathrm{cl}}_A \overline{a}\,\overline{b} & \text{[by Symmetry]} \\ & \Leftrightarrow & \overline{c} \perp\!\!\!\perp^{\mathrm{cl}}_A \overline{a} \text{ and } \overline{c} \perp\!\!\!\perp^{\mathrm{cl}}_{A\cup\overline{a}} \overline{b} & \text{[by Transitivity]} \\ & \Leftrightarrow & \overline{a} \perp\!\!\!\perp^{\mathrm{cl}}_A \overline{c} \text{ and } \overline{b} \perp\!\!\!\perp^{\mathrm{cl}}_{A\cup\overline{a}} \overline{c} & \text{[by Symmetry].} \end{array}$$

□

**Corollary 6.3.21.** If $\overline{a} \perp\!\!\!\perp^{\mathrm{cl}}_A \overline{b}$ and $\overline{a}\,\overline{b} \perp\!\!\!\perp^{\mathrm{cl}}_A \overline{c}$, then $\overline{a} \perp\!\!\!\perp^{\mathrm{cl}}_A \overline{b}\,\overline{c}$.

*Proof.*

$$\begin{array}{cl} \overline{a} \perp\!\!\!\perp^{\mathrm{cl}}_A \overline{b} \text{ and } \overline{a}\,\overline{b} \perp\!\!\!\perp^{\mathrm{cl}}_A \overline{c} & \\ \Downarrow & \\ \overline{a} \perp\!\!\!\perp^{\mathrm{cl}}_A \overline{b} \text{ and } \overline{b}\,\overline{a} \perp\!\!\!\perp^{\mathrm{cl}}_A \overline{c} & \\ \Downarrow & \\ \overline{a} \perp\!\!\!\perp^{\mathrm{cl}}_A \overline{b} \text{ and } \overline{a} \perp\!\!\!\perp^{\mathrm{cl}}_{A\cup\overline{b}} \overline{c} & \text{[by Corollary 6.3.20]} \\ \Downarrow & \\ \overline{a} \perp\!\!\!\perp^{\mathrm{cl}}_A \overline{b}\,\overline{c} & \text{[by Transitivity].} \end{array}$$

□

**Proposition 6.3.22.** If $\overline{a} \perp\!\!\!\perp^{\mathrm{cl}}_B \overline{a}$, then $\overline{a} \perp\!\!\!\perp^{\mathrm{cl}}_B \overline{b}$ for any $\overline{b} \in M$.

*Proof.* If $\dim(\overline{a}/B\cup\overline{a}) = \dim(\overline{a}/B)$, then $\dim(\overline{a}/B) = 0$ because $\dim(\overline{a}/B\cup\overline{a}) = 0$. So $\emptyset$ is basis for $\overline{a}$ over $B$ and hence $\overline{a} \subseteq \mathrm{cl}(B \cup \emptyset) = \mathrm{cl}(B)$. Let now $\overline{b} \in M$, by Lemma 6.3.18 we have that $\overline{b} \perp\!\!\!\perp^{\mathrm{cl}}_B \mathrm{cl}(B)$ and hence by Lemma 6.3.15 and Lemma 6.3.19 we can conclude that $\overline{a} \perp\!\!\!\perp^{\mathrm{cl}}_B \overline{b}$.

□

As announced we conclude this section with the definition of a pregeometry with the independence property.

**Definition 6.3.23.** Let $(\mathcal{M}, \mathrm{cl})$ be a pregeometry in the signature $\mathcal{L}$. We say that $(\mathcal{M}, \mathrm{cl})$ has the independence property if for every $A \subseteq M$ and $D_0 \subseteq_{\mathrm{fin}} M$ independent the following conditions hold:

i) $\mathrm{cl}(\emptyset) \neq \emptyset$;

ii) $\mathrm{cl}(D_0) \neq \bigcup_{D \subsetneq D_0} \mathrm{cl}(D)$;

iii) $|\mathrm{cl}(A)| \leq \aleph_0 + |A| + |\mathcal{L}|$.

From what we showed in Section 5.6, it follows that any model of the formal theory of algebraically closed fields is a pregeometry with the independence property with respect to the algebraic closure operator. In Section 6.6 we will see that the same holds for the formal theory of vector spaces over a fixed field $\mathbb{K}$ with respect to the span pregeometry.

## 6.4 Pregeometry Atomic Absolute Independence Logic

In this section we define the system Pregeometry Atomic Absolute Independence Logic (PGAAIndL) and then prove its soundness and completeness. The syntax and deductive apparatus of this system are the same as those of AAIndL.

### 6.4.1 Semantics

As we saw in the previous section, in any pregeometry we can define an absolute notion of independence. This notion generalizes both the notions of linear independence and algebraic independence. It is then the natural candidate for the atom $\perp(\overline{x})$.

Let $T$ be a first-order theory with infinite models and cl a unary second order operator on models of $T$ such that for every $\mathcal{M} \models T$ we have that $(\mathcal{M}, \mathrm{cl})$ is a pregeometry with the independence property.

**Definition 6.4.1.** Let $\mathcal{M} \models T$ and $s : \mathrm{dom}(s) \to M$ with $\overline{x} \subseteq \mathrm{dom}(s) \subseteq \mathrm{Var}$. We say that $\mathcal{M}$ satisfies $\perp(\overline{x})$ under $s$, in symbols $\mathcal{M} \models_s \perp(\overline{x})$, if for every $x \in \overline{x}$ we have that $s(x) \notin \mathrm{cl}(\{s(z) \mid z \in \overline{x}\} \setminus \{s(x)\})$.

**Definition 6.4.2.** Let $\Sigma$ be a set of atoms and let $s$ be such that the set of variables occurring in $\Sigma$ is included in $\mathrm{dom}(s)$. We say that $\mathcal{M}$ satisfies $\Sigma$ under $s$, in symbols $\mathcal{M} \models_s \Sigma$, if $\mathcal{M}$ satisfies every atom in $\Sigma$ under $s$.

**Definition 6.4.3.** Let $\Sigma$ be a set of atoms. We say that $\perp(\overline{x})$ is a logical consequence of $\Sigma$, in symbols $\Sigma \models \perp(\overline{x})$, if for every $\mathcal{M} \models T$ and $s$ such that the set of variables occurring in $\Sigma \cup \{\perp(\overline{x})\}$ is included in $\mathrm{dom}(s)$ we have that

$$\text{if } \mathcal{M} \models_s \Sigma \text{ then } \mathcal{M} \models_s \perp(\overline{x}).$$

### 6.4.2 Soundness and Completeness

**Theorem 6.4.4.** Let $\Sigma$ be a set of atoms, then

$$\Sigma \models \perp(\overline{x}) \text{ if and only if } \Sigma \vdash \perp(\overline{x}).$$

[The deductive system to which we refer has been defined in Section 2.3.3.]

*Proof.* ($\Leftarrow$) Nothing to prove.

($\Rightarrow$) Suppose $\Sigma \nvdash \perp(\overline{x})$. Notice that if this is the case then $\overline{x} \neq \emptyset$. Indeed if $\overline{x} = \emptyset$ then $\Sigma \vdash \perp(\overline{x})$ because by rule ($a_2$.) $\vdash \perp(\emptyset)$.

We can assume that $\overline{x}$ is injective. This is without loss of generality because clearly $\mathcal{M} \models_s \perp(\overline{x})$ if and only if $\mathcal{M} \models_s \perp(\pi\overline{x})$, where $\pi : \mathrm{Var}^{<\omega} \to \mathrm{Var}^{<\omega}$ is the function that eliminates repetitions in finite sequences of variables. Let then $\overline{x} = (x_{j_0}, ..., x_{j_{n-1}}) \neq \emptyset$ be injective.

Let $\kappa > |\mathcal{L}| + \aleph_0$ where $\mathcal{L}$ is the signature of $T$. The theory $T$ has infinite models so by the Löwenheim-Skolem theorem there is a structure $\mathcal{M}$ such that $\mathcal{M} \models T$ and $|M| = \kappa$. We then have that $\dim(M) = \kappa$ because by property iii) of Definition 6.3.23 for every $A \subseteq M$ such that $|A| < \kappa$ we have that $\mathrm{cl}(A) \leq \aleph_0 + |A| + |\mathcal{L}| < \kappa$.

Let then $\{a_i \mid i \in \kappa\}$ be an injective enumeration of a basis $B$ for $\mathcal{M}$ and $\{w_i \mid i \in \omega\}$ an injective enumeration of $\text{Var} \setminus \{x_{j_0}\}$. Let $s$ be the following assignment:

$$s(w_i) = a_i$$

and

$$s(x_{j_0}) = e \qquad \text{if } \overline{x} = \{x_{j_0}\}$$
$$s(x_{j_0}) = d \qquad \text{if } \overline{x} \neq \{x_{j_0}\},$$

where $e \in \text{cl}(\emptyset)$ and $d$ is such that $d \in \text{cl}(\{s(x_{j_1}), ..., s(x_{j_{n-1}})\})$ but $d \notin \text{cl}(D)$ for every $D \subsetneq \{s(x_{j_1}), ..., s(x_{j_{n-1}})\}$. Notice that $e$ and $d$ do exist because of properties i) and ii) of Definition 6.3.23.

We claim that $\mathcal{M} \not\models_s \perp(\overline{x})$. This is immediate because either

$$s(x_{j_0}) = e \quad \text{or} \quad s(x_{j_0}) = d,$$

and

$$e \in \text{cl}(\emptyset) \subseteq \text{cl}(\{s(x) \mid x \in \overline{x}\} \setminus \{s(x_{j_0})\})$$
$$d \in \text{cl}(\{s(x_{j_1}), ..., s(x_{j_{n-1}})\}) = \text{cl}(\{s(x) \mid x \in \overline{x}\} \setminus \{s(x_{j_0})\}).$$

Let now $\perp(\overline{v}) \in \Sigma$, we want to show that $\mathcal{M} \models_s \perp(\overline{v})$. As before, we assume, without loss of generality, that $\overline{v}$ is injective. Notice that if $\overline{v} = \emptyset$, then $\mathcal{M} \models_s \perp(\overline{v})$. Thus let $\overline{v} = (v_{h_0}, ..., v_{h_{c-1}}) \neq \emptyset$.

**Case 1.** $x_{j_0} \notin \overline{v}$.

Let $w_{r_i} = v_{h_i}$ for every $i \in \{0, ..., c-1\}$, we then have that

$$\{s(v_{h_0}) = s(w_{r_0}) = a_{r_0}, ..., s(v_{h_{c-1}}) = s(w_{r_{c-1}}) = a_{r_{c-1}}\}$$

is independent.

**Case 2.** $x_{j_0} \in \overline{v}$.

**Subcase 1.** $\overline{x} \setminus \overline{v} \neq \emptyset$.

Notice that $\overline{x} \neq \{x_{j_0}\}$ because if not then $\overline{x} \setminus \overline{v} = \{x_{j_0}\}$ and so $x_{j_0} \notin \overline{v}$. Hence $s(x_{j_0}) = d$.

Let $(\overline{v} \setminus \{x_{j_0}\}) \cap \overline{x} = \left\{v_{h'_0}, ..., v_{h'_{b-1}}\right\}$, $\overline{v} \setminus \overline{x} = \left\{v_{h''_0}, ..., v_{h''_{t-1}}\right\}$, $w_{r'_i} = v_{h'_i}$ for every $i \in \{0, ..., b-1\}$ and $w_{r''_i} = v_{h''_i}$ for every $i \in \{0, ..., t-1\}$.

Suppose now that the set $\left\{a_{r'_0}, ..., a_{r'_{b-1}}, d, a_{r''_0}, ..., a_{r''_{t-1}}\right\}$ is dependent. The set $\left\{a_{r'_0}, ..., a_{r'_{b-1}}, a_{r''_0}, ..., a_{r''_{t-1}}\right\}$ is independent, so there are three cases.

Case 1. $a_{r''_l} \in \text{cl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}, d, a_{r''_0}, ..., a_{r''_{l-1}}, a_{r''_{l+1}}, ..., a_{r''_{t-1}}\right\})$.

If this is the case, then

$$a_{r''_l} \in \text{cl}(\left\{s(x_{j_1}), ..., s(x_{j_{n-1}}), a_{r''_0}, ..., a_{r''_{l-1}}, a_{r''_{l+1}}, ..., a_{r''_{t-1}}\right\})$$

because $d \in \text{cl}(\{s(x_{j_1}), ..., s(x_{j_{n-1}})\})$. This is absurd though because the set $\left\{s(x_{j_1}), ..., s(x_{j_{n-1}}), a_{r''_0}, ..., a_{r''_{t-1}}\right\}$ is made of distinct elements of the basis $B$ and so it is independent.

Case 2. $d \in \text{cl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}, a_{r''_0}, ..., a_{r''_{t-1}}\right\})$.

Notice that

$$d \notin \text{cl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}\right\})$$

because $\left\{a_{r'_0}, ..., a_{r'_{b-1}}\right\} \subsetneq \left\{s(x_{j_1}), ..., s(x_{j_{n-1}})\right\}$ and $d$ has been chosen such that $d \in \mathrm{cl}(\left\{s(x_{j_1}), ..., s(x_{j_{n-1}})\right\})$ but $d \notin \mathrm{cl}(D)$ for every $D \subsetneq \left\{s(x_{j_1}), ..., s(x_{j_{n-1}})\right\}$.

Thus there is $l \leq t-1$ such that

$$d \in \mathrm{cl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}, a_{r''_0}, ..., a_{r''_{l-1}}\right\} \cup \left\{a_{r''_l}\right\}) \setminus \mathrm{cl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}, a_{r''_0}, ..., a_{r''_{l-1}}\right\})$$

and then by the Exchange Principle we have that

$$a_{r''_l} \in \mathrm{cl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}, a_{r''_0}, ..., a_{r''_{l-1}}\right\} \cup \{d\}).$$

Thus we have that $a_{r''_l} \in \mathrm{cl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}, d, a_{r''_0}, ..., a_{r''_{l-1}}, a_{r''_{l+1}}, ..., a_{r''_{t-1}}\right\})$, which is impossible as we saw in Case 1.

Case 3. $a_{r'_c} \in \mathrm{cl}(\left\{a_{r'_0}, ..., a_{r'_{c-1}}, a_{r'_{c+1}}, ..., a_{r'_{b-1}}, d, a_{r''_0}, ..., a_{r''_{t-1}}\right\})$.

Notice that

$$a_{r'_c} \notin \mathrm{cl}(\left\{a_{r'_0}, ..., a_{r'_{c-1}}, a_{r'_{c+1}}, ..., a_{r'_{b-1}}, a_{r''_0}, ..., a_{r''_{t-1}}\right\}).$$

Thus by the Exchange Principle we have that $d \in \mathrm{cl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}, a_{r''_0}, ..., a_{r''_{t-1}}\right\})$, which is impossible as we saw in Case 2.

**Subcase 2.** $\overline{x} \subseteq \overline{v}$.

This case is not possible. Suppose indeed it is, then by rule (c$_2$.) we can assume that $\overline{v} = \overline{x}\ \overline{v}'$ with $\overline{v}' \subseteq \mathrm{Var} \setminus \overline{x}$. Thus by rule (b$_2$.) we have that $\Sigma \vdash \perp(\overline{x})$ which is absurd.

□

## 6.5 Pregeometry Atomic Independence Logic

In this section we define the system Pregeometry Atomic Independence Logic (PGAIndL) and then prove its soundness and completeness. The syntax and deductive apparatus of this system are the same as those of AIndL.

### 6.5.1 Semantics

The intuition behind the atom $\overline{x} \perp \overline{y}$ in this context is that if some elements from $\overline{x}$ are independent, then they are also independent over $\overline{y}$. This generalizes the interpretation that we gave of the independence atom in the context of algebraically closed fields.

Let $T$ be a first-order theory with infinite models and cl a unary second order operator on models of $T$ such that for every $\mathcal{M} \models T$ we have that $(\mathcal{M}, \mathrm{cl})$ is a pregeometry with the independence property.

**Definition 6.5.1.** Let $\mathcal{M} \models T$ and $s : \mathrm{dom}(s) \to M$ with $\overline{x}\,\overline{y} \subseteq \mathrm{dom}(s) \subseteq \mathrm{Var}$. We say that $\mathcal{M}$ satisfies $\overline{x} \perp \overline{y}$ under $s$, in symbols $\mathcal{M} \models_s \overline{x} \perp \overline{y}$, if $\dim(s(\overline{x})/s(\overline{y})) = \dim(s(\overline{x}))$.

Notice that, because of Proposition 6.3.14, the condition that we used in the above definition is equivalent to the intuitive condition that we mentioned at the beginning of the section.

**Definition 6.5.2.** Let $\Sigma$ be a set of atoms and let $s$ be such that the set of variables occurring in $\Sigma$ is included in $\mathrm{dom}(s)$. We say that $\mathcal{M}$ satisfies $\Sigma$ under $s$, in symbols $\mathcal{M} \models_s \Sigma$, if $\mathcal{M}$ satisfies every atom in $\Sigma$ under $s$.

**Definition 6.5.3.** Let $\Sigma$ be a set of atoms. We say that $\overline{x} \perp \overline{y}$ is a logical consequence of $\Sigma$, in symbols $\Sigma \models \overline{x} \perp \overline{y}$, if for every $\mathcal{M} \models T$ and $s$ such that the set of variables occurring in $\Sigma \cup \{\overline{x} \perp \overline{y}\}$ is included in $\mathrm{dom}(s)$ we have that

$$\text{if } \mathcal{M} \models_s \Sigma \text{ then } \mathcal{M} \models_s \overline{x} \perp \overline{y}.$$

### 6.5.2 Soundness and Completeness

**Theorem 6.5.4.** Let $\Sigma$ be a set of atoms, then

$$\Sigma \models \overline{x} \perp \overline{y} \text{ if and only if } \Sigma \vdash \overline{x} \perp \overline{y}.$$

[The deductive system to which we refer has been defined in Section 2.4.3.]

*Proof.* ($\Leftarrow$) ($a_3$.) Obvious.

($b_3$.)

$$\begin{aligned} \mathcal{M} \models_s \overline{x} \perp \overline{y} \quad &\Longrightarrow \quad s(\overline{x}) \downarrow^{\mathrm{cl}}_{\emptyset} s(\overline{y}) \\ &\Longrightarrow \quad s(\overline{y}) \downarrow^{\mathrm{cl}}_{\emptyset} s(\overline{x}) \quad \text{[By Lemma 6.3.19]} \\ &\Longrightarrow \quad \mathcal{M} \models_s \overline{y} \perp \overline{x}. \end{aligned}$$

($c_3$.)

$$\begin{aligned} \mathcal{M} \models_s \overline{x} \perp \overline{y}\,\overline{z} \quad &\Longrightarrow \quad s(\overline{x}) \downarrow^{\mathrm{cl}}_{\emptyset} s(\overline{y}\,\overline{z}) \\ &\Longrightarrow \quad s(\overline{x}) \downarrow^{\mathrm{cl}}_{\emptyset} s(\overline{y}) \quad \text{[By Lemma 6.3.15]} \\ &\Longrightarrow \quad \mathcal{M} \models_s \overline{x} \perp \overline{y}. \end{aligned}$$

($d_3$.)

$$\begin{gathered} \mathcal{M} \models_s \overline{x} \perp \overline{y} \text{ and } \mathcal{M} \models_s \overline{x}\,\overline{y} \perp \overline{z} \\ \Downarrow \\ s(\overline{x}) \downarrow^{\mathrm{cl}}_{\emptyset} s(\overline{y}) \text{ and } s(\overline{x})s(\overline{y}) \downarrow^{\mathrm{cl}}_{\emptyset} s(\overline{z}) \\ \Downarrow \\ s(\overline{x}) \downarrow^{\mathrm{cl}}_{\emptyset} s(\overline{y})s(\overline{z}) \qquad \text{[By Corollary 6.3.21]} \\ \Downarrow \\ \mathcal{M} \models_s \overline{x} \perp \overline{y}\,\overline{z}. \end{gathered}$$

($e_3$.) Suppose that $\mathcal{M} \models_s x \perp x$, then $s(x) \downarrow^{\mathrm{cl}}_{\emptyset} s(x)$ and so by Proposition 6.3.22 we have that $s(x) \downarrow^{\mathrm{cl}}_{\emptyset} s(\overline{y})$ for any $\overline{y} \in \mathrm{Var}$.

($f_3$.) Obvious.

($\Rightarrow$) Suppose $\Sigma \nvdash \overline{x} \perp \overline{y}$. Notice that if this is the case then $\overline{x} \neq \emptyset$ and $\overline{y} \neq \emptyset$. Indeed if $\overline{y} = \emptyset$ then $\Sigma \vdash \overline{x} \perp \overline{y}$ because by rule ($a_3$.) $\vdash \overline{x} \perp \emptyset$. Analogously if $\overline{x} = \emptyset$ then $\Sigma \vdash \overline{x} \perp \overline{y}$ because by rule ($a_3$.) $\vdash \overline{y} \perp \emptyset$ and so by rule ($b_3$.) $\vdash \emptyset \perp \overline{y}$.

We can assume that $\overline{x}$ and $\overline{y}$ are injective. This is without loss of generality because clearly $\mathcal{M} \models_s \overline{x} \perp \overline{y}$ if and only if $\mathcal{M} \models_s \pi\overline{x} \perp \pi\overline{y}$, where $\pi : \mathrm{Var}^{<\omega} \to \mathrm{Var}^{<\omega}$ is the function that eliminates repetitions in finite sequences of variables.

Furthermore we can assume that $\overline{x} \perp \overline{y}$ is minimal, in the sense that if $\overline{x}' \subseteq \overline{x}$, $\overline{y}' \subseteq \overline{y}$ and $\overline{x}'\,\overline{y}' \neq \overline{x}\,\overline{y}$, then $\Sigma \vdash \overline{x}' \perp \overline{y}'$. This is for two reasons.

i) If $\overline{x} \perp \overline{y}$ is not minimal we can always find a minimal atom $\overline{x}^* \perp \overline{y}^*$ such that $\Sigma \nvdash \overline{x}^* \perp \overline{y}^*$, $\overline{x}^* \subseteq \overline{x}$ and $\overline{y}^* \subseteq \overline{y}$ −− just keep deleting elements of $\overline{x}$ and $\overline{y}$ until you obtain the desired property or until both $\overline{x}^*$ and $\overline{y}^*$ are singletons, in which case, due to the trivial independence rule (a$_3$.), $\overline{x}^* \perp \overline{y}^*$ is a minimal statement.

ii) For any $\overline{x}' \subseteq \overline{x}$ and $\overline{y}' \subseteq \overline{y}$ we have that if $\mathcal{M} \not\models_s \overline{x}' \perp \overline{y}'$ then $\mathcal{M} \not\models_s \overline{x} \perp \overline{y}$, for every $\mathcal{M}$ and $s$.

Let indeed $\overline{x} = \overline{x}'\overline{x}''$ and $\overline{y} = \overline{y}'\overline{y}''$, then

$$\begin{array}{lcll} \mathcal{M} \models_s \overline{x}'\overline{x}'' \perp \overline{y}'\overline{y}'' & \Longrightarrow & s(\overline{x}')s(\overline{x}'') \mathop{\downarrow}^{\text{cl}}_{\emptyset} s(\overline{y}')s(\overline{y}'') & \\ & \Longrightarrow & s(\overline{x}')s(\overline{x}'') \mathop{\downarrow}^{\text{cl}}_{\emptyset} s(\overline{y}') & \text{[By Lemma 6.3.15]} \\ & \Longrightarrow & s(\overline{x}') \mathop{\downarrow}^{\text{cl}}_{\emptyset} s(\overline{y}') & \text{[By Corollary 6.3.20]} \\ & \Longrightarrow & \mathcal{M} \models_s \overline{x}' \perp \overline{y}'. & \end{array}$$

Let then $\overline{x} = (x_{j_0}, ..., x_{j_{n-1}})$ and $\overline{y} = (y_{k_0}, ..., y_{k_{m-1}})$ be injective and such that $\overline{x} \perp \overline{y}$ is minimal.

Let $V = \{v \in \text{Var} \mid \Sigma \vdash v \perp v\}$ and $W = \text{Var} \setminus V$. We claim that $\overline{x}, \overline{y} \not\subseteq V$. We prove it only for $\overline{x}$, the other case is symmetrical. Suppose that $\overline{x} \subseteq V$, then for every $x \in \overline{x}$ we have that $\Sigma \vdash x \perp x$ so by rule (e$_3$.), (b$_3$.) and (d$_3$.)

$$\Sigma \vdash \overline{y} \perp x_{j_0} \quad \text{and} \quad \Sigma \vdash \overline{y}\, x_{j_0} \perp x_{j_1} \Rightarrow \quad \Sigma \vdash \overline{y} \perp x_{j_0}\, x_{j_1},$$

$$\Sigma \vdash \overline{y} \perp x_{j_0}\, x_{j_1} \quad \text{and} \quad \Sigma \vdash \overline{y}\, x_{j_0}\, x_{j_1} \perp x_{j_2} \Rightarrow \quad \Sigma \vdash \overline{y} \perp x_{j_0}\, x_{j_1}\, x_{j_2},$$

$$\vdots$$

$$\Sigma \vdash \overline{y} \perp x_{j_0} \cdots x_{j_{n-2}} \quad \text{and} \quad \Sigma \vdash \overline{y}\, x_{j_0} \cdots x_{j_{n-2}} \perp x_{j_{n-1}} \Rightarrow \quad \Sigma \vdash \overline{y} \perp \overline{x}.$$

Hence by rule (b$_3$.) $\Sigma \vdash \overline{x} \perp \overline{y}$.

Thus $\overline{x} \cap W \neq \emptyset$ and $\overline{y} \cap W \neq \emptyset$. Without loss of generality suppose that $x_{j_0} \in W$ and $y_{k_0} \in W$. Let $\overline{x} \cap W = \overline{x}' = (x_{j'_0}, ..., x_{j'_{n'-1}}) = (x_{j_0}, ..., x_{j'_{n'-1}}) \neq \emptyset$ and $\overline{y} \cap W = \overline{y}' = (y_{k'_0}, ..., y_{k'_{m'-1}})$. Notice that $\overline{x}' \cap \overline{y}' = \emptyset$. Indeed let $z \in \overline{x}' \cap \overline{y}'$, then by rules (b$_3$.) and (c$_3$.) we have that $\Sigma \vdash z \perp z$. Thus $z \in V$, a contradiction.

Let $\kappa > |\mathcal{L}| + \aleph_0$ where $\mathcal{L}$ is the signature of $T$. The theory $T$ has infinite models so by the Löwenheim-Skolem theorem there is a structure $\mathcal{M}$ such that $\mathcal{M} \models T$ and $|M| = \kappa$. Then we have that $\dim(M) = \kappa$ because by property iii) of Definition 6.3.23 for every $A \subseteq M$ such that $|A| < \kappa$ we have that $\text{cl}(A) \leq \aleph_0 + |A| + |\mathcal{L}| < \kappa$.

Let then $\{a_i \mid i \in \kappa\}$ be an injective enumeration of a basis $B$ for $\mathcal{M}$ and $\{w_i \mid i \in \lambda\}$ be an injective enumeration of $W \setminus \{x_{j_0}\}$. Let $s$ be the following assignment:

i) $s(v) = e$ for every $v \in V$,

ii) $s(w_i) = a_i$ for every $i \in \lambda$,

iii) $s(x_{j_0}) = d$,

where $e \in \text{cl}(\emptyset)$ and $d$ is such that

$$d \in \text{cl}(\left\{s(x_{j'_1}), ..., s(x_{j'_{n'-1}}), s(y_{k'_0}), ..., s(y_{k'_{m'-1}})\right\})$$

but $d \notin \text{cl}(D)$ for every $D \subsetneq \left\{s(x_{j'_1}), ..., s(x_{j'_{n'-1}}), s(y_{k'_0}), ..., s(y_{k'_{m'-1}})\right\}$. Notice that $e$ and $d$ do exist because of properties i) and ii) of Definition 6.3.23.

We claim that $\mathcal{M} \not\models_s \overline{x}' \perp \overline{y}'$, as noticed this implies that $\mathcal{M} \not\models_s \overline{x} \perp \overline{y}$.

First we show that the set $\{s(x') \mid x' \in \overline{x}'\}$ is independent. By construction $s(x_{j_0}) \notin \text{cl}(\{s(x') \mid x' \in \overline{x}'\} \setminus \{s(x_{j_0})\})$. Let then $i \in \{1, ..., n'-1\}$ and suppose that $s(x_{j'_i}) \in \text{cl}(\left\{s(x_{j_0}), ..., s(x_{j'_{i-1}}), s(x_{j'_{i+1}}), ..., s(x_{j'_{n-1}})\right\})$.

The set $\left\{s(x_{j'_1}), ..., s(x_{j'_{i-1}}), s(x_{j'_{i+1}}), ..., s(x_{j'_{n-1}})\right\}$ is independent, so

$$s(x_{j'_i}) \in \text{cl}(\left\{s(x_{j'_1}), ..., s(x_{j'_{i-1}}), s(x_{j'_{i+1}}), ..., s(x_{j'_{n-1}})\right\} \cup \{s(x_{j_0})\})$$

but

$$s(x_{j'_i}) \notin \text{cl}(\left\{s(x_{j'_1}), ..., s(x_{j'_{i-1}}), s(x_{j'_{i+1}}), ..., s(x_{j'_{n-1}})\right\}).$$

Hence by the Exchange Principle

$$s(x_{j_0}) \in \text{cl}(\left\{s(x_{j'_1}), ..., s(x_{j'_{n-1}})\right\}),$$

a contradiction. Thus $\dim(s(\overline{x}')) = |\{s(x') \mid x' \in \overline{x}'\}|$.

We now show that $\left\{s(x_{j'_1}), ..., s(x_{j'_{n-1}})\right\}$ is a basis for $\{s(x') \mid x' \in \overline{x}'\}$ over $\{s(y') \mid y' \in \overline{y}'\}$.

As we noticed above $\overline{x}' \cap \overline{y}' = \emptyset$, so by properties of our assignment $s(\overline{x}') \cap s(\overline{y}') = \emptyset$. Thus, by Lemma 6.3.11, $\left\{s(x_{j'_1}), ..., s(x_{j'_{n-1}})\right\}$ is independent over $\{s(y') \mid y' \in \overline{y}'\}$, also $\left\{s(x_{j_0}), ..., s(x_{j'_{n-1}})\right\} \subseteq \text{cl}(s(\overline{y}') \cup \left\{s(x_{j'_1}), ..., s(x_{j'_{n-1}})\right\})$ because $s(x_{j_0}) \in \text{cl}(\left\{s(x_{j'_1}), ..., s(x_{j'_{n'-1}}), s(y_{k'_0}), ..., s(y_{k'_{m'-1}})\right\})$.

Hence

$$\dim(s(\overline{x}')/s(\overline{y}')) = |\left\{s(x_{j'_1}), ..., s(x_{j'_{n-1}})\right\}| = \dim(s(\overline{x}')) - 1.$$

Let now $\overline{v} \perp \overline{w} \in \Sigma$, we want to show that $\mathcal{M} \models_s \overline{v} \perp \overline{w}$. As before, we assume, without loss of generality, that $\overline{v}$ and $\overline{w}$ are injective. Notice also that if $\overline{v} = \emptyset$ or $\overline{w} = \emptyset$, then $\mathcal{M} \models_s \overline{v} \perp \overline{w}$. Thus let $\overline{v}, \overline{w} \neq \emptyset$.

**Case 1.** $\overline{v} \subseteq V$ or $\overline{w} \subseteq V$.

Suppose that $\overline{v} \subseteq V$, the other case is symmetrical, then $s(\overline{v}) \subseteq \text{cl}(\emptyset)$. Thus $\dim(s(\overline{v})/s(\overline{w})) = 0 = \dim(s(\overline{v}))$.

**Case 2.** $\overline{v} \not\subseteq V$ and $\overline{w} \not\subseteq V$.

Let $\overline{v} \cap W = \overline{v}' \neq \emptyset$ and $\overline{w} \cap W = \overline{w}' \neq \emptyset$.

Notice that

$$s(\overline{v}) \mathop{\downarrow}^{\text{cl}}_{\emptyset} s(\overline{w}) \text{ if and only if } s(\overline{v'}) \mathop{\downarrow}^{\text{cl}}_{\emptyset} s(\overline{w'})$$

Left to right holds in general. As for the other direction, suppose that $s(\overline{v}') \downarrow^{\text{cl}}_{\emptyset} s(\overline{w}')$. If $u \in \overline{v}\,\overline{w} \setminus \overline{v'}\,\overline{w'}$, then $s(u) = e \in \text{cl}(\emptyset)$. Thus

$$
\begin{gathered}
s(\overline{v}') \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{cl}}_{\emptyset} s(\overline{w}') \text{ and } s(\overline{v}')\, s(\overline{w}') \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{cl}}_{\emptyset} \mathrm{cl}(\emptyset) \quad \text{[By Lemma 6.3.18 ]} \\
\Downarrow \\
s(\overline{v}') \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{cl}}_{\emptyset} s(\overline{w}') \cup \mathrm{cl}(\emptyset) \\
\Downarrow \\
s(\overline{v}') \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{cl}}_{\emptyset} s(\overline{w}') \cup (\mathrm{cl}(\emptyset) \cap s(\overline{w})) \\
\Downarrow \\
s(\overline{v}') \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{cl}}_{\emptyset} s(\overline{w}).
\end{gathered}
$$

So

$$
\begin{gathered}
s(\overline{w}) \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{cl}}_{\emptyset} s(\overline{v}') \text{ and } s(\overline{w})\, s(\overline{v}') \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{cl}}_{\emptyset} \mathrm{cl}(\emptyset) \quad \text{[By Lemma 6.3.18 ]} \\
\Downarrow \\
s(\overline{w}) \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{cl}}_{\emptyset} s(\overline{v}') \cup \mathrm{cl}(\emptyset) \\
\Downarrow \\
s(\overline{w}) \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{cl}}_{\emptyset} s(\overline{v}') \cup (\mathrm{cl}(\emptyset) \cap s(\overline{v})) \\
\Downarrow \\
s(\overline{w}) \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{cl}}_{\emptyset} s(\overline{v}).
\end{gathered}
$$

**Subcase 2.1.** $x_{j_0} \notin \overline{v}'\, \overline{w}'$.

Notice that $\overline{v}' \cap \overline{w}' = \emptyset$, so by properties of our assignment $s(\overline{v}') \cap s(\overline{w}') = \emptyset$. Thus by Lemma 6.3.11 it follows that $\dim(s(\overline{v}')/s(\overline{w}')) = \dim(s(\overline{v}'))$.

**Subcase 2.2.** $x_{j_0} \in \overline{v}'\, \overline{w}'$.

**Subsubcase 2.2.1.** $(\overline{x}'\, \overline{y}') \setminus (\overline{v}'\, \overline{w}') \neq \emptyset$.

Let $\overline{v}'\, \overline{w}' \setminus \{x_{j_0}\} \cap \overline{x}'\, \overline{y}' = \left\{u_{h'_0}, ..., u_{h'_{b-1}}\right\}$, $\overline{v}'\, \overline{w}' \setminus \overline{x}'\, \overline{y}' = \left\{u_{h''_0}, ..., u_{h''_{t-1}}\right\}$, $w_{r'_i} = u_{h'_i}$ for every $i \in \{0, ..., b-1\}$ and $w_{r''_i} = u_{h''_i}$ for every $i \in \{0, ..., t-1\}$

Suppose now that the set $\left\{a_{r'_0}, ..., a_{r'_{b-1}}, d, a_{r''_0}, ..., a_{r''_{t-1}}\right\}$ is dependent. There are three cases.

Case 1. $a_{r''_l} \in \mathrm{cl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}, d, a_{r''_0}, ..., a_{r''_{l-1}}, a_{r''_{l+1}}, ..., a_{r''_{t-1}}\right\})$.

If this is the case, then

$$
a_{r''_l} \in \mathrm{cl}(\left\{s(x_{j'_1}), ..., s(x_{j'_{n'-1}}), s(y_{k'_0}), ..., s(y_{k'_{m'-1}}), a_{r''_0}, ..., a_{r''_{l-1}}, a_{r''_{l+1}}, ..., a_{r''_{t-1}}\right\})
$$

because $d \in \mathrm{cl}(\left\{s(x_{j'_1}), ..., s(x_{j'_{n'-1}}), s(y_{k'_0}), ..., s(y_{k'_{m'-1}})\right\})$. This is absurd though because the set $\left\{s(x_{j'_1}), ..., s(x_{j'_{n'-1}}), s(y_{k'_0}), ..., s(y_{k'_{m'-1}}), a_{r''_0}, ..., a_{r''_{t-1}}\right\}$ is made of distinct elements of the basis $B$ and so it is independent.

Case 2. $d \in \mathrm{cl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}, a_{r''_0}, ..., a_{r''_{t-1}}\right\})$.

Notice that

$$
d \notin \mathrm{cl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}\right\})
$$

because $\left\{a_{r'_0}, ..., a_{r'_{b-1}}\right\} \subsetneq \left\{s(x_{j'_1}), ..., s(x_{j'_{n'-1}}), s(y_{k'_0}), ..., s(y_{k'_{m'-1}})\right\}$ and $d$ has been chosen such that $d \in \mathrm{cl}(\left\{s(x_{j'_1}), ..., s(x_{j'_{n'-1}}), s(y_{k'_0}), ..., s(y_{k'_{m'-1}})\right\})$ but $d \notin \mathrm{cl}(D)$ for every $D \subsetneq \left\{s(x_{j'_1}), ..., s(x_{j'_{n'-1}}), s(y_{k'_0}), ..., s(y_{k'_{m'-1}})\right\}$.

Thus there is $l \leq t-1$ such that

$$d \in \mathrm{cl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}, a_{r''_0}, ..., a_{r''_{l-1}}\right\} \cup \left\{a_{r''_l}\right\}) \setminus \mathrm{cl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}, a_{r''_0}, ..., a_{r''_{l-1}}\right\})$$

and then by the Exchange Principle we have that

$$a_{r''_l} \in \mathrm{cl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}, a_{r''_0}, ..., a_{r''_{l-1}}\right\} \cup \{d\}).$$

Thus we have that $a_{r''_l} \in \mathrm{cl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}, d, a_{r''_0}, ..., a_{r''_{l-1}}, a_{r''_{l+1}}, ..., a_{r''_{t-1}}\right\})$, which is impossible as we saw in Case 1.

Case 3. $a_{r'_c} \in \mathrm{cl}(\left\{a_{r'_0}, ..., a_{r'_{c-1}}, a_{r'_{c+1}}, ..., a_{r'_{b-1}}, d, a_{r''_0}, ..., a_{r''_{t-1}}\right\})$.

Notice that

$$a_{r'_c} \notin \mathrm{cl}(\left\{a_{r'_0}, ..., a_{r'_{c-1}}, a_{r'_{c+1}}, ..., a_{r'_{b-1}}, a_{r''_0}, ..., a_{r''_{t-1}}\right\}).$$

Thus by the Exchange Principle we have that $d \in \mathrm{cl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}, a_{r''_0}, ..., a_{r''_{t-1}}\right\})$, which is impossible as we saw in Case 2.

We can then conclude that the set $\left\{a_{r'_0}, ..., a_{r'_{b-1}}, d, a_{r''_0}, ..., a_{r''_{t-1}}\right\}$ is independent. Clearly $\{s(v') \mid v' \in \overline{v}'\} \cup \{s(w') \mid w' \in \overline{w}'\} = \left\{a_{r'_0}, ..., a_{r'_{b-1}}, d, a_{r''_0}, ..., a_{r''_{t-1}}\right\}$. Furthermore, as we noticed above, $s(\overline{v}') \cap s(\overline{w}') = \emptyset$. Hence by Lemma 6.3.11 we have that $\dim(s(\overline{v}')/s(\overline{w}')) = \dim(s(\overline{v}'))$.

**Subsubcase 2.2.2.** $\overline{x}'\, \overline{y}' \subseteq \overline{v}'\, \overline{w}'$.

As shown in Theorem 3.4.4, this case is not possible.

□

## 6.6 Pregeometry in Vector spaces

In this section firstly we define the formal theory of vector spaces over a fixed field $\mathbb{K}$ and see how the span operator determines a pregeometry with the independence property. Then we show that the semantic condition for the independence atom obtained via instantiation of the case analyzed in this chapter to the formal theory of vector spaces over a fixed field $\mathbb{K}$ with respect to the span pregeometry is equivalent to the one given in Chapter 4 and thus conclude that the vector spaces systems that we studied in Chapter 4 are particular cases of the systems treated in this one.

Notice that this last result, if paired with the analogous one in Section 5.6 and the trivial remark that an algebraic closure operator that satisfies the Exchange Principle determines a pregeometry, shows that the pregeometric systems generalize both the vector spaces and the algebraically closed fields ones and hence that this chapter can be considered as a proper generalization of both the first and the second half of Chapter 4.

Let $\mathbb{K}$ be a field and $\mathcal{L} = \{+, 0\} \cup \{k : k \in K\}$, where $+$ is a binary function symbol, 0 is a constant and $k$ is a unary function symbol for each $k \in K$. Let $\mathrm{VS}_{\mathbb{K}}$ be the theory which consists of the axioms for additive commutative groups plus the following axioms:

i) $\forall x \forall y\, r(x+y) = r(x) + r(y)$ for $r \in K$;

ii) $\forall x \ (r+s)(x) = r(x) + s(x) \quad$ for $r, s \in K$;

iii) $\forall x \ r(s(x)) = rs(x) \quad$ for $r, s \in K$;

iv) $\forall x \ 1(x) = x$.

Any vector space $\mathbb{V}$ over $\mathbb{K}$ can be seen as model $\mathcal{V}$ of $\mathrm{VS}_{\mathbb{K}}$ by interpreting $k(a)$ as $ka$ and any model $\mathcal{V}$ of $\mathrm{VS}_{\mathbb{K}}$ can be seen as a vector space $\mathbb{V}$ over $\mathbb{K}$ by defining $ka$ as $k(a)$.

Let $\langle\rangle : U(\mathrm{Mod}(\mathrm{VS}_{\mathbb{K}})) \to \mathbf{V}$ be such that for every $\mathcal{V} \models \mathrm{VS}_{\mathbb{K}}$ we have $\langle\rangle(V)(A) = \langle A \rangle$ where $A \subseteq V$ and, as usual, $\langle A \rangle$ denotes the subspace spanned by $A$ of the vector space $\mathbb{V}$ corresponding to the model $\mathcal{V}$. Then $\langle\rangle$ is a unary second order operator on models of $\mathrm{VS}_{\mathbb{K}}$.

**Theorem 6.6.1.** Let $\mathcal{V} \models \mathrm{VS}_{\mathbb{K}}$, then $(\mathcal{V}, \langle\rangle)$ is a pregeometry with the independence property.

*Proof.* Let $\mathcal{V} \models \mathrm{VS}_{\mathbb{K}}$. Among the conditions defining a pregeometry we only show that the Exchange Principle is satisfied. Suppose that $a \in \langle A \cup \{b\} \rangle \setminus \langle A \rangle$ then there exists $\bar{c} \in K$ and $d \neq 0 \in K$ such that

$$a = \sum_{i=0}^{n-1} c_i a_i + db,$$

so

$$b = \frac{a}{d} - \sum_{i=0}^{n-1} \frac{c_i}{d} a_i.$$

Hence $b \in \langle A \cup \{a\} \rangle$.

We now show that the pregeometry has the independence property.

i) $\langle \emptyset \rangle = \{0\}$.

ii) Let $D_0 = \{d_0, ..., d_{n-1}\} \subseteq V$ be an independent set. Clearly

$$\sum_{i=0}^{n-1} d_i \in \langle D_0 \rangle.$$

Suppose that there exists $j \in \{0, ..., n-1\}$ such that

$$\sum_{i=0}^{n-1} d_i \in \langle d_0, ..., d_{j-1}, d_{j+1}, ..., d_{n-1} \rangle,$$

then there exists $\bar{c} \in K$ such that

$$\sum_{i=0}^{n-1} d_i = \sum_{\substack{i=0 \\ i \neq j}}^{n-1} c_i d_i.$$

Hence

$$d_j = \sum_{\substack{i=0 \\ i \neq j}}^{n-1} c_i d_i - \sum_{\substack{i=0 \\ i \neq j}}^{n-1} d_i,$$

a contradiction.

iii) Let $A \subseteq V$, then $|\langle A \rangle| \leq \aleph_0 + |A| \leq \aleph_0 + |A| + |\mathcal{L}|$.

$\square$

**Lemma 6.6.2.** Let $\mathbb{V}$ be a vector space over the field $\mathbb{K}$ and $\mathcal{V}$ the corresponding model of $\mathrm{VS}_{\mathbb{K}}$. Let $\overline{a}, \overline{b} \in V$, then

$$\dim(\overline{a}/\overline{b}) = \dim(\overline{a}) \text{ if and only if } \langle\overline{a}\rangle \cap \langle\overline{b}\rangle = \{0\}$$

where $\dim(\overline{a}/\overline{b})$ and $\dim(\overline{a})$ are computed in the pregeometry $(\mathcal{V}, \langle\rangle)$.

*Proof.* ($\Rightarrow$) Let $\overline{a}' = \{a_0, ..., a_{k-1}\}$ be a basis for $\overline{a}$, then $\langle\overline{a}'\rangle = \langle\overline{a}\rangle$. Suppose there exists $c \in \langle\overline{a}\rangle \cap \langle\overline{b}\rangle$ such that $c \neq 0$, then $c \in \langle\overline{a}'\rangle \cap \langle\overline{b}\rangle$ and so there exists $\overline{c}, \overline{d} \in K$ such that

$$0 \neq \sum_{i=0}^{k-1} c_i a_i = c = \sum_{i=0}^{m-1} d_i b_i \neq 0,$$

where $\overline{b} = \{b_0, ..., b_{m-1}\}$. Let $j \in \{0, ..., k-1\}$ be such that $c_j \neq 0$, then

$$a_j = \sum_{i=0}^{m-1} \frac{d_i}{c_j} b_i - \sum_{\substack{i=0 \\ i \neq j}}^{k-1} \frac{c_i}{c_j} a_i.$$

So

$$a_j \in \langle\overline{b} \cup \{a_0, ..., a_{j-1}, a_{j+1}, ..., a_{k-1}\}\rangle$$

and hence $\overline{a}'$ is not independent over $\overline{b}$.

($\Leftarrow$) Suppose there exists $\overline{a}' = \{a_0, ..., a_{k-1}\}$ independent such that $\overline{a}'$ is not independent over $\overline{b} = \{b_0, ..., b_{m-1}\}$, then there exists $j \in \{0, ..., k-1\}$ such that

$$a_j \in \langle\overline{b} \cup \{a_0, ..., a_{j-1}, a_{j+1}, ..., a_{k-1}\}\rangle.$$

So there exists $\overline{c}, \overline{d} \in K$ such that

$$a_j = \sum_{i=0}^{m-1} d_i b_i + \sum_{\substack{i=0 \\ i \neq j}}^{k-1} c_i a_i$$

and then

$$\sum_{i=0}^{m-1} d_i b_i = a_j - \sum_{\substack{i=0 \\ i \neq j}}^{k-1} c_i a_i.$$

Notice now that $\sum_{i=0}^{m-1} d_i b_i \neq 0$ because otherwise $a_j \in \langle\{a_0, ..., a_{j-1}, a_{j+1}, ..., a_{k-1}\}\rangle$, hence $\langle\overline{a}\rangle \cap \langle\overline{b}\rangle \neq 0$.

□

We denote by $\models^{\mathrm{VS}}$ the satisfaction relation of the systems VSADL, VSAAIndL and VSAIndL and with ${}^{\langle\rangle}\!\models^{\mathrm{VS}_{\mathbb{K}}}$ the satisfaction relation of the systems PGADL, PGAAIndL and PGAIndL relative to the theory $\mathrm{VS}_{\mathbb{K}}$ and the pregeometric operator $\langle\rangle$. From what we showed in this section it follows directly the following theorem.

**Theorem 6.6.3.** Let $\mathbb{K}$ be a field, $\mathbb{V}$ a vector space over it and $\mathcal{V}$ the corresponding model of $\mathrm{VS}_{\mathbb{K}}$. Let $s : \mathrm{dom}(s) \to V$ with $\overline{x}\,\overline{y} \subseteq \mathrm{dom}(s) \subseteq \mathrm{Var}$. Then the following hold:

i) $\mathbb{V} \models_s^{\mathrm{VS}} =(\overline{x}, \overline{y})$ if and only if $\mathcal{V}^{\langle\rangle} \models_s^{\mathrm{VS}_{\mathbb{K}}} =(\overline{x}, \overline{y})$;

ii) $\mathbb{V} \models_s^{\mathrm{VS}} \perp(\overline{x})$ if and only if $\mathcal{V}^{\langle\rangle} \models_s^{\mathrm{VS}_{\mathbb{K}}} \perp(\overline{x})$;

iii) $\mathbb{V} \models_s^{\mathrm{VS}} \overline{x} \perp \overline{y}$ if and only if $\mathcal{V}^{\langle\rangle} \models_s^{\mathrm{VS}_{\mathbb{K}}} \overline{x} \perp \overline{y}$.

□

## 6.7 Pregeometry Atomic Absolute Conditional Independence Logic

In this section we define the system Pregeometry Atomic Absolute Conditional Independence Logic (PGAACIndL) and then prove its soundness. The syntax and deductive apparatus of this system are the same as those of AACIndL.

### 6.7.1 Semantics

The interpretation of the atom $\perp_{\overline{z}}(\overline{x})$ in the context of pregeometries is a relativization of the interpretation given for its unconditional counterpart. The intuition behind this atom then is that the elements in $\overline{x}$ are independent over $\overline{z}$.

Let $T$ be a first-order theory with infinite models and cl a unary second order operator on models of $T$ such that for every $\mathcal{M} \models T$ we have that $(\mathcal{M}, \mathrm{cl})$ is a pregeometry with the independence property.

**Definition 6.7.1.** Let $\mathcal{M} \models T$ and $s : \mathrm{dom}(s) \to M$ with $\overline{x}\,\overline{z} \subseteq \mathrm{dom}(s) \subseteq \mathrm{Var}$. We say that $\mathcal{M}$ satisfies $\perp_{\overline{z}}(\overline{x})$ under $s$, in symbols $\mathcal{M} \models_s \perp_{\overline{z}}(\overline{x})$, if for every $x \in \overline{x}$ we have that $s(x) \notin \mathrm{cl}(s(\overline{z}) \cup (\{s(u) \mid u \in \overline{x}\} \setminus \{s(x)\}))$.

**Definition 6.7.2.** Let $\Sigma$ be a set of atoms and let $s$ be such that the set of variables occurring in $\Sigma$ is included in $\mathrm{dom}(s)$. We say that $\mathcal{M}$ satisfies $\Sigma$ under $s$, in symbols $\mathcal{M} \models_s \Sigma$, if $\mathcal{M}$ satisfies every atom in $\Sigma$ under $s$.

**Definition 6.7.3.** Let $\Sigma$ be a set of atoms. We say that $\perp_{\overline{z}}(\overline{x})$ is a logical consequence of $\Sigma$, in symbols $\Sigma \models \perp_{\overline{z}}(\overline{x})$, if for every $\mathcal{M} \models T$ and $s$ such that the set of variables occurring in $\Sigma \cup \{\perp_{\overline{z}}(\overline{x})\}$ is included in $\mathrm{dom}(s)$ we have that

$$\text{if } \mathcal{M} \models_s \Sigma \text{ then } \mathcal{M} \models_s \perp_{\overline{z}}(\overline{x}).$$

### 6.7.2 Soundness

**Theorem 6.7.4.** Let $\Sigma$ be a set of atoms, then

$$\Sigma \vdash \perp_{\overline{z}}(\overline{x}) \quad \Rightarrow \quad \Sigma \models \perp_{\overline{z}}(\overline{x}).$$

[The deductive system to which we refer has been defined in Section 2.5.3.]

*Proof.* ($a_4$.) Obvious.

($b_4$.) Suppose that $\mathcal{M} \models_s \perp_{\overline{z}}(\overline{x}\,\overline{y})$, then for every $v \in \overline{x}\,\overline{y}$ we have that $s(v) \notin \mathrm{cl}(s(\overline{z}) \cup (s(\overline{x}\,\overline{y}) \setminus \{s(v)\}))$. In particular for every $x \in \overline{x}$ we have that $s(x) \notin \mathrm{cl}(s(\overline{z}) \cup (s(\overline{x}\,\overline{y}) \setminus \{s(x)\}))$ and so $s(x) \notin \mathrm{cl}(s(\overline{z}) \cup (s(\overline{x}) \setminus \{s(x)\}))$ because $s(\overline{z}) \cup (s(\overline{x}) \setminus \{s(x)\})$ is a subset of $s(\overline{z}) \cup (s(\overline{x}\,\overline{y}) \setminus \{s(x)\})$.

(c$_4$.) Suppose that $\mathcal{M} \models_s \perp_{\overline{z},\overline{u}}(\overline{x})$, then for every $x \in \overline{x}$ we have that $s(x) \notin \mathrm{cl}(s(\overline{z}\,\overline{u}) \cup (s(\overline{x}) \setminus \{x\}))$ and so $s(x) \notin \mathrm{cl}(s(\overline{z}) \cup (s(\overline{x}) \setminus \{x\}))$ because $s(\overline{z}) \cup (s(\overline{x}) \setminus \{x\})$ is a subset of $s(\overline{z}\,\overline{u}) \cup (s(\overline{x}) \setminus \{x\})$.

(d$_4$.) Suppose that $\mathcal{M} \not\models_s \perp_{\overline{z},\overline{u}}(\overline{x})$, then there exists $x \in \overline{x}$ such that $s(x) \in \mathrm{cl}(s(\overline{z}) \cup s(\overline{u}) \cup (s(\overline{x}) \setminus \{s(x)\}))$. Suppose now that $\mathcal{M} \models_s \perp_{\overline{z}}(\overline{x})$ and let $s(\overline{u}) = \{a_0, ..., a_{n-1}\}$, then there exists $j \in \{0, ..., n-1\}$ such that $s(x) \in \mathrm{cl}(s(\overline{z}) \cup \{a_0, ..., a_j\} \cup (s(\overline{x}) \setminus \{s(x)\}))$ but $s(x) \notin \mathrm{cl}(s(\overline{z}) \cup \{a_0, ..., a_{j-1}\} \cup (s(\overline{x}) \setminus \{s(x)\}))$. Thus by the Exchange Principle $a_j \in \mathrm{cl}(s(\overline{x}) \cup s(\overline{z}) \cup \{a_0, ..., a_{j-1}\})$ and so $a_j \in \mathrm{cl}(s(\overline{x}) \cup s(\overline{z}) \cup (s(\overline{u}) \setminus \{a_j\}))$. Hence $\mathcal{M} \not\models_s \perp_{\overline{x},\overline{z}}(\overline{u})$.

(e$_4$.) Suppose that $\mathcal{M} \models \perp_{\overline{x}}(\overline{x})$, then for every $x \in \overline{x}$ we have that $s(x) \notin \mathrm{cl}(s(\overline{x}) \cup (s(\overline{x}) \setminus \{s(x)\})) = \mathrm{cl}(s(\overline{x}))$, a contradiction. Thus everything follows, in particular we can conclude that $\mathcal{M} \models \perp_{\overline{z}}(\overline{y})$.

(f$_4$.) Obvious.

□

As for its abstract version, it is at present not known whether a completeness theorem holds for this system, but in the light of what is known about conditional independence in database theory it seems that this is not the case (at least with respect to a finite axiomatization). A clue in the direction of a negative answer is the interdefinability of this atom and the atom $\overline{x} \perp_{\overline{z}} \overline{y}$ that we will show in Section 6.9.

## 6.8 Pregeometry Atomic Conditional Independence Logic

In this section we define the system Pregeometry Atomic Conditional Independence Logic (PGACIndL) and then prove its soundness. The syntax and deductive apparatus of this system are the same as those of ACIndL.

### 6.8.1 Semantics

The intuition behind the atom $\overline{x} \perp_{\overline{z}} \overline{y}$ in this context is that if some elements from $\overline{x}$ are independent over $\overline{z}$, then they are also independent over $\overline{z}\,\overline{y}$. This relativizes the interpretation that we gave of the unconditional independence atom.

Let $T$ be a first-order theory with infinite models and cl a unary second order operator on models of $T$ such that for every $\mathcal{M} \models T$ we have that $(\mathcal{M}, \mathrm{cl})$ is a pregeometry with the independence property.

**Definition 6.8.1.** Let $\mathcal{M} \models T$ and $s : \mathrm{dom}(s) \to M$ with $\overline{x}\,\overline{y}\,\overline{z} \subseteq \mathrm{dom}(s) \subseteq \mathrm{Var}$. We say that $\mathcal{M}$ satisfies $\overline{x} \perp_{\overline{z}} \overline{y}$ under $s$, in symbols $\mathcal{M} \models_s \overline{x} \perp_{\overline{z}} \overline{y}$, if $\dim(s(\overline{x})/s(\overline{z}) \cup s(\overline{y})) = \dim(s(\overline{x})/s(\overline{z}))$.

Notice that, because of Proposition 6.3.14, the condition that we used in the above definition is equivalent to the intuitive condition that we mentioned at the beginning of the section.

**Definition 6.8.2.** Let $\Sigma$ be a set of atoms and let $s$ be such that the set of variables occurring in $\Sigma$ is included in $\mathrm{dom}(s)$. We say that $\mathcal{M}$ satisfies $\Sigma$ under $s$, in symbols $\mathcal{M} \models_s \Sigma$, if $\mathcal{M}$ satisfies every atom in $\Sigma$ under $s$.

**Definition 6.8.3.** Let $\Sigma$ be a set of atoms. We say that $\overline{x} \perp_{\overline{z}} \overline{y}$ is a logical consequence of $\Sigma$, in symbols $\Sigma \models \overline{x} \perp_{\overline{z}} \overline{y}$, if for every $\mathcal{M} \models T$ and $s$ such that the set of variables occurring in $\Sigma \cup \{\overline{x} \perp_{\overline{z}} \overline{y}\}$ is included in $\mathrm{dom}(s)$ we have that

$$\text{if } \mathcal{M} \models_s \Sigma \text{ then } \mathcal{M} \models_s \overline{x} \perp_{\overline{z}} \overline{y}.$$

### 6.8.2 Soundness

**Theorem 6.8.4.** Let $\Sigma$ be a set of atoms, then

$$\Sigma \vdash \overline{x} \perp_{\overline{z}} \overline{y} \quad \Rightarrow \quad \Sigma \models \overline{x} \perp_{\overline{z}} \overline{y}.$$

[The deductive system to which we refer has been defined in Section 2.6.3.]

*Proof.* ($a_5$.) $\dim(s(\overline{x})/s(\overline{x})) = 0 = \dim(s(\overline{x})/s(\overline{x}) \cup s(\overline{y}))$, thus $\mathcal{M} \models_s \overline{x} \perp_{\overline{x}} \overline{y}$.

($b_5$.)

$$\begin{array}{rcll} \mathcal{M} \models_s \overline{x} \perp_{\overline{z}} \overline{y} & \Longrightarrow & s(\overline{x}) \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{cl}}_{s(\overline{z})} s(\overline{y}) & \\ & \Longrightarrow & s(\overline{y}) \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{cl}}_{s(\overline{z})} s(\overline{x}) & \text{[By Lemma 6.3.19]} \\ & \Longrightarrow & \mathcal{M} \models_s \overline{y} \perp_{\overline{z}} \overline{x}. & \end{array}$$

($c_5$.)

$$\begin{array}{rcll} \mathcal{M} \models_s \overline{x}\,\overline{x}' \perp_{\overline{z}} \overline{y}\,\overline{y}' & \Longrightarrow & s(\overline{x})s(\overline{x}') \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{cl}}_{s(\overline{z})} s(\overline{y})s(\overline{y}') & \\ & \Longrightarrow & s(\overline{x})s(\overline{x}') \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{cl}}_{s(\overline{z})} s(\overline{y}) & \text{[By Lemma 6.3.15]} \\ & \Longrightarrow & s(\overline{x}) \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{cl}}_{s(\overline{z})} s(\overline{y}) & \text{[By Corollary 6.3.20]} \\ & \Longrightarrow & \mathcal{M} \models_s \overline{x} \perp_{\overline{z}} \overline{y}. & \end{array}$$

($d_5$.) Suppose that $\mathcal{M} \models \overline{x} \perp_{\overline{z}} \overline{y}$, then $s(\overline{x}) \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{cl}}_{s(\overline{z})} s(\overline{y})$ and so $s(\overline{x}) \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{cl}}_{s(\overline{z})} s(\overline{y})s(\overline{z})$ because $\dim(s(\overline{x})/s(\overline{z}) \cup s(\overline{y})) = \dim(s(\overline{x})/s(\overline{z}) \cup (s(\overline{y}) \cup s(\overline{z})))$. Furthermore $s(\overline{z}) \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{cl}}_{s(\overline{z}),s(\overline{x})} s(\overline{y})s(\overline{z})$ because $\dim(s(\overline{z})/s(\overline{z}) \cup s(\overline{x}) \cup s(\overline{y}) \cup s(\overline{z})) = 0 = \dim(s(\overline{z})/s(\overline{z}) \cup s(\overline{x}))$. Hence by Corollary 6.3.20 we have that $s(\overline{x})s(\overline{z}) \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{cl}}_{s(\overline{z})} s(\overline{y})s(\overline{z})$. Thus $\mathcal{M} \models_s \overline{x}\,\overline{z} \perp_{\overline{z}} \overline{y}\,\overline{z}$.

($e_5$.)

$$\begin{array}{ccc} \mathcal{M} \models_s \overline{x} \perp_{\overline{z}} \overline{y} & & \mathcal{M} \models_s \overline{u} \perp_{\overline{z},\overline{x}} \overline{y} \\ \Downarrow & & \Downarrow \\ s(\overline{x}) \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{cl}}_{s(\overline{z})} s(\overline{y}) & & s(\overline{u}) \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{cl}}_{s(\overline{z}),s(\overline{x})} s(\overline{y}) \end{array}$$

$$\begin{array}{cl} \Downarrow & \\ s(\overline{x})s(\overline{u}) \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{cl}}_{s(\overline{z})} s(\overline{y}) & \text{[By Corollary 6.3.20]} \\ \Downarrow & \\ s(\overline{u})s(\overline{x}) \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{cl}}_{s(\overline{z})} s(\overline{y}) & \\ \Downarrow & \\ s(\overline{u}) \mathop{\smile\hskip -0.9em ^| \ }^{\mathrm{cl}}_{s(\overline{z})} s(\overline{y}) & \text{[By Corollary 6.3.20]} \\ \Downarrow & \\ \mathcal{M} \models_s \overline{u} \perp_{\overline{z}} \overline{y} & \end{array}$$

(f$_5$.)

$$
\begin{array}{ccll}
\mathcal{M} \models \overline{y} \perp_{\overline{z}} \overline{y} & & \mathcal{M} \models_s \overline{z}\,\overline{x} \perp_{\overline{y}} \overline{u} & \\
\Downarrow & & \Downarrow & \\
s(\overline{y}) \mathop{\perp\!\!\!\perp}^{\mathrm{cl}}_{s(\overline{z})} s(\overline{y}) & & s(\overline{z})s(\overline{x}) \mathop{\perp\!\!\!\perp}^{\mathrm{cl}}_{s(\overline{y})} s(\overline{u}) & \\
\Downarrow & & \Downarrow & \\
s(\overline{y}) \mathop{\perp\!\!\!\perp}^{\mathrm{cl}}_{s(\overline{z})} s(\overline{y}) & & s(\overline{x}) \mathop{\perp\!\!\!\perp}^{\mathrm{cl}}_{s(\overline{y}),s(\overline{z})} s(\overline{u}) & \text{[By Corollary 6.3.20]} \\
\Downarrow & & \Downarrow & \\
s(\overline{y}) \mathop{\perp\!\!\!\perp}^{\mathrm{cl}}_{s(\overline{z})} s(\overline{u}) & & s(\overline{x}) \mathop{\perp\!\!\!\perp}^{\mathrm{cl}}_{s(\overline{y}),s(\overline{z})} s(\overline{u}) & \text{[By Proposition 6.3.22]} \\
\Downarrow & & \Downarrow & \\
s(\overline{y}) \mathop{\perp\!\!\!\perp}^{\mathrm{cl}}_{s(\overline{z})} s(\overline{u}) & & s(\overline{x}) \mathop{\perp\!\!\!\perp}^{\mathrm{cl}}_{s(\overline{z}),s(\overline{y})} s(\overline{u}) & \\
& \Downarrow & & \\
& s(\overline{x}) \mathop{\perp\!\!\!\perp}^{\mathrm{cl}}_{s(\overline{z})} s(\overline{u}) & \text{[By what we showed in (e}_5\text{.)]} & \\
& \Downarrow & & \\
& \mathcal{M} \models_s \overline{x} \perp_{\overline{z}} \overline{u} & &
\end{array}
$$

(g$_5$.)

$$
\begin{array}{cl}
\mathcal{M} \models_s \overline{x} \perp_{\overline{z}} \overline{y} \text{ and } \mathcal{M} \models_s \overline{x}\,\overline{y} \perp_{\overline{z}} \overline{u} & \\
\Downarrow & \\
s(\overline{x}) \mathop{\perp\!\!\!\perp}^{\mathrm{cl}}_{s(\overline{z})} s(\overline{y}) \text{ and } s(\overline{x})s(\overline{y}) \mathop{\perp\!\!\!\perp}^{\mathrm{cl}}_{s(\overline{z})} s(\overline{u}) & \\
\Downarrow & \\
s(\overline{x}) \mathop{\perp\!\!\!\perp}^{\mathrm{cl}}_{s(\overline{z})} s(\overline{y})s(\overline{u}) & \text{[By Corollary 6.3.21]} \\
\Downarrow & \\
\mathcal{M} \models_s \overline{x} \perp_{\overline{z}} \overline{y}\,\overline{u} &
\end{array}
$$

(h$_5$.) Obvious.

□

As we noticed at the beginning of Chapter 2, in the context of database theory, the unconditional independence atom is known to be non finitely axiomatizable [32]. This does not prove that also the present system is incomplete (with respect to a finite set of axioms), but it is a strong clue that the system is too complex to admit such a strong form of completeness.

## 6.9 Relations between PGAACIndL and PGACIndL

In this section we study the relations, under the given semantics, between the different kinds of independence atoms that we used in the present chapter. We will see that not only the absolute conditional independence atom is definable in terms of the conditional independence one but also the other way around, giving rise to a case of interdefinability.

**Lemma 6.9.1.** Let $(\mathcal{M}, \mathrm{cl})$ be a pregeometry and $A, C \subseteq M$. The following are equivalent:

i) $a \notin \mathrm{cl}(C \cup (A \setminus \{a\}))$ for every $a \in A$;

ii) $A \cap \mathrm{cl}(C) = \emptyset$ and $\dim(a/C \cup (A \setminus \{a\})) = \dim(a/C)$ for every $a \in A$.

*Proof.* ii) $\Rightarrow$ i) Suppose that there exists $a \in A$ such that $a \in \mathrm{cl}(C \cup (A \setminus \{a\}))$ and $a \notin \mathrm{cl}(C)$, then $\dim(a/C) = 1$ and $\dim(a/C \cup (A \setminus \{a\})) = 0$ because $a \in \mathrm{cl}(C \cup ((A \setminus \{a\}) \cup \emptyset))$.

i) $\Rightarrow$ ii) If there exists $a \in A$ such that $a \in \mathrm{cl}(C)$, then $a \in \mathrm{cl}(C \cup (A \setminus \{a\}))$. If there exists $a \in A$ such that $\dim(a/C \cup (A \setminus \{a\})) \neq \dim(a/C)$, then $\dim(a/C \cup (A \setminus \{a\})) = 0$ and $\dim(a/C) = 1$, so $a \in \mathrm{cl}(C \cup ((A \setminus \{a\}) \cup \emptyset)) = \mathrm{cl}(C \cup (A \setminus \{a\}))$. $\square$

From the above lemma it follows directly the following characterization of PGAACIndL in terms of PGACIndL.

**Theorem 6.9.2.** Let $s$ be an assignment such that the set of variables occurring in $\Sigma \cup \{\perp_{\overline{z}}(\overline{x})\}$ is included in $\mathrm{dom}(s)$. Let $x \in \overline{x}$, we denote by $\overline{x} -_X x$ any enumeration of the set $\{x' \in \overline{x} \mid \mathcal{M} \not\models_s x' = x\}$. Then

$$\mathcal{M} \models_s \perp_{\overline{z}}(\overline{x}) \iff \mathcal{M} \models_s x \perp_{\overline{z}} \overline{x} -_X x \text{ and } \mathcal{M} \not\models_s x \perp_{\overline{z}} x, \text{ for all } x \in \overline{x}.$$

$\square$

Clearly we have that

$$\mathcal{M} \models_s \perp_{\emptyset}(\overline{x}) \text{ iff } \mathcal{M} \models_s \perp(\overline{x}) \text{ and } \mathcal{M} \models_s \overline{x} \perp_{\emptyset} \overline{y} \text{ iff } \mathcal{M} \models_s \overline{x} \perp \overline{y}.$$

So as a particular case of the result above we have the following characterization of PGAAIndL in terms of PGAIndL.

**Theorem 6.9.3.** Let $s$ be an assignment such that the set of variables occurring in $\Sigma \cup \{\perp(\overline{x})\}$ is included in $\mathrm{dom}(s)$. Let $x \in \overline{x}$, we denote by $\overline{x} -_X x$ any enumeration of the set $\{x' \in \overline{x} \mid \mathcal{M} \not\models_s x' = x\}$. Then

$$\mathcal{M} \models_s \perp(\overline{x}) \iff \mathcal{M} \models_s x \perp \overline{x} -_X x \text{ and } \mathcal{M} \not\models_s x \perp x, \text{ for all } x \in \overline{x}.$$

$\square$

Finally from Lemma 6.3.14 it follows directly the following characterization of PGACIndL in terms of PGAACIndL.

**Theorem 6.9.4.** Let $s$ be an assignment such that the set of variables occurring in $\Sigma \cup \{\overline{x} \perp_{\overline{z}} \overline{y}\}$ is included in $\mathrm{dom}(s)$. Then

$$\mathcal{M} \models_s \overline{x} \perp_{\overline{z}} \overline{y} \iff \text{ for all } \overline{x}' \subseteq \overline{x} \text{ if } \mathcal{M} \models_s \perp_{\overline{z}}(\overline{x}') \text{ then } \mathcal{M} \models_s \perp_{\overline{z},\overline{y}}(\overline{x}').$$

$\square$

# Chapter 7

# $\omega$-Stable Theories and Strongly Minimal Sets

In this chapter we develop the basics of stability theory. The background that we build here will allow us to define the forking independence relation in the next chapter. We will define forking only in $\omega$-stable theories and not in theories satisfying more general forms of stability. This is why we focus on this class of theories in what follows.

First of all, we will review some basic facts about types and spaces of complete types. Secondly, we will define what a stable theory is and, in particular, what an $\omega$-stable theory is. Then, we will introduce the notions of Morley rank and Morley degree and talk about the monster models. Finally, we will define minimal and strongly minimal sets and show that in any $\aleph_0$-saturated model of an $\omega$-stable theory we can always find strongly minimal sets.

## 7.1 Types

**Definition 7.1.1.** Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $A \subseteq M$. Let $p$ be a set of $\mathcal{L}_A$-formulas in the free variables $v_0, ..., v_{n-1}$. We call $p$ an *n-type* if $p \cup \mathrm{Th}_A(\mathcal{M})$ is satisfiable. We say that $p$ is a *complete n-type* if $\phi \in p$ or $\neg\phi \in p$ for all $\mathcal{L}_A$-formulas $\phi$ with free variables from $v_0, ..., v_{n-1}$. We let $S_n^{\mathcal{M}}(A)$ denote the set of all complete $n$-types.

We sometimes refer to incomplete types as *partial types.* Also, we often write $p(v_0, .., v_{n-1})$ to stress that $p$ is an $n$-type. By the Compactness Theorem, we could replace "satisfiable" by "finitely satisfiable" in the definition above.

If $\mathcal{M}$ is an $\mathcal{L}$-structure, $A \subseteq M$ and $\overline{a} = (a_0, ..., a_{n-1}) \in M^n$, we let $\mathrm{tp}^{\mathcal{M}}(\overline{a}/A) = \{\phi(v_0, ..., v_{n-1}) \in \mathcal{L}_A \mid \mathcal{M} \models \phi(a_0, ..., a_{n-1})\}$. Then $\mathrm{tp}^{\mathcal{M}}(\overline{a}/A)$ is a complete $n$-type. We write $\mathrm{tp}^{\mathcal{M}}(\overline{a})$ for $\mathrm{tp}^{\mathcal{M}}(\overline{a}/\emptyset)$.

**Definition 7.1.2.** If $p$ is an $n$-type over $A$, we say that $\overline{a} \in M^n$ *realizes* $p$ if $\mathcal{M} \models \phi(\overline{a})$ for all $\phi \in p$. If $p$ is not realized in $\mathcal{M}$ we say that $\mathcal{M}$ *omits* $p$.

The following proposition gives an equivalent definition of the notion of complete type.

**Proposition 7.1.3.** $p \in S_n^{\mathcal{M}}(A)$ if and only if there is an elementary extension $\mathcal{N}$ of $\mathcal{M}$ and $\overline{a} \in N^n$ such that $p = \text{tp}^{\mathcal{N}}(\overline{a}/A)$.

*Proof.* See [29, Corollary 4.1.4].

□

There is a natural topology on the space of complete $n$-types $S_n^{\mathcal{M}}(A)$. For $\phi$ an $\mathcal{L}_A$-formula with free variables from $v_0, ..., v_{n-1}$, let

$$[\phi] = \left\{ p \in S_n^{\mathcal{M}}(A) \mid \phi \in p \right\}.$$

If $p$ is a complete type and $\phi \vee \psi \in p$, then $\phi \in p$ or $\psi \in p$. Thus $[\phi \vee \psi] = [\phi] \cup [\psi]$. Similarly, $[\phi \wedge \psi] = [\phi] \cap [\psi]$.

The *Stone topology* on $S_n^{\mathcal{M}}(A)$ is the topology generated by taking the sets $[\phi]$ as basic open sets. For complete types $p$, exactly one of $\phi$ and $\neg\phi$ is in $p$. Thus $[\phi] = S_n^{\mathcal{M}}(A) \setminus [\neg\phi]$ is also closed.

**Lemma 7.1.4.** i) $S_n^{\mathcal{M}}(A)$ is compact.

ii) $S_n^{\mathcal{M}}(A)$ is totally disconnected.

*Proof.* i) It suffices to show that every cover of $S_n^{\mathcal{M}}(A)$ by basic open sets has a finite subcover. Suppose to the contrary that $C = \{[\phi_i(\overline{v})] \mid i \in I\}$ is a cover of $S_n^{\mathcal{M}}(A)$ by basic open sets with no finite subcover. Let

$$\Gamma = \{\neg\phi_i(\overline{v}) \mid i \in I\}.$$

We claim that $\Gamma \cup \text{Th}_A(\mathcal{M})$ is satisfiable. Let $I_0$ be a finite subset of $I$, because there is no finite subcover of $C$ there is a type $p$ such that

$$p \notin \bigcup_{i \in I_0} [\phi_i].$$

Let $\mathcal{N}$ be an elementary extension of $\mathcal{M}$ containing a realization $\overline{a}$ of $p$. Then

$$\mathcal{N} \models \text{Th}_A(\mathcal{M}) \cup \bigwedge_{i \in I_0} \neg\phi_i(\overline{a}).$$

We have shown that $\Gamma$ is finitely satisfiable and hence, by the Compactness Theorem, satisfiable.

Let $\mathcal{N}$ be an elementary extension of $\mathcal{M}$ and let $\overline{a} \in N$ realizes $\Gamma$. Then

$$\text{tp}^{\mathcal{N}}(\overline{a}/A) \in S_n^{\mathcal{M}}(A) \setminus \bigcup_{i \in I} [\phi_i(\overline{v})],$$

which is a contradiction.

ii) If $p \neq q$ there is a formula $\phi$ such that $\phi \in p$ and $\neg\phi \in q$. Thus $[\phi]$ is a basic clopen set separating $p$ and $q$.

□

**Lemma 7.1.5.** If $A \subseteq B \subseteq M$ and $p \in S_n^{\mathcal{M}}(B)$, let $p|A$ be the set of $\mathcal{L}_A$-formulas in $p$. Then $p|A \in S_n^{\mathcal{M}}(A)$ and $p \mapsto p|A$ is a continuous map from $S_n^{\mathcal{M}}(B)$ onto $S_n^{\mathcal{M}}(A)$.

*Proof.* See [29, Lemma 4.1.9].

□

**Definition 7.1.6.** We say that $p \in S_n^{\mathcal{M}}(A)$ is *isolated* if $\{p\}$ is an open subset of $S_n^{\mathcal{M}}(A)$.

**Proposition 7.1.7.** Let $p \in S_n^{\mathcal{M}}(A)$. The following are equivalent.

i) $p$ is isolated.

ii) $\{p\} = [\phi(\overline{v})]$ for some $\mathcal{L}_A$-formula $\phi(\overline{v})$. In this case we say that $\phi(\overline{v})$ isolates $p$.

iii) There is an $\mathcal{L}_A$-formula $\phi(\overline{v}) \in p$ such that for all $\mathcal{L}_A$-formulas $\psi(\overline{v})$, $\psi(\overline{v}) \in p$ if and only if
$$\mathrm{Th}_A(\mathcal{M}) \models \forall \overline{v}(\phi(\overline{v}) \to \psi(\overline{v})).$$

*Proof.* i) ⇒ ii) If $X$ is open, then
$$X = \bigcup_{i \in I} [\phi_i]$$
for some collection of formulas $(\phi_i \mid i \in I)$. So if $\{p\}$ is open, then $\{p\} = [\phi]$ for some formula $\phi$.

ii) ⇒ iii) Suppose that $\{p\} = [\phi(\overline{v})]$ and let $\psi(\overline{v}) \in p$. We claim that $\mathrm{Th}_A(\mathcal{M}) \models \forall \overline{v}(\phi(\overline{v}) \to \psi(\overline{v}))$. Suppose not, then being the theory complete we have that $\mathrm{Th}_A(\mathcal{M}) \models \neg\forall \overline{v}(\phi(\overline{v}) \to \psi(\overline{v}))$. Now clearly $\mathcal{M}_A \models \mathrm{Th}_A(\mathcal{M})$ so $\mathcal{M}_A \models \neg\forall \overline{v}(\phi(\overline{v}) \to \psi(\overline{v}))$, thus there exists $\overline{a} \in M$ such that $\mathcal{M} \models \phi(\overline{a}) \wedge \neg\psi(\overline{a})$. Let $q = \mathrm{tp}^{\mathcal{M}}(\overline{a}/A) \in S_n^{\mathcal{M}}(A)$. Because $\phi(\overline{v}) \in q$, $q = p$. But then $\neg\psi(\overline{v}) \wedge \psi(\overline{v}) \in q$ which is a contradiction.

If, on the other hand, $\psi(\overline{v}) \notin p$, then $\neg\psi(\overline{v}) \in p$ and, by the argument above, $\mathrm{Th}_A(\mathcal{M}) \models \forall \overline{v}(\phi(\overline{v}) \to \neg\psi(\overline{v}))$. Now $\phi(\overline{v}) \in p \in S_n^{\mathcal{M}}(A)$ so there exists $\mathcal{M} \prec \mathcal{N}$ and $\overline{a} \in N$ such that $\mathcal{N} \models \mathrm{Th}_A(\mathcal{M}) \cup \{\phi(\overline{a})\}$, hence $\mathrm{Th}_A(\mathcal{M}) \not\models \forall \overline{v}(\phi(\overline{v}) \to \psi(\overline{v}))$.

iii) ⇒ i) We claim that $[\phi(\overline{v})] = \{p\}$. Clearly, $p \in [\phi(\overline{v})]$. Suppose that $q \in [\phi(\overline{v})]$ and $\psi(\overline{v})$ is an $\mathcal{L}_A$-formula. If $\psi(\overline{v}) \in p$, then $\mathrm{Th}_A(\mathcal{M}) \models \forall \overline{v}(\phi(\overline{v}) \to \psi(\overline{v}))$ and so $\psi(\overline{v}) \in q$ because $\phi(\overline{v}) \in q$. On the other hand, if $\psi(\overline{v}) \notin p$, then $\neg\psi(\overline{v}) \in p$ and, by the argument above, $\neg\psi(\overline{v}) \in q$ so $\psi(\overline{v}) \notin q$. Thus $q = p$.

□

So far we considered types of a structure, it is also possible to define a notion of type of a theory.

**Definition 7.1.8.** Let $T$ be an $\mathcal{L}$-theory. Let $p$ be a set of $\mathcal{L}$-formulas in the free variables $v_0, ..., v_{n-1}$. We call $p$ an *$n$-type* if $p \cup T$ is satisfiable. We say that $p$ is a *complete $n$-type* if $\phi \in p$ or $\neg\phi \in p$ for all $\mathcal{L}$-formulas $\phi$ with free variables from $v_0, ..., v_{n-1}$. We let $S_n(T)$ denote the set of all complete $n$-types.

Notice that if $T$ is a complete theory and $\mathcal{M} \models T$, then $S_n(T) = S_n^{\mathcal{M}}(\emptyset)$. Indeed if $p \in S_n^{\mathcal{M}}(\emptyset)$ then $\mathrm{Th}(\mathcal{M}) \cup p$ is satisfiable and so $T \cup p$ is also satisfiable because clearly $T \subseteq \mathrm{Th}(\mathcal{M})$. On the other hand if $p \in S_n(T)$ then $T \cup p$ is satisfiable and so $\{\phi \in \mathcal{L} \mid T \models \phi\} \cup p$ is also satisfiable. Hence $p \in S_n^{\mathcal{M}}(\emptyset)$

because if $T$ is complete then for every $\mathcal{M} \models T$ we have $\{\phi \in \mathcal{L} \mid T \models \phi\} = \mathrm{Th}(\mathcal{M})$.

We define a topology on $S_n(T)$ by letting

$$[\phi] = \{p \in S_n(T) \mid \phi \in p\}$$

as basic open sets. As for $S_n^{\mathcal{M}}(A)$ the space $S_n(T)$ is compact and totally disconnected.

For $p$ a complete type, we say that $p$ is isolated in $S_n(T)$ if $\{p\}$ is open in $S_n(T)$. An analogous of Proposition 7.1.7 holds for isolated types of a theory. Furthermore we can omit an isolated type only if we do not witness the isolating formula.

**Proposition 7.1.9.** If $\phi(\overline{v})$ isolates $p$, then $p$ is realized in any model of $T \cup \{\exists \overline{v} \phi(\overline{v})\}$. In particular, if $T$ is complete, then every isolated type is realized in any model of $T$.

*Proof.* If $\mathcal{M} \models T$ and $\mathcal{M} \models \phi(\overline{a})$, then $\overline{a}$ realizes p. If $T$ is complete and $p$ is a type then there exists $\mathcal{M} \models T$ such that $\mathcal{M}$ realizes $p$. Thus $\mathcal{M} \models \exists \overline{v} \phi(\overline{v})$ and so $T \models \exists \overline{v} \phi(\overline{v})$.

□

## 7.2 $\omega$-Stable Theories

Let $\mathcal{M}$ be a model in a countable signature $\mathcal{L}$, $\kappa$ an infinite cardinal and $A \subseteq M$ with $|A| = \kappa$, then

$$\kappa \leq |S_n^{\mathcal{M}}(A)| \leq 2^\kappa,$$

indeed $|\{\mathrm{tp}(\overline{a}/A) \mid \overline{a} \in A^n\}| = \kappa$ and $|\mathcal{P}(\mathcal{L}_A)| = 2^\kappa$.

**Theorem 7.2.1.** Let $T$ be a complete theory in a countable language, $\mathcal{M} \models T$ and $A \subseteq M$ with $A$ countable. If $|S_n^{\mathcal{M}}(A)| < 2^{\aleph_0}$, then

i) the set of isolated types in $S_n^{\mathcal{M}}(A)$ are dense;

ii) $|S_n^{\mathcal{M}}(A)| \leq \aleph_0$.

*Proof.* i) Let $P = \{p \in S_n^{\mathcal{M}}(A) \mid p \text{ is isolated}\}$. We want to show that for every $\phi \in \mathcal{L}_A$ such that $[\phi] \neq \emptyset$ we have that $[\phi] \cap P \neq \emptyset$. Suppose to the contrary that there exists $\phi \in \mathcal{L}_A$ such that $[\phi] \neq \emptyset$ and $[\phi] \cap P = \emptyset$.

We build a binary tree of formulas $T = \{\phi_\sigma \mid \sigma \in 2^{<\omega}\}$ such that:

a) for every $\phi_\sigma \in T$, $[\phi_\sigma] \neq \emptyset$ and $[\phi_\sigma] \cap P = \emptyset$;

b) if $\sigma \subseteq \tau$, then $\phi_\tau \models \phi_\sigma$;

c) $\phi_{\sigma,\, i} \models \neg \phi_{\sigma,\, 1-i}$.

Let $\phi_\emptyset = \phi$. Suppose now that $[\phi_\sigma] \neq \emptyset$ and $[\phi_\sigma] \cap P = \emptyset$, then for every $p \in S_n^{\mathcal{M}}(A)$ we have that $[\phi_\sigma] \neq \{p\}$, so there exist $q_0, q_1 \in [\phi_\sigma]$ with $q_0 \neq q_1$. Thus there exists $\psi \in \mathcal{L}_A$ such that $\psi \in q_0$ and $\neg\psi \in q_1$ and hence $q_0 \in [\phi_\sigma \wedge \psi]$ and $q_1 \in [\phi_\sigma \wedge \neg\psi]$. Furthermore $[\phi_\sigma \wedge \psi]$ and $[\phi_\sigma \wedge \neg\psi]$ contain no isolated type, indeed they are both contained in $[\phi_\sigma]$ and $[\phi_\sigma]$ by hypothesis contains no isolated type. Let then $\phi_{\sigma,0} = \phi_\sigma \wedge \psi$ and $\phi_{\sigma,1} = \phi_\sigma \wedge \neg\psi$.

Let $f : \omega \to 2$. By b) we have that

$$..., \ \phi_{f|2} \models \phi_{f|1}, \ \phi_{f|1} \models \phi_{f|0},$$

so

$$[\phi_{f|0}] \supseteq [\phi_{f|1}] \supseteq [\phi_{f|2}] \supseteq ... \ .$$

By a), for every $n \in \omega$ there is

$$p_n \in [\phi_{f|n}] = \bigcap_{i=0}^{n} [\phi_{f|i}].$$

But then by the fact that in $S_n^{\mathcal{M}}(A)$ the basic opens are clopen and that $S_n^{\mathcal{M}}(A)$ is compact it follows that there is

$$p_f \in \bigcap_{i \in \omega} [\phi_{f|i}].$$

If $f \neq g$ then there exists $m \in \omega$ such that $f|m = g|m$ but $f(m) \neq g(m)$. By c),

$$\phi_{f|m+1} = \phi_{f|m, \ f(m)} \models \neg\phi_{f|m, \ 1-f(m)} = \neg\phi_{f|m, \ g(m)} = \neg\phi_{g|m, \ g(m)} = \neg\phi_{g|m+1}.$$

Thus

$$\begin{aligned} p_f \in \bigcap_{i \in \omega} [\phi_{f|i}] \quad &\Longrightarrow \quad p_f \in [\phi_{f|m+1}] \\ &\Longrightarrow \quad \phi_{f|m+1} \in p_f \\ &\Longrightarrow \quad \neg\phi_{g|m+1} \in p_f \\ &\Longrightarrow \quad \phi_{g|m+1} \notin p_f. \end{aligned}$$

Notice though that $\phi_{g|m+1} \in p_g$, hence $p_f \neq p_g$.

The function $f \mapsto p_f$ is then a one-to-one function from $2^\omega$ into $S_n^{\mathcal{M}}(A)$, so $|S_n^{\mathcal{M}}(A)| = 2^{\aleph_0}$ which is a contradiction.

ii) First we prove a lemma.

**Lemma 7.2.2.** If $|[\phi]| > \aleph_0$, there is an $\mathcal{L}_A$ formula $\psi$ such that $|[\phi \wedge \psi]| > \aleph_0$ and $|[\phi \wedge \neg\psi]| > \aleph_0$.

*Proof.* Suppose not. Let $p = \{\psi(\overline{v}) \mid |[\phi \wedge \psi]| > \aleph_0\}$. Notice that for each $\psi$ either $\psi \in p$ or $\neg\psi \in p$ but not both. Indeed both cannot be there because this would contradict the hypothesis and if $|[\phi \wedge \psi]| \leq \aleph_0$ and $|[\phi \wedge \neg\psi]| \leq \aleph_0$ then $|[\phi]| = |[\phi \wedge \psi] \cup [\phi \wedge \neg\psi]| = |[\phi \wedge \psi]| + |[\phi \wedge \neg\psi]| \leq \aleph_0$ which is absurd.

Furthermore $p$ is satisfiable. Suppose that $\psi_0, ..., \psi_{n-1} \in p$, then either

$$\psi_0 \wedge \cdots \wedge \psi_{n-1} \in p \ \text{ or } \ \neg(\psi_0 \wedge \cdots \wedge \psi_{n-1}) \in p.$$

If the first, then $|[\phi \wedge (\psi_0 \wedge \cdots \wedge \psi_{n-1})]| > \aleph_0$; so, clearly, there exists a type $q$ such that $\phi \wedge (\psi_0 \wedge \cdots \wedge \psi_{n-1}) \in q$ and hence $\{\psi_0, ..., \psi_{n-1}\} \cup \mathrm{Th}_A(\mathcal{M})$ is satisfiable.

If the second, then $|[\phi \wedge \neg(\psi_0 \wedge \cdots \wedge \psi_{n-1})]| > \aleph_0$. Suppose that for all $i \in \{0, ..., n-1\}$ we have that $|[\neg\psi_i]| \leq \aleph_0$, then

$$|[\phi \wedge \neg(\psi_0 \wedge \cdots \wedge \psi_{n-1})]| \leq |[\neg(\psi_0 \wedge \cdots \wedge \psi_{n-1})]| = |[\neg\psi_0]| \cup \cdots \cup |[\neg\psi_{n-1}]| \leq \aleph_0$$

which is a contradiction. Thus there exists an $i \in \{0, ..., n-1\}$ such that $|[\neg\psi_i]| > \aleph_0$ but then $\psi_i \wedge \neg\psi_i \in p$ which is absurd.

Hence we conclude that $p \in S_n^{\mathcal{M}}(A)$. Notice that

$$[\phi] = \bigcup_{\psi \notin p} [\phi \wedge \psi] \cup \{p\}\,.$$

For all $\psi \notin p$ we have that $|[\phi \wedge \psi]| \leq \aleph_0$ and clearly $|p| \leq \aleph_0$ because $|\mathcal{L}_A| \leq \aleph_0$, so $[\phi]$ is the union of at most $\aleph_0$ sets each of size at most $\aleph_0$ and thus $|[\phi]| \leq \aleph_0$, a contradiction.

□

Suppose that $|S_n^{\mathcal{M}}(A)| > \aleph_0$, we want to show that $|S_n^{\mathcal{M}}(A)| = 2^{\aleph_0}$. Because $|S_n^{\mathcal{M}}(A)| > \aleph_0$ and there are only countably many $\mathcal{L}_A$ formulas, there is a formula $\phi$ such that $|[\phi]| > \aleph_0$.

We build a binary tree of formulas $T = \{\phi_\sigma \mid \sigma \in 2^{<\omega}\}$ such that:

a) $|[\phi_\sigma]| > \aleph_0$;

b) if $\sigma \subseteq \tau$, then $\phi_\tau \models \phi_\sigma$;

c) $\phi_{\sigma,\, i} \models \neg\phi_{\sigma,\, 1-i}$.

Let $\phi_\emptyset = \phi$. Given $\phi_\sigma$, where $|[\phi_\sigma]| > \aleph_0$, by the lemma we can find $\psi$ such that $|[\phi_\sigma \wedge \psi]| > \aleph_0$ and $|[\phi_\sigma \wedge \neg\psi]| > \aleph_0$. Let then $\phi_{\sigma,0} = \phi_\sigma \wedge \psi$ and $\phi_{\sigma,1} = \phi_\sigma \wedge \neg\psi$.

As in i), for each $f \in 2^\omega$ there is a

$$p_f \in \bigcap_{i \in \omega} [\phi_{f|i}],$$

and if $f \neq g$, then $p_f \neq p_g$. Thus $|S_n^{\mathcal{M}}(A)| = 2^{\aleph_0}$.

□

**Definition 7.2.3.** Let $T$ be a complete theory in a countable language with infinite models and let $\kappa$ be an infinite cardinal. We say that T is *$\kappa$-stable* if whenever $\mathcal{M} \models T$, $A \subseteq M$ and $|A| = \kappa$, then $|S_n^{\mathcal{M}}(A)| = \kappa$.

**Definition 7.2.4.** We say that $T$ is *stable* if $T$ is $\kappa$-stable for some infinite cardinal $\kappa$. We say that $T$ is *superstable* if $T$ is $\kappa$-stable for all $\kappa \geq 2^{\aleph_0}$. Finally, we say that $T$ is *unstable* if $T$ is not $\kappa$-stable for every infinite cardinal $\kappa$.

Complete theories can be classified in function of their stability and instability. The next theorem shows that complete theories can be partitioned in three classes.

**Theorem 7.2.5.** Let $T$ be a complete theory in a countable language. Then one the following holds:

1. $T$ is unstable;
2. $T$ is superstable;
3. $T$ is $\kappa$-stable if and only if $\kappa^{\aleph_0} = \kappa$.

*Proof.* See [5, Theorem 4.36]. □

Among all the stable theories, $\aleph_0$-stable theories play a key role. For historical reasons we will refer to these theories as $\omega$-stable theories.

**Theorem 7.2.6.** Let $T$ be a complete theory in a countable language. If $T$ is $\omega$-stable, then $T$ is $\kappa$-stable for all infinite cardinals $\kappa$.

*Proof.* Let $\mathcal{M} \models T$, $A \subseteq M$ and $|A| = \kappa$. First we prove a lemma.

**Lemma 7.2.7.** If $|[\phi]| > \kappa$, there is an $\mathcal{L}_A$ formula $\psi$ such that $|[\phi \wedge \psi]| > \kappa$ and $|[\phi \wedge \neg\psi]| > \kappa$.

*Proof.* The proof of this lemma is a straightforward adaptation of the proof of Lemma 7.2.2. Just notice that the union of at most $\kappa$ sets each of size at most $\kappa$ is at most $\kappa$. □

Suppose now that $|S_n^{\mathcal{M}}(A)| > \kappa$, because there are only $\kappa$ formulas with parameters from $A$, there is an $\mathcal{L}_A$ formula $\phi_\emptyset$ such that $|[\phi_\emptyset]| > \kappa$. Using the same technique used in Theorem 7.2.1 ii) we can build a binary tree of formulas $T = \{\phi_\sigma \mid \sigma \in 2^{<\omega}\}$ such that:

i) $|[\phi_\sigma]| > \kappa$;

ii) if $\sigma \subseteq \tau$, then $\phi_\tau \models \phi_\sigma$;

iii) $\phi_{\sigma,\, i} \models \neg\phi_{\sigma,\, 1-i}$.

Let $A_0$ be the set of all parameters from $A$ occurring in any formula $\phi_\sigma$. The tree $T$ is a a countable object and clearly each of its node contains only finitely many parameters, thus the set $A_0$ is countable. Then arguing as in Theorem 7.2.1 ii) we can prove that $S_n^{\mathcal{M}}(A_0) = 2^{\aleph_0}$, contradicting the $\omega$-stability of $T$. □

**Definition 7.2.8.** Let $\kappa$ be an infinite cardinal. We say that $\mathcal{M} \models T$ is $\kappa$-*saturated* when for all $A \subseteq M$ if $|A| < \kappa$ and $p \in S_n^{\mathcal{M}}(A)$, then $p$ is realized in $\mathcal{M}$. We say that $\mathcal{M}$ is *saturated* if it is $|M|$-saturated.

**Theorem 7.2.9.** Let $\kappa$ be a regular cardinal. If $T$ is $\kappa$-stable then there is a saturated $\mathcal{M} \models T$ with $|M| = \kappa$. Indeed, if $\mathcal{M}_0 \models T$ with $|M_0| = \kappa$, then there is a saturated elementary extension $\mathcal{M}$ of $\mathcal{M}_0$ with $|M| = \kappa$.

In particular, if T is $\omega$-stable, then there are saturated models of size $\kappa$ for all regular cardinals $\kappa$.

*Proof.* See [29, Theorem 4.3.15]. □

**Definition 7.2.10.** We say that $\mathcal{M} \models T$ is $\kappa$-*universal* if for all $\mathcal{N} \models T$ with $|N| < \kappa$ there is an elementary embedding of $\mathcal{N}$ into $\mathcal{M}$. We say that $\mathcal{M}$ is *universal* if it is $|M|^+$ universal.

**Lemma 7.2.11.** Let $\kappa \geq \aleph_0$. If $\mathcal{M}$ is $\kappa$-saturated, then $\mathcal{M}$ is $\kappa^+$-universal.

*Proof.* See [29, Theorem 4.3.17]. □

## 7.3 Morley Rank

Let $T$ be a complete theory with infinite models.

**Definition 7.3.1.** Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $\phi(\overline{v})$ an $\mathcal{L}_M$ formula. We will define $\mathrm{RM}^{\mathcal{M}}(\phi)$, the *Morley rank* of $\phi$ in $\mathcal{M}$. First, we inductively define $\mathrm{RM}^{\mathcal{M}}(\phi) \geq \alpha$ for $\alpha$ an ordinal.

i) $\mathrm{RM}^{\mathcal{M}}(\phi) \geq 0$ if and only if $\phi(\mathcal{M}) \neq \emptyset$;

ii) if $\alpha$ is limit ordinal, $\mathrm{RM}^{\mathcal{M}}(\phi) \geq \alpha$ if and only if $\mathrm{RM}^{\mathcal{M}}(\phi) \geq \beta$ for all $\beta < \alpha$;

iii) for any ordinal $\alpha$, $\mathrm{RM}^{\mathcal{M}}(\phi) \geq \alpha + 1$ if and only if there are $\mathcal{L}_M$ formulas $\psi_1(\overline{v}), \psi_2(\overline{v}), ...$ such that $\psi_1(\mathcal{M}), \psi_2(\mathcal{M}), ...$ is an infinite family of pairwise disjoint subsets of $\phi(\mathcal{M})$ and $\mathrm{RM}^{\mathcal{M}}(\psi_i) \geq \alpha$ for all $i$.

If $\phi(\mathcal{M}) = \emptyset$, then $\mathrm{RM}^{\mathcal{M}}(\phi) = -1$. If $\phi(\mathcal{M}) \geq \alpha$ but $\phi(\mathcal{M}) \not\geq \alpha + 1$, then $\mathrm{RM}^{\mathcal{M}}(\phi) = \alpha$. If $\mathrm{RM}^{\mathcal{M}}(\phi) \geq \alpha$ for all ordinals $\alpha$, then $\mathrm{RM}^{\mathcal{M}}(\phi) = \infty$.

Given $\mathcal{M} \prec \mathcal{N}$ and $\phi$ an $\mathcal{L}_M$ formula, it is possible that $\mathrm{RM}^{\mathcal{M}}(\phi) \neq \mathrm{RM}^{\mathcal{N}}(\phi)$, for $\aleph_0$-saturated models this is not the case.

**Proposition 7.3.2.** Let $\mathcal{M} \models T$ be an $\mathcal{L}$-structure, $\phi$ an $\mathcal{L}_M$ formula and $\mathcal{N}_0$ and $\mathcal{N}_1$ be $\aleph_0$-saturated elementary extensions of $\mathcal{M}$. Then $\mathrm{RM}^{\mathcal{N}_0}(\phi) = \mathrm{RM}^{\mathcal{N}_1}(\phi)$.

*Proof.* See [29, Lemma 6.2.4]. □

Proposition 7.3.2 allows us to define the *Morley rank* of $\phi$ in a way that does not depend on which model contains the parameters occurring in $\phi$.

**Definition 7.3.3.** Let $\mathcal{M} \models T$ be an $\mathcal{L}$-structure and $\phi$ an $\mathcal{L}_M$ formula, we define $\mathrm{RM}(\phi)$, the Morley rank of $\phi$, to be $\mathrm{RM}^{\mathcal{N}}(\phi)$, where $\mathcal{N}$ is any $\aleph_0$-saturated elementary extension of $\mathcal{M}$.

Morley rank allows us to define a notion of "dimension" for definable sets.

**Definition 7.3.4.** Let $\mathcal{M} \models T$ and $X \subseteq M^n$ be such that $X = \phi(\mathcal{M})$, for $\phi$ an $\mathcal{L}_M$ formula. We define $\mathrm{RM}(X)$, the *Morley rank* of $X$, to be $\mathrm{RM}(\phi)$.

In particular, if $\mathcal{M} \models T$ is $\aleph_0$-saturated and $X \subseteq M^n$ is definable, then $\mathrm{RM}(X) \geq \alpha + 1$ if and only if we can find $Y_0, Y_1, ...$ pairwise disjoint definable subsets of $X$ of Morley rank at least $\alpha$.

The next lemma shows that Morley rank has some basic properties that we would want for a good notion of dimension.

**Lemma 7.3.5.** Let $\mathcal{M} \models T$ be an $\mathcal{L}$-structure and let $X$ and $Y$ be definable subsets of $M^n$.

i) If $X \subseteq Y$, then $\mathrm{RM}(X) \leq \mathrm{RM}(Y)$.

ii) $\mathrm{RM}(X \cup Y)$ is the maximum of $\mathrm{RM}(X)$ and $\mathrm{RM}(Y)$.

iii) If $X$ is nonempty, then $\mathrm{RM}(X) = 0$ if and only if $X$ is finite.

*Proof.* Let $\mathcal{N}$ an $\aleph_0$-saturated elementary extension of $\mathcal{M}$, $X = \phi(\mathcal{M})$ and $Y = \psi(\mathcal{M})$.

i) We prove by induction on $\alpha$ that if $\phi(\mathcal{M}) \subseteq \psi(\mathcal{M})$, then

$$\mathrm{RM}^{\mathcal{N}}(\phi) \geq \alpha \implies \mathrm{RM}^{\mathcal{N}}(\psi) \geq \alpha.$$

Notice that if $\mathcal{M}$ is a model of a complete theory and $\mathcal{M} \prec \mathcal{N}$ then we have that if $\phi(\mathcal{M}) \subseteq \psi(\mathcal{M})$ then $\phi(\mathcal{N}) \subseteq \psi(\mathcal{N})$.

Base case). If $\mathrm{RM}^{\mathcal{N}}(\phi) \geq 0$, then $\phi(\mathcal{N}) \neq \emptyset$ and so $\psi(\mathcal{N}) \neq \emptyset$ because $\phi(\mathcal{N}) \subseteq \psi(\mathcal{N})$. Hence $\mathrm{RM}^{\mathcal{N}}(\psi) \geq 0$.

Limit case). Let $\alpha$ be limit, then

$$\begin{aligned} \mathrm{RM}^{\mathcal{N}}(\phi) \geq \alpha &\implies \mathrm{RM}^{\mathcal{N}}(\phi) \geq \beta \text{ for all } \beta < \alpha \\ &\implies \mathrm{RM}^{\mathcal{N}}(\psi) \geq \beta \text{ for all } \beta < \alpha \\ &\implies \mathrm{RM}^{\mathcal{N}}(\psi) \geq \alpha. \end{aligned}$$

Inductive case). Suppose the claim true for $\alpha$ and $\mathrm{RM}^{\mathcal{N}}(\phi) \geq \alpha + 1$, then there are $\mathcal{L}_N$ formulas $\psi_0, \psi_1, ...$ such that $\psi_0(\mathcal{N}), \psi_1(\mathcal{N}), ...$ is an infinite sequence of pairwise disjoint subsets of $\phi(\mathcal{N})$ and $\mathrm{RM}^{\mathcal{N}}(\psi_i) \geq \alpha$ for all $i \in \omega$. Notice now that $\phi(\mathcal{N}) \subseteq \psi(\mathcal{N})$, thus $\psi_0(\mathcal{N}), \psi_1(\mathcal{N}), ...$ is an infinite sequence of pairwise disjoint subsets of $\psi(\mathcal{N})$ such that $\mathrm{RM}^{\mathcal{N}}(\psi_i) \geq \alpha$ for all $i \in \omega$ and hence $\mathrm{RM}^{\mathcal{N}}(\psi) \geq \alpha + 1$.

ii) See [29, Lemma 6.2.7].

iii) First of all notice that $\phi(\mathcal{N}) \neq \emptyset$ because $\phi(\mathcal{M}) \neq \emptyset$, so $\mathrm{RM}^{\mathcal{N}}(\phi) \geq 0$. Notice also that if $\mathcal{M}$ is a model of a complete theory and $\mathcal{M} \prec \mathcal{N}$ then $\phi(\mathcal{M})$ is finite if and only if $\phi(\mathcal{N})$ is finite.

Suppose now that $\phi(\mathcal{M})$ is finite, then $\phi(\mathcal{N})$ is also finite and so it cannot be partitioned in infinitely many nonempty sets. Hence $\mathrm{RM}^{\mathcal{N}}(\phi) \ngeq 1$. Thus $\mathrm{RM}^{\mathcal{N}}(\phi) = 0$.

On the other side suppose that $\phi(\mathcal{M})$ is infinite, then $\phi(\mathcal{N})$ is also infinite. Let $\{a_0, a_1, ...\}$ be an injective enumeration of $\phi(\mathcal{N})$, then $\{a_0\}, \{a_0\}, ...$ is an infinite sequence of pairwise disjoint nonempty definable subsets of $\phi(\mathcal{N})$. Thus $\mathrm{RM}^{\mathcal{N}}(\phi) \geq 1$.

□

**Definition 7.3.6.** A theory $T$ is called *totally transcendental* if for all $\mathcal{M} \models T$ and $\mathcal{L}_M$-formulas $\phi$ we have that $\mathrm{RM}(\phi) < \infty$.

## 7.4 The Monster Model

The definition we just gave of Morley rank is rather awkward because even if a formula has parameters from $\mathcal{M} \models T$ we need to work in a $\aleph_0$-saturated elementary extension of $\mathcal{M}$ to calculate the Morley rank. To simplify proofs, we will then frequently adopt the expository device of assuming that we are working in a fixed, very large, saturated model of $T$.

Let $\mathfrak{M} \models T$ be saturated of cardinality $\kappa$, where $\kappa$ is "very large". We call $\mathfrak{M}$ the *monster model* of $T$. If $\mathcal{M} \models T$ and $|M| \leq \kappa$, then by Lemma 7.2.11 there is an elementary embedding of $\mathcal{M}$ into $\mathfrak{M}$. Moreover, if $\mathcal{M} \prec \mathfrak{M}$, $f : \mathcal{M} \to \mathcal{N}$ is elementary and $|N| < \kappa$ we can find $j : \mathcal{N} \to \mathfrak{M}$ elementary such that $j|M$ is the identity. Thus, if we focus our attention on models of $T$ of cardinality less than $\kappa$, we can view all models as elementary submodels of $\mathfrak{M}$.

There are several problems with this approach. First, we really want to prove theorems about all the models of $T$, not just the small ones. But if there are arbitrarily large saturated models of $T$, then we can prove something about all models of $T$ by proving it for larger and larger monster models. Second, and more problematic, for general theories $T$ there may not be any saturated models.

Notice though that if we restrict our attention to $\omega$-stable theories this is not a problem because by Theorem 7.2.9 there are saturated models of $T$ of cardinality $\kappa$ for each regular cardinal $\kappa$.

For the remainder we then make the following assumptions:

1. $\mathfrak{M}$ is a large saturated model of $T$;
2. all $\mathcal{M} \models T$ that we consider are elementary submodels of $\mathfrak{M}$ and $|M| < |\mathfrak{M}|$;
3. all sets $A$ of parameters that we consider are subsets of $\mathfrak{M}$ with $|A| < |\mathfrak{M}|$;
4. if $\phi(\overline{v}, \overline{a})$ is a formula with parameters, we assume $\overline{a} \in \mathfrak{M}$;
5. we write $\mathrm{tp}(\overline{a}/A)$ for $\mathrm{tp}^{\mathfrak{M}}(\overline{a}/A)$ and $S_n(A)$ for $S_n^{\mathfrak{M}}(A)$.

## 7.5 Morley Degree and Ranks of Types

If $X$ is a definable set of Morley rank $\alpha$, then we cannot partition $X$ into infinitely many pairwise disjoint definable subsets of Morley rank $\alpha$. Indeed, there is a number $d$ such that $X$ cannot be partitioned into more than $d$ definable sets of Morley rank $\alpha$.

**Proposition 7.5.1.** Let $\phi$ be an $\mathcal{L}_{\mathfrak{M}}$-formula with $\mathrm{RM}(\phi) = \alpha$ for some ordinal $\alpha$. Then there is a natural number $d$ such that if $\psi_0, ..., \psi_{n-1}$ are $\mathcal{L}_{\mathfrak{M}}$-formulas such that $\psi_0(\mathfrak{M}), ..., \psi_{n-1}(\mathfrak{M})$ are disjoint subsets of $\phi(\mathfrak{M})$ and $\mathrm{RM}(\psi_i) = \alpha$ for all $i \in \{0, ..., n-1\}$, then $n \leq d$.

We call $d$ the *Morley degree* of $\phi$ and write $\deg_M(\phi) = d$.

*Proof.* We build $S \subseteq 2^{<\omega}$ and $\{\phi_\sigma \mid \sigma \in S\}$ with the following properties.

i) $\phi_\emptyset = \phi$

ii) If $\sigma \in S$ and $\tau \subseteq \sigma$, then $\tau \in S$;

iii) $\mathrm{RM}(\phi_\sigma) = \alpha$ for all $\sigma \in S$

iv) If $\sigma \in S$ there are two cases to consider. If there is an $\mathcal{L}_{\mathfrak{M}}$-formula $\psi$ such that $\mathrm{RM}(\phi_\sigma \wedge \psi) = \mathrm{RM}(\phi_\sigma \wedge \neg\psi) = \alpha$, then $\sigma, 0$ and $\sigma, 1$ are in $S$, $\phi_{\sigma,0}$ is $\phi_\sigma \wedge \psi$ and $\phi_{\sigma,1}$ is $\phi_\sigma \wedge \neg\psi$. If there is no such $\psi$, then no $\tau \supset \sigma$ is in $S$.

The set $S$ is a binary tree, furthermore it is finite. Suppose indeed that $S$ is infinite, then, by König's Lemma, there is $f : \omega \to 2$ such that $f|n \in S$ for all $n \in \omega$. Let $\psi_n$ be the formula $\phi_{f|n} \wedge \neg\phi_{f|n+1}$ for $n = 1, 2, ...$ .

**Claim.** $\mathrm{RM}(\psi_n) = \alpha$ for every $n = 1, 2, ...$ and $\psi_1(\mathfrak{M}), \psi_2(\mathfrak{M}), ...$ are pairwise disjoint subsets of $\phi(\mathfrak{M})$.

*Proof.* Let $n \geq 1$. By construction $\phi_{f|n+1} = \phi_{f|n,\, f(n)} = \phi_{f|n} \wedge \psi$ for $\mathcal{L}_{\mathfrak{M}}$-formula $\psi$ such that $\mathrm{RM}(\phi_{f|n} \wedge \psi) = \mathrm{RM}(\phi_{f|n} \wedge \neg\psi) = \alpha$. Thus $\psi_n = \phi_{f|n} \wedge \neg\phi_{f|n+1} = \phi_{f|n} \wedge \neg(\phi_{f|n} \wedge \psi)$ and so $\psi_n \equiv \phi_{f|n} \wedge \neg\psi$. Then $\psi_n(\mathfrak{M}) = (\phi_{f|n} \wedge \neg\psi)(\mathfrak{M})$, so by Lemma 7.3.5 i) $\mathrm{RM}(\psi_n) = \mathrm{RM}(\phi_{f|n} \wedge \neg\psi) = \alpha$.

Let now $m, n \in \omega$ with $m > n$ and suppose that there exists $\overline{a} \in \mathfrak{M}$ such that $\mathfrak{M} \models \psi_m(\overline{a}) \wedge \psi_n(\overline{a})$. Notice that $\psi_m = \phi_{f|m} \wedge \neg\phi_{f|m+1}$. Furthermore, by construction we have that $\models \phi_{f|m} \rightarrow \phi_{f|n+1}$ and so that

$$\mathfrak{M} \models \psi_n(\overline{a}) \wedge \phi_{f|n+1}(\overline{a}).$$

As before $\phi_{f|n+1} = \phi_{f|n} \wedge \psi$ for $\mathcal{L}_{\mathfrak{M}}$-formula $\psi$ such that $\mathrm{RM}(\phi_{f|n} \wedge \psi) = \mathrm{RM}(\phi_{f|n} \wedge \neg\psi) = \alpha$ and $\psi_n \equiv \phi_{f|n} \wedge \neg\psi$. Hence

$$\mathfrak{M} \models (\phi_{f|n} \wedge \neg\psi)(\overline{a}) \wedge (\phi_{f|n} \wedge \psi)(\overline{a}),$$

a contradiction. Thus $\psi_m(\mathfrak{M}) \cap \psi_n(\mathfrak{M}) = \emptyset$.

□

By the claim above $\mathrm{RM}(\phi) \geq \alpha + 1$, a contradiction. Hence $S$ is finite.

Let $S_0 = \{\sigma \in S \mid \tau \notin S \text{ for all } \tau \supset \sigma\}$ be the terminal nodes of the tree S. Let $d = |S_0|$ and let $\{\psi_0, ..., \psi_{d-1}\}$ be an enumeration of $\{\phi_\sigma \mid \sigma \in S_0\}$. Then, $\mathrm{RM}(\psi_i) = \alpha$ for all $i$, $\phi(\mathfrak{M})$ is the disjoint union of of $\psi_0(\mathfrak{M}), ..., \psi_{d-1}(\mathfrak{M})$ and, for each $i$, there is no formula $\chi$ such that $\mathrm{RM}(\psi_i \wedge \chi) = \mathrm{RM}(\psi_i \wedge \neg\chi) = \alpha$.

Suppose that $\theta_0, ..., \theta_{n-1}$ are $\mathcal{L}_{\mathfrak{M}}$-formulas of Morley rank $\alpha$ such that the sequence $\theta_0(\mathfrak{M}), ..., \theta_{n-1}(\mathfrak{M})$ is a sequence of pairwise disjoint subsets of $\phi(\mathfrak{M})$. We claim that $n \leq d$. By our choice of $\psi_0, ..., \psi_{d-1}$, for each $i < d$, there is at most one $j < n$ such that $\mathrm{RM}(\psi_i \wedge \theta_j) = \alpha$. Thus if $n > d$, there is $\hat{j} < n$ such that $\mathrm{RM}(\psi_i \wedge \theta_{\hat{j}}) < \alpha$ for all $i < d$.

Notice now that

$$\mathfrak{M} \models \theta_{\hat{j}} \leftrightarrow \bigvee_{i=0}^{d-1} \psi_i \wedge \theta_{\hat{j}},$$

because $\phi(\mathfrak{M})$ is the disjoint union of of $\psi_0(\mathfrak{M}), ..., \psi_{d-1}(\mathfrak{M})$. Thus by Lemma 7.3.5 $\mathrm{RM}(\theta_{\hat{j}}) < \alpha$, a contradiction.

□

We extend the definitions of Morley rank and degree from formulas to types.

**Definition 7.5.2.** If $p \in S_n(A)$, then $\mathrm{RM}(p) = \inf\{\mathrm{RM}(\phi) \mid \phi \in p\}$. If $\mathrm{RM}(p)$ is an ordinal, then $\deg_M(p) = \inf\{\deg_M(\phi) \mid \phi \in p \text{ and } \mathrm{RM}(\phi) = \mathrm{RM}(p)\}$.

Notice that if $p \in S_n(A)$ with $n \neq 0$, then $\mathrm{RM}(p) \neq -1$. Also $\mathrm{RM}(p)$ is an ordinal if and only if $\mathrm{RM}(p) < \infty$ if and only if there exists $\phi \in p$ such that $\mathrm{RM}(\phi) < \infty$.

If $\mathrm{RM}(p) < \infty$, then $(\mathrm{RM}(p), \deg_M(p))$ is the minimum element of $\{(\mathrm{RM}(\phi), \deg_M(\phi)) \mid \phi \in p\}$ in the lexicographic order. For each type $p$ with $\mathrm{RM}(p) < \infty$, we can find a formula $\phi_p \in p$ such that $(\mathrm{RM}(p), \deg_M(p)) = (\mathrm{RM}(\phi_p), \deg_M(\phi_p))$.

**Lemma 7.5.3.** If $p, q \in S_n(A)$, $\mathrm{RM}(p), \mathrm{RM}(q) < \infty$ and $p \neq q$, then $\phi_p \neq \phi_q$.

*Proof.* If $p \neq q$, then there is a formula $\psi$ such that $\psi \in p$ and $\neg\psi \in q$. Because $\phi_p \wedge \psi \in p$ and $\mathrm{RM}(\phi_p)$ is minimal,

$$\mathrm{RM}(\phi_p \wedge \psi) = \mathrm{RM}(\phi_p) = \mathrm{RM}(p).$$

Similarly

$$\mathrm{RM}(\phi_q \wedge \neg\psi) = \mathrm{RM}(\phi_q) = \mathrm{RM}(q).$$

If $\phi_p = \phi_q$, then

$$\mathrm{RM}(\phi_p \wedge \psi) = \mathrm{RM}(\phi_p \wedge \neg\psi) = \mathrm{RM}(\phi_p).$$

Thus, $\deg_M(\phi_p \wedge \psi) < \deg_M(\phi_p)$, contradicting our choice of $\phi_p$.

□

**Theorem 7.5.4.** If $T$ is $\omega$-stable, then $T$ is totally transcendental. Conversely, if $\mathcal{L}$ is countable and $T$ is totally transcendental, then $T$ is $\omega$-stable.

*Proof.* See [29, Theorem 6.2.14].

□

**Definition 7.5.5.** If $A \subseteq \mathfrak{M}$ and $\overline{a} \in \mathfrak{M}$, we write $\mathrm{RM}(\overline{a})$ for $\mathrm{RM}(\mathrm{tp}(\overline{a}))$ and $\mathrm{RM}(\overline{a}/A)$ for $\mathrm{RM}(\mathrm{tp}(\overline{a})/A)$.

## 7.6 Strongly Minimal Sets

**Definition 7.6.1.** Let $\mathcal{M}$ be an $\mathcal{L}$-structure and let $D \subseteq M^n$ be an infinite definable set. We say that $D$ is *minimal* in $\mathcal{M}$ if for every definable $Y \subseteq D$ either $Y$ is finite or $D \setminus Y$ is finite. If $\phi(\overline{v}, \overline{a})$ is the formula that defines $D$, then we also say that $\phi(\overline{v}, \overline{a})$ is minimal.

We say that $D$ and $\phi$ are *strongly minimal* if $\phi$ is minimal in any elementary extension $\mathcal{N}$ of $\mathcal{M}$.

We say that a theory $T$ is strongly minimal if the formula $v = v$ is strongly minimal (i.e. if $\mathcal{M} \models T$, then $M$ is strongly minimal).

**Lemma 7.6.2** (Exchange Principle)**.** Let $D \subseteq M$ be strongly minimal, $A \subseteq D$ and $a, b \in D$. If $a \in \mathrm{acl}(A \cup \{b\}) \setminus \mathrm{acl}(A)$, then $b \in \mathrm{acl}(A \cup \{a\})$.

*Proof.* See [29, Lemma 6.1.4].

□

**Theorem 7.6.3.** Let $T$ be a strongly minimal theory. If $\mathcal{M}, \mathcal{N} \models T$, then $\mathcal{M} \cong \mathcal{N}$ if and only if $\dim(M) = \dim(N)$.

*Proof.* See [29, Theorem 6.1.11].

□

**Corollary 7.6.4.** If $T$ is a strongly minimal theory, then $T$ is $\kappa$-categorical for $\kappa \geq \aleph_1$.

*Proof.* Let $\mathcal{M}, \mathcal{N} \models T$ with $|M| = |N| = \kappa \geq \aleph_1$, then $\dim(M) = \kappa = \dim(N)$ and so by Theorem 7.6.3 we have that $\mathcal{M} \cong \mathcal{N}$.

□

In $\omega$-stable theories, we can always find minimal formulas.

**Lemma 7.6.5.** Let $T$ be $\omega$-stable.

1. If $\mathcal{M} \models T$, then there is a minimal formula in $\mathcal{M}$.
2. If $\mathcal{M} \models T$ is $\aleph_0$-saturated and $\phi(\overline{v}, \overline{a})$ is a minimal formula in $\mathcal{M}$, then $\phi(\overline{v}, \overline{a})$ is strongly minimal.

*Proof.* See [29, Lemma 6.1.13].

□

Using Lemma 7.6.5 it is possible to prove the following theorem.

**Theorem 7.6.6.** Let $T$ be a complete theory in a countable language with infinite models and let $\kappa$ be an uncountable cardinal. If $T$ is $\kappa$-categorical, then $T$ is $\omega$-stable.

*Proof.* See [29, Theorem 6.1.18].

□

This last theorem can be used to show that several well-known theories are $\omega$-stable. In Sections 8.6 and 8.7 we will apply it to deduce the $\omega$-stability of the formal theory of infinite vector spaces over a fixed countable field $\mathbb{K}$ and the $\omega$-stability of ACF.

# Chapter 8

# Forking Independence in $\omega$-Stable Theories

The notion of forking was introduced by Saharon Shelah [36] in order to study stable theories. With this notion it is possible to define a fine independence relation, the forking independence relation, which is a ternary relation between a tuple of elements and a pair of subsets of a model of a stable theory.

In this last chapter we analyze the forking independence relation in $\omega$-stable theories. The restriction to $\omega$-stable theories allows for an easier formulation of the notion of forking and a consequently easier study of the forking independence relation.

This form of independence, although being a generalization of the forms of independence studied in the previous chapters, is rather different from them from an abstract point of view. Indeed, in this case the ternary independence relation is defined directly, and not via an absolute notion of independence which in turn is defined via the negation of a notion of dependence, as happens for the other cases.

For this reason the atoms that will lead the scene here will be the independence and the conditional independence atoms. Indeed, we will not deal with the absolute independence atoms and furthermore, we will define the dependence atom in function of the conditional independence one.

An interesting result is that the case of dependence that we consider here reduces to the case of dependence that we dealt with in Chapter 5.

The systems that we are going to study are: $\omega$-Stable Independence Logic, $\omega$-Stable Conditional Independence Logic and $\omega$-Stable Atomic Dependence Logic.

## 8.1 Forking in $\omega$-Stable Theories

In this section firstly we define the notion of forking in $\omega$-stable theories and see that in these theories non-forking extensions always exist. Then we define the forking independence relation and prove some fundamental results about it.

Let $T$ be an $\omega$-stable theory.

**Definition 8.1.1.** Let $\mathcal{M} \models T$, $A \subseteq B \subseteq M$, $p \in S_n(A)$, $q \in S_n(B)$ and $p \subseteq q$. If $\mathrm{RM}(q) < \mathrm{RM}(p)$, we say that $q$ is a *forking extension* of $p$ and that $q$ forks over $A$. If $\mathrm{RM}(q) = \mathrm{RM}(p)$, we say that $q$ is a *non-forking extension* of $p$.

**Theorem 8.1.2** (Existence of non-forking extensions)**.** Suppose that $p \in S_n(A)$ and $A \subseteq B$.

i) There is $q \in S_n(B)$ a non-forking extension of $p$.

ii) There are at most $\deg_M(p)$ non-forking extensions of $p$ in $S_n(B)$, and, if $\mathcal{M}$ is an $\aleph_0$-saturated model with $A \subseteq M$, there are exactly $\deg_M(p)$ non-forking extensions of $p$ in $S_n(M)$.

iii) There is at most one $q \in S_n(B)$, a non-forking extension of $p$ with $\deg_M(p) = \deg_M(q)$. In particular, if $\deg_M(p) = 1$, then $p$ has a unique non-forking extension in $S_n(B)$.

*Proof.* See [29, Theorem 6.3.2].

□

**Definition 8.1.3.** We say that $\overline{a}$ is *independent* from $B$ over $A$ if $\mathrm{tp}(\overline{a}/A \cup B)$ is a non-forking extension of $\mathrm{tp}(\overline{a}/A)$. We write $\overline{a} \mathop{\smile\!\!\!\!\!\!|}\,^{\mathrm{frk}}_A B$.

This notion of independence as many desirable properties.

**Lemma 8.1.4** (Monotonicity)**.** If $\overline{a} \mathop{\smile\!\!\!\!\!\!|}\,^{\mathrm{frk}}_A B$ and $C \subseteq B$, then $\overline{a} \mathop{\smile\!\!\!\!\!\!|}\,^{\mathrm{frk}}_A C$.

*Proof.* $A \subseteq A \cup C \subseteq A \cup B$ so $\mathrm{tp}(\overline{a}/A) \subseteq \mathrm{tp}(\overline{a}/A \cup C) \subseteq \mathrm{tp}(\overline{a}/A \cup B)$ and hence $\mathrm{RM}(\overline{a}/A) \geq \mathrm{RM}(\overline{a}/A \cup C) \geq \mathrm{RM}(\overline{a}/A \cup B)$. Thus if $\mathrm{RM}(\overline{a}/A) = \mathrm{RM}(\overline{a}/A \cup B)$, then $\mathrm{RM}(\overline{a}/A) = \mathrm{RM}(\overline{a}/A \cup C)$.

□

**Lemma 8.1.5** (Transitivity)**.** $\overline{a} \mathop{\smile\!\!\!\!\!\!|}\,^{\mathrm{frk}}_A \overline{b}\,\overline{c}$ if and only if $\overline{a} \mathop{\smile\!\!\!\!\!\!|}\,^{\mathrm{frk}}_A \overline{b}$ and $\overline{a} \mathop{\smile\!\!\!\!\!\!|}\,^{\mathrm{frk}}_{A \cup \overline{b}} \overline{c}$.

*Proof.* $\mathrm{RM}(\overline{a}/A \cup \overline{b}\,\overline{c}) \leq \mathrm{RM}(\overline{a}/A \cup \overline{b}) \leq \mathrm{RM}(\overline{a}/A)$, so $\mathrm{RM}(\overline{a}/A) = \mathrm{RM}(\overline{a}/A \cup \overline{b}\,\overline{c})$ if and only if $\mathrm{RM}(\overline{a}/A) = \mathrm{RM}(\overline{a}/A \cup \overline{b})$ and $\mathrm{RM}(\overline{a}/A \cup \overline{b}) = \mathrm{RM}(\overline{a}/A \cup \overline{b}\,\overline{c})$

□

**Lemma 8.1.6** (Finite Basis)**.** $\overline{a} \mathop{\smile\!\!\!\!\!\!|}\,^{\mathrm{frk}}_A B$ if and only if $\overline{a} \mathop{\smile\!\!\!\!\!\!|}\,^{\mathrm{frk}}_A B_0$ for all finite $B_0 \subseteq B$.

*Proof.* ($\Rightarrow$) Follows from Monotonicity.

($\Leftarrow$) Suppose that $\overline{a} \not\mathop{\smile\!\!\!\!\!\!|}\,^{\mathrm{frk}}_A B$. Then, there is $\phi(\overline{v}) \in \mathrm{tp}(\overline{a}/A \cup B)$ with $\mathrm{RM}(\phi) < \mathrm{RM}(\overline{a}/A)$. Let $B_0$ be a finite subset of $B$ such that $\phi$ is an $\mathcal{L}_{A \cup B_0}$-formula. Then $\overline{a} \not\mathop{\smile\!\!\!\!\!\!|}\,^{\mathrm{frk}}_A B_0$.

□

**Lemma 8.1.7** (Symmetry)**.** If $\overline{a} \mathop{\smile\!\!\!\!\!\!|}\,^{\mathrm{frk}}_A \overline{b}$, then $\overline{b} \mathop{\smile\!\!\!\!\!\!|}\,^{\mathrm{frk}}_A \overline{a}$.

*Proof.* See [29, Lemma 6.3.19].

□

**Corollary 8.1.8.** $\overline{a}\,\overline{b} \mathop{\smile\!\!\!\!\!\!|}\,^{\mathrm{frk}}_A C$ if and only if $\overline{a} \mathop{\smile\!\!\!\!\!\!|}\,^{\mathrm{frk}}_A C$ and $\overline{b} \mathop{\smile\!\!\!\!\!\!|}\,^{\mathrm{frk}}_{A \cup \overline{a}} C$.

*Proof.* Because forking occurs over a finite subset, it suffices to assume that $C$ is a finite sequence $\overline{c}$.

$$\begin{array}{rcll} \overline{a}\,\overline{b} \mathop{\smile\!\!\!\!\!\!\mid}^{\text{frk}}_A \overline{c} & \Leftrightarrow & \overline{c} \mathop{\smile\!\!\!\!\!\!\mid}^{\text{frk}}_A \overline{a}\,\overline{b} & \text{[by Symmetry]} \\ & \Leftrightarrow & \overline{c} \mathop{\smile\!\!\!\!\!\!\mid}^{\text{frk}}_A \overline{a} \text{ and } \overline{c} \mathop{\smile\!\!\!\!\!\!\mid}^{\text{frk}}_{A\cup\overline{a}} \overline{b} & \text{[by Transitivity]} \\ & \Leftrightarrow & \overline{a} \mathop{\smile\!\!\!\!\!\!\mid}^{\text{frk}}_A \overline{c} \text{ and } \overline{b} \mathop{\smile\!\!\!\!\!\!\mid}^{\text{frk}}_{A\cup\overline{a}} \overline{c} & \text{[by Symmetry].} \end{array}$$

□

**Corollary 8.1.9.** If $\overline{a} \mathop{\smile\!\!\!\!\!\!\mid}^{\text{frk}}_A \overline{b}$ and $\overline{a}\,\overline{b} \mathop{\smile\!\!\!\!\!\!\mid}^{\text{frk}}_A \overline{c}$, then $\overline{a} \mathop{\smile\!\!\!\!\!\!\mid}^{\text{frk}}_A \overline{b}\,\overline{c}$.

*Proof.*

$$\begin{array}{cl} \overline{a} \mathop{\smile\!\!\!\!\!\!\mid}^{\text{frk}}_A \overline{b} \text{ and } \overline{a}\,\overline{b} \mathop{\smile\!\!\!\!\!\!\mid}^{\text{frk}}_A \overline{c} & \\ \Downarrow & \\ \overline{a} \mathop{\smile\!\!\!\!\!\!\mid}^{\text{frk}}_A \overline{b} \text{ and } \overline{b}\,\overline{a} \mathop{\smile\!\!\!\!\!\!\mid}^{\text{frk}}_A \overline{c} & \\ \Downarrow & \\ \overline{a} \mathop{\smile\!\!\!\!\!\!\mid}^{\text{frk}}_A \overline{b} \text{ and } \overline{a} \mathop{\smile\!\!\!\!\!\!\mid}^{\text{frk}}_{A\cup\overline{b}} \overline{c} & \text{[by Corollary 8.1.8]} \\ \Downarrow & \\ \overline{a} \mathop{\smile\!\!\!\!\!\!\mid}^{\text{frk}}_A \overline{b}\,\overline{c} & \text{[by Transitivity].} \end{array}$$

□

**Corollary 8.1.10.** For any $\overline{b}$, $\overline{b} \mathop{\smile\!\!\!\!\!\!\mid}^{\text{frk}}_A \text{acl}(A)$.

*Proof.* Let $\overline{c} \in \text{acl}(A)$, then by Lemma 5.1.3 there exists $\phi(\overline{v}, \overline{w})$ and $\overline{a} \in A$ such that $\mathfrak{M} \models \phi(\overline{c}, \overline{a})$ and $|\phi(\mathfrak{M}, \overline{a})| < \infty$. Thus $\text{RM}(\phi(\overline{v}, \overline{a})) = 0$. Furthermore $\phi(\overline{v}, \overline{a}) \in \text{tp}(\overline{c}/A) \subseteq \text{tp}(\overline{c}/A, \overline{b})$, so $\text{RM}(\overline{c}/A, \overline{b}) = \text{RM}(\overline{c}/A)$. Thus $\overline{c} \mathop{\smile\!\!\!\!\!\!\mid}^{\text{frk}}_A \overline{b}$ and, by symmetry, $\overline{b} \mathop{\smile\!\!\!\!\!\!\mid}^{\text{frk}}_A \overline{c}$. Hence by finite basis $\overline{b} \mathop{\smile\!\!\!\!\!\!\mid}^{\text{frk}}_A \text{acl}(A)$.

□

**Proposition 8.1.11.** If $\overline{a} \mathop{\smile\!\!\!\!\!\!\mid}^{\text{frk}}_B \overline{a}$, then $\overline{a} \mathop{\smile\!\!\!\!\!\!\mid}^{\text{frk}}_B \overline{b}$ for any $\overline{b} \in \mathfrak{M}$.

*Proof.* If $\text{RM}(\overline{a}/B \cup \overline{a}) = \text{RM}(\overline{a}/B)$, then $\text{RM}(\overline{a}/B) = 0$ because $\text{RM}(\overline{a}/B \cup \overline{a}) = 0$. So there exists $\phi(\overline{v}) \in \text{tp}(\overline{a}/B)$ such that $|\phi(\mathfrak{M})| < \infty$ and thus by Lemma 5.1.3 we have that $\overline{a} \subseteq \text{acl}(B)$. Let now $\overline{b} \in \mathfrak{M}$, by Corollary 8.1.10 we have that $\overline{b} \mathop{\smile\!\!\!\!\!\!\mid}^{\text{cl}}_B \text{cl}(B)$ and hence by Lemma 8.1.4 and Lemma 8.1.7 we can conclude that $\overline{a} \mathop{\smile\!\!\!\!\!\!\mid}^{\text{cl}}_B \overline{b}$.

□

## 8.2 Theories with the FRK-Independence Property

In this section firstly we state a fundamental theorem that shows how in any strongly minimal set of a model of an $\omega$-stable theory the notion of Morley rank and the pregeometric notion of dimension coincide (with respect to a particular pregeometry). Then we define a particular class of $\omega$-stable theories, what we call the theories with the FRK-independence property. These theories are just $\omega$-stable theories whose monster model contains a strongly minimal set that has properties similar to the ones asked to a pregeometry in order to be a pregeometry with the independence property.

Combining the above mentioned theorem with these properties we will be able to prove a completeness theorem for the forking independence system that we define in the next section. Indeed the theorem will allow us to use the same strategy that we used in Chapter 6 and the properties that we ask will make this strategy succeed.

If $\mathcal{M}$ is a model of an $\omega$-stable theory and $F = \phi(\mathcal{M}, \overline{c})$ is a strongly minimal set we can define a pregeometry $(F, \mathrm{cl}_{\overline{c}})$ by defining $\mathrm{cl}_{\overline{c}}(X) = \mathrm{acl}(\overline{c} \cup X) \cap F$.

**Theorem 8.2.1.** Let $T$ be an $\omega$-stable theory, $F = \phi(\mathcal{M}, \overline{c})$ a strongly minimal set, $\overline{a} \in F$ and $A \subseteq F$. Then $\mathrm{RM}(\mathrm{tp}(\overline{a}/A \cup \overline{c})) = \dim(\overline{a}/A)$, where $\dim(\overline{a}/A)$ is computed in the pregeometry $(F, \mathrm{cl}_{\overline{c}})$.

*Proof.* Adaptation of [29, Theorem 6.2.19]. See also [33, Section 1.5]. □

**Definition 8.2.2.** Let $T$ be an $\omega$-stable theory. We say that $T$ has the FRK-independence property if there exists a strongly minimal set $F \subseteq \mathfrak{M}$ defined over the empty set of parameters such that the following conditions hold:

i) $\dim(F) = \aleph_0$;

ii) $\mathrm{cl}(\emptyset) \neq \emptyset$;

iii) if $D_0 \subseteq_{\mathrm{fin}} F$ is independent, then $\mathrm{cl}(D_0) \neq \bigcup_{D \subsetneq D_0} \mathrm{cl}(D)$;

where $(F, \mathrm{cl})$ is the pregeometry $\mathrm{cl}(X) = \mathrm{acl}(X) \cap F$.

As we saw in Lemma 7.6.5 in every model of an $\omega$-stable theory we can find a minimal set and if the model is also $\aleph_0$-saturated then this set is not only minimal but strongly minimal. For this reason, asking further conditions to be satisfied by this (strongly) minimal set seems to be a meaningful thing to do.

## 8.3 $\omega$-Stable Atomic Independence Logic

In this section we define the system $\omega$-Stable Atomic Independence Logic ($\omega$SAIndL) and then prove its soundness and completeness. The syntax and deductive apparatus of this system are the same as those of AIndL.

### 8.3.1 Semantics

Let $T$ be an $\omega$-stable theory with the FRK-independence property.

**Definition 8.3.1.** Let $\mathcal{M} \models T$ and $s : \mathrm{dom}(s) \to M$ with $\overline{x}\,\overline{y} \subseteq \mathrm{dom}(s) \subseteq \mathrm{Var}$. We say that $\mathcal{M}$ satisfies $\overline{x} \perp \overline{y}$ under $s$, in symbols $\mathcal{M} \models_s \overline{x} \perp \overline{y}$, if $s(\overline{x}) \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{frk}}_{\emptyset} s(\overline{y})$, that is the type $\mathrm{tp}(s(\overline{x})/s(\overline{y}))$ is a non-forking extension of $\mathrm{tp}(s(\overline{x})/\emptyset)$.

**Definition 8.3.2.** Let $\Sigma$ be a set of atoms and let $s$ be such that the set of variables occurring in $\Sigma$ is included in $\mathrm{dom}(s)$. We say that $\mathcal{M}$ satisfies $\Sigma$ under $s$, in symbols $\mathcal{M} \models_s \Sigma$, if $\mathcal{M}$ satisfies every atom in $\Sigma$ under $s$.

**Definition 8.3.3.** Let $\Sigma$ be a set of atoms. We say that $\overline{x} \perp \overline{y}$ is a logical consequence of $\Sigma$, in symbols $\Sigma \models \overline{x} \perp \overline{y}$, if for every $\mathcal{M} \models T$ and $s$ such that the set of variables occurring in $\Sigma \cup \{\overline{x} \perp \overline{y}\}$ is included in $\mathrm{dom}(s)$ we have that

$$\text{if } \mathcal{M} \models_s \Sigma \text{ then } \mathcal{M} \models_s \overline{x} \perp \overline{y}.$$

### 8.3.2 Soundness and Completeness

**Theorem 8.3.4.** Let $\Sigma$ be a set of atoms, then

$$\Sigma \models \overline{x} \perp \overline{y} \text{ if and only if } \Sigma \vdash \overline{x} \perp \overline{y}.$$

[The deductive system to which we refer has been defined in Section 2.4.3.]

*Proof.* ($\Leftarrow$) ($a_3$.) Obvious.

($b_3$.)

$$\begin{array}{lll} \mathcal{M} \models_s \overline{x} \perp \overline{y} & \Longrightarrow & s(\overline{x}) \downarrow^{\text{frk}}_{\emptyset} s(\overline{y}) \\ & \Longrightarrow & s(\overline{y}) \downarrow^{\text{frk}}_{\emptyset} s(\overline{x}) \quad \text{[By Lemma 8.1.7]} \\ & \Longrightarrow & \mathcal{M} \models_s \overline{y} \perp \overline{x}. \end{array}$$

($c_3$.)

$$\begin{array}{lll} \mathcal{M} \models_s \overline{x} \perp \overline{y}\,\overline{z} & \Longrightarrow & s(\overline{x}) \downarrow^{\text{frk}}_{\emptyset} s(\overline{y})s(\overline{z}) \\ & \Longrightarrow & s(\overline{x}) \downarrow^{\text{frk}}_{\emptyset} s(\overline{y}) \quad \text{[By Lemma 8.1.4]} \\ & \Longrightarrow & \mathcal{M} \models_s \overline{x} \perp \overline{y}. \end{array}$$

($d_3$.)

$$\begin{array}{c} \mathcal{M} \models_s \overline{x} \perp \overline{y} \text{ and } \mathcal{M} \models_s \overline{x}\,\overline{y} \perp \overline{z} \\ \Downarrow \\ s(\overline{x}) \downarrow^{\text{frk}}_{\emptyset} s(\overline{y}) \text{ and } s(\overline{x})s(\overline{y}) \downarrow^{\text{frk}}_{\emptyset} s(\overline{z}) \\ \Downarrow \\ s(\overline{x}) \downarrow^{\text{frk}}_{\emptyset} s(\overline{y})s(\overline{z}) \qquad \text{[By Corollary 8.1.9]} \\ \Downarrow \\ \mathcal{M} \models_s \overline{x} \perp \overline{y}\,\overline{z}. \end{array}$$

($e_3$.) Suppose that $\mathcal{M} \models_s x \perp x$, then $s(x) \downarrow^{\text{cl}}_{\emptyset} s(x)$ and so by Proposition 8.1.11 we have that $s(x) \downarrow^{\text{cl}}_{\emptyset} s(\overline{y})$ for any $\overline{y} \in \text{Var}$.

($f_3$.) Obvious.

($\Rightarrow$) Suppose $\Sigma \nvdash \overline{x} \perp \overline{y}$. Notice that if this is the case then $\overline{x} \neq \emptyset$ and $\overline{y} \neq \emptyset$. Indeed if $\overline{y} = \emptyset$ then $\Sigma \vdash \overline{x} \perp \overline{y}$ because by rule ($a_3$.) $\vdash \overline{x} \perp \emptyset$. Analogously if $\overline{x} = \emptyset$ then $\Sigma \vdash \overline{x} \perp \overline{y}$ because by rule ($a_3$.) $\vdash \overline{y} \perp \emptyset$ and so by rule ($b_3$.) $\vdash \emptyset \perp \overline{y}$.

We can assume that $\overline{x}$ and $\overline{y}$ are injective. This is without loss of generality because clearly $\mathcal{M} \models_s \overline{x} \perp \overline{y}$ if and only if $\mathcal{M} \models_s \pi\overline{x} \perp \pi\overline{y}$, where $\pi : \text{Var}^{<\omega} \to \text{Var}^{<\omega}$ is the function that eliminates repetitions in finite sequences of variables.

Furthermore we can assume that $\overline{x} \perp \overline{y}$ is minimal, in the sense that if $\overline{x}' \subseteq \overline{x}$, $\overline{y}' \subseteq \overline{y}$ and $\overline{x}'\,\overline{y}' \neq \overline{x}\,\overline{y}$, then $\Sigma \vdash \overline{x}' \perp \overline{y}'$. This is for two reasons.

i) If $\overline{x} \perp \overline{y}$ is not minimal we can always find a minimal atom $\overline{x}^* \perp \overline{y}^*$ such that $\Sigma \nvdash \overline{x}^* \perp \overline{y}^*$, $\overline{x}^* \subseteq \overline{x}$ and $\overline{y}^* \subseteq \overline{y}$ $--$ just keep deleting elements of $\overline{x}$ and $\overline{y}$ until you obtain the desired property or until both $\overline{x}^*$ and $\overline{y}^*$ are singletons, in which case, due to the trivial independence rule ($a_3$.), $\overline{x}^* \perp \overline{y}^*$ is a minimal statement.

ii) For any $\overline{x}' \subseteq \overline{x}$ and $\overline{y}' \subseteq \overline{y}$ we have that if $\mathcal{M} \not\models_s \overline{x}' \perp \overline{y}'$ then $\mathcal{M} \not\models_s \overline{x} \perp \overline{y}$, for every $\mathcal{M}$ and $s$.

Let indeed $\overline{x} = \overline{x}'\overline{x}''$ and $\overline{y} = \overline{y}'\overline{y}''$, then

$$\begin{array}{llll} \mathcal{M} \models_s \overline{x}'\,\overline{x}'' \perp \overline{y}'\overline{y}'' & \Longrightarrow & s(\overline{x}')s(\overline{x}'') \mathop{\smile\!\!\!\!\!\!|}_{\emptyset}^{\text{frk}} s(\overline{y}')s(\overline{y}'') & \\ & \Longrightarrow & s(\overline{x}')s(\overline{x}'') \mathop{\smile\!\!\!\!\!\!|}_{\emptyset}^{\text{frk}} s(\overline{y}') & \text{[By Lemma 8.1.4]} \\ & \Longrightarrow & s(\overline{x}') \mathop{\smile\!\!\!\!\!\!|}_{\emptyset}^{\text{frk}} s(\overline{y}') & \text{[By Corollary 8.1.8]} \\ & \Longrightarrow & \mathcal{M} \models_s \overline{x}' \perp \overline{y}'. & \end{array}$$

Let then $\overline{x} = (x_{j_0}, ..., x_{j_{n-1}})$ and $\overline{y} = (y_{k_0}, ..., y_{k_{m-1}})$ be injective and such that $\overline{x} \perp \overline{y}$ is minimal.

Let $V = \{v \in \text{Var} \mid \Sigma \vdash v \perp v\}$ and $W = \text{Var} \setminus V$. We claim that $\overline{x}, \overline{y} \not\subseteq V$. We prove it only for $\overline{x}$, the other case is symmetrical. Suppose that $\overline{x} \subseteq V$, then for every $x \in \overline{x}$ we have that $\Sigma \vdash x \perp x$ so by rule (e$_3$.), (b$_3$.) and (d$_3$.)

$$\Sigma \vdash \overline{y} \perp x_{j_0} \quad \text{and} \quad \Sigma \vdash \overline{y}\,x_{j_0} \perp x_{j_1} \Rightarrow \Sigma \vdash \overline{y} \perp x_{j_0}\,x_{j_1},$$

$$\Sigma \vdash \overline{y} \perp x_{j_0}\,x_{j_1} \quad \text{and} \quad \Sigma \vdash \overline{y}\,x_{j_0}\,x_{j_1} \perp x_{j_2} \Rightarrow \Sigma \vdash \overline{y} \perp x_{j_0}\,x_{j_1}\,x_{j_2},$$

$$\vdots$$

$$\Sigma \vdash \overline{y} \perp x_{j_0} \cdots x_{j_{n-2}} \quad \text{and} \quad \Sigma \vdash \overline{y}\,x_{j_0} \cdots x_{j_{n-2}} \perp x_{j_{n-1}} \Rightarrow \Sigma \vdash \overline{y} \perp \overline{x}.$$

Hence by rule (b$_3$.) $\Sigma \vdash \overline{x} \perp \overline{y}$.

Thus $\overline{x} \cap W \neq \emptyset$ and $\overline{y} \cap W \neq \emptyset$. Without loss of generality suppose that $x_{j_0} \in W$ and $y_{k_0} \in W$. Let $\overline{x} \cap W = \overline{x}' = (x_{j'_0}, ..., x_{j'_{n'-1}}) = (x_{j_0}, ..., x_{j'_{n'-1}}) \neq \emptyset$ and $\overline{y} \cap W = \overline{y}' = (y_{k'_0}, ..., y_{k'_{m'-1}})$. Notice that $\overline{x}' \cap \overline{y}' = \emptyset$. Indeed let $z \in \overline{x}' \cap \overline{y}'$, then by rules (b$_3$.) and (c$_3$.) we have that $\Sigma \vdash z \perp z$. Thus $z \in V$, a contradiction.

The theory $T$ has the FRK-independence property so by property i) of Definition 8.2.2 there exists a strongly minimal set $F \subseteq \mathfrak{M}$ defined over the empty set of parameters such that $\dim(F) = \aleph_0$, where, as in Definition 8.2.2, $\dim(F)$ is computed in the pregeometry $(F, \text{cl})$ defined as $\text{cl}(X) = \text{acl}(X) \cap F$.

Let $\{a_i \mid i \in \omega\}$ be an injective enumeration of a basis $B$ for $F$ and $\{w_i \mid i \in \lambda\}$ be an injective enumeration of $W \setminus \{x_{j_0}\}$. Let $s$ be the following assignment:

i) $s(v) = e$ for every $v \in V$,

ii) $s(w_i) = a_i$ for every $i \in \lambda$,

iii) $s(x_{j_0}) = d$,

where $e \in \text{cl}(\emptyset)$ and $d$ is such that

$$d \in \text{cl}(\left\{s(x_{j'_1}), ..., s(x_{j'_{n'-1}}), s(y_{k'_0}), ..., s(y_{k'_{m'-1}})\right\})$$

but $d \notin \text{cl}(D)$ for every $D \subsetneq \left\{s(x_{j'_1}), ..., s(x_{j'_{n'-1}}), s(y_{k'_0}), ..., s(y_{k'_{m'-1}})\right\}$. Notice that $e$ and $d$ do exist because of properties ii) and iii) of Definition 8.2.2.

Notice that by Theorem 8.2.1 for every $\overline{x}, \overline{y} \in \text{Var}$, we have that

$$s(\overline{x}) \mathop{\perp}_{\emptyset}^{\text{frk}} s(\overline{y}) \text{ if and only if } s(\overline{x}) \mathop{\perp}_{\emptyset}^{\text{cl}} s(\overline{y}). \qquad (\star)$$

We claim that $\mathfrak{M} \not\models_s \overline{x}' \perp \overline{y}'$, as noticed this implies that $\mathfrak{M} \not\models_s \overline{x} \perp \overline{y}$.

First we show that the set $\{s(x') \mid x' \in \overline{x}'\}$ is independent. By construction $s(x_{j_0}) \notin \mathrm{cl}(\{s(x') \mid x' \in \overline{x}'\} \setminus \{s(x_{j_0})\})$. Let then $i \in \{1, ..., n'-1\}$ and suppose that $s(x_{j'_i}) \in \mathrm{cl}(\Big\{s(x_{j_0}), ..., s(x_{j'_{i-1}}), s(x_{j'_{i+1}}), ..., s(x_{j'_{n-1}})\Big\})$.

The set $\Big\{s(x_{j'_1}), ..., s(x_{j'_{i-1}}), s(x_{j'_{i+1}}), ..., s(x_{j'_{n-1}})\Big\}$ is independent, so

$$s(x_{j'_i}) \in \mathrm{cl}(\Big\{s(x_{j'_1}), ..., s(x_{j'_{i-1}}), s(x_{j'_{i+1}}), ..., s(x_{j'_{n-1}})\Big\} \cup \{s(x_{j_0})\})$$

but

$$s(x_{j'_i}) \notin \mathrm{cl}(\Big\{s(x_{j'_1}), ..., s(x_{j'_{i-1}}), s(x_{j'_{i+1}}), ..., s(x_{j'_{n-1}})\Big\}).$$

Hence by the Exchange Principle

$$s(x_{j_0}) \in \mathrm{cl}(\Big\{s(x_{j'_1}), ..., s(x_{j'_{n-1}})\Big\}),$$

a contradiction. Thus $\dim(s(\overline{x}')) = |\,\{s(x') \mid x' \in \overline{x}'\}\,|$.

We now show that $\Big\{s(x_{j'_1}), ..., s(x_{j'_{n-1}})\Big\}$ is a basis for $\{s(x') \mid x' \in \overline{x}'\}$ over $\{s(y') \mid y' \in \overline{y}'\}$.

As we noticed above $\overline{x}' \cap \overline{y}' = \emptyset$, so by properties of our assignment $s(\overline{x}') \cap s(\overline{y}') = \emptyset$. Thus, by Lemma 6.3.11, $\Big\{s(x_{j'_1}), ..., s(x_{j'_{n-1}})\Big\}$ is independent over $\{s(y') \mid y' \in \overline{y}'\}$, also $\Big\{s(x_{j_0}), ..., s(x_{j'_{n-1}})\Big\} \subseteq \mathrm{cl}(s(\overline{y}') \cup \Big\{s(x_{j'_1}), ..., s(x_{j'_{n-1}})\Big\})$ because $s(x_{j_0}) \in \mathrm{cl}(\Big\{s(x_{j'_1}), ..., s(x_{j'_{n'-1}}), s(y_{k'_0}), ..., s(y_{k'_{m'-1}})\Big\})$.

Hence $\dim(s(\overline{x}')/s(\overline{y}')) = |\,\Big\{s(x_{j'_1}), ..., s(x_{j'_{n-1}})\Big\}\,| = \dim(s(\overline{x}')) - 1$. So $s(\overline{x}') \not\downarrow^{\mathrm{cl}}_{\emptyset} s(\overline{y}')$ and then by $(\star)$ we have that $s(\overline{x}') \not\downarrow^{\mathrm{frk}}_{\emptyset} s(\overline{y}')$, that is $\mathfrak{M} \not\models_s \overline{x}' \perp \overline{y}'$.

Let now $\overline{v} \perp \overline{w} \in \Sigma$, we want to show that $\mathfrak{M} \models_s \overline{v} \perp \overline{w}$. As before, we assume, without loss of generality, that $\overline{v}$ and $\overline{w}$ are injective. Notice also that if $\overline{v} = \emptyset$ or $\overline{w} = \emptyset$, then $\mathfrak{M} \models_s \overline{v} \perp \overline{w}$. Thus let $\overline{v}, \overline{w} \neq \emptyset$.

**Case 1.** $\overline{v} \subseteq V$ or $\overline{w} \subseteq V$.

Suppose that $\overline{v} \subseteq V$, the other case is symmetrical, then $s(\overline{v}) \subseteq \mathrm{cl}(\emptyset)$. Thus $\dim(s(\overline{v})/s(\overline{w})) = 0 = \dim(s(\overline{v}))$. So $s(\overline{v}) \downarrow^{\mathrm{cl}}_{\emptyset} s(\overline{w})$ and hence by $(\star)$ we have that $s(\overline{v}) \downarrow^{\mathrm{frk}}_{\emptyset} s(\overline{w})$, that is $\mathfrak{M} \models_s \overline{v} \perp \overline{w}$.

**Case 2.** $\overline{v} \not\subseteq V$ and $\overline{w} \not\subseteq V$.

Let $\overline{v} \cap W = \overline{v}' \neq \emptyset$ and $\overline{w} \cap W = \overline{w}' \neq \emptyset$.

Notice that

$$s(\overline{v}) \mathop{\downarrow}^{\mathrm{frk}}_{\emptyset} s(\overline{w}) \text{ if and only if } s(\overline{v'}) \mathop{\downarrow}^{\mathrm{frk}}_{\emptyset} s(\overline{w'}). \qquad (\star\star)$$

Left to right holds in general. As for the other direction, suppose that $s(\overline{v}') \downarrow^{\mathrm{frk}}_{\emptyset} s(\overline{w}')$. If $u \in \overline{v}\,\overline{w} \setminus \overline{v'}\,\overline{w'}$, then $s(u) = e \in \mathrm{cl}(\emptyset)$. Thus

$$
\begin{array}{c}
s(\overline{v}') \mathop{\smile\!\!\!\!\!\!\big|}\nolimits^{\text{frk}}_{\emptyset} s(\overline{w}') \\
\Downarrow \\
s(\overline{v}') \mathop{\smile\!\!\!\!\!\!\big|}\nolimits^{\text{cl}}_{\emptyset} s(\overline{w}') \text{ and } s(\overline{v}')\, s(\overline{w}') \mathop{\smile\!\!\!\!\!\!\big|}\nolimits^{\text{cl}}_{\emptyset} \text{cl}(\emptyset) \quad \text{[By } (\star) \text{ and Lemma 6.3.18 ]} \\
\Downarrow \\
s(\overline{v}') \mathop{\smile\!\!\!\!\!\!\big|}\nolimits^{\text{cl}}_{\emptyset} s(\overline{w}') \cup \text{cl}(\emptyset) \\
\Downarrow \\
s(\overline{v}') \mathop{\smile\!\!\!\!\!\!\big|}\nolimits^{\text{cl}}_{\emptyset} s(\overline{w}') \cup (\text{cl}(\emptyset) \cap s(\overline{w})) \\
\Downarrow \\
s(\overline{v}') \mathop{\smile\!\!\!\!\!\!\big|}\nolimits^{\text{cl}}_{\emptyset} s(\overline{w}).
\end{array}
$$

So

$$
\begin{array}{c}
s(\overline{w}) \mathop{\smile\!\!\!\!\!\!\big|}\nolimits^{\text{cl}}_{\emptyset} s(\overline{v}') \text{ and } s(\overline{w})\, s(\overline{v}') \mathop{\smile\!\!\!\!\!\!\big|}\nolimits^{\text{cl}}_{\emptyset} \text{cl}(\emptyset) \quad \text{[By Lemma 6.3.18 ]} \\
\Downarrow \\
s(\overline{w}) \mathop{\smile\!\!\!\!\!\!\big|}\nolimits^{\text{cl}}_{\emptyset} s(\overline{v}') \cup \text{cl}(\emptyset) \\
\Downarrow \\
s(\overline{w}) \mathop{\smile\!\!\!\!\!\!\big|}\nolimits^{\text{cl}}_{\emptyset} s(\overline{v}') \cup (\text{cl}(\emptyset) \cap s(\overline{v})) \\
\Downarrow \\
s(\overline{w}) \mathop{\smile\!\!\!\!\!\!\big|}\nolimits^{\text{cl}}_{\emptyset} s(\overline{v}) \\
\Downarrow \\
s(\overline{w}) \mathop{\smile\!\!\!\!\!\!\big|}\nolimits^{\text{frk}}_{\emptyset} s(\overline{v}). \qquad \text{[By } (\star)\text{]}
\end{array}
$$

**Subcase 2.1.** $x_{j_0} \notin \overline{v}'\, \overline{w}'$.

Notice that $\overline{v}' \cap \overline{w}' = \emptyset$, so by properties of our assignment $s(\overline{v}') \cap s(\overline{w}') = \emptyset$. Thus, by Lemma 6.3.11, it follows that $\dim(s(\overline{v}')/s(\overline{w}')) = \dim(s(\overline{v}'))$. So $s(\overline{v}') \mathop{\smile\!\!\!\!\!\!\big|}\nolimits^{\text{cl}}_{\emptyset} s(\overline{w}')$ and then by $(\star)$ we have that $s(\overline{v}') \mathop{\smile\!\!\!\!\!\!\big|}\nolimits^{\text{frk}}_{\emptyset} s(\overline{w}')$ which by $(\star\star)$ implies that $s(\overline{v}) \mathop{\smile\!\!\!\!\!\!\big|}\nolimits^{\text{frk}}_{\emptyset} s(\overline{w})$, that is $\mathfrak{M} \models_s \overline{v} \perp \overline{w}$.

**Subcase 2.2.** $x_{j_0} \in \overline{v}'\, \overline{w}'$.

**Subsubcase 2.2.1.** $\overline{x}'\, \overline{y}' \setminus \overline{v}'\, \overline{w}' \neq \emptyset$.

Let $\overline{v}'\, \overline{w}' \setminus \{x_{j_0}\} \cap \overline{x}'\, \overline{y}' = \left\{u_{h'_0}, ..., u_{h'_{b-1}}\right\}$, $\overline{v}'\, \overline{w}' \setminus \overline{x}'\, \overline{y}' = \left\{u_{h''_0}, ..., u_{h''_{t-1}}\right\}$, $w_{r'_i} = u_{h'_i}$ for every $i \in \{0, ..., b-1\}$ and $w_{r''_i} = u_{h''_i}$ for every $i \in \{0, ..., t-1\}$

Suppose now that the set $\left\{a_{r'_0}, ..., a_{r'_{b-1}}, d, a_{r''_0}, ..., a_{r''_{t-1}}\right\}$ is dependent. There are three cases.

Case 1. $a_{r''_l} \in \text{cl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}, d, a_{r''_0}, ..., a_{r''_{l-1}}, a_{r''_{l+1}}, ..., a_{r''_{t-1}}\right\})$.

If this is the case, then

$$a_{r''_l} \in \text{cl}(\left\{s(x_{j'_1}), ..., s(x_{j'_{n'-1}}), s(y_{k'_0}), ..., s(y_{k'_{m'-1}}), a_{r''_0}, ..., a_{r''_{l-1}}, a_{r''_{l+1}}, ..., a_{r''_{t-1}}\right\})$$

because $d \in \text{cl}(\left\{s(x_{j'_1}), ..., s(x_{j'_{n'-1}}), s(y_{k'_0}), ..., s(y_{k'_{m'-1}})\right\})$. This is absurd though because the set $\left\{s(x_{j'_1}), ..., s(x_{j'_{n'-1}}), s(y_{k'_0}), ..., s(y_{k'_{m'-1}}), a_{r''_0}, ..., a_{r''_{t-1}}\right\}$ is made of distinct elements of the basis $B$ and so it is independent.

Case 2. $d \in \text{cl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}, a_{r''_0}, ..., a_{r''_{t-1}}\right\})$.

Notice that

$$d \notin \text{cl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}\right\})$$

because $\left\{a_{r'_0}, ..., a_{r'_{b-1}}\right\} \subsetneq \left\{s(x_{j'_1}), ..., s(x_{j'_{n'-1}}), s(y_{k'_0}), ..., s(y_{k'_{m'-1}})\right\}$ and $d$ has been chosen such that $d \in \mathrm{cl}(\left\{s(x_{j'_1}), ..., s(x_{j'_{n'-1}}), s(y_{k'_0}), ..., s(y_{k'_{m'-1}})\right\})$ but $d \notin \mathrm{cl}(D)$ for every $D \subsetneq \left\{s(x_{j'_1}), ..., s(x_{j'_{n'-1}}), s(y_{k'_0}), ..., s(y_{k'_{m'-1}})\right\}$.

Thus there is $l \leq t-1$ such that

$$d \in \mathrm{cl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}, a_{r''_0}, ..., a_{r''_{l-1}}\right\} \cup \left\{a_{r''_l}\right\}) \setminus \mathrm{cl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}, a_{r''_0}, ..., a_{r''_{l-1}}\right\})$$

and then by the Exchange Principle we have that

$$a_{r''_l} \in \mathrm{cl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}, a_{r''_0}, ..., a_{r''_{l-1}}\right\} \cup \{d\}).$$

Thus we have that $a_{r''_l} \in \mathrm{cl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}, d, a_{r''_0}, ..., a_{r''_{l-1}}, a_{r''_{l+1}}, ..., a_{r''_{t-1}}\right\})$, which is impossible as we saw in Case 1.

Case 3. $a_{r'_c} \in \mathrm{cl}(\left\{a_{r'_0}, ..., a_{r'_{c-1}}, a_{r'_{c+1}}, ..., a_{r'_{b-1}}, d, a_{r''_0}, ..., a_{r''_{t-1}}\right\})$.

Notice that

$$a_{r'_c} \notin \mathrm{cl}(\left\{a_{r'_0}, ..., a_{r'_{c-1}}, a_{r'_{c+1}}, ..., a_{r'_{b-1}}, a_{r''_0}, ..., a_{r''_{t-1}}\right\}).$$

Thus by the Exchange Principle we have that $d \in \mathrm{cl}(\left\{a_{r'_0}, ..., a_{r'_{b-1}}, a_{r''_0}, ..., a_{r''_{t-1}}\right\})$, which is impossible as we saw in Case 2.

We can then conclude that the set $\left\{a_{r'_0}, ..., a_{r'_{b-1}}, d, a_{r''_0}, ..., a_{r''_{t-1}}\right\}$ is independent. Clearly $\{s(v') \mid v' \in \overline{v}'\} \cup \{s(w') \mid w' \in \overline{w}'\} = \left\{a_{r'_0}, ..., a_{r'_{b-1}}, d, a_{r''_0}, ..., a_{r''_{t-1}}\right\}$. Furthermore, as we noticed above, $s(\overline{v}') \cap s(\overline{w}') = \emptyset$. Thus by Lemma 6.3.11 we have that $\dim(s(\overline{v}')/s(\overline{w}')) = \dim(s(\overline{v}'))$. So $s(\overline{v}') \mathop{\smile\!\!\!\!\!\!|}\nolimits^{\mathrm{cl}}_{\emptyset} s(\overline{w}')$ and then by $(\star)$ we have that $s(\overline{v}') \mathop{\smile\!\!\!\!\!\!|}\nolimits^{\mathrm{frk}}_{\emptyset} s(\overline{w}')$ which by $(\star\star)$ implies that $s(\overline{v}) \mathop{\smile\!\!\!\!\!\!|}\nolimits^{\mathrm{frk}}_{\emptyset} s(\overline{w})$, that is $\mathfrak{M} \models_s \overline{v} \perp \overline{w}$.

**Subsubcase 2.2.2.** $\overline{x}' \, \overline{y}' \subseteq \overline{v}' \, \overline{w}'$.

As shown in Theorem 3.4.4, this case is not possible.

□

## 8.4 ω-Stable Atomic Conditional Independence Logic

In this section we define the system ω-Stable Atomic Conditional Independence Logic (ωSACIndL) and then prove its soundness. The syntax and deductive apparatus of this system are the same as those of ACIndL.

### 8.4.1 Semantics

Let $T$ an ω-stable theory.

**Definition 8.4.1.** Let $\mathcal{M} \models T$ and $s : \mathrm{dom}(s) \to M$ with $\overline{x}\,\overline{y}\,\overline{z} \subseteq \mathrm{dom}(s) \subseteq \mathrm{Var}$. We say that $\mathcal{M}$ satisfies $\overline{x} \perp_{\overline{z}} \overline{y}$ under $s$, in symbols $\mathcal{M} \models_s \overline{x} \perp_{\overline{z}} \overline{y}$, if $s(\overline{x}) \mathop{\smile\!\!\!\!\!\!|}\nolimits^{\mathrm{frk}}_{s(\overline{z})} s(\overline{y})$, that is the type $\mathrm{tp}(s(\overline{x})/s(\overline{z}) \cup s(\overline{y}))$ is a non-forking extension of $\mathrm{tp}(s(\overline{x})/s(\overline{z}))$.

**Definition 8.4.2.** Let $\Sigma$ be a set of atoms and let $s$ be such that the set of variables occurring in $\Sigma$ is included in $\mathrm{dom}(s)$. We say that $\mathcal{M}$ satisfies $\Sigma$ under $s$, in symbols $\mathcal{M} \models_s \Sigma$, if $\mathcal{M}$ satisfies every atom in $\Sigma$ under $s$.

**Definition 8.4.3.** Let $\Sigma$ be a set of atoms. We say that $\overline{x} \perp_{\overline{z}} \overline{y}$ is a logical consequence of $\Sigma$, in symbols $\Sigma \models \overline{x} \perp_{\overline{z}} \overline{y}$, if for every $\mathcal{M} \models T$ and $s$ such that the set of variables occurring in $\Sigma \cup \{\overline{x} \perp_{\overline{z}} \overline{y}\}$ is included in $\mathrm{dom}(s)$ we have that

$$\text{if } \mathcal{M} \models_s \Sigma \text{ then } \mathcal{M} \models_s \overline{x} \perp_{\overline{z}} \overline{y}.$$

### 8.4.2 Soundness

**Theorem 8.4.4.** Let $\Sigma$ be a set of atoms, then

$$\Sigma \vdash \overline{x} \perp_{\overline{z}} \overline{y} \quad \Rightarrow \quad \Sigma \models \overline{x} \perp_{\overline{z}} \overline{y}.$$

[The deductive system to which we refer has been defined in Section 2.6.3.]

*Proof.* ($a_5$.) $\mathrm{RM}(s(\overline{x})/s(\overline{x})) = 0 = \mathrm{RM}(s(\overline{x})/s(\overline{x}) \cup s(\overline{y}))$, thus $\mathcal{M} \models_s \overline{x} \perp_{\overline{x}} \overline{y}$.

($b_5$.)

$$\begin{array}{rcll} \mathcal{M} \models_s \overline{x} \perp_{\overline{z}} \overline{y} & \Longrightarrow & s(\overline{x}) \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{frk}}_{s(\overline{z})} s(\overline{y}) & \\ & \Longrightarrow & s(\overline{y}) \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{frk}}_{s(\overline{z})} s(\overline{x}) & \text{[By Lemma 8.1.7]} \\ & \Longrightarrow & \mathcal{M} \models_s \overline{y} \perp_{\overline{z}} \overline{x}. & \end{array}$$

($c_5$.)

$$\begin{array}{rcll} \mathcal{M} \models_s \overline{x}\,\overline{x}' \perp_{\overline{z}} \overline{y}\,\overline{y}' & \Longrightarrow & s(\overline{x})s(\overline{x}') \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{frk}}_{s(\overline{z})} s(\overline{y})s(\overline{y}') & \\ & \Longrightarrow & s(\overline{x})s(\overline{x}') \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{frk}}_{s(\overline{z})} s(\overline{y}) & \text{[By Lemma 8.1.4]} \\ & \Longrightarrow & s(\overline{x}) \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{frk}}_{s(\overline{z})} s(\overline{y}) & \text{[By Corollary 8.1.8]} \\ & \Longrightarrow & \mathcal{M} \models_s \overline{x} \perp_{\overline{z}} \overline{y}. & \end{array}$$

($d_5$.) Suppose that $\mathcal{M} \models \overline{x} \perp_{\overline{z}} \overline{y}$, then $s(\overline{x}) \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{frk}}_{s(\overline{z})} s(\overline{y})$ and so $s(\overline{x}) \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{frk}}_{s(\overline{z})} s(\overline{y})s(\overline{z})$ because $\mathrm{RM}(s(\overline{x})/s(\overline{z}) \cup s(\overline{y})) = \mathrm{RM}(s(\overline{x})/s(\overline{z}) \cup (s(\overline{y}) \cup s(\overline{z})))$. Furthermore $s(\overline{z}) \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{frk}}_{s(\overline{z}),s(\overline{x})} s(\overline{y})s(\overline{z})$ because $\mathrm{RM}(s(\overline{z})/s(\overline{z}) \cup s(\overline{x}) \cup s(\overline{y}) \cup s(\overline{z})) = 0 = \mathrm{RM}(s(\overline{z})/s(\overline{z}) \cup s(\overline{x}))$. Hence by Corollary 8.1.8 we have that $s(\overline{x})s(\overline{z}) \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{frk}}_{s(\overline{z})} s(\overline{y})s(\overline{z})$.

($e_5$.)

$$\begin{array}{ccc} \mathcal{M} \models_s \overline{x} \perp_{\overline{z}} \overline{y} & & \mathcal{M} \models_s \overline{u} \perp_{\overline{z},\overline{x}} \overline{y} \\ \Downarrow & & \Downarrow \\ s(\overline{x}) \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{frk}}_{s(\overline{z})} s(\overline{y}) & & s(\overline{u}) \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{frk}}_{s(\overline{z}),s(\overline{x})} s(\overline{y}) \end{array}$$

$$\begin{array}{cl} \Downarrow & \\ s(\overline{x})s(\overline{u}) \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{frk}}_{s(\overline{z})} s(\overline{y}) & \text{[By Corollary 8.1.8]} \\ \Downarrow & \\ s(\overline{u})s(\overline{x}) \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{frk}}_{s(\overline{z})} s(\overline{y}) & \\ \Downarrow & \\ s(\overline{u}) \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{frk}}_{s(\overline{z})} s(\overline{y}) & \text{[By Corollary 8.1.8]} \\ \Downarrow & \\ \mathcal{M} \models_s \overline{u} \perp_{\overline{z}} \overline{y} & \end{array}$$

(f$_5$.)

$$\begin{array}{ccc}
\mathcal{M} \models \overline{y} \perp_{\overline{z}} \overline{y} & & \mathcal{M} \models_s \overline{z}\,\overline{x} \perp_{\overline{y}} \overline{u} \\
\Downarrow & & \Downarrow \\
s(\overline{y}) \mathop{\smile\!\!\!\!\!\!\mid}^{\text{frk}}_{s(\overline{z})} s(\overline{y}) & & s(\overline{z})s(\overline{x}) \mathop{\smile\!\!\!\!\!\!\mid}^{\text{frk}}_{s(\overline{y})} s(\overline{u}) \\
\Downarrow & & \Downarrow \\
s(\overline{y}) \mathop{\smile\!\!\!\!\!\!\mid}^{\text{frk}}_{s(\overline{z})} s(\overline{y}) & & s(\overline{x}) \mathop{\smile\!\!\!\!\!\!\mid}^{\text{frk}}_{s(\overline{y}),s(\overline{z})} s(\overline{u}) \quad \text{[By Corollary 8.1.8]} \\
\Downarrow & & \Downarrow \\
s(\overline{y}) \mathop{\smile\!\!\!\!\!\!\mid}^{\text{frk}}_{s(\overline{z})} s(\overline{u}) & & s(\overline{x}) \mathop{\smile\!\!\!\!\!\!\mid}^{\text{frk}}_{s(\overline{y}),s(\overline{z})} s(\overline{u}) \quad \text{[By Proposition 8.1.11]} \\
\Downarrow & & \Downarrow \\
s(\overline{y}) \mathop{\smile\!\!\!\!\!\!\mid}^{\text{frk}}_{s(\overline{z})} s(\overline{u}) & & s(\overline{x}) \mathop{\smile\!\!\!\!\!\!\mid}^{\text{frk}}_{s(\overline{z}),s(\overline{y})} s(\overline{u}) \\
& \Downarrow & \\
& s(\overline{x}) \mathop{\smile\!\!\!\!\!\!\mid}^{\text{frk}}_{s(\overline{z})} s(\overline{u}) & \text{[By what we showed in (e}_5\text{.)]} \\
& \Downarrow & \\
& \mathcal{M} \models_s \overline{x} \perp_{\overline{z}} \overline{u} &
\end{array}$$

(g$_5$.)

$$\begin{array}{cc}
\mathcal{M} \models_s \overline{x} \perp_{\overline{z}} \overline{y} \text{ and } \mathcal{M} \models_s \overline{x}\,\overline{y} \perp_{\overline{z}} \overline{u} & \\
\Downarrow & \\
s(\overline{x}) \mathop{\smile\!\!\!\!\!\!\mid}^{\text{frk}}_{s(\overline{z})} s(\overline{y}) \text{ and } s(\overline{x})s(\overline{y}) \mathop{\smile\!\!\!\!\!\!\mid}^{\text{frk}}_{s(\overline{z})} s(\overline{u}) & \\
\Downarrow & \\
s(\overline{x}) \mathop{\smile\!\!\!\!\!\!\mid}^{\text{frk}}_{s(\overline{z})} s(\overline{y})s(\overline{u}) & \text{[By Corollary 8.1.9]} \\
\Downarrow & \\
\mathcal{M} \models_s \overline{x} \perp_{\overline{z}} \overline{y}\,\overline{u} &
\end{array}$$

(h$_5$.) Obvious.

□

The same considerations that we made for the pregeometric version of this system in Section 6.8.2 apply in this context. That is, in the light of what is known about conditional independence in database theory, it seems that it is not possible to find a finite and complete axiomatization for these atoms.

## 8.5 $\omega$-Stable Atomic Dependence Logic

As known [18], in dependence logic the dependence atom is expressible in terms of the conditional independence atom. Indeed given a FO-structure $\mathcal{M}$ and an appropriate $X$ we have that

$$\mathcal{M} \models_X =\!(\overline{x}, \overline{y}) \text{ if and only if } \mathcal{M} \models_X \overline{y} \perp_{\overline{x}} \overline{y}.$$

This remark justifies the condition that we use in the following definition of the system $\omega$-Stable Atomic Dependence Logic ($\omega$SADL). The syntax and deductive apparatus of this system are the same as those of ADL.

### 8.5.1 Semantics

Let $T$ an $\omega$-stable theory such that for every $\mathcal{M} \models T$ we have $\mathrm{acl}(\emptyset) \neq \emptyset$.

**Definition 8.5.1.** Let $\mathcal{M} \models T$ and $s : \mathrm{dom}(s) \to M$ with $\overline{x}\,\overline{y} \subseteq \mathrm{dom}(s) \subseteq \mathrm{Var}$. We say that $\mathcal{M}$ satisfies $=(\overline{x}, \overline{y})$ under $s$, in symbols $\mathcal{M} \models_s =(\overline{x}, \overline{y})$, if $s(\overline{y}) \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{frk}}_{s(\overline{x})} s(\overline{y})$.

**Definition 8.5.2.** Let $\Sigma$ be a set of atoms and let $s$ be such that the set of variables occurring in $\Sigma$ is included in $\mathrm{dom}(s)$. We say that $\mathcal{M}$ satisfies $\Sigma$ under $s$, in symbols $\mathcal{M} \models_s \Sigma$, if $\mathcal{M}$ satisfies every atom in $\Sigma$ under $s$.

**Definition 8.5.3.** Let $\Sigma$ be a set of atoms. We say that $=(\overline{x}, \overline{y})$ is a logical consequence of $\Sigma$, in symbols $\Sigma \models\ =(\overline{x}, \overline{y})$, if for every $\mathcal{M} \models T$ and $s$ such that the set of variables occurring in $\Sigma \cup \{= (\overline{x}, \overline{y})\}$ is included in $\mathrm{dom}(s)$ we have that

$$\text{if } \mathcal{M} \models_s \Sigma \text{ then } \mathcal{M} \models_s =(\overline{x}, \overline{y}).$$

### 8.5.2 Characterization of $\omega$SADL in terms of ACADL

Let $\models^{\mathrm{fork}}$ and $\models^{\mathrm{acl}}$ denote the satisfaction relation of $\omega$SADL and ACADL respectively.

**Theorem 8.5.4.** Let $\mathcal{M} \models T$ and $s : \mathrm{dom}(s) \to M$ with $\overline{x}\,\overline{y} \subseteq \mathrm{dom}(s) \subseteq \mathrm{Var}$, then

$$\mathcal{M} \models^{\mathrm{fork}}_s =(\overline{x}, \overline{y}) \text{ if and only if } \mathcal{M} \models^{\mathrm{acl}}_s =(\overline{x}, \overline{y}).$$

*Proof.*

$$\begin{aligned}
\mathcal{M} \models^{\mathrm{fork}}_s =(\overline{x}, \overline{y}) &\iff \mathcal{M} \models_s \overline{y} \perp_{\overline{x}} \overline{y} \\
&\iff s(\overline{y}) \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{frk}}_{s(\overline{x})} s(\overline{y}) \\
&\iff \mathrm{RM}(\mathrm{tp}(s(\overline{y})/s(\overline{x}) \cup s(\overline{y}))) = \mathrm{RM}(\mathrm{tp}(s(\overline{y})/s(\overline{x}))) \\
&\iff \mathrm{RM}(\mathrm{tp}(s(\overline{y})/s(\overline{x}))) = 0 \\
&\iff \exists \phi(\overline{v}) \in \mathrm{tp}(s(\overline{y})/s(\overline{x})) \text{ s.t. } |\phi(\mathcal{M})| < \infty \\
&\iff s(\overline{y}) \in \mathrm{acl}(s(\overline{x})) \quad \text{[By Lemma 5.1.3]} \\
&\iff \forall y \in \overline{y}\ s(y) \in \mathrm{acl}(s(\overline{x})) \\
&\iff \mathcal{M} \models^{\mathrm{acl}}_s =(\overline{x}, \overline{y}).
\end{aligned}$$

□

### 8.5.3 Soundness and Completeness

We know show that the system $\omega$SADL is sound and complete.

**Theorem 8.5.5.** Let $\Sigma$ be a set of atoms, then

$$\Sigma \models =(\overline{x}, \overline{y}) \text{ if and only if } \Sigma \vdash =(\overline{x}, \overline{y}).$$

[The deductive system to which we refer has been defined in Section 2.2.3.]

*Proof.* ($\Leftarrow$) Because of Theorem 8.5.4, this reduces to the soundness proof of Theorem 5.2.5.

($\Rightarrow$) Suppose $\Sigma \nvdash =(\overline{x}, \overline{y})$. Let $V = \{z \in \mathrm{Var} \mid \Sigma \vdash =(\overline{x}, z)\}$ and $W = \mathrm{Var} \backslash V$.

Let $\kappa > |\mathcal{L}| + \aleph_0$ where $\mathcal{L}$ is the signature of $T$. The theory $T$ has infinite models so by the Löwenheim-Skolem Theorem there is a structure $\mathcal{M}$ such that $\mathcal{M} \models T$ and $|M| = \kappa$. Notice now that if $|M| = \kappa$ then for every $m \in M$ we have that $\mathrm{acl}(\{m\}) \neq M$ because $|\mathrm{acl}(\{m\})| \leq |\mathcal{L}| + \aleph_0$.

Let then $a \in \mathrm{acl}(\emptyset)$, $b \in M \setminus \mathrm{acl}(\{a\})$ and let $s$ be the following assignment:

$$s(v) = \begin{cases} a & \text{if } v \in V \\ b & \text{if } v \in W. \end{cases}$$

Because of Theorem 8.5.4, the arguments used in Theorem 5.2.5 show that $\mathcal{M} \models_s \Sigma$ and $\mathcal{M} \not\models_s {=}(\overline{x}, \overline{y})$.

□

## 8.6 Forking in Vector Spaces

In this section firstly we define the formal theory of infinite vector spaces over a fixed field $\mathbb{K}$ and see that with respect to this theory the model-theoretic operator of algebraic closure coincides with the span operator. Then we show that if the field is countable then the theory is $\omega$-stable and has the FRK-independence property. Finally, combining the results of this section with the ones of Section 6.6, we conclude that, if we restrict to infinite vector spaces over a countable field, all the possible formulations of vector spaces systems are equivalent.

We denote by $\mathrm{VS}_{\mathbb{K}}^{\mathrm{inf}}$ the theory

$$\mathrm{VS}_{\mathbb{K}} \cup \left\{ \exists x_0 \cdots \exists x_{n-1} \bigwedge_{\substack{i,j=0 \\ i \neq j}}^{n-1} x_i \neq x_j : n \in \omega \right\}.$$

**Proposition 8.6.1.** The theory $\mathrm{VS}_{\mathbb{K}}^{\mathrm{inf}}$ is strongly minimal.

*Proof.* See [29, Example 8.1.10].

□

**Proposition 8.6.2.** Let $\mathcal{V} \models \mathrm{VS}_{\mathbb{K}}^{\mathrm{inf}}$ and $A \subseteq V$, then $\mathrm{acl}(A) = \langle A \rangle$, where $\langle A \rangle$ denotes the subspace spanned by $A$ of the vector space $\mathbb{V}$ corresponding to the model $\mathcal{V}$.

*Proof.* See [29, Example 8.1.10].

□

For the rest of the section let $\mathbb{K}$ be a countable field.

**Proposition 8.6.3.** The theory $\mathrm{VS}_{\mathbb{K}}^{\mathrm{inf}}$ is $\omega$-stable.

*Proof.* By Proposition 8.6.1 the theory $\mathrm{VS}_{\mathbb{K}}^{\mathrm{inf}}$ is strongly minimal and so, by Corollary 7.6.4, it is $\kappa$-categorical for $\kappa \geq \aleph_1$. Thus by Theorem 7.6.6 we have that $T$ is $\omega$-stable. Notice indeed that if the field $\mathbb{K}$ is countable then the signature of the theory is also countable and so we are under the hypotheses of Theorem 7.6.6.

□

**Proposition 8.6.4.** The theory $\mathrm{VS}_{\mathbb{K}}^{\mathrm{inf}}$ has the FRK-independence property.

*Proof.* Let $\mathbb{V}$ be a vector space over $\mathbb{K}$ such that $\mathrm{DIM}(\mathbb{V}) = \aleph_0$, where $\mathrm{DIM}(\mathbb{V})$ denotes the dimension of $\mathbb{V}$, and let $\mathcal{V}$ be the corresponding model of $\mathrm{VS}_{\mathbb{K}}^{\mathrm{inf}}$. By Proposition 8.6.1, the set

$$\{v \in V \mid \mathcal{V} \models x = x\,(v)\} = V$$

is strongly minimal. Thus the pregeometry $(V, \mathrm{cl})$ of Definition 8.2.2, defined as $\mathrm{cl}(X) = \mathrm{acl}(X) \cap V$, is just the pregeometry $(V, \mathrm{acl})$ which in turn by Proposition 8.6.2 is just the pregeometry $(V, \langle\rangle)$.

Let $D_0$ be a finite independent set in the pregeometry $(V, \mathrm{cl})$, then we have the following:

i) $\dim_{\mathrm{cl}}(V) = \dim_{\mathrm{acl}}(V) = \mathrm{DIM}(\mathbb{V}) = \aleph_0$;

ii) $\mathrm{cl}(\emptyset) = \mathrm{acl}(\emptyset) = \langle\emptyset\rangle = \{0\}$;

iii) $\mathrm{cl}(D_0) = \mathrm{acl}(D_0) = \langle D_0\rangle \neq \bigcup_{D \subsetneq D_0} \langle D\rangle$, see Theorem 6.6.1.

□

We denote with ${}^{\langle\rangle}\models^{\mathrm{VS}_{\mathbb{K}}^{\mathrm{inf}}}$ and ${}^{\mathrm{acl}}\models^{\mathrm{VS}_{\mathbb{K}}^{\mathrm{inf}}}$ the satisfaction relation of the systems PGACIndL, PGAIndL and PGADL relative to the theory $\mathrm{VS}_{\mathbb{K}}^{\mathrm{inf}}$ and the pregeometric operators $\langle\rangle$ and acl respectively. We denote by ${}^{\mathrm{frk}}\models^{\mathrm{VS}_{\mathbb{K}}^{\mathrm{inf}}}$ the satisfaction relation of the systems $\omega$SACIndL, $\omega$SAIndL and $\omega$SADL relative to the theory $\mathrm{VS}_{\mathbb{K}}^{\mathrm{inf}}$. By what we showed in this section, Theorem 8.2.1 and Theorem 8.5.4 it follows directly the following theorem.

**Theorem 8.6.5.** Let $\mathcal{V} \models \mathrm{VS}_{\mathbb{K}}^{\mathrm{inf}}$ and $s : \mathrm{dom}(s) \to V$ with $\overline{x}\,\overline{y}\,\overline{z} \subseteq \mathrm{dom}(s) \subseteq \mathrm{Var}$, then

$$\mathcal{V}\ {}^{\langle\rangle}\models_s^{\mathrm{VS}_{\mathbb{K}}^{\mathrm{inf}}} \overline{x} \perp_{\overline{z}} \overline{y} \quad \text{iff} \quad \mathcal{V}\ {}^{\mathrm{acl}}\models_s^{\mathrm{VS}_{\mathbb{K}}^{\mathrm{inf}}} \overline{x} \perp_{\overline{z}} \overline{y} \quad \text{iff} \quad \mathcal{V}\ {}^{\mathrm{frk}}\models_s^{\mathrm{VS}_{\mathbb{K}}^{\mathrm{inf}}} \overline{x} \perp_{\overline{z}} \overline{y},$$

$$\mathcal{V}\ {}^{\langle\rangle}\models_s^{\mathrm{VS}_{\mathbb{K}}^{\mathrm{inf}}} =\!(\overline{x}, \overline{y}) \quad \text{iff} \quad \mathcal{V}\ {}^{\mathrm{acl}}\models_s^{\mathrm{VS}_{\mathbb{K}}^{\mathrm{inf}}} =\!(\overline{x}, \overline{y}) \quad \text{iff} \quad \mathcal{V}\ {}^{\mathrm{frk}}\models_s^{\mathrm{VS}_{\mathbb{K}}^{\mathrm{inf}}} =\!(\overline{x}, \overline{y}).$$

□

In particular, noticing once again that

$$\mathcal{M} \models_s \overline{x} \perp_{\emptyset} \overline{y} \quad \text{if and only if} \quad \mathcal{M} \models_s \overline{x} \perp \overline{y},$$

we have the following corollary.

**Corollary 8.6.6.** Let $\mathcal{V} \models \mathrm{VS}_{\mathbb{K}}^{\mathrm{inf}}$ and $s : \mathrm{dom}(s) \to V$ with $\overline{x}\,\overline{y} \subseteq \mathrm{dom}(s) \subseteq \mathrm{Var}$, then

$$\mathcal{V}\ {}^{\langle\rangle}\models_s^{\mathrm{VS}_{\mathbb{K}}^{\mathrm{inf}}} \overline{x} \perp \overline{y} \quad \text{iff} \quad \mathcal{V}\ {}^{\mathrm{acl}}\models_s^{\mathrm{VS}_{\mathbb{K}}^{\mathrm{inf}}} \overline{x} \perp \overline{y} \quad \text{iff} \quad \mathcal{V}\ {}^{\mathrm{frk}}\models_s^{\mathrm{VS}_{\mathbb{K}}^{\mathrm{inf}}} \overline{x} \perp \overline{y}.$$

□

As in Section 6.6, we denote by $\models^{\mathrm{VS}}$ the satisfaction relation of the systems VSADL, VSAAIndL and VSAIndL. If we combine Theorem 8.6.5, Corollary 8.6.6 and Theorem 6.6.3 we obtain the following result.

**Theorem 8.6.7.** Let $\mathbb{V}$ be an infinite vector space over $\mathbb{K}$ and $\mathcal{V}$ the corresponding model of $\mathrm{VS}_{\mathbb{K}}^{\mathrm{inf}}$. Let $s : \mathrm{dom}(s) \to V$ with $\overline{x}\,\overline{y} \subseteq \mathrm{dom}(s) \subseteq \mathrm{Var}$. Then

$$\mathbb{V} \models_s^{\mathrm{VS}} \overline{x} \perp \overline{y} \quad \text{if and only if} \quad \mathcal{V}\ {}^{\mathrm{frk}}\models_s^{\mathrm{VS}_{\mathbb{K}}^{\mathrm{inf}}} \overline{x} \perp \overline{y},$$

$$\mathbb{V} \models_s^{\mathrm{VS}} =\!(\overline{x}, \overline{y}) \quad \text{if and only if} \quad \mathcal{V}\ {}^{\mathrm{frk}}\models_s^{\mathrm{VS}_{\mathbb{K}}^{\mathrm{inf}}} =\!(\overline{x}, \overline{y}).$$

□

## 8.7 Forking in Algebraically Closed Fields

In this last section firstly we prove that the theory ACF is $\omega$-stable. Then we see that ACF has the FRK-independence property. Finally, combining the results of this section with the ones of Section 5.6, we conclude that all the possible formulations of algebraically closed fields systems are equivalent.

**Proposition 8.7.1.** The theory ACF is strongly minimal.

*Proof.* See [29, Corollary 3.2.9].

□

**Proposition 8.7.2.** The theory ACF is $\omega$-stable.

*Proof.* By Proposition 8.7.1 the theory ACF is strongly minimal and so, by Corollary 7.6.4, it is $\kappa$-categorical for $\kappa \geq \aleph_1$. Thus by Theorem 7.6.6 we have that $T$ is $\omega$-stable.

□

**Proposition 8.7.3.** The theory ACF has the FRK-independence property.

*Proof.* Let $\mathbb{K}$ be an algebraically closed field such that $\mathrm{trdg}(\mathbb{K}/\mathbb{P}) = \aleph_0$, where $\mathbb{P}$ denotes the prime field of $\mathbb{K}$, and let then $\mathcal{K}$ be the corresponding model of ACF. By Proposition 8.7.1, the set

$$\{k \in K \mid \mathcal{K} \models x = x\,(k)\} = K$$

is strongly minimal. Thus the pregeometry $(K, \mathrm{cl})$ of Definition 8.2.2, defined as $\mathrm{cl}(X) = \mathrm{acl}(X) \cap K$, is just the pregeometry $(K, \mathrm{acl})$.

Let $D_0$ be a finite independent set in the pregeometry $(K, \mathrm{cl})$, we then have the following:

i) $\dim_{\mathrm{cl}}(K) = \dim_{\mathrm{acl}}(K) = \mathrm{trdg}(\mathbb{K}/\mathbb{P}) = \aleph_0$, by Lemma 5.6.3;

ii) $\mathrm{cl}(\emptyset) = \mathrm{acl}(\emptyset) = \{a \in K \mid a \text{ is algebraic over } \mathbb{P}\} \neq \emptyset$;

iii) $\mathrm{cl}(D_0) = \mathrm{acl}(D_0) \neq \bigcup_{D \subsetneq D_0} \mathrm{acl}(D)$, see Theorem 5.6.2.

□

We denote with ${}^{\mathrm{acl}}\!\models^{\mathrm{ACF}}$ the satisfaction relation of the systems PGACIndL, PGAIndL and PGADL relative to the theory ACF and the pregeometric operator acl. We denote by ${}^{\mathrm{frk}}\!\models^{\mathrm{ACF}}$ the satisfaction relation of the systems $\omega$SACIndL, $\omega$SAIndL and $\omega$SADL relative to the theory ACF. By what we showed in this section, Theorem 8.2.1 and Theorem 8.5.4 it follows directly the following theorem.

**Theorem 8.7.4.** Let $\mathcal{K} \models$ ACF and $s : \mathrm{dom}(s) \to K$ with $\overline{x}\,\overline{y}\,\overline{z} \subseteq \mathrm{dom}(s) \subseteq \mathrm{Var}$, then

$$\mathcal{K}\ {}^{\mathrm{acl}}\!\models_s^{\mathrm{ACF}} \overline{x} \perp_{\overline{z}} \overline{y} \quad \text{iff} \quad \mathcal{K}\ {}^{\mathrm{frk}}\!\models_s^{\mathrm{ACF}} \overline{x} \perp_{\overline{z}} \overline{y},$$

$$\mathcal{K}\ {}^{\mathrm{acl}}\!\models_s^{\mathrm{ACF}} {=}(\overline{x}, \overline{y}) \quad \text{iff} \quad \mathcal{K}\ {}^{\mathrm{frk}}\!\models_s^{\mathrm{ACF}} {=}(\overline{x}, \overline{y}).$$

□

In particular, we have the following corollary.

**Corollary 8.7.5.** Let $\mathcal{K} \models$ ACF and $s : \mathrm{dom}(s) \to K$ with $\overline{x}\,\overline{y} \subseteq \mathrm{dom}(s) \subseteq \mathrm{Var}$, then

$$\mathcal{K}\ {}^{\mathrm{acl}}\!\models^{\mathrm{ACF}}_{s} \overline{x} \perp \overline{y} \quad \text{iff} \quad \mathcal{K}\ {}^{\mathrm{frk}}\!\models^{\mathrm{ACF}}_{s} \overline{x} \perp \overline{y},$$

As in Section 5.6, we denote by $\models^{\mathrm{ACF}}$ the satisfaction relation of the systems ACFADL, ACFAAIndL and ACFAIndL. If we combine Theorem 8.7.4, Corollary 8.7.5 and Theorem 5.6.5 we obtain the following result.

**Theorem 8.7.6.** Let $\mathbb{K}$ be an algebraically closed field and $\mathcal{K}$ the corresponding model of ACF. Let $s : \mathrm{dom}(s) \to K$ with $\overline{x}\,\overline{y} \subseteq \mathrm{dom}(s) \subseteq \mathrm{Var}$. Then

$$\mathbb{K} \models^{\mathrm{ACF}}_{s} \overline{x} \perp \overline{y} \quad \text{if and only if} \quad \mathcal{K}\ {}^{\mathrm{acl}}\!\models^{\mathrm{ACF}}_{s} \overline{x} \perp \overline{y},$$

$$\mathbb{K} \models^{\mathrm{ACF}}_{s} {=}(\overline{x}, \overline{y}) \quad \text{if and only if} \quad \mathcal{K}\ {}^{\mathrm{acl}}\!\models^{\mathrm{ACF}}_{s} {=}(\overline{x}, \overline{y}).$$

□

# Chapter 9

# Conclusions

In this chapter, we recap the results of this work, draw some conclusions, and hint at possible extensions and generalizations of the present study.

## 9.1 Analysis Report

As clearly stated in Chapter 1, the aim of this work was to frame the algebraic and model-theoretic dependence and independence concepts in the more general theory of dependence logic. In particular, the open question that we faced was whether the kind of dependence and independence relations studied in dependence logic arise also in algebra and geometric model theory.

The chosen strategy was to interpret the dependence and independence atoms in each of the relevant contexts, and then to verify if these interpretations are sound and complete with respect to the deductive systems that characterize the behavior of the atoms in abstract terms.

We addressed the issue in increasing order of generality. Firstly, we considered the linear and algebraic dependence and independence notions of linear algebra and field theory. Secondly, we considered the notions of dependence and independence definable in function of the model-theoretic operator of algebraic closure. Then, we considered the dependence and independence notions definable in a pregeometry. Finally, we studied the forking independence relation in $\omega$-stable theories.

In all these cases we have been able to prove a soundness and completeness result answering positively the motivating question of this thesis.

Apart from their mathematical interest, these results support the claims of Väänänen and Galliani in [16], putting the exact and authoritative concepts of dependence and independence occurring in mathematics and formal mathematics under the wide wing of dependence logic.

## 9.2 Future Work

The positive results that we obtained in this thesis pave the way for further studies on the relations between geometric model theory and dependence logic. Here we draw some possible lines of research.

1. Deeper analysis of the conditional independence atoms.
2. Analysis of the forking independence relation in more general classes of theories: superstable theories, stable theories, simple theories.
3. Analysis of the so-called abstract independence relations.
4. Analysis of independence relations in abstract elementary classes.
5. Analysis of other forms of (in)dependence: inclusion, orthogonality.
6. Extension of the analysis to wider fragments of dependence logic.
7. Solutions of specific open questions in classification theory via tools from dependence logic.

# Bibliography


[1] Hans Adler. A geometric introduction to forking and thorn-forking. *Journal of Mathematical Logic*, 09(1), 2009.

[2] William Ward Armstrong. Dependency structures of data base relationships. In *IFIP Congress*, pages 580–583, 1974.

[3] Alan Baker. *Transcendental Number Theory.* Cambridge University Press, 1975.

[4] John T. Baldwin. First-order theories of abstract dependence relations. *Annals of of Pure arid Applied Logic*, 26:215–243, 1984.

[5] John T. Baldwin. *Fundamentals of Stability Theory.* Springer, 1988.

[6] Catriel Beeri, Martin Dowd, Ronald Fagin, and Richard Statman. On the structure of Armstrong relations for functional dependencies. *J. ACM*, 31(1):30–46, 1984.

[7] Catriel Beeri, Ronald Fagin, and John H. Howard. A complete axiomatization for functional and multivalued dependencies in database relations. In *Proceedings of the 1977 ACM SIGMOD international conference on Management of data*, SIGMOD '77, pages 47–61, New York, NY, USA, 1977. ACM.

[8] Andreas Blass and Yuri Gurevich. Henkin quantifiers and complete problems. *Annals of of Pure arid Applied Logic*, 32(1986):1–16.

[9] Maurizio Cailotto. Algebra e geometria lineari e quadratiche, 2009. Available at http://www.math.unipd.it/ maurizio/m2m/AGLQ910pp.pdf.

[10] Enrique Casanovas. Pregeometries and minimal types, 2008. Available at http://www.ub.edu/modeltheory/documentos/pregeometries.pdf.

[11] Herbert Enderton. Finite partially ordered quantifiers. *Z. Math. Logik Grundlagen Math.*, 16:393–397, 1970.

[12] Fredrik Engström. Generalized quantifiers in dependence logic. *Journal of Logic, Language and Information*, 21(3):299–324, 2012.

[13] R. Fagin. Functional dependencies in a relational database and propositional logic. *IBM J. Res. Dev.*, 21(6):534–544, 1977.

[14] Pietro Galliani. Multivalued dependence logic and independence logic, 2011. Non-classical Modal and Predicate Logics 2011, Guangzhou, China (extended abstract).

[15] Pietro Galliani. Inclusion and exclusion dependencies in team semantics, on some logics of imperfect information. *Annals of Pure and Applied Logic*, 163(1):68–84, 2012.

[16] Pietro Galliani and Jouko Väänänen. On dependence logic. To appear.

[17] Dan Geiger, Azaria Paz, and Judea Pearl. Axioms and algorithms for inferences involving probabilistic independence. *Information and Computation*, 91(1):128–141, 1991.

[18] Erich Grädel and Jouko Väänänen. Dependence, Independence, and Incomplete Information. In *Proceedings of 15th International Conference on Database Theory, ICDT 2012*, 2012.

[19] Erich Grädel and Jouko Väänänen. Dependence and Independence. *Studia Logica*, 101(2):399–410, 2013.

[20] Victor Harnik and Leo Harrington. Fundamentals of forking. *Annals of Pure and Applied Logic*, 26(1):245–286, 1984.

[21] Leo Henkin. Some remarks on infinitely long formulas. In *Infinitistic Methods (Proc. Sympos. Foundations of Math., Warsaw, 1959)*, pages 167–183. Pergamon, 1961.

[22] Christian Herrmann. Corrigendum: Corrigendum to "On the undecidability of implications between embedded multivalued database dependencies" [inform. and comput. 122 (1995) 221235]. *Information and Computation*, 122(2):221–235, 1995.

[23] Christian Herrmann. On the undecidability of implications between embedded multivalued database dependencies. *Information and Computation*, 204(12):1847–1851, 2006.

[24] Jaakko Hintikka and Gabriel Sandu. Informational independence as a semantical phenomenon. In *Logic, Methodology and Philosophy of Science VIII*, pages 571–589. NorthHolland, 1989.

[25] Wilfrid Hodges. Compositional semantics for a logic of imperfect information. *Logic Journal of IGPL*, 05:539–563, 1997.

[26] Wilfrid Hodges and Jouko Väänänen. Dependence of variables construed as an atomic formula. *Annals of Pure and Applied Logic*, 161:817–828, 2010.

[27] Wilbur John Walkoe Jr. Finite partially-ordered quantification. *J. Symb. Log.*, 35(4):535–555, 1970.

[28] Juha Kontinen and Jouko Väänänen. Axiomatizing first order consequences in dependence logic. *Annals of Pure and Applied Logic*. To appear.

[29] David Marker. *Introduction to model theory*. Springer, 2002.

[30] James S. Milne. Fields and galois theory (v4.21), 2008. Available at www.jmilne.org/math/.

[31] Pavel Naumov and Brittany Nicholls. R.E. axiomatization of conditional independence, 2013. Available at http://www2.mcdaniel.edu/pnaumov/papers/2013.tark.nn.pdf.

[32] D. Stott Parker, Jr. and Kamran Parsaye-Ghomi. Inferences involving embedded multivalued dependencies and transitive dependencies. In *Proceedings of the 1980 ACM SIGMOD international conference on Management of data*, SIGMOD '80, pages 52–57, New York, NY, USA, 1980. ACM.

[33] Anand Pillay. *Geometric Stability Theory*. Oxford University Press, 1996.

[34] Philipp Rothmaler. *Introduction to Model Theory*. Gordon and Breach Science Publishers, 2000.

[35] Yehoshua Sagiv and Scott F. Walecka. Subset dependencies and a completeness result for a subclass of embedded multivalued dependencies. *J. ACM*, 29(1):103–117, 1982.

[36] Saharon Shelah. *Classification Theory: and the Number of Non-Isomorphic Models*. Elsevier, 1990.

[37] Jouko Väänänen. *Dependence logic*. Cambridge University Press, 2007.

# Index