On The Modal Logics of Some Set-Theoretic Constructions

MSc Thesis *(Afstudeerscriptie)*

written by

Tanmay C. Inamdar
(born January 16th, 1991 in Pune, India)

under the supervision of **Prof Dr Benedikt Löwe**, and submitted to the Board of Examiners in partial fulfillment of the requirements for the degree of

MSc in Logic

at the *Universiteit van Amsterdam.*

Date of the public defense:
August 10, 2013

Members of the Thesis Committee:
Dr Jakub Szymanik
Dr Benno van den Berg
Prof Dr Stefan Geschke
Dr Yurii Khomskii
Prof Dr Benedikt Löwe
Prof Dr Jouko Väänänen

INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Contents

Chapter 1

Introduction

The paper is self contained. It uses forcing - this can be eliminated easily but for me this has no point.

Saharon Shelah
Around Classification Theory of Models

In set theory, there are various transformations between models. In particular, *forcing*, *inner models*, and *ultrapowers* occupy a fundamental place in modern set theory. Each of these play a different role. For example, forcing and inner models are typically used to establish the *consistency of statements* and the *consistency strength of statements*, and ultrapowers are typically used to define various *large cardinal notions*, which play the role of a *barometer for consistency strength* of statements.

Each of these techniques however, can be seen as a process for starting with one model of set theory, and obtaining another. Indeed, it is this aspect of these techniques that we are interested in in this thesis. Each such method of transforming models of set theory lends itself to analysis by the techniques of *modal logic* [Ham03, HL08], which is the general study of the logic of processes. It is a recent trend in set theory that research has focussed on these modal aspects of models of set theory. This is partly due to philosophical concerns, such as Hamkins's *multiverse view* [Ham09, Ham11], Woodin's *conditional platonism* [Woo04], Friedman's *inner model hypothesis* [Fri06], but also due to mathematical concerns, such as to account for the curious fact that, in some sense, these techniques that we have mentioned are essentially the only known techniques that set theorists have to prove independence results.

Concretely, if we fix a particular technique of model-transformation, we may reasonably ask of a given model of set theory questions of the following nature: "which statements are always true in all models that we shall construct by using this technique?"; "which statements can we always change the truth value of in any model that we shall construct by using this technique?" etc. Questions of the first sort are the topic of study of the area of set theory which is known as *absoluteness*, whereas questions of the second sort are the topic of study of the area of set theory known as *resurrection*. However, in both these cases, the questions we are asking talk about *specific sentences in the language of set theory*. That is, while the answers to these questions change depending on the type of model-transformation technique that we are considering, they are not purely questions about these techniques.

In this thesis, we are (for the most part) not interested in this interplay between a model-transformation technique and sentences in the language of set theory, but instead, in the *purely modal* side of these techniques. That is, we are interested in understanding the general principles that are true of these techniques when they are seen as processes. As an example of the kind of questions that we shall concern ourselves with, consider: "If φ is a statement that is true in some model that we construct by using this technique, and ψ is another statement that is true in some model that we construct by using this technique, then is it the case that we can construct a model where both φ and ψ are true by using this technique?", or "If φ is true of all models that we shall construct by using this technique, is φ already true?". Note that the answers to these questions do not depend on what φ and ψ are, but only on the nature of these model-transformation techniques.

These questions were first considered by Hamkins in [Ham03]. In particular, Hamkins showed that by interpreting the modal $\Box$ operator by "in all forcing extensions" and the $\Diamond$ operator by "in some forcing extension" one could interpret modal logic in set theory in a very natural way, and using this interpretation, study the technique of forcing through the modal lens. Hamkins used this interpretation to express certain forcing axioms known as *maximality principles*. These axioms were meant to capture the essence of models where a lot of forcing had already occurred, or to quote Hamkins, "*anything forceable and not subsequently unforceable is true*", and relativisations of 'forceable' to specific types of forcing notions. It is easily seen that modal logic provides an elegant way of expressing these statements using the scheme $\Diamond\Box\varphi \to \varphi$. Hamkins also gave a lower bound of $\mathsf{S4.2}$ for the modal logic that arises from forcing, *the modal logic of forcing*, in this paper. Hamkins's work on maximality principles has had many follow ups, the earliest ones being [Lei04] and [HW05].

The first paper devoted entirely to the modal logic of forcing was [HL08]. In particular, they were able to show that the modal logic of forcing is $\mathsf{S4.2}$. They also studied various generalisations of the modal logic of forcing, such as the modal logic of forcing with parameters, and developed some techniques which modularise the process of calculating the modal logics of set-theoretic constructions.

In addition to this, in [HL08], various relativisations of modal logic of forcing were also considered. For example, if we fix a definable class of partial orders $\mathcal{P}$, and a definition for it, we may interpret the $\Box$ operator as "in all forcing extensions obtained by forcing with a partial order in $\mathcal{P}$" and the $\Diamond$ operator as "in some forcing extensions obtained by forcing with a partial order in $\mathcal{P}$" and ask what the modal logic so obtained, denoted by $\mathsf{ML}_{\mathcal{P}}$, is. This line of investigation is the main topic of study of [HLL], where for many natural classes $\mathcal{P}$, upper and lower bounds are given for their modal logic. We continue this line of enquiry in this thesis. In particular, we take $\mathcal{P}$ to be the class of ccc-partial orders, and we study their corresponding modal logic, $\mathsf{ML}_{\mathsf{ccc}}$. We are able to improve the upper bound for $\mathsf{ML}_{\mathsf{ccc}}$ which was obtained in [HL08]. In order to do this, we generalise the method found there from the case of a single ω_1-tree to the case of an arbitrary finite number of ω_1-trees. Along the way, we obtain a characterisation of *Aronszajn trees* to which a branch can be added by ccc forcing which is interesting in its own right, and which also raises some questions of independent interest.

Another different direction that we pursue is that of looking at a different technique for relating models, namely that of *taking definable-with-parameters inner models*. The germs of this endeavour can be found in [HL13], where the modal logic of the relation of being a *forcing ground*[1] is studied. We are able to compute the exact modal logic of this relation, though this modal theory was not one

[1]This is the converse of the relation of being a forcing extension.

which had been considered in this area before. We obtain this theory by adding an extra axiom to the well-studied modal theory S4.2 which captures the property of $\mathbf{L}$, Gödel's constructible universe, being in a sense the minimal model of ZFC. Our proofs strongly rely on the results from [HL13].

A third concern of ours relates to a technical question raised by a mistake in the literature. In [HL08], a proof of the main theorem contained a gap as was pointed out by Jakob Rittberg in [Rit10]. As Hamkins and Löwe had given two proofs of this main theorem (a detailed version of the second proof can be found in [HLL]), the result continued to hold, but the gap that Rittberg pointed out was interesting in its own right: in order to construct an arbitrarily large family of mutually independent buttons and switches (see Section 3.4 in Chapter 3), Hamkins and Löwe implicitly assumed that in any generic extension M of $\mathbf{L}$, for any natural number $n > 0$, if $\aleph_n^{\mathbf{L}}$ is still a cardinal in M, then there is a generic extension of M in which $\aleph_n^{\mathbf{L}}$ is not a cardinal any more, but no other cardinals are collapsed. As Rittberg pointed out, the standard partial order for collapsing cardinals, the Lévy collapse, requires some cardinal arithmetic asssumptions to ensure that no other cardinals are collapsed. Indeed, for the case of $n = 2$ already, if $2^{\aleph_0} > \aleph_2$ in M, then the Lévy collapse, $\mathrm{Lev}(\aleph_1, \aleph_2)$, can be shown to collapse $\aleph_3$. As Hamkins and Löwe had not specified any method for collapsing cardinals which behaved in the way they desired, their proof had a gap. However, this does not rule out the possibility that there are other partial orders different from the Lévy collapse which have this behaviour.

Question 1. *Let $n > 1$ be a natural number. M be a generic extension of $\mathbf{L}$ sucn that $M \vDash$"$\aleph_n^{\mathbf{L}}$ is a cardinal". Then, is there a generic extension N of M such that $N \vDash$"$\aleph_n^{\mathbf{L}}$ is not a cardinal" and such that for all other natural numbers $m > 1$, if $M \vDash$"$\aleph_m^{\mathbf{L}}$ is a cardinal", then $N \vDash$"$\aleph_m^{\mathbf{L}}$ is a cardinal"?*

While researching this question, we found that similar questions had already been considered by Abraham in his PhD thesis. In particular, in [Abr83], he had given a method for collapsing the second uncountable cardinal in any model of set theory without collapsing any other cardinals. We give an exposition of his intricate method.

The organisation of the thesis is as follows: in Chapter 2 we introduce some notation that we will use throughout the thesis. We will also mention what we assume of the reader. In Chapter 3 we show formally how, given a relation between models of set theory satisfying certain properties, modal logic can be interpreted in set theory. We then go over the basic techniques that are used to calculate this modal logic. While most proofs are not hard, the most tricky issue we face while doing this interpretation is the metamathematical one of formalising these statements in the appropriate language.

In Chapter 4, we study the modal logic of inner models. In particular, we define a certain modal theory, prove some characterisation results for it, and then piggyback on the results of [HL13] to show that this modal theory is exactly the one corresponding to the modal logic of inner models. The result which allows us to do this is that the relation 'being a forcing ground' is an initial segment of the relation 'being a definable-with-parameters inner model'.

In Chapter 5, we study the modal logic of ccc forcing. We introduce a class of frames which have not been studied in the literature before, and show how the modal logic of ccc forcing, $\mathsf{ML}_{\mathsf{ccc}}$, is contained in the modal logic characterised by these frames. We use this to show that $\mathsf{ML}_{\mathsf{ccc}}$ is not contained in a certain natural modal theory. Our main tools for showing that $\mathsf{ML}_{\mathsf{ccc}}$ is contained in the aforementioned modal logic is the analysis of the effect on ccc forcing on the *Aronszajn-ness of ω_1-trees*. We also prove a (to us) surprising negative result which we found while attempting to generalise the techniques of this chapter.

In Chapter 6, we give an exposition of Abraham's technique to collapse $\aleph_2$. We start off by showing why the standard partial order does not work, and then explain how Abraham gets around this obstacle. We also discuss some obstacles with generalising his techniques.

Personal Remarks and Acknowledgements

There are many people who have wittingly or unwittingly helped in the production of this thesis, but the contribution of one of them stands out. If at all this thesis can be thought of as a labour of love, it would be the love that Benedikt Löwe has for the idea of ensuring that I get a Masters degree. He has somehow managed to overcome increasingly bizarre situations in his unrelenting quest to successfully supervise another student. Hopefully, he is unaware of the worst of them.

He is also the man who shoulders the most blame (a sizeable portion of the rest being shouldered by S. P. Suresh) for me deciding to become a mathematician, a logician, and a set theorist. It seems strange now that it was only three and a half years back that I first received an email from him informing me that he would like to invite me to Amsterdam to teach me some set theory. I believe the gist of my answer was "Why would you bother?". Even now, a few years later, I still don't quite understand it.

In the meanwhile, however, I have learnt much from him, on matters set-theoretic and otherwise. He has always been willing to answer ridiculously vague questions[2] in as erudite a way as possible, and has always tried to facilitate my development as a set theorist in any way he can. I have also learnt much from him about academia (though unfortunately, not about deadlines), and recently, also about grammar and bibliography management. The most recent thing that I learnt because of him was how to draw diagrams in LaTeX, and I hope the reader enjoys each and every one of them thoroughly.

I would also like to thank the members of my committee, Dr Jakub Szymanik, Dr Benno van den Berg, Prof Dr Stefan Geschke, Dr Yurii Khomskii, Prof Dr Jouko Väänänen, for first agreeing to be on my committe, and then, allowing me to submit after my deadline, and while I am at it, also my supervisor, Prof Dr Benedikt Löwe, for reading my thesis in his free time even though I did not keep to the commitment I had made to him about when I would finish the writing. In addition to this, Jouko provided some helpful pointers to the literature which unfortunately did not come to much good, because the chapter which I needed them for turned out to be filled with lies. Yurii also provided some nice conversations and easy answers to questions that I thought were hard in Barcelona.

Other people who have contributed in some academic way to this thesis are Mirna Džamonja, Joel David Hamkins, Paul Larson, Philipp Schlicht, and those poor, poor souls who had to sit through all of the Set Theory Lunches that I talked in. Lastly, I would like to thank the people involved in accidentally mailing me a copy of [Jud93]. It promises to be an invaluable reference.

On the non-academic side, I would like to thank all my friends in Amsterdam for the many (safe and unsafe) bike rides, Sunday dinners, science fiction movies and bad puns that we have shared. The occasional trip into the canals notwithstanding, it has been quite a pleasant time, and often have the lights on the Goodyear Blimp mentioned O'Shea Jackson's side business at the end of the evening [Jac93]. I would like to thank the wonderful people at Cafe Frieda for always having a table for the solitary set theorist, and for their great taste in music. I would also like to thank

[2] One of my emails to him started thus: "In the true spirit of asking questions without thinking about them so much myself, here's a few more..."

the Evert Willem Beth Foundation for the scholarship without which I would surely not have come to Amsterdam, and Tanja Kassenaar and Ulle Endriss for support bureaucratic going above and beyond the call of duty.

Lastly, I would like to thank my family for the support, financial and otherwise.

Chapter 2

Preliminaries

For the most part, all of the notation that are used in this thesis is standard. In particular, we follow [Jec03] for set-theoretic notation, and [BdRV02] for modal-logical notation.

As far as possible, we have tried to keep each chapter as self-contained as possible. This chapter, where we give the set-theoretic and modal-logical background, and Chapter 3, where the basic theory of the modal logic of set-theoretic constructions is exponded, are sufficient background to read Chapter 5 and Chapter 4, which form the core of the thesis. In particular, neither of these chapters depends on the other.

There are two common threads that run throughout this thesis: modal logic, and forcing. Even in Chapter 4 where we discuss the modal logic of inner models, the set-theoretic side of the main proof still relies heavily on forcing. Similarly, even in Chapter 6, where an exposition is given of a paper that was written well before the connections between modal logic and set theory that this thesis primarily concerns itself with were discovered, we only found this paper when doing background research on a question that can most naturally be expressed with the language of modal logic.

In any chapter where modal techniques going beyond what is discussed in this chapter are required, sections of the relevant chapter are devoted to these techniques. All of the techniques that we use from modal logic are (properly) contained in Chapters 1, 2 and 4 of [BdRV02].

When it comes to set theory, this thesis demands more from the reader. In particular, it is assumed that the reader is familiar with the first fifteen Chapters of [Jec03]. Barring Chapter 6 where an exposition is given of a research paper, the only results that might not be taught in a graduate course on forcing that are used are Grigorieff's Theorem, Theorem 101, the Laver-Woodin Theorem, Theorem 99, and a theorem of Abraham and Shelah, Theorem 130. In the case of the first of these, a proof is not given as it can be found in [Jec03, Chapter 15], whereas a proof is provided of the second one. A complete proof of the third result (or even its corollary which we use, Corollary 131) would have required far too much space, and unless the proof were sufficiently detailed, not added anything to the reader's understanding. Hence, we skipped its proof as well. We also use a modification of a model of Reitz [Rei07] which was built by class forcing using Easton support. Explaining the basics of class forcing would have involved much work, so it is assumed the reader has Chapter 15 of [Jec03] closeby for definitions and basic results. All of the other set-theoretic techniques that are used are those that would probably be covered in a graduate course on forcing. For example, the only types of iterated forcing that are used are two-step iterations. Nonetheless, basic facts that are used are, as far as possible, explicitly stated, even though the reader is often referred to [Jec03] for the proofs. In Chapter 6, we use two theorems without supplying a proof. Neither of these are theorems of Abraham, and while one of them does not add much to the understanding of the paper, the proof of the other would require the development of quite some background.

2.1 Notation

All trees, frames, partial orders and Boolean algebras grow upwards. If $(S, \leq)$ is a pre-order, then if it is clear from context the order shall not be mentioned. That is, we will often refer to S, which we call the *carrier set* of this pre-order, being a pre-order if there is no scope for confusion. Similarly, we will often refer to Boolean algebras only by their carrier sets $\mathbb{B}$, implicitly assuming that they have an ordering $\leq$, a bottommost element 0 and a topmost element 1. We will also sometimes use the equivalent characterisation of Boolean algebras in terms of a join $\vee$ and their meet $\wedge$. When talking about the powerset algebra of some set S, we shall instead refer to the ordering by the symbol $\subseteq$ to refer to the containment relation, the bottommost element by the emptyset symbol, $\emptyset$, the topmost element by the set S, the join by the union symbol, $\cup$ and the meet by the intersection symbol $\cap$.

Given a pre-order $(S, \leq)$, there is a natural equivalence relation on this structure given by $s \equiv y \leftrightarrow x \leq y \leq x$. Taking the quotient of the pre-order by this equivalence relation gives a partial order. This partial order shall be denoted by $([S]_\equiv, \leq_\equiv)$, and refer to it as the *quotient partial order* of $(S, \leq)$. Also, for $x \in S$, the equivalence class of x is denoted by $[x]_\equiv$. When talking about pre-orders, we shall refer to this equivalence relation as "*the natural equivalence relation*". If $[x]_\equiv = [y]_\equiv$, we say that the nodes x and y are *equivalent*. A collection C of equivalent nodes of a pre-order is called a *cluster*. Call C a *complete cluster* if it is empty, or if there is a node $p \in S$ such that $C = [p]_\equiv$. It is easy to see that a pre-order is obtained from its quotient partial order by adding at each point the corresponding complete cluster of the pre-order.

If $(S, \leq)$ is a pre-order, and $x, y \in S$, we say that x and y are *comparable* if $x \leq y$ or $y \leq x$. We shall say that they are *incomparable* if this is not the case. Also, if there is a $z \in S$ such that $x \leq z$ and $y \leq z$ as well, then we say that x and y are *compatible*, denoted $x \parallel y$. If this is not the case, we say that they are *incompatible*, denoted $x \perp y$.

If $(S, \leq)$ is a partial order, then a node $t \in S$ is a *maximal node* or *extremal node* of S if the only $s \in S$ such that $t \leq s$ is t itself. A node $t \in S$ is a *penultimate node* if there is exactly one other element $s \in S$ such that $t \leq s$. Such an element is called a *coatom* in the literature, though we do not use this term in this thesis. A node $t \in S$ such that for each $s \in S$, $t \geq s$ is called the *bottom node* of S. Note that this implies that any partial order can have at most one bottom node.

Maximal clusters, penultimate clusters, bottom clusters etc are defined in an analogous way.

A *tree* is a partial order T with a bottom node (which we call the *root*) such that if $p, q \in T$ are incomparable, then they are incompatible. It is easy to see that the converse always holds for any partial order. In Chapter 5 we shall be interested in specific types of trees, and we explicate there exactly what we expect of them.

A *linear pre-order* is a pre-order where any two nodes are comparable. A *directed* pre-order $(S, \leq)$ is a pre-order such that for any $x, y \in S$, there is a $z \in S$ such that $x \leq z$ and $y \leq z$. Any linear pre-order is directed.

Let $(S, \leq)$ be a partial order, and let $p, q \in S$. Then p is an *immediate successor* of q if $q \leq p$, and if for very $r \in S$ different from these two elements, if r is comparable with both p and q, then either $r < q$ or $p < r$. Also, we define intervals for partial orders in the standard way, with the standard notation for open, closed, half-closed half-open intervals etc.

If $(S, \leq)$ is a partial order, and $p, q \in S$, a *path* between p and q is a sequence $\langle p_0, p_1, \ldots, p_n \rangle$ of some finite length such that for each i, $0 \leq i < n$, p_{i+1} is an immediate successor of p_i, and $p_0 = p$ and $q = p_n$. Note that this implies that no vertex occurs twice in a path between two nodes, since we are in a partial order firstly, and because no node is an immediate successor of itself. Also, if $p = q$, then we call the path from p to p a *trivial* path. We also define a subpath relation between paths in the obvious way.

Suborders, subalgebras, complete subalgebras, homomorphisms, isomorphisms are all defined as in the standard literature, and each of them preserves *all of the structure* (including constants) of the object in question. An embedding is an injective homomorphism. Also, if A, B are two objects of the same type which are isomorphic, then we denote this as $A \cong B$. We also use the symbols $\uplus$ and $\biguplus$ for disjoint unions.

2.2 Forcing Background

All models of set theory that we speak about in this thesis are transitive well-founded models. Except for one case, in Section 4.3 in Chapter 4, all forcings are set-forcings. For most of this thesis, partial orders are used for forcing. The general principle is this: in specific forcing constructions, partial orders are more convenient, whereas to prove more structural results Boolean algebras are more convenient.

The one place in this thesis where we have deviated from standard notation is the following: since a large percentage of the partial orders that are used for forcing in this thesis are trees, the forcing order is chosen so that when restricted to trees it will agree with the tree-order. That is, we work Jerusalem-style, so $p \geq q$ actually does imply that p is stronger than q. This causes some confusion (especially when it comes to working with Boolean algebras), but hopefully we have managed to avoid this by forcing with Boolean algebras as little as possible.

We do not go into the definitions of basic forcing theory here, such as the definitions of a name, canonical name, nice name, the semantic and syntactic forcing relations and their ZFC-provable equivalence, or the preservation of the axioms of ZFC by passing to a generic extension. It is assumed that the reader is familiar with the standard method of formalising forcing in ZFC, where the model does not need to be assumed to a countable transitive model. It is also assumed that the reader is aware of the standard methods by which a model M of ZFC can prove results about its generic extension by means of the Boolean truth value of any statement or by means of maximal antichains.

Definition 2. Let $\mathbb{P}$ be a partially ordered set. Then $\mathbb{P}$ is a *forcing poset* if the following hold:

(i) There is an element $0 \in \mathbb{P}$ such that for each $p \in \mathbb{P}$, $0 \leq p$. In this case, we say that 0 is the *least element* of $\mathbb{P}$;

(ii) For all $p, q \in \mathbb{P}$, if $p \not\leq q$, then there is an $r \in \mathbb{P}$ such that $p \leq r$, and $q \perp r$. In this case, we say that $\mathbb{P}$ is separative.

We shall sometimes simple call a forcing poset a poset if the context of forcing is clear. When $\mathbb{P}$ is a forcing poset, any $p \in \mathbb{P}$ is a *condition*, and for any statement φ such that p forces φ, we write that $p \Vdash \varphi$. If each condition $p \in \mathbb{P}$ forces φ, then we write $\Vdash_\varphi$.

When talking about Boolean algebras in the context of forcing, they will be *atomless* and *complete*. That is, if we mention that $\mathbb{B}$ is a Boolean algebra, then it is implicitly assumed that:

(i) For any $b \in \mathbb{B} \setminus \{0\}$, there is an $a \in \mathbb{B}$ such that $0 < a < b$.

(ii) For any $X \subseteq \mathbb{B}$, there is an $a \in \mathbb{B}$ such that for each $x \in X$, $x \leq a$, and further, for all other b having this property, $a \leq b$.

In the rest of this section, we mention the basic concepts and facts about forcing that we shall use.

Definition 3. A collection S of finite sets is called a Δ-*system* if there is a finite set R, called the *root* of the Δ-system, such that for any distinct $X, Y \in S$, $X \cap Y = R$.

Lemma 4. *(Δ-System Lemma) Let S be an uncountable collection of finite sets. Then there is an uncountable subset T of S such that T is a Δ-system.*

Definition 5. Let M be a model of set theory and $\mathbb{P} \in M$ a forcing poset.

(i) A set $A \subseteq \mathbb{P}$ in M is called an *antichain* if for all $p, q \in A$, $p \perp q$. It is *maximal* if for all $B \subseteq \mathbb{P}$ in M, $A \subsetneq B$ implies that B is not an antichain.

(ii) A set $D \subseteq \mathbb{P}$ in M is called a *dense* set of $\mathbb{P}$ in M if for all $p \in \mathbb{P}$, there is a $q \geq p$ in M such that $q\ inD$. It is called *dense open* if further, for each $p \in \mathbb{P}$ such that $p \in D$, for each $q \in \mathbb{P}$ such that $q \geq p$, $q \in D$.

Definition 6. Let M be a model of set theory, and $\mathbb{P}$ a forcing poset in M. A set $G \subseteq \mathbb{P}$ is M-*generic* for $\mathbb{P}$ if any of the following equivalent conditions are met:

(i) For each maximal antichain $A \in M$, $G \cap A \neq \emptyset$.

(ii) For each dense set $D \in M$, $G \cap D \neq \emptyset$.

(iii) For each dense open set $D \in M$, $G \cap D \neq \emptyset$.

Theorem 7. *Let M be a model of set theory, and $\mathbb{P}$ a forcing poset in M. Let G be M-generic for po. Then $M[G]$ is the smallest transitive model of set theory which has the same ordinals as M, and contains M and G.*

Proposition 8. *Let M, N be transitive class models of set theory with the same ordinals. Then $M = N$ if and only if they have the same sets of ordinals.*

Definition 9. Given two partial orders $\mathbb{P}, \mathbb{Q}$, a dense embedding of $\mathbb{P}$ in $\mathbb{Q}$ is an embedding $f : \mathbb{P} \to \mathbb{Q}$ such that for each $q \in \mathbb{Q}$, there is a $p \in \mathbb{P}$ such that $q \leq f(p)$.

Proposition 10. *Let M be a model of set theory, and in M, let $\mathbb{P}$ and $\mathbb{Q}$ be forcing posets and $f : \mathbb{P} \to \mathbb{Q}$ a dense embedding of $\mathbb{P}$ into $\mathbb{Q}$. Let G be M-generic for $\mathbb{Q}$. Then $f^{-1}[G]$ is M-generic for $\mathbb{P}$.*

Theorem 11. *Let $\mathbb{P}$ be a forcing poset. Then there is a canonical complete atomless Boolean algebra $\mathbb{B}$ (called the completion or Boolean completion of $\mathbb{P}$) such that $\mathbb{P}$ densely embeds into $\mathbb{B} \setminus \{1\}$, the partial order of non-unit elements of the Boolean algebra.*

Proposition 12. *Let $V \subseteq W$ be models of set theory. Let $\mathbb{P} \in V$ be a forcing poset. Let G be W-generic for $\mathbb{P}$. Then G is V-generic for $\mathbb{P}$ as well.*

Recall that for any model V of set theory and λ an ordinal, V_λ denote the set of elements of V of rank less than λ.

Proposition 13. *Let $\mathbb{P} \in V$ be a partial order. Let G be V-generic for $\mathbb{P}$. Then for λ large enough, $V_\lambda[G]$ is a generic extension of V_λ, and $V_\lambda[G] = (V[G])_\lambda$.*

Definition 14. Let $\mathbb{P}$ be a forcing poset. Let κ be an uncountable regular cardinal.

(i) $\mathbb{P}$ has the *countable chain condition* (abbreviated as ccc) if for any $A \subseteq \mathbb{P}$ which is an antichain, A is countable.

(ii) $\mathbb{P}$ is *Knaster* if for any uncountable subset $A \subseteq \mathbb{P}$, there is an uncountable set $B \subseteq A$ such that for all $p, q \in B$, $p \parallel q$.

(iii) $\mathbb{P}$ has the κ-cc if for any $A \subset \mathbb{P}$ which is an antichain of $\mathbb{P}$, $|A| < \kappa$.

(iv) $\mathbb{P}$ is κ-closed if for every ordinal $\lambda < \kappa$, for every strictly increasing, every increasing chain

$$p_0 \leq p_1 \leq \dots p_\alpha \leq p_{\alpha+1} \leq \dots \ (\alpha < \lambda)$$

of elements of $\mathbb{P}$, there is an element $p \in \mathbb{P}$ such that for all $\alpha < \lambda$, $p_\alpha \leq p$. If $\kappa = \aleph_1$, then $\mathbb{P}$ is *σ-closed.* If the above condition holds for all ordinals less than κ, then $\mathbb{P}$ is *$< \kappa$-closed.*

(v) $\mathbb{P}$ is κ-distributive if the intersection of κ-many open dense sets of $\mathbb{P}$ is open dense. If $\kappa = \aleph_0$, then $\mathbb{P}$ is *σ-distributive.* If this condition holds for all cardinals less than κ, then $\mathbb{P}$ is *$< \kappa$-distributive.*

Proposition 15. *Let $\mathbb{P}$ be a forcing poset. Let κ be a regular cardinal. Then $\mathbb{P}$ is κ-distributive iff forcing with $\mathbb{P}$ does not add any new κ-sized subsets of ordinals. If $\mathbb{P}$ is $< \kappa^+$-closed, then it is κ-distributive.*

Corollary 16. *Let $\mathbb{P}$ be a forcing poset. If $\mathbb{P}$ is σ-distributive, forcing with $\mathbb{P}$ cannot collapse $\aleph_1$.*

Proposition 17. *Let $\mathbb{P}$ be a partial order and κ a cardinal such that $\mathbb{P}$ has the κ-cc. Then by forcing with $\mathbb{P}$ no cardinals greater than or equal to κ are collapsed. In particular, if $\mathbb{P}$ has size less than κ, then forcing with $\mathbb{P}$ cannot collapse ck. Al any partial order with the ccc does not collapse any cardinals.*

2.2.1 Cohen Reals

For us, 'reals' will be logician's reals, that is, elements of the Baire space, ω^ω, of ω-length sequences of natural numbers.

Recall that the forcing poset Coh to add a *Cohen real* is the following:

(i) The carrier set of Coh is $\omega^{<\omega}$, the set of finite sequences of natural numbers;

(ii) If $p, q \in$ Coh, then $p \geq q$ if q is an initial segment of p, denoted $q \preccurlyeq p$.

Also, if $p \in$ Coh, then by $|p|$ we denote the length of p.

Proposition 18. *The partial order* Coh *is Knaster.*

If G is a generic for the Cohen poset, then if we define $c = \bigcup\{p \in \text{Coh} \mid p \in G\}$, it is easy to see by defining the right dense sets that c is an elment of the Baire space as well, and hence a real, called a Cohen real.

Definition 19. Let V be a model of set theory. Let c be a real. Then c is *Cohen over V*, if for each $D \in V$ such that $V \vDash$"D is a dense open subset of Coh", there is a finite initial segment d of c such that $d \in D$.

Notice that as this poset consists of *finite sequences* of ω_1, any two transitive models of set theory compute Coh in the same way. That is, for any two transitive models M, N of set theory, $\text{Coh}^M = \text{Coh}^N$. Also, by coding the dense subsets for Coh by real numbers in a suitable way, it can be shown that they remain dense subsets for Coh even in outer models:

Theorem 20. *Let $V \subseteq W$ be models of set theory.*

(i) Then $\text{Coh}^V = \text{Coh}^W$.

(ii) If c is Cohen over W, then c is Cohen over V.

(iii) In particular, if V' is a model of set theory and c is Cohen over V', then c is Cohen over **L**.

2.2.2 Product Forcing

Definition 21. Let M be a model of set theory and let $\mathbb{P}, \mathbb{Q} \in M$ be forcing posets. The *product* of these posets, $\mathbb{P} \times \mathbb{Q}$ is the forcing poset defined as follows:

(i) The elements of $\mathbb{P} \times \mathbb{Q}$ are pairs (p, q) such that $p \in \mathbb{P}$ and $q \in \mathbb{Q}$;

(ii) The order is given as follows: if $(p_1, q_1), (p_2, q_2) \in \mathbb{P} \times \mathbb{Q}$ are two conditions, then

$$(p_1, q_1) \leq (p_2, q_2) \text{ if and only if } p_1 \leq p_2 \text{ and } q_1 \leq q_2.$$

Note that the from the above definition it follows that $\mathbb{P} \times \mathbb{Q} \cong \mathbb{Q} \times \mathbb{P}$.

Definition 22. Let M be a model of set theory and let $\mathbb{P}, \mathbb{Q} \in M$ be forcing posets. Let G be a subset of $\mathbb{P} \times \mathbb{Q}$. Define the *projections* of G on $\mathbb{P}$ and $\mathbb{Q}$ as follows:

$$G_1 \triangleq \{p \in \mathbb{P} \mid \exists q \in \mathbb{Q}[(p, q) \in G]\}$$
$$G_2 \triangleq \{p \in \mathbb{P} \mid \exists p \in \mathbb{P}[(p, q) \in G]\}.$$

Proposition 23. *Let M be a model of set theory and let $\mathbb{P}, \mathbb{Q} \in M$ be forcing posets. Let G be a subset of $\mathbb{P} \times \mathbb{Q}$. Then the following are equivalent:*

(i) G is M-generic for $\mathbb{P} \times \mathbb{Q}$;

(ii) G_1 is M-generic for $\mathbb{P}$ and G_2 is $M[G_1]$-generic for $\mathbb{Q}$.

Moreover, if this is the case, then $M[G] = M[G_1][G_2]$.

Proposition 24. *Let κ be a regular cardinal. Let $\mathbb{P}, \mathbb{Q}$ be forcing posets such that $\mathbb{P}$ is κ-closed. Let $V[G]$ be a generic extension by $\mathbb{P} \times \mathbb{Q}$. Let G_1 be the projection of G on $\mathbb{Q}$. Then if $S \in V[G]$ is a subset of κ, then $S \in V[G_1]$.*

Proposition 25. *Let $\mathbb{P}$ be a Knaster poset and $\mathbb{Q}$ a poset with the countable chain condition. Then $\mathbb{P} \times \mathbb{Q}$ has the countable chain condition.*

Definition 26. Let I be an index set. Let $\mathbb{P}$ be a forcing poset whose bottom element is 0, and whose ordering is given by $\leq_{\mathbb{P}}$. The *finite support I-product* of $\mathbb{P}$ is defined to be the partial order $\prod_I \mathbb{P}$ defined as follows:

(i) The elements of $\prod_I \mathbb{P}$ consist of functions $f : I \to \mathbb{P}$ such that $\{i \in I \mid f(i) \neq 0\}$ is a finite set. This set is called the *support* of f, denoted $\mathrm{supp}(f)$.

(ii) The ordering is given as follows: if f, g are two elements of the partial order, $f \leq g$ if $\mathrm{supp}(f) \subseteq \mathrm{supp}(g)$ and for all $i \in \mathrm{supp}(f)$, $f(i) \leq_{\mathbb{P}} g(i)$.

For any $p \in \prod_I \mathbb{P}$, for any $S \subseteq I$, the *projection of p to S*, denoted p_S, is defined as follows:

(i) $\mathrm{supp}(p_S) = \mathrm{supp}(p) \cap S$;

(ii) For $i \in S$, $p_S(i) = p(i)$.

For any subset of $S \subseteq I$, the *projection of $\prod_I \mathbb{P}$ to S*, denoted $(\prod_I \mathbb{P})\restriction S$, is defined to be the suborder of $\mathbb{P}$ given by $\{p_S \mid p \in \mathbb{P}\}$.

Note that since $\mathbb{P}$ is a forcing poset, then $\prod_I \mathbb{P}$, and $(\prod_I \mathbb{P})\restriction S$ are both seen to be forcing posets. Also, it is clear that $(\prod_I \mathbb{P})\restriction S \cong \prod_S \mathbb{P}$. We shall henceforth identify the two.

Definition 27. Let $\mathbb{P}$ be a forcing poset, and I and index set. Let G be V-generic for $\prod_I \mathbb{P}$. Let $S \subseteq I$. Then $G\restriction S$ is defined to be the set $\{g_S \mid g \in G\}$.

Proposition 28. *Let $\mathbb{P}$ be a forcing poset, and I and index set. Let G be V-generic for $\prod_I \mathbb{P}$. Let $S \subseteq I$. Then $G\restriction S$ is generic for $\prod_S \mathbb{P}$.*

Proposition 29. *Let $\mathbb{P}$ be a Knaster poset. Let I be any index set. Then $\prod_I \mathbb{P}$ is Knaster.*

2.2.3 Iterated Forcing

Definition 30. Let M be a model of set theory and $\mathbb{P} \in M$. Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name such that in any forcing extension by $\mathbb{P}$, the interpretation of $\dot{\mathbb{Q}}$ is a forcing poset. Then we can define the *iteration* of $\mathbb{P}$ and $\dot{\mathbb{Q}}$, $\mathbb{P} * \dot{\mathbb{Q}}$, as follows:

(i) $(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}}$ if $p \in \mathbb{P}$ and $\Vdash_{\mathbb{P}} \dot{q} \in \mathbb{Q}$;

(ii) $(p_1, \dot{q}_1) \leq (p_2, \dot{q}_2)$ if $p_1 \leq p_2$ and $p_2 \Vdash \dot{q}_1 \leq \dot{q}_2$.

Theorem 31. *Let M be a model of set theory and $\mathbb{P} \in M$ a forcing poset. Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name such that in any forcing extension by $\mathbb{P}$, the interpretation of $\dot{\mathbb{Q}}$ is a forcing poset.*

(i) *Let G be M-generic for $\mathbb{P}$, and let $\mathbb{Q} = \dot{\mathbb{Q}}^G$ be the interpretation of $\dot{\mathbb{Q}}$ in $M[G]$. Let H be $M[G]$-generic for $\mathbb{Q}$. Then*

$$G * H = \{(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}} \mid p \in G \text{ and } \dot{q}^G \in H\}$$

is M-generic for $\mathbb{P} * \dot{\mathbb{Q}}$, and $M[G * H] = M[G][H]$.

(ii) *Let K be M-generic for $\mathbb{P} * \dot{\mathbb{Q}}$. Then*

$$G = \{p \in \mathbb{P} \mid (p, \dot{q}) \in K\}$$

is M-generic on $\mathbb{P}$, and

$$H = \{\dot{q}^G \mid \exists p \in \mathbb{P}[(p, \dot{q}) \ inK]\}$$

is $M[G]$-generic for $\mathbb{Q} = \dot{\mathbb{Q}}^G$, and $K = G * H$.

Iterated forcing can also be done with Boolean algebras, and the Boolean algebra analogues of the above theorem hold true in this case as well. We shall need these facts in Chapter 4.

Definition 32. Let M be a model of set theory, $\mathbb{B} \in M$ a complete atomless Boolean algebra, and $\dot{cba} \in M^{\mathbb{B}}$ a name such that $\|\dot{\mathbb{C}}$ is a complete atomless Boolean algebra$\|_{\mathbb{B}} = 1_{\mathbb{B}}$. The *iteration* of $\mathbb{B}$ and $\mathbb{C}$, denoted $\mathbb{B} * \dot{\mathbb{C}}$, is the complete atomless Boolean algebra $\mathbb{D}$ obtained as follows:

(i) Let D be the set of all $\dot{c} \in V^{\mathbb{B}}$ such that $\|\dot{c} \in \dot{\mathbb{C}}\| = 1_{\mathbb{B}}$. The carrier set of $\mathbb{D}$ is then the set D quotiented by the following equivalence relation

$$\dot{c}_1 \equiv \dot{c}_2 \text{ if and only if } \|\dot{c}_1 = \dot{c}_2\| = 1_{\mathbb{B}}.$$

(ii) If $\dot{c}_1, \dot{c}_2 \in \mathbb{D}$, then $\dot{c}_1 +_{\mathbb{D}} \dot{c}_2$ is the unique $\dot{c} \in \mathbb{D}$ such that $\|\dot{c} = \dot{c}_1 +_{\mathbb{B}} \dot{c}_2\| = 1_{\mathbb{B}}$. The operations $\cdot_{\mathbb{D}}$ and $-_{\mathbb{D}}$ are defined similarly. For the ordering as well,

$$\dot{c}_1 \leq \dot{c}_2 \text{ if and only if } \|\dot{c}_1 \leq \dot{c}_2\| = 1_{\mathbb{B}}.$$

Proposition 33. *With these operations $\mathbb{D}$ is indeed a complete atomless Boolean algebra. Further, $\mathbb{B}$ embeds in $\mathbb{D}$ as a complete subalgebra.*

Theorem 34. *Let M be a model of set theory, and let $\mathbb{B}$ and $\mathbb{D}$ be complete atomless Boolean algebras such that $\mathbb{B}$ is a complete subalgera of $\mathbb{D}$. Then in $M^{\mathbb{B}}$, there is a $\dot{\mathbb{C}}$ such that*

$$\|\dot{\mathbb{C}} \textit{ is a complete Boolean algebra}\| = 1_{\mathbb{B}},$$

and such that $\mathbb{D} = \mathbb{B} * \dot{\mathbb{C}}$.

Corollary 35. *Let M be a model of set theory. If $M[G]$ and $M[H]$ are generic extensions of M such that $M[G] \subset M[H]$, then $M[H]$ is a generic extension of $M[G]$.*

Theorem 36. *Let κ be a regular uncountable cardinal. Let $\mathbb{P}, \mathbb{Q}$ be forcing posets. Then $\mathbb{P} * \dot{\mathbb{Q}}$ has the κ-cc iff $\Vdash_{\mathbb{P}}$"$\dot{\mathbb{Q}}$ has the κ-cc".*

2.3 Modal Logic Background

This section contains a concise introduction to the parts of modal logic that will be used in this thesis. Unless explicitly mentioned, proofs of all of the statements mentioned in this section can be found in any of the standard textbooks [BdRV02, CZ97]. The following are some well-studied modal axioms:

$$\begin{array}{rl}
\mathsf{K} & \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \\
\mathsf{Dual} & \neg\Diamond\varphi \leftrightarrow \Box\neg\varphi \\
\mathsf{T} & \Box\varphi \to \varphi \\
\mathsf{4} & \Box\varphi \to \Box\Box\varphi \\
.2 & \Diamond\Box\varphi \to \Box\Diamond\varphi \\
.3 & (\Diamond\varphi \wedge \Diamond\psi) \to \Diamond[(\varphi \wedge \Diamond\psi) \vee (\Diamond\varphi \wedge \psi)] \\
\mathsf{5} & \Diamond\Box\varphi \to \varphi
\end{array}$$

In this thesis a *modal theory*, will be a collection of modal axioms which are closed under modus ponens, necessitation, and substitution. Any such modal theory is then completely described by the axioms which generate it, and this is how modal theories will be referred to in this thesis. All the modal theories discussed in this thesis will in fact be *normal*. That is, the axioms K and Dual axioms will be in this theory. Hence, whenever we talk about a modal theory, the reader should assume that we are talking about a theory which has all of these properties.

The following are some well-studied modal theories:

$$\begin{aligned}
\mathsf{KDual} &= \mathsf{K} + \mathsf{Dual} \\
\mathsf{TKDual} &= \mathsf{K} + \mathsf{Dual} + \mathsf{T} \\
\mathsf{S4} &= \mathsf{K} + \mathsf{Dual} + \mathsf{T} + 4 \\
\mathsf{S4.2} &= \mathsf{K} + \mathsf{Dual} + \mathsf{T} + 4 + .2 \\
\mathsf{S4.3} &= \mathsf{K} + \mathsf{Dual} + \mathsf{T} + 4 + .3 \\
\mathsf{S5} &= \mathsf{K} + \mathsf{Dual} + \mathsf{T} + 4 + 5.
\end{aligned}$$

We point out that in the literature KDual and TKDual are often referred to simply as K and T respectively. It is not very hard to see that $\mathsf{S4} \vdash 5 \to .3$ and $\mathsf{S4} \vdash .3 \to .2$, and therefore these theories are linear, in the sense that:

$$\mathsf{KDual} \subseteq \mathsf{TKDual} \subseteq \mathsf{S4} \subseteq \mathsf{S4.2} \subseteq \mathsf{S4.3} \subseteq \mathsf{S5}.$$

The standard notion of semantics for modal logic is that of Kripke models, which consist of an underlying set called the *set of worlds* or *nodes*, and an order on the elements of this set, called the *accessibility relation* (the two of these together are called a *frame*), and for each element of the set of nodes, a valuation for each of the propositional variables. The modal operators are then interpreted using the ordering. Given a Kripke model, its frame is referred to as the *underlying frame* of the model.

If F is a frame, a modal assertion is *valid for F* if it is true at all worlds of all the Kripke models having F as a frame. If $\mathcal{C}$ is a class of frames, a modal theory is *sound with respect to $\mathcal{C}$* if every statement from the theory is valid on every frame in $\mathcal{C}$. A modal theory is *complete with respect to* $\mathcal{C}$ if every statement which is valid for every frame in $\mathcal{C}$ is in the theory, or equivalently, if for any statement φ not in the theory, there is a model based on a frame in $\mathcal{C}$ and a node in the model which satisfies $\neg\varphi$. Note that this is called *weakly complete* in [BdRV02]. A modal logic is *characterised by* $\mathcal{C}$ if it is both sound and complete with respect to $\mathcal{C}$. If M is a Kripke model, w a node of the frame, and φ a modal formula, if φ is true at w then we write $M, w \Vdash \varphi$. Otherwise we write $M, w \nVdash \varphi$. If φ is valid on M, we write this as $M \vDash \varphi$.

Another very important notion that we will use throughout the thesis is the following:

Definition 37. Let $M = (W, R, V)$ and $M' = (W', R', V')$ be two Kripke models. A non-empty relation $Z \subseteq W \times W'$ is called a *bisimulation* between M and M' if the following conditions are satisfied:

(i) If wZw', then w and w' satisfy the same proposition letters;

(ii) If wZw' and Rwv, then there exists $v' \in W'$ such that vZv' and $R'w'v'$;

(iii) If wZw' and $R'w'v'$, then there exists $v \in W$ such that vZv' and Rwv.

In this case, we say that M and M' are *bisimilar*. Also, if $w \in W$ and $w' \in W'$ are such that there is a bisimulation Z between M and M' such that wZw', then we say that w and w' are *bisimilar nodes*.

Proposition 38. *Let M and M' be bisimilar Kripke models, and let $w \in W$ and $w' \in W'$ be bisimilar nodes. Then for each formula φ, $M, w \Vdash \varphi$ if and only if $M', w' \Vdash \varphi$. Consequently, if two Kripke models are bisimular, then the set of all formulas which are valid on them is exactly the same.*

Proposition 39. *Let M and M' be bisimilar Kripke models. Let M'' be another Kripke model. Then M and M'' are bisimilar if and only if M'' and M' are bisimilar.*

The modal theories we have mentioned are characterised by some very natural classes of finite frames. In Chapter 4, we shall see a method for proving such results, the method of *canonical models* (see Definition 71).

Theorem 40. *The modal logic* KDual *is characterised by the class of all finite frames.*

Theorem 41. *The modal logic* TKDual *is characterised by the class of all reflexive frames.*

Theorem 42. *The modal logic* $\mathsf{S4}$ *is characterised by the class of finite pre-orders.*

Theorem 43. *The modal logic* $\mathsf{S5}$ *is characterised by the classs of finite equivalence relations with one equivalence class.*

Theorem 44. *The modal logic* $\mathsf{S4.3}$ *is characterised by the class of finite linear pre-order frames.*

The first half of the following theorem is from [HL08], whereas the second half is standard.

Theorem 45. *The modal logic* $\mathsf{S4.2}$ *is characterised by the class of finite pre-Boolean algebras as well as by the class of finite directed pre-orders.*

Using these characterisation results, we can prove for example that each of the containments between these logics that we mentioned above is strict.

Definition 46. Let $M = (W, R, V)$ be a Kripke frame. Let w be a node in M. Then the *submodel of M generated by w* (also called the *submodel of M rooted at w*), $M[w] = (W', R', V')$, is the following Kripke frame:

(i) The set of worlds is the smallest set $W' \subseteq W$ such that $w \in W'$, and for all $w' \in W'$, if $v \in W$ is such that $Rw'v$, then $v \in W'$;

(ii) The relation R' is the restriction of R to W';

(iii) The valuation V' is the restriction of V to W'.

If F is a Kripke frame, and w is a node in F, we can similarly define the *subframe of F generated by w* (also called the *subframe of F rooted at w*, $F[w]$.

Proposition 47. *Let M be a Kripke frame, and w a node in M. Then for each node v in $M[w]$, for each formula φ,*

$$M, v \Vdash \varphi \text{ iff } M[w], v \Vdash \varphi.$$

Theorem 48. *Let $\mathcal{F}$ be a class of frames, and Λ a modal theory. Let $\mathcal{F}'$ be the class of all rooted subframes of frames in $\mathcal{F}$. Then $\mathcal{F}'$ is sound and complete with respect to Λ if and only if $\mathcal{F}$ is.*

Corollary 49. *Let $\mathcal{F}$ be a class of frames such that the class of all rooted subframes of frames in $\mathcal{F}$ is contained in $\mathcal{F}$. Let Λ be a modal theory. Then $\mathcal{F}$ characterises Λ iff the subclass of all rooted frames in $\mathcal{F}$ characterises Λ.*

We note that all of the classes of frames that we will consider in this thesis are closed under rooted subframes.

Definition 50. Let (W, R) be a rooted finite directed partial order. Let r be the root of W, and t the top node. Then the *unravelling* of (W, R) is the partial order $(U, \leq_U)$ obtained as follows:

(i) U is the set of all paths from r to w, for $w \in W$.

(ii) If $u, v \in U$, then $u \leq_U v$ if u is a subpath of v.

Further, if $M = (W, R, V)$ is a Kripke model based on (W, R), then the unravelling of M' is the following Kripke model:

(i) The underlying frame of M' is the unravelling $(U, \leq_U)$ of (W, R).

(ii) The valuation V' is defined as follows: If $u \in U$ is a path from r to some $w \in W$, then for each propositional variable p,

$$M, w \Vdash p \text{ iff } M', u \Vdash p.$$

Apart from these basic results, whenever more specific results from modal logic are needed in some chapter, we will prove those results in that chapter.

Chapter 3

Interpreting Modal Logic in Set Theory

This chapter serves as an introduction to the modal logics of set-theoretic constructions. In Section 3.1, we show formally how, given a relation Γ between models of set theory, modal logic can be interpreted in set theory using Γ. We then give an approach to calculating this modal logic by splitting up the task into one of finding upper and lower bounds for it. The lower bounds involve proving general structural results about Γ, which we shall see in Section 3.2, whereas the upper bounds involve finding special models M of set theory such that the subrelation of Γ generated by M can be 'nicely described', which we shall see in Section 3.3. The operative definition of 'nicely described' requires the notion of a Γ-*labelling*, Definition 56, of a frame over a model of set theory. The usefulness of this notion is witnessed by the next theorem, Theorem 57, which shows how Γ-labellings of frames over a given model of set theory allow us to give upper bounds on the Γ-interpretation of modal logic of this model. In the next section, Section 3.4 we establish some useful techniques for obtaining upper bounds in this way. Finally, in the last section, Section 3.5, we consider the example of ccc-forcing and use it to demonstrate the how our techniques work.

3.1 The Basic Setup

Let Γ be a relation between models of set theory. In this section we show how modal logic can be interpreted in set theory using this relation. In order to do this, let us first develop some basic vocabulary, and then fix a suitable language in which to do our investigation.

Definition 51. If $(M, N) \in \Gamma$, then we say that N is a Γ-*extension* of M. Further, the smallest collection of models of set theory which contains M and is closed under the Γ-extension relation is called the *multiverse of* M *generated by* Γ.

3.1.1 von Neumann-Bernays-Gödel Set Theory

The metatheory we work in in this thesis is von Neumann-Bernays-Gödel set theory, NBG, as opposed to ZFC. The main difference between NBG and ZFC is that the former is a two-sorted theory, with one sort for sets, and another sort for classes. Nonetheless, NBG is a conservative extension of ZFC, and hence, any statement in the language of ZFC which NBG proves can be proved in ZFC itself.

That is, the language $\mathcal{L}_{\mathsf{NBG}}$ of NBG consists of set variables, which we denote by lower case letters, and class variables, which we denote by upper case letters. The logical symbols of NBG are quantifiers, and the relations $=$ and $\in$. If A and B are classes, then it cannot be the case that $A \in B$. However, one may represent a class by a set, and each set is a class. The language of ZFC, $\mathcal{L}_{\in}$, has the same symbols as L_{NBG}, except that it has no class variables.

The axioms of NBG are the following (note that the names of those axioms which only deal with sets are in lower case, and those which deal with classes are in upper case):

(i) The following axioms of ZFC: extensionality, union, pairing, powerset, infinity;

(ii) A version of Extensionality for classes, which says that $\forall A \forall B[\forall x(x \in A) \leftrightarrow (x \in B)] \rightarrow A = B$.

(iii) A version of Foundation, which says that each non-empty class contains an element which it is disjoint from;

(iv) Size, which says that a class C is a set if and only if there is no class-bijection between it and the class of all classes V;

(v) A version of Comprehension, which says that for any formula φ which does not contain any class-quantifiers, there is a class A such that $x \in A \leftrightarrow \varphi(x)$.

Further, the last item here is the only axiom scheme of NBG, and it can be shown to be equivalent to a conjunction of finitely many instances of it. Hence, NBG is finitely axiomatisable. See [Men97] for a detailed introduction to NBG.

3.1.2 The Interpretation

In order to interpret modal logic in set theory, we need to make some assumptions about Γ which allow us to express the modalities in the language of set theory. Indeed, it was this motivation that guided our choice of NBG as a metatheory. Given our choice of metatheory (and our motivation of wanting to express the modalities in this language), it is then natural to restrict our choice of Γ to those relations between models of set theory which are definable in NBG.

That is, we assume that there is a predicate $\varphi(X, Y)$ in the language $\mathcal{L}_{\mathsf{NBG}}$ with two free class variables X and Y such that if $M \vDash \mathsf{NBG}$, then for any class N,

$$N \text{ is a } \Gamma\text{-extension of } M \text{ iff } N \text{ has the same ordinals as } M, \text{ and } N \vDash \mathsf{NBG}, \text{ and } \varphi(M, N).$$

Now, for the interpretation itself. Recall that the language of propositional modal logic, $\mathcal{L}_{\Box}$, has propositional variables, logical connectives and the modal operators $\Diamond$ and $\Box$. Now, for any relation between models of set theory, Γ, which is definable in NBG in the above sense, and any model M of NBG, any assignment $p_i \mapsto \psi_i$ of propositional variables to $\mathcal{L}_{\mathsf{NBG}}$-statements in M recursively extends to a Γ-translation $T : \mathcal{L}_{\Box} \rightarrow \mathcal{L}_{\mathsf{NBG}}$ as follows:

$$\begin{aligned} T(p_i) &= \psi_i \\ T(\eta \wedge \theta) &= T(\eta) \wedge T(\theta) \\ T(\neg \eta) &= \neg T(\eta) \\ T(\Box \eta) &= \Box_{\Gamma} T(\eta), \end{aligned}$$

where the last line means that $T(\eta)$ holds in any Γ-extension of M (note that our assumptions imply that this is $\mathcal{L}_{\mathsf{NBG}}$-expressible). In this case, we say that $T(\eta)$ is Γ-*necessary* in M.

More generally, an $\mathcal{L}_{\mathsf{NBG}}$ statement φ is Γ-*necessary* over a model of set theory, written $\Box_\Gamma\varphi$, if φ holds in every Γ-extension of the model. An $\mathcal{L}_{\mathsf{NBG}}$ statement φ is Γ-*possible* over a model of set theory, written $\Diamond_\Gamma\varphi$, if φ holds in some Γ-extension of the model. We call these the Γ-*modalities.* It is easily seen that these modalities satisfy the requirements to define a normal modal logic:

$$\begin{aligned} \mathsf{K} \quad & \Box_\Gamma(\varphi \to \psi) \to (\Box_\Gamma\varphi \to \Box_\Gamma\psi), \\ \mathsf{Dual} \quad & \neg\Diamond_\Gamma\varphi \leftrightarrow \Box_\Gamma\neg\varphi. \end{aligned}$$

An important point to be made here is that in the case that Γ is in fact definable in $\mathcal{L}_\in$, that is, definable in the language of ZFC, and that the assignment $p_i \mapsto \psi_i$ sends propositional variables to $\mathcal{L}_\in$-statements, then we can similarly make a translation $T' : \mathcal{L}_\Box \to \mathcal{L}_\in$. Here, by definable we mean the following: for any $\mathcal{L}_\in$-statement ψ, there is an $\mathcal{L}_\in$-statement φ such that

$$M \models \varphi(\ulcorner\psi\urcorner) \text{ if and only if for each } \Gamma\text{-extension } N \text{ of } M, N \models \psi,$$

where $\ulcorner\psi\urcorner$ is a code for the formula ψ.

To clarify, in this thesis, we will be interested in three types of Γ:

(i) If $\mathcal{P}$ is some definable class of forcing notions, with some fixed definition, then the relation N is a forcing extension of M by some forcing poset $\mathbb{P} \in \mathcal{P}^M$ (in this case we say that N is a $\mathcal{P}$-extension of M).

(ii) If $\mathcal{P}$ is as above, then the relation M is a forcing extension of N by some forcing poset $\mathbb{P} \in \mathcal{P}^N$ (in this case we say that N is a $\mathcal{P}$-ground of M, or that N is a $\bar{\mathcal{P}}$-extension of M).

(iii) N is a definable inner model of M with a formula whose parameters are in M.

Of these three, in the first two cases, this modality is indeed $\mathcal{L}_\in$-definable. This is easiest to see in the first case: Here, $\Box_\Gamma T(\eta)$ holds in M if and only if

$$M \models \forall\mathbb{B} \in \mathcal{P}, \|T(\eta)\| = 1_{\mathbb{B}},$$

and clearly the above formula can be expressed in ZFC.

In the case of the second one this follows by the Laver-Woodin Theorem, Theorem 99, which says that for any model M of ZFC, there is a formula φ defining a class of parameters P, and a Σ_2 formula ψ such that all grounds of M are definable with this formula ψ from a parameter in P.

Therefore, any results that we prove about the modal logic of $\mathcal{P}$-extensions or about the modal logic of $\mathcal{P}$-grounds can be proved using ZFC alone. On the other hand, when talking about the modal logic of inner models, we have no such way to $\mathcal{L}_\in$-define the modalities, and in this case, all of the results that we shall prove shall be in NBG. From now on, we do not focus on these metamathematical matters, in the hope that the reader will use the above heuristic to gauge which results can be proved in ZFC alone, and which require NBG.

Now, having done such a translation, we define the NBG-*provable modal logic of* Γ *over* M as follows:

$$\mathsf{ML}_\Gamma^M \triangleq \{\varphi \in \mathcal{L}_\Box \mid M \models T(\varphi) \text{ for all } \Gamma\text{-translations } T : \mathcal{L}_\Box \to \mathcal{L}_{\mathsf{NBG}}\}.$$

Further, we can define the NBG-*provable modal logic of* Γ as follows:

$$\mathsf{ML}_\Gamma \triangleq \{\varphi \in \mathcal{L}_\Box \mid \mathsf{NBG} \text{ proves } T(\varphi) \text{ for all } \Gamma\text{-translations } T : \mathcal{L}_\Box \to \mathcal{L}_{\mathsf{NBG}}\}.$$

In the case that Γ is the relation of being a $\mathcal{P}$-extension for some definable $\mathcal{P}$ whose definition we have fixed as well, then we define the ZFC-*provable modal logic of* $\mathcal{P}$ *over* M as follows:

$$\mathsf{ML}^M_\mathcal{P} \triangleq \{\varphi \in \mathcal{L}_\Box \mid M \vDash T(\varphi) \text{ for all } \Gamma\text{-translations } T : \mathcal{L}_\Box \to \mathcal{L}_\in\},$$

and the ZFC-*provable modal logic of* $\mathcal{P}$ as follows:

$$\mathsf{ML}_\mathcal{P} \triangleq \{\varphi \in \mathcal{L}_\Box \mid \mathsf{ZFC} \text{ proves } T(\varphi) \text{ for all } \Gamma\text{-translations } T : \mathcal{L}_\Box \to \mathcal{L}_\in\}.$$

We can similarly define the ZFC-*provable modal logic of* $\mathcal{P}$-*grounds over* M, $\mathsf{ML}^M_{\bar{\mathcal{P}}}$, and the ZFC-*provable modal logic of* $\mathcal{P}$-*grounds*, $\mathsf{ML}_{\bar{\mathcal{P}}}$.

Before we proceed, we make an important clarification. In the case that Γ is a definable relation in some way, for example when it arises from the relation of taking $\mathcal{P}$-extensions for some fixed definable class of poset (with a fixed definition), then this definition needs to be interpreted appropriately in each model of set theory. For example if Γ is obtained from the relation of $\mathcal{P}$-extensions, where $\mathcal{P}$ is the class of all ccc-posets, then we first fix a formula φ which defines this class, and then $(M, N) \in \Gamma$ iff there is a $\mathbb{P} \in M$ such that $M \vDash \varphi(\mathbb{P})$ and N is the generic extension of M by $\mathbb{P}$.

To see why this is relevant, note that if $T \in M$ is a Suslin tree, then there is a ccc-poset $\mathbb{Q} \in M$ such that in any generic extension N of M by $\mathbb{Q}$, $N \vDash$"T is not ccc". See the Specialisation Theorem, Theorem 136, in Chapter 5 for details.

3.2 Lower Bounds

Having fixed our language, we can now try to understand the procedure of computing ML_Γ for some Γ. The approach we take is to split the problem up into two parts, namely to establish separately upper and lower bounds for ML_Γ. The techniques required for these two parts are quite different, and we now discuss these.

In the case of lower bounds, general results about Γ are required.

Definition 52. Let Γ be a relation on models of set theory.

(i) We say that Γ is *reflexive* if for each model M of set theory, $(M, M) \in \Gamma$. In the case when Γ is obtained from some class of posets $\mathcal{P}$, this corresponds to a trivial forcing poset being in $\mathcal{P}$.

(ii) We say that Γ is *transitive* if for any models M_1, M_2, M_3 such that $(M_1, M_2) \in \Gamma$ and $(M_2, M_3) \in \Gamma$, $(M_1, M_3) \in \Gamma$ as well. This corresponds to $\mathcal{P}$ being closed under finite iterations.

(iii) We say that Γ is *directed* if for any models M_1, M_2, M_3 such that $(M_1, M_2) \in \Gamma$ and $(M_1, M_3) \in \Gamma$, there is a model M_4 such that $(M_2, M_4) \in \Gamma$ and $(M_3, M_4) \in \Gamma$. The corresponding property of $\mathcal{P}$ is that for any two posets $\mathbb{P}_1, \mathbb{P}_2 \in \mathcal{P}$, there is a third poset $\mathbb{P}_3 \in \mathcal{P}$ such that the Boolean completions of $\mathbb{P}_1$ and $\mathbb{P}_2$ both embed as complete subalgebras into the Boolean completion of $\mathbb{P}_3$ (we say that in this case $\mathbb{P}_3$ *absorbs* both $\mathbb{P}_1$ and $\mathbb{P}_2$).

There are many natural examples of $\mathcal{P}$ for which the corresponding Γ is reflexive, transitive and directed. For example, when $\mathcal{P}$ is the class of all forcing notions, or the class of Knaster posets. This can easily seen by noticing that any forcing (Knaster) poset stays a forcing (Knaster) poset in any forcing (Knaster) extension. As observed in [HLL], any $\mathcal{P}$ which is transitive, persistent, and closed under products gives rise to a directed Γ, where $\mathcal{P}$ is *persistent* if in any model V of set theory, for any $\mathbb{P}, \mathbb{Q} \in \mathcal{P}^V$, $\mathbb{Q} \in \mathcal{P}^{V^{\mathbb{P}}}$. While reflexivity and transitivity are easily satisfied for many natural classses of posets, directedness is much harder. In particular, as observed in [HL08] and [HLL], directedness does not hold for any class of posets which contains all ccc posets and is contained in the class of ω_1-preserving posets. This can be seen by noticing that given a Suslin tree T, the two ccc posets which respectively add a branch to T and specialise T cannot be simultaneously absorbed by any ω_1-preserving poset, see Theorem 67 for details. Hence, when $\mathcal{P}$ is the class of **ccc** posets, or proper posets, or stationary-set preserving posets etc, Γ is not directed.

The importance of these structural properties is easy to see. The structures whose modal logic we are studying can be seen as Kripke frames with universes of set theory at the nodes, and the relations between them given by elements of Γ. It is then obvious that if this relation is, say, reflexive then if M is a model such that all of its Γ-successors satisfy a certain sentence, them M itself must satisy this sentence. The next proposition expresses this sort of reasoning.

Proposition 53. *Let Γ be a relation between models of set theory.*

(i) If Γ is reflexive, then S *is valid for* ML_Γ*.*

(ii) If Γ is transitive, then $\mathsf{4}$ *is valid for* ML_Γ*.*

(iii) If Γ is directed, then .2 *is valid for* ML_Γ*.*

Proof. Trivial. □

In this sense, proving general structural results about Γ allows us to give lower bounds for ML_Γ. In the case of $\mathcal{P}$-extensions for natural classes of posets, it is usually easy to see which of these structural properties are possessed by this relation. On the other hand, it is harder to give lower bounds for the $\mathcal{P}$-ground relation. For example, no answer is known to the following fundamental question from [Rei06]:

Question 54. *Let $\mathcal{P}$ be the class of all forcing posets. Is the relation of being a $\mathcal{P}$-ground directed (this modal logic is called the* modal logic of grounds*)?*

A model M of set theory for which the above is true is said to satisfy the axiom of *downwards directedness of grounds* (DDG). In all universes in which the truth value of DDG has been calculated, it has been found to be true.

Indeed, in [HL13], Hamkins and Löwe prove the following theorem (we give a proof of this in Chapter 4):

Theorem 55. *If* ZFC *is consistent, then there is a model M of* ZFC *for which the modal logic obtained of $\mathcal{P}$-grounds (with $\mathcal{P}$ being the class of all forcing posets) is exactly* S4.2*.*

If the answer to the above question is 'yes', then this result would show that the modal logic of grounds is exactly S4.2. That is, the only obstacle in the path of showing that the modal logic of grounds is S4.2 is that the best known lower bounds for this modal logic are S4, and that there is no ZFC-proof of DDG. Indeed, in their proof, Hamkins and Löwe pick a model of ZFC which is a model of DDG, and such that its ground models have enough structure to attain this lower bound.

We also point out that one might also ask variations of Question 54 about subclasses of the class of all posets. For example, for the relation of ccc-grounds, or grounds via proper forcing etc.

3.3 Upper Bounds

In this section we explain our main technique to obtain upper bounds for ML_Γ. Before we do this, we fix some notation:

The next definition is a minor generalisation of a notion which was first defined in [HLL]. In particular, in that paper, the Γ studied were those which arose from transitive forcing classes, whereas our definition works for any Γ.

Definition 56. Let $(F, \leq_F)$ be a Kripke frame. A Γ-*labelling* of F for a model of set theory W is an assignment to each node w in F a set-theoretic statement Φ_w such that

(i) The statements Φ_w form a mutually exclusive partition of truth in the multiverse of W generated by Γ. That is, if W' is in the multiverse of W generated by Γ, then W' satisfies exactly one of the Φ_w.

(ii) Any W' in the multiverse of W generated by Γ in which Φ_w is true satisfies $\Diamond_\Gamma \Phi_u$ if and only if $w \leq_F u$.

(iii) If w_0 is an initial element of F, then $W \vDash \Phi_{w_0}$.

The formal assertion of these properties is called the Jankov-Fine formula for F (cf. [HL08]). The next theorem, which is also a similar minor generalisation of a theorem from [HLL] (which itself generalises a result from [HL08]), is our main technique to calculate upper bound for ML_Γ. The same proof as in [HLL] suffices.

Theorem 57. *Suppose that $w \mapsto \Phi_w$ is a Γ-labelling of a finite Kripke frame F for a model of set theory W and that w_0 is an initial element of F. Then for any model M based on F, there is an assignment of the propositional variables to set theoretic assertions $p \mapsto \psi_p$ such that for any modal assertion $\varphi(p_0, p_1, \ldots, p_n)$,*

$$(M, w_0) \Vdash \varphi(p_0, p_1, \ldots, p_n) \text{ iff } W \vDash \varphi(\psi_{p_0}, \psi_{p_1}, \ldots, \psi_{p_n}).$$

In particular, any modal assertion that fails at w_0 in M also fails in W under this Γ-interpretation. Consequently, the modal logic of Γ over W is contained in the modal logic of assertions valid in F.

Proof. Let $w \mapsto \Phi_w$ be a labelling of F for W. Let M be a Kripke model with frame F. For each propositional variable p, consider the formula

$$\psi_p \triangleq \bigvee \{\Phi_w \mid (M, w) \Vdash p\}.$$

Now, we prove the following more uniform claim that whenever W' is in the multiverse generated by Γ of W, and $W' \vDash \Phi_w$, then

$$(M, w) \Vdash \varphi(p_0, p_1, \ldots, p_n) \text{ iff } W' \vDash \varphi(\psi_{p_0}, \psi_{p_1}, \ldots, \psi_{p_n}).$$

We prove this for all W' simultaneously by induction on the complexity of φ. The atomic case follows trivially from the definition of the ψ_p. The Boolean combinations also follow immediately from this. What is left then is to prove this for the case when there are new modal operators involved. If $W' \vDash \Diamond\varphi(\psi_{p_0}, \psi_{p_1}, \ldots, \psi_{p_n})$ then there is a W'' such that $(W', W'') \in \Gamma$ such that $W'' \vDash \varphi(\psi_{p_0}, \psi_{p_1}, \ldots, \psi_{p_n})$. In this case, W'' must satisfy some Φ_u, and then, by the induction hypothesis, $(M, u) \Vdash \varphi(p_0, p_1, \ldots, p_n)$. As Φ_u is true in a Γ-extension of W', where Φ_w is true, it follows that $w \leq_F u$, and hence, $(M, w) \Vdash \Diamond\varphi(p_0, p_1, \ldots, p_n)$.

For the other direction, if $(M, w) \Vdash \Diamond\varphi(p_0, p_1, \ldots, p_n)$, then there is a u such that $w \leq_F u$ and $(M, u) \Vdash \varphi(p_0, p_1, \ldots, p_n)$. Hence, by the induction hypothesis, any W' in the multiverse generated by Γ of W with Φ_u must satisfy $\varphi(\psi_{p_0}, \psi_{p_1}, \ldots, \psi_{p_n})$. Now, for any W' where Φ_w is true, there is an Γ-extension where Φ_u is true, it follows that any such W' will satisfy $\Diamond\varphi(\psi_{p_0}, \psi_{p_1}, \ldots, \psi_{p_n})$. The rest is immediate. □

Hence, proving upper bounds amounts to being able to describe all of the Γ-extensions of a model of set theory by statements of set theory, as well as being able to describe which of these Γ-extensions can be reached by Γ from the other Γ-extensions. A natural way to this is to pick a well-understood object in some model of set theory M, and then describe the behaviour of this object in all Γ-extensions of M. We shall see in Theorem 67 a small example of such a method, and in Chapter 5 we shall see a much more technically involved attempt at using such a method.

3.4 Buttons and Switches

In this section we discuss a slightly different approach to labellings. Whereas the one we just discussed involved understanding the combinatorial properties of a collection of objects in great detail, the approach we present here is less ad hoc.

In Section 2.3 of Chapter 2, we say that many standard modal logics, are characterised by some simple classes of frames. Often, these simple classes have an easy to describe branching structure. The control statements we define now are an attempt to find this sort of branching structure in statements of set theory. We note that in [HL08], where this concept was first isolated, and where the ancestors of the theorems we will talk about in this section were first proved, these ancestor-theorems were always stated as an equivalence between a modal logic being 'consistent with' some control statements and being contained in some standard modal logic. See for example [HL08, Theorem 11].

We will now stop mentioning Γ if it is clear from context. Also, in all that follows in this thesis, our Γ will be reflexive and transitive, so for a model N to be in the multiverse of M generated by Γ will be the same as for it to be an Γ-extension of M.

Let Γ be a reflexive transitive relation between models of set theory. Let M be a model of set theory. A set-theoretic statement ψ is a (Γ-)*switch* in M if both s and $\neg s$ are necessarily possible. Modally, this amounts to $M \vDash \Box(\Diamond\psi \wedge \Diamond\neg\psi)$. We will say that a switch is 'on' in N if $N \vDash \varphi$ is true. On the other hand, we will say that it is 'off' if $N \vDash \neg\varphi$. A set-theoretic statement ψ is a (Γ-)*button* in M if it is necessarily possibly necessary. Modally, this amounts to $M \vDash \Box\Diamond\Box\psi$. A button ψ is said to be *pushed* if $\Box\psi$ holds. Otherwise it is *unpushed*. A button is said to be *pure* in M if whenever it is true, it become necessarily true. Modally, this amounts to $M \vDash \Box(\psi \rightarrow \Box\psi)$. To each button ψ we can associate a corresponding pure button $\Box\psi$. Often, pure buttons are more convenient from the point of view of obtaining labellings.

A sequence of set-theoretic statements $\varphi_1, \varphi_2, \ldots, \varphi_n$ is said to be a (Γ-)*ratchet* of length n in M if each is an unpushed pure button, each of them necessarily implies the previous, and each can be pushed without pushing the next. Modally, this amounts to:

$$\begin{aligned}
&\neg\varphi_i \\
&\Box(\varphi_i \to \Box\varphi_i) \\
&\Box(\varphi_{i+1} \to \varphi_i) \\
&\Box[\neg\varphi_{i+1} \to \Diamond(\varphi_i \wedge \neg\varphi_{i+1})].
\end{aligned}$$

Given such a ratchet, the *value* of the ratchet in a Γ-extension N of M is the largest i such that $N \vDash \varphi_i$.

Note that a ratchet is unidirectional: The value of the ratchet can only increase as we take Γ-extensions. This allows us to divide all the Γ-extensions of M into levels, where a model N is in level i if the value of the ratchet is i in N. We can also generalise this notion to infinite lengths: A transfinite sequence of set-theoretic statements $\langle\varphi_\alpha \mid \alpha < \delta\rangle$ (possibly involving parameters) is said to be a ratchet of length δ in M if each is an unpushed pure button, each necessarily implies the previous, and each can be pushed without pushing the next. The ratchet is said to be *uniform* if there is a formula $\varphi(x)$ with one free variable such that $\varphi_\alpha = \varphi(\alpha)$. Every ratchet of a finite length is uniform. The ratchet is said to be continuous if for every limit ordinal $\lambda < \delta$ the statement φ_λ is equivalent to $\forall\alpha < \lambda\varphi_\alpha$. Any uniform ratchet can be made continuous by reindexing, by replacing φ_β by the statement $\forall\alpha < \beta\varphi_{\alpha+1}$. A *long* ratchet is a uniform ratchet $\langle\varphi_\alpha \mid \alpha \in \mathrm{ORD}\rangle$ of length ORD in M, with the additional property that no Γ-extension of M satisfies all of the φ_α. Hence, each Γ-extension of M has a well-defined ordinal ratchet value.

We stress here that each of the statements that we are talking about are set-theoretic statements. That is, they are expressible in $\mathcal{L}_{\mathsf{NBG}}$ or $\mathcal{L}_\in$, depending on the context. A collection $\mathcal{S}$ of statements are said to be (Γ-)*independent* in M if, necessarily, each of them can be 'realised' without affecting the truth value of the others. Here, by realising we mean the following:

(i) Realising a button amounts to pushing it. So for a button to be independent of a collection of statements means that we can push it without affecting the truth values of the other statements.

(ii) Realising a switch amounts to switching it on and off. So for a switch to be independent of a collection of statements means that we can switch it on and off without affecting the truth values of the other statements.

(iii) Realising a ratchet is to increase the value of the ratchet by one. So for a ratchet to be independent of a collection of statements means that we can increase its value by one without affecting the truth values of the other statements.

The value of these control statements is that they allow us to modularise the process of labelling frames. That is, we can break down the process of labelling many standard classes of frames to one of finding a suitable collection of control statements with some suitable dependencies holding between them. We shall now see some theorems (all of which are from [HLL], except Theorem 61, which is from [HL08]) which support this claim. We do not give proofs for all of them, but only the ones that we shall use.

Theorem 58. *If Γ is a reflexive transitive relation between models of set theory having arbitrarily large finite independent families of switches over a model of set theory M, then ML_Γ^M (and hence, ML_Γ as well) is contained in the modal theory* S5*.*

Theorem 59. *If Γ is a reflexive transitive relation between models of set theory having arbitrarily long finite ratchets mutually independent with arbitrarily large finite families of switches over a model of set theory M, then ML_Γ^M (and hence, ML_Γ as well) is contained in the modal theory* S4.3*.*

Proof. Note that by Theorem 44, any modal statement which is not in S4.3 fails on a Kripke model whose underlying frame is a finite linear pre-order. Then, by Theorem 57, if we can Γ-label each finite linear pre-order, we will be done. Now, let F be such a frame. Notice that each such frame consists of a finite increasing sequence of n clusters of mutually accessible worlds. That is, the kth cluster consists of n_k many worlds $w_0^k, w_1^k, \ldots w_{n_k-1}^k$ and the order is given by $w_m^k \leq w_n^l$ iff $k \leq l$. By adding dummy copies of worlds in each cluster (this does not affect truth in the Kripke model), we may assume that each of the clusters have the same size, and further, that this size is 2^m for some natural number m.

Let $r_1, r_2, \ldots r_n$ be a ratchet of length n for M which is mutually independent with the m switches $s_0, s_1, \ldots s_{m-1}$. We may assume that all of the switches are off. Let $\bar{r}_k$ be the statement that the ratchet value is exactly k. For each $j < 2^m$, let $\bar{s}_j$ be the statement that the pattern of switches is exactly that of the m binary digits of j. For example, if $m = 3$ and $j = 2$, then $\bar{s}_j$ is the statement $\neg s_0 \wedge s_1 \wedge \neg s_2$. Now, with the world w_j^k where $k < n$ and $j < 2^m$, we associate the statement $\Phi_{w_j^k} = \bar{r}_k \wedge \bar{s}_j$.

Now, since the ratchet value can never decrease, and any pattern of switches can be realised without affecting the ratchet value, it is easy to see that if N is in the multiverse of M generated by Γ, and $N \vDash \Phi_{w_j^k}$, then $N \vDash \Phi_{w_i^l}$ if and only if $k \leq l$. Also, since the ratchet value is 0 at M and all the switches are off, it follows that $M \vDash \Phi_{w_0^0}$. Hence, we have shown how to Γ-label any finite linear pre-order over M with these control statement, so the conclusions follow.

□

Often it is the case that when we can build ratchets, we can actually build a long ratchet. The following theorem tells us that in such a scenario, we can get by without considering switches:

Theorem 60. *If Γ is a reflexive transitive relation between models of set theory having a long ratchet over a model of set theory M, then ML_Γ^M (and hence, ML_Γ as well) is contained within the modal theory* S4.3*.*

Proof. Let $\langle r_\alpha \mid \alpha \in \mathrm{ORD}\rangle$ be a long ratchet over M. As we have already mentioned, we can obtain a continuous ratchet from this one, so let us assume that this ratchet is itself continuous. By the prevous theorem, it suffices to show that there are arbitrarily long finite ratchets independent from arbitrarily large finite families of switches over M. To do this, we divide the ordinals into blocks of length ω as follows: Each ordinal can be uniquely expressed in the form $\omega \cdot \alpha + k$ where $k < \omega$. We can think of this ordinal as being the kth element of the αth block of ordinals.

Now, let s_i be the statement that if the current ratchet value is exactly $\omega \cdot \alpha + k$, then the ith binary bit of k is 1. Let q_α be the statement that the current ratchet value lies in the interval $[\omega \cdot \alpha, \omega \cdot \alpha + \omega)$. It is easy to see that the q_α for a ratchet themselves, and the s_i are a collection of switches which are mutually independet of this ratchet. □

The next theorem, which is from [HL08], was used by Hamkins and Löwe to show that the modal logic obtained from the class of all forcing posets is exactly $\mathsf{S4.2}$ in the paper that started this investigation of the modal logics associated with set-theoretic constructions. We do not give its proof because in Chapter 4, in the proof of Theorem 95, we shall see a slightly more complicated version of this argument.

Theorem 61. *If Γ is a reflexive transitive relation between models of set theory having arbitrarily large finite families of mutually independent buttons and switches over a model M, then ML_Γ^M (and hence, ML_Γ as well) is contained within the modal theory $\mathsf{S4.2}$.*

Corollary 62. *If Γ is a reflexive transitive relation between models of set theory having arbitrarily large finite families of mutually independent buttons and switches over a model M, then ML_Γ^M (and hence, ML_Γ as well) is exactly the modal theory $\mathsf{S4.2}$.*

Proof. By Proposition 53, it is clear that $\mathsf{S4.2} \subseteq \mathsf{ML}_\Gamma$. On the other hand, by the previous theorem, $\mathsf{ML}_\Gamma^M \subseteq \mathsf{S4.2}$. Hence, we have that $\mathsf{S4.2} \subseteq \mathsf{ML}_\Gamma \subseteq \mathsf{ML}_\Gamma^M \subseteq \mathsf{S4.2}$, and the result follows. □

We now present a method from [HLL] which shows that in the case of Γ which are derived from some definable class of forcing posets $\mathcal{P}$, there is a natural way to extract a Γ-ratchet over $\mathbf{L}$ from $\mathcal{P}$ by 'measuring the distance' of any other model in the multiverse so obtained from $\mathbf{L}$.

Definition 63. A forcing notion $\mathbb{P}$ has *essential size* δ if the complete Boolean algebra corresponding to $\mathbb{P}$ has size δ and $\mathbb{P}$ is not equivalent to any poset $\mathbb{Q}$ whose Boolean completion has smaller size.

Definition 64. If W is a generic extension of $\mathbf{L}$, then the *forcing distance of W from* $\mathbf{L}$ is defined as the least $\mathbf{L}$-cardinal δ such that $W = \mathbf{L}[G]$ where G is $\mathbb{P}$-generic over $\mathbf{L}$ for a forcing notion $\mathbb{P}$ of essential size δ. In such a case, we write $\mathrm{fd}_\mathbf{L} = \delta$.

Now, it is natural to expect that if we have such a notion of distance from $\mathbf{L}$, then the distance can only increase if we go to any further forcing extensions. This is the content of the next lemma:

Lemma 65. *If $M \vDash \mathrm{fd}_\mathbf{L} = \delta$ and H is $\mathbb{Q}$-generic over M for some $\mathbb{Q} \in M$, then $M[H] \vDash \mathrm{fd}_\mathbf{L} \geq \delta$.*

Proof. Let $M[H] = \mathbf{L}[G]$ for some G which is $\mathbb{P}$-generic over $\mathbf{L}$ for some $\mathbb{P} \in \mathbf{L}$, and let $\mathbb{B}$ be its Boolean completion in $\mathbf{L}$. Since $\mathbf{L} \subseteq M \subseteq M[H] = \mathbf{L}[G]$, by Grigorieff's Theorem (this is Theorem 101 in Chapter 4), there is a complete subalgebra $\mathbb{C}$ of $\mathbb{B}$ such that $M = \mathbf{L}[\mathbb{C} \cap G]$. But now, we had assumed that $M \vDash \mathrm{fd}_\mathbf{L} = \delta$, and therefore, $|\mathbb{C}|^\mathbf{L} \geq \delta$. Since $\mathbb{C}$ is a complete subalgebra of $\mathbb{B}$, it follows that $|\mathbb{B}|^\mathbf{L} \geq \delta$. □

Theorem 66. *Suppose that Γ is obtained from a reflexive transitive class of forcing posets $\mathcal{P}$ with the property that there is a definable proper class C of regular cardinals in $\mathbf{L}$ such that in any forcing extension $\mathbf{L}[G]$ by a forcing notion in $\mathcal{P}$ of essential size δ, and any larger $\lambda \in C$, there is a forcing notion in $\mathcal{P}^{\mathbf{L}[G]}$ having essential size λ. Then $\mathsf{ML}_\Gamma^\mathbf{L}$ (and hence, ML_Γ as well) is contained in $\mathsf{S4.3}$.*

Proof. We construct a long ratchet. For each ordinal α, let w_α be the statement "$\mathrm{fd}_\mathbf{L}$ is larger than the αth element of C". By the previous lemma, each of the w_α is an unpushed button in $\mathbf{L}$. Our assumptions about Γ and C ensure that in any set-forcing extension $\mathbf{L}[G]$, if w_α is not satisfied, then we can force $w_\alpha \wedge w_{\alpha+1}$ with some poset in $\mathcal{P}$. Here, we use the fact that if δ is a regular cardinal, and $|\mathbb{B}|^\mathbf{L} < \delta$ and G is generic for $\mathbb{B}$ over $\mathbf{L}$, and $\mathbb{C} \in L[G]$ is such that $|\mathbb{C}|^{\mathbf{L}[G]} = \delta$, then $|\mathbb{B} * \dot{\mathbb{C}}|^\mathbf{L} = \delta$, which can easily be seen by looking at the definition of $\mathbb{B} * \dot{\mathbb{C}}$. Lastly, since any set-forcing extension is obtained by a poset of *some* size, no such extension can satisfy w_α. Therefore, $\langle w_\alpha \mid \alpha \in \mathrm{ORD} \rangle$ is indeed a long ratchet. Now, we apply Theorem 60 to reach our conclusion. $\square$

3.5 An Example: ccc Forcing

In this section, we give a demonstration of our techniques by proving a theorem from [HLL] (though the lower bound of $\mathsf{S4}$ and that $\mathsf{S4.2}$ is not an upper bound was already shown in [HL08]) which gives some upper and lower bounds for the modal logic obtained from ccc forcing.

Theorem 67. *$\mathsf{S4}$ is a valid principle for ML_{ccc}, whereas $\mathsf{S4.2}$ is not a valid principle for ML_{ccc}. Also, ML_{ccc} is contained in $\mathsf{S4.3}$.*

Proof. To see that $\mathsf{S4}$ is valid for ML_{ccc}, notice that the class of ccc posets is reflexive and transitive, and hence by Proposition 53, $\mathsf{S4}$ is valid for ML_{ccc}.

For the second part, we give a labelling of the two element fork frame which is a binary tree with a single root which has two successors. It is then easy to find a modal statement φ which is in $\mathsf{S4.2}$, but not true on a Kripke model based on this frame.

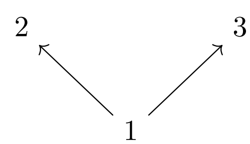

Figure 3.1: The two element fork

The labelling is done as follows: let φ be the statement "The $<_{\mathbf{L}}$-least Suslin tree in $\mathbf{L}$ is non-special Aronszajn". Let ψ be the statement "The $<_{\mathbf{L}}$-least Suslin tree in $\mathbf{L}$ is not Aronszajn". Let χ be the statement "The $<_{\mathbf{L}}$-least Suslin tree in $\mathbf{L}$ is special Aronszajn". Here, by 'special' we mean this: An ω_1-tree is special iff no ccc-poset can add a branch to it. This is *not* the standard definition, but is weaker than it. However, by the Specialisation Theorem, Theorem 136, of Chapter 5, we can specialise (in the traditional sense, and hence in our sense as well) any Aronszajn tree in a ccc-way. On the other hand, by definition, there is a ccc-poset adding a branch to any non-special Aronszajn tree. Hence, $\varphi \to \Diamond\psi$ and $\varphi \to \Diamond\chi$, and $\psi \to \Box\psi$ and $\chi \to \Box\chi$. It is easy to see then that we can label this three element frame with these statements.

For the third part, we use Theorem 66. In particular, we define C to be the class of uncountable successor cardinals of $\mathbf{L}$. We note that for each $\delta \in C$, the forcing poset Coh_δ, which adds δ many Cohen reals has essential size δ in $\mathbf{L}$ and in all forcing extensions $\mathbf{L}[G]$ which are obtained by set-forcing of essential size less than δ. Hence, the class of ccc posets satisfies all the requirements of Theorem 66, and hence the conclusion follows. $\square$

Chapter 4

The Modal Logic of Inner Models

In this chapter we shall investigate the modal logic of inner models. In particular, in Section 4.1, we shall augment the modal theory $\mathsf{S4.2}$ with an extra axiom to obtain the modal theory $\mathsf{S4.2Top}$, see Definition 85. We will then show that this theory is sound and complete with respect to a certain class of frames $\mathcal{L}$, and that the modal logic of inner models of any model of set theory contains $\mathsf{S4.2Top}$. Next, we will relate the modal logic of inner models with the modal logic of grounds in Section 4.2. We will then, in Section 4.3, exhibit a certain model of set theory whose modal logic of inner models is exactly $\mathsf{S4.2Top}$, hence reaching our conclusion. This model will be a modification of the 'bottomless model' of Reitz [Rei07], which was also used by Hamkins and Löwe in their study of the modal logic of grounds [HL13], and indeed, our approach to this chapter is to modify the proof of their main theorem to the context of inner models.

4.1 Modal Logic

In this section, we shall prove the results from modal logic that we shall need in this chapter. We use the method of canonical models [BdRV02, Chapter 4].

4.1.1 Canonical Models

First, we remind the reader of the basic results in the theory of canonical models. The proofs of all of the results that we mention can be found in [BdRV02, Chapter 4.2]. Let Λ be a normal modal theory.

Definition 68. A set of formulas Σ is Λ-*consistent* if there is a node in a Kripke model on which $\Sigma \cup \Lambda$ is valid. A set of formulas Σ is *maximal* Λ-*consistent* if Σ is Λ-consistent and any set of fomulas properly containing Σ is Λ-inconsistent. If Σ is maximal Λ-consistent, we say that it is an Λ-MCS.

Proposition 69. *Let Λ be a modal theory and Σ a Λ-MCS. Then:*

(i) *Σ is closed under modus ponens, necessitation and substitution;*

(ii) *$\Lambda \subseteq \Sigma$;*

(iii) *For all fomulas φ, either $\varphi \in \Sigma$ or $\neg\varphi \in \Sigma$;*

(iv) *For all formulas* φ *and* ψ, $\varphi \vee \psi \in \Sigma$ *iff* $\varphi \in \Sigma$ *or* $\psi \in \Sigma$.

Lemma 70. *(Lindenbaum Lemma) Let* Λ *be a modal theory. If* Δ *is* Λ*-consistent then there is* Σ*, a* Λ-MCS*, such that* $\Delta \subseteq \Sigma$*.*

Definition 71. The *canonical model* $\mathcal{M}^\Lambda$ for a modal theory Λ is the Kripke model given by the triple $(\mathcal{W}^\Lambda, \mathcal{R}^\Lambda, \mathcal{V}^\Lambda)$, where

(i) $\mathcal{W}^\Lambda$ is the set of all the Λ-MCS;

(ii) $\mathcal{R}^\Lambda$ is the binary relation on $\mathcal{W}^\Lambda$ defined by $\mathcal{R}^\Lambda wv$ if for all formulas ψ, if $\psi \in v$, then $\Diamond\psi \in w$. $\mathcal{R}^\Lambda$ is called the *canonical relation*;

(iii) $\mathcal{V}^\Lambda$ is the valuation defined by $\mathcal{V}^\Lambda(p) = \{w \in \mathcal{W}^\Lambda \mid p \in w\}$ for all propositional variables p. $\mathcal{V}^\Lambda$ is called the *canonical valuation.*

Also, we call the underlying frame $\mathcal{F}$ of $\mathcal{M}$ the *canonical frame* of the logic.

Lemma 72. *For any modal theory* Λ*,* $\mathcal{R}^\Lambda wv$ *iff for all formulas* φ*, if* $\Box\varphi \in w$ *then* $\varphi \in v$*.*

Lemma 73. *(Existence Lemma) For any modal theory* Λ*, and any node* $w \in \mathcal{W}^\Lambda$*, if* $\Diamond\varphi \in w$*, then there is a node* $v \in \mathcal{W}^\Lambda$ *such that* $\mathcal{R}^\Lambda wv$ *and* $\varphi \in v$*.*

Lemma 74. *(Truth Lemma) For any modal theory* Λ*, and any formula* φ*,* $\mathcal{M}^\Lambda, w \Vdash \varphi$ *iff* $\varphi \in w$*.*

Theorem 75. *(Canonical Model Theorem) Any modal theory* Λ *is complete with respect to its canonical model. That is, for any* $\varphi \notin \Lambda$*, there is a node* $w \in \mathcal{W}$ *such that* $\mathcal{M}, w \Vdash \neg\varphi$*.*

From now on, whenever the modal theory is clear from the context, it will not be explicitly mentioned in the notation of its canonical model. We will also not mention it when talking about consistent sets, with the understanding that in this context, we will, unless explicitly mentioned otherwise, always be talking about Λ-consistency where Λ is the modal theory.

Another important notion that we shall need while proving chracterisation theorems is the following:

Definition 76. A set of formulas Σ is *subformula closed* if for all formulas φ, ψ

(i) If $\varphi \vee \psi \in \Sigma$, then $\varphi, \psi \in \Sigma$;

(ii) If $\varphi \wedge \psi \in \Sigma$, then $\varphi, \psi \in \Sigma$;

(iii) If $\neg\varphi \in \Sigma$, then $\varphi \in \Sigma$;

(iv) If $\Diamond\varphi \in \Sigma$, then $\varphi \in \Sigma$;

(v) If $\Box\varphi \in \Sigma$, then $\varphi \in \Sigma$.

Definition 77. Let $M = (W, R, V)$ be a Kripke model, and Σ a subformula-closed set. Let $\leftrightsquigarrow_\Sigma$ be the equivalence relation on the states of M defined by:

$$w \leftrightsquigarrow_\Sigma v \text{ iff for all } \varphi \in \Sigma, [M, w \Vdash \varphi \text{ iff } M, v \Vdash \varphi.]$$

Note that $\leftrightsquigarrow_\Sigma$ is an equivalence relation. The equivalence class of a node w of a Kripke model M by this equivalence relation will be denoted $|w|_\Sigma$. The map $w \mapsto |w|_\Sigma$ is called the *natural map.*

Let $W_\Sigma = \{|w|_\Sigma \mid w \in W\}$. Let M^f_Σ be any model (W^f, R^f, V^f) such that:

(i) $W^f = W_\Sigma$;

(ii) If Rwv, then $R^f|w|_\Sigma|v|_\Sigma$;

(iii) If $R^f|w|_\Sigma|v|_\Sigma$, then for all $\Diamond\varphi \in \Sigma$, if $M, v \Vdash \varphi$, then $M, w \Vdash \Diamond\varphi$;

(iv) $V^f(p) = \{|w|_\Sigma \mid M, w \Vdash p\}$ for all propositional variables p.

Then M^f is called the *filtration* of M through Σ. We will often supress subscripts in the notation.

Theorem 78. *Let Σ be a subformula-closed set. Let M be a Kripke model and M^f a filtration of M through Σ. Let Λ be a set of formulas which is valid in M. Then for all formulas $\varphi \in \Lambda \bigcup \Sigma$, and all nodes $w \in M$,*

$$M, w \Vdash \varphi \text{ iff } M^f, |w|_\Sigma \Vdash \varphi.$$

In particular, all the formulas of Λ are still valid in M^f.

Proposition 79. *Let Σ be a finite subformula closed set. Let M be a Kripke model, and let Λ be a set of formulas which are valid on M. Then, if M^f is a filtration of M through Σ, then M^f is finite Kripke model such that all the formulas of Λ are still valid in M.*

4.1.2 Characterisation Theorems

In this section, we use the techniques developed in the previous subsection to prove some characterisation theorems for some modal logics. Recall that a modal theory characterises a class of frames if it is both sound and complete with respect to this class of frames. We remind the reader of the following modal axioms which we have already discussed:

K	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
Dual	$\neg\Diamond\varphi \leftrightarrow \Box\neg\varphi$
S	$\Box\varphi \to \varphi$
4	$\Box\varphi \to \Box\Box\varphi$
.2	$\Diamond\Box\varphi \to \Box\Diamond\varphi$

Theorem 80. *Let $\Lambda \supseteq \mathsf{S4.2}$ be a modal theory. Let $\mathcal{M} = (\mathcal{W}, \mathcal{R}, \mathcal{V})$ be its canonical model. Then $\mathcal{F}$ is reflexive, transitive, and directed.*

Proof. Let $w \in \mathcal{F}$. Then as $\Box\varphi \to \varphi \in w$, it follows by Lemma 72 that $\varphi \in w$ for every φ such that $\Box\varphi \in w$. Hence $w\mathcal{R}w$, and we have reflexivity.

For transitivity, let $u, v, w \in \mathcal{F}$ be such that $u\mathcal{R}v\mathcal{R}w$. Then for every φ such that $\Box\varphi \in u$, $\Box\Box\varphi \in u$ (by applying the Truth Lemma first, and then since u contains all instances of 4 and is closed under modus ponens, and then applying the Truth Lemma again), hence $\Box\varphi \in v$, hence $\varphi \in w$, where again we use Lemma 72. Summing up, for every formula φ, $\Box\varphi \in u$ implies $\varphi \in w$. Hence, by Lemma 72, $u\mathcal{R}w$. This gives us transitivity.

For directedness, let $u\mathcal{R}v$ and $u\mathcal{R}w$. If

$$\Phi = \{\varphi \mid \Box\varphi \in v \text{ or } \Box\varphi \in w\}$$

is Λ-consistent, then by the Lindenbaum Lemma, Lemma 70, there must be $u' \in \mathcal{W}$, a Λ-MCS, containing Φ. Further, by Lemma 72, $v\mathcal{R}u'$ and $w\mathcal{R}u'$, hence ensuring directedness. Towards a contradiction then, assume that there is a finite subset $\Phi' \subseteq \Phi$ such that Φ' is Λ-inconsistent. Let $\Phi' = \Psi_1 \cup \Psi_2$ where

$$\Psi_1 = \{\varphi \mid \Box\varphi \in v\}$$

and

$$\Psi_2 = \{\varphi \mid \Box\varphi \in w\}.$$

Note (by the Truth Lemma) that $v \Vdash \Box(\wedge\Psi_1)$. Hence, by the definition of $\mathcal{R}$, $v \Vdash \Diamond\Box(\wedge\Psi_1)$. But, by the Truth Lemma, for every formula φ,

$$u \Vdash \Diamond\Box\varphi \to \Box\Diamond\varphi.$$

Hence, using the fact that the Λ-MCS corresponding to u is closed under modus ponens and then by an application of the Truth Lemma, $u \Vdash \Box\Diamond(\wedge\Psi_1)$. In particular, by Lemma 72, $w \Vdash \Diamond(\wedge\Psi_1)$. Therefore, by the Existence Lemma, there is a node z such that $w\mathcal{R}z$ and $z \Vdash \wedge\Psi_1$. But $w \Vdash \Box(\wedge\Psi_2)$, so, by Lemma 72, $z \Vdash \wedge\Psi_2$ as well. Summing up, $z \Vdash (\wedge\Psi_1) \wedge (\wedge\Psi_2)$. Hence Φ' is Λ-consistent (note that all of our nodes force Λ), and we have a contradiction. Hence, $\mathcal{F}$ is directed, and we are done. □

We note that given the frequency of the usage of the results from Section 4.1.1 in the above proof, it is inconvenient to keep citing all of the results we use at each step. Therefore, in what follows, we will only mention the results from that section if we are using them in way that is not obvious.

Corollary 81. *The class of finite directed pre-orders characterises* S4.2*.*

Proof. Soundness is trivial. For completeness, we have done most of the hard work already. Let φ be a formula which is not in S4.2. Then there is a node w in the canonical model $\mathcal{M} = (\mathcal{W}, \mathcal{R}, \mathcal{V})$ of S4.2 such that $\varphi \notin w$. Let $\mathcal{M}[w]$ be the submodel of $\mathcal{M}$ generated by w. Let $\mathcal{M}' = (\mathcal{W}', \mathcal{R}', \mathcal{V}')$ be the model obtained from $\mathcal{M}[w]$ by filtrating by the finite subformula-closed set

$$\Phi = \{\psi \mid \psi \text{ is a subformula of } \varphi\}.$$

Clearly, $\mathcal{F}'$ is finite, and as $\mathcal{F}$ is reflexive, transitive and directed, so is $\mathcal{M}[w]$, and so is $\mathcal{M}'$. Hence, we have obtained a model $\mathcal{M}'$ based on a finite directed pre-order such that $\mathcal{M}' \vDash \mathsf{S4.2} \wedge \neg\varphi$, so we are done. □

Definition 82. A *baled tree* is a partial order $(P, \leq)$ such that there is a highest node $b \in P$ such that for every $p \in P$, $p \leq b$, and further, $(P \setminus \{b\}, \leq)$ is a tree. A *baled pre-tree* is a pre-order such that its quotient by the natural equivalence relation yields a baled tree.

It is clear that any finite directed partial order can be unravelled to get a finite baled tree, and that any finite directed pre-order can be unravelled to get a baled pre-tree. Also, note that for a finite baled tree, we can define a join operation: we define the join of two nodes, denoted $x \vee y$ if x and y are these nodes, to be the least node above them (where x is above y is $x \geq y$). This is unambiguous because if b is the topmost node of F, then for any two elements $e, f \in F$, either $e \vee f$ is e, or f, or b because $(P \setminus \{b\}, \leq)$ is a tree. Hence, either one of the elements is below the other, or their join is b. We can also extend this join operation to a Kripke model whose underlying frame is a baled tree by saying that the join of two worlds is the world whose position in the baled tree corresponds to the join of the positions of the elements.

We note that if F is a baled pre-tree, then we can also similarly define a cluster-join operation which, given two clusters, returns a cluster. This operation is defined as follows: if e, f are two clusters of F, then $e \vee f = d$ where d is the complete cluster such that if $[e]_\equiv, [f]_\equiv, [d]_\equiv$ are the natural equivalence classes of e, f, d respectively, then $[e]_\equiv \vee [f]_\equiv = [d]_\equiv$. We pick d to be a complete cluster, so combining this with the previous discussion about the unambiguity of the join operation on baled trees, we see that this cluster-join operation on baled pre-trees is unambiguous as well. We can also similarly define this cluster-join operation on Kripke models based on baled pre-trees in the obvious way.

Lemma 83. *Let M be a model based on a finite directed pre-order. Then there is a model M' based on a finite pre-Boolean algebra such that M is bisimilar to M'.*

Proof. Let $F = (W, \leq)$ be the frame underlying M and V the valuation on F. We can, without any loss of generality, assume that F is a baled pre-tree. Now, we first consider the case where F is a baled tree. Let r be the root of F. We call a node $w \in W$ *non-trivial* if it is not the root. Let S be the set of non-trivial nodes of F. Let n be the cardinality of S. We show that there is a model M' whose frame is the Boolean algebra on n generators which is bisimilar to M. We represent this Boolean algebra as the Boolean algebra $\mathbb{B}$ of all subsets of n.

Now, fix a bijective map $g : \{1, 2, \dots n\} \to S$. For each $a \subseteq \{1, 2, \dots, n\}$, let

$$w_a \triangleq \bigvee \{g(e) \mid e \in a\}.$$

Above, we define that $w_\emptyset = r$ for notational convenience. Also, note that we are taking the join of worlds of the Kripke model. Note that by the discussion above, since F is a finite baled tree, the above definition is unambiguous. Now, we define a valuation V' on the nodes of $\mathbb{B}$ in the following way:

$$V'(a) \triangleq \{p \mid (M, w_a) \Vdash p\}$$

That is, we place a copy of w_a at position a in the Boolean algebra. This model that we have defined is the model M' on the Boolean algebra on n generators that we claim is bisimilar to M.

Before we prove this claim, we note that for a $w \in W$, there may be multiple $a \subseteq \{1, 2, \dots n\}$ such that $w = w_a$. For example, if t is the topmost node of F, and i is such that $g(i) = t$, then for any a such that $i \in a$, $w_a = t$.

Now, we show that M' is bisimilar to M. Towards this, we claim that if $w = w_a$, then

$$\{w_b \mid b \supseteq a\} = \{v \in W \mid v \geq w\}.$$

The reason why this suffices is this: Let T be the relation between M and M' given by wTa iff $w = w_a$. Then this is a bisimulation between M and M':

(i) If wTa, then $w = w_a$, and so a and w have the same valuation by the definition of the valuation V'.

(ii) Let wTa and $v \geq w$. Then by the claim, there is a $b \supseteq a$ such that $v = w_b$. Then, by the definition of T, vTb.

(iii) Let wTa and $b \supseteq a$. Then by the claim, there is a $v \geq w$ such that $v = w_b$. Then, by the definition of T, vTb.

Hence, we only have to prove our claim.

For the case that $a = \emptyset$, this is clear, since $\{w_{\{e\}} \mid e \in \{1, 2, \ldots, n\}\} = S$, and as this node itself is reflexive, it follows that

$$\{w_b \mid b \supseteq \emptyset\} = W = \{v \in W \mid v \geq r\}.$$

Now, suppose that $w = w_a$. Let $b \supseteq a$. Let $v = w_b$. Then

$$v = w_b = \bigvee\{g(e) \mid e \in b\} \geq \bigvee\{g(e) \mid e \in a\} = w_a = w,$$

hence

$$\{w_b \mid b \supseteq a\} \subseteq \{v \in W \mid v \geq w\}.$$

For the other direction, let $v > w$ (the case that $v = w$ is obvious). Then $v = g(e)$ for some $e \in \{1, 2, \ldots n\}$. Also, for any $b \subseteq \{1, 2, \ldots, n\}$ such that $e \in b$,

$$w_b = \bigvee\{g(c) \mid c \in b\} \geq v > w = w_a = \bigvee\{g(c) \mid c \in a\},$$

which is not possible if $e \in a$. Now, let $b = a \cup \{e\}$. Then

$$w_b = \bigvee\{g(c) \mid c \in b\} = g(e) \vee (\bigvee\{g(c) \mid c \in a\}) = v \vee (\bigvee\{g(c) \mid c \in a\}) = v \vee w_a = v \vee w = v.$$

Hence,

$$\{w_b \mid b \supseteq a\} \supseteq \{v \in W \mid v \geq w\}.$$

Therefore,

$$\{w_b \mid b \supseteq a\} = \{v \in M \mid v \geq w\}$$

holds, and so we have that M and M' are bisimilar.

Now, for the case that F is a baled pre-tree, the argument is similar, except that in all cases, nodes will be replaced by clusters. In particular, r will be the root cluster of F, S will be defined as the set of all non-trivial clusters of F, n will be its size, the frame of M' will the pre-Boolean algebra whose quotient is the Boolean algebra on a set of n elements, $g : \{1, 2, \ldots, n\} \to S$ will be a bijective map, for $a \subseteq \{1, 2, \ldots, n\}$, w_a will be the cluster-join of the $g(e)$ for $e \in a$. Also, instead of a single node at point a, we will place the cluster w_a, and the bottom cluster of the pre-Boolean algebra will be r, and so on. All the proofs are exactly the same as we have just shown except that all occurrences of 'node' need to be replaced by 'cluster', with the requisite changes to the bisimulation and the proofs. □

Corollary 84. *The class of finite pre-Boolean algebras characterises* S4.2.

Proof. Again, soundness is trivial. For completeness, let $\varphi \notin$ S4.2. By Corollary 81, let M be a model based on a finite directed pre-order such that φ is not valid on M. By Lemma 83, let M' be a model based on a finite pre-Boolean algebra which is bisimilar to M. Then clearly φ is not valid on M' as well. □

However, the axioms of S4.2 are not sufficient to capture the modal logic of inner models. In particular, we seek to model the property of **L** that it is an inner model of every model of NBG and that it has no non-trivial inner models itself. This suggests the need for an axiom to capture this notion of a single reflexive node on top of the entire frame. Towards this end, we introduce the following axiom:

$$\mathsf{Top}\quad \Diamond((\Box\varphi \leftrightarrow \varphi) \wedge (\Box\neg\varphi \leftrightarrow \neg\varphi))^1.$$

Definition 85. The modal theory S4.2Top is defined to be the smallest normal modal theory containing S4.2 and Top.

Definition 86. Let $(P, \leq)$ be a pre-order. We say that $(P, \leq)$ has a *single top node* if there is a unique node $t \in P$ such that for all $s \in P$, $s \leq t$. We call t the *top node* of $(P, \leq)$.

Note that the above definition implies that in a pre-order with a single top node, if t is the top node, then the equivalence class of t by the natural equivalence relation is a singleton set, consisting of t.

Theorem 87. *The class of finite directed pre-orders with a single top node characterises* S4.2Top*.*

Proof. Again, soundness is trivial. For completeness, let φ be a formula which is not in S4.2Top. Let $\mathcal{M}' = (\mathcal{W}', \mathcal{R}', \mathcal{V}')$ be the canonical model of S4.2Top. Let $w \in \mathcal{F}'$ be a node such that $\varphi \notin w$. Let $\mathcal{M}'[w]$ be the submodel of $\mathcal{M}'$ generated by w. Let $\mathcal{M} = (\mathcal{W}, \mathcal{R}, \mathcal{V})$ be the model obtained from $\mathcal{M}'[w]$ by filtrating by the finite subformula-closed set

$$\Phi = \{\psi \mid \psi \text{ is a subformula of } \varphi\}.$$

We claim that $\mathcal{F}$ has a single top node. That is, a node t such that for all $s \in \mathcal{F}$, $s\mathcal{R}t$, and further, $t\mathcal{R}s$ iff $t = s$.

It is clear that $\mathcal{F}$ is finite. Also, as S4.2Top $\supseteq$ S4.2, it follows by Theorem 80 that $\mathcal{F}'$ is reflexive, transitive and directed. Therefore, $\mathcal{F}$ is reflexive, transitive and directed as well. Now, let $T \subseteq \mathcal{W}$ be a maximal cluster. Suppose towards a contradiction that T is *not* a singleton. Now, the nodes of $\mathcal{W}'$ are S4.2Top-MCS. Therefore, if $u, v \in \mathcal{W}'$ are distinct, then there is a formula φ such that it is forced by one of them, but not forced by the other. As $\mathcal{M}$ is obtained from $\mathcal{M}'$ by taking a generated submodel and then by filtrating it by a finite subformula-closed set, it follows that there is a formula ψ such that it is true of atleast one node in T, but not true of all nodes in T. Then, for every node $s \in T$,

$$\mathcal{M}, s \Vdash \Diamond((\Box\psi \leftrightarrow \psi) \wedge (\Box\neg\psi \leftrightarrow \neg\psi)).$$

Therefore, there is some node $t \in T$ such that

$$\mathcal{M}, t \Vdash (\Box\psi \leftrightarrow \psi) \wedge (\Box\neg\psi \leftrightarrow \neg\psi).$$

[1] In discussions with Nick Bezhanishvili we found out that the choice of this axiom is canonical in that it is equivalent over the theory S4.2 to the *cofinal subframe formula* for the one element frame of a single reflexive node. A cofinal subframe formula for a frame F is a formula φ_F such that for any other frame F', φ_F is valid for F' if and only if F is a cofinal subframe of F'. That is, for any node v of F, there is a node w of F and a *p-morphism* (also sometimes called a *bounded morphism* in the literature) $\Phi : F' \to F$ such that $w \in \text{ran}(\Phi)$. It is important that we are talking about frames and p-morphisms (which is the natural notion of morphism for frames), since for example, a cofinal subgraph of a finite directed graph is merely a terminal node. See [CZ97] for a discussion of these concepts.

Thus, $\mathcal{M}, t \Vdash \psi$ iff for every node $t' \in T$, $\mathcal{M}, t' \Vdash \psi$, and $\mathcal{M}, t \Vdash \neg\psi$ iff for every node $t' \in T$, $\mathcal{M}, t' \Vdash \neg\psi$ (since T is a cluster). As ψ was picked so that it is true of atleast one node in T but not true of all nodes in T, this is a contradiction. Hence, T must be a singleton. Therefore, having been given a formula $\varphi \notin \mathsf{S4.2Top}$, we have obtained a model $\mathcal{M}$, based on a finite, directed pre-order with a single top node, such that $\mathcal{M} \vDash \mathsf{S4.2Top} \wedge \neg\varphi$. The result follows. □

Definition 88. A partial order $(P, \leq)$ is called a *sharp pre-order* if it is a pre-order, and there is a unique node $t \in P$ such that for each $s \in P$, $s \leq t$, and the pre-order $(P \setminus \{t\}, \leq)$ is a directed pre-order.

Lemma 89. *Let M be a model based on a finite directed pre-order with a single top node. Then there is a model M' based on a finite sharp pre-order which is bisimilar to M.*

Proof. Let t be the top node of M. Then we obtain M' from M by adding an extra node t' above t which has the same valuation as t. It is easy to see that M and M' are bisimilar, where the bisimulation is the identity on all the nodes of M which are not t, and $t \in M$ is matched to both t and t' in M'. □

Corollary 90. *The class of finite sharp pre-orders characterises* $\mathsf{S4.2Top}$.

Proof. Soundness is trivial. For completeness, let $\varphi \notin \mathsf{S4.2Top}$. Then by Theorem 87, there is a model M based on a finite directed pre-order with a single top node such that φ is not valid on M. We can then use the previous corollary to obtain a model M' based on a finite sharp pre-order which is bisimilar to M. Clearly then, φ is not valid on M'. □

Definition 91. A frame F is called an *inverted lollipop* if it is a finite pre-Boolean algebra with a single extra node above all of the other elements. By $\mathcal{L}$ we denote the class of inverted lollipops.

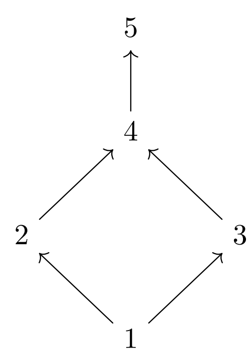

Figure 4.1: An inverted lollipop

Lemma 92. *Let M be a model based on a finite directed pre-order with a single top node. Then there is a model M' whose underlying frame is an inverted lollipop such that M is bisimilar to M'.*

Proof. By Lemma 89, we can, without losing any generality, assume that M is based on a finite sharp pre-order. Let $F = (W, R)$ be the underlying frame of M, and let t be the top node. Let $\bar{W} = W \setminus \{t\}$. Let $\bar{R} = R \cap \bar{W} \times \bar{W}$. Let $\bar{F} = (\bar{W}, \bar{R})$. Then $\bar{F}$ is a finite directed pre-order. Also, if

$\bar{M}$ is the model with frame $\bar{F}$ obtained from M by restricting the valuation of M to the nodes in $\bar{W}$, then $\bar{M}$ is a model based on a finite directed pre-order.

By Lemma 83, we can get a model M'' based on a finite pre-Boolean algebra which is bisimilar to $\bar{M}$. M' is then obtained by adding a copy of t above the topmost cluster of M''. It is easy to see that M' is an inverted lollipop and that it is bisimilar to M. □

Theorem 93. *The class of inverted lollipops characterises* S4.2Top.

Proof. Straightforward. □

We now introduce some variants of the control statements which we defined in Chapter 3. In particular, these control statements will be parametrised by a pure button in the sense that the nature of their behaviour depends on whether the button has been pushed or not. This is meant to reflect the behaviour of a sharp pre-Boolean algebra or an inverted lollipop, or in general, a sharp pre-order, of having a single node above all other elements.

Definition 94. Let Γ be a relation between models of set theory. Let M be a model of set theory. Let φ be a fixed pure button in M.

(i) A statement ψ is called a (Γ-)*φ-switch* if, for any model M such that $M \models \neg\varphi$, there are models N_1, N_2 such that $(M, N_1) \in \Gamma$ and $(M, N_2) \in \Gamma$, and $N_1 \models \neg\varphi \wedge \psi$ and $N_2 \models \neg\varphi \wedge \neg\psi$. That is, ψ is a conditional switch, which behaves as a switch so long as φ is not true. Modally, this amounts to $\Box(\neg\varphi \to \Diamond(\neg\varphi \wedge \psi) \wedge \Diamond(\neg\varphi \wedge \neg\psi))$.

(ii) A statement ψ is called a (Γ-)*φ-button* if, necessarily, $\varphi \to \psi$, and for any model M such that $M \models \neg\varphi \wedge \neg\psi$, there is a model N such that $(M, N) \in \Gamma$ and $N \models \neg\varphi \wedge \Box\psi$. That is, ψ is a conditional button, conditioned on whether or not φ is true. Modally, this amounts to $\Box((\neg\varphi) \to (\neg\varphi \wedge \psi))$. ψ is called a *pur φ-button* if in addition to this, necessarily, whenever ψ is true, it is necessarily true. Modally, this amounts to $\Box(\psi \to \Box\psi)$.

In the above cases, we say that φ is a *fatal button* for the statement ψ.

As usual, when Γ is clear from context, we will not explicitly mention it. Also, if the fatal button is clear from context, then we do not explicitly mention that it is parametrising the control statements. Realising a φ-switch in this case is switching it on or off without affecting the value of φ. Realising a φ-button is pushing it without affecting the value of φ. The independence of these control statements is defined in the obvious way. We make the trivial observation that if a collection $\mathcal{S}$ of statements is independent, and φ is a fatal button for $\mathcal{S}$, then if $\mathcal{S}$ contains a button, then for each $\psi \in \mathcal{S}$ distinct from some button in $\mathcal{S}$, it is Γ-necessarily not the case that $\psi \to \varphi$.

An example of these control statements for the ground model modality (this also works with inner models) is the following: for each natural number n, let φ_n assert that there are no Cohen subsets of $\aleph_n^{\mathbf{L}}$, and let $\mathcal{S} = \{\varphi_n \mid n \in \omega\}$. Then it is clear that $\mathcal{S}$ is a family of $\mathsf{V} = \mathsf{L}$-buttons. In Section 4.3 we shall see another example of a family $\mathcal{S}$ and a fatal button for this family. We shall use these statements to label all inverted lollipops. The theorem which will let us do this is the following:

Theorem 95. *If Γ is a reflexive transitive relation between models of set theory and M is a model of set theory such that there is a pure button φ in M, and arbitrarily large finite families of mutually independent unpushed φ-buttons and φ-switches over M, then the valid principles of ML_Γ^M are contained within* S4.2Top.

Proof. We show that the hypotheses allows us to label all inverted lollipops. The conclusion then follows from the conjunction of Theorem 93 and Theorem 57.

Let Γ be a reflexive transitive relation between models of set theory having arbitrarily large finite families of mutually independent buttons and switches over M.

Let L be a frame which is an inverted lollipop. Let F be the quotient partial order of L under the natural equivalence relation. Then F is a finite Boolean algebra with a single extra node on top (F is a *sharp* partial order in our terminology). Therefore, the partial order of non-maximal elements of F is isomorphic to the powerset algebra $\mathcal{P}(A)$ for some finite set A. We fix such a set A.

With each element $a \in F$ which is not maximal, there is associated a complete cluster $w_1^a, w_2^a, \ldots w_{k_a}^a$ of worlds of L. By adding dummy nodes to each cluster, we may assume that there is some natural number m such that for each non-maximal $a \in F$, the sizes k_a of the complete clusters at node a are the same, and equal to 2^m. Also, suppose that F has size $2^n + 1$, that is, the size of A is n, so there are n atoms in the Boolean algebra of non-maximal elements of F. We can therefore think of the Boolean algebra of non-maximal elements of F as the worlds w_j^a where $a \subseteq A$, and $j < 2^m$, with the order obtained by $w_j^a \leq w_i^c$ if and only if $a \subseteq c$. Also, since F consists exactly of this pre-Boolean algebra and a single extra node above every element of it, we can consider F as being made up of worlds w_j^a where $a \subseteq A$, and $j < 2^m$, with the order obtained by $w_j^a \leq w_i^c$ if and only if $a \subseteq c$, and a node t with $w_j^a \leq t$ for each a.

Associate with each element $i \in A$ an unpushed pure φ-button b_i such that the collection $\{b_i \mid i \in A\}$ form a mutually independent family with m-many φ-switches $s_0, s_1, \ldots s_{m-1}$. For $j < 2^m$, let $\bar{s}_j$ be the assertion that the pattern of switches corresponds to the binary digits of j. We associate the node w_j^a with the assertion

$$\Phi_{w_j^a} = \neg\varphi \wedge (\bigwedge_{i \in a} b_i) \wedge \bar{s}_j,$$

and we associate the node t with the assertion $\Phi_t = \varphi$. Clearly, we can assume that all of the switches are off. Now, if W is a model in the multiverse of Γ generated by M, and $W \vDash \Phi_{w_j^a}$, then by the mutual independence of buttons and switches combined with our remark that pushing these buttons cannot push the fatal button φ, we see that $W \vDash \Phi_{w_r^c}$ if and only if $a \subseteq c$.

Also, for any model W in the multiverse of M generated by Γ, if $W \vDash \neg\varphi$, then as φ is itself a button, it follows that $W \vDash \Diamond\Phi_t$. Therefore, if $W \vDash \Phi_{w_j^a}$, then $W \vDash \Phi_t$. Also, since all of these buttons and switches are off in M, we have $M \vDash \Phi_0^{\emptyset}$. Thus, we have provided a Γ-labelling of this frame for W, hence demonstrating that we can label all inverted lollipops. The result follows. □

Before we move on to the next section, we see a justification for considering the theory S4.2Top.

Theorem 96. *For any model V of* NBG*, the modal logic of inner models always contains* S4.2Top*.*

Proof. The axioms K, Dual, S and 4 are easily seen to hold. For .2 and Top, notice that every inner model M of V has $\mathbf{L}$ as an inner model. Hence, if $M \vDash \Box\varphi$ for some φ, then $\mathbf{L} \vDash \varphi$. Consequently, for every inner model N of M, $N \vDash \Diamond\varphi$. Further, $\mathbf{L}$ has no proper inner model, and hence for every φ, $\mathbf{L} \vDash \Box\varphi \leftrightarrow \varphi$. Therefore, $V \vDash \Diamond((\Box\varphi \leftrightarrow \varphi) \wedge (\Box\neg\varphi \leftrightarrow \neg\varphi))$. □

Hence, in order to show that the modal logic of inner models is exactly S4.2Top, we only need to find a model whose modal logic of inner models contains S4.2Top. We show how to do so in what follows.

4.2 Relating the Modal Logic of Inner Models to the Modal Logic of Grounds

Our aim in this chapter was to study the modal logic of the relation of being a definable inner model. So far, we have seen that this moda logic is contains the theory S4.2Top. In order to show that there is a model whose modal logic is exactly S4.2Top, we will first show that the relation of being an inner model has, as an initial segment, a relation that we understand better, the relation of being a forcing ground. Once we have shown this, we will be able to the results from [HL13], where a model was constructed whose modal logic of grounds is exactly S4.2, to obtain our main result.

We fix some notation. Let Γ_{IM} be the following relation:

$$(M, N) \in \Gamma_{\mathrm{IM}} \text{ iff } N \text{ is a definable inner model of } M,$$

and let $\Gamma_{\bar{\mathcal{P}}}$ be the relation:

$$(M, N) \in \Gamma_{\bar{\mathcal{P}}} \text{ iff } N \text{ is a forcing ground of } M.$$

In order to show that $\Gamma_{\bar{\mathcal{P}}}$ is an initial segment of Γ_{IM}, we split the task into two parts, one where we show that $\Gamma_{\bar{\mathcal{P}}}$ is contained in Γ_{IM}, and one where we show that the former is actually an initial segment of the later.

4.2.1 The Laver-Woodin Theorem

The first task is achieved by the Laver-Woodin Theorem, which shows that if $(M, N) \in \Gamma_{\bar{\mathcal{P}}}$, then $(M, N) \in \Gamma_{\mathrm{IM}}$. Our treatment follows [WDR12]. We note that while we talk about models of NBG everywhere, all of the proofs go through with ZFC itself.

Definition 97. (Hamkins) Let δ be an uncountable regular cardinal. Let M be a transitive class model of NBG.

(i) Then M is said to have the *δ-covering property* if for every $\sigma \subset M$ with $|\sigma| < \delta$, there is a $\tau \in M$ such that $|\tau| < \delta$ and $\sigma \subseteq \tau$.

(ii) The pair M is said to have the *δ-approximation property* if for every cardinal κ such that $\mathrm{cf}(\kappa) \geq \delta$ and every $\subseteq$-increasing sequence of sets $\langle \tau_\alpha \mid \alpha < \kappa \rangle$ from M, $\cup\tau_\alpha \in M$.

The next lemma shows that if V is a model of set theory, and $V[G]$ is a forcing extension of it, then V has these properties in $V[G]$ for all cardinals which are large enough.

Lemma 98. *Let V be a model of* NBG*. Let δ be an uncountable regular cardinal. Let $\mathbb{P} \in V$ be a poset of size less than δ. Let G be a V-generic filter for $\mathbb{P}$. Then V has the δ-covering and δ-approximation properties in $V[G]$.*

Proof. We first show the δ-covering property. Let σ be a $\mathbb{P}$-name and $p \in \mathbb{P}$ a condition such that $p \Vdash$"$\sigma \subset V$ and $|\sigma| < \delta$". Now, let

$$S = \{\lambda < \delta \mid \exists q \geq p[q \Vdash |\sigma| = \lambda]\}.$$

Now, if $q_1, q_2 \geq p$ and λ_1, λ_2 are such that $q_i \Vdash |\sigma| = \lambda_i$ for $\lambda_1 \neq \lambda_2$, then it follows that $q_1 \perp q_2$. Therefore, as $|\mathbb{P}| < \delta$, it follows that $|S| < \delta$. Let $\gamma = \sup S$. Then by the regularity of δ, it follows that $\gamma < \delta$ (since $S \subset \delta$). In this case, let $\dot{f}$ be a name such that $p \Vdash$"$\dot{f} : \gamma \to \sigma$ is a surjection". Then if

$$\tau = \{x \mid \exists q \geq p \exists \alpha < \gamma [q \Vdash \dot{f}(\alpha) = x]\}.,$$

it is clear that $\tau \in V$, $\sigma \subseteq \tau$ and $|\tau| < \delta$, hence establishing the δ-covering property.

Now, for δ-approximation. Let $p \in \mathbb{P}$ be a condition such that

$$p \Vdash \text{"cf}(\kappa) \geq \delta \text{ and } \langle \tau_\alpha \mid \alpha < \kappa \rangle \text{ is a } \subseteq \text{-increasing sequence of sets from } V\text{"}.$$

For each $\alpha < \kappa$, let $p_\alpha \geq p$ be a condition which decides the value of τ_α. Since $|\mathbb{P}| < \delta \leq \text{cf}(\kappa)$, it follows that there must be some $q \in \mathbb{P}$ such that for cofinally many $\alpha < \kappa$, $q = p_\alpha$. Since

$$p \Vdash \text{"}\langle \tau_\alpha \mid \alpha < \kappa \rangle \text{ is a } \subseteq \text{-increasing sequence of sets from } V\text{"},$$

it follows that q decides the value of $\cup \tau_\alpha$, and hence, $\cup \tau_\alpha \in V$. □

Theorem 99. *(Laver [Lav07], Woodin) Let δ be in N a regular uncountable cardinal. Let M, N be transitive class models of* NBG *such that both satisfy the δ-covering and δ-approximation property. Suppose that $\delta^+ = (\delta^+)^M = (\delta^+)^N$, and that $N \cap \mathcal{P}(\delta) = M \cap \mathcal{P}(\delta)$.*

(i) Then $M = N$.

(ii) In particular, M is Σ_2-definable from $M \cap \mathcal{P}(\delta)$.

Proof. (i) We show by induction on ordinals γ that for all $A \subseteq \gamma$,

$$A \in M \iff A \in N.$$

If $\gamma \leq \delta$, this is clear. Hence, assume that $\gamma > \delta$. Then, by the induction hypothesis, M and N have the same cardinals $\leq \gamma$. Also, if γ is not a cardinal in these models, then in both the models, there is a bijection between γ and $|\gamma|$, which allows us to conclude by applying the induction hypothesis on $|\gamma|$ that the powerset of γ is the same in both models. Hence, we may assume that γ is a cardinal in both M and N.

(a) $\text{cf}(\gamma) \geq \delta$. Then $A \in M$ iff $A \cap \alpha \in M$ for each $\alpha < \gamma$. The forward direction is clear, and for the reverse direction, we use the δ-approximation property for the sequence of sets $\langle A \cap \alpha \mid \alpha < \gamma \rangle$. Therefore, by using the induction hypothesis on $A \cap \alpha$ for $\alpha < \gamma$, we see that we are done.

(b) $\gamma > \delta$, $\text{cf}(\gamma) < \delta$, and $|A| < \delta$. We will use the δ-approximation property to find a set $S \supseteq A$ such that $S \in M \cap N$. We do this by using the δ-covering property as follows: define increasing sequences $\langle E_\alpha \mid \alpha < \delta \rangle$ and $\langle F_\alpha \mid \alpha < \delta \rangle$ of subsets of γ such that

(1) $|E_\alpha|, |F_\alpha| < \delta$;

(2) $A \subseteq E_0$;

(3) $E_\alpha \subseteq F_\alpha$;

(4) $\bigcup_{\alpha < \beta} F_\alpha \subseteq E_\beta$;

(5) $E_\alpha \in M$ and $F_\alpha \in N$.

Then

$$S = \bigcup_{\alpha<\gamma} E_\alpha = \bigcup_{\alpha<\gamma} F_\alpha$$

clearly satisfies the requirements that $A \subseteq S$ and $S \in M \cap N$. Now, let θ be the ordertype of S, and let $\pi : S \to \theta$ be the Mostowski collapse of S. As subsets of ordinals can only be collapsed in one way, it follows that $\pi \in M \cap N$. Now, it is clear that $|S| \leq \delta$, and hence, $\theta < \delta^+$. But now, the hypothesis of the theorem tell us that

$$\delta^+ = (\delta^+)^M = (\delta^+)^N.$$

Applying now the induction hypothesis to δ^+ and $\pi(A) \subset \delta^+$, it follows that

$$A \in M \iff \pi[A] \in M \iff \pi[A] \in N \iff A \in N.$$

(c) $\gamma > \delta$, $\mathrm{cf}(\gamma) < \delta$, and $|A| \geq \delta$. We claim that $A \in M$ iff

$(1)_M$ $A \cap \alpha \in M$ for all $\alpha < \gamma$;

$(2)_M$ For every $\sigma \subseteq \gamma$ such that $|\sigma| < \delta$ and $\sigma \in M$, $A \cap \sigma \in M$,

and analogously,$A \in N$ iff $(1)_N$ and $(2)_N$. If we could show this, then by the induction hypothesis and Case (b), we would be done.

The forward direction is obvious, so assume $(1)_M$ and $(2)_M$. Fix first a large θ with $\mathrm{cf}(\theta) > \gamma$ and a formula defining M which is absolute in V_θ. Now, define an increasing chain $\langle X_\alpha \mid \alpha < \delta\rangle$ of elementary substructures of V_θ and an increasing chain $\langle Y_\alpha \mid \alpha < \delta\rangle$ of subsets of $V_\theta \cap M$ such that:

(1) $|X_\alpha|, |Y_\alpha| < \delta$;

(2) $A \in X_0$;

(3) $\sup(X_0 \cap \gamma) = \gamma$;

(4) $X_\alpha \cap M \subseteq Y_\alpha$;

(5) $Y_\alpha \in M$;

(6) $\bigcup_{\alpha<\beta}(Y_\alpha \cup X_\alpha) \subseteq X_\beta$.

To do this, use the Downward Löwenheim-Skolem Theorem to get the X_α, and the δ-covering property on M to get Y_α. Let $X = \bigcup_{\alpha<\delta} X_\alpha$ and let $Y = \bigcup_{\alpha<\delta} Y_\alpha$. Then $X \prec V_\theta$ and $Y = X \cap M \prec V_\theta \cap M$.

Since $Y_\alpha \in M$ and $|Y_\alpha| < \delta$, it follows by assumption $(2)_M$ that for each $\alpha < \delta$, $A \cap Y_\alpha \in M$. Then, by the δ-approximation property, it follows that $A \cap Y \in M$.

Now, for any $\alpha \in Y \cap \gamma$, notice that $A \cap \alpha \in Y$ because $A \in X$ and $\alpha \in X$, and since $X \prec V_\theta$, $A \cap \alpha \in X$. Also, by $(1)_M$, $A \cap \alpha \in M$. Hence, $A \cap \alpha \in X \cap M = Y$. Also, for every $b \in Y$, if $b \cap Y = (A \cap Y) \cap \alpha$, then $Y \vDash b = A \cap \alpha$, and therefore, $b = A \cap \alpha$. This is because $Y \cap \alpha \in M$, and $Y \prec V_\theta \cap M$.

Therefore, the sequence $\langle A \cap \alpha \mid \alpha \in Y \cap \gamma\rangle$ is definable in M with parameters $\gamma, Y, A \cap Y$. In particular, this sequence belongs to M, and it follows then that $A = \bigcup_{\alpha<\gamma} A_\gamma$ is in M.

(ii) This part follows: $A \in M$ if there is a large regular cardinal θ, and an $N \subset V_\theta$ which is a model of $\mathsf{NBG}-\mathsf{Powerset}$ satisfying δ-covering and δ-approximation and such that $M \cap \mathcal{P}(\delta) = N \cap \mathcal{P}(\delta)$ and $A \in N$. This is a Σ_2 statement.

□

Corollary 100. *The relation $\Gamma_{\bar{\mathcal{P}}}$ is contained in Γ_{IM}.* □

4.2.2 Grigorieff's Theorem

Now, we shall appeal to a theorem of Grigorieff to accomplish the second task, namely, show that $\Gamma_{\bar{\mathcal{P}}}$ is an *initial segment* of Γ_{IM}.

Theorem 101. *(Grigorieff) Let V be a model of* NBG. *Let $\mathbb{B} \in V$ be a complete atomless Boolean algebra. Let G be V-generic for $\mathbb{B}$ and $V[G]$ the corresponding generic extension. Let M be an inner model of $V[G]$ such that $V \subseteq M \subseteq V[G]$. Then there is a complete atomless Boolean subalgebra $\mathbb{C}$ of $\mathbb{B}$ in V such that $M = V[\mathbb{C} \cap G]$.*

Corollary 102. *Let V be a model of* NBG. *Let $M \subseteq V$ be an inner model. If M is not a ground of V, then there is no inner model of M which is a ground of V. That is, $\Gamma_{\bar{\mathcal{P}}}$ is an initial segment of Γ_{IM}.*

Proof. Let $N \subseteq M$ be an inner model of M and a ground of V. Let $\mathbb{B} \in N$ be a complete atomless Boolean algebra and G an N-generic for this Boolean algebra such that $V = N[G]$. Then by Grigorieff's theorem, there is $\mathbb{C}$, a complete subalgebra of $\mathbb{B}$, in N and H an N-generic for $\mathbb{C}$ such that $M = N[H]$. Then by Corollary 35, it follows that V is a generic extension of M as well. The second part follows. □

4.3 An Interesting Model

In this section, we construct a model of set theory which is a slight modification of a model from [Rei06][2]. Our aim in constructing this model is to find a model such that the modal logic of inner models of this model is exactly S4.2Top.

Definition 103. Let γ be a regular cardinal. The forcing poset $\mathrm{Add}(\gamma)$ which adds a *Cohen subset of* γ is the following:

(i) $p \in \mathrm{Add}(\gamma)$ if p is a function such that $\mathrm{dom}(p) \subseteq \gamma$, $\mathrm{range}(p) \subseteq \{0,1\}$and $|p| < \gamma$.

(ii) If $p, q \in \mathrm{Add}(\gamma)$, then $p \leq q$ if $p \subseteq q$.

Note that for the case that $\gamma = \omega$, $\mathrm{Add}(\omega)$ is the same as Coh. The next two lemmas will come in handy in our main proof. Their proofs are standard.

Lemma 104. *Let $M \subseteq N$ be models of set theory. Let γ be an infinite regular cardinal in N. In N, let $S \subseteq \gamma$. Then if S is a Cohen subset of γ over M, then for each ordinal $\alpha < \gamma$, $S \cap \alpha \in M$.* □

Lemma 105. *Let γ be a regular cardinal. Then $|\mathrm{Add}(\gamma)| = 2^{<\gamma}$. Therefore, if $2^{<\gamma} = \gamma$, then $|\mathrm{Add}(\gamma)| = \gamma$.* □

We now define the class-forcing poset which we shall use to construct the model we want.

[2]The modification here is that in both, [Rei06] and [HL13], the class forcing which was used added a Cohen subset to each regular cardinal of **L**. However, as we were unable to prove for this model that an analogue of Theorem 108 holds, we modified their construction to one for which we could do so.

Definition 106. Let $\text{Succ}^{\mathbf{L}}$ denote the class of infinite successor cardinals in $\mathbf{L}$. Define in $\mathbf{L}$ the following (class-sized) poset with Easton support:

$$\mathbb{P} \triangleq \prod_{\gamma \in \text{Succ}^{\mathbf{L}}} \text{Add}(\gamma).$$

That is, $p \in \mathbb{P}$ if

(i) p is a class function such that $\text{dom}(p) = \text{Succ}^{\mathbf{L}}$;

(ii) For each $\gamma \in \text{Succ}^{\mathbf{L}}$, $p(\gamma) \in \text{Add}(\gamma)$;

(iii) For each such p, for each regular cardinal γ, $|\{\lambda \in \text{Succ}^{\mathbf{L}} \mid p(\lambda) \neq 0\} \cap \gamma| < \gamma$ (the class $\{\gamma \in \text{Succ}^{\mathbf{L}} \mid p(\gamma) \neq 0\}$ is called to be the *support* of p).

The ordering is defined by $p \leq q$ if $p \subseteq q$.

Also, for each $p \in \mathbb{P}$ and each $\gamma \in \text{Succ}^{\mathbf{L}}$, we can decompose p into three parts:

$$\begin{aligned} p_{<\gamma} &= p \restriction [0, \gamma); \\ p_{\gamma} &= p \restriction [\gamma, \gamma]; \\ p_{>\gamma} &= p \restriction (\gamma, \infty). \end{aligned}$$

Using this decomposition, for each $\gamma \in \text{Succ}^{\mathbf{L}}$, we can decompose $\mathbb{P}$ into three parts:

$$\begin{aligned} \mathbb{P}_{<\gamma} &= \{p_{<\gamma} \mid p \in \mathbb{P}\}; \\ \mathbb{P}_{\gamma} &= \{p_{\gamma} \mid p \in \mathbb{P}\}; \\ \mathbb{P}_{>\gamma} &= \{p_{>\gamma} \mid p \in \mathbb{P}\}. \end{aligned}$$

It is clear that $\mathbb{P} \cong \mathbb{P}_{<\gamma} \times \mathbb{P}_{\gamma} \times \mathbb{P}_{>\gamma}$.

Proposition 107. *($\mathsf{V} = \mathsf{L}$) Let $\gamma \in \text{Succ}^{\mathbf{L}}$. Then*

(i) $\mathbb{P}_{>\gamma}$ is $\leq \gamma$-closed;

(ii) $\mathbb{P}_{<\gamma}$ has size less than γ.

Proof. The first part is trivial. For the second part, let $\gamma = \kappa^+$. We use the fact that the GCH is true in $\mathbf{L}$. The result then follows from the following chain of equivalences:

$$|\mathbb{P}_{<\gamma}| = \prod_{\lambda \in \text{Succ}^{\mathbf{L}}}^{\lambda<\gamma} |\text{Add}(\lambda)| = \prod_{\lambda \in \text{Succ}^{\mathbf{L}}}^{\lambda<\gamma} 2^{<\lambda} = \prod_{\lambda \in \text{Succ}^{\mathbf{L}}}^{\lambda<\gamma} \lambda \leq \prod_{\lambda \in \text{Succ}^{\mathbf{L}}}^{\lambda<\gamma} \kappa \leq \kappa \times \kappa = \kappa < \gamma.$$

□

Theorem 108. *($\mathsf{V} = \mathsf{L}$) Let γ be an infinite successor cardinal. Let $\mathbb{Q}_{\gamma} = \mathbb{P}_{<\gamma} \times \mathbb{P}_{>\gamma}$. Then forcing with $\mathbb{Q}_{\gamma}$ does not add a Cohen subset of γ.*

Proof. Since $\mathbb{Q}_\gamma = \mathbb{P}_{<\gamma} \times \mathbb{P}_{>\gamma}$ and $\mathbb{P}_{>\gamma}$ is $\leq \gamma$-closed, by Proposition 24, we only need to show that forcing with $\mathbb{P}_{<\gamma}$ does not add a Cohen subset of γ. Suppose towards a contradiction that this is not so. Let $\mathbf{L}[G]$ be a generic extension by $\mathbb{P}_{<\gamma}$ such that $S \in \mathbf{L}[G]$ is a Cohen subset of γ. Therefore, for each $\alpha < \gamma$, $S \cap \alpha \in \mathbf{L}$. Now, by the previous proposition, $|\mathbb{P}_{<\gamma}| < \gamma$. Hence, by Lemma 98, it follows that $\mathbf{L}$ has the γ-approximation property in $\mathbf{L}[G]$. But then, $S = \bigcup_{\alpha<\gamma}(S \cap \alpha)$, and $\langle S \cap \alpha \mid \alpha < \gamma\rangle$ is a $\subseteq$-increasing sequence of length γ of elements of $\mathbf{L}$, and hence, $S \in \mathbf{L}$, which is a contradiction. Therefore, forcing with $\mathbb{Q}_\gamma$ does not add any Cohen subsets of γ. □

Definition 109. Let G be $\mathbf{L}$-generic for $\mathbb{P}$, and let $M^{\mathcal{R}} \triangleq L[G]$. Let γ be an infinite successor cardinal. Then $G_{>\gamma} = G \cap \mathbb{P}_{>\gamma}$. Let

$$\varphi^{\mathcal{R}} \triangleq \forall \kappa \in \mathrm{Succ}^{\mathbf{L}} \exists G \subseteq \kappa (G \text{ is an } \mathbf{L}\text{-Cohen subset of } \kappa).$$

Let

$$\psi^{\mathcal{R}} \triangleq \exists \mathbb{B}[(\mathbb{B} \text{ is a complete atomless Boolean algebra }) \wedge (\|\varphi^{\mathcal{R}}\|_{\mathbb{B}} = 1_{\mathbb{B}})].$$

Clearly, $M^{\mathcal{R}} \vDash \varphi^{\mathcal{R}}$, and for any model N, N is a ground of $M^{\mathcal{R}}$ if and only if $N \vDash \psi^{\mathcal{R}}$.

Hence, if N is an inner model of $M^{\mathcal{R}}$ such that $N \vDash \neg\psi^{\mathcal{R}}$, then by Corollary 102, no further inner model N' of N can be a model of $\psi^{\mathcal{R}}$. That is, $M^{\mathcal{R}} \vDash \Box(\neg\psi^{\mathcal{R}} \to \Box(\neg\psi^{\mathcal{R}}))$. Hence, the statement $\neg\psi^{\mathcal{R}}$ is a pure button.

We now prove an interesting property of this model which we shall use in the next section, namely that every ground of this model itself has a non-trivial ground.

Lemma 110. *Let N be a ground of $M^{\mathcal{R}}$. Then there is an infinite successor cardinal γ such that $N \supseteq \mathbf{L}[G_{>\gamma}]$. In particular, $\mathbf{L}[G_{>\gamma}]$ is a ground (and hence, a definable inner model) of N.*

Proof. Towards a contradiction, suppose this is not so. Let $\mathbb{Q} \in N$ be a forcing poset and let H be $\mathbb{Q}$-generic over N such that $M^{\mathcal{R}} = N[H]$. For some infinite successor cardinal γ large enough, let τ be a name for $G_{>\gamma}$. Let $p \in \mathbb{Q}$ be such that

$$p \Vdash \text{“}\tau \text{ is } \mathbb{P}_{>\gamma}\text{-generic over } \mathbf{L}.$$

By assumption, $G_{>\gamma}$ is a class-function which is not in N, but in a forcing extension of N by $\mathbb{Q}$ (which is a set). Therefore, for any $q \geq p$, q can decide only a set-sized initial segment of $G_{>\gamma}$. However, for every $\beta > \alpha$, there is a $r \geq p$ such that r decides $G_{>\gamma}\restriction(\gamma, \beta)$. Therefore, we can form a class-length strictly increasing chain of conditions in $\mathbb{Q}$, thus contadicting that it is a set. □

In [HL13], Hamkins and Löwe studied a model which was a slight modification of $M^{\mathcal{R}}$, and proved a similar property of this model. They used this property to show that .2 is valid for the modal logic of grounds of $M^{\mathcal{R}}$, and hence that the modal logic of grounds of this model is exactly S4.2.

4.4 The Modal Logic of Inner Models

We are now in a position to calculate the modal logic of $M^{\mathcal{R}}$.

Lemma 111. *In $M^{\mathcal{R}}$, there are arbitrarily large finite families of mutually independent $\neg\psi^{\mathcal{R}}$-switches and $\neg\psi^{\mathcal{R}}$-buttons. Consequently, the modal logic of inner models of $M^{\mathcal{R}}$ is exactly* S4.2Top.

Proof. For each natural number n, let b_n be the statement "there is no $\mathbf{L}$-generic subset of $\aleph_n^{\mathbf{L}}$". Clearly, each of the b_n is a $\neg\psi^{\mathcal{R}}$-button. Also, partition the regular cardinals above $\aleph_\omega$ into $\aleph_0$-many classes, $\langle\Gamma_n\rangle_{n\in\omega}$ such that each class contains unboundedly many cardinals. Enumerate each class as $\Gamma_n = \{\gamma_\alpha^n \mid \alpha \in \mathrm{ORD}\}$. Let s_n be the statement "the least α such that there is an $\mathbf{L}$-generic subset of γ_α^n is even". To see that each of the s_n are a $\neg\psi^{\mathcal{R}}$-switch, notice that any inner model of $M^{\mathcal{R}}$ such that $M^{\mathcal{R}} \vDash \psi^{\mathcal{R}}$ is in fact a ground model of $M^{\mathcal{R}}$. Hence, it has a ground of the form $\mathbf{L}[G^\alpha]$ for α large enough. One can then choose a ground where any specific cardinals do not have an $\mathbf{L}$-generic subset, hence allowing us to obtain any possible combination of the switchboard. To see that all of these switches and buttons are mutually independent, we appeal to Theorem 108.

Also, it is clear that $\neg\psi^{\mathcal{R}}$ is a fatal button for all of these statements. We then appeal to Theorem 95 and Theorem 93 to see that we are done. □

And finally, our main theorem:

Theorem 112. *If* NBG *is consistent, then the* NBG*-provable modal logic of inner models is exactly* S4.2Top. □

Chapter 5

The Modal Logic of ccc Forcing

In this chapter, we improve upon the upper bounds for $\mathsf{ML}_{\mathsf{ccc}}$ that were obtained in Theorem 67 by providing labellings for some Kripke frames. The labellings are a natural generalisation of the labelling of the two element fork that was used to show that $\mathsf{S4.2}$ is not valid for $\mathsf{ML}_{\mathsf{ccc}}$ in Theorem 67. In particular, we use ω_1-trees for these labellings as well.

In Section 5.2 we study two classes of frames, *spiked pre-Boolean algebras*, Definition 113, and *topless pre-Boolean algebras*, Definition 116, each of which contain the two element fork frame. Of these, the first one is a notion that has not been considered in the literature before, whereas the second has been. We give upper and lower bounds with respect to some standard modal theories for both the modal theories that are characterised by these frames in the hierarchy of normal modal logics upto containment. In particular, both these theories occupy a similar place in this hierarchy with respect to many familiar logics. We then show that these two theories are not the same by showing that one of them is not contained in the other.

In Section 5.3 we describe the labelling of one of these classes of frames. This requires the existence of an arbitrarily large finite number of ω_1-trees with some specific properties. In Section 5.3 we appeal to a theorem of Abraham and Shelah to obtain these trees in a model of $\diamondsuit_{\omega_1}$. We do not, however, prove this result or its corollary that we use.

The existence of these labellings depends on being able to understand when a tree can be specialised or made non-Aronszajn without adding a branch or specialising another. We study these properties in Section 5.4. Using the results from this section, we show in Section 5.5 that these labellings are complete. Finally, in the last section, Section 5.7, we discuss generalisations of this method and some questions related to the modal logic of ccc forcing.

5.1 Preliminaries

Recall that a partial order $(T, \leq)$ is called a *tree* if for each element $t \in T$, the set $\{s \in T \mid s < t\}$ is well-ordered and if there is a unique node $r \in T$ such that for every $t \in T$, $r \leq t$. In this case, r is called the *root* of T. For an element $t \in T$, the *level* or *height* of t, $\mathrm{lev}_T(t)$, is defined to be the ordertype of the set $\{s \in T \mid s < t\}$. We may omit mentioning T here if it is clear from context. Given an ordinal α, the *αth level* of T, T_α is defined to be the set $\{t \in T \mid \mathrm{lev}_T(t) = \alpha\}$. The height of the tree T is defined to be the least ordinal λ such that $T_\lambda = \emptyset$. In such a case, we call T a *λ-tree*.

In this report we will only talk about ω_1-trees. An ω_1-tree is *normal* if the following hold:

(i) Every $t \in T$ has $\aleph_0$-many immediate successors.

(ii) If $\alpha < \omega_1$ is a limit ordinal, and if $s, t \in T_\alpha$, then $\{p \in T \mid p < s\} \neq \{p \in T \mid p < t\}$.

Given $p \in T$, the tree T_p is defined as follows:

$$T_p \stackrel{\Delta}{=} \{q \in T \mid p \leq q\}.$$

While this terminology might seem to cause conflict with the definition of the αth level of T, this will in practice not be so since elements of the tree will usually be denoted by letters of the English alphabet and ordinals by letters of the Greek alphabet.

A *branch* of T is a downwards-closed subset b of T such that for any $s, t \in b$, $s \parallel t$. That is, a branch is a downwards-closed linearly ordered subset of T. If T is an ω_1-tree, then a branch b is *cofinal* if its ordertype is ω_1. By default, a branch is assumed to be cofinal.

Henceforth, all (ω_1-)trees that will be mentioned will be normal unless specified otherwise. We will also assume that their levels are countable. This justifies the choice of wording of the next definition. An ω_1-tree T is *Aronszajn* if it has no cofinal branches. It is *Suslin* if all of its antichains are countable. Hence, a Suslin tree is a ccc (forcing) poset. A *special* (ω_1-)tree will be an Aronszajn tree to which no ccc poset can add a branch to by forcing. Note that this is *not* the standard definition, which asserts that an ω_1-tree is special if it is the union of countably many antichains. However, this definition is weaker, in the sense that every tree which is special in the 'traditional' sense is special in this sense as well. It follows that standard results about special trees such as the fact that any Aronszajn tree can be specialised by a ccc poset, and the standard method of constructing a special ω_1-tree can be used when speaking about special trees in this sense as well. The matter is discussed in some more detail in Section 5.3.

Let S and T be two trees of the same height. Then

$$T \otimes S \stackrel{\Delta}{=} \{(x, y) \in T \times S \mid \mathrm{lev}_T(x) = \mathrm{lev}_S(y)\}.$$

This set with the co-ordinatewise ordering is clearly also a tree (we call this tree the *tensor product* of the composite trees). Further, the poset we obtain from this tree can be embedded as a dense subset of the poset obtained from the product $T \times S$ in a canonical way. This implies in particular that the largest size of an antichain in $T \times S$ is the same as the largest size of an antichain in $T \otimes S$ (and vice versa). Also, this operation is clearly abelian, in the sense that $S \otimes T$ is isomorphic to $T \otimes S$. It is also associative, in the sense that if T_1, T_2, T_3 are Aronszajn trees, then $(T_1 \otimes T_2) \otimes T_3$ is isomorphic to $T_1 \otimes (T_2 \otimes T_3)$. From this, we can, without any ambiguity, define the tensor product of a finite set of trees as follows: if $X = \{T_1, T_2 \cdots T_n\}$ is a set of trees, then

$$\bigotimes X \stackrel{\Delta}{=} T_1 \otimes T_2 \otimes \cdots \otimes T_n.$$

5.2 Frames

We now introduce the class of frames that we will label by ccc-forcing. As far as we are aware, these frames have not been considered before in the literature of either modal logic or intuitionistic logic. Indeed, the only reason for our discovery of these frames was that we were able to label them by ccc posets.

Definition 113. A partial order $(\mathbb{S}, \leq)$ is called a *spiked Boolean algebra* if the following hold:

(i) There is a set $T \subseteq \mathbb{S}$ of maximal nodes of $\mathbb{S}$ such that $\mathbb{S} \setminus T$ is a complete Boolean algebra. We call T the *set of spikes* of $\mathbb{S}$, and Boolean algebra $\mathbb{S} \setminus T$ as the *underlying or corresponding Boolean algebra* of T.

(ii) For each $n \in \mathbb{S}$ which is a penultimate node of $\mathbb{S}$, there is exactly one $t \in T$ such that $n \leq t$.

Figure 5.1: The spiked Boolean algebras with two extreme nodes (left) and with three extreme nodes (right)

Note that if $\mathbb{S}$ is a spiked Boolean algebra, then the definition is unambiguous because the set of spikes is unique: If T_1 and T_2 are two distinct sets of spikes, then if $t \in T_1 \setminus T_2$, then there is a $t' \in T_2$ such that $t \leq t'$. This contradicts the assumption that t is a maximal node of $\mathbb{S}$.

Only finite spiked Boolean algebras are considered in this thesis, and they can be imagined as follows: they are complete Boolean algebras on an n-element set such that each element on the penultimate level (recall that our frames grow upwards) has, in addition to the topmost node of the Boolean algebra, an extra neighbour, to which none of the other nodes on the penultimate level have access to. We will usually talk about a spiked Boolean algebra in terms of the number of its extremal nodes, or in terms of the cardinality of the base set (this is one less than the number of extremal nodes).

Just as a finite Boolean algebra can be represented as the collection of all subsets of some fixed finite set, spiked Boolean algebras can also be represented in the following way: The spiked Boolean algebra on an n-element set (or equivalently, with $(n+1)$-many extremal nodes) can be represented by triples (a, b, c) of subsets of an n-element set (we take this to be $\{1, 2, \ldots, n\}$ here for convenience) with the following properties:

(i) $a \uplus b \uplus c = \{1, 2, \ldots, n\}$

(ii) $|a| \leq 1$.

(iii) $a \neq \emptyset \implies c = \emptyset$,

and the relation $\leq$ between the nodes is given by $(a, b, c) \leq (d, e, f)$ if:

(i) The first co-ordinate increases. That is, $a \subseteq d$.

(ii) The second co-ordinate increases. That is, $b \subseteq e$.

(iii) The third co-ordinate decreases. That is, $c \supseteq f$.

The set of spikes correspond to the nodes where $a \neq \emptyset$, and the underlying Boolean algebra corresponds to those nodes where $a = \emptyset$. This is clearly seen to be isomorphic to the Boolean algebra on an n-element set. It is clear then that the spiked Boolean algebra on an n-element set has $n + 2^n$ nodes.

Definition 114. A *spiked pre-Boolean algebra* is a pre-order such that its quotient by the natural equivalence relation is a spiked Boolean algebra.

Definition 115. The modal theory $\mathsf{S4sBA}$ is defined to be the set of all modal assertions which are true on all Kripke models whose frame is a spiked pre-Boolean algebra.

Thus, this logic is by definition complete with respect to spiked pre-Boolean algebras.

5.2.1 Topless pre-Boolean Algebras

We now introduce a second class of frames.

Definition 116. A partial order $\mathbb{T}$ is called a *topless Boolean algebra* if, the partial order obtained by adding a single node above all the elements of $\mathbb{T}$ is a Boolean algebra. A *topless pre-Boolean algebra* is a partial pre-order such that its quotient partial order is a topless Boolean algebra.

Just as a finite Boolean algebra can be represented as the set of all subsets of a given finite set, a finite topless Boolean algebra can be represented as the collection of all *proper* subsets of a given finite set.

Definition 117. The modal theory $\mathsf{S4tBA}$ is defined to be the collection of all modal assertions which are true in all Kripke models whose frame is a finite topless pre-Boolean algebra.

Thus, by definition, this logic is complete with respect to the class of finite topless pre-Boolean algebras. This is a fairly natural logic, having previously been studied in the context of the modal logic of forcing as well as that of intuitionistic logic. In particular, this logic is the smallest modal companion of what is known as *Medvedev's Logic*, which is a well-known intermediate logic.[1] Also, in the context of the modal logic of forcing, Hamkins, Leibman and Löwe also showed that the modal logic of ω_1-preserving forcing is contained in $\mathsf{S4tBA}$.

A proof of the next theorem can be found in [HLL].

Theorem 118. *The logic* $\mathsf{S4tBA}$ *is properly contained in* $\mathsf{S4.2}$.

It is clear that $\mathsf{S4}$ is valid in any topless pre-Boolean algebra, and so it follows that $\mathsf{S4} \subseteq \mathsf{S4tBA}$. In fact, this containment is strict because all topless pre-Boolean algebras satisfy the principle that whenever three mutually incompatible assertions are possibly necessary, then it is possible to exclude one of them without deciding between the other two. The underlying property of topless Boolean algebras being:

[1] An *intermediate logic* is a propositional logic between intuitionistic logic and classical logic. A modal logic Λ is called the modal companion of an intermediate logic Λ' if Λ consists exactly of the Gödel translations of the formulas in Λ'. In such a case, if Λ is characterised by a class of frames $\mathcal{C}$ in the sense of the Kripke semantics for modal logic, then Λ' is characterised by the same class of frames in the sense of the Kripke semantics for intuitionistic logic. Medvedev's Logic is known to be characterised by the class of finite topless pre-Boolean algebras and to not be finitely axiomatisable [Gab70], [MSS79]. See [BdJ06] for more on intuitionistic logic and for undefined terms. We note that the results on Medvedev's Logic naturally raise the question of the finite axiomatisability of $\mathsf{S4sBA}$. We do not know the answer to this question.

Proposition 119. *Let $\mathbb{T}$ be a finite topless Boolean algebra. Let s be an element in $\mathbb{B}$ and u, v, w be elements of $\mathbb{B}$ such that no two of them have a join, and $s \leq u, v, w$. Then there is an element t in the algebra such that $s \leq t \leq u, v$ but such that it is not the case that $t \leq w$.*

Proof. We identify $\mathbb{T}$ with the topless Boolean algebra of proper subsets of some finite set S. Let s, u, v, w be as in the hypothesis. Then $s \subseteq u \cap v \cap w$. No two among u, v, w have a join, and this implies that the union of any two of them is S. In particular, this implies that they are all distinct and incomparable. Let $c \in S$ be such that $c \notin w$. This implies that $c \in u$ and $c \in v$. This implies that $u \cap v \nsubseteq w$. It is also clear that $s \subseteq u \cap v$. Then $u \cap v$ is the element t that we are looking for. □

It is easy to see that this result can then be carried over to the case of a finite topless pre-Boolean algebra as well.

Corollary 120. *Let $\mathbb{T}$ be a finite topless pre-Boolean algebra. Let s be an element in $\mathbb{T}$ and u, v, w be elements of $\mathbb{T}$ such that no two of them have a join, and $s \leq u, v, w$. Then there is an element $t \in \mathbb{T}$ such that $s \leq t \leq u, v$ but such that it is not the case that $t \leq w$.*

Proof. We know that $s \leq u, v, w$ and that no two of u, v, w have a join. Therefore, we can conclude that s, u, v, w all lie in different equivalence classes of the natural equivalence relation on $\mathbb{T}$. Therefore, if we consider the finite topless Boolean algebra $[\mathbb{T}]_\equiv$, then $[s]_\equiv, [u]_\equiv, [v]_\equiv, [w]_\equiv$ all satisfy the hypothesis of the previous proposition. Hence, we can find an element $t \in \mathbb{B}$ such that $[t]_\equiv \leq_\equiv [u]_\equiv, [v]_\equiv$ but such that it is not the case that $[t]_\equiv \leq [w]_\equiv$. This t suffices for the result. □

This is not valid for S4: using the characterisation of S4, Theorem 42, consider the frame that is a tree with a root and three leaves and then assign a suitable valuation to it. We can easily find a modal formula which is true on all topless Boolean algebras and which is not valid on this frame. Hence, we can summarise the situation as:

$$\mathsf{S4} \subsetneq \mathsf{S4tBA} \subsetneq \mathsf{S4.2} \subsetneq \mathsf{S4.3}.$$

5.2.2 Spiked pre-Boolean Algebras

A similar result also holds for S4sBA.

Theorem 121. *The logic* S4sBA *is properly contained in* S4.2.

Proof. We argue that S4sBA is contained in S4.2 by showing that every Kripke model $\mathcal{M}$ based on a finite pre-Boolean algebra is bisimilar to a model $\mathcal{M}'$ which is based on a finite spiked pre-Boolean algebra. Consider first the case when the frame of $\mathcal{M}$ is actually a finite Boolean algebra. In this case, $\mathcal{M}'$ is obtained by adding above each penultimate node of $\mathcal{M}$ an extra reflexive node which is not accessible to any other penultimate nodes of the Boolean algebra, and which has the same valuation as the topmost node of the Boolean algebra. Hence, each of the penultimate nodes can see exactly two bisimilar nodes, one of which is the topmost node of the Boolean algebra, and the other is a 'spike' in the sense that no other penultimate node can access it. Clearly then, there is a simple bisimulation between $\mathcal{M}'$ and $\mathcal{M}$ which is the identity bisimulation on the nodes of the Boolean algebra which are not maximal, and all of the maximal nodes of $\mathcal{M}'$ (this includes the topmost node of the Boolean algebra) are bisimilar to the topmost node of $\mathcal{M}$. Hence, we see that for any formula φ, it is true on a node in $\mathcal{M}$ iff it is true on $\mathcal{M}'$, and hence modal truth is preserved by this frame transformation.

Now, in the case that $\mathcal{M}$ is a finite pre-Boolean algebra, one can do something similar, except that all of the penultimate clusters get an extra cluster above them which is an exact copy of the topmost cluster on the Boolean algebra. Here too, the bisimulation is the identity on all of the old clusters of $\mathcal{M}'$, and all of the new clusters are bisimilar to the topmost cluster of $\mathcal{M}$. Here too, it follows that modal truth is preserved by this transformation.

To show that the containment is strict, notice that spiked Boolean algebras are not directed, whereas Boolean algebras are, and so one can easily construct a formula which is valid on a frame only if the frame is directed. For example, one can assign different truth values to a statement on some two different extremal nodes of the spiked Boolean algebras. $\square$

It is also easy to see that $\mathsf{S4}$ is valid in $\mathsf{S4sBA}$. It follows that $\mathsf{S4} \subseteq \mathsf{S4sBA}$. Here too, the containment can be proven to be strict. The corresponding observation about Kripke models based on a spiked pre-Boolean algebra is that whenever two mutually incompatible assertions are possible, then the node itself is an element of the Boolean algebra. Hence, if there are two such worlds, each of which satisfy that there are some two mutually incompatible assertions that are possible, then these two worlds must have a join. The underlying property in this case being:

Proposition 122. *Let $\mathbb{S}$ be a finite spiked Boolean algebra. Let s, t, u, v, x, y be elements of $\mathbb{S}$ such that $s \leq u, v$ and $t \leq x, y$, and such that u and v do not have a join, and such that x and y do not have a join either. Then there is an element $w \in \mathbb{S}$ such that $s, t \leq w$.*

Proof. Identify $\mathbb{S}$ with the collection of triples (a, b, c) of subsets of $\{1, 2, \dots, n\}$ such that:

(i) $a \uplus b \uplus c = \{1, 2, \dots, n\}$

(ii) $|a| \leq 1$.

(iii) $a \neq \emptyset \implies c = \emptyset$,

and the relation between the nodes is given by $(a, b, c)\mathcal{R}(d, e, f)$ if:

(i) The first co-ordinate increases. That is, $a \subseteq d$.

(ii) The second co-ordinate increases. That is, $b \subseteq e$.

(iii) The third co-ordinate decreases. That is, $c \supseteq f$.

Also, in order to simplify notation, for $r \in \mathbb{B}$, denote the corresponding triples as (r_1, r_2, r_3). Now, since s and t both see pairs of elements which do not have a join, it follows that both s and t must be elements of the corresponding Boolean algebra. That is, $s_0 = t_0 = \emptyset$. It is then easy to see that there is an element w in the Boolean algebra (their join) such that $x, t \leq t$. $\square$

Using this, by similar arguments as were used in Corollary 120, we get the following:

Corollary 123. *Let $\mathbb{S}$ be a finite spiked pre-Boolean algebra. Let s, t, u, v, x, y be elements of $\mathbb{S}$ such that $s \leq u, v$ and $t \leq x, y$, and such that u and v do not have a join, and such that x and y do not have a join either. Then there is an element $w \in \mathbb{S}$ such that $s, t \leq w$.* $\square$

This too is not true in S4: using the characterisation of S4, Theorem 42, consider the full binary tree of height 3, and define a suitable valuation on it. This situation too can be encapsulated by a modal formula, though we leave it for the reader to piece together the actual formulas using the techniques that follow. The situation then can be summarised as follows:

$$\mathsf{S4} \subsetneq \mathsf{S4sBA} \subsetneq \mathsf{S4.2} \subsetneq \mathsf{S4.3}.$$

Hence, S4tBA and S4sBA occupy a similar position with respect to these natural modal logics. We shall now prove that S4tBA is not contained in S4sBA. However, the exact relation between S4sBA and S4tBA is unclear.

Question 124. *Is* S4sBA *contained in* S4tBA*?*

Question 125. *Is* S4sBA *finitely axiomatisable?*

5.2.3 Comparison of Topless pre-Boolean Algebras and Spiked pre-Boolean Algebras

In this section, we shall separate the theories S4sBA and S4tBA. Our motivation for doing so is this: recall that in Theorem 67, we showed that $\mathsf{ML}_{\mathsf{ccc}}$ is not contained in S4.2 by ccc-labelling the two element fork frame. Since the two element fork is a topless Boolean algebra, and since S4tBA is a naturally occurring modal logic and has been studied in the context of the modal logic of forcing, it is natural to conjecture that $\mathsf{ML}_{\mathsf{ccc}}$ is contained in S4tBA.

However, as we will show, spiked Boolean algebras, which are also a generalisation of the two element fork, paint a more accurate picture of $\mathsf{ML}_{\mathsf{ccc}}$, and in particular, $\mathsf{ML}_{\mathsf{ccc}}$ is indeed contained in S4sBA. This will also imply that $\mathsf{ML}_{\mathsf{ccc}}$ is not contained in S4tBA by the main theorem of this section.

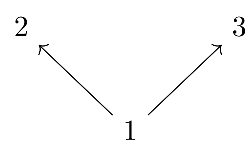

Figure 5.2: The two element fork, which is both, a spiked Boolean algebra and a topless Boolean algebra

The primary observation that we use to show that S4tBA is not contained in S4sBA is this: Corollary 120 is not true of spiked Boolean algebras. Consider the spiked Boolean algebra with exactly 3 extremal nodes, u, v and w, where w is the topmost element of the corresponding Boolean algebra. No two of them have a join, but if t is an element such that $t \leq u$ and $t \leq v$, then t must be the bottom element of the Boolean algebra, and hence $t \leq w$.

What is then needed to give the proof is to show how to use this observation to find a modal formula which distinguishes between spiked Boolean algebras and topless Boolean algebras.

Definition 126. Let S be a finite set of modal formulas. Let $T \subseteq S$. The formula χ_T^S is defined as follows:

$$\chi_T^S \triangleq \bigwedge (\{\psi \mid \psi \in T\} \cup \{\neg\psi \mid \psi \in S \setminus T\}).$$

The formula θ_T^S is defined as follows:

$$\theta_T^S \triangleq \chi_T^S \wedge \Box\chi_T^S.$$

If S is clear from the context, it will not be mentioned in these formulas. That is, if the context is clear, the notation for these formulas will be χ_T and θ_T.

It is clear that if T_1 and T_2 are distinct subsets of S, then $(\theta_{T_1} \wedge \theta_{T_2}) \leftrightarrow \bot$. The reason for defining such formulas is the following: Let S be a set of atomic formulas and (W, R, V) a finite reflexive transitive Kripke frame. Now, if $T \subseteq S$ and $u \in W$ is such that $u \Vdash \theta_T$, and $v \in W$ is such that uRv, then $v \Vdash \theta_T$ as well. If $T' \subseteq S$ is distinct from T and $w \in W$ is a node such that $w \Vdash \theta_{T'}$, then there is no $v \in W$ such that uRv and wRv. In particular, were the frame to be an algebra with some sort of (necessarily partial) 'join' operation defined, then such u and w would have no join in the algebra. This, together with our previous observation about the difference between the join operation on spiked Boolean algebras and topless Boolean algebras will help us obtain the promised formula.

Let p, q, r be distinct atomic propositions. Let $\psi_1 = \theta_{\{p\}}$, $\psi_2 = \theta_{\{q\}}$ and $\psi_3 = \theta_{\{r\}}$ (S in this case would be $\{p, q, r\}$). Consider the formula

$$\psi \triangleq (\bigwedge_{i \le 3} \Diamond\psi_i) \to \Diamond(\Diamond\psi_i \wedge \Diamond\psi_2 \wedge \neg\Diamond\psi_3).$$

Lemma 127. *Let $\mathbb{T}$ be any finite topless pre-Boolean algebra. Then ψ is valid on $\mathbb{T}$. Hence, $\psi \in sftba$.*

Proof. Let V be a valuation on $\mathbb{T}$ and let h be a node of $\mathbb{T}$ such that $h \Vdash \bigwedge_{i \le 3} \Diamond\psi_i$. Then there are nodes d, e, f such that $h \le d, e, f$ and $d \Vdash \psi_1$, $e \Vdash \psi_2$ and $f \Vdash \psi_3$. It is also clear that no two of these nodes can have a join in the algebra. By Corollary 120, we can find a node g such that $g \le d, e$ and $g \not\le f$. If $g \Vdash \bigwedge_{i \le 3} \Diamond\psi_i$, we can repeat this step again. As the frame is finite, after some iterations we reach a node h' such that $h' \Vdash \Diamond\psi_i \wedge \Diamond\psi_2 \wedge \neg\Diamond\psi_3$. As the relation is transitive, we are done. $\Box$

Lemma 128. *Let $\mathbb{S}$ be the spiked Boolean algebra with 3 extremal nodes, the extremal nodes being u, v, w where w is the top node of the corresponding Boolean algebra. Then there is a valuation V on $\mathbb{S}$ such that $(\mathbb{S}, V) \not\Vdash \psi$. Hence, $\psi \notin \mathsf{S4sBA}$.*

Proof. Let V be any valuation on $\mathbb{S}$ such that $u \Vdash p \wedge \neg q \wedge \neg r$, $v \Vdash \neg p \wedge q \wedge \neg r$ and $w \Vdash \neg p \wedge \neg q \wedge r$. Let t be the root of $\mathbb{S}$. Clearly, $t \Vdash \bigwedge_{i \le 3} \Diamond\psi_i$. However, it is equally clear that there is no node $s \in \mathbb{S}$ such that $s \Vdash \Diamond\psi_i \wedge \Diamond\psi_2 \wedge \neg\Diamond\psi_3$. $\Box$

Corollary 129. *It is not the case that $\mathsf{S4tBA}$ is contained in $\mathsf{S4sBA}$.* $\Box$

5.3 Labelling Frames

The aim of this section is to describe our strategy for labelling all spiked Boolean algebras. First, we recall the ccc-labelling we have previously seen, that in Theorem 67.

Let M be a model of set theory, and $T \in V$ be an ω_1-tree. Then all of the ccc-extensions of M can be described in the following way:

$$\varphi_{\text{nA}} \triangleq T \text{ is non-Aronszajn;}$$
$$\varphi_{\text{Sp}} \triangleq T \text{ is special;}$$
$$\varphi_{\text{Su}} \triangleq T \text{ is non-special Aronszajn.}$$

These statements have the following properties, which were used to label the two element fork frame in Theorem 67 where T was taken to be the $<_{\mathbf{L}}$-least Suslin tree:

$$\varphi_{\text{nA}} \to \Box\varphi_{\text{nA}},$$
$$\varphi_{\text{nA}} \to \Box\varphi_{\text{nA}},$$
$$\varphi_{\text{Su}} \to (\Diamond\varphi_{\text{nA}}) \wedge (\Diamond\varphi_{\text{Sp}}).$$

The labellings that are provided in this chapter are generalisations of this labelling to the case of n-many ω_1-trees, albeit with some restrictions on the treees. In particular, our starting combinatorial structure is provided by the following theorem of Abraham-Shelah [AS93]:

Theorem 130. *($\Diamond_{\omega_1}$) Let* Sp *be a collection of non-empty finite subsets of ω_1 closed under supersets, and let* Su *be those non-empty finite sets $e \subset \omega_1$ which are not in* Sp*. Then there is a sequence of trees $\langle T^\gamma \mid \gamma < \omega_1 \rangle$ such that for a finite set $e \subset \omega_1$,*

(i) If $e \in \text{Sp}$, $T^e \triangleq \bigotimes_{\gamma \in e} T^\gamma$ is special.

(ii) If $e \in \text{Su}$, $T^e \triangleq \bigotimes_{\gamma \in e} T^\gamma$ is Suslin.

Corollary 131. *($\Diamond_{\omega_1}$) For every natural number n, there are Suslin trees $T^1, T^2, \dots, T^n$ such that if $i, j \in \{1, 2, \dots, n\}$ then $T_i \otimes T_j$ is special.*

Now, let us fix a natural number n and Suslin trees $T^1, T^2, \dots, T^n$ in $\mathbf{L}$ with properties as above. It is clear then that in any generic extension of $\mathbf{L}$ where $\omega_1^{\mathbf{L}}$ is not collapsed, each of the T^i is either special, non-special Aronszajn, or non-Aronszajn. We can therefore describe all of the ccc-extensions of $\mathbf{L}$ based on the properties that these trees possess in this extension.

Definition 132. Let n be a fixed natural number, and let $T^1, T^2, \dots, T^n$ be trees as above. Let M be a ccc-extension of $\mathbf{L}$. Let $(\text{nA}, \text{Sp}, \text{Su})$ be a triple of subsets of $\{1, 2, \dots, n\}$ such that $\text{nA} \uplus \text{Sp} \uplus \text{Su} = \{1, 2, \dots, n\}$. We say that $(\text{nA}, \text{Sp}, \text{Su})$ is a *description of M* if the following holds: For each i, $1 \leq we \leq n$,

(i) $i \in \text{nA}$ iff T^i is not Aronszajn in M;

(ii) $i \in \text{Sp}$ iff T^i is special in M;

(iii) $i \in \text{Su}$ iff T^i is non-special Aronszajn in M.

Thus, it is clear that each ccc-extension M of $\mathbf{L}$ has a description by some triple. In fact, due to the properties of our trees, we shall see in Proposition 133 that all of the ccc-extensions of $\mathbf{L}$ can be described by triples $(\text{nA}, \text{Sp}, \text{Su})$ of subsets of $\{1, 2, \dots, n\}$ with the following properties:

(i) $\mathrm{nA} \uplus \mathrm{Sp} \uplus \mathrm{Su} = \{1, 2, \ldots, n\}$;

(ii) $|\mathrm{nA}| \leq 1$;

(iii) $\mathrm{nA} \neq \emptyset \implies \mathrm{Su} = \emptyset$.

If n is clear from context, we call a triple of subsets of $\{1, 2, \ldots, n\}$ which satisfies these properties a *valid triple*.

Further, by Corollary 144, it follows that for each triple $(\mathrm{nA}, \mathrm{Sp}, \mathrm{Su})$ with these properties, there is a ccc-extension of $\mathbf{L}$ whose description is exactly this triple.

We shall then see in Lemma 145 that if M is a ccc-extension whose description is $(\mathrm{nA}, \mathrm{Sp}, \mathrm{Su})$, then for any other valid triple $(\mathrm{nA}', \mathrm{Sp}', \mathrm{Su}')$ there is a ccc extension N of M which is described by the triple $(\mathrm{nA}', \mathrm{Sp}', \mathrm{Su}')$ if and only if:

(i) The first co-ordinate increases. That is, $\mathrm{nA} \subseteq \mathrm{nA}'$.

(ii) The second co-ordinate increases. That is, $\mathrm{Sp} \subseteq \mathrm{Sp}'$.

(iii) The third co-ordinate decreases. That is, $\mathrm{Su} \supseteq \mathrm{Su}'$.

This is clearly reminiscent of the spiked Boolean algebra on an n-element set, and we label this partial order using statements that we derive from the behaviour of these trees in Theorem 146

We shall then see in Corollary 148 that adding an arbitrarily large number of $\mathbf{L}$-Cohen reals does not change the truth value of any of the statements that we use to label the topless Boolean algebras. Hence, arbitrarily large clusters can also be added at each node of the frames, that is, any finite spiked pre-Boolean algebra can be labelled by ccc forcing, hence giving the main theorem, Theorem 150.

5.4 Adding Branches and Specialising

In this section, we prove some results about ω_1-trees. The main concern is to show that under certain conditions, a non-special Aronszajn tree can be made special or non-Aronszajn without disturbing the non-special Aronszajn-ness of other trees. Before we do this, we first prove some basic results about ω_1-trees which will come in handy later.

The following are some basic properties of the tensor product operation that we are studying:

Proposition 133. *Let T and S be ω_1-trees. Then*

(i) $T \otimes S$ is non-Aronszajn iff T and S are non-Aronszajn.

(ii) In general, if T is non-Aronszajn, then $T \otimes S$ is non-special, special or non-Aronszajn iff S is as well.

(iii) If T is special, then $T \otimes S$ is special.

(iv) If $T \otimes S$ is special in a model M, and in some model N, $M \subset N$ such that $(\omega_1^1)^M = (\omega_1)^N$, T is non-Aronszajn, then S is special in N.

Proof. Trivial. □

We will also need the following folklore fact about adding branches to ω_1-trees:

Lemma 134. *Let $\mathbb{P}$ be a poset such that $\mathbb{P} \times \mathbb{P}$ is ccc. Let T be an ω_1-tree. Then forcing with $\mathbb{P}$ does not add any new branches to T.*

Proof. Assume towards a contradiction that $\dot{b}$ is a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}}$"$\dot{b}$ is a new branch of T". Then for each $p \in \mathbb{P}$ and $\alpha < \omega_1$, there is a $\bar{\alpha} \geq \alpha$ and p_α and q_α which decide $b \restriction \bar{\alpha}$ incompatibly (because $\dot{b}$ is a new branch of T). Using this, one inductively constructs a sequence $\langle (p_\alpha, q_\alpha) \mid \alpha < \omega_1 \rangle$ of elements of $\mathbb{P} \times \mathbb{P}$ satisfying the following:

(i) For $\alpha < \beta$, $p_\alpha < p_\beta, q_\beta$.

(ii) For $\beta < \omega_1$, there is some $\delta \geq \gamma$ such that p_β and q_β decide $\dot{b} \restriction \delta$ incompatibly, where

$$\gamma = \sup\{\lambda \mid p_\alpha \text{ decides } \dot{b} \restriction \lambda \text{ for some } \alpha < \beta\}.$$

It is then clear that this sequence is an uncountable antichain of $\mathbb{P} \times \mathbb{P}$ because for any $\alpha < \omega_1$, $p_\alpha \perp q_\alpha$, and for any $\alpha < \beta < \omega_1$, $p_\alpha < p_\beta, q_\beta$. □

5.4.1 The Specialisation Theorem

The primary task of this section is to understand how Aronszajn trees can be specialised, and made non-Aronszajn. For the former, a theorem of Baumgartner [Bau70] provides a canonical method for ccc-specialising an Aronszajn tree. Our treatment follows [Jec03, Chapter 16]. Note that this poset and the results of this subsection talk about specialising ω_1-trees in the 'traditional sense', but that this implies that they are special in our sense as well.

Lemma 135. *Let T be an Aronszajn tree and A an uncountable collection of pairwise disjoint finite subsets of T. Then there are $S, S' \in A$ such that $S \cup S'$ is an antichain of T.*

Proof. Assume towards a contradiction that the claim is false. Therefore, for each distinct $S, S' \in A$, $S \cup S'$ is not an antichain of T. Without loss of generality, assume that $|A| = \aleph_1$. Also, assume that there is a natural number n such that for each $S \in A$, $|S| = n$. We can do this because A is uncountable, so there must be a natural number n such that uncountably many elements of A have size n. For each $S \in A$, fix an enumeration of S. Thus, for each i, $1 \leq i \leq n$, $S(i)$ is the ith element of S in this enumeration. Lastly, fix a uniform ultrafilter $\mathcal{U}$ on A. That is, an ultrafilter with domain A such that for any $X \in \mathcal{U}$, $|X| = \aleph_1$. Such an ultrafilter can be found using the Axiom of Choice. Now, by assumpton, for each distinct $S, S' \in A$, $S \cup S'$ is not an antichain T, so there is an $x \in S$ and $x' \in S'$ such that x and x' are comparable. Therefore, for each $S \in A$ and $x \in S$ we can consider the set

$$Y_{x,k} \triangleq \{S' \in A \mid x \parallel S'(k)\}$$

for each k, $1 \leq k \leq n$. By assumption, for each $S' \neq S$, there is a $k \leq n$ and an $x \in S$ such that $S' \in Y_{x,k}$. Therefore,

$$A = \bigcup_{x \in S} \bigcup_{k=1}^{n} Y_{x,k}$$

for each $S \in A$. We note that both of the unions are indexed by finite sets, and so we can appeal to the ultrafilter $\mathcal{U}$ to give us, for each $S \in A$, an $x_S \in S$ and a k_S, $1 \leq k_S \leq n$ such that $Y_{x_S,k_S} \in \mathcal{U}$.

Since A is uncountable, there is a $1 \leq k \leq n$ such that $B = \{S \mid k_S = k\}$ is uncountable. We will now show that for any two $S, S' \in B$, the elements x_S and $x_{S'}$ must be comparable, and show that this implies that T is Aronszajn.

So, let $x = x_S$ and $x' = x_{S'}$. Now, $Y_{x,k}$ and $Y_{x',k}$ are both in $\mathcal{U}$, so $Y = Y_{x,k} \cap Y_{x',k}$ is in $\mathcal{U}$ as well. If $S'' \in Y$, then $S''(k)$ is then comparable with both x and x'. Now, recall that by our hypothesis, any two elements of A are disjoint. Hence, any two elements of Y are disjoint. Hence, for any distinct $S_1, S_2 \in Y$, $S_1(k) \neq S_2(k)$. Now finally, recall that $\mathcal{U}$ was a *uniform* ultrafilter, so Y is uncountable. Therefore, there are uncountably many distinct elements of T which are comparable with both x and y. In particular, there is an element z whose height is larger than the heights of x and y and which is comparable with both of them, and hence, x and y must be comparable.

Therefore, for any two distinct $S, S' \in B$, x_S and $x_{S'}$ are (distinct and) comparable, and since B is uncountable, this gives us an uncountable chain in T, which implies that T is not Aronszajn, which is a contradiction. □

Now, recall Baumgartner's poset $\mathcal{S}(T)$ that consists of functions p such that

(i) $\mathrm{dom}(p)$ is a finite subset of T;

(ii) $\mathrm{ran}(p) \subset \omega$;

(iii) If $x, y \in \mathrm{dom}(p)$ are such that $x \parallel y$, then $p(x) \neq p(y)$,

and ordered such that $p \geq q$ if $p \supseteq q$. At this point, the reader should take note that the ordering of forcing notions is Jerusalem-style, so stronger conditions are actually larger in the poset.

The importance of this poset is because of the following theorem, which, in addition to showing that an Aronszajn tree can be ccc-specialised, also states that to know whether T is Aronszajn or not is the same as knowing whether $\mathcal{S}(T)$ is ccc or not. Such a characterisation (in terms of the ccc-ness of a poset) of the Aronszajn-ness of T will be particularly useful once we have proved the Antichain Lemma, Lemma 140 at the end of this section.

Theorem 136. *(Specialisation Theorem) Let T be an ω_1-tree. The following are equivalent:*

(i) T is Aronszajn.

(ii) $\mathcal{S}(T)$ has the ccc.

Proof. For (ii) implies (i), we use contraposition. Let b be an uncountable branch of T, and n a natural number. For each $p \in b$, let f_p be the function whose domain is $\{p\}$ and $f_p(p) = n$. Consider the following subset of $\mathcal{S}(T)$:

$$A \stackrel{\Delta}{=} \{f_p \mid p \in b\}.$$

Clearly A is uncountable, and if $g \in \mathcal{S}(T)$ is such that $g \supset f_p$, then $p \in \mathrm{dom}(g)$, and $g(p) = n$. Clearly then, for two distinct $p, q \in b$, it cannot be the case that g extends both f_p and f_q, as this would imply that $\{p, q\} \subseteq g^{-1}(n)$, contradicting the third clause in the definition of $\mathcal{S}(T)$. Hence, A is an uncountable antichain of $\mathcal{S}(T)$.

For (i) implies (ii) Let W be an uncountable subset of $\mathcal{S}(T)$. Refine W in the following way:

(i) Using the Δ-System Lemma, obtain an uncountable $V \subseteq W$ such that there is a finite set $S \subset T$ such that for any two distinct $p, q \in V$, $\mathrm{dom}(p) \cap \mathrm{dom}(q) = S$.

(ii) Now, V is uncountable, whereas for each $p \in V$, $p \restriction S$ is a finite subset of ω, hence has only countably many possible values. Therefore, obtain an uncountable $U \subseteq V$ such that for any two $p, q \in U$, $p \restriction S = q \restriction S$.

Now, appeal to Lemma 135 to get elements $p, q \in U$ such that $(\mathrm{dom}(p) \setminus S)) \cup (\mathrm{dom}(q) \setminus S)$ is an antichain. Finally, we claim that $p \cup q$ is an element of $\mathcal{S}(T)$ which extends both p and q, and so W is not an antichain of $\mathcal{S}(T)$, hence allowing us to conclude that $\mathcal{S}(T)$ has the ccc.

So, suppose that $p \cup q$ is not an element of $\mathcal{S}(T)$. This is only possible if condition (iii) is not satisfied. But if $x, y \in \mathrm{dom}(p \cup q)$ are such that $x \parallel y$, then it follows (because of the way we chose p and q) that it cannot be the case that $x \in \mathrm{dom}(p) \setminus S$ and $y \in \mathrm{dom}(q) \setminus S$. In particular, either both of them are in $\mathrm{dom}(p)$ or both of them are in $\mathrm{dom}(q)$. In either case, since p and q are both elements of $\mathcal{S}(T)$, it cannot be the case then that $(p \cup q)(x) = (p \cup q)(y)$. Hence, $p \cup q \in \mathcal{S}(T)$, and the result follows. □

Corollary 137. *Let T be an Aronszajn tree. Then $\mathcal{S}(T) \times \mathcal{S}(T)$ has the ccc, and hence, $\mathcal{S}(T)$ cannot add a branch to an ω_1-tree.*

Proof. By the above theorem, if T is Aronszajn, then $\mathcal{S}(T)$ is ccc. But now, if we force with $\mathcal{S}(T)$, then T is special, and in particular, Aronszajn. It follows then that $\Vdash_{\mathcal{S}(T)}$"$\mathcal{S}(T)$ has the ccc", and hence by Theorem 36, $\mathcal{S}(T) \times \mathcal{S}(T)$ has the ccc. The rest is immediate by Lemma 134. □

5.4.2 The Subtree Theorem

We turn next to the problem of adding a branch to a non-special Aronszajn tree. Note that such a tree need not necessarily be Suslin[2]. Indeed, one can construct using $\diamondsuit_{\omega_1}$ a Suslin trees T such that for any two nodes p, q on the same level of T, T_p can be specialised while ensuring that T_q remains Suslin and vice-versa, see for example [DJ74] or [AS93] for a construction of such a Suslin tree[3]. This implies that there can be Aronszajn trees with a special subtree and a Suslin subtree. Such a tree is then non-special, but also clearly not Suslin.

Therefore, a tree being non-special Aronszajn only gives information that there is *some* ccc poset which adds a branch to this tree, but does not a priori say anything about what the poset is. However, it is clear that if T is an Aronszajn tree such that there is a node $p \in T$ such that T_p is Suslin, then T must be non-special, since a branch can always be added to T by forcing with T_p. The next theorem shows that such a situation must occur if T is a non-special Aronszajn tree. We point out that the one of the (non-obvious) directions of the theorem is proved in [BJ95, Lemma 9.7.19], which the authors attribute to [HS85].

Theorem 138. *(Subtree Theorem) Let T be an Aronszajn tree. Then the following are equivalent[4]:*

(i) There is a ccc poset adding a branch to T.

[2]There are, however, homogeneity conditions that can be imposed on the tree so that the two are equivalent; see Section 5.7.

[3]The requisite condition that needs to be imposed on T_p and T_q is that $T_p \otimes T_q$ is Suslin, and then the Antichain Lemma, Lemma 140, allows us to show the preservation results.

[4]It should be mentioned that Lemma 9.7.17 of [BJ95] says that the negation of (ii) above implies that the specialising poset $\mathcal{S}(T)$ is Knaster. Clearly, if a tree T is such that $\mathcal{S}(T)$ is Knaster, a branch cannot be added to it by ccc forcing. However, we are not fully convinced that the proof as given is correct, and have not been able to supply a proof ourselves, so we do not add the extra item to this chain of equivalences.

(ii) There is an uncountable subset S of T such that no uncountable $S' \subseteq S$ is an antichain.

(iii) There is a node $p \in T$ such that T_p is Suslin

Proof. For (i) implies (ii), let $\mathbb{P}$ be a ccc poset, $p \in \mathbb{P}$ a condition, and τ a $\mathbb{P}$-name such that $p \Vdash$"τ is a branch of T". Let

$$A \stackrel{\Delta}{=} \{t \in T \mid \exists q \geq p(q \Vdash \check{t} \in \tau)\}.$$

As τ is a name for a confinal branch of this tree, A is uncountable. If $B \subseteq A$ were an uncountable antichain, then for each $t \in B$ pick an element p_t such that $p_t \Vdash \check{t} \in \tau$. Then the set $S = \{p_t \mid t \in B\}$ is clearly an uncountable antichain of $\mathbb{P}$, contradicting that $\mathbb{P}$ is ccc. Hence no uncountable subset of A can be an antichain.

For (ii) implies (iii), let S be such a subset of T. If S is not dense above any node $p \in T$, then consider the set $S' \stackrel{\Delta}{=} \{p \in S \mid S \cap T_p = \emptyset\}$. Note that if $p \in S'$, then $S' \cap T_p = \emptyset$, and as T is a tree, this implies that for any $p, p' \in S'$, $p \perp p'$. Hence, S' is an antichain of T. Also, for every $p \in S$, there is a $p' \in S'$ such that $p' \geq p$. Hence, S' bounds the elements of S from above. Now, if S' were countable, this would imply that all elements of S are bounded by some level α in T. But as T is an ω_1-tree, and in particular, all the levels of T are countable, this implies that S is countable. Hence, S' must be an uncountable antichain of T, which contradicts the choice of S.

Consequently, let $p \in T$ be such that S is dense above p. Suppose that T_p is not a Suslin tree, and let $A \subset T_p$ be an uncountable antichain. Then as S is dense above p, we can extend each element of A to an element in S, obtaining a set S', which is an uncountable subset of S which is an antichain, which is not possible. Hence T_p must be Suslin.

The remaining direction, (iii) implies (i) is obvious. □

5.4.3 The Antichain Lemma

So far, we have shown that there is a ccc-way to specialise Aronszajn trees, and that branches can be added to non-special Aronszajn trees by a fairly well-understood poset. While this was quite crucial to the endeavour, even more crucial is being able to do both of these things with one tree, without affecting the another tree. While this is harder to understand with Aronszajn trees in general, this is easier for the case of Suslin trees as we shall now see. The next definition and lemma are from [Lar99].

Definition 139. Let $M \subset N$ be models of ZFC. Let T be a tree in M, and $A \in N \setminus M$ an antichain of T. Call A a *deep antichain* of T with respect to M if for all maximal antichains B of T in M, there is an element of A above some element of B in T.

Note that a deep antichain of a tree must be unbounded in the levels, and hence uncountable if ω_1 is not collapsed. Also note that if a poset adds a generic branch through a Suslin tree, it adds a deep antichain as well. In fact, as any antichain in a Suslin tree is bounded by some level of the tree, any poset that destroys the Suslinity of some tree adds a deep antichain to the tree.

Lemma 140. *(Antichain Lemma) Let T be an ω_1-tree and $\mathbb{P}$ a forcing poset. If forcing with $\mathbb{P}$ can add a deep antichain on T with respect to the ground model, then forcing with T can put an ω_1-antichain through $\mathbb{P}$.*

Proof. Let τ be a name and $p \in \mathbb{P}$ a condition such that $p \Vdash$ "τ is a deep antichain of T". Consider the set

$$D \triangleq \{\alpha \in T \mid \exists q \in \mathbb{P}, q \geq p \text{ and } q \Vdash \check{\alpha} \in \tau\}.$$

If D is not dense above any node of T, then the set

$$E \triangleq \{\alpha \in T \mid \alpha \in D \text{ and } \forall \beta > \alpha(\beta \notin D)\}$$

is a maximal antichain of T such that no node of τ is above any element of E, which is not possible. Hence, D is dense below some node $q \in T$. Now clearly, a generic path through T which contains q must meet D densely, and hence, $\aleph_1$-many times. Let this set be $\{\beta_\gamma \mid \gamma < \omega_1\}$. For each $\gamma < \omega_1$, fix an element $p_\gamma \in \mathbb{P}$ such that $p_\gamma \geq p$ and $p_\gamma \Vdash \beta_\gamma \in \tau$. Then the set $\{p_\gamma \mid \gamma < \omega_1\}$ is an uncountable antichain of $\mathbb{P}$ since $p \Vdash$ "τ is a deep antichain of T" and each of these elements is above p. □

As mentioned above, deep antichains are easy to come by when the Suslinity of some tree has been destroyed. This is also why the Subtree Theorem is useful for us; it tells us that if a non-special Aronszajn tree is specialised, then there is a subtree of it to which a deep antichain has been added. This allows the Antichain Lemma to 'retaliate' against this poset in a way that we will see below.

Before we see such an example, we point out the 'can' in the formulation of the Antichain Lemma. The proof itself shows that if $\mathbb{P}$ adds a deep antichain to T, then there is a node $p \in T$ such that any generic branch of T through p adds an uncountable antichain to $\mathbb{P}$. The next two lemmas show that this can be rephrased in terms of a generic branch through T_p, which is often more convenient for our purposes.

Lemma 141. *Let T be an ω_1-tree. If $p \in T$ is a node, then adding a generic for T_p adds a generic for T. In fact, if b' is the T_p-generic branch, and $V[b']$ is the generic extension so obtained (i.e. V is the ground model containing T), then there is an object $b \in V[b']$ which is T-generic over V and such that $V[b'] = V[b]$. That is, a T_p-generic extension is also a T-generic extension.*

Proof. The first part follows from the observation that every dense open subset of T contains a dense open subset of T_p as well. For the second part, b is obtained from b' by prefixing the branch (of length $\mathrm{ht}(p)$) $b'' \triangleq \{q \in T \mid q < p\}$ to b'. Clearly, b'' is in the ground model, and so b and b' are inter-definable from the other over the ground model, so $V[b'] = V[b]$. This branch is generic by the first part. □

Corollary 142. *Let T be an ω_1-tree and $\mathbb{P}$ a poset which adds a deep antichain to T. Then there is a node $p \in T$ such that adding a generic branch to T_p adds an uncountable antichain to $\mathbb{P}$.* □

A sample usage of the Antichain Lemma is the following:

Proposition 143. *Let S and T be Aronszajn trees such that $S \otimes T$ is special. Let $p \in S$ be a node such that S_p is a Suslin tree. Then $\mathcal{S}(T)$ does not destroy the Suslinity of S_p.*

Proof. Towards a contradiction, assume the contrary. Then $\mathcal{S}(T)$ adds a deep antichain to S_p. Then, there is a $q \in S_p$ such that forcing with S_q adds an uncountable antichain to $\mathcal{S}(T)$, so it adds a cofinal branch of T. But it also trivially adds a cofinal branch to S. Hence, by Proposition 133, it adds a cofinal branch to $S \otimes T$, hence collapsing ω_1. But as S_p is Suslin, this is not possible. □

The reason for the hypothesis that there is a node $p \in S$ such that S_p is Suslin is this: the Subtree Theorem tells us that this is exactly what happens when S is a non-special Aronszajn tree.

Corollary 144. *Let S and T be Aronszajn trees such that $S \otimes T$ is special. Further, assume that S is non-special Aronszajn. Then after forcing with $\mathcal{S}(T)$, S is still non-special Aronszajn.*

Proof. By Corollary 137, it is clear that $\mathcal{S}(T)$ cannot add a branch to S. Hence, S will be Aronszajn in that extension. On the other hand, the above proposition tells us that it cannot destroy the Suslinity of any Suslin subtree of S. □

5.5 Labelling Kripke Frames

In this section, for each n, we give a complete labelling for all the finite spiked Boolean algebras over $\mathbf{L}$ following the method we described in Section 5.3.

Fix a natural number n, and using $\diamondsuit_{\omega_1}$, let $\langle T^1, T^2, \dots, T^n \rangle$ be a sequence of Suslin trees in $\mathbf{L}$ such that if $i, j \in \{1, 2, \dots, n\}$ are distinct, then $T^i \otimes T^j$ is special. In fact, choose the sequence of T^i such that they are the $<_{\mathbf{L}}$-least such sequence. This will ensure that all the sentences that we will use in the labelling can be expressed in ZFC.

The following result shows the exact accessibility relation that holds between valid triples in the ccc-multiverse of $\mathbf{L}$:

Lemma 145. *Let M be a ccc-extension of $\mathbf{L}$, and let the description of M be (a, b, c). Let (d, e, f) be some other valid triple. Then there is a ccc-extension N of M such that the description of N is (d, e, f) if an only if:*

(i) The first co-ordinate increases. That is, $a \subseteq d$.

(ii) The second co-ordinate increases. That is, $b \subseteq e$.

(iii) The third co-ordinate decreases. That is, $c \supseteq f$.

Proof. It is clear that if T^i is special in M, that is, if $i \in b$, then in any ccc-extension of M, T^i remains special. Similarly, if T^i is non-Aronszajn in M, that is, if $i \in b$, then in any ccc-extension of M, T^i remains non-Aronszajn. Hence, it is clear the conditions on (d, e, f) are necessary.

For sufficiency, suppose that (d, e, f) satisfies these conditions. If $(a, b, c) = (d, e, f)$, then there is nothing to be done, so assume that this is not so. If $d \neq \emptyset$, then by Proposition 133, there is exactly one $i \in \{1, 2, \dots, n\}$ such that $d = \{i\}$, and $f = \emptyset$ and $e = \{1, 2, \dots, n\} \setminus \{i\}$. Also, as the two triples are distinct, $a = \emptyset$, and in particular, $i \in c$, and so T^i is non-special Aronszajn in M. In this case, there is a ccc poset which adds a branch to T^i, and it is easy to see that after forcing with this poset, T^i is not Aronszajn, and this implies that the description of the generic extension is exactly (d, e, f).

On the other hand, if $d = \emptyset$, then by Corollary 144, we can one by one specialise each T^i for $i \in e \setminus b$ without specialising or adding a branch to any other $j \in f$, and it is easy to see that after finitely many steps we get a ccc-extension whose description is exactly (d, e, f). □

We now show how we can give a complete labelling over $\mathbf{L}$ for the spiked Boolean algebra on an n-element set, $(\mathbb{S}, \leq)$, using these Suslin trees. We use our representation of $\mathbb{S}$ as a set of triples (a, b, b) of subsets of $\{1, 2, \dots, n\}$ of the following form (see Section 5.2):

(i) $a \uplus b \uplus c = \{1, 2, \ldots, n\}$;

(ii) $|a| \leq 1$;

(iii) $a \neq \emptyset \implies c = \emptyset$;

with the relation $\leq$ between the nodes is given by $(a, b, c) \leq (d, e, f)$ if:

(i) The first co-ordinate increases. That is, $a \subseteq d$;

(ii) The second co-ordinate increases. That is, $b \subseteq e$;

(iii) The third co-ordinate decreases. That is, $c \supseteq f$;

Note that the triples used in this representation are the same as what we call valid triples of subsets of $\{1, 2, \ldots, n\}$.

We recall here the definition of a labelling: A Γ-labelling of a frame F for a model of set theory W is an assignment to each node w in F a set-theoretic statement Φ_w such that

(i) The statements Φ_w form a mutually exclusive partition of truth in the multiverse of W generated by Γ. That is, if W' is in the multiverse of W generated by Γ, then W' satisfies exactly one of the Φ_w.

(ii) Any W' in the multiverse of W generated by Γ in which Φ_w is true satisfies $\Diamond\Phi_u$ if and only if $w \leq_F u$.

(iii) If w_0 is an initial element of F, then $W \vDash \Phi_{w_0}$.

For every valid triple (a, b, c), let $\varphi_{(a,b,c)}$ be the statement expressing that for the the $<_{\mathbf{L}}$-least sequence $\langle T^1, T^2, \ldots T^n \rangle$ of Suslin trees such that the product of any two distinct members of them is special, the following holds:

(i) $i \in a$ iff T^i is not Aronszajn;

(ii) $i \in b$ iff T^i is special;

(iii) $i \in c$ iff T^i is non-special Aronszajn.

The map $(a, b, c) \mapsto \varphi_{(a,b,c)}$ is then the labelling of $\mathbb{S}$. To see that this is indeed a labelling, note that:

(i) The initial element of $\mathbb{S}$ is represented by $(\emptyset, \emptyset, \{1, 2, \ldots, n\})$, and it is clear that $\mathbf{L} \vDash \varphi_{(\emptyset,\emptyset,\{1,2,\ldots,n\})}$;

(ii) In any ccc-extension M of $\mathbf{L}$, there is exactly one valid triple (a, b, c) which describes it, and hence, $M \vDash \varphi_{(a,b,c)}$;

(iii) If M is a ccc-extension of $\mathbf{L}$ which satisfies $\varphi_{(a,b,c)}$, then the description of M is (a, b, c), and hence, for any other triple (d, e, f), by Lemma 145, there is a ccc-extension N of M satisfying $\varphi_{(d,e,f)}$ if and only if $(a, b, c) \leq (d, e, f)$ in $\mathbb{S}$.

Theorem 146. *For each natural number n, there is a ccc-labelling of the finite spiked Boolean algebra on an n-element set for* $\mathbf{L}$. $\square$

5.6 Switches

In this section we show that Cohen forcing allows us to add arbitrarily large finite clusters at each node of our Kripke frames.

Definition 147. Let $\mathbb{P}$ be a forcing poset. We say that $\mathbb{P}$ is *productively ccc* if for any ccc poset $\mathbb{Q}$, $\Vdash_{\mathbb{P}}$"$\mathbb{Q}$ is ccc", or equivalently, $\mathbb{P} \times \mathbb{Q}$ is ccc.

Note that above, these two definitions are equivalent by Theorem 36. Any countable, Knaster, or σ-centered poset is productively ccc. In particular, for any regular cardinal κ, the poset $\prod_{\kappa} \mathrm{Coh}$ which adds κ-many Cohen reals is Knaster, and hence productively ccc.

Proposition 148. *Let $\mathbb{P}$ be a productively ccc poset. Let T be a non-special Aronszajn tree. Then $\Vdash_{\mathbb{P}}$"T is a non-special Aronszajn tree".*

Proof. Since $\mathbb{P}$ is productively ccc, $\mathbb{P} \times \mathbb{P}$ is ccc, and hence, by Lemma 134, forcing with $\mathbb{P}$ does not add any branches to T. On the other hand, if $p \in T$ is such that T_p is a Suslin subtree (by the Subtree Theorem, such a p must exist), then $\mathbb{P} \times T_p$ has the ccc, and hence, by Theorem 36, $\Vdash_{\mathbb{P}}$"T_p is Suslin". Therefore, after forcing with $\mathbb{P}$, T remains a non-special Aronszajn tree. $\square$

Hence, in any generic extension of $\mathbf{L}$, we can add as many Cohen reals as we like without adding a branch to any Aronszajn tree, and without specialising any non-special Aronszajn tree. Also, by Theorem 20, all of these reals must be Cohen over $\mathbf{L}$ as well. This informs our choice of switches in the next lemma.

Lemma 149. *Any finite spiked pre-Boolean algebra can be ccc-labelled over $\mathbf{L}$. Consequently, $\mathsf{ML}^{\mathbf{L}}_{\mathsf{ccc}}$ is included in $\mathsf{S4sBA}$.*

Proof. Let $\mathbb{S}$ be a spiked pre-Boolean algebra. Let n be the number of nodes of the partial order of $\mathbb{S}$ obtained by taking the quotient of this pre-order by the natural equivalence relation, and by adding dummy nodes if required, let each of the clusters of $\mathbb{S}$ have size 2^m.

For $j < m$, let s_j be the statement "If p is the unique natural number such that there are $\aleph_{\omega \cdot \alpha + p}$-many $\mathbf{L}$-Cohen reals for some ordinal α, then the remainder of p when divided by m is j".

Also, recall the statements $\varphi_{(a,b,c)}$ for (a, b, c) a valid triple of subsets of $\{1, 2, \ldots, n\}$ which we used to prove Theorem 129. Each of the s_m's are independent switches, and one only needs to add the relevant number of Cohen reals to flip a certain switch 'on' or 'off' without affecting the other switches. Further, by Proposition 148, each of these switches can be flipped on or of without affecting the truth values of any of the $\varphi_{(a,b,c)}$ that we used to label the spiked Boolean algebra on an n-element set.

Now, we represent $\mathbb{S}$ in the following way: each element of $\mathbb{S}$ can be represented by a pair $((a, b, c), j)$ whose first co-ordinal is a valid triple of subsets of $\{1, 2, \ldots, n\}$, and whose second co-ordinate represents a position in the cluster. The relation between the nodes can be given by:

$$((a, b, c), j) \leq ((d, e, f), k) \text{ iff } (a, b, c) \leq (d, e, f),$$

where the ordering on the right hand side is the one on the representation of the spiked Boolean algebra on an n-element set.

The labelling is given by: $((a, b, c), j) \mapsto \varphi_{(a,b,c)} \wedge s_j$, and this labelling is easily seen to be complete. $\square$

Theorem 150. *If* ZFC *is consistent, then the* ZFC*-provable modal logic of ccc forcing,* $\mathsf{ML}_{\mathsf{ccc}}$*, is included in* S4sBA*.* □

Corollary 151. *If* ZFC *is consistent, then the* ZFC*-provable modal logic of ccc-forcing,* $\mathsf{ML}_{\mathsf{ccc}}$*, does not include* S4tBA*.*

Proof. Follows from the above theorem and Corollary 129. □

5.7 Generalisations

In this section, we discuss the scope of generalising our methods to the case of arbitrarily large finite sets of Aronszajn trees, such that there are fewer restrictions than the trees, so as to enable us to label a larger class of frames. The most natural such generalisation is the following:

For any natural number n, appealing to Theorem 130, start with a sequence of Suslin trees $T^1, T^2, \ldots, T^n$ such that if $i, j, k \in \{1, 2, \ldots, n\}$ are distinct, then $T_i \otimes T_j \otimes T_k$ is special, and such that all other tensor products are Suslin. In this case, it is no longer true that the tensor product of any two distinct trees is special, and so we would like to obtain an analogue of Corollary 144 which would also be able to handle this case. Let us see the difficulties with doing this:

It is clear by Lemma 134 that if T, S are both non-special Aronszajn, then forcing with $\mathcal{S}(T)$ does not add a branch to S, so all we need to ensure is that it does not specialise S either. Since S is non-special Aronszajn, it follows by the Subtree Theorem, that there is a $q \in S$ such that S_q is Suslin. Now, if T does specialise S, then in particular, it specialises S_q. Appealing to the Antichain Lemma, we get an $r \in S_q$, such that forcing with S_r adds a branch to T. Hence, by Proposition 133, we see that by forcing with S_q we have added a branch to $T \otimes S$.

Now, *if* $T \otimes S$ is special, *then* this would collapse ω_1, which is not possible since S_r is Suslin. Hence, we can specialise T while leaving S non-special Aronszajn if $T \otimes S$ is special. But if this is not the case (by Proposition 133, this implies that $T \otimes S$ is non-special Aronszajn), then we do not know how to prove the corresponding result.

On the other hand, if $T \otimes S$ were not just non-special Aronszajn, but Suslin (this implies that T and S are themselves Suslin), then it is clear that after forcing with S, T is still Suslin, and since a generic for S_r is the same as a generic for S which contains r (see Lemma 141), it follows that after forcing with S_r, T must still be Suslin, and hence $\mathcal{S}(T)$ cannot specialise S.

This suggests a way to deal with the obstacle we face. Namely, start with trees such that if they are non-special Aronszajn, then they are Suslin.

Definition 152. Let T be an ω_1-tree. We say that T is *very homogenous* if for any two $p, q \in T$, $T_p \cong T_q$.

The following is an example of a very homogenous tree. All of the proofs can be found in [Lar99].

Definition 153 (Coherent ω_1-trees)**.** An ω_1-tree is *coherent* if there exists a collection of is a collection of maps $\{\pi_{\alpha_0\alpha_1} \mid \mathrm{lev}(\alpha_0) = \mathrm{lev}(\alpha_1)\}$ satisfying the following:

(i) $\pi_{\alpha_0\alpha_1}$ is a level preserving isomorphism from T_{α_0} to T_{α_1} and $\pi_{\alpha_0\alpha_0}$ is the identity function.

(ii) (Commutativity) For all $\alpha_0, \alpha_1, \alpha_2$ on the same level of T, and all $\beta \geq \alpha_0$, $\pi_{\alpha_1\alpha_2}(\pi_{\alpha_0\alpha_1}(\beta)) = \pi_{\alpha_0\alpha_2}(\beta)$.

(iii) (Coherence) Let $\alpha_0, \alpha_1, \beta_0, \beta_1$ be such that $\pi_{\alpha_0\alpha_1}(\beta_0) = \beta_1$. Then $\pi_{\beta_0\beta_1} = \pi_{\alpha_0\alpha_1} \upharpoonright T_{\beta_0}$.

(iv) (Transitivity) If β_0, β_1 are two nodes at some limit level of T, then there are α_0, α_1 such that $\pi_{\alpha_0\alpha_1}(\beta_0) = \beta_1$.

Note that coherent ω_1-trees are also called *strongly homogenous trees* in the literature. These trees have been studied quite extensively in the recent literature [LT02, Tod11]. These were also the trees that Todorcevic used in his proof of Shelah's result that adding a Cohen real adds a Suslin tree [She84].

Theorem 154. *There is a coherent Aronszajn tree.*

Proposition 155. *($\Diamond_{\omega_1}$) There is a coherent Suslin tree.*

Theorem 156. *Let T be a coherent ω_1-tree. Then it is very homogenous.*

Now, by the Subtree Theorem, it follows that a very homogenous tree is non-special Aronszajn if and only if it is Suslin. This suggests that the argument that we suggested above should work with very homogenous trees, assuming that we can find very homogenous trees T, S such that $T \otimes S$ is Suslin. However, this is not possible.

Proposition 157. *Let T, S be very homogenous trees. Then $T \otimes S$ is very homogenous.*

Proof sketch. Let $p, q \in T$ and $v, w \in S$ be some nodes. Let $\pi : T_p \cong T_q$ and $\nu S_u \cong S_w$ be automorphisms between their subtrees. Then $(\pi, \nu) : (T \otimes S)_{(p,v)}) \cong (T \otimes S)_{(q,w)})$ is an automorphism as well. □

Theorem 158. *Let T and S be very homogenous ω_1-trees. Then $T \otimes S$ cannot be Suslin. In particular, if T and S are homogenously Suslin trees, then their product cannot be Suslin.*

Proof. Towards a contradiction, assume this is not so, and let T, S be very homogenous ω_1-trees such that $T \otimes S$ is Suslin. Then $T \otimes S$ has the ccc, and so $\Vdash_T$"S has the ccc". Then, by appealing to Proposition 133, this implies that $\Vdash_T$"$T \otimes$ is non-special Aronszajn". But then the Subtree Theorem tells us that in this generic extension, $T \otimes S$ has a Suslin subtree, and since $T \otimes S$ is very homogenous, that $T \otimes S$ is Suslin. However, we can easily construct an uncountable antichain of $T \otimes S$: since T is non-Aronszajn, let A be an uncountable antichain of T, and for each $p \in A$, let $q_p \in S$ be chosen such that $(p, q_p) \in T \otimes S$. Then $B = \{(p, q_p) \mid p \in A\}$ is clearly seen to be an uncountable antichain of $T \otimes S$. □

Hence, a different method might be required to generalise these results.

5.8 Questions

The aim of this chapter was to understand the modal logic of ccc forcing, and while we have improved on the existing upper bounds, we are nowhere close to a full computation of $\mathsf{ML}_{\mathsf{ccc}}$.

Question 159. *What is the modal logic of ccc forcing?*

Note that in [HL08], Hamkins and Löwe conjecture that the answer to this question is $\mathsf{S4}$. Another related question which we have not dealt with in this thesis is the following:

Question 160. (MA_{ω_1}) *What is the modal logic of ccc forcing?*

In [HL08], Hamkins and Löwe observe that since MA_{ω_1} implies that the product of ccc posets is ccc, $\mathsf{S4.2}$ is valid for the above logic. They also observe that an answer to this question would be $\mathsf{S4.2}$ if the following question has a positive answer:

Question 161. *In the Solovay-Tennenbaum model $\mathbf{L}[G]$ of MA_{ω_1}, are there arbitrarily large finite families of independent ccc-buttons and switches?*

Curiously, there are no known large families of mutually independent ccc-buttons.

Question 162. (ZFC) *Are there arbitrarily large finite families of mutually independent ccc-buttons?*

A weaker version of this question is the following:

Question 163. ($\mathsf{ZFC} + \mathsf{V} = \mathsf{L}$) *Are there arbitrarily large finite families of mutually independent ccc-buttons?*

One can also ask these questions for any of the other classes of forcing posets that are regularly used in set theory. [HLL] contains all of the known results on this topic.

The Subtree Theorem raises the following interesting question:

Question 164. *Let $\mathcal{P}$ be a class of forcing posets contained in the class of ω_1-preserving posets. Can one characterise the class of Aronszajn trees T such that there is a $\mathbb{P} \in \mathcal{P}$ such that forcing with $\mathbb{P}$ adds a branch to T without collapsing ω_1?*[5]

A related result can be found in [LT01], where Larson and Todorcevic construct assuming $\diamondsuit_{\omega_1}$ an Aronszajn tree such that forcing with the tree destroys a stationary subset of ω_1 but does not collapse ω_1. Larson [Lar13] modifed this construction to show that one can further ensure that if an ω_1-preserving poset adds a branch to this tree, then it must destroy a stationary set. Hence, this is an example of an Aronszajn tree such that no proper poset adds a branch to it, but there is an ω_1-preserving poset which adds a branch to it. These trees might perhaps also give a positive answer to Question 163.

An analysis of the standard method of proving that a forcing poset is proper suggests the following question:

Question 165. *Let T be an Aronszajn tree such that there is a proper poset which adds a branch to T. Then does T have a Suslin subtree?*

A positive answer to this question would show that all of the results in this chapter also extend to the modal logic of proper forcing. We also do not know the answer to the following stronger version of this question:

Question 166. *Let T be an Aronszajn tree such that there is a poset which adds a branch to T and does not destroy any stationary subsets of ω_1. Does T have a Suslin subtree?*

Another related question is this: if it is indeed the case that for any Aronszajn tree T, if no ccc poset adds a branch to T, then $\mathcal{S}(T)$ is Knaster, then the next two questions each generalise the previous two questions:

[5] Questions similar in spirit to this one have been asked on Mathoverflow by Erin K. Carmody [Car] Paul McKenney [McK].

Question 167. *Let $\mathbb{P}$ be a Knaster poset. Let $\mathbb{Q}$ be a proper poset. Then is it the case that $\Vdash_{\mathbb{Q}}$"$\mathbb{P}$ has the ccc"?*

Question 168. *Let $\mathbb{P}$ be a Knaster poset. Let $\mathbb{Q}$ be a stationary-set-preserving poset. Then is it the case that $\Vdash_{\mathbb{Q}}$"$\mathbb{P}$ has the ccc"?*

These questions might perhaps be related to the research program on separating various statements which lie between MA_{ω_1} and Suslin's Hypothesis [TV87, LT01, Woo10]. For example, one could classify how ccc a partial order is by the kind of posets that do not destroy its ccc-ness.

Another line of questions is this: the following axiom is known as $\mathcal{C}^2$ in the literature: "The product of any two ccc posets is ccc". This is known to follow from MA_{ω_1}. By Theorem 36, this axiom is equivalent to the statement "Every ccc poset remains ccc after forcing with a ccc poset". For any $\mathcal{P}$, a class of forcing posets which is contained in the class of ω_1-preserving posets, one could formulate the following statement, which we call $\mathcal{C}^{\mathcal{P}}$: "Every ccc poset remains ccc after forcing with any poset in $\mathcal{P}$".

Question 169. *Let $\mathcal{P}$ be a class of forcing posets which is contained in the class of ω_1-preserving posets, and which strictly contains the class of all ccc posets. Is $\mathcal{C}^{\mathcal{P}}$ consistent?*

One may also try to characterise other combinatorial obects of size ω_1 based on the kind of posets which affect their behaviour, for example towers. We note that such a study has already been done for gaps, see [Sch93] for details.

Another area of investigation is the following: we know that if T is special in the traditional sense, that is, if $T = \bigcup_{i\in\omega} A_i$, where each of the A_i is an antichain of T, then one cannot add a branch to T without collapsing ω_1. A reasonable question to ask is if the converse holds too. Namely,

Question 170. *Let T be an Aronszajn tree such that there is no ω_1-preserving poset $\mathbb{P}$ which adds a cofinal branch to T. Are there countably many antichains A_i for $i \in \omega$ such that $T = \bigcup_{i\in\omega} A_i$?*

All of the questions above can also be asked in the context ω_1-trees in general. We note that a generalisation of the notion of 'special' to ω_1-trees can be found in [Tod82].

Another interesting line of investigation related to the modal logic of ccc forcing is that of investigating whether specific statements are ccc-buttons or ccc-switches. The following two were at one stage contenders for being a topic of study in this thesis:

Question 171. *Is Suslin's Hypothesis a ccc-switch?*

Question 172. *Is MA_{ω_1} a ccc-switch?*

Note that by results of Shelah [She84], one direction of both these questions is true. Namely, in any model M of set theory, if we force with the Cohen poset to get $M[r]$, then in $M[r]$, there is a Suslin tree, and hence, Suslin's Hypothesis, and therefore MA_{ω_1} as well, does not hold.

We note that the naïve approach to restoring Martin's approach does not work above. In particular, Farah proved in [Far96] that forcing with a Suslin tree can never give a model of MA_{ω_1}. Larson improved this in [Lar99] by showing that forcing with an Aronszajn tree which does not collapse ω_1 can never give a model of MA_{ω_1}. Larson also mentions the following question, which he attributes to Woodin, which is similar in spirit, though not strictly related to ccc-forcing.:

Question 173. *($2^{\aleph_0} > \aleph_1 + \neg\mathsf{MA}_{\omega_1}$) Let $\mathbb{P}$ be a forcing poset such that the generic is a real. Can forcing with $\mathbb{P}$ give a model of MA_{ω_1}?*

We point out that in the case that $\neg\mathsf{MA}_{\omega_1}$ does not hold, the answer to this question is positive by results of Carlson and Laver [CL89], who showed that adding a Sacks real to a model of PFA gives a model where MA_{om} continues to hold.

Question 174. *($2^{\aleph_0} > \aleph_1 + \neg\mathsf{MA}_{\omega_1}$) Let $\mathbb{P}$ be a ccc poset of size less than the continuum. Can forcing with $\mathbb{P}$ give a model of MA_{ω_1}?*

Also, Theorem 158 raises the following question:

Question 175. *If they exist, are all homogenous Suslin trees isomorphic? If they exist, how many non-isomorphic homogenous Suslin trees must there be?*

Chapter 6

Collapsing $\aleph_2$

In this chapter, we give an exposition of Abraham's method [Abr83] for collapsing $\aleph_2$. Recall from our discussion in the introduction, that Hamkins and Löwe, in one of their proofs that the modal logic of forcing is exactly S4.2, assumed that for each natural number $n \leq 1$, the statements b_n="$\aleph_n^{\mathbf{L}}$ is not a cardinal" form a mutually independent collection of buttons. However, they did not indicate which forcing posets they would use to push these buttons.

The standard posets to collapse cardinals are the following:

Definition 176. Let λ be a regular cardinal, and let $\kappa > \lambda$ be another cardinal. The *Leṽy collapse* poset for collapsing κ to λ, $\mathrm{Lev}(\lambda, \kappa)$, is the following:

(i) A function p is in $\mathrm{Lev}(\lambda, \kappa)$ if $\mathrm{dom}(p) \subseteq \lambda$, $|\mathrm{dom}(p)| < \lambda$, and $\mathrm{ran}(p) \subseteq \kappa$;

(ii) If $p, q \in \mathrm{Lev}(\lambda, \kappa)$ then $p \leq q$ if $p \subseteq q$.

The following properties of this poset are easy to see:

Proposition 177. *Let λ be a regular cardinal, and let $\kappa > \lambda$ be another cardinal.*

(i) If G is generic for $\mathrm{Lev}(\lambda, \kappa)$, then $\cup G$ is a surjection from λ onto κ. Hence, forcing with $\mathrm{Lev}(\lambda, \kappa)$ collapses κ to λ.

(ii) $\mathrm{Lev}(\lambda, \kappa)$ is $< \lambda$-closed, and hence does not collapse any cardinal $\leq \lambda$;

(iii) $|\mathrm{Lev}(\lambda, \kappa)| = \kappa^{<\lambda}$, and hence, if $\kappa^{<\lambda} = \kappa$, then no cardinals larger than κ are collapsed.

(iv) Therefore, if $\kappa^{<\lambda} = \kappa$, then $\mathrm{Lev}(\lambda, \kappa)$ collapses κ, and no other cardinals.

It is now easy to see that b_1 is indeed a button and independent of all of the other statements, as the poset $\mathrm{Lev}(\aleph_0, \aleph_1)$ has size $\aleph_1^{<\aleph_0} = \aleph_1$.

However, as Rittberg pointed out in [Rit10], the statement b_2 need not be independent of b_3. In particular, if $2^{\aleph_0} > \aleph_2$, then $\mathrm{Lev}(\aleph_1, \aleph_2)$ collapses $2^{\aleph_0}$ to $\aleph_1$, and hence, collapses $\aleph_3$. We shall see a proof in Section 6.2.

However, this does not rule out the possibility that b_2 is indeed independent of each of the statements b_n for $n \neq 2$, or even that the proof by Hamkins and Löwe is not incorrect, but merely incomplete. Recall:

Question 1. *Let $n > 1$ be a natural number. M be a generic extension of* $\mathbf{L}$ *sucn that $M \vDash$"$\aleph_n^{\mathbf{L}}$ is a cardinal". Then is there a generic extension N of M such that $N \vDash$"$\aleph_n^{\mathbf{L}}$ is not a cardinal" and such that for all other natural numbers $m > 1$, if $M \vDash$"$\aleph_m^{\mathbf{L}}$ is a cardinal" then $N \vDash$"$\aleph_m^{\mathbf{L}}$ is a cardinal"?*

While researching this question, we found that Abraham had already thought of questions of a similar nature, and in particular, had given a proof that in any model of set theory, the second uncountable cardinal can always be collapsed without collapsing any other cardinals. In this chapter, we shall see a proof of this result. The organisation is as follows:

In Section 6.1 we shall see some basic results which will be used in Abraham's proof. In Section 6.2 we shall see a proof that if the continuum is larger than $\aleph_2$, then $\mathrm{Lev}(\aleph_1, \aleph_2)$ necessarily collapses the continuum, and hence $\aleph_3$ as well. In Section 6.3 we shall describe Abraham's idea. In order to prove that $\aleph_1$ is not collapsed, Abraham uses a covering theorem of Shelah's. We see this theorem in Section 6.4. In Section 6.5, we see the poset that Abraham uses, and why it collapses $\aleph_2$, but does not collapse any cardinals larger than $\aleph_2$. In Section 6.6 we see a proof that $\aleph_1$ is not collapsed by this poset, hence completing the proof. Abraham's theorem only answers the case $n = 2$ of Question 1, and in Section 6.7 we discuss attempts to generalise Abraham's construction and other related questions.

6.1 Basic Results about Cohen Reals and Elementary Submodels

In this section we prove some basic results about Cohen reals and elementary submodels that we shall use later in this chapter.

6.1.1 Cohen Reals

Recall that the poset Coh to add a *Cohen real* is the following:

(i) The carrier set of Coh is $\omega^{<\omega}$, the set of finite sequences of natural numbers;

(ii) If $p, q \in \mathrm{Coh}$, then $p \geq q$ if q is an initial segment of p, denoted $q \preccurlyeq p$.

Definition 178. The poset $\mathbb{C}$ is the finite support ω_1-product of Coh. For any $\gamma < \omega_1$, $\mathbb{C}_{\geq\gamma}$ is defined as the suborder of $\mathbb{C}$ which consists of functions whose support is a subset of $[\gamma, \omega_1)$, that is, $\mathbb{C}\restriction[\gamma, \omega_1)$. The poset $\mathbb{C}_{<\gamma}$ is defined as the suborder of $\mathbb{C}$ which consists of functions whose support is a subset of $[0, \gamma)$ that is, $\mathbb{C}\restriction[0, \gamma)$.

As Coh is Knaster, it follows by Proposition 29 that $\mathbb{C}$ is Knaster as well, and in particular ccc. Also, it is easy to see that for any $\gamma < \omega_1$, $\mathbb{C} \cong \mathbb{C}_{<\gamma} \times \mathbb{C}_{\geq\gamma}$. Also, as ω_1 is an *indecomposable ordinal* (that is, if $\alpha, \beta < \omega_1$ are such that the ordinal sum $\alpha + \beta = \omega_1$, then $\beta = \omega_1$), it is clear that $\mathbb{C} \cong \mathbb{C}_{\geq\gamma}$ for any $\gamma < \omega_1$.

Proposition 179. *Let M be a model of set theory, and let G be M-generic for $\mathbb{C}$, then $f = \bigcup G$ has the following properties:*

(i) $f : \omega_1 \to \omega^\omega$, that is, f is a function whose domain is ω_1, and for each $\alpha \in \omega_1$, $f(\alpha) \in \omega^\omega$.

(ii) For each $\gamma < \omega_1$, $f(\gamma)$ is a Cohen real over M.

Proof sketch. The results follow once we make the following observations:

(i) For any $\alpha < \omega_1$, and $n \in \omega$, the set

$$D_n^\alpha = \{p \in \mathbb{C} \mid \alpha \in \operatorname{supp}(p) \text{ and } |p(\alpha)| \geq n\}$$

is a dense open subset of $\mathbb{C}$.

(ii) For any $\gamma < \omega_1$, for any S an open dense set of Coh in M, the set

$$D_\gamma^S = \{p \in \mathbb{P} \mid \gamma \in \operatorname{supp}(p) \text{ and } p(\gamma) \in S\}$$

is a dense open subset of $\mathbb{C}$.

$\square$

Corollary 180. *Let M be a model of set theory and G be M-generic for $\mathbb{C}$. Let $f = \bigcup G$. Then for any $\gamma \leq \eta < \omega_1$, $f(\eta)$ is a Cohen real over $M[G_{<\gamma}]$.* $\square$

Proposition 181. *Let M be a model of set theory and G be M-generic for $\mathbb{C}$. Let $X \subseteq M$ be a set in $M[G]$ such that*

$$M[G] \vDash \text{“}X \textit{ is countable”}.$$

Then there is a $Y \in M$ such that

$$M \vDash \text{“}Y \textit{ is countable”},$$

and $M[G] \vDash X \subseteq Y$.

Proof sketch. If $\dot{X}$ is a name for X, and $p \in \mathbb{C}$ a condition and $\dot{F}$ a $\mathbb{C}$-name such that

$$p \Vdash \text{“}\dot{F} : \aleph_0 \to X \text{ is a bijection”},$$

then for each $n \in \aleph_0$, the set

$$X_n \triangleq \{y \in V \mid \exists q \geq p[q \Vdash \dot{F}(n) = \check{y}]\}$$

is countable as $\mathbb{C}$ has the ccc, and hence $Y = \bigcup X_n$ is the countable set in M that we wanted. $\square$

Corollary 182. *Let M be a model of set theory and G be M-generic for $\mathbb{C}$. Let $X \in M[G]$ be such that*

$$M[G] \vDash \text{“}X \textit{ is countable”}.$$

Then there is $\gamma < \omega_1$ such that $X \in M[G_{<\gamma}]$, and

$$M[G_{<\gamma}] \vDash \text{“}X \textit{ is countable”}.$$

Proof sketch. By the previous proposition, there is, in M, a countable set Y such that $M[G] \vDash X \subseteq Y$. If we then pick a nice name $\dot{X}$ for X as a subset of Y, then

$$\dot{X} = \bigcup_{y \in Y} \{\check{y}\} \times A_y,$$

where each A_y is an antichain of $\mathbb{C}$. Since $\mathbb{C}$ has the ccc, there is a $\gamma < \omega_1$ such that for each $y \in Y$, for each $p \in A_y$, $\operatorname{supp}(p) \subseteq [0, \gamma)$. The result follows. $\square$

Proposition 183. *Let G be M-generic for $\mathbb{C}$. Then $M[G] \vDash 2^{\aleph_0} = (2^{\aleph_0})^V$.*

6.1.2 Elementary submodels

We end this section with a discussion of elementary submodels of large fragments of set theory. When discussing Abraham's proof, we shall talk about an elementary substructures $M \prec (V_\lambda, \in, <, \ldots)$ "for some large ordinal λ". In practice, we can take λ to be some ordinal larger than the cardinality of the set $\mathcal{P}(\mathcal{P}(\gamma))$, where γ is any ordinal which is larger than the ranks of all the objects we talk about in our proof. The $<$ is some unspecified well-ordering of V_λ which allows us to do inductive constructions in M. The extra unspecified parameters are those that we use in the proof. The only thing we will require of λ will be that V_λ is a model of some large fragment of ZFC, and that it contains enough information about all of the objects that we are interested in. Such a λ can be found by the Montague-Lévy Reflection Principle. In this case, if $A \in M$ and B is a set which is definable from some formula $\varphi(x)$ which only involves parameters from A, then we can assume that $B \in M$ by picking λ large enough as to be a model of $\exists y[\forall x[x \in y \leftrightarrow \varphi(x)]$. In particular, any object which is definable without any parameters can be assumed to be in M. For example, all the natural numbers, ω, ω_1, ω_2 etc. It is however important to note that M need not be transitive. For example, if M is countable, then clearly even though $\omega_1 \in M$, $\omega_1 \not\subset M$.

On the other hand, if S is some countable set and $S \in M$, then as $V_\lambda \models$"S is countable", $M \models$"S is countable". In particular, there is a $f \in M$ which M sees as witnessing this countability. That is, $M \models$"$f : \omega \to S$ is surjective". In this case, since $\omega \subseteq M$, and since $V_\lambda \models \forall n \in \omega \exists y[f(n) = y]$, and since the formulas in the language of set theory which express the sentences "f is a function", "n is in the domain of f", "$f(n) = y$" are absolute, it follows that firstly $M \models \forall n \in \omega \exists y[f(n) = y]$, and therefore, for each natural number n, there is a $y \in M$ such that $M \models f(n) = y$. But in this case, by appealing to the absoluteness, it follows that $V_\lambda \models f(n) = y$. The following fact captures the argument that we have given:

Proposition 184. *Let $M \prec V_\lambda$ for λ large enough. If $A, B \in M$ and $f : A \to B$ is a function, then there is a $g : A \to B$ such that $g \in M$. Further, if f is injective, surjective, bijective etc., then a g can be found with the same property. If $A \subseteq M$, then* $\mathrm{range}(g) \subseteq M$. *In particular, if $S \in M$ has a size which is some definable cardinal λ (for example $\aleph_n$ for some natural number n), then S has the same size in M. If in addition $\lambda \subseteq M$, then $S \subset M$.*

Hence, it is not always true that if $S \in M$ then $S \subseteq M$. The opposite direction always holds: for any $S \subseteq M$, $S \in M$. A useful corollary of this fact is the following:

Corollary 185. *Let $M \prec V_\lambda$ for λ large enough be a countable elementary substructure. Then $M \cap \aleph_1 \subset M$.*

A detailed exposition of countable elementary submodels can be found in [JW97].

6.2 The Lévy Collapse

Before we see Abraham's technique, we shall first see why the standard technique does not work. Recall that our goal is to show that in any generic extension M of $\mathbf{L}$ where $\aleph_2^{\mathbf{L}}$ is a cardinal, there is a further generic extension N in which $\aleph_2^{\mathbf{L}}$ is collapsed, but no other $\aleph_m^{\mathbf{L}}$. We show that in some generic extensions of $\mathbf{L}$, the standard poset for collapsing $\aleph_2$ does not work.

This standard forcing poset for collapsing $\aleph_2$ is the poset $\mathrm{Lev}(\aleph_1, \aleph_2)$ which consists of functions p such that:

(i) $\mathrm{dom}(p)$ is a countable subset of $\aleph_1$;

(ii) $\mathrm{ran}(p)$ is a subset of $\aleph_2$;

(iii) $p \geq q$ if $p \supseteq q$.

If G is V-generic for $\mathrm{Lev}(\aleph_1, \aleph_2)$, then it is easily seen that $f \triangleq \bigcup G$ is a function such that $\mathrm{ran}(f) = \aleph_1$, and $\mathrm{dom}(f) = \aleph_2$. Hence,

$$V[G] \vDash \text{“}f : (\aleph_1)^V \to (\aleph_2)^V \text{ is surjective”},$$

and hence, by using the axiom of choice, we can obtain a $h \in V[G]$ such that

$$V[G] \vDash \text{“}h : (\aleph_2)^V \to (\aleph_1)^V \text{ is injective”},$$

and since $V[G] \vDash (\aleph_1)^V \subset (\aleph_2)^V$, it follows that $V[G] \vDash |\aleph_1| = |\aleph_2|$.

Also, it is easily seen that $\mathbb{P}$ is σ-closed, and so by Proposition 15, we see that for any $h \in V[G]$ such that

$$V[G] \vDash h : \aleph_0 \to \mathrm{ORD},$$

we have $h \in V$. It follows that $(\aleph_1)^V$ cannot have been collapsed when going from V to $V[G]$, and hence, $V[G] \vDash (\aleph_1)^V = \aleph_1$.

Hence, forcing with $\mathrm{Lev}(\aleph_1, \aleph_2)$ collapses $\aleph_2$, and does not collapse $\aleph_1$. Further, if we assume $2^{\aleph_0} \leq \aleph_2$, then

$$|\mathrm{Lev}(\aleph_1, \aleph_2)| = \aleph_2^{<\aleph_1} = \aleph_1 \times \aleph_2^{\aleph_0} = \aleph_1 \times \aleph_2 = \aleph_2.$$

Therefore, in this case, $\mathrm{Lev}(\aleph_1, \aleph_2)$ trivially has the $\aleph_3$-cc , and so by Theorem 17, forcing with $\mathrm{Lev}(\aleph_1, \aleph_2)$ cannot collapse any cardinals from $\aleph_3$ onwards.

However, if $2^{\aleph_0} > \aleph_2$, then

$$|\mathrm{Lev}(\aleph_1, \aleph_2)| = \aleph_2^{<\aleph_1} = \aleph_1 \times \aleph_2^{\aleph_0} = \aleph_2^{\aleph_0} \leq (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0} \leq \aleph_2^{\aleph_0}.$$

Hence,

$$|\mathrm{Lev}(\aleph_1, \aleph_2)| = \aleph_2^{\aleph_0} = 2^{\aleph_0} > \aleph_2.$$

Before we show that this implies that $\mathrm{Lev}(\aleph_1, \aleph_2)$ collapses cardinals other than $\aleph_2$, we make the following observation: since we used no properties of $\aleph_2$ in the definition of $\mathrm{Lev}(\aleph_1, \aleph_2)$, and in the arguments that followed, we could also have defined $\mathrm{Lev}(\aleph_1, S)$ for any arbitrary set S, and if $|S| \geq \aleph_1$, then the generic G would have given us a surjection from $\aleph_1$ onto S, and by appealing to the axiom of choice, we could have obtained an injection from S into $\aleph_1$. Then, by the Bernstein-Cantor-Schröder Theorem, we would have gotten a bijection between $\aleph_1$ and S in $V[G]$. Hence, to show that some cardinal other than $\aleph_2$ is collapsed by this forcing, it suffices to prove that there is a set S whose cardinality is larger than $\aleph_2$ whose cardinality is changed by forcing with this poset, and this itself can be shown if we can find a generic for the poset $\mathrm{Lev}(\aleph_1, S)$ in $V[G]$.

Now, we use this strategy to show that if $2^{\aleph_0} > \aleph_2$, then some cardinal other than $\aleph_2$ is collapsed by $\mathrm{Lev}(\aleph_1, \aleph_2)$:

Proposition 186. $\mathrm{Lev}(\aleph_1, \aleph_2^{\aleph_0})$ *can be embedded as a dense subset of* $\mathrm{Lev}(\aleph_1, \aleph_2)$. *Hence, if* $\aleph_2^{\aleph_0} > \aleph_2$, *then forcing with* $\mathrm{Lev}(\aleph_1, \aleph_2)$ *collapses* $\aleph_3$ *as well.*

Proof. Note that we can see $\aleph_1$ as being composed of $\aleph_1$-many blocks of size $\aleph_0$. That is, fix a partition $\aleph_1 = \bigcup_{\alpha\in\omega_1} P_\alpha$. Then, by σ-closedness of $\text{Lev}(\aleph_1, \aleph_2)$, for any $p \in \text{Lev}(\aleph_1, \aleph_2)$, we can find $q \geq p$ such that if $\gamma \in \text{dom}(q)$ and $\alpha \in \omega_1$ is such that $\gamma \in P_\alpha$, then for all $P_\alpha \subseteq \text{dom}(q)$. That is, the domain of q is the union of countably many blocks from this partition. We call such a q *saturated*. Hence, the set of all saturated elements of $\text{Lev}(\aleph_1, \aleph_2)$ is a dense subset of $\text{Lev}(\aleph_1, \aleph_2)$.

But clearly, for each saturated element q we can get a function $\bar{q}$ whose domain is a countable subset of $\aleph_1$ and whose range is a subset of $\aleph_2^{\aleph_0}$ as follows:

(i) $\text{dom}(\bar{q}) = \{\alpha \mid \text{there is a } \gamma \in \text{dom}(q) \text{ such that } \gamma \in P_\alpha\}$.

(ii) For $\alpha \in \text{dom}(\bar{q})$, $q'(\alpha) = q \restriction P_\alpha$.

Since q is saturated, for each $\alpha \in \text{dom}(\bar{q})$, $\bar{q}(\alpha) : \aleph_0 \to \aleph_2$. Hence, this $^-$ operation is well-defined, and for each saturated q, it is the case that $\bar{q} \in \text{Lev}(\aleph_1, \aleph_2^{\aleph_0})$ since $\text{dom}(q)$ is countable, and hence $\text{dom}(\bar{q})$ is countable as well.

Also, if $p \geq q$ are both saturated, then for any $\alpha \in \text{dom}(\bar{q})$, $\alpha \in \text{dom}(\bar{p})$ as well, and further, since q is saturated, for any $\alpha \in \text{dom}(\bar{q})$, we have that $\bar{q}(\alpha) : \aleph_0 \to \aleph_2$, and then because $p \supseteq q$, it follows that $\bar{q}(\alpha) = \bar{p}(\alpha)$. Hence, $\bar{p} \supseteq \bar{q}$, and hence $\bar{p} \geq \bar{q}$ in $\text{Lev}(\aleph_1, \aleph_2^{\aleph_0})$.

To summarise, the poset of saturated elements is a dense suborder of $\text{Lev}(\aleph_1, \aleph_2)$, and further, this poset can be embedded into $\text{Lev}(\aleph_1, \aleph_2^{\aleph_0},)$ using the $^-$ operation. But also, it is easy to see that for any $r \in \text{Lev}(\aleph_1, \aleph_2^{\aleph_0})$, we can find an $s \in \text{Lev}(\aleph_1, \aleph_2,)$ such that $\bar{s} = r$. Hence, the poset of saturated elements of $\text{Lev}(\aleph_1, \aleph_2)$ is isomorphic to $\text{Lev}(\aleph_1, \aleph_2^{\aleph_0})$, and since it is a dense suborder of $\text{Lev}(\aleph_1, \aleph_2)$, it follows that adding a generic for $\text{Lev}(\aleph_1, \aleph_2)$ adds a generic for $\text{Lev}(\aleph_1, \aleph_2^{\aleph_0})$. Therefore, forcing with $\text{Lev}(\aleph_1, \aleph_2)$ adds a bijection between $\aleph_1$ and $\aleph_2^{\aleph_0}$. However, if the latter is larger than $\aleph_2$, then clearly $\aleph_3$ is collapsed in this extension.

□

6.3 Abraham's Idea

We have seen how $\text{Lev}(\aleph_1, \aleph_2^{\aleph_0})$ can be densely embedded into $\text{Lev}(\aleph_1, \aleph_2)$. Hence, when $\aleph_2^{\aleph_0} > \aleph_2$, $\text{Lev}(\aleph_1, \aleph_2)$ collapses cardinals other than $\aleph_2$. We now start with explaining Abraham's method for collapsing the second uncountable cardinal in ay model of set theory.

Now, since we want to show that in *any* model we can collapse $\aleph_2$, we can clearly not assume that $\aleph_2^{\aleph_0} \leq \aleph_2$. On the other hand, we know that in $\mathbf{L}$, the GCH is true, and in particular, $\aleph_2^{\aleph_0} = \aleph_2$. However, we are in an arbitrary model of set theory, so it is possible that $(\aleph_2^{\aleph_0})^{\mathbf{L}} \neq \aleph_2^{\aleph_0}$. We can however get around this trick by picking a model W which is intermediate between $\mathbf{L}$ and V and which does compute $\aleph_2$ correctly. The next fact lets us do this. We do not prove it as its proof would involve a lengthy detour into the theory of constructibility. Its proof can be found in Chapter 13 of [Jec03]:

Proposition 187. *Let V be a model of set theory. Let $\kappa = \aleph_2^V$. Then there is a $A \subseteq \aleph_2$ in V such that $\aleph_2^{\mathbf{L}[A]} = \aleph_2^V$. Further, $\mathbf{L}[A] \vDash 2^{\aleph_0} \leq \aleph_2$.*

Hence, in $\mathbf{L}[A]$, $|\text{Lev}(\aleph_1, \aleph_2)| = \aleph_2$. Also, an $\mathbf{L}[A]$-generic for $(\text{Lev}(\aleph_1, \aleph_2))^{\mathbf{L}[A]}$ adds a bijection between $\aleph_2^{\mathbf{L}[A]}$ and $\aleph_1^{\mathbf{L}[A]}$. However, $\aleph_2^{\mathbf{L}[A]} = \aleph_2$, and hence $\aleph_1^{\mathbf{L}[A]} = \aleph_1$. Also, since $\mathbf{L}[A] \subseteq V$, any V-generic for $(\text{Lev}(\aleph_1, \aleph_2))^{\mathbf{L}[A]}$ is also an $\mathbf{L}[A]$-generic, and hence, if we force in V with $(\text{Lev}(\aleph_1, \aleph_2))^{\mathbf{L}[A]}$, then we add a bijection between $\aleph_1^V$ and $\aleph_2^V$. In particular, forcing with this poset collapses $\aleph_2^V$, and since $V \vDash |(\text{Lev}(\aleph_1, \aleph_2))^{\mathbf{L}[A]}| = \aleph_2$, no cardinals larger than $\aleph_2$ can be collapsed by this poset.

This does not mean that we are done however. In particular, in $\mathbf{L}[A]$, $(\text{Lev}(\aleph_1, \aleph_2))^{\mathbf{L}[A]}$ is σ-closed, and hence forcing with $(\text{Lev}(\aleph_1, \aleph_2))^{\mathbf{L}[A]}$ *over* $\mathbf{L}[A]$ does not collapse $\aleph_1$. But V might have many more countable sequences of ordinals. Hence, it is possible that $(\text{Lev}(\aleph_1, \aleph_2))^{\mathbf{L}[A]}$ is not σ-closed in V, and maybe even not σ-distributive. Therefore, there is no way to guarantee that by forcing over V with the poset $(\text{Lev}(\aleph_1, \aleph_2))^{\mathbf{L}[A]}$, we do not collapse $\aleph_1$.

Abraham gets around this obstacle by first adding $\aleph_1$-many Cohen reals by finite support to V to obtain a generic extension $V[G]$, and then applying the above idea to use the poset $\text{Lev}(\aleph_1, \aleph_2)$ as defined in some suitable inner model to collapse $\aleph_2$. The preparatory forcing allows us to show that this poset is σ-distributive, by allowing us to approximate countable elementary substructures of $V_\lambda[G]$ for some large λ in some model intermediate between V and $V[G]$.

The structure of the rest of this chapter is as follows: in Section 6.4 we prove a covering theorem which will let us do these approximations. In Section 6.5, we describe Abraham's construction in detail and prove some basic facts about it. In Section 6.6 we prove that this construction does not collapse $\aleph_1$.

6.4 Facts about $\mathcal{P}_\kappa(\lambda)$

Definition 188. Let κ be an uncountable regular cardinal.

(i) For $\lambda \geq \kappa$, $\mathcal{P}_\kappa(\lambda) \triangleq \{S \subseteq \lambda \mid |X| < \kappa\}$.

(ii) A set $S \subseteq \mathcal{P}_\kappa(\lambda)$ is *unbounded* if for each $X \in \mathcal{P}_\kappa(\lambda)$, there is a $X' \in S$ such that $X \subseteq X'$.

(iii) S is *closed* if for $\gamma < \kappa$, for any sequence of elements $\langle X_\alpha \mid \alpha < \gamma\rangle$ of S, $\bigcup_{\alpha\in\gamma} X_\alpha \in S$.

(iv) We say that S is a *club subset of* $\mathcal{P}_\kappa(\lambda)$ if it is both closed and unbounded in $\mathcal{P}_\kappa(\lambda)$.

(v) If $T \subseteq \mathcal{P}_\kappa(\lambda)$ is such that its intersection with all club subsets of $\mathcal{P}_\kappa(\lambda)$ is non-empty, then we say that T is a *stationary* subsets of $\mathcal{P}_\kappa(\lambda)$.

(vi) For $f : [\lambda]^{<\omega} \to \mathcal{P}_\kappa(\lambda)$, we say that $X \subseteq \lambda$ is *closed under* f if $\bigcup f''[X]^{<\omega} \subseteq X$.

(vii) Given a function $f : [\lambda]^{<\omega} \to \mathcal{P}_\kappa(\lambda)$, we define the set

$$C_f \triangleq \{X \subseteq \mathcal{P}_\kappa(\lambda) \mid \text{ X is closed under } f\}.$$

The proofs of the following basic facts can be found in Chapter 8 of [Jec03]:

Proposition 189. *Let κ be an uncountable regular cardinal, and let $\lambda \geq \kappa$ be another cardinal.*

(i) For any $f : [\lambda]^{<\omega} \to \mathcal{P}_\kappa(\lambda)$, C_f is a club.

(ii) If S is a club in $\mathcal{P}_\kappa(\lambda)$, then for some $f : [\lambda]^{<\omega} \to \mathcal{P}_\kappa(\lambda)$, $C_f \subseteq S$.

(iii) If γ is a cardinal such that $\kappa \leq \gamma \leq \kappa$, and if S is a club in $\mathcal{P}_\kappa(\lambda)$, then the set

$$S{\restriction}\gamma \triangleq \{X \cap \gamma \mid X \in S\}$$

contains a club subset of $\mathcal{P}_\kappa(\gamma)$.

Let $W \subseteq W'$ be models of set thoery with the same ordinals and such that W is transitive in W'. For any cardinal κ, we denote by κ^{+n} the n-th successor cardinal of κ. We can naturally relativise this to particular models of set theory. For example, if κ is a cardinal in W, then $(\kappa^{+n})^W$ is the nth successor cardinal of κ in W.

The following is a theorem which Abraham attributes to Shelah and mentions has been generalised by Shelah in [She82, Chapter 13].

Theorem 190. *Let κ be a regular cardinal in W'. If $(\kappa^{+n})^W = (\kappa^{+n})^{W'}$ for some $n < \omega$, then in W':*

(i) $\mathcal{P}_\kappa(\kappa^{+n}) \cap W$ is a stationary subset of $\mathcal{P}_\kappa(\kappa^{+n})$.

(ii) For any cardinal λ and $X \in \mathcal{P}_\kappa(W'_\lambda)$, there is an elementary substructure $M \prec (W'_\lambda)$ of size less than κ such that $X \subseteq M$ and $M \cap \kappa^{+n} \in W$.

(iii) If $\kappa = \mu^+$ and $\mu^{<\mu} = \mu$ in W', then for any cardinal λ and any $X \in \mathcal{P}_\kappa(W'_\lambda)$, there is an elementary substructure $M \prec W'_\lambda$ such that $X \subseteq M$, $M \cap \kappa^{+n} \in W$, and further, for every subset Y of $M \cap \kappa^{+n}$ of size less than μ in W, $Y \in M$.

6.5 Abraham's Construction

For the rest of this chapter we fix the model of set theory V that we are in. We want to show that there is a poset $\mathbb{Q}$ in V such that forcing over V with $\mathbb{Q}$ collapses exactly $\aleph_2$. Clearly, if $2^{\aleph_0} \leq \aleph_2$, then we could just use $\mathrm{Lev}(\aleph_1, \aleph_0)$. Therefore, we assume that $2^{\aleph_0} > \aleph_2$.

Using Proposition 187, fix a subset $A \subseteq \aleph_2$ such that $\aleph_2 = \aleph_2^{\mathbf{L}[A]}$. Note that this also implies that $\aleph_1^{\mathbf{L}[A]} = \aleph_1$. Let Coh and $\mathbb{C}$ be respectively the Cohen poset and the finite support ω_1-product of the Cohen poset as defined in $\mathbf{L}[A]$. We note that though it does not make a difference whether these posets are defined in $\mathbf{L}[A]$ or in V (because *any* two models compute Coh in the same way because the definition involves only finite strings of natural numbers, and because $\mathbf{L}[A]$ computes ω_1 correctly, that is, $\omega_1^{\mathbf{L}[A]} = \omega_1^V$, so both these models compute $\mathbb{C}$ correctly), we shall use the fact that $\mathbb{C} \in \mathbf{L}[A]$, and in particular, that there is a formula defining $\mathbb{C}$ using A and $\omega_1^{\mathbf{L}} = \omega_1$ as parameters. To be precise:

Fact 191. *Let W be a model of set theory such that $V \subseteq W$, and let λ be some large ordinal such that $M \prec W_\lambda$ is an elementary substructure, and $A \in M$. In W, let $\bar{M}$ be the transitive collapse of M, and $\pi : M \to \bar{M}$ the collapsing map. Then $\mathbb{C} \in M$, and $\pi(\mathbb{C}) = \mathbb{C} \restriction \pi(\aleph_1)$. Also, if W contains a V-generic subset G for $\mathbb{C}$, and if $G \in M$, then $\pi(G) = G \cap \pi(\mathbb{C})$.*

Now, let G be V-generic for $\mathbb{C}$. Then G is $\mathbf{L}[A]$-generic for $\mathbb{C}$ as well. As $\mathbf{L}[A] \vDash |\mathbb{C}| \leq 2^{\aleph_0}$, and since $\mathbf{L}[A] \vDash 2^{\aleph_0} \leq \aleph_2$, it follows that $\mathbf{L}[A] \vDash |\mathbb{C}| \leq \aleph_2$. Hence, by Proposition 183, it follows that $\mathbf{L}[A, G] \vDash 2^{\aleph_0} \leq \aleph_2$.

Now, let $\mathbb{P}$ be the Lévy collapse poseter *as defined in* $\mathbf{L}[A, G]$. That is,

$$\mathbb{P} \stackrel{\Delta}{=} (\mathrm{Lev}(\aleph_1, \aleph_2))^{\mathbf{L}[A,G]}.$$

Since $\mathbf{L}[A, G] \vDash 2^{\aleph_0} \leq \aleph_2$, it follows that

$$\mathbf{L}[A, G] \vDash |(\mathrm{Lev}(\aleph_1, \aleph_2))^{\mathbf{L}[A,G]}| = \aleph_2,$$

and since $\aleph_2^{\mathbf{L}[A,G]} = (\aleph_2)^V$, it follows that $V \models |\mathbb{P}| = \aleph_2$. Hence, by forcing with $\mathbb{P}$, we *cannot collapse cardinals larger than* $\aleph_2$.

On the other hand, $\mathbb{P}$ is σ-closed in $\mathbf{L}[A,G]$, but $V[G]$ clearly contains more countable sets of ordinals (since $\mathbf{L}[A,G] \models 2^{\aleph_0} \leq \aleph_2$), so $\mathbb{P}$ need not be σ-closed in $V[G]$. However, we can still show that $\mathbb{P}$ is σ-distributive in $V[G]$, and therefore, $\aleph_1$ is not collapsed. We do so in the next section.

6.6 $\aleph_1$ is not collapsed

This entire section is devoted to a proof of the next lemma.

Lemma 192. *$\mathbb{P}$ is σ-distributive in $V[G]$. That is, forcing with $\mathbb{P}$ does not add any new countable subsets of $V[G]$.*

Proof. Let $\dot{F} \in V[G]$ be a $\mathbb{P}$-name and $p_0 \in \mathbb{P}$ a condition such that $p_0 \Vdash \dot{F} : \aleph_0 \to \mathrm{ORD}$. We would like to find a $p \geq p_0$ deciding all the values of $\dot{F}$. Now, since $\mathbb{P}$ has size $\aleph_2$ in $V[G]$, let $E = \langle p_\alpha \mid \alpha < \aleph_2 \rangle$ be an enumeration of $\mathbb{P}$ in $\mathbf{L}[A,G]$. Now, using part (ii) of Theorem 190 for the case of $W = \mathbf{L}[A,G]$ and $W' = V[G]$, $n = 1$ and $\kappa = \aleph_1$, obtain a countable model $M \prec V_\lambda[G]$ such that:

(i) λ is large enough so that $V_\lambda[G] = (V[G])_\lambda$ (this is possible by Proposition 13), and F, $\mathbb{P}$, E, A, p_0, $\aleph_1$, $\aleph_2$, $G \in V_\lambda[G]$.

(ii) $M \cap \aleph_2 \in \mathbf{L}[A,G]$.

Let $\bar{M}$ be the transitive collapse of M, and $\pi : M \to \bar{M}$ the collapsing function. We make the following observations:

(i) As $M \cap \aleph_2$ is a set of ordinals, $M \cap \aleph_2$ can be collapsed in exactly one way. Further, as $M \cap \aleph_2 \in \mathbf{L}[A,G]$, it follows that $\pi(M \cap \aleph_2) \in \mathbf{L}[A,G]$.

(ii) By a similar argument, $A \subseteq \aleph_2$ is also a set of ordinals such that $A \in \mathbf{L}[A,G]$, and hence $\pi(A) \in \mathbf{L}[A,G]$.

(iii) $\mathbb{C}$ was defined in $\mathbf{L}[A]$, and hence is definable by a formula whose parameters consist of A and $\omega_1^{\mathbf{L}[A]}$. Now, since $\mathbf{L}[A]$ and V compute $\aleph_2$ correctly, they also compute $\aleph_1$ correctly, and in particular $\omega_1^V = \omega_1^{\mathbf{L}[A]}$. Further, $V[G]$ is obtained from V by forcing with a poset which does not collapse any cardinals, and hence,

$$\omega_1 = \omega_1^{V[G]} = \omega_1^V = \omega_1^{\mathbf{L}[A]}.$$

It follows then that the same formula which defines $\mathbb{C}$ in $\mathbf{L}[A]$ from A and $\omega_1^{\mathbf{L}[A]}$ can be used to define $\pi(\mathbb{C})$ from $\pi(A)$ and $\pi(\omega_1) = \pi(\omega_1^{\mathbf{L}[A]})$. Therefore, $\pi(\mathbb{C}) \in \mathbf{L}[A,G]$ as $\pi(A)$ and $\pi(\aleph_1)$ are in $\mathbf{L}[A,G]$, but further, by Fact 191, $\pi(\mathbb{C}) = \mathbb{C} \restriction \pi(\aleph_1)$, and $\pi(G) = G \cap \pi(\mathbb{C})$.

(iv) $\mathbb{P}$ was defined in $\mathbf{L}[A,G]$ using $\aleph_0, \aleph_1, \aleph_2$ as parameters. However, $\mathbf{L}[A]$, $\mathbf{L}[A,G]$, V, and $V[G]$ all compute these parameters in the same way. Hence, we can define $\pi(\mathbb{P})$ from the parameters $\pi(A), \pi(\aleph_0), \pi(\aleph_1)$ and $\pi(\aleph_2)$. Since all of these parameters are in $\mathbf{L}[A,G]$ (because they are ordinals), it follows that $\pi(\mathbb{P}) \in \mathbf{L}[A,G]$.

(v) Let $\pi(E) = \langle p^*_\alpha \mid \alpha < \pi(\aleph_2)\rangle$. Then this is an enumeration of $\pi(\mathbb{P})$ in $\bar{M}$. Further, since E was an enumeration of $\mathbb{P}$ of length $\aleph_2$ in $\mathbf{L}[A, G]$, and $\pi(A), \pi(\aleph_2), \pi(G)$ are in $\mathbf{L}[A, G]$, it follows that $\pi(E) \in \mathbf{L}[A, G]$.

(vi) Since M is countable, $\bar{M}$ is countable as well, and hence, $\pi(\aleph_2)$ is countable. Hence, we can fix a bijection $h : \aleph_0 \to \pi(\aleph_2)$ in $\mathbf{L}[A, G]$.

(vii) So far, we have shown that the image under π of each of the elements A, $M \cap \aleph_2$, $\mathbb{C}$, G, $\mathbb{P}$, E, $\aleph_0$, $\aleph_1$, $\aleph_2$ is in $\mathbf{L}[A, G]$. Also, G is V-generic for $\mathbb{C}$, and hence, $\mathbf{L}[A]$-generic as well. Therefore, we have shown that a finite set (the set of transitive collapses of each of these elements) is contained in the generic extension of $\mathbf{L}[A]$ by $\mathbb{C}$, and now we can appeal to Corollary 182 over the model $\mathbf{L}[A]$ to give a $\gamma' < \omega_1$ such that this finite set is already in $\mathbf{L}[A, G_{<\gamma'}]$.

(viii) Now, since $\bar{M}$ is countable, we can appeal to Corollary 182 over the model V to give a $\gamma < \omega_1$ such that $\gamma' < \gamma$ such $\bar{M} \in V[G_{<\gamma}]$. Note that this also implies that the finite set from the previous item is already in $\mathbf{L}[A, G_{<\gamma}]$.

(ix) Also, by Corollary 180, if $g = \bigcup\{c \in \mathbb{C} \mid c \in G\}$, then $g : \omega_1 \to \omega^\omega$ such that for each $\alpha < \beta < \omega_1$, $g(\beta)$ is Cohen over $V[G_{<\alpha}]$. Then, if we let $d = g(\gamma)$, then d is Cohen over $V[G_{<\gamma}]$.

Now, we inductively define a sequence $\langle p'_i \mid i \in \omega\rangle$ of elements of $\pi(\mathbb{P})$ in the following way: $p'_0 = \pi(p_0)$, for each natural number n, $p'_n \leq p'_{n+1}$ so that

$$\begin{aligned} p'_{n+1} &= p^*_{h(d(n))} \text{ if } p^*_{h(d(n))} \geq p'_n, \\ &= p'_n \qquad \text{otherwise.} \end{aligned}$$

Since this is an inductive process such that all of the parameters d,h, $\pi(\mathbb{P})$, $\pi(p_0)$,$\pi(E)$ are in $\mathbf{L}[A, G]$, it follows that $\langle p'_i \mid i \in \omega\rangle \in \mathbf{L}[A, G]$. Hence, $\langle \pi^{-1}(p'_i) \mid i \in \omega\rangle \in \mathbf{L}[A, G]$ is an increasing sequence in $\mathbb{P}$. Using the σ-closedness of $\mathbb{P}$ in $\mathbf{L}$, let $p = \cup_{i\in\omega}\pi^{-1}(p'_i) \in \mathbb{P}$.

We now claim that for each $n < \omega$, there is an ordinal α such that $p \Vdash \dot{F}(n) = \alpha$. That is, p decides all values of $\dot{F}$.

In order to prove this claim, we prove that the sequence $\langle p'_i \mid i \in \omega\rangle \in \mathbf{L}[A, G]$ is $\bar{M}$-generic over $\pi(\mathbb{P})$. The reason why this would suffice is the following:

For any $n < \omega$, and any condition $p' \in \pi(\mathbb{P})$ such that $p' \geq \pi(p_0)$, since $p_0 \Vdash \dot{F} : \aleph_0 \to \mathrm{ORD}$, by the definability of forcing, $\pi(p_0) \Vdash \pi(\dot{F}) : \aleph_0 \to \mathrm{ORD}$. Hence, there is a $p'' \geq p'$ and an ordinal α such that $p'' \Vdash \pi(\dot{F})(n) = \alpha$. That is, the set

$$D_n \triangleq \{p' \in \pi(\mathbb{P}) \mid \text{ for some ordinal } \alpha, p' \Vdash \pi(\dot{F})(n) = \alpha\},$$

defined in $\bar{M}$, is a dense open subset of $\pi(\mathbb{P})$ below $\pi(p_0)$ in $\bar{M}$. It follows then that there is some $k < \omega$ such that $p'_k \in D_n$, and thus, $(p'_k \Vdash \pi(\dot{F})(n) = \alpha)^{\bar{M}}$ for some ordinal α. Thus, by the definability of forcing, $(\pi^{-1}(p'_k) \Vdash \dot{F}(n) = \pi^{-1}(\alpha))^M$, and then, because M is an elementary substructure of $V_\lambda[G]$ for some large λ, it follows that $\pi^{-1}(p'_k) \Vdash \dot{F}(n) = \pi^{-1}(\alpha)$. Therefore, p decides the value of $\dot{F}(n)$.

Hence, all we need to do now is to show that the sequence $\langle p'_i \mid i \in \omega\rangle \in \mathbf{L}[A, G]$ is $\bar{M}$-generic over $\pi(\mathbb{P})$. In order to do this, we use the fact that d is Cohen over $V[G_{<\alpha}]$, and that $\bar{M} \in V[G_{<\alpha}]$, and so are the previously mentioned finite set of parameters. Let $\mathbb{C}(\gamma)$ denote the poset $\mathbb{C}\restriction\{\gamma\}$. We note that $\mathbb{C}(\gamma)$ is the same as the Cohen poset Coh. The reason for our notation is that we want to show that for each dense open subset $D \in \bar{M}$ of $\pi(\mathbb{P})$, the following is a dense open subset of $\mathbb{C}(\gamma)$ in $V[G_{<\gamma}]$:

$$\{c \in \mathbb{C}\gamma \mid c \Vdash \exists k < \omega[p'_k \in D]\}.$$

Towards this, let $c \in \mathbb{C}\gamma$. Let $l = \text{length}(c)$. Then $c \Vdash d \restriction l = c$. Now, if we know $d \restriction l$, then we can perform our inductive construction (which only uses $d(k)$ to construct p'_{k+1} if it has access to p'_k, h, $\pi(\mathbb{P})$, and $\pi(E)$) of $\langle p'_i \mid i \in \omega \rangle$ upto stage l. That is, we can construct $\langle p'_i \mid i \leq l \rangle$ already. In particular, there is some $p' \in \pi(\mathbb{P})$ such that $c \Vdash p'_l = p'$. Using the density of D in $\bar{M}$, pick $p'' \geq p'$ be in D. Then there is an $\alpha < \pi(\aleph_2)$ such that $p'' = p^*_\alpha$. Let $i = h(m)$ for some $m < \omega$, and finally, let $q' = q\hat{\ }m$. Then $q' \geq q$ is an element of $\mathbb{C}(\gamma)$, and $q' \Vdash d(l) = m$. Hence, $q' \Vdash p'_{l+1} = p''$, and consequently, $q' \Vdash p'_{l+1} \in D$. Therefore, the set

$$\{c \in \mathbb{C}\gamma \mid c \Vdash \exists k < \omega[p'_k \in D]\}$$

is a dense open subset of $\mathbb{C}(\gamma)$ in $V[G_{<\gamma}]$, and hence we are done. □

Theorem 193. *Let V be a model of set theory. Then there is a poset $\mathbb{Q} \in V$ such that forcing with $\mathbb{Q}$ collapses $\aleph_2$, and does not collapse any other cardinals.* □

Corollary 194. *In any model M of set theory, if $\aleph_2^{\mathbf{L}}$ is still a cardinal, then there is a generic extension N of M such that in N, $\aleph_2^{\mathbf{L}}$ is not a cardinal, and further, for any other natural number $n > 0$, if $\aleph_n^{\mathbf{L}}$ is a cardinal in M, then it remains a cardinal in N. Consequently, the statement b_2 is a button which is mutually independent of all other buttons b_n for $n \neq 2$.*

6.7 Generalisations and Questions

The theorem we have just seen naturally raises the following question:

Question 195. *(Abraham) In any model of set theory, is there always a poset such that forcing with it collapses $\aleph_3$ and does not collapse any other cardinals?*

Also, given the motivating question of the chapter, the following weaker version of the above question also seems relevant:

Question 196. *Let M be a model of set theory such that $\aleph_3^{\mathbf{L}}$ is still a cardinal. Then is there a poset such that forcing with it collapses $\aleph_3^{\mathbf{L}}$ but does not collapse any other cardinals?*

Note that in this case, if $\aleph_3^{\mathbf{L}} \neq (\aleph_3)^M$, then this question has an answer using either the standard Lévy collapse if $\aleph_3^{\mathbf{L}} = (\aleph_1)^M$ or Abraham's construction if $\aleph_3^{\mathbf{L}} = (\aleph_2)^M$. Hence, for the remaining case, Theorem 190 might be applicable.

The problem with generalising Abraham's method in the most naïve way, that is, by increasing all of the indices by 1 is this: it was crucial in his argument that the preparatory forcing was done by adding Cohen reals. In the last step, this aspect of Cohen reals, that any initial segment of a Cohen real is in the ground model, was used in a crucial way in the last density argument. But the standard way to add a subset S of ω_1 such that for each ordinal $\alpha < \omega_1$, $S \cap \alpha$ is in the ground model (call this a *new* subset of ω_1) is $\text{Add}(\omega_1)$, which collapses the continuum.

Generalising Abraham's argument in this way would then require showing that in any model of set theory, there is always a poset which adds a new subset of ω_1 without collapsing any cardinals.

Theorem 197. *(Todorcevic [Tod82]) If it is consistent with* ZFC *that there is an inaccessible cardinal, then it is consistent with* ZFC *that there is no poset which adds a new subset of* ω_1 *without collapsing either* $\aleph_1$ *or* $\aleph_2$*.*

The model that Todorcevic used was the Mitchell model of [Mit72] which was used to show the independence of the Kurepa Hypothesis, where an inaccessible cardinal is collapsed to become $\aleph_2$. This raises the following question:

Question 198. *(Abraham-Shelah [AS83]) Is the innaccessible cardinal necessary in the above theorem?*

Note that because in the Mitchell model an inaccessible cardinal is collapsed to $\aleph_2$, for each natural number $n > 2$, the statement b_n is already pushed. Hence, the theorem of Todorcevic should not discourage us from trying to answer a revised version of Question 1:

Question 199. *Let* $n > 2$ *be a natural number.* M *be a generic extension of* $\mathbf{L}$ *sucn that* $M \vDash$"$\aleph_n^{\mathbf{L}}$ *is a cardinal". Then, is there a generic extension* N *of* M *such that* $N \vDash$"$\aleph_n^{\mathbf{L}}$ *is not a cardinal" and such that for all other natural numbers* $m > 2$*, if* $M \vDash$"$\aleph_m^{\mathbf{L}}$ *is a cardinal", then* $N \vDash$"$\aleph_m^{\mathbf{L}}$ *is a cardinal"?*

Another interesting phenomenon which is witnessed by the theorems of Abraham and Todorcevic is this: many forcing constructions in set theory require cardinal arithmetic assumptions so as to not have unintended consequences, such as $\mathrm{Lev}(\aleph_1, \aleph_2)$, $\mathrm{Add}(\omega_1)$ etc. Also, in iterated forcing constructions, cardinal arithmetic requirements are needed to do the bookkeeping, for example, in the standard iterated forcing construction of a model of MA_{ω_1}, CH is assumed in the ground model for this purpose. In both of these cases, can we either show that these requirements can be done away with or are necessary?

Question 200. *Let* M *be a model of set theory such that* $M \vDash 2^{\aleph_0} > \aleph_1 + \neg\mathsf{MA}_{\omega_1}$*. Is there a generic extension* N *of* M *obtained without collapsing the continuum such that* $N \vDash \mathsf{MA}_{\omega_1}$*?*

The above question can also be asked for various other statements which are obtained by iterated forcing, for example Suslin's hypothesis. Also note that it is a weaker form of Question 172 and Question 174.

Bibliography

[Abr83] Uri Abraham. On forcing without the continuum hypothesis. *Journal of Symbolic Logic*, pages 658–661, 1983.

[AS83] Uri Abraham and Saharon Shelah. Forcing closed unbounded sets. *Journal of Symbolic Logic*, pages 643–657, 1983.

[AS93] Uri Abraham and Saharon Shelah. A Δ^2_2 well-order of the reals and incompactness of $L(Q^{MM})$. *Annals of Pure and Applied Logic*, 59(1):1–32, 1993.

[Bau70] James E Baumgartner. *Results and independence proofs in combinatorial set theory.* PhD thesis, University of California, Berkeley, 1970.

[BdJ06] Nick Bezhanishvili and Dick de Jongh. *Intuitionistic logic*, volume PP-2006-25. Institute for Logic, Language and Computation (ILLC), University of Amsterdam, 2006.

[BdRV02] Patrick Blackburn, Maarten de Rijke, and Yde Venema. *Modal logic*, volume 53. Cambridge University Press, 2002.

[BJ95] Tomek Bartoszynski and Haim Judah. *Set theory.* AK Peters Wellesley, 1995.

[Car] Erin K Carmody. Which notions of forcing add a cofinal branch to an ω_1-tree? MathOverflow. URL:http://mathoverflow.net/q/93658 (version: 2012-04-13).

[CL89] Tim Carlson and Richard Laver. Sacks reals and Martin's axiom. *Fundamenta Mathematicae*, 133(2):161–168, 1989.

[CZ97] Alexander Chagrov and Michael Zakharyaschev. *Modal Logic, volume 35 of Oxford Logic Guides.* Clarendon Press, Oxford, 1997.

[DJ74] Keith J Devlin and Håvard Johnsbråten. *The Souslin problem.* Springer-Verlag New York, 1974.

[Far96] Ilijas Farah. OCA and towers in $\mathcal{P}(\mathbb{N})/fin$. *Commentationes Mathematicae Universitatis Carolinae*, 37(4):861–866, 1996.

[Fri06] Sy-David Friedman. Internal consistency and the inner model hypothesis. *Bulletin of Symbolic Logic*, pages 591–600, 2006.

[Gab70] Dov M Gabbay. The decidability of the Kreisel-Putnam system. *Journal of Symbolic Logic*, 35(3):431–437, 1970.

[Ham03] Joel David Hamkins. A simple maximality principle. *Journal of Symbolic Logic*, 68(2):527–550, 2003.

[Ham09] Joel David Hamkins. Some second order set theory. In R. Ramanujam and Sundar Sarukkai, editors, *Logic and Its Applications*, volume 5378 of *Lecture Notes in Computer Science*, pages 36–50. Springer Berlin Heidelberg, 2009.

[Ham11] Joel David Hamkins. The set-theoretic multiverse. *arXiv preprint arXiv:1108.4223*, 2011.

[HL08] Joel Hamkins and Benedikt Löwe. The modal logic of forcing. *Transactions of the American Mathematical Society*, 360(4):1793–1817, 2008.

[HL13] Joel David Hamkins and Benedikt Löwe. Moving up and down in the generic multiverse. In Kamal Lodaya, editor, *Logic and Its Applications*, volume 7750 of *Lecture Notes in Computer Science*, pages 139–147. Springer Berlin Heidelberg, 2013.

[HLL] Joel David Hamkins, George Leibman, and Benedikt Löwe. Structural connections between a forcing class and its modal logic. *to appear in the Israel Journal of Mathematics*.

[HS85] Leo Harrington and Saharon Shelah. Some exact equiconsistency results in set theory. *Notre Dame Journal of Formal Logic*, 26(2):178–188, 1985.

[HW05] Joel D Hamkins and W Hugh Woodin. The necessary maximality principle for ccc forcing is equiconsistent with a weakly compact cardinal. *Mathematical Logic Quarterly*, 51(5):493–498, 2005.

[Jac93] O'Shea 'Ice Cube' Jackson. It was a good day, 1993.

[Jec03] Thomas Jech. *Set theory: the third millennium edition*. Springer, 2003.

[Jud93] Haim Judah, editor. *Set theory of the reals*, volume 6 of *Israel Mathematical Conference Proceedings*. American Mathematical Society, 1993.

[JW97] Winfried Just and Martin Weese. *Discovering modern set theory II: Set-theoretic tools for every mathematician*. Amer. Math. Soc., Providence, RI, 1997.

[Lar99] Paul Larson. An $\mathbb{S}$-max variation for one Souslin tree. *Journal of Symbolic Logic*, pages 81–98, 1999.

[Lar13] Paul Larson. Personal communication, 2013.

[Lav07] Richard Laver. Certain very large cardinals are not created in small forcing extensions. *Ann. Pure Appl. Logic*, pages 1–6, 2007.

[Lei04] George Leibman. *Consistency strengths of maximality principles*. PhD thesis, PhD thesis, The Graduate Center of the City University of New York, 2004.

[LT01] Paul Larson and Stevo Todorcevic. Chain conditions in maximal models. *Fundamenta Mathematicae*, 168(1):77–104, 2001.

[LT02] Paul Larson and Stevo Todorcevic. Katetov's problem. *Transactions of the American Mathematical Society*, 354(5):1783–1791, 2002.

[McK] Paul McKenney. Which ω_1-trees are proper? MathOverflow. URL:http://mathoverflow.net/q/129470 (version: 2013-05-02).

[Men97] Elliot Mendelson. *Introduction to mathematical logic*. CRC press, 1997.

[Mit72] William Mitchell. Aronszajn trees and the independence of the transfer property. *Annals of Mathematical Logic*, 5(1):21–46, 1972.

[MSS79] L Maksimova, V Shehtman, and D Skvortsov. The impossibility of a finite axiomatization of Medvedev's logic of finitary problems. *Soviet Math. Dokl*, 20:394–398, 1979.

[Rei06] Jonas Reitz. *The ground axiom*. PhD thesis, The Graduate Center of the City University of New York, 2006.

[Rei07] Jonas Reitz. The ground axiom. *Journal of Symbolic Logic*, 72(4):1299–1317, 2007.

[Rit10] Colin Jakob Rittberg. The modal logic of forcing. Master's thesis, Westfälische Wilhelms-Universität Munster, 2010.

[Sch93] Marion Scheepers. Gaps in ω^ω. In *Israel Mathematical Conference Proceedings*, volume 6, pages 439–561, 1993.

[She82] Saharon Shelah. *Proper forcing*, volume 940 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1982.

[She84] Saharon Shelah. Can you take Solovay's inaccessible away? *Israel Journal of mathematics*, 48(1):1–47, 1984.

[Tod82] Stevo Todorcevic. Some combinatorial properties of trees. *Bulletin of the London Mathematical Society*, 14(3):213–217, 1982.

[Tod11] Stevo Todorcevic. Forcing with a coherent Souslin tree. *available at homepage*, 2011.

[TV87] Stevo Todorcevic and Boban Velickovic. Martin's axiom and partitions. *Compositio Mathematica*, 63(3):391–408, 1987.

[WDR12] W Hugh Woodin, Jacob Davis, and Daniel Rodríguez. The HOD dichotomy. In James Cummings and Ernest Schimmerling, editors, *Appalachian Set Theory: 2006-2012*, volume 406 of *London Mathematical Society Lecture Notes*. Cambridge University Press, 2012.

[Woo04] W Hugh Woodin. Set Theory after Russell: The journey back to Eden. In *One Hundred Years of Russell's Paradox*, volume 6 of *De Gruyter Series in Logic and its Applications*. Walter de Gruyter & Co., Berlin, 2004.

[Woo10] W Hugh Woodin. *The axiom of determinacy, forcing axioms, and the nonstationary ideal*, volume 1 of *De Gruyter Series in Logic and Its Applications*. Walter de Gruyter & Co., Berlin, 2010.