Algebraic Canonicity in Non-Classical Logics

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

This thesis is a study of the notion of canonicity (as is understood e.g. in modal logic) from an algebraic viewpoint. The main conceptual contribution of this thesis is a better understanding of the connection between the Jónsson-style canonicity proof and the canonicity-via-correspondence. The main results of this thesis include an ALBA-aided Jónsson-style canonicity proof for inductive inequalities in distributive modal logic and a Jónsson-style canonicity proof for a certain fragment of the distributive modal μ-calculus.

Contents

Chapter 1

Introduction

Canonicity theory is a fruitful research area originating in modal logic, which is applied to the proof of completeness results. Its main research question can be roughly stated as follows: Under which syntactic restrictions, for a formula or an inequality φ, does the following preservation hold:

$$\mathbb{S} \models \varphi \quad \Rightarrow \quad \mathbb{S}^\delta \models \varphi$$

where, on the algebraic side, $\mathbb{S}$ is an algebra and $\mathbb{S}^\delta$ is its canonical extension, and on the frame side, $\mathbb{S}$ is a frame with topological structure (e.g. a descriptive general frame), and $\mathbb{S}^\delta$ is its underlying frame.

The relevance of canonicity theory is that, on the relational side, we can get completeness results for the normal modal logic $\Lambda(\varphi)$ generated by φ with respect to the class of Kripke frames validating φ. Thanks to the duality between algebras and Kripke frames, this same question can be transferred to algebras and be studied in a purely algebraic and order-topological way.

Before moving on to the motivation of our thesis, let us briefly explain in the next section the link between completeness and canonicity.

1.1 Completeness via Canonicity

It is well known from general model theory that there is a Galois connection between (modal) logics, understood as deductively closed sets of modal formulas, and the classes of their modally defined frames. This Galois connection is defined by associating any deductively closed set of modal formulas Λ with the class $Fr(\Lambda)$ of frames which validate every formula in Λ, and by associating any class K of Kripke frames with the deductively closed set $Th(\mathsf{K})$ of modal formulas which are valid on every frame in K.

If a modal formula φ is canonical, then the normal modal logic $\Lambda(\varphi)$ generated by φ is complete with respect to $Fr(\Lambda(\varphi))$. The definition of canonicity provides a uniform method to proving this completeness result, namely the canonical model construction. Indeed, the model-theoretic version of the definition of canonicity for φ means that the canonical frame associated with $\Lambda(\varphi)$ belongs to $Fr(\Lambda(\varphi))$.

From the perspective of duality theory, the canonical frame can be associated with two algebras: the first one is the Lindenbaum-Tarski algebra; indeed, the canonical frame is the ultrafilter frame of the Lindenbaum-Tarski algebra. The second one is the complex algebra associated with the canonical frame.

This 3-way relationship can be encoded in purely algebraic terms, and completely reconstructed in the algebraic setting, thanks to the notion of canonical extension. Indeed, the complex algebra of the canonical frame is the canonical extension of the Lindenbaum-Tarski algebra. Hence, given the fact that the validity of formulas/inequalities is preserved and reflected by the duality between a frame and its complex algebra, and given that, by its definition, the Lindenbaum-Tarski algebra of $\Lambda(\varphi)$ validates exactly the formulas in $\Lambda(\varphi)$, the crucial step in the completeness proof mentioned above can be equivalently reformulated by the preservation condition displayed at the beginning. We refer the reader to [3] for further discussion on this.

1.2 Overview of Existing Literature

The canonicity results in the literature belong to two main approaches:

The *Jónsson-style* approach to canonicity (Jónsson [21], Ghilardi and Meloni [19], Gehrke, Nagahashi and Venema [17], Suzuki [24]) is purely algebraic, pursues canonicity independently from correspondence (we will come back to discussing correspondence below), and relies on canonical extensions and on the fact that the algebraic operations interpreting the logical connectives can be extended to operations on the canonical extension. The canonicity results are then obtained via a study of the order-theoretic properties of the compositions of these extended operations. The work in [19] and in [21] is mutually independent, while [17] builds on [21] and recognizes its similarity to [19]; the paper [24] builds on the work of [19] and works constructively. Chapter 3 will be dedicated to illustrating this methodology.

The *canonicity-via-correspondence* approach, initiated by van Benthem [25], Sambin and Vaccaro [23], and also developed by Goranko and Vakarelov [20], Conradie and Palmigiano [10] and Bezhanishvili and Hodkinson [2], is relational and also topological, is closely related to the well-known proof of canonicity via d-persistence (cf. Proposition 5.85 in [3]), and its crucial tools are Esakia lemma, intersection lemma and Ackermann lemma. The strategy of this approach can be illustrated by the following U-shaped argument:

$$
\begin{array}{ccc}
\mathbb{G} \Vdash \varphi & & \mathbb{F} \Vdash \varphi \\
\Updownarrow & & \\
\mathbb{F} \Vdash_{\mathbb{G}} \varphi & & \Updownarrow \\
\Updownarrow & & \\
\mathbb{F} \vDash_{\mathbb{G}} \mathrm{FO}(\varphi) & \Leftrightarrow & \mathbb{F} \vDash \mathrm{FO}(\varphi).
\end{array}
$$

In the diagram above, $\mathbb{G} = (\mathbb{F}, \tau)$ is the descriptive general frame with $\mathbb{F}$ as its underlying frame, and τ is its additional topological structure. The U-shaped argument relies on the existence of a first-order sentence FO(φ), the *first-order correspondent* of φ, which holds of $\mathbb{F}$ as a first-order model iff φ is valid on $\mathbb{F}$, as shown in the right-hand side of the diagram. On the left-hand side, by definition, φ is valid on $\mathbb{G}$ iff φ is satisfied on $\mathbb{F}$ with respect to every admissible valuation (as the notation $\Vdash_{\mathbb{G}}$ intends to represent). The proof is finished if an analogous correspondence-type result holds restricted to admissible valuations, as represented by the vertical equivalence in the lower left-hand side of the diagram. Indeed, the bottom equivalence always holds, since the fact that FO(φ) holds does not depend on admissible valuations, or, in other words, FO(φ) cannot distinguish between $\mathbb{F}$ and $\mathbb{G}$.

As outlined by the U-shaped argument, this second approach focuses mainly on correspondence, and its main contributions have been the design of algorithms, such as the Sahlqvist-van Benthem algorithm ([22], [25]), and more recently SQEMA [7], for computing the first-order correspondents of large classes of formulas. The core of the Sahlqvist-van Benthem algorithm is the well-known "minimal valuation argument" (see [3] and also [11] for a discussion).

The algorithm SQEMA is based on the recognition that the Ackermann lemma [1] encodes the so called "minimal valuation argument", see [6] for a discussion. Another interesting contribution of this line of research is the syntactic characterization of the class of the so called "inductive" [20] and "recursive" [8] formulas. Inductive formulas, in particular, extend Sahlqvist formulas in the sense that formulas in each class both have a first-order correspondent and are canonical. Intuitively, both the inductive formulas and the recursive formulas are classes of formulas where the minimal valuation argument, encoded in the form of the Ackermann lemma, is guaranteed to succeed.

Recent developments in this line of research involve modal μ-calculus. In [29], Sahlqvist correspondence has been extended to modal μ-formulas on a classical base. In particular, the class of modal μ-formulas identified as Sahlqvist is rather large, as it includes even more than the inductive formulas, and fixed points are allowed on critical branches (see Definition 2.38). These results have been extended to the modal μ-calculus on an intuitionistic base in [5]. In [2], some canonicity results for μ-formulas with respect to clopen semantics in the classical case have been achieved for a fragment of μ-formulas which is much more restricted than the class of formulas defined in [29]. In particular, fixed points are not allowed on critical branches.

In [10], the two lines of research come together. Indeed, the correspondence results can be achieved algebraically by the algorithm ALBA, which is the algebraic counterpart of SQEMA in the distributive case. In [9], a purely algebraic proof of canonicity in non-distributive setting is given, via a version of ALBA which is sound on non-distributive lattices. Also, the results in [5] are based on an adaptation of ALBA for the intuitionistic modal μ-calculus.

1.3 Open Issues, and Aim of this Thesis

The state-of-the-art up to ALBA reflects a kind of division of labor between the two approaches: namely, algebras are used for studying canonicity independently from correspondence, and Kripke frames and general frames are used for canonicity-via-correspondence. The research yielding to ALBA made clear that correspondence, and hence canonicity-via-correspondence, can also be developed on algebras. Moreover, together with [11], it made clear that algebraic correspondence is grounded on exactly the same order-theoretic principles which guarantee the purely algebraic canonicity. However, what is striking is that, even if the two treatments use exactly the same order-theoretic principles and exactly the same setting of perfect algebras, they look so radically different, to the point that their relationship can be qualified as a mystery. So it is natural to try and clarify how these two approaches are related to each other.

In [30], an analysis was given of the Jónsson-style canonicity proof of [17] for Sahlqvist inequalities in distributive modal logic. This analysis was motivated by an attempt at extending the Jónsson-style canonicity method to the inductive inequalities. As a result of this analysis, a close connection emerged between the **n**-trick—the key tool in the canonicity proof in [17]—and the Ackermann lemma, which, as mentioned above, is the key tool in the SQEMA-type algorithms. However, in [30], the attempt at giving a Jónsson-style proof of canonicity for inductive inequality failed, and it was observed that the Jónsson strategy for proving canonicity of Sahlqvist formulas in the language of distributive modal logic cannot be straightforwardly extended to inductive formulas as defined in [10] (see Section 3.2 below for more details on this issue). So these remarks actually made the situation even more mysterious: from the point of view of correspondence, both Sahlqvist and inductive formulas are designed to guarantee that the minimal valuation argument, as encoded by the Ackermann lemma, succeeds; moreover, the Ackermann lemma and the **n**-trick are essentially one and the same thing. So why should the intimate similarity between Sahlqvist and inductive formulas break down when it comes to Jónsson-style canonicity? Hopefully, by solving this question, we could gain enough tools and insights to also extend the Sahlqvist canonicity results obtained in [2] to the modal μ-calculus on non-classical base.

The aim of this thesis is to address these open issues: firstly, to clarify the relationship between the two approaches to canonicity, and, by doing this, to extend the Jónsson-style canonicity proof to the inductive inequalities; secondly, to improve the canonicity results for μ-formulas.

As a brief summary, the main contributions of this thesis are listed in the following section.

1.4 Main Contributions

The contributions of this thesis include, but are not limited to:

- an analysis of the Jónsson-style canonicity proof for Sahlqvist inequalities in distributive modal logic in [17] (Chapter 3), which puts an emphasis on the compositional structure of σ-expanding terms. The proof in [17] is refined by removing one tricky lemma (Lemma 5.11 in [17]), and an explanation of why this method does not generalize to inductive inequalities is also given.

- a generalized definition of canonical extensions of maps (Chapter 6) whose co-domain need not be the set of clopen elements, and conditions on the compositional structure of maps which guarantee these maps to be contracting or expanding. These are the tools to be able to extend the Jónsson-style canonicity treatment to inequalities in the expanded language for correspondence introduced in [10].

- a new methodology combining the two approaches on canonicity (Chapter 7), aimed at giving an ALBA-aided Jónsson-style canonicity proof for inductive inequalities in the language of distributive modal logic. Especially when restricted to Sahlqvist inequalities, we recognize the two approaches as two faces of the same coin: see Section 7.3 for further discussion.
- a Jónsson-style canonicity proof for the Sahlqvist fragment of distributive modal μ-calculus (Chapter 8), generalizing the Sahlqvist formulas of modal μ-calculus in classical setting defined in [2].

1.5 Structure of the Thesis

The structure of this thesis is the following:

In Chapter 2, the relevant preliminaries on distributive modal logic and its algebraic semantics are given, as well as the definitions of Sahlqvist and inductive inequalities.

In Chapter 3, the Jónsson-style canonicity proof for Sahlqvist inequalities in distributive modal logic given in [17] is reviewed. The outline of the proof is sketched, and the two key tools of the proof, i.e. the **n**-trick and the contracting or expanding terms, are pointed out. In Section 3.1.3, a refinement of the proof is given which shows that one tricky lemma (Lemma 5.11 in [17]) is actually not needed. Then an analysis is given that the two tools above are not enough to prove canonicity for inductive inequalities.

In Chapter 4, the Ackermann lemma is reformulated and proved in a purely algebraic setting.

In Chapter 5, the ALBA-style canonicity proof is reformulated in an algebraic way, and the algebraic idea behind this approach is explained.

The following three chapters are the main part of this thesis:

In Chapter 6, a generalized version of canonical extension is defined as an approximation method for computing values of functions on arbitrary elements based on their values on clopen elements, and conditions for maps to be contracting or expanding are given.

In Chapter 7, the results on contracting and expanding terms are generalized to the expanded language $\mathcal{L}^{++}$ of distributive modal logic containing the adjoints and residuals for all connectives, and the proof is given that ALBA inequalities are canonical. From this, the canonicity of inductive inequalities straightforwardly follows, given that the ALBA inequalities are by definition those on which the ALBA algorithm is guaranteed to succeed, and in [10], ALBA has been proved to succeed on all inductive inequalities. The proof strategy combines the Jónsson approach with the ALBA approach, and makes full use of the ALBA algorithm. In this ALBA-aided Jónsson-style approach, the manipulation steps in ALBA are recognized as admissible manipulations aimed at equivalently transforming the inequality into some desired shape, to which the proof strategy in the Jónsson-style canonicity proof for Sahlqvist inequalities can be applied. A parallel comparison between the two approaches to canonicity is also given.

In Chapter 8, a Jónsson-style canonicity proof is given for the Sahlqvist fragment of distributive modal μ-calculus, generalizing the Sahlqvist formulas of modal μ-calculus on a classical base, as defined in [2].

In Chapter 9, the main results of this thesis are summarized and two future directions are mentioned.

Chapter 2

Preliminaries

In this chapter, we review very concisely the preliminary definitions and facts; we refer the reader to [10], [12], [16], [17] and [30] for a more expanded discussion.

2.1 Distributive Modal Logic

Distributive modal logic (DML, cf. Definition 2.1 in [17]) is introduced in [17], and further discussed in [10] and [30]. Algebraically, distributive modal logic is interpreted in the bounded distributive lattice setting, and in its signature, we have the fragment of propositional logic with $\wedge, \vee, \top, \bot$ as well as modalities $\Diamond, \Box$, and modalities $\rhd, \lhd$ as weak forms of negation. Intuitively, $\Diamond, \Box, \rhd, \lhd$ mean "it is possible that", "it is necessary that", "it is impossible that", "it is possibly not the case that ", respectively. Since the logic does not have the deduction detachment theorem, hence entailment cannot be computed by theoremhood, we need to use sequents of the form $\alpha \Rightarrow \beta$ to formalize the axiomatization of distributive modal logic. Distributive modal logic has both a relational semantics and an algebraic semantics, but here we only work with the algebraic semantics. We recall some preliminary definitions of the basic language $\mathcal{L}$ of distributive modal logic from [17] and [10], with some slight revision.

Given a set Prop of proposition variables, the *formulas* of distributive modal logic are defined as follows:

$$\alpha ::= p \mid \top \mid \bot \mid \alpha \wedge \beta \mid \alpha \vee \beta \mid \Diamond\alpha \mid \Box\alpha \mid \rhd\alpha \mid \lhd\alpha,$$

where $p \in \mathsf{Prop}$. We denote the set of all proposition variables occurring in α as $\mathsf{Prop}(\alpha)$. In the algebraic setting, we also call a formula α a *term*, as the syntax of a term function in an algebra. We denote the set of all formulas as $\mathcal{L}_{term}$.

For the reasons mentioned early on, DML is introduced in the form of *sequents* $\alpha \Rightarrow \beta$, where α and β are formulas defined above, and $\Rightarrow$ is a meta-level implication capturing entailment. In the algebraic setting, we prefer to use the *inequality* notation $\alpha \leq \beta$ for sequents. We denote the set of all inequalities as $\mathcal{L}_{\leq}$. We will also work with *quasi-inequalities* of the form $\alpha_1 \leq \beta_1 \& \ldots \& \alpha_n \leq \beta_n \Rightarrow \alpha \leq \beta$ for convenience of discussion, where & and $\Rightarrow$ are meta-level conjunction and implication, respectively. Let us denote the set of all quasi-inequalities as $\mathcal{L}_{quasi}$.

Definition 2.1 (**Distributive Modal Logic**)**.** A *distributive modal logic* (DML) is a set of inequalities containing the following axioms:

$$p \Rightarrow p \qquad \bot \Rightarrow p \qquad p \Rightarrow \top \qquad p \wedge (q \vee r) \Rightarrow (p \wedge q) \vee (p \wedge r)$$
$$p \Rightarrow p \vee q \qquad q \Rightarrow p \vee q \qquad p \wedge q \Rightarrow p \qquad p \wedge q \Rightarrow q$$
$$\Diamond(p \vee q) \Rightarrow \Diamond p \vee \Diamond q \qquad \Diamond \bot \Rightarrow \bot$$
$$\Box p \wedge \Box q \Rightarrow \Box(p \wedge q) \qquad \top \Rightarrow \Box\top$$
$$\rhd p \wedge \rhd q \Rightarrow \rhd(p \vee q) \qquad \top \Rightarrow \rhd \bot$$
$$\lhd(p \wedge q) \Rightarrow \lhd p \vee \lhd q \qquad \lhd\top \Rightarrow \bot$$

and closed under the following inference rules:

$$\frac{\alpha \Rightarrow \beta \quad \beta \Rightarrow \gamma}{\alpha \Rightarrow \gamma} \qquad \frac{\alpha \Rightarrow \beta}{\alpha(\gamma/x) \Rightarrow \beta(\gamma/x)} \qquad \frac{\alpha \Rightarrow \gamma \quad \beta \Rightarrow \gamma}{\alpha \vee \beta \Rightarrow \gamma} \qquad \frac{\gamma \Rightarrow \alpha \quad \gamma \Rightarrow \beta}{\gamma \Rightarrow \alpha \wedge \beta}$$

$$\frac{\alpha \Rightarrow \beta}{\Diamond\alpha \Rightarrow \Diamond\beta} \qquad \frac{\alpha \Rightarrow \beta}{\Box\alpha \Rightarrow \Box\beta} \qquad \frac{\alpha \Rightarrow \beta}{\rhd\beta \Rightarrow \rhd\alpha} \qquad \frac{\alpha \Rightarrow \beta}{\lhd\beta \Rightarrow \lhd\alpha}$$

where $p, q, r \in \mathsf{Prop}$ and $\alpha, \beta, \gamma \in \mathcal{L}_{term}$.

Similar to classical modal logic, we use K_{DML} to denote the smallest distributive modal logic and $\mathrm{K}_{DML}{+}\Gamma$ to denote the smallest distributive modal logic containing the axioms in Γ, where Γ is a set of inequalities.

2.2 Distributive Modal Algebras

As is well known, given an algebra $\mathbb{A}$, a formula $\alpha(p_1, \ldots, p_n)$ induces an n-ary function $\alpha^{\mathbb{A}} : \mathbb{A}^n \to \mathbb{A}$. Given an assignment $h : \mathsf{Prop} \to \mathbb{A}$, the formula α is then interpreted as the element $[\alpha]_h^{\mathbb{A}} = \alpha^{\mathbb{A}}(h(p_1), \ldots, h(p_n)) \in \mathbb{A}$. Then satisfaction with respect to assignments and validity of sequents $\alpha \Rightarrow \beta$ in $\mathbb{A}$ are respectively defined in terms of the inequality $[\alpha]_h^{\mathbb{A}} \leq [\beta]_h^{\mathbb{A}}$ being true in $\mathbb{A}$, and of the inequality $\alpha^{\mathbb{A}} \leq \beta^{\mathbb{A}}$ being true pointwise in the function space $[\mathbb{A}^n \to \mathbb{A}]$ (see Section 2.6 for more details). Hence, we prefer to use the inequality notation instead of the sequent notation in the algebraic setting. The formulas and sequents in DML are interpreted over a distributive modal algebra (cf. Definition 2.9 in [17]) defined as follows:

Definition 2.2 (**Distributive Modal Algebra**)**.** A *distributive modal algebra* (DMA) is an algebra $\mathbb{A} = (A, \vee, \wedge, \bot, \top, \Diamond, \Box, \rhd, \lhd)$, where $(A, \vee, \wedge, \bot, \top)$ is a *bounded distributive lattice* (BDL) with unary operations $\Diamond, \Box, \rhd, \lhd$ satisfying the following identities:

$$\Diamond(a \vee b) = \Diamond a \vee \Diamond b \qquad \Diamond \bot = \bot \qquad \Box(a \wedge b) = \Box a \wedge \Box b \qquad \Box\top = \top$$
$$\rhd(a \vee b) = \rhd a \wedge \rhd b \qquad \rhd \bot = \top \qquad \lhd(a \wedge b) = \lhd a \vee \lhd b \qquad \lhd\top = \bot$$

The underlying BDL of a DMA $\mathbb{A}$ is denoted by $\mathbb{D}_{\mathbb{A}}$. Therefore we use the notation $\mathbb{A} = (\mathbb{D}_{\mathbb{A}}, \Diamond, \Box, \rhd, \lhd)$.

Since, in the setting of DMA, we have both order-preserving and order-reversing operations, it will be convenient to introduce the following terminology about order duals (cf. Definition 2.10 in [17]):

Definition 2.3 (**Dual Lattice**)**.** Given a BDL $\mathbb{A} = (A, \vee, \wedge, \bot, \top)$, we denote the *dual lattice* $(A, \wedge, \vee, \top, \bot)$ as $\mathbb{A}^\partial$. We also use the notation $\mathbb{A}^1 = \mathbb{A}$.

An *n-order type* ϵ is an element of $\{1, \partial\}^n$, and its i-th coordinate is denoted as ϵ_i. We omit n when it is clear from the context. We use ϵ^∂ to denote the *dual order type* of ϵ where $\epsilon_i^\partial = 1$ (resp. ∂) if $\epsilon_i = \partial$ (resp. 1). Given an order type ϵ, we denote $\mathbb{A}^\epsilon$ as the algebra $\mathbb{A}^{\epsilon_1} \times \ldots \times \mathbb{A}^{\epsilon_n}$.

2.3 Canonical Extensions

Definition 2.4 (**Canonical Extension of BDL**, cf. Definition 2.12 in [17])**.** The *canonical extension* of a BDL $\mathbb{A}$ is a complete BDL $\mathbb{A}^\delta$ containing $\mathbb{A}$ as a sublattice, such that

1. (*denseness*) every element of $\mathbb{A}^\delta$ can be expressed both as a join of meets and as a meet of joins of elements from $\mathbb{A}$;
2. (*compactness*) for all $S, T \subseteq \mathbb{A}$ with $\bigwedge S \leq \bigvee T$ in $\mathbb{A}^\delta$, there exist finite sets $F \subseteq S$ and $G \subseteq T$ such that $\bigwedge F \leq \bigvee G$.

It can be shown that the canonical extension of a BDL is unique up to isomorphism (for a proof, see e.g. Theorem 1 in [18]). It turns out that the canonical extension of a BDL is a perfect BDL (cf. Definition 2.14 in [17]), as defined below.

Definition 2.5 (**Perfect Lattice**)**.** We call a BDL $\mathbb{A}$ *perfect*, if it is complete, completely distributive and completely join generated by the set $J^\infty(\mathbb{A})$ of the completely join-irreducible elements of $\mathbb{A}$ as well as completely meet generated by the set $M^\infty(\mathbb{A})$ of the completely meet-irreducible elements of $\mathbb{A}$.

Theorem 2.6. *If $\mathbb{A}$ is a BDL, then $\mathbb{A}^\delta$ is perfect.*

An element $x \in \mathbb{A}^\delta$ is *closed* (resp. *open*), if it is the meet (resp. join) of a subset of $\mathbb{A}$. The set of closed and open elements of $\mathbb{A}^\delta$ are denoted by $K(\mathbb{A}^\delta)$ and $O(\mathbb{A}^\delta)$, respectively. It is easy to see that the denseness condition in the definition of canonical extension implies $J^\infty(\mathbb{A}^\delta) \subseteq K(\mathbb{A}^\delta)$ and $M^\infty(\mathbb{A}^\delta) \subseteq O(\mathbb{A}^\delta)$ (cf. Page 9 in [17]).

It is easy to check that the following properties about the interaction among canonical extension, order duals, and products hold:

Lemma 2.7.

1. *$(\mathbb{A}^\partial)^\delta \cong (\mathbb{A}^\delta)^\partial$;*
2. *$(\mathbb{A}^n)^\delta \cong (\mathbb{A}^\delta)^n$;*
3. *$(\mathbb{A}^\epsilon)^\delta \cong (\mathbb{A}^\delta)^\epsilon$;*
4. *$K((\mathbb{A}^\partial)^\delta) = O(\mathbb{A}^\delta)^\partial$;*

5. $O((\mathbb{A}^\partial)^\delta) = K(\mathbb{A}^\delta)^\partial$;

6. $K((\mathbb{A}^n)^\delta) = (K(\mathbb{A}^\delta))^n$;

7. $O((\mathbb{A}^n)^\delta) = (O(\mathbb{A}^\delta))^n$.

2.3.1 Canonical Extensions of Maps

Let $\mathbb{A}, \mathbb{B}$ be BDLs. A given map $f : \mathbb{A} \to \mathbb{B}$ can be extended to a map : $\mathbb{A}^\delta \to \mathbb{B}^\delta$ in two canonical ways. Let f^σ and f^π respectively denote the σ and π-*extension* of f (cf. Definition 2.15 in [17]) defined as follows:

Definition 2.8 (**σ- and π-Extension**). Given $f : \mathbb{A} \to \mathbb{B}$, for all $u \in \mathbb{A}^\delta$, we define

$$f^\sigma(u) = \bigvee\{\bigwedge\{f(a) : a \in \mathbb{A} \text{ and } x \leq a \leq y\} : K(\mathbb{A}^\delta) \ni x \leq u \leq y \in O(\mathbb{A}^\delta)\}$$
$$f^\pi(u) = \bigwedge\{\bigvee\{f(a) : a \in \mathbb{A} \text{ and } x \leq a \leq y\} : K(\mathbb{A}^\delta) \ni x \leq u \leq y \in O(\mathbb{A}^\delta)\}$$

Notice that the order of the codomain is all that matters; indeed, by this definition, for maps $f : \mathbb{A} \to \mathbb{B}$ and $g : \mathbb{A}^\partial \to \mathbb{B}$ where $f(a) = g(a)$ for all $a \in \mathbb{A}$, their canonical extensions are the same in the sense that $f^\sigma(u) = g^\sigma(u)$ and $f^\pi(u) = g^\pi(u)$ for all $u \in \mathbb{A}^\delta$. However, if we take the order dual of the codomain, then for maps $f : \mathbb{A} \to \mathbb{B}$ and $g : \mathbb{A} \to \mathbb{B}^\partial$ where $f(a) = g(a)$ for all $a \in \mathbb{A}$, we have $f^\sigma(u) = g^\pi(u)$ and $f^\pi(u) = g^\sigma(u)$ for all $u \in \mathbb{A}^\delta$.

Remark 2.9. For maps which are order-preserving, the definition of canonical extensions can be simplified as follows:

Proposition 2.10 (cf. Remark 2.17 in [17]). *If $f : \mathbb{A} \to \mathbb{B}$ is order-preserving, then for all $u \in \mathbb{A}^\delta$,*

$$f^\sigma(u) = \bigvee\{\bigwedge\{f(a) : x \leq a \in \mathbb{A}\} : u \geq x \in K(\mathbb{A}^\delta)\}$$
$$f^\pi(u) = \bigwedge\{\bigvee\{f(a) : y \geq a \in \mathbb{A}\} : u \leq y \in O(\mathbb{A}^\delta)\}.$$

For $f : \mathbb{A} \to \mathbb{B}$, by defining $f^\partial : \mathbb{A}^\partial \to \mathbb{B}^\partial$ where $f^\partial(a) = f(a)$ for all $a \in \mathbb{A}$, we have that $(f^\partial)^\sigma = (f^\pi)^\partial$ and $(f^\partial)^\pi = (f^\sigma)^\partial$.

2.3.2 Continuity Properties

In this subsection, we define some continuity properties which will be used in the Jónsson-style canonicity proofs. Here $\mathbb{A}$ and $\mathbb{B}$ are arbitrary BDLs.

Definition 2.11 (**Upper and Lower Continuity**). Given any map $f : \mathbb{A}^\delta \to \mathbb{B}^\delta$, we say that:

1. f is *upper continuous* (UC), if for all $u \in \mathbb{A}^\delta$ and all $q \in J^\infty(\mathbb{B}^\delta)$, if $q \leq f(u)$ then there exist some x, y such that $K(\mathbb{A}^\delta) \ni x \leq u \leq y \in O(\mathbb{A}^\delta)$ and $q \leq f(v)$ for all $x \leq v \leq y$;

2. f is *lower continuous* (LC), if for all $u \in \mathbb{A}^\delta$ and all $n \in M^\infty(\mathbb{B}^\delta)$, if $n \geq f(u)$ then there exist some x, y such that $K(\mathbb{A}^\delta) \ni x \leq u \leq y \in O(\mathbb{A}^\delta)$ and $n \geq f(v)$ for all $x \leq v \leq y$.

Theorem 2.12 (cf. Theorem 2.15 in [16])**.** *Given any map $f : \mathbb{A} \to \mathbb{B}$,*

1. *$f^\sigma : \mathbb{A}^\delta \to \mathbb{B}^\delta$ is the largest UC extension of f to $\mathbb{A}^\delta$;*
2. *$f^\pi : \mathbb{A}^\delta \to \mathbb{B}^\delta$ is the smallest LC extension of f to $\mathbb{A}^\delta$.*

For order-preserving maps, the definitions of UC and LC can be simplified to the following:

Proposition 2.13. *Given any order-preserving map $f : \mathbb{A}^\delta \to \mathbb{B}^\delta$,*

1. *f is UC iff for all $u \in \mathbb{A}^\delta$ and all $q \in J^\infty(\mathbb{B}^\delta)$, if $q \leq f(u)$ then there exists some $x \in K(\mathbb{A}^\delta)$ such that $x \leq u$ and $q \leq f(x)$;*
2. *f is LC iff for all $u \in \mathbb{A}^\delta$ and all $n \in M^\infty(\mathbb{B}^\delta)$, if $n \geq f(u)$ then there exists some $y \in O(\mathbb{A}^\delta)$ such that $y \geq u$ and $n \geq f(y)$.*

For order-preserving maps, the following definitions will also be needed in the following chapters. In fact, these definitions also make sense for arbitrary maps, similarly to the definition of UC and LC, but in this thesis we will only use them for order-preserving maps.

Definition 2.14 (**Strong Upper and Lower Continuity**)**.** For any order-preserving map $f : \mathbb{A}^\delta \to \mathbb{B}^\delta$,

1. f is *strongly upper continuous* (SUC), if for all $u \in \mathbb{A}^\delta$ and all $q \in K(\mathbb{B}^\delta)$, if $q \leq f(u)$ then there exists some $x \in K(\mathbb{A}^\delta)$ such that $x \leq u$ and $q \leq f(x)$;
2. f is *strongly lower continuous* (SLC), if for all $u \in \mathbb{A}^\delta$ and all $n \in O(\mathbb{B}^\delta)$, if $n \geq f(u)$ then there exists some $y \in O(\mathbb{A}^\delta)$ such that $y \geq u$ and $n \geq f(y)$.

Definition 2.15 (**Scott and Dual Scott Continuity**)**.** For any order-preserving map $f : \mathbb{A}^\delta \to \mathbb{B}^\delta$,

1. f is *Scott continuous*, if for all $u \in \mathbb{A}^\delta$ and all $q \in J^\infty(\mathbb{B}^\delta)$, if $q \leq f(u)$ then there exists some $x \in J^\infty(\mathbb{A}^\delta)$ such that $x \leq u$ and $q \leq f(x)$;
2. f is *dual Scott continuous*, if for all $u \in \mathbb{A}^\delta$ and all $n \in M^\infty(\mathbb{B}^\delta)$, if $n \geq f(u)$ then there exists some $y \in M^\infty(\mathbb{A}^\delta)$ such that $y \geq u$ and $n \geq f(y)$.

2.3.3 Canonical Extensions of DMAs

Since each unary operation in a DMA is either meet-preserving, join-preserving, join-reversing (i.e. turning joins into meets), or meet-reversing (i.e. turning meets into joins), each of them is *smooth* in the sense that $\Box^\sigma = \Box^\pi$, $\Diamond^\sigma = \Diamond^\pi$,$\lhd^\sigma = \lhd^\pi$,$\rhd^\sigma = \rhd^\pi$ (for a proof, see Lemma 2.25 in [16]). Therefore, we can define the canonical extension of a DMA (cf. Definition 2.19 and Definition 2.20 in [17]) as follows:

Definition 2.16 (**Canonical Extension of DMA**)**.** Given any DMA $\mathbb{A} = (\mathbb{D}_\mathbb{A}, \Diamond, \Box, \rhd, \lhd)$, the canonical extension $\mathbb{A}^\delta$ of $\mathbb{A}$ is defined as $\mathbb{A}^\delta = (\mathbb{D}^\delta_\mathbb{A}, \Diamond^\sigma, \Box^\sigma, \rhd^\sigma, \lhd^\sigma) = (\mathbb{D}^\delta_\mathbb{A}, \Diamond^\pi, \Box^\pi, \rhd^\pi, \lhd^\pi)$. We also use $(\mathbb{D}^\delta_\mathbb{A}, \Diamond, \Box, \rhd, \lhd)$ to denote $\mathbb{A}^\delta$.

Definition 2.17 (**Perfect DMA**)**.** A DMA $\mathbb{A} = (\mathbb{D}_{\mathbb{A}}, \Diamond, \Box, \rhd, \lhd)$ is *perfect*, if $\mathbb{D}_{\mathbb{A}}$ is a perfect BDL, and the operations satisfy the following complete distribution properties:

$$\begin{array}{ll} \Diamond(\bigvee X) = \bigvee \Diamond(X) & \rhd(\bigvee X) = \bigwedge \rhd(X) \\ \Box(\bigwedge X) = \bigwedge \Box(X) & \lhd(\bigwedge X) = \bigvee \lhd(X) \end{array}$$

Theorem 2.18 (cf. Lemma 2.21 in [17])**.** *For any DMA $\mathbb{A}$, its canonical extension $\mathbb{A}^\delta$ is perfect.*

Then every $\alpha \in \mathcal{L}_{term}$ is interpreted in $\mathbb{A}^\delta$ as a composition of operations in $\mathbb{A}^\delta$ as a DMA; but on the other hand, as a term function $\alpha^{\mathbb{A}}$, it can be extended via σ- or π-extension. The following definitions (cf. Definition 5.2 in [17]) compare these functions:

Definition 2.19 (**Stable, Expanding and Contracting Terms**)**.** Let $\alpha \in \mathcal{L}_{term}$, and $\lambda \in \{\sigma, \pi\}$. We say that

- α is λ-*stable* if $\alpha^{\mathbb{A}^\delta} = (\alpha^{\mathbb{A}})^\lambda$ for all DMA $\mathbb{A}$;
- α is λ-*expanding* if $\alpha^{\mathbb{A}^\delta} \leq (\alpha^{\mathbb{A}})^\lambda$ for all DMA $\mathbb{A}$;
- α is λ-*contracting* if $\alpha^{\mathbb{A}^\delta} \geq (\alpha^{\mathbb{A}})^\lambda$ for all DMA $\mathbb{A}$.

The same definitions apply more in general than terms:

Definition 2.20 (**Stable, Expanding and Contracting Maps**)**.** Given BDLs $\mathbb{A}$, $\mathbb{B}$ and maps $f^{\mathbb{A}} : \mathbb{A} \to \mathbb{B}$, $f^{\mathbb{A}^\delta} : \mathbb{A}^\delta \to \mathbb{B}^\delta$ where $f^{\mathbb{A}^\delta}$ extends $f^{\mathbb{A}}$, $\lambda \in \{\sigma, \pi\}$. We say that

- $f^{\mathbb{A}^\delta}$ is λ-*stable* if $f^{\mathbb{A}^\delta} = (f^{\mathbb{A}})^\lambda$;
- $f^{\mathbb{A}^\delta}$ is λ-*expanding* if $f^{\mathbb{A}^\delta} \leq (f^{\mathbb{A}})^\lambda$;
- $f^{\mathbb{A}^\delta}$ is λ-*contracting* if $f^{\mathbb{A}^\delta} \geq (f^{\mathbb{A}})^\lambda$.

2.4 Adjoints and Residuals

The algebraic and algorithmic correspondence theory (cf. [10]) makes crucial use of an expanded language, whose additional connectives are interpreted as the adjoints of the original connectives. In this section, we give the relevant preliminaries on adjoints and residuals, in order to define the expanded languages of distributive modal logic. In this section, $\mathbb{C}$ and $\mathbb{C}'$ are complete lattices (in fact, these definitions can be given for arbitrary lattices, but in this thesis, we only need them for complete lattices), and we will often use subscripts to indicate the underlying lattice/algebra of the order/meet/join. For the proofs of the propositions in this section, see [12].

Definition 2.21 (**Adjoint Pair**)**.** The monotone maps $f : \mathbb{C} \to \mathbb{C}'$ and $g : \mathbb{C}' \to \mathbb{C}$ form an *adjoint pair* (notation: $f \dashv g$), if for every $x \in \mathbb{C}, y \in \mathbb{C}'$,

$$f(x) \leq_{\mathbb{C}'} y \text{ iff } x \leq_{\mathbb{C}} g(y).$$

Whenever $f \dashv g$, f is called the *left adjoint* of g and g the *right adjoint* of f. We also say f is a left adjoint and g is a right adjoint.

An important property of adjoint pairs of maps is that if a map has a left (resp. right) adjoint, then the adjoint is unique and can be computed pointwise from the map itself and the order relation on the lattices. This means that having a left (resp. right) adjoint is an intrinsically order-theoretic property of maps.

Proposition 2.22. *For monotone maps $f : \mathbb{C} \to \mathbb{C}'$ and $g : \mathbb{C}' \to \mathbb{C}$ such that $f \dashv g$, for every $x \in \mathbb{C}, y \in \mathbb{C}'$,*

1. $f(x) = \bigwedge_{\mathbb{C}'}\{y \in \mathbb{C}' : x \leq_{\mathbb{C}} g(y)\}$;
2. $g(y) = \bigvee_{\mathbb{C}}\{x \in \mathbb{C} : f(x) \leq_{\mathbb{C}'} y\}$.

Proposition 2.23. *For any map $f : \mathbb{C} \to \mathbb{C}'$,*

1. *f is completely join-preserving iff it has a right adjoint;*
2. *f is completely meet-preserving iff it has a left adjoint.*

Definition 2.24 (Residual Pair). For n-ary maps $f : \mathbb{C}^n \to \mathbb{C}$ and $g : \mathbb{C}^n \to \mathbb{C}$, they form a *residual pair* in the ith coordinate (notation: $f \dashv_i g$), if for all $x_1, \ldots, x_n, y \in \mathbb{C}$,

$$f(x_1, \ldots, x_i, \ldots, x_n) \leq_{\mathbb{C}} y \quad \text{iff} \quad x_i \leq_{\mathbb{C}} g(x_1, \ldots, y, \ldots, x_n).$$

Whenever $f \dashv_i g$, f is called the *left residual* of g in the ith coordinate and g the *right residual* of f in the ith coordinate. We also say f is a left residual and g is a right residual.

Similar to the adjoint case, we have the following:

Proposition 2.25. *For $f : \mathbb{C}^n \to \mathbb{C}$ and $g : \mathbb{C}^n \to \mathbb{C}$ such that $f \dashv_i g$, for all $x_1, \ldots, x_n, y \in \mathbb{C}$,*

1. $f(x_1, \ldots, x_i, \ldots, x_n) = \bigwedge_{\mathbb{C}}\{y \in \mathbb{C} : x_i \leq_{\mathbb{C}} g(x_1, \ldots, y, \ldots, x_n)\}$;
2. $g(x_1, \ldots, y, \ldots, x_n) = \bigvee_{\mathbb{C}}\{x_i \in \mathbb{C} : f(x_1, \ldots, x_i, \ldots, x_n) \leq_{\mathbb{C}} y\}$.

Proposition 2.26. *For any n-ary map $f : \mathbb{C}^n \to \mathbb{C}$,*

1. *f is completely join-preserving in the i-th coordinate iff it has a right residual in that same coordinate;*
2. *f is completely meet-preserving in the i-th coordinate iff it has a left residual in that same coordinate.*

Notice that the following fact is easy to see:

Proposition 2.27. *For an adjoint pair $f \dashv g$ where $f : \mathbb{C} \to \mathbb{C}'$ and $g : \mathbb{C}' \to \mathbb{C}$, consider $f^\partial : \mathbb{C}^\partial \to \mathbb{C}'^\partial$ and $g^\partial : \mathbb{C}'^\partial \to \mathbb{C}^\partial$ such that $f(x) = f^\partial(x)$ for all $x \in \mathbb{C}$ and $g(y) = g^\partial(y)$ for all $y \in \mathbb{C}'$, then $g^\partial \dashv f^\partial$.*

The fact above plays an important role when considering the compositional structure of Sahlqvist terms. Now we give some examples of adjoints and residuals, some of which will be part of the expanded language in the following sections:

Example 2.28. Given a perfect DMA $\mathbb{A}$, the operations $\Diamond : \mathbb{A} \to \mathbb{A}$ and $\Box : \mathbb{A} \to \mathbb{A}$ are completely join- and meet-preserving, respectively, and therefore are left and right adjoints, respectively. Hence there are operations $\blacksquare : \mathbb{A} \to \mathbb{A}$ and $\blacklozenge : \mathbb{A} \to \mathbb{A}$ such that for every $x, y \in \mathbb{A}$,

$$\begin{gathered} \Diamond x \leq_{\mathbb{A}} y \text{ iff } x \leq_{\mathbb{A}} \blacksquare y, \\ x \leq_{\mathbb{A}} \Box y \text{ iff } \blacklozenge x \leq_{\mathbb{A}} y. \end{gathered}$$

Therefore, by Proposition 2.22, $\blacklozenge x = \bigwedge_{\mathbb{A}}\{y \in \mathbb{A} : x \leq_{\mathbb{A}} \Box y\}$ and $\blacksquare y = \bigvee_{\mathbb{A}}\{x \in \mathbb{A} : \Diamond x \leq_{\mathbb{A}} y\}$.

Example 2.29. Given a perfect DMA $\mathbb{A}$, the operations $\rhd : \mathbb{A}^\partial \to \mathbb{A}$ and $\lhd : \mathbb{A}^\partial \to \mathbb{A}$ are completely meet- and join-preserving, respectively (note that meet in $\mathbb{A}^\partial$ is the same as join in $\mathbb{A}$, and vice versa), and therefore have left and right adjoints, respectively. Hence there are operations $\blacktriangleright : \mathbb{A} \to \mathbb{A}^\partial$ and $\blacktriangleleft : \mathbb{A} \to \mathbb{A}^\partial$ such that for every $x, y \in \mathbb{A}$,

$$\begin{gathered} x \leq_{\mathbb{A}} \rhd y \text{ iff } \blacktriangleright x \leq_{\mathbb{A}^\partial} y \text{ iff } y \leq_{\mathbb{A}} \blacktriangleright x, \\ \lhd x \leq_{\mathbb{A}} y \text{ iff } x \leq_{\mathbb{A}^\partial} \blacktriangleleft y \text{ iff } \blacktriangleleft y \leq_{\mathbb{A}} x. \end{gathered}$$

Therefore, by Proposition 2.22, $\blacktriangleright x = \bigwedge_{\mathbb{A}^\partial}\{y \in \mathbb{A}^\partial : x \leq_{\mathbb{A}} \rhd y\} = \bigvee_{\mathbb{A}}\{y \in \mathbb{A} : x \leq_{\mathbb{A}} \rhd y\}$ and $\blacktriangleleft y = \bigvee_{\mathbb{A}^\partial}\{x \in \mathbb{A}^\partial : \lhd x \leq_{\mathbb{A}} y\} = \bigwedge_{\mathbb{A}}\{x \in \mathbb{A} : \lhd x \leq_{\mathbb{A}} y\}$.

Example 2.30. Given a perfect DMA $\mathbb{A}$, the binary operations $\wedge : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ and $\vee : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ are completely join and meet preserving in each coordinate, respectively, and therefore have right and left residuals, respectively. Hence there are binary operations $\to : \mathbb{A}^\partial \times \mathbb{A} \to \mathbb{A}$ and $- : \mathbb{A} \times \mathbb{A}^\partial \to \mathbb{A}$ such that for every $x, y, z \in \mathbb{A}$,

$$\begin{gathered} x \wedge y \leq_{\mathbb{A}} z \text{ iff } x \leq_{\mathbb{A}} y \to z, \\ x - y \leq_{\mathbb{A}} z \text{ iff } x \leq_{\mathbb{A}} y \vee z. \end{gathered}$$

Therefore, by Proposition 2.25, $y \to z = \bigvee_{\mathbb{A}}\{x \in \mathbb{A} : x \wedge y \leq_{\mathbb{A}} z\}$ and $x - y = \bigwedge_{\mathbb{A}}\{z \in \mathbb{A} : x \leq_{\mathbb{A}} y \vee z\}$.

Example 2.31. Given a perfect DMA $\mathbb{A}$, consider the following binary operations $\mathbf{n} : \mathbb{A} \times \mathbb{A}^\partial \to \mathbb{A}$ and $\mathbf{l} : \mathbb{A}^\partial \times \mathbb{A} \to \mathbb{A}$ such that for all $x, y \in \mathbb{A}$,

$$\mathbf{n}(x, y) := \begin{cases} \bot & \text{if } x \leq_{\mathbb{A}} y \\ \top & \text{if } x \not\leq_{\mathbb{A}} y, \end{cases}$$

$$\mathbf{l}(x, y) := \begin{cases} \bot & \text{if } x \not\leq_{\mathbb{A}} y \\ \top & \text{if } x \leq_{\mathbb{A}} y, \end{cases}$$

then there are binary operations $\mathbf{m} : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ and $\mathbf{k} : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ such that for all $x, y \in \mathbb{A}$,

$$\mathbf{m}(x, y) := \begin{cases} y & \text{if } y = \top \\ x & \text{if } y \neq \top, \end{cases}$$

$$\mathbf{k}(x, y) := \begin{cases} x & \text{if } x = \bot \\ y & \text{if } x \neq \bot, \end{cases}$$

such that for all $x, y, z \in \mathbb{A}$,

$$\mathbf{k}(x,y) \leq_{\mathbb{A}} z \text{ iff } x \leq_{\mathbb{A}} \mathbf{l}(y,z),$$
$$\mathbf{n}(x,y) \leq_{\mathbb{A}} z \text{ iff } x \leq_{\mathbb{A}} \mathbf{m}(y,z),$$

therefore $\mathbf{n}$ is a left residual in its first coordinate and $\mathbf{l}$ is a right residual in its second coordinate.

In fact, we can see that $\mathbf{m}(x,y) = x \vee \mathbf{l}(\top, y)$ and $\mathbf{k}(x,y) = y \wedge \mathbf{n}(x, \bot)$.

Example 2.32 (**Δ-Adjoints**)**.** Consider the *diagonal map* $\Delta : \mathbb{C} \to \mathbb{C} \times \mathbb{C}$, defined by the assignment $x \mapsto (x,x)$. The defining clauses of the least upper bound and greatest lower bound can be equivalently restated by saying that the operations $\vee$ and $\wedge$ are the left and the right adjoints of the diagonal map Δ, respectively:

$$x \leq_{\mathbb{C}} y \wedge z \text{ iff } x \leq_{\mathbb{C}} y \text{ and } x \leq_{\mathbb{C}} z, \text{ iff } \Delta(x) \leq_{\mathbb{C}\times\mathbb{C}} (y,z),$$
$$y \vee z \leq_{\mathbb{C}} x \text{ iff } y \leq_{\mathbb{C}} x \text{ and } z \leq_{\mathbb{C}} x, \text{ iff } (y,z) \leq_{\mathbb{C}\times\mathbb{C}} \Delta(x).$$

Sometimes we will refer to $\wedge$ and $\vee$ collectively as Δ-*adjoints*.

2.5 The Distributive Modal Languages $\mathcal{L}^+$ and $\mathcal{L}^{++}$

In this section, we expand our distributive modal language to a generalized setting, which will be used in the canonicity proofs.

We expand our basic language $\mathcal{L}$ with special variables, called *nominals* (NOM) and *conominals* ($\mathsf{CO\text{-}NOM}$), which range over the set of completely join-irreducible ($J^\infty(\mathbb{A}^\delta)$) and completely meet-irreducible ($M^\infty(\mathbb{A}^\delta)$) elements of $\mathbb{A}^\delta$, as well as with some new operations introduced in Example 2.28, 2.29, 2.30, 2.31 in the previous section. The formulas of the expanded language $\mathcal{L}^+_{term}$ are defined as follows:

$$\alpha ::= \bot \mid \top \mid p \mid i \mid m \mid \alpha \wedge \beta \mid \alpha \vee \beta \mid \Diamond\alpha \mid \Box\alpha \mid \rhd\alpha \mid \lhd\alpha \mid \alpha \to \beta \mid \alpha - \beta \mid \blacklozenge\alpha \mid \blacksquare\alpha \mid \blacktriangleleft\alpha \mid \blacktriangleright\alpha$$

where $p \in \mathsf{Prop}$, $i \in \mathsf{NOM}$ and $m \in \mathsf{CO\text{-}NOM}$. $\mathcal{L}^+_{\leq}$, $\mathcal{L}^+_{quasi}$ are defined similarly. A formula $\alpha \in \mathcal{L}^+_{term}$ is *pure*, if no proposition variable occur in it. Note that the expanded language can only be interpreted in $\mathbb{A}^\delta$, since $\mathbb{A}$ is not a "tense algebra" in general.

One crucial technical trick for our study is to be able to incorporate the meta-level symbols & and $\Rightarrow$ as well as the inequality relation $\leq$ into our language. We will achieve this by defining an expanded language $\mathcal{L}^{++}$ in which terms of the form $\mathbf{n}(\alpha,\beta)$ and $\mathbf{l}(\alpha,\beta)$ are allowed as well. We incorporate & as $\wedge$ for truth values, $\Rightarrow$ as $\mathbf{l}$ for truth values, and $\leq$ as $\mathbf{l}$ for terms. $\mathbf{n}$ can also be seen as $\nleq$ or $\not\Rightarrow$. Although it is not necessary to consider the inequalities $\mathcal{L}^{++}_{\leq}$ and quasi-inequalities $\mathcal{L}^{++}_{quasi}$ in $\mathcal{L}^{++}$, we will still find it convenient to use them.

2.6 Algebraic Semantics of $\mathcal{L}$, $\mathcal{L}^+$ and $\mathcal{L}^{++}$

In this section, we define the algebraic semantics of distributive modal logic for all the three languages $\mathcal{L}$, $\mathcal{L}^+$ and $\mathcal{L}^{++}$:

2.6.1 Interpretation of Connectives

In this subsection we give the interpretation for the connectives in $\mathcal{L}$, $\mathcal{L}^+$ and $\mathcal{L}^{++}$.

Definition 2.33. For any DMA $\mathbb{A} = (A, \vee, \wedge, \bot, \top, \Diamond, \Box, \rhd, \lhd)$, the connectives $\vee, \wedge, \bot, \top, \Diamond, \Box, \rhd, \lhd$ in $\mathcal{L}$ are interpreted as the binary, nullary and unary functions on $\mathbb{A}$, respectively, denoted with the same notations. The symbols $\mathbf{n}, \mathbf{l}$ are interpreted as in Example 2.31. Note that, contrary to the additional connectives in $\mathcal{L}^+$, the symbols $\mathbf{n}, \mathbf{l}$ can be interpreted also in arbitrary DMA $\mathbb{A}$.

The symbols $\blacklozenge, \blacksquare, \blacktriangleleft, \blacktriangleright, \rightarrow, -$ can only be interpreted in the canonical extension $\mathbb{A}^\delta$ of DMA $\mathbb{A}$. Their interpretations are already given in Example 2.28, 2.29, 2.30.

In order to distinguish the syntax from the semantic interpretation, we can add a superscript $\mathbb{A}$ (or $\mathbb{A}^\delta$) to the connectives, but we often omit it when it is clear from the context.

We prefer to see all maps as monotone functions for convenience. Therefore, we interpret the connectives as maps $\mathbb{A}^\epsilon \to \mathbb{A}$ (or $(\mathbb{A}^\delta)^\epsilon \to \mathbb{A}^\delta$), where $\mathbb{A}$ is any DMA. Here ϵ is

- $(1, 1)$, for $\vee, \wedge$;
- (1), for $\Diamond, \Box, \blacklozenge, \blacksquare$;
- (∂), for $\rhd, \lhd, \blacktriangleright, \blacktriangleleft$;
- $(1, \partial)$, for $\mathbf{n}, -$;
- $(\partial, 1)$, for $\mathbf{l}, \rightarrow$.

Sometimes, we also consider their *dual maps*, especially when proving the contracting and expanding results for Sahlqvist terms (see e.g. Theorem 7.17). We take $\Box : \mathbb{A}^\delta \to \mathbb{A}^\delta$ and $\blacklozenge : \mathbb{A}^\delta \to \mathbb{A}^\delta$ as an example: by taking $\Box^\partial : (\mathbb{A}^\delta)^\partial \to (\mathbb{A}^\delta)^\partial$ and $\blacklozenge^\partial : (\mathbb{A}^\delta)^\partial \to (\mathbb{A}^\delta)^\partial$ such that $\Box^\partial u = \Box u$ and $\blacklozenge^\partial u = \blacklozenge u$ for all $u \in \mathbb{A}^\delta$, by Proposition 2.27, we have that $\Box^\partial \dashv \blacklozenge^\partial$, hence by Proposition 2.23, $\Box^\partial$ is completely join-preserving in $(\mathbb{A}^\delta)^\partial$ (which is equivalent to completely meet-preserving in $\mathbb{A}^\delta$), and $\blacklozenge^\partial$ is completely meet-preserving in $(\mathbb{A}^\delta)^\partial$.

Therefore, by taking the dual map $f^\partial : (\mathbb{A}^\epsilon)^\partial \to \mathbb{A}^\partial$ of $f : \mathbb{A}^\epsilon \to \mathbb{A}$, f^∂ and f are set-theoretically the same, but right (resp. left) adjoints/residuals become left (resp. right) adjoints/residuals, completely meet (resp. join)-preserving become completely join (resp. meet)-preserving, and $(f^\partial)^\sigma = f^\pi$, $(f^\partial)^\pi = f^\sigma$.

2.6.2 Algebraic Semantics

In this subsection, we give the formal definition of the algebraic semantics for $\mathcal{L}$, $\mathcal{L}^+$ and $\mathcal{L}^{++}$. We start with $\mathcal{L}$, and then generalize it to $\mathcal{L}^+$ and $\mathcal{L}^{++}$.

Definition 2.34. For any DMA $\mathbb{A}$, an *assignment* on $\mathbb{A}$ is a map $h : \mathsf{Prop} \to \mathbb{A}$. The interpretation of a formula $\alpha \in \mathcal{L}_{term}$ in $\mathbb{A}$ under the assignment h is defined as follows:

$$
\begin{array}{rclrcl}
[p]_h^{\mathbb{A}} & = & h(p) & & & \\
[\bot]_h^{\mathbb{A}} & = & \bot^{\mathbb{A}} & [\Diamond\beta]_h^{\mathbb{A}} & = & \Diamond^{\mathbb{A}}[\beta]_h^{\mathbb{A}} \\
[\top]_h^{\mathbb{A}} & = & \top^{\mathbb{A}} & [\Box\beta]_h^{\mathbb{A}} & = & \Box^{\mathbb{A}}[\beta]_h^{\mathbb{A}} \\
[\beta \wedge \gamma]_h^{\mathbb{A}} & = & [\beta]_h^{\mathbb{A}} \wedge^{\mathbb{A}} [\gamma]_h^{\mathbb{A}} & [\rhd\beta]_h^{\mathbb{A}} & = & \rhd^{\mathbb{A}}[\beta]_h^{\mathbb{A}} \\
[\beta \vee \gamma]_h^{\mathbb{A}} & = & [\beta]_h^{\mathbb{A}} \vee^{\mathbb{A}} [\gamma]_h^{\mathbb{A}} & [\lhd\beta]_h^{\mathbb{A}} & = & \lhd^{\mathbb{A}}[\beta]_h^{\mathbb{A}}.
\end{array}
$$

We will also interpret $\alpha(\vec{p})$ as a *term function* $\alpha^{\mathbb{A}} : \mathbb{A}^n \to \mathbb{A}$ such that for all $\vec{a} \in \mathbb{A}$, $\alpha^{\mathbb{A}}(\vec{a}) = [\alpha]_h^{\mathbb{A}}$ where $h(\vec{p}) = \vec{a}$.

For any DMA $\mathbb{A}$ and any assignment h on $\mathbb{A}$, any inequality $\alpha \leq \beta$ in $\mathcal{L}_{\leq}$, we denote

$$\mathbb{A}, h \vDash \alpha \leq \beta,$$

if $[\alpha]_h^{\mathbb{A}} \leq [\beta]_h^{\mathbb{A}}$ in $\mathbb{A}$. For any quasi-inequality $\alpha_1 \leq \beta_1 \& \ldots \& \alpha_n \leq \beta_n \Rightarrow \alpha \leq \beta$ in $\mathcal{L}_{quasi}$, we denote

$$\mathbb{A}, h \vDash \alpha_1 \leq \beta_1 \& \ldots \& \alpha_n \leq \beta_n \Rightarrow \alpha \leq \beta,$$

if whenever $[\alpha_i]_h^{\mathbb{A}} \leq [\beta_i]_h^{\mathbb{A}}$ in $\mathbb{A}$ for all $1 \leq i \leq n$ we also have $[\alpha]_h^{\mathbb{A}} \leq [\beta]_h^{\mathbb{A}}$ in $\mathbb{A}$. We denote

$$\mathbb{A} \vDash \alpha \leq \beta, \text{ } or \text{ } \alpha^{\mathbb{A}} \leq \beta^{\mathbb{A}},$$

and say $\alpha \leq \beta$ is *valid* on $\mathbb{A}$, if for all assignments h on $\mathbb{A}$, we have $\mathbb{A}, h \vDash \alpha \leq \beta$. Validity of quasi-inequalities is defined similarly.

The definitions above can be extended to $\mathcal{L}^{+}$ and $\mathcal{L}^{++}$. Recall that $\mathcal{L}^{+}$- and $\mathcal{L}^{++}$-formulas are interpreted in $\mathbb{A}^{\delta}$. An assignment in this setting is a map $h : \mathsf{Prop} \cup \mathsf{NOM} \cup \mathsf{CO\text{-}NOM} \to \mathbb{A}^{\delta}$ such that $h(i) \in J^{\infty}(\mathbb{A}^{\delta})$ for any $i \in \mathsf{NOM}$ and $h(m) \in M^{\infty}(\mathbb{A}^{\delta})$ for any $m \in \mathsf{CO\text{-}NOM}$. For the interpretation of formulas, the following clauses are added to the ones above:

$$
\begin{array}{rclrcl}
[i]_h^{\mathbb{A}^\delta} & = & h(i) & & & \\
[m]_h^{\mathbb{A}^\delta} & = & h(m) & [\blacklozenge\beta]_h^{\mathbb{A}^\delta} & = & \blacklozenge^{\mathbb{A}^\delta}[\beta]_h^{\mathbb{A}^\delta} \\
[\beta \to \gamma]_h^{\mathbb{A}^\delta} & = & [\beta]_h^{\mathbb{A}^\delta} \to^{\mathbb{A}^\delta} [\gamma]_h^{\mathbb{A}^\delta} & [\blacksquare\beta]_h^{\mathbb{A}^\delta} & = & \blacksquare^{\mathbb{A}^\delta}[\beta]_h^{\mathbb{A}^\delta} \\
[\beta - \gamma]_h^{\mathbb{A}^\delta} & = & [\beta]_h^{\mathbb{A}^\delta} -^{\mathbb{A}^\delta} [\gamma]_h^{\mathbb{A}^\delta} & [\blacktriangleright\beta]_h^{\mathbb{A}^\delta} & = & \blacktriangleright^{\mathbb{A}^\delta}[\beta]_h^{\mathbb{A}^\delta} \\
[\mathbf{n}(\beta, \gamma)]_h^{\mathbb{A}^\delta} & = & \mathbf{n}^{\mathbb{A}^\delta}([\beta]_h^{\mathbb{A}^\delta}, [\gamma]_h^{\mathbb{A}^\delta}) & [\blacktriangleleft\beta]_h^{\mathbb{A}^\delta} & = & \blacktriangleleft^{\mathbb{A}^\delta}[\beta]_h^{\mathbb{A}^\delta}. \\
[\mathbf{l}(\beta, \gamma)]_h^{\mathbb{A}^\delta} & = & \mathbf{l}^{\mathbb{A}^\delta}([\beta]_h^{\mathbb{A}^\delta}, [\gamma]_h^{\mathbb{A}^\delta}) & & &
\end{array}
$$

Term functions, interpretation of inequalities and quasi-inequalities and validity can be defined similarly to the $\mathcal{L}$ case. In the setting of $\mathcal{L}^{++}$, we say that an assignment $h : \mathsf{Prop} \cup \mathsf{NOM} \cup \mathsf{CO\text{-}NOM} \to \mathbb{A}^{\delta}$ is *admissible*, if $h(p) \in \mathbb{A}$ for any $p \in \mathsf{Prop}$. Then we can speak of an inequality $\alpha \leq \beta$ being *admissibly valid* (notation: $\mathbb{A}^{\delta} \vDash_{\mathbb{A}} \alpha \leq \beta$), if for all admissible assignments h on $\mathbb{A}^{\delta}$, we have $\mathbb{A}^{\delta}, h \vDash \alpha \leq \beta$. For quasi-inequalities, the definition of admissible validity is similar.

It is easy to see that for $\alpha \leq \beta \in \mathcal{L}_{\leq}$, $\mathbb{A}^{\delta} \vDash_{\mathbb{A}} \alpha \leq \beta$ iff $\mathbb{A} \vDash \alpha \leq \beta$.

Definition 2.35 (**Soundness and Completeness**). A deductively closed class Λ of inequalities in $\mathcal{L}_{\leq}$ is *sound* with respect to a class C of DMAs, if $\mathbb{A} \vDash \alpha \leq \beta$ for all $\mathbb{A} \in \mathsf{C}$ and all $\alpha \leq \beta \in \Lambda$.

A class Λ of inequalities in $\mathcal{L}_{\leq}$ is *complete* with respect to a class C of DMAs, if whenever $\mathbb{A} \vDash \alpha \leq \beta$ for all $\mathbb{A} \in \mathsf{C}$, we have $\alpha \leq \beta \in \Lambda$.

Definition 2.36 (**Canonicity**)**.** A class of DMAs is *canonical* if it closed under taking canonical extensions. An inequality, formula, set of formulas, or set of inequalities, is called canonical, if the class of DMAs defined by the inequality, formula, set of formulas, or set of inequalities, is canonical.

Therefore, in our setting, we say that an inequality $\alpha \leq \beta$ is canonical, if the class of DMAs defined by $\alpha \leq \beta$ is closed under taking canonical extensions, i.e., for all DMAs $\mathbb{A}$, we have

$$\mathbb{A} \models \alpha \leq \beta \quad \Rightarrow \quad \mathbb{A}^\delta \models \alpha \leq \beta.$$

2.7 Sahlqvist and Inductive Inequalities

In this section we define Sahlqvist and inductive inequalities in the basic language $\mathcal{L}$. Note that many definitions apply also to $\mathcal{L}^+$ and $\mathcal{L}^{++}$, therefore we define them in the expanded language $\mathcal{L}^{++}$ whenever possible. Here we only give the definitions and examples of Sahlqvist and inductive inequalities; the intuition behind these definitions is discussed in [10]. Further discussions on Sahlqvist and inductive inequalities in distributive modal logic can be found in [10], [17] and [30].

Definition 2.37 (**Signed Generation Tree**)**.** A *positive (resp. negative) signed generation tree* for a term $s \in \mathcal{L}^{++}_{term}$ is defined as follows:

- The root node $+s$ (resp. $-s$) is the root node of the positive (resp. negative) generation tree of s signed with $+$ (resp. $-$).
- If a node is labelled with $\vee, \wedge, \Box, \Diamond, \blacksquare, \blacklozenge$, assign the same sign to its child node(s).
- If a node is labelled with $\lhd$, $\rhd$, $\blacktriangleleft$, $\blacktriangleright$, assign the opposite sign to its child node.
- If a node is labelled with $\rightarrow, \mathbf{l}$, assign the opposite sign to its left child node and the same sign to its right child node.
- If a node is labelled with $-, \mathbf{n}$, assign the same sign to its left child node and the opposite sign to its right child node.

We say that a node in the signed generation tree is *positive* (resp. *negative*), if it is signed $+$ (resp. $-$).

We use the notation $+s$ (resp. $-s$) for the positive (resp. negative) signed generation tree of the term $s \in \mathcal{L}^{++}_{term}$ and $\star\alpha \prec *s$ to denote $\star\alpha$ is a *subtree* of $*s$, where $\star, * \in \{+, -\}$. Sometimes we will refer to the symbols such as $+s$ and $-s$ as "terms", and we will write e.g. $+s \in \mathcal{L}^{++}_{term}$ even if this is an abuse of notation. We denote $\alpha \prec s$ if α is a *subterm* of s.

Definition 2.38. Given an order type $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$, a term $s(p_1, \ldots, p_n) \in \mathcal{L}^{++}_{term}$ and $* \in \{+, -\}$,

- an *ϵ-critical node* in the signed generation tree $*s$ is a leaf node $+p_i$ if $\epsilon_i = 1$ or $-p_i$ if $\epsilon_i = \partial$; sometimes we will say that this occurrence of p_i in $*s$ *agrees with* ϵ;
- an *ϵ-critical branch* in the tree is a branch terminating in an ϵ-critical node;

- $*s$ *agrees with* ϵ if every occurrence of every proposition variable in $*s$ agrees with ϵ. We also call $*s$ ϵ-*uniform*;

- $*s$ is said to be *uniform*, if there exists an ϵ such that $*s$ is ϵ-*uniform*.

The following definitions are given for $\mathcal{L}$ only:

Definition 2.39 (**Universal and Choice Nodes**)**.** For $* \in \{+, -\}$, a node in the signed generation tree of a term $*s \in \mathcal{L}_{term}$ is

- a *choice node*, if it is either positive and labelled $\vee, \Diamond, \lhd$, or negative and labelled $\wedge, \Box, \rhd$;

- a *universal node*, if it is either positive and labelled $\Box, \rhd$, or negative and labelled $\Diamond, \lhd$.

An explanation why those nodes are called "universal" or "choice" has been given and discussed in [10].

Definition 2.40 (**Sahlqvist Inequality**)**.** Given an order type $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$, $* \in \{+, -\}$, the signed generation tree $*s$ of a term $s(p_1, \ldots, p_n) \in \mathcal{L}_{term}$ is ϵ-*Sahlqvist*, if on every ϵ-critical branch with leaf labelled p_i, $1 \leq i \leq n$, no choice node has a universal node as ancestor. Here we say that $+s$ is ϵ-*left Sahlqvist*, and $-s$ is ϵ-*right Sahlqvist*. An inequality $s \leq t$ is ϵ-Sahlqvist if the trees $+s$ and $-t$ are both ϵ-Sahlqvist. An inequality $s \leq t$ is Sahlqvist if it is ϵ-Sahlqvist for some ϵ. If $*s$ is ϵ-Sahlqvist and agrees with ϵ, then we say that $*s$ is ϵ-*uniform Sahlqvist*. ϵ-uniform left Sahlqvist and ϵ-uniform right Sahlqvist are defined similarly.

Example 2.41. $\Diamond\Box p \leq \Box\Diamond p$, which is the distributive counterpart of the Geach axiom $\Diamond\Box p \rightarrow \Box\Diamond p$, is ϵ-Sahlqvist for $\epsilon = (1)$.

In classical modal logic, a correspondence and canonicity theorem analogous to Sahlqvist has been given for an extended class of formulas called *inductive formulas* by Goranko and Vakarelov in [20]. In the distributive modal logic setting, the class of Sahlqvist inequalities has also been extended to inductive inequalities as defined in [10].

Definition 2.42 (**Inductive Inequality**)**.** Given an order type ϵ, and an irreflexive and transitive relation Ω on the proposition variables $p_1, \ldots, p_n$, $* \in \{+, -\}$, the signed generation tree of a term $*s(p_1, \ldots, p_n) \in \mathcal{L}_{term}$ is (Ω, ϵ)-*inductive* if, on every ϵ-critical branch with leaf labelled p_i, $1 \leq i \leq n$, every choice node c with a universal node as ancestor is binary, hence labelled with $\star(\alpha \circ \beta)$, where $\star \in \{+, -\}$, and moreover:

1. $\star\alpha$ is of order type ϵ^∂;

2. $p_j <_\Omega p_i$ for every p_j occurring in α.

Clearly, the conditions above imply that the ϵ-critical branches all run through β. We will refer to Ω as the *dependency order* on the variables. (Ω, ϵ)-*left inductive* and (Ω, ϵ)-*right inductive* terms are defined similarly to ϵ-left Sahlqvist and ϵ-right Sahlqvist. An inequality $s \leq t$ is (Ω, ϵ)-*inductive* if the trees $+s$ and $-t$ are both (Ω, ϵ)-inductive. An inequality $s \leq t$ is *inductive* if it is (Ω, ϵ)-inductive for some Ω and ϵ.

Example 2.43. Consider the inequality $\Box(p \vee \lhd q) \wedge \Box q \leq \Diamond(p \wedge q)$ in $\mathcal{L}_{\leq}$, it is (Ω, ϵ)-inductive with $q \leq_\Omega p$, $\epsilon_p = 1$ and $\epsilon_q = 1$, but it is not ϵ-Sahlqvist for any ϵ.

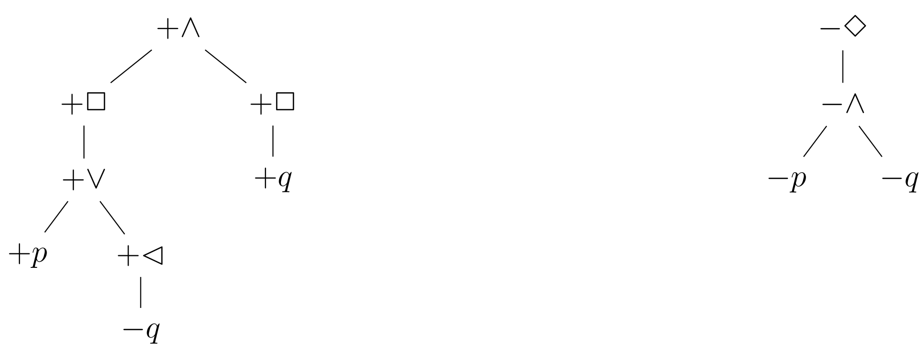


Figure 2.1: Signed generation tree for $\Box(p \vee \lhd q) \wedge \Box q \leq \Diamond(p \wedge q)$

Chapter 3

Jónsson-Style Canonicity

In this chapter we review the Jónsson-style canonicity proof for Sahlqvist inequalities in distributive modal logic, as given in [17], with slight revisions. In Subsection 3.1.3, we discuss a refinement of this proof, and then in Section 3.2, we briefly discuss the difficulties in generalizing this method to inductive inequalities, and propose some directions for a solution.

3.1 Jónsson-Style Canonicity for Sahlqvist Inequalities

Let $\mathbb{A}$ be any DMA, let $\mathbb{A}^\delta$ be its canonical extension and $\alpha \leq \beta$ be an inequality in the basic distributive modal language. In [17], it was shown that if $\alpha \leq \beta$ is a Sahlqvist inequality, then it is preserved under taking canonical extensions:

$$\mathbb{A} \models \alpha \leq \beta \quad \Rightarrow \quad \mathbb{A}^\delta \models \alpha \leq \beta.$$

The following theorem states the Jónsson-style canonicity for Sahlqvist inequalities given in [17].

Theorem 3.1 (cf. Theorem 5.1 in [17]). *Every Sahlqvist inequality $\alpha \leq \beta$ is canonical.*

Proof. We need to show that $\mathbb{A} \models \alpha \leq \beta \Rightarrow \mathbb{A}^\delta \models \alpha \leq \beta$ for any DMA $\mathbb{A}$. The proof consists of the motivation of the following chain:

$$\begin{array}{lll}
 & \mathbb{A} \models \alpha \leq \beta & \\
\Longleftrightarrow & \alpha^{\mathbb{A}} \leq \beta^{\mathbb{A}} & \text{(by definition)} \\
\Longleftrightarrow & \alpha_1^{\mathbb{A}} \leq \beta_1^{\mathbb{A}} \vee \gamma^{\mathbb{A}} & \text{(see Lemma 3.2 below)} \\
\Longleftrightarrow & \left(\alpha_1^{\mathbb{A}}\right)^\sigma \leq \left(\beta_1^{\mathbb{A}} \vee \gamma^{\mathbb{A}}\right)^\sigma & \text{(by the definition of } \sigma\text{-extension)} \\
\Longrightarrow & \left(\alpha_1^{\mathbb{A}}\right)^\sigma \leq \left(\beta_1^{\mathbb{A}}\right)^\pi \vee \left(\gamma^{\mathbb{A}}\right)^\sigma & \text{(see Lemma 3.3 below)} \\
\Longrightarrow & \alpha_1^{\mathbb{A}^\delta} \leq \beta_1^{\mathbb{A}^\delta} \vee \gamma^{\mathbb{A}^\delta} & \text{(see Lemma 3.4 and 3.5 below)} \\
\Longleftrightarrow & \alpha^{\mathbb{A}^\delta} \leq \beta^{\mathbb{A}^\delta} & \text{(see Lemma 3.2 below)} \\
\Longleftrightarrow & \mathbb{A}^\delta \models \alpha \leq \beta & \text{(by definition).}
\end{array}$$

The lemmas needed in this proof are listed below. They rely on Definitions 2.8 and 2.19.

Lemma 3.2 (cf. Lemma 5.14 in [17])**.** *Every ϵ-Sahlqvist inequality $\alpha \leq \beta$ is equivalent to an inequality $\alpha_1 \leq \beta_1 \vee \gamma$ where $+\alpha_1$ is ϵ'-uniform left Sahlqvist, $-\beta_1$ is ϵ'-uniform right Sahlqvist, and $+\gamma$ is ϵ'-uniform, for some ϵ' extending ϵ.*

Lemma 3.3 (cf. Lemma 5.11 in [17])**.** *If $f, g : \mathbb{A} \to \mathbb{B}$ are maps between BDLs such that f is order-preserving and g is order-reversing then $(f \vee g)^\sigma \leq f^\sigma \vee g^\pi$.*

Lemma 3.4 (cf. Lemma 5.5 in [17], with slight revision)**.** *Every uniform term in $\mathcal{L}^n_{term}$ is both σ-contracting and π-expanding.*

Lemma 3.5 (cf. Lemma 5.10 in [17], with slight revision)**.** *Every ϵ-uniform left Sahlqvist term is σ-expanding, and every ϵ-uniform right Sahlqvist term is π-contracting.*

□

There are two key steps to the proof of Theorem 3.1: the first step transforms the inequality $\alpha \leq \beta$ into some inequality $\alpha_1 \leq \beta_1 \vee \gamma$ such that α_1, β_1 and γ are uniform terms and satisfy certain additional conditions (Lemma 3.2); the second step shows that left (resp. right) Sahlqvist terms are σ-expanding (resp. π-contracting) (Lemma 3.5), and that uniform terms are σ-contracting (Lemma 3.4).

One of the reasons why this proof is not point-free is that it depends on Lemma 3.3, and in the proof of this lemma, completely join-irreducible elements (which are points in algebraic disguise) are used. In Subsection 3.1.3 we will show that this lemma is not really needed. However, having been able to dispense with it does not completely eliminate the use of points. Indeed, completely join-irreducible elements play an essential role in the definition of upper continuity and Scott continuity (see Definitions 2.11 and 2.15), hence the proof is still not point-free.

In what follows we look into the details of these issues.

3.1.1 Minimal Collapse Algorithm, n-Trick, and Ackermann Lemma

We are set to show that every ϵ-Sahlqvist inequality $\alpha \leq \beta$ is equivalent to some inequality $\alpha_1 \leq \beta_1 \vee \gamma$, where α_1 is ϵ'-uniform left Sahlqvist, β_1 is ϵ'-uniform right Sahlqvist, and γ is ϵ'-uniform for some ϵ' extending ϵ. Towards this end, we use the $\mathbf{n}$ operation in Example 2.31 to separate the ϵ^∂-parts from $+\alpha$ and $-\beta$ to form γ.

First of all, we recall the definition of the $\mathbf{n}$-term:

Definition 3.6 (**n-Term**)**.** For a DMA $\mathbb{A}$, we define the binary operation $\mathbf{n}^\mathbb{A} : \mathbb{A} \times \mathbb{A}^\partial \to \mathbb{A}$ by

$$\mathbf{n}^\mathbb{A}(a, b) := \begin{cases} \bot & \text{if } a \leq b \\ \top & \text{if } a \nleq b. \end{cases}$$

It is easy to check that $\mathbf{n}^\mathbb{A} : \mathbb{A} \times \mathbb{A}^\partial \to \mathbb{A}$ is an operator, preserving binary joins in the first coordinate and turning binary meets (in $\mathbb{A}$) into binary joins (in $\mathbb{A}$) in the second coordinate.

We denote the language $\mathcal{L}$ expanded with $\mathbf{n}$ as $\mathcal{L}^n$. The languages $\mathcal{L}^n_{term}$, $\mathcal{L}^n_{\leq}$ and $\mathcal{L}^n_{quasi}$ can be defined similarly to what is done in Section 2.5, and stable, expanding and contracting terms can be defined similarly to Definition 2.19.

Then we use the following $\mathbf{n}$-trick to separate the ϵ^∂-part, which we also sometimes will call $\mathbf{n}$-part, from the Sahlqvist "skeleton" (for further details, see discussion below Lemma 3.8). The idea used in its proof is very close to the "minimal valuation" argument in the van Benthem's strategy (see [26]) for correspondence. This is because, as mentioned early on, the Ackermann lemma encodes the minimal valuation argument (see [6] for more discussion), and because there is a striking similarity between Lemma 3.7 and Lemma 3.9 below.

Lemma 3.7 (**n-Trick**)**.** *Let terms $\alpha, \beta, s \in \mathcal{L}^n_{term}$ and z be a new variable not occurring in α or β, then:*

1. *if $+s \prec +\alpha$ and $+s \prec -\beta$, then $\mathbb{A} \models \alpha \leq \beta$ iff $\mathbb{A} \models \alpha(z/s) \leq \beta(z/s) \vee \mathbf{n}(z, s)$;*
2. *if $-s \prec +\alpha$ and $-s \prec -\beta$, then $\mathbb{A} \models \alpha \leq \beta$ iff $\mathbb{A} \models \alpha(z/s) \leq \beta(z/s) \vee \mathbf{n}(s, z)$.*

Proof. See Lemma 2.1.5. and Corollary 2.1.6. in [30]. □

Repeated applications of the Lemma 3.7 make it possible to transform Sahlqvist inequalities into uniform Sahlqvist inequalities, following what is called "the minimal collapse algorithm" in [30]. This is the content of the following lemma, which is obtained by exhaustive application of the $\mathbf{n}$-trick:

Lemma 3.8 (cf. Lemma 5.14 in [17])**.** *Given an inequality $\alpha \leq \beta$ in $\mathcal{L}^n_{\leq}$, let $+s_i \prec +\alpha$ for $1 \leq i \leq m$, $-s'_i \prec +\alpha$ for $1 \leq i \leq m'$, $+t_i \prec -\beta$ for $1 \leq i \leq l$ and $-t'_i \prec -\beta$ for $1 \leq i \leq l'$ be all the subterms of $+\alpha$ and $-\beta$ which are maximal ϵ^∂-subterms. The following are equivalent:*

1. *$\alpha^{\mathbb{A}} \leq \beta^{\mathbb{A}}$;*
2. *$\alpha_1^{\mathbb{A}} \leq \beta_1^{\mathbb{A}} \vee \gamma^{\mathbb{A}}$, where $\alpha_1 = \alpha(\vec{z}/\vec{s}, \vec{z'}/\vec{s'})$, $\beta_1 = \beta(\vec{w}/\vec{t}, \vec{w'}/\vec{t'})$, $\gamma = \bigvee_{i=1}^{m} \mathbf{n}(z_i, s_i) \vee \bigvee_{i=1}^{m'} \mathbf{n}(s'_i, z'_i) \vee \bigvee_{i=1}^{l} \mathbf{n}(w_i, t_i) \vee \bigvee_{i=1}^{l'} \mathbf{n}(t'_i, w'_i)$.*

In addition, α_1 is ϵ'-uniform left Sahlqvist, β_1 is ϵ'-uniform right Sahlqvist and γ is ϵ'-uniform, where ϵ' is ϵ extended by defining $\vec{z}, \vec{w}$ to be of order type 1 and $\vec{z'}, \vec{w'}$ of order type ∂.

The lemma above is a reformulation of Lemma 3.2, showing how to obtain α_1, β_1 and γ in an effective way. In item 2. above, the γ is the ϵ^∂-part, which has been effectively separated from the ϵ-part α_1 and β_1, which we call the Sahlqvist "skeleton".

In fact, it is easy to see that the following universal[1] versions of Ackermann lemmas are equivalent to the $\mathbf{n}$-trick.

[1] The statement of Lemma 3.9 is referred to as "universal" because when applied to a logical setting, it is about the validity of term inequalities on algebras. In Chapter 4, we will treat the existential version, which, when applied to a logical setting, is about the satisfaction of term inequalities.

Lemma 3.9 (Universal Ackermann Lemmas)**.** *Let $\mathbb{A}$ be a DMA, $\alpha \leq \beta$ be an inequality in $\mathcal{L}^n_{\leq}$, and z be a new variable which does not occur in both α and β. Then:*

1. *if $+s \prec +\alpha$ and $+s \prec -\beta$, then $\mathbb{A} \models \alpha \leq \beta$ iff $\mathbb{A} \models z \leq s \Rightarrow \alpha(z/s) \leq \beta(z/s)$;*
2. *if $-s \prec +\alpha$ and $-s \prec -\beta$, then $\mathbb{A} \models \alpha \leq \beta$ iff $\mathbb{A} \models s \leq z \Rightarrow \alpha(z/s) \leq \beta(z/s)$.*

The equivalence between e.g. items 1. of Lemma 3.8 and Lemma 3.9 relies on the equivalence between the following conditions, for any DMA $\mathbb{A}$ and any assignment h on $\mathbb{A}$:

1. $\mathbb{A}, h \models \alpha(z/s) \leq \beta(z/s) \vee \mathbf{n}(z, s)$;
2. $\mathbb{A}, h \models z \leq s \Rightarrow \alpha(z/s) \leq \beta(z/s)$.

Indeed, if $z \nleq s$ is true under h, then the two conditions are trivially true. If the $z \leq s$ is true, then both conditions hold iff $\alpha(z/s) \leq \beta(z/s)$ is true under h.

This shows that the **n**-trick encodes the universal version of the Ackermann lemma which is a statement in the meta-language, into the object-level syntax by means of the binary operation **n**.

3.1.2 Contracting and Expanding Terms

After extracting all the ϵ^∂-parts in $+\alpha$ and $-\beta$, we get to an inequality of the form $\alpha_1 \leq \beta_1 \vee \gamma$, where α_1 is ϵ'-uniform left Sahlqvist, β_1 is ϵ'-uniform right Sahlqvist, and γ is ϵ'-uniform for some ϵ' extending ϵ. Therefore, the next thing to show is that α_1, β_1 and γ are contracting and expanding as required by Lemma 3.4 and Lemma 3.5.

The fact that the term γ is σ-contracting follows from its being uniform and from the following lemma:

Lemma 3.10 (cf. Lemma 5.5 in [17], with slight revision)**.** *Every uniform term in $\mathcal{L}^n_{term}$ is both σ-contracting and π-expanding.*

Finally, we are left to show that uniform left (resp. right) Sahlqvist terms are σ-expanding (resp. π-contracting). Without loss of generality, we consider left Sahlqvist terms here. By Theorem 2.12, f^σ is the largest UC map extending f. Hence, in order to show that a term $t \in \mathcal{L}^n_{term}$ is σ-expanding, i.e., $t^{\mathbb{A}^\delta} \leq (t^{\mathbb{A}})^\sigma$, it suffices to show that $t^{\mathbb{A}^\delta}$ is a UC map. Recall that, for order-preserving maps,

- upper continuity is a condition of the form "for all $q \in J^\infty(\mathbb{B}^\delta)$ and $u \in \mathbb{A}^\delta$ there exists an $x \in K(\mathbb{A}^\delta) \ldots$", which we abbreviate with the symbol $(\forall J \exists K)$;
- Scott continuity is a condition of the form "for all $q \in J^\infty(\mathbb{B}^\delta)$ and $u \in \mathbb{A}^\delta$ there exists an $x \in J^\infty(\mathbb{A}^\delta) \ldots$", which we abbreviate with the symbol $(\forall J \exists J)$;
- strong upper continuity is a condition of the form "for all $q \in K(\mathbb{B}^\delta)$ and $u \in \mathbb{A}^\delta$ there exists an $x \in K(\mathbb{A}^\delta) \ldots$", which we abbreviate with the symbol $(\forall K \exists K)$.

Hence, the core of the Jónsson-style canonicity is that, for order-preserving maps $f : \mathbb{A} \to \mathbb{B}$, $g : \mathbb{B} \to \mathbb{C}$ and their σ-extensions $f^\sigma : \mathbb{A}^\delta \to \mathbb{B}^\delta$ and $g^\sigma : \mathbb{B}^\delta \to \mathbb{C}^\delta$, there are two immediately available sufficient conditions on f and g which guarantee that $g^\sigma f^\sigma$ is UC:

- f^σ satisfies UC ($\forall J \exists K$) and g^σ satisfies Scott continuity ($\forall J \exists J$);
- f^σ satisfies SUC ($\forall K \exists K$) and g^σ satisfies UC ($\forall J \exists K$).

Since the σ-extensions are always UC, the first case holds when g is join-preserving in each coordinate (here g is also called an *operator*) (Corollary 3.12), and the second case holds when and f is meet-preserving (Corollary 3.14).

Hence, we can make the term t be σ-expanding by requiring the inner operations to be meet-preserving and the outer operations to be operators. In fact, this is one aspect of the idea behind "universal" and "choice": all non-universal ones are operators, and all non-choice ones are meet-preserving, therefore by forbidding choice nodes to appear in the scope of universal nodes, the compositional structure of t guarantees that the outer components are operators and the inner components are meet-preserving (note that for any negative node, the codomain of its corresponding function is $\mathbb{A}^\partial$). Now we list the relevant results without proofs.

Theorem 3.11 (cf. Theorem 5.6 in [17], with slight revision)**.** *If the BDL map $g : \mathbb{B}^\epsilon \to \mathbb{C}$ is an operator, then g^σ is Scott continuous.*

Corollary 3.12 (cf. Corollary 5.7 in [17], with slight revision)**.** *If $f : \mathbb{A} \to \mathbb{B}^\epsilon$ is any map between BDLs and $g : \mathbb{B}^\epsilon \to \mathbb{C}$ is an operator, then $g^\sigma f^\sigma \leq (gf)^\sigma$.*

Theorem 3.13 (cf. Theorem 5.8 in [17])**.** *If the BDL map $f : \mathbb{A} \to \mathbb{B}$ is meet-preserving, then f^σ is SUC.*

Corollary 3.14 (cf. Corollary 5.9 [17])**.** *If $f : \mathbb{A} \to \mathbb{B}$ is meet-preserving and $g : \mathbb{B} \to \mathbb{C}$ is order-preserving, then $g^\sigma f^\sigma \leq (gf)^\sigma$.*

Therefore, by an induction on the term structure, the following lemma can be proved for Sahlqvist terms:

Lemma 3.15 (cf. Lemma 5.10 in [17], with slight revision)**.** *Every ϵ-uniform left (resp. right) Sahlqvist term is σ-expanding (resp. π-contracting).*

3.1.3 A Proof Refinement

In this subsection we will give a refinement to the proof above, showing that the Lemma 3.3 is not needed for this proof. The strategy is to use $\mathbf{n}$ to put the uniform left Sahlqvist term α_1 and uniform right Sahlqvist term β_1 together to get $\mathbf{n}(\alpha_1, \beta_1)$, then transform $\alpha_1 \leq \beta_1 \vee \gamma$ into $\mathbf{n}(\alpha_1, \beta_1) \leq \gamma$, and show that $\mathbf{n}(\alpha_1, \beta_1)$ is σ-expanding. The proof chain becomes the following (here we only mark the parts which are different from the proof we discussed above):

$$
\begin{aligned}
& \mathbb{A} \models \alpha \leq \beta \\
\iff\quad & \alpha^{\mathbb{A}} \leq \beta^{\mathbb{A}} \\
\iff\quad & \alpha_1^{\mathbb{A}} \leq \beta_1^{\mathbb{A}} \vee \gamma^{\mathbb{A}} \\
\iff\quad & (\mathbf{n}(\alpha_1, \beta_1))^{\mathbb{A}} \leq \gamma^{\mathbb{A}} && \text{(see Lemma 3.16 below)} \\
\iff\quad & \left((\mathbf{n}(\alpha_1, \beta_1))^{\mathbb{A}}\right)^{\sigma} \leq \left(\gamma^{\mathbb{A}}\right)^{\sigma} && \text{(by the definition of the } \sigma\text{-extension)} \\
\implies\quad & (\mathbf{n}(\alpha_1, \beta_1))^{\mathbb{A}^{\delta}} \leq \left(\gamma^{\mathbb{A}}\right)^{\sigma} && \text{(see Lemma 3.18 below)} \\
\implies\quad & (\mathbf{n}(\alpha_1, \beta_1))^{\mathbb{A}^{\delta}} \leq \gamma^{\mathbb{A}^{\delta}} \\
\iff\quad & \alpha_1^{\mathbb{A}^{\delta}} \leq \beta_1^{\mathbb{A}^{\delta}} \vee \gamma^{\mathbb{A}^{\delta}} && \text{(see Lemma 3.16 below)} \\
\iff\quad & \alpha^{\mathbb{A}^{\delta}} \leq \beta^{\mathbb{A}^{\delta}} \\
\iff\quad & \mathbb{A}^{\delta} \models \alpha \leq \beta.
\end{aligned}
$$

In the following we are going to prove the additional lemmas needed in this refined proof.

Lemma 3.16 (The Second $\mathbf{n}$-Trick)**.** *For any terms $\alpha, \beta, \gamma \in \mathcal{L}^n_{term}$ such that γ is of the form $\bigvee \mathbf{n}(s,t)$, for any DMA $\mathbb{A}$, $\alpha_1^{\mathbb{A}} \leq \beta_1^{\mathbb{A}} \vee \gamma^{\mathbb{A}}$ iff $(\mathbf{n}(\alpha_1, \beta_1))^{\mathbb{A}} \leq \gamma^{\mathbb{A}}$.*

Proof. We need to show that for any assignment h,

$$[\alpha_1]^{\mathbb{A}}_h \leq [\beta_1]^{\mathbb{A}}_h \vee [\gamma]^{\mathbb{A}}_h \text{ iff } [\mathbf{n}(\alpha_1, \beta_1)]^{\mathbb{A}}_h \leq [\gamma]^{\mathbb{A}}_h.$$

Let h be any assignment. First of all, we notice that γ is of the form $\bigvee \mathbf{n}(s,t)$, therefore $[\gamma]^{\mathbb{A}}_h = \top$ or $\bot$.

$\Rightarrow$: If $[\gamma]^{\mathbb{A}}_h = \bot$, then $[\alpha_1]^{\mathbb{A}}_h \leq [\beta_1]^{\mathbb{A}}_h \vee [\gamma]^{\mathbb{A}}_h = [\beta_1]^{\mathbb{A}}_h \vee \bot = [\beta_1]^{\mathbb{A}}_h$, therefore $[\mathbf{n}(\alpha_1, \beta_1)]^{\mathbb{A}}_h = \bot \leq [\gamma]^{\mathbb{A}}_h$. If $[\gamma]^{\mathbb{A}}_h = \top$, then $[\mathbf{n}(\alpha_1, \beta_1)]^{\mathbb{A}}_h \leq \top = [\gamma]^{\mathbb{A}}_h$.

$\Leftarrow$: If $[\gamma]^{\mathbb{A}}_h = \bot$, then $[\mathbf{n}(\alpha_1, \beta_1)]^{\mathbb{A}}_h = \bot$, therefore $[\alpha_1]^{\mathbb{A}}_h \leq [\beta_1]^{\mathbb{A}}_h \leq [\beta_1]^{\mathbb{A}}_h \vee [\gamma]^{\mathbb{A}}_h$. If $[\gamma]^{\mathbb{A}}_h = \top$, then $[\alpha_1]^{\mathbb{A}}_h \leq \top = [\beta_1]^{\mathbb{A}}_h \vee \top = [\beta_1]^{\mathbb{A}}_h \vee [\gamma]^{\mathbb{A}}_h$. □

In order to prove that $\mathbf{n}(\alpha_1, \beta_1)$ is σ-expanding, we need the following lemma:

Lemma 3.17 (cf. Lemma 5.15 in [17])**.** $(\mathbf{n}^{\mathbb{A}})^{\sigma} = \mathbf{n}^{\mathbb{A}^{\delta}}$*, i.e. $\mathbf{n}$ is σ-stable.*

Then we come to the lemma stating that $\mathbf{n}(\alpha_1, \beta_1)$ is σ-expanding. In this proof, the key feature is that the $\mathbf{n}$-term is an operator.

Lemma 3.18. *If α_1 is ϵ-left Sahlqvist and β_1 is ϵ-right Sahlqvist, then $\mathbf{n}(\alpha_1, \beta_1)$ is σ-expanding.*

Proof. Since $\mathbf{n}^{\mathbb{A}} : \mathbb{A} \times \mathbb{A}^{\partial} \to \mathbb{A}$ is an operator, by taking $\alpha_1^{\mathbb{A}} : \mathbb{A}^{\epsilon} \to \mathbb{A}$ and $\beta_1^{\mathbb{A}} : (\mathbb{A}^{\epsilon})^{\partial} \to \mathbb{A}$ (therefore $(\beta_1^{\mathbb{A}})^{\partial} : \mathbb{A}^{\epsilon} \to \mathbb{A}^{\partial}$), we have that $\mathbf{n}^{\mathbb{A}}$, $\alpha_1^{\mathbb{A}}$ and $(\beta_1^{\mathbb{A}})^{\partial}$ are all order-preserving, so $(\mathbf{n}(\alpha_1, \beta_1))^{\mathbb{A}} = \mathbf{n}^{\mathbb{A}}(\alpha_1^{\mathbb{A}}, (\beta_1^{\mathbb{A}})^{\partial})$. By Corollary 3.12 and Lemma 3.17, we have $((\mathbf{n}(\alpha_1, \beta_1))^{\mathbb{A}})^{\sigma} \geq (\mathbf{n}^{\mathbb{A}})^{\sigma}(\alpha_1^{\mathbb{A}}, (\beta_1^{\mathbb{A}})^{\partial})^{\sigma} = \mathbf{n}^{\mathbb{A}^{\delta}}((\alpha_1^{\mathbb{A}})^{\sigma}, ((\beta_1^{\mathbb{A}})^{\partial})^{\sigma}) = \mathbf{n}^{\mathbb{A}^{\delta}}((\alpha_1^{\mathbb{A}})^{\sigma}, ((\beta_1^{\mathbb{A}})^{\pi})^{\partial})$. Since α_1 is ϵ-left Sahlqvist and β_1 is ϵ-right Sahlqvist, by Lemma 3.15, α_1 is σ-expanding and β_1 is π-contracting, i.e., $(\alpha_1^{\mathbb{A}})^{\sigma} \geq \alpha_1^{\mathbb{A}^{\delta}}$ and $(\beta_1^{\mathbb{A}})^{\pi} \leq \beta_1^{\mathbb{A}^{\delta}}$, i.e. $((\beta_1^{\mathbb{A}})^{\pi})^{\partial} \geq (\beta_1^{\mathbb{A}^{\delta}})^{\partial}$. So we get $((\mathbf{n}(\alpha_1, \beta_1))^{\mathbb{A}})^{\sigma} \geq \mathbf{n}^{\mathbb{A}^{\delta}}((\alpha_1^{\mathbb{A}})^{\sigma}, ((\beta_1^{\mathbb{A}})^{\pi})^{\partial}) \geq \mathbf{n}^{\mathbb{A}^{\delta}}(\alpha_1^{\mathbb{A}^{\delta}}, (\beta_1^{\mathbb{A}^{\delta}})^{\partial}) = (\mathbf{n}(\alpha_1, \beta_1))^{\mathbb{A}^{\delta}}$. Therefore $\mathbf{n}(\alpha_1, \beta_1)$ is σ-expanding. □

Remark 3.19. By this refinement, we can see that in the Jónsson-style canonicity proof of [17], the key point is to separate the σ-contracting part (i.e. the ϵ^∂-part) from the σ-expanding part (i.e. the ϵ-part). To guarantee this, we need some syntactic restrictions on these two parts. For the σ-contracting part, uniformity is enough; however, for the σ-expanding part, additional compositional restrictions on the syntactic shape of the terms are needed, namely operators as the outer skeleton and meet-preserving maps as the inner structure. However, since all uniform terms in $\mathcal{L}^n_{term}$ are σ-contracting, this signature is not the right setting to a deeper understanding of this property. Indeed, in Chapter 6, we will show that in the expanded language $\mathcal{L}^{++}$, σ-contracting terms are exactly those enjoying the so called "closed Esakia condition", as defined in Definition 6.6.

3.2 Why the n-Trick alone Fails on Inductive Inequalities, and Ideas for a Solution

In this section we discuss why the attempt in [30] to prove the canonicity of inductive inequalities using the **n**-trick fails. For simplicity of discussion, we use the original proof strategy in [17] without the second **n**-trick.

For an ϵ-inductive inequality $\alpha \leq \beta$, similar to the Sahlqvist case, we use the same minimal collapse algorithm to get to an inequality $\alpha_1 \leq \beta_1 \vee \gamma$, where α_1, γ are ϵ'-uniform and β_1 is ϵ'^∂-uniform. However, by an example in [30], we have that after executing the minimal collapse algorithm, the resulting α_1 and β_1 might not be inductive terms:

Example 3.20 (cf. Example 5.1.3 in [30])**.** For $\alpha = \Box(\lhd x \vee y)$, it is (Ω, ϵ)-left inductive for $\epsilon = (1, 1)$ and $x \leq_\Omega y$. After the minimal collapse algorithm, the resulting formula is $\alpha_1 = \Box(z \vee y)$ for order type $\epsilon' = (1, 1, 1)$. However, this is not (Ω', ϵ')-left inductive for any Ω'.

In the Jónsson-style canonicity proof which makes use of the **n**-trick, the crucial thing for the resulting α_1 is that α_1 is ϵ'-uniform and σ-expanding. Therefore, even if α_1 is not inductive, we still hope to show that it is σ-expanding. However, by our existing tools, we cannot show this.

Indeed, as we have already seen in the previous section, to prove that $g^\sigma f^\sigma \leq (gf)^\sigma$, there are two possibilities, namely Corollary 3.12 and Corollary 3.14: the first one is to make f^σ satisfy UC ($\forall J \exists K$) and g^σ satisfy Scott continuity ($\forall J \exists J$), and the second one is to make f^σ satisfy SUC ($\forall K \exists K$) and g^σ satisfy UC ($\forall J \exists K$). Therefore, for any order-preserving maps f and g, the first case holds when g is an operator, and the second case holds when f is meet-preserving.

For $\alpha = \Box(z \vee y)$, we cannot employ the first possibility, since in general $\Box$ is not an operator.

We cannot employ the second one either, since $\vee$ is not always SUC (in fact, the following example is essentially the same as Example 2.26 in [16]).

Indeed, consider a DMA $\mathbb{A}$ and its canonical extension $\mathbb{A}^\delta$, where the underlying lattice of $\mathbb{A}^\delta$ is an infinite Boolean algebra, and $K(\mathbb{A}^\delta) \cup O(\mathbb{A}^\delta) \neq \mathbb{A}^\delta$. Then consider $\mathbb{A}^\delta \times \mathbb{A}^\delta$, $(x, y) \in \mathbb{A}^\delta \times \mathbb{A}^\delta$ where x is the complement of y and $x, y \notin K(\mathbb{A}^\delta) \cup O(\mathbb{A}^\delta)$, then for $z = \top \in K(\mathbb{A}^\delta)$, we have $z \leq x \vee y$, but there is no $(u, v) \in K(\mathbb{A}^\delta \times \mathbb{A}^\delta)$ such that $z \leq u \vee v$ (i.e., $u \vee v = \top$) and $(u, v) \leq (x, y)$.

Therefore, with the tools about contracting and expanding terms we have employed up to now, just using the **n**-trick and minimal collapse algorithm is not enough to show the canonicity of inductive inequalities.

Notice that the order-theoretic properties of the basic logic connectives that we have used up to now are either completely join- or meet-preserving or reversing in each coordinate, or they are completely join- or meet-preserving or reversing in the product where they are defined as binary operations. It is well known that in the context of complete lattices (actually, perfect distributive lattices) in which we are, these properties can be equivalently stated in terms of the existence of the appropriate adjoint and residuals of each connective. Hence, the language expansion $\mathcal{L}^+$, which we have already seen in Section 2.5, will hopefully provide us with a better array of tools with which more syntactic manipulations will be possible, aimed at reaching a shape in which one of the two possibilities mentioned above, namely Corollary 3.12 and Corollary 3.14, can be applied. In fact, in some sense, using the syntactic machinery already introduced for correspondence, also for canonicity proof, is after all in line with the spirit of the **n**-trick: indeed, the **n**-trick consists in expanding the language with a new term which makes it possible to express at the level of terms more things that could be expressed before.

The next two chapters report on the ALBA-style canonicity-via-correspondence proof. Then in Chapter 6, we start building the formal tools for the result in Chapter 7, where canonicity is proven for all ALBA inequalities (i.e. those inequalities on which ALBA succeeds) in the language of distributive modal logic using a mixed method.

Chapter 4

Algebraic Ackermann Lemmas

In this chapter, we are going to build some of the tools which will be needed to prove the ALBA-aided Jónsson-style canonicity as well as the ALBA-style canonicity of the inductive inequalities. In particular, we are going to focus on the proof of the purely algebraic version of the topological Ackermann lemmas, which are the algebraic counterparts of the frame-theoretic proofs of Lemmas 9.3 and 9.4 in [10]. The proofs in this chapter are non-trivial reformulations of those in [10], for instance, the dependency order of the lemmas is different from the frame-theoretic proof. Moreover, they are an improvement in the sense that they hold in a point-free setting. We prove the existential versions of the algebraic Ackermann lemmas, both the plain version (Lemma 4.1 and 4.2) and the topological version (Lemma 4.4 and 4.5), in an algebraic way. In this chapter we are mainly working with the language $\mathcal{L}^+$.

As discussed in the introduction, a proof strategy alternative to the Jónsson-style canonicity is the so called "canonicity-via-correspondence". This strategy is based on finding a first-order sentence that the validity of an inequality is equivalent to. If this first-order condition exists, then, as discussed in the introduction, the admissible validity implies validity, which is enough to prove that an inequality is canonical. The algorithm ALBA provides a way to calculate the first-order correspondent ALBA$(\varphi \leq \psi)$ of a given inequality $\varphi \leq \psi$, and canonicity is proved at the same time, as the following U-shaped argument shows (notice that the meaning of ALBA$(\varphi \leq \psi)$ here is different from Chapter 5):

$$
\begin{array}{ccc}
\mathbb{A}^\delta \vDash_{\mathbb{A}} \varphi \leq \psi & & \mathbb{A}^\delta \vDash \varphi \leq \psi \\
\Updownarrow & & \Updownarrow \\
\mathbb{A}^\delta \vDash_{\mathbb{A}} \mathrm{ALBA}(\varphi \leq \psi) & \Leftrightarrow & \mathbb{A}^\delta \vDash \mathrm{ALBA}(\varphi \leq \psi)
\end{array}
$$

The Ackermann lemmas are the crucial steps in the ALBA-based correspondence proof, which is represented in the right-hand side of the diagram above. The topological version of the Ackermann lemmas is the crucial step of the left-hand equivalence, and the plain version of the Ackermann lemmas is the crucial step of the right-hand equivalence. Below we state the plain version, in the sense that no topology is involved.

Lemma 4.1 (Plain Right-handed Ackermann Lemma). *Let α be such that $p \notin \mathsf{Prop}(\alpha)$, $\beta_1(p), \ldots, \beta_n(p)$ be positive in p and $\gamma_1(p), \ldots, \gamma_n(p)$ negative in p, and let $\vec{q}, \vec{j}, \vec{m}$ be all the proposition variables,* *nominals, co-nominals, respectively, occurring in* $\alpha, \beta_1(p), \ldots, \beta_n(p)$, $\gamma_1(p), \ldots, \gamma_n(p)$ *other than* p. *Then for all* $\vec{u} \in \mathbb{A}^\delta, \vec{x} \in J^\infty(\mathbb{A}^\delta), \vec{y} \in M^\infty(\mathbb{A}^\delta)$, *the following are equivalent:*

1. $\beta_i^{\mathbb{A}^\delta}(\vec{u}, \vec{x}, \vec{y}, \alpha^{\mathbb{A}^\delta}(\vec{u}, \vec{x}, \vec{y})) \leq \gamma_i^{\mathbb{A}^\delta}(\vec{u}, \vec{x}, \vec{y}, \alpha^{\mathbb{A}^\delta}(\vec{u}, \vec{x}, \vec{y}))$ *for* $1 \leq i \leq n$;

2. *There exists* $u_0 \in \mathbb{A}^\delta$ *such that* $\alpha^{\mathbb{A}^\delta}(\vec{u}, \vec{x}, \vec{y}) \leq u_0$ *and* $\beta_i^{\mathbb{A}^\delta}(\vec{u}, \vec{x}, \vec{y}, u_0) \leq \gamma_i^{\mathbb{A}^\delta}(\vec{u}, \vec{x}, \vec{y}, u_0)$ *for* $1 \leq i \leq n$.

Lemma 4.2 (Plain Left-handed Ackermann Lemma). *Let* α *be such that* $p \notin \mathsf{Prop}(\alpha)$, $\beta_1(p), \ldots, \beta_n(p)$ *be negative in* p *and* $\gamma_1(p), \ldots, \gamma_n(p)$ *positive in* p, *and let* $\vec{q}, \vec{j}, \vec{m}$ *be all the proposition variables, nominals, co-nominals, respectively, occurring in* $\alpha, \beta_1(p), \ldots, \beta_n(p)$, $\gamma_1(p), \ldots, \gamma_n(p)$ *other than* p. *Then for all* $\vec{u} \in \mathbb{A}^\delta, \vec{x} \in J^\infty(\mathbb{A}^\delta), \vec{y} \in M^\infty(\mathbb{A}^\delta)$, *the following are equivalent:*

1. $\beta_i^{\mathbb{A}^\delta}(\vec{u}, \vec{x}, \vec{y}, \alpha^{\mathbb{A}^\delta}(\vec{u}, \vec{x}, \vec{y})) \leq \gamma_i^{\mathbb{A}^\delta}(\vec{u}, \vec{x}, \vec{y}, \alpha^{\mathbb{A}^\delta}(\vec{u}, \vec{x}, \vec{y}))$ *for* $1 \leq i \leq n$;

2. *There exists* $u_0 \in \mathbb{A}^\delta$ *such that* $u_0 \leq \alpha^{\mathbb{A}^\delta}(\vec{u}, \vec{x}, \vec{y})$ *and* $\beta_i^{\mathbb{A}^\delta}(\vec{u}, \vec{x}, \vec{y}, u_0) \leq \gamma_i^{\mathbb{A}^\delta}(\vec{u}, \vec{x}, \vec{y}, u_0)$ *for* $1 \leq i \leq n$.

Proof. The two lemmas are order variants, so we only discuss the right-handed version here. The direction from top to bottom holds by taking u_0 to be $\alpha^{\mathbb{A}^\delta}(\vec{u}, \vec{x}, \vec{y})$. The other direction holds by the monotonicity of $\beta_i^{\mathbb{A}^\delta}(p)$ and the antitonicity of $\gamma_i^{\mathbb{A}^\delta}(p)$ in p. $\square$

However, if we try to adapt these lemmas to the "admissible" case, i.e. the case in which the valuations for all proposition variables and u_0 are restricted to $\mathbb{A}$, then there would be some problem with u_0. Indeed, the element u_0 we take needs to be in $\mathbb{A}$, but taking $u_0 = \alpha^{\mathbb{A}^\delta}(\vec{u}, \vec{x}, \vec{y})$ might not be an admissible choice, since α is a term in the expanded language $\mathcal{L}^+$, therefore it might contain nominals, co-nominals and expanded operators under which the subalgebra $\mathbb{A}$ might not be closed, therefore the value of α might be out of the subalgebra $\mathbb{A}$.

Therefore, the topological versions of the Ackermann lemmas (Lemma 4.4 and Lemma 4.5) are needed, which will depend on the syntactic shape of the formulas in such a way that compactness can be applied to find a suitable admissible element $u_0 \in \mathbb{A}$ (in the following we are going to use a_0 to denote this element).

In order to prove the new topological version of Ackermann lemmas, we first introduce the following definitions (in fact, this very definition applies as it stands to $\mathcal{L}^{++}$; however, in this chapter we are only interested in $\mathcal{L}^+$):

Definition 4.3 (Syntactically Closed and Open Formulas).

1. A formula in $\mathcal{L}^+$ is *syntactically closed* if all occurences of nominals, $\blacktriangleleft, \blacklozenge, -$ are positive, and all occurences of co-nominals, $\blacktriangleright, \blacksquare, \rightarrow$ are negative;

2. A formula in $\mathcal{L}^+$ is *syntactically open* if all occurences of nominals, $\blacktriangleleft, \blacklozenge, -$ are negative, and all occurences of co-nominals, $\blacktriangleright, \blacksquare, \rightarrow$ are positive.

The intuition behind these definitions is that the value of a syntactically open (resp. closed) formula under an admissible assignment is always an open (resp. closed) element in $\mathbb{A}^\delta$, i.e., in $O(\mathbb{A}^\delta)$ (resp. $K(\mathbb{A}^\delta)$), therefore we can apply compactness to obtain an admissible element a_0. In Section 7.1, the syntactic closeness and openness will play an important role in proving the intersection lemmas.

The proofs of the topological Ackermann lemmas given below are improved by being purely algebraic and point-free:

Lemma 4.4 (Right-handed Topological Ackermann Lemma). *Let α be syntactically closed, $p \notin \mathsf{Prop}(\alpha)$, let $\beta_1(p), \ldots, \beta_n(p)$ be syntactically closed and positive in p, and let $\gamma_1(p), \ldots, \gamma_n(p)$ be syntactically open and negative in p, and let $\vec{q}, \vec{j}, \vec{m}$ be all the proposition variables, nominals, co-nominals, respectively, occurring in $\alpha, \beta_1(p), \ldots, \beta_n(p)$, $\gamma_1(p), \ldots, \gamma_n(p)$ other than p. Then for all $\vec{a} \in \mathbb{A}, \vec{x} \in J^\infty(\mathbb{A}^\delta), \vec{y} \in M^\infty(\mathbb{A}^\delta)$, the following are equivalent:*

1. $\beta_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, \alpha^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y})) \leq \gamma_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, \alpha^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}))$ *for* $1 \leq i \leq n$;
2. *There exists* $a_0 \in \mathbb{A}$ *such that* $\alpha^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}) \leq a_0$ *and* $\beta_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, a_0) \leq \gamma_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, a_0)$ *for* $1 \leq i \leq n$.

Lemma 4.5 (Left-handed Topological Ackermann Lemma). *Let α be syntactically open, $p \notin \mathsf{Prop}(\alpha)$, let $\beta_1(p), \ldots, \beta_n(p)$ be syntactically closed and negative in p, and let $\gamma_1(p), \ldots, \gamma_n(p)$ be syntactically open and positive in p, and let $\vec{q}, \vec{j}, \vec{m}$ be all the proposition variables, nominals, co-nominals, respectively, occurring in $\alpha, \beta_1(p), \ldots, \beta_n(p)$, $\gamma_1(p), \ldots, \gamma_n(p)$ other than p. Then for all $\vec{a} \in \mathbb{A}, \vec{x} \in J^\infty(\mathbb{A}^\delta), \vec{y} \in M^\infty(\mathbb{A}^\delta)$, the following are equivalent:*

1. $\beta_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, \alpha^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y})) \leq \gamma_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, \alpha^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}))$ *for* $1 \leq i \leq n$;
2. *There exists* $a_0 \in \mathbb{A}$ *such that* $a_0 \leq \alpha^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y})$ *and* $\beta_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, a_0) \leq \gamma_i^{\mathbb{A}^\delta}(\vec{a}, \vec{x}, \vec{y}, a_0)$ *for* $1 \leq i \leq n$.

To prove the topological Ackermann lemmas, we need some properties of the modalities in the expanded language, which will be proved in the following. We can notice that in the algebraic proof, the dependence order of the following lemmas is different from the order of those in the frame-theoretic proof in [10]. Notice also that no complete join-irreducibles or complete meet-irreducibles are substantially used in the proofs, which means that the statements hold in a point-free setting as well.

The following statement immediately follows from the definition of the closed and open elements of $\mathbb{A}^\delta$, as well as the fact that $\Box^{\mathbb{A}^\delta} = (\Box^{\mathbb{A}})^\sigma, \Diamond^{\mathbb{A}^\delta} = (\Diamond^{\mathbb{A}})^\pi, \rhd^{\mathbb{A}^\delta} = (\rhd^{\mathbb{A}})^\sigma, \lhd^{\mathbb{A}^\delta} = (\lhd^{\mathbb{A}})^\pi$.

Lemma 4.6. *For all $c \in K(\mathbb{A}^\delta)$ and $o \in O(\mathbb{A}^\delta)$,*

1. $\Box c \in K(\mathbb{A}^\delta)$;
2. $\Diamond o \in O(\mathbb{A}^\delta)$;
3. $\rhd o \in K(\mathbb{A}^\delta)$;

4. $\lhd c \in O(\mathbb{A}^\delta)$.

Lemma 4.7. *For all $c \in K(\mathbb{A}^\delta)$ and $o \in O(\mathbb{A}^\delta)$,*

1. $\blacklozenge c \in K(\mathbb{A}^\delta)$;
2. $\blacksquare o \in O(\mathbb{A}^\delta)$;
3. $\blacktriangleleft o \in K(\mathbb{A}^\delta)$;
4. $\blacktriangleright c \in O(\mathbb{A}^\delta)$;
5. $c - o \in K(\mathbb{A}^\delta)$;
6. $c \to o \in O(\mathbb{A}^\delta)$.

Proof. 1. By denseness, $\blacklozenge c = \bigwedge\{o \in O(\mathbb{A}^\delta) : \blacklozenge c \leq o\}$. Let $Y = \{o \in O(\mathbb{A}^\delta) : \blacklozenge c \leq o\}$ and $X = \{a \in \mathbb{A} : \blacklozenge c \leq a\}$. To show that $\blacklozenge c \in K(\mathbb{A}^\delta)$, it is enough to show that $\bigwedge X = \bigwedge Y$. Since clopens are opens, $X \subseteq Y$, so $\bigwedge Y \leq \bigwedge X$. In order to show that $\bigwedge X \leq \bigwedge Y$, it suffices to show that for every $o \in Y$ there exists some $a \in X$ such that $a \leq o$. Let $o \in Y$, i.e., $\blacklozenge c \leq o$. By adjunction, $c \leq \Box o$. Since $c \in K(\mathbb{A}^\delta)$, and $\Box o = \Box^\pi o = \bigvee\{\Box a : a \in \mathbb{A}$ and $a \leq o\}$, and $\Box a \in \mathbb{A} \subseteq O(\mathbb{A}^\delta)$, we may apply compactness and get that $c \leq \Box a_1 \vee \cdots \vee \Box a_n$ for some $a_1, \ldots, a_n \in \mathbb{A}$ s.t. $a_1, \ldots, a_n \leq o$. Let $a = a_1 \vee \cdots \vee a_n \leq o$. By the monotonicity of $\Box$ we have $c \leq \Box a_1 \vee \cdots \vee \Box a_n \leq \Box a$, and hence $\blacklozenge c \leq a$. 2. 3. and 4. are order-variants of 1.

5. By denseness, $c - o = \bigwedge\{o' \in O(\mathbb{A}^\delta) : c - o \leq o'\}$. Let $Y = \{o' \in O(\mathbb{A}^\delta) : c - o \leq o'\}$ and $X = \{a \in \mathbb{A} : c - o \leq a\}$. To show that $c - o \in K(\mathbb{A}^\delta)$, it is enough to show that $\bigwedge X = \bigwedge Y$. Since clopens are opens, $X \subseteq Y$, so $\bigwedge Y \leq \bigwedge X$. In order to show that $\bigwedge X \leq \bigwedge Y$, it suffices to show that for every $o' \in Y$ there exists some $a \in X$ such that $a \leq o'$. Consider any $o' \in Y$, then $c - o \leq o'$, since $c \in K(\mathbb{A}^\delta)$ and $o, o' \in O(\mathbb{A}^\delta)$, there are $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \subseteq \mathbb{A}$ such that $c = \bigwedge \mathcal{A}_1, o = \bigvee \mathcal{A}_2$ and $o' = \bigvee \mathcal{A}_3$, then by adjunction we have $\bigwedge \mathcal{A}_1 \leq \bigvee \mathcal{A}_2 \vee \bigvee \mathcal{A}_3$, then by compactness we have that there is a finite $\mathcal{A}'_3 \subseteq \mathcal{A}_3$ such that $\bigwedge \mathcal{A}_1 \leq \bigvee \mathcal{A}_2 \vee \bigvee \mathcal{A}'_3$, so by adjunction again we have $\bigwedge \mathcal{A}_1 - \bigvee \mathcal{A}_2 \leq \bigvee \mathcal{A}'_3$, i.e. $c - o \leq \bigvee \mathcal{A}'_3$. Take $a = \bigvee \mathcal{A}'_3$, then $a \in \mathbb{A}, c - o \leq a$, so $a \in X$ and also $a = \bigvee \mathcal{A}'_3 \leq \bigvee \mathcal{A}_3 = o'$. 6. is order-variant of 5. $\Box$

The following kind of lemma is first proved in [13], and further used in [23].

Lemma 4.8 (Esakia Lemmas)**.** *Let $\mathcal{C} = \{c_i : i \in I\} \subseteq K(\mathbb{A}^\delta)$ be downward-directed, $\mathcal{O} = \{o_i : i \in I\} \subseteq O(\mathbb{A}^\delta)$ be upward-directed, then*

1. $\Box \bigvee \mathcal{O} = \bigvee\{\Box o : o \in \mathcal{O}\}$;
2. $\Diamond \bigwedge \mathcal{C} = \bigwedge\{\Diamond c : c \in \mathcal{C}\}$;
3. $\rhd \bigwedge \mathcal{C} = \bigvee\{\rhd c : c \in \mathcal{C}\}$;
4. $\lhd \bigvee \mathcal{O} = \bigwedge\{\lhd o : o \in \mathcal{O}\}$.

Proof. 1. By denseness, $\Box\bigvee\mathcal{O} = \bigvee\{c \in K(\mathbb{A}^\delta) : c \leq \Box\bigvee\mathcal{O}\}$. Let $Y = \{c \in K(\mathbb{A}^\delta) : c \leq \Box\bigvee\mathcal{O}\}$ and $X = \{\Box o : o \in \mathcal{O}\}$, then it suffices to show that $\bigvee Y = \bigvee X$. $\bigvee X \leq \Box\bigvee\mathcal{O} = \bigvee Y$ immediately follows from the monotonicity of $\Box$. For the other direction it suffices to show that for every $c \in Y$ there exists some $o \in \mathcal{O}$ such that $c \leq \Box o$. Fix $c \in Y$, i.e., assume $c \leq \Box\bigvee\mathcal{O}$, by adjunction, we get $\blacklozenge c \leq \bigvee\mathcal{O}$. Then by Lemma 4.7, $\blacklozenge c \in K(\mathbb{A}^\delta)$, by compactness there is a finite $\mathcal{O}_0 \subseteq \mathcal{O}$ such that $\blacklozenge c \leq \bigvee\mathcal{O}_0$, therefore by upward-directedness of $\mathcal{O}$, there is an $o \in \mathcal{O}$ such that $\bigvee\mathcal{O}_0 \leq o$, therefore $\blacklozenge c \leq o$, again by adjunction we have $c \leq \Box o$. 2. 3. and 4. are order-variants of 1. $\Box$

As an immediate consequence of the previous lemma, since any singleton is both upward-directed and downward-directed, we can get the following corollary:

Corollary 4.9. *For all $c \in K(\mathbb{A}^\delta)$ and $o \in O(\mathbb{A}^\delta)$,*

1. $\Box o \in O(\mathbb{A}^\delta)$;
2. $\Diamond c \in K(\mathbb{A}^\delta)$;
3. $\rhd c \in O(\mathbb{A}^\delta)$;
4. $\lhd o \in K(\mathbb{A}^\delta)$.

Notice that in the proof of Lemma 4.8, the fact that the operations $\Box^{\mathbb{A}^\delta}, \Diamond^{\mathbb{A}^\delta}, \rhd^{\mathbb{A}^\delta}, \lhd^{\mathbb{A}^\delta}$ are σ- or π-extensions of the corresponding operations in $\mathbb{A}$ plays no rule, hence the same proof applies also to $\blacksquare, \blacklozenge, \blacktriangleright, \blacktriangleleft$, respectively, and yields the following Esakia lemmas for the new operations:

Lemma 4.10 (Esakia Lemmas)**.** *Let $\mathcal{C} = \{c_i : i \in I\} \subseteq K(\mathbb{A}^\delta)$ be downward-directed, $\mathcal{O} = \{o_i : i \in I\} \subseteq O(\mathbb{A}^\delta)$ be upward-directed, then*

1. $\blacksquare\bigvee\mathcal{O} = \bigvee\{\blacksquare o : o \in \mathcal{O}\}$;
2. $\blacklozenge\bigwedge\mathcal{C} = \bigwedge\{\blacklozenge c : c \in \mathcal{C}\}$;
3. $\blacktriangleright\bigwedge\mathcal{C} = \bigvee\{\blacktriangleright c : c \in \mathcal{C}\}$;
4. $\blacktriangleleft\bigvee\mathcal{O} = \bigwedge\{\blacktriangleleft o : o \in \mathcal{O}\}$;
5. $\bigwedge\mathcal{C} - \bigvee\mathcal{O} = \bigwedge\{c - o : c \in \mathcal{C}, o \in \mathcal{O}\}$;
6. $\bigwedge\mathcal{C} \rightarrow \bigvee\mathcal{O} = \bigvee\{c \rightarrow o : c \in \mathcal{C}, o \in \mathcal{O}\}$.

Proof. 5. By denseness, $\bigwedge\mathcal{C} - \bigvee\mathcal{O} = \bigwedge\{o' \in O(\mathbb{A}^\delta) : \bigwedge\mathcal{C} - \bigvee\mathcal{O} \leq o'\}$. Let $Y = \{o' \in O(\mathbb{A}^\delta) : \bigwedge\mathcal{C} - \bigvee\mathcal{O} \leq o'\}$ and $X = \{c - o : c \in \mathcal{C}, o \in \mathcal{O}\}$. To show that $\bigwedge\mathcal{C} - \bigvee\mathcal{O} = \bigwedge\{c - o : c \in \mathcal{C}, o \in \mathcal{O}\}$, it is enough to show that $\bigwedge X = \bigwedge Y$. Since $-$ is monotone in the first coordinate and antitone in the second coordinate, we have $\bigwedge Y \leq c - o$ for all $c - o \in X$, so $\bigwedge Y \leq \bigwedge X$. In order to show that $\bigwedge X \leq \bigwedge Y$, it suffices to show that for every $o' \in Y$ there exists some $c - o \in X$ such that $c - o \leq o'$. Consider any $o' \in Y$, then $\bigwedge\mathcal{C} - \bigvee\mathcal{O} \leq o'$, by adjunction we have $\bigwedge\mathcal{C} \leq \bigvee\mathcal{O} \vee o'$, since $\mathcal{C} \subseteq K(\mathbb{A}^\delta)$, $\mathcal{O} \subseteq O(\mathbb{A}^\delta)$ and $o' \in O(\mathbb{A}^\delta)$, by compactness, there are finite $\mathcal{C}' \subseteq \mathcal{C}$ and $\mathcal{O}' \subseteq \mathcal{O}$ such that $\bigwedge\mathcal{C}' \leq \bigvee\mathcal{O}' \vee o'$, then by downward-directedness of $\mathcal{C}$ and upward-directedness of $\mathcal{O}$ we have that there are $c \in \mathcal{C}$ and $o \in \mathcal{O}$ such that $c \leq \bigwedge\mathcal{C}'$ and $\bigvee\mathcal{O}' \leq o$, so $c \leq \bigwedge\mathcal{C}' \leq \bigvee\mathcal{O}' \vee o' \leq o \vee o'$, so by adjunction again we have $c - o \leq o'$, and also $c - o \in X$. 6. is order-variant of 5. $\Box$

Lemma 4.11. *Let $\varphi(p)$ be syntactically closed and $\psi(p)$ syntactically open, and let $\vec{p}, \vec{j}, \vec{m}$ be all the proposition variables, nominals, co-nominals, respectively, occuring in $\varphi(p)$ and $\psi(p)$ other than p, $\vec{a} \in \mathbb{A}, \vec{x} \in J^{\infty}(\mathbb{A}^{\delta}), \vec{y} \in M^{\infty}(\mathbb{A}^{\delta})$, $c \in K(\mathbb{A}^{\delta}), o \in O(\mathbb{A}^{\delta})$, then*

1. *If $\varphi(p)$ is positive in p, then $\varphi^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}, c) \in K(\mathbb{A}^{\delta})$;*
2. *If $\psi(p)$ is negative in p, then $\psi^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}, c) \in O(\mathbb{A}^{\delta})$;*
3. *If $\varphi(p)$ is negative in p, then $\varphi^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}, o) \in K(\mathbb{A}^{\delta})$;*
4. *If $\psi(p)$ is positive in p, then $\psi^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}, o) \in O(\mathbb{A}^{\delta})$.*

Proof. By simultaneous induction on φ and ψ, using Lemma 4.6, 4.7 and 4.9. □

Lemma 4.12 (Intersection lemma)**.** *Let $\varphi(p)$ be syntactically closed, $\psi(p)$ be syntactically open, and let $\vec{p}, \vec{j}, \vec{m}$ be all the proposition variables, nominals, co-nominals, respectively, occuring in $\varphi(p)$ and $\psi(p)$ other than p, $\vec{a} \in \mathbb{A}, \vec{x} \in J^{\infty}(\mathbb{A}^{\delta}), \vec{y} \in M^{\infty}(\mathbb{A}^{\delta})$, $\{c_i : i \in I\} \subseteq K(\mathbb{A}^{\delta})$ be downward-directed, $\{o_i : i \in I\} \subseteq O(\mathbb{A}^{\delta})$ be upward-directed, then*

1. *If $\varphi(p)$ is positive in p, then $\varphi^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}, \bigwedge\{c_i : i \in I\}) = \bigwedge\{\varphi^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}, c_i) : i \in I\}$;*
2. *If $\psi(p)$ is negative in p, then $\psi^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}, \bigwedge\{c_i : i \in I\}) = \bigvee\{\psi^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}, c_i) : i \in I\}$;*
3. *If $\varphi(p)$ is negative in p, then $\varphi^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}, \bigvee\{o_i : i \in I\}) = \bigwedge\{\varphi^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}, o_i) : i \in I\}$;*
4. *If $\psi(p)$ is positive in p, then $\psi^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}, \bigvee\{o_i : i \in I\}) = \bigvee\{\psi^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}, o_i) : i \in I\}$.*

Proof. We prove by simultaneous induction on φ and ψ. It is easy to see that φ cannot be m and the outmost connective of φ cannot be $\blacksquare, \blacktriangleright, \rightarrow$, and similarly, ψ cannot be i and the outmost connective of ψ cannot be $\blacklozenge, \blacktriangleleft, -$. In the proof we omit $\vec{a}, \vec{x}, \vec{y}$ and the superscript $\mathbb{A}^{\delta}$, just write $\varphi(a)$ and $\psi(a)$ where p is assigned a.

1. For the basic cases where $\varphi = \bot, \top, p, q, i$ and $\psi = \bot, \top, p, q, m$, easy.
2. For the case where $\varphi(p) = \varphi_1(p) \wedge \varphi_2(p)$,

   If $\varphi(p)$ is positive in p, then $\varphi_1(p)$ and $\varphi_2(p)$ are syntactically closed and positive in p, by induction hypothesis, $\varphi_1(\bigwedge\{c_i : i \in I\}) = \bigwedge\{\varphi_1(c_i) : i \in I\}$ and $\varphi_2(\bigwedge\{c_i : i \in I\}) = \bigwedge\{\varphi_2(c_i) : i \in I\}$, so $\varphi(\bigwedge\{c_i : i \in I\}) = \varphi_1(\bigwedge\{c_i : i \in I\}) \wedge \varphi_2(\bigwedge\{c_i : i \in I\}) = \bigwedge\{\varphi_1(c_i) : i \in I\} \wedge \bigwedge\{\varphi_2(c_i) : i \in I\} = \bigwedge\{\varphi_1(c_i) \wedge \varphi_2(c_i) : i \in I\} = \bigwedge\{\varphi(c_i) : i \in I\}$. The negative case is order-variant of the positive case.

   For the case where $\psi(p) = \psi_1(p) \vee \psi_2(p)$, similar to $\varphi(p) = \varphi_1(p) \wedge \varphi_2(p)$ case.
3. For the case where $\varphi(p) = \varphi_1(p) \vee \varphi_2(p)$,

   If $\varphi(p)$ is positive in p, then $\varphi_1(p)$ and $\varphi_2(p)$ are syntactically closed and positive in p, by induction hypothesis, $\varphi_1(\bigwedge\{c_i : i \in I\}) = \bigwedge\{\varphi_1(c_i) : i \in I\}$ and $\varphi_2(\bigwedge\{c_i : i \in I\}) = \bigwedge\{\varphi_2(c_i) : i \in I\}$, so $\varphi(\bigwedge\{c_i : i \in I\}) = \varphi_1(\bigwedge\{c_i : i \in I\}) \vee \varphi_2(\bigwedge\{c_i : i \in I\}) = \bigwedge\{\varphi_1(c_i) : i \in I\} \vee \bigwedge\{\varphi_2(c_i) : i \in I\} = \bigwedge\{\varphi_1(c_i) \vee \varphi_2(c_{i'}) : i, i' \in I\}$, while $\bigwedge\{\varphi(c_i) : i \in I\} = \bigwedge\{\varphi_1(c_i) \vee \varphi_2(c_i) : i \in I\}$, so it suffices to show that $\bigwedge\{\varphi_1(c_i) \vee \varphi_2(c_{i'}) : i, i' \in I\} =$

$\bigwedge\{\varphi_1(c_i) \vee \varphi_2(c_i) : i \in I\}$, i.e., $\bigwedge\{\varphi_1(c_i) \vee \varphi_2(c_{i'}) : i, i' \in I\} \leq \bigwedge\{\varphi_1(c_i) \vee \varphi_2(c_i) : i \in I\}$ and $\bigwedge\{\varphi_1(c_i) \vee \varphi_2(c_i) : i \in I\} \leq \bigwedge\{\varphi_1(c_i) \vee \varphi_2(c_{i'}) : i, i' \in I\}$. The first inequation is easy, and for the second inequation, it suffices to show that for all $i, i' \in I$ there is an $i'' \in I$ such that $\varphi_1(c_{i''}) \vee \varphi_2(c_{i''}) \leq \varphi_1(c_i) \vee \varphi_2(c_{i'})$. Now fix $i, i' \in I$, then by the downward-directedness of $\{c_i : i \in I\}$ we have that there is an $i'' \in I$ such that $c_{i''} \leq c_i$ and $c_{i''} \leq c_{i'}$, then by the monotonicity of $\varphi_1(p)$ and $\varphi_2(p)$ we have that $\varphi_1(c_{i''}) \leq \varphi_1(c_i)$ and $\varphi_2(c_{i''}) \leq \varphi_2(c_{i'})$, therefore $\varphi_1(c_{i''}) \vee \varphi_2(c_{i''}) \leq \varphi_1(c_i) \vee \varphi_2(c_{i'})$. The negative case is order-variant of the positive case.

For the case where $\psi(p) = \psi_1(p) \wedge \psi_2(p)$, similar to $\varphi(p) = \varphi_1(p) \vee \varphi_2(p)$ case.

4. For the case where $\varphi(p) = \Box\varphi_1(p)$,

   If $\varphi(p)$ is positive in p, then $\varphi_1(p)$ is syntactically closed and positive in p, by induction hypothesis, $\varphi_1(\bigwedge\{c_i : i \in I\}) = \bigwedge\{\varphi_1(c_i) : i \in I\}$, therefore by the complete distributivity of $\Box$ and $\wedge$ we have that $\varphi(\bigwedge\{c_i : i \in I\}) = \Box\varphi_1(\bigwedge\{c_i : i \in I\}) = \Box \bigwedge\{\varphi_1(c_i) : i \in I\} = \bigwedge\{\Box\varphi_1(c_i) : i \in I\} = \bigwedge\{\varphi(c_i) : i \in I\}$. The negative case is order-variant of the positive case.

   For the cases where $\varphi(p) = \rhd\varphi_1(p)$, $\psi(p) = \Diamond\psi_1(p), \lhd\psi_1(p)$, similar to $\varphi(p) = \Box\varphi_1(p)$ case.

5. For the case where $\varphi(p) = \Diamond\varphi_1(p)$,

   If $\varphi(p)$ is positive in p, then $\varphi_1(p)$ is syntactically closed and positive in p, by induction hypothesis, $\varphi_1(\bigwedge\{c_i : i \in I\}) = \bigwedge\{\varphi_1(c_i) : i \in I\}$. We define $\mathcal{C} = \{\varphi_1(c_i) : i \in I\}$, then by Lemma 4.11, $\mathcal{C} \subseteq K(\mathbb{A}^\delta)$, while for any $\varphi_1(c_i), \varphi_1(c_{i'}) \in \mathcal{C}$, we have that $i, i' \in I$, by the downward-directedness of $\{c_i : i \in I\}$ we have that there is an $i'' \in I$ such that $c_{i''} \leq c_i \wedge c_{i'}$, since $\varphi_1(p)$ is positive in p, we have that $\varphi_1(c_{i''}) \leq \varphi_1(c_i \wedge c_{i'}) \leq \varphi_1(c_i)$ and similarly $\varphi_1(c_{i''}) \leq \varphi_1(c_{i'})$, so there is an $i'' \in I$ such that $\varphi_1(c_{i''}) \leq \varphi_1(c_i) \wedge \varphi_1(c_{i'})$, therefore $\mathcal{C}$ is downward-directed. By Lemma 4.8, we have that $\varphi(\bigwedge\{c_i : i \in I\}) = \Diamond\varphi_1(\bigwedge\{c_i : i \in I\}) = \Diamond \bigwedge\{\varphi_1(c_i) : i \in I\} = \bigwedge\{\Diamond\varphi_1(c_i) : i \in I\} = \bigwedge\{\varphi(c_i) : i \in I\}$. The negative case is order-variant of the positive case.

   For the cases where $\varphi(p) = \lhd\varphi_1(p), \blacklozenge\varphi_1(p), \blacktriangleleft \varphi_1(p)$, $\psi(p) = \Box\psi_1(p), \rhd\psi_1(p), \blacksquare\psi_1(p), \blacktriangleright \psi_1(p)$, similar to $\varphi(p) = \Diamond\varphi_1(p)$ case.

6. For the case where $\varphi(p) = \varphi_1(p) - \varphi_2(p)$,

   If $\varphi(p)$ is positive in p, then $\varphi_1(p)$ is syntactically closed and positive in p, while $\varphi_2(p)$ is open and negative in p. By induction hypothesis, $\varphi_1(\bigwedge\{c_i : i \in I\}) = \bigwedge\{\varphi_1(c_i) : i \in I\}$ and $\varphi_2(\bigwedge\{c_i : i \in I\}) = \bigvee\{\varphi_2(c_i) : i \in I\}$. Therefore, $\varphi(\bigwedge\{c_i : i \in I\}) = \varphi_1(\bigwedge\{c_i : i \in I\}) - \varphi_2(\bigwedge\{c_i : i \in I\}) = \bigwedge\{\varphi_1(c_i) : i \in I\} - \bigvee\{\varphi_2(c_i) : i \in I\}$, and $\bigwedge\{\varphi(c_i) : i \in I\} = \bigwedge\{\varphi_1(c_i) - \varphi_2(c_i) : i \in I\}$. By an argument similar to the $\varphi(p) = \Diamond\varphi_1(p)$ case, we have that $\mathcal{C} = \{\varphi_1(c_i) : i \in I\}$ is downward-directed and $\mathcal{O} = \{\varphi_2(c_i) : i \in I\}$ is upward-directed, so by Lemma 4.10, we have that $\varphi(\bigwedge\{c_i : i \in I\}) = \bigwedge\{\varphi_1(c_i) : i \in I\} - \bigvee\{\varphi_2(c_i) : i \in I\} = \bigwedge\{\varphi_1(c_i) - \varphi_2(c_{i'}) : i, i' \in I\}$, so to show that $\varphi(\bigwedge\{c_i : i \in I\}) = \bigwedge\{\varphi(c_i) : i \in I\}$, it suffices to show that $\bigwedge\{\varphi_1(c_i) - \varphi_2(c_{i'}) : i, i' \in I\} = \bigwedge\{\varphi_1(c_i) - \varphi_2(c_i) : i \in I\}$. For this step, the proof is similar to the $\varphi(p) = \varphi_1(p) \vee \varphi_2(p)$ case. The negative case is order-variant of the positive case.

   For the case where $\psi(p) = \psi_1(p) \rightarrow \psi_2(p)$, similar to $\varphi(p) = \varphi_1(p) - \varphi_2(p)$ case.

$\square$

Corollary 4.13. *Let φ be syntactically closed and ψ syntactically open, and let $\vec{p}, \vec{j}, \vec{m}$ be all the proposition variables, nominals, co-nominals, respectively, occuring in φ and ψ, $\vec{a} \in \mathbb{A}, \vec{x} \in J^{\infty}(\mathbb{A}^{\delta}), \vec{y} \in M^{\infty}(\mathbb{A}^{\delta})$, then*

1. $\varphi^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}) \in K(\mathbb{A}^{\delta})$;
2. $\psi^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}) \in O(\mathbb{A}^{\delta})$.

Proof of Topological Ackermann Lemma. We prove the right-handed topological Ackermann Lemma, and the left-handed topological Ackermann Lemma is similar.

$\Leftarrow$: By the monotonicity of $\beta_i(p)$ and antitonicity of $\gamma_i(p)$ in p for $1 \leq i \leq n$, together with $\alpha^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}) \leq a_0$ we have that $\beta_i^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}, \alpha^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y})) \leq \beta_i^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}, a_0) \leq \gamma_i^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}, a_0) \leq \gamma_i^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}, \alpha^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}))$.

$\Rightarrow$: Suppose that $\beta_i^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}, \alpha^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y})) \leq \gamma_i^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}, \alpha^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}))$ for $1 \leq i \leq n$, then by Corollary 4.13, $\alpha^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}) \in K(\mathbb{A}^{\delta})$, therefore by the definition of $K(\mathbb{A}^{\delta})$ we have that $\alpha^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}) = \bigwedge \mathcal{U}$, where $\mathcal{U} = \{u \in \mathbb{A} : \alpha^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}) \leq u\}$, making it the meet of a downward-directed set of clopen elements. Therefore, for all $1 \leq i \leq n$, $\beta_i^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}, \bigwedge \mathcal{U}) \leq \gamma_i^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}, \bigwedge \mathcal{U})$. Since β_i is syntactically closed and positive in p and γ_i is syntactically open and negative in p, by Lemma 4.12 we have that $\bigwedge\{\beta_i^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}, u) : u \in \mathcal{U}\} \leq \bigvee\{\gamma_i^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}, u) : u \in \mathcal{U}\}$ for $1 \leq i \leq n$. Again by Corollary 4.13, $\beta_i^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}, u) \in K(\mathbb{A}^{\delta})$ and $\gamma_i^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}, u) \in O(\mathbb{A}^{\delta})$ for $1 \leq i \leq n$, therefore by compactness, there is a finite set $\mathcal{U}_0 \subseteq \mathcal{U}$ such that $\bigwedge\{\beta_i^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}, u) : u \in \mathcal{U}_0\} \leq \bigvee\{\gamma_i^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}, u) : u \in \mathcal{U}_0\}$ for $1 \leq i \leq n$. Then take $a_0 = \bigwedge \mathcal{U}_0 \in \mathbb{A}$, we have that $\alpha^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}) = \bigwedge \mathcal{U} \leq \bigwedge \mathcal{U}_0 = a_0$ and by the monotonicity of $\beta_i(p)$ and antitonicity of $\gamma_i(p)$ in p for $1 \leq i \leq n$, we have $\beta_i^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}, a_0) \leq \beta_i^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}, u)$ and $\gamma_i^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}, u) \leq \gamma_i^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}, a_0)$ for all $u \in \mathcal{U}_0$ and $1 \leq i \leq n$, therefore $\beta_i^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}, a_0) \leq \bigwedge\{\beta_i^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}, u) : u \in \mathcal{U}_0\} \leq \bigvee\{\gamma_i^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}, u) : u \in \mathcal{U}_0\} \leq \gamma_i^{\mathbb{A}^{\delta}}(\vec{a}, \vec{x}, \vec{y}, a_0)$ for $1 \leq i \leq n$. $\square$

Chapter 5

ALBA-Style Canonicity

In this chapter we review the proof of canonicity of ALBA-inequalities which has been given in [10]. Notice that even if we have refined the proof of the topological Ackermann lemmas so that it is now point-free, the overall proof of canonicity-via-correspondence relies on the existence of enough points and hence is certainly not point-free. However, these results, which in [10] were formulated in a frame-theoretic setting, are reformulated here in a purely algebraic setting, replacing descriptive general frames and their underlying Kripke frames with their dual DMAs and their canonical extensions, respectively.

5.1 The Algorithm ALBA

We recall the **A**ckermann's **L**emma **B**ased **A**lgorithm (ALBA) formulated in [10], with slight revision, and give an example showing how it works. In our setting, we limit ourselves to eliminate all proposition variables, but we do not extensively compute the first-order correspondent in the final step of the algorithm, since this requires introducing the relational semantics for DML, which we do not use in this thesis.

The discussion below relies on the definition of signed generation tree in Section 2.7. ALBA proceeds in three stages:

1. **Preprocessing and first approximation**:

   In the generation tree of $+\varphi$ and $-\psi$,

   (a) Apply the distribution rules:

      i. Push down $+\Diamond, +\wedge$ and $-\rhd$, by distributing them over nodes labelled with $+\vee$ which are not below universal nodes, and

      ii. Push down $-\Box, -\vee$ and $+\lhd$, by distributing them over nodes labelled with $-\wedge$ which are not below universal nodes.

   (b) Apply the splitting rules:

$$\frac{\alpha \leq \beta \wedge \gamma}{\alpha \leq \beta \quad \alpha \leq \gamma} \qquad \frac{\alpha \vee \beta \leq \gamma}{\alpha \leq \gamma \quad \beta \leq \gamma}$$

(c) Apply the monotone and antitone variable-elimination rules:

$$\frac{\alpha(p) \leq \beta(p)}{\alpha(\bot) \leq \beta(\bot)} \qquad \frac{\beta(p) \leq \alpha(p)}{\beta(\top) \leq \alpha(\top)}$$

for $\beta(p)$ positive in p and $\alpha(p)$ negative in p.

We denote by Preprocess($\varphi \leq \psi$) the finite set $\{\varphi_i \leq \psi_i\}_{i\in I}$ of inequalities obtained after the exhaustive application of the previous rules. Then we apply the following first approximation rule to every inequality in Preprocess($\varphi \leq \psi$) :

$$\frac{\varphi_i \leq \psi_i}{i_0 \leq \varphi_i \quad \psi_i \leq m_0}$$

Here, i_0 and m_0 are special nominals and co-nominals. Now we get a set of inequalities $\{i_0 \leq \varphi_i, \psi_i \leq m_0\}_{i\in I}$.

2. **The reduction-elimination cycle**:

In this stage, for each $\{i_0 \leq \varphi_i, \psi_i \leq m_0\}$, we apply the following rules together with the splitting rules in the previous stage to eliminate all the proposition variables in $\{i_0 \leq \varphi_i, \psi_i \leq m_0\}$:

(a) Residuation rules:

$$\frac{\alpha \wedge \beta \leq \gamma}{\alpha \leq \beta \rightarrow \gamma} \qquad \frac{\alpha \leq \beta \vee \gamma}{\alpha - \beta \leq \gamma} \qquad \frac{\Diamond\alpha \leq \beta}{\alpha \leq \blacksquare\beta} \qquad \frac{\alpha \leq \Box\beta}{\blacklozenge\alpha \leq \beta} \qquad \frac{\lhd\alpha \leq \beta}{\blacktriangleleft \beta \leq \alpha} \qquad \frac{\alpha \leq \rhd\beta}{\beta \leq \blacktriangleright \alpha}$$

(b) Approximation rules:

$$\frac{i \leq \Diamond\alpha}{j \leq \alpha \quad i \leq \Diamond j} \qquad \frac{\Box\alpha \leq m}{\alpha \leq n \quad \Box n \leq m} \qquad \frac{i \leq \lhd\alpha}{\alpha \leq m \quad i \leq \lhd m} \qquad \frac{\rhd\alpha \leq m}{i \leq \alpha \quad \rhd i \leq m}$$

The nominals and co-nominals introduced by the approximation rules must not occur in the system before applying the rule.

(c) The Ackermann rules. These two rules are the core of ALBA, since their application eliminates proposition variables. In fact, all the preceding steps are aimed at reaching a shape in which the rules can be applied. Notice that an important feature of these rules is that they are executed on the whole set of inequalities, and not on a single inequality.

The right-handed Ackermann rule:

$$\text{The system} \begin{cases} \alpha_1 \leq p \\ \vdots \\ \alpha_n \leq p \\ \beta_1 \leq \gamma_1 \\ \vdots \\ \beta_m \leq \gamma_m \end{cases} \text{is replaced by} \begin{cases} \beta_1((\alpha_1 \vee \ldots \vee \alpha_n)/p) \leq \gamma_1((\alpha_1 \vee \ldots \vee \alpha_n)/p) \\ \vdots \\ \beta_m((\alpha_1 \vee \ldots \vee \alpha_n)/p) \leq \gamma_m((\alpha_1 \vee \ldots \vee \alpha_n)/p) \end{cases}$$

where:

i. p does not occur in $\alpha_1, \ldots, \alpha_n$;

ii. Each β_i is positive, and each γ_i negative in p, for $1 \leq i \leq m$.

The left-handed Ackermann rule:

$$\text{The system}\begin{cases} p \leq \alpha_1 \\ \vdots \\ p \leq \alpha_n \\ \beta_1 \leq \gamma_1 \\ \vdots \\ \beta_m \leq \gamma_m \end{cases} \text{is replaced by} \begin{cases} \beta_1((\alpha_1 \wedge \ldots \wedge \alpha_n)/p) \leq \gamma_1((\alpha_1 \wedge \ldots \wedge \alpha_n)/p) \\ \vdots \\ \beta_m((\alpha_1 \wedge \ldots \wedge \alpha_n)/p) \leq \gamma_m((\alpha_1 \wedge \ldots \wedge \alpha_n)/p) \end{cases}$$

where:

i. p does not occur in $\alpha_1, \ldots, \alpha_n$;

ii. Each β_i is negative, and each γ_i positive in p, for $1 \leq i \leq m$.

3. **Output**: If in the previous stage, for some $\{i_0 \leq \varphi_i, \psi_i \leq m_0\}$, the algorithm gets stuck, i.e. some proposition variables cannot be eliminated by the application of the reduction rules, then the algorithm halts and output "failure". Otherwise, each initial tuple $\{i_0 \leq \varphi_i, \psi_i \leq m_0\}$ of inequalities after the first approximation has been reduced to a set of pure inequalities Reduce($\varphi_i \leq \psi_i$), and then the output is a set of quasi-inequalities $\{\&\text{Reduce}(\varphi_i \leq \psi_i) \Rightarrow i_0 \leq m_0 : \varphi_i \leq \psi_i \in \text{Preprocess}(\varphi \leq \psi)\}$.

We give the following example to show how ALBA works:

Example 5.1. Consider the inequality $\Diamond((\Box p \wedge q) \vee \Box(\rhd p \vee \Box(\rhd q \vee r))) \leq \rhd p \vee \rhd q \vee \rhd r \vee r$, we can see that this inequality is (Ω, ϵ)-inductive (see Definition 2.42) for the dependence order $p <_\Omega q <_\Omega r$ and order type $\epsilon = (1, 1, 1)$.

Stage 1:

By the distribution rules and splitting rules, we get

$\{\Diamond(\Box p \wedge q) \leq \rhd p \vee \rhd q \vee \rhd r \vee r, \Diamond\Box(\rhd p \vee \Box(\rhd q \vee r)) \leq \rhd p \vee \rhd q \vee \rhd r \vee r\}$;

then by the monotone and antitone variable elimination rule, we get

$\{\Diamond(\Box\top \wedge \top) \leq \rhd\top \vee \rhd\top \vee \rhd r \vee r, \Diamond\Box(\rhd p \vee \Box(\rhd q \vee r)) \leq \rhd p \vee \rhd q \vee \rhd r \vee r\}$;

then by the approximation rule,we get

$\{i_0 \leq \Diamond(\Box\top \wedge \top), \rhd\top \vee \rhd\top \vee \rhd r \vee r \leq m_0\}, \{i_0 \leq \Diamond\Box(\rhd p \vee \Box(\rhd q \vee r)), \rhd p \vee \rhd q \vee \rhd r \vee r \leq m_0\}$.

Stage 2:

For the first set of inequalities $\{i_0 \leq \Diamond(\Box\top \wedge \top), \rhd\top \vee \rhd\top \vee \rhd r \vee r \leq m_0\}$, by splitting rule we get

$\{i_0 \leq \Diamond(\Box\top \wedge \top), \rhd\top \leq m_0, \rhd r \leq m_0, r \leq m_0\}$;

by lefthanded Ackermann rule, we get

$\{i_0 \leq \Diamond(\Box\top \wedge \top), \rhd\top \leq m_0, \rhd m_0 \leq m_0\}$, which is pure.

For the second set of inequalities $\{i_0 \leq \Diamond\Box(\rhd p \vee \Box(\rhd q \vee r)), \rhd p \vee \rhd q \vee \rhd r \vee r \leq m_0\}$,

by splitting rule we get

$\{i_0 \leq \Diamond\Box(\rhd p \vee \Box(\rhd q \vee r)), \rhd p \leq m_0, \rhd q \leq m_0, \rhd r \leq m_0, r \leq m_0\}$;

by approximation rule we get

$\{i_0 \leq \Diamond i_1, i_1 \leq \Box(\rhd p \vee \Box(\rhd q \vee r)), i_2 \leq p, i_3 \leq q, i_4 \leq r, \rhd i_2 \leq m_0, \rhd i_3 \leq m_0, \rhd i_4 \leq m_0, r \leq m_0\}$;

by residuation rule we get

$\{i_0 \leq \Diamond i_1, \blacklozenge i_1 \leq \rhd p \vee \Box(\rhd q \vee r), i_2 \leq p, i_3 \leq q, i_4 \leq r, \rhd i_2 \leq m_0, \rhd i_3 \leq m_0, \rhd i_4 \leq m_0, r \leq m_0\}$;

then we use right-handed Ackermann rule to eliminate p:

$\{i_0 \leq \Diamond i_1, \blacklozenge i_1 \leq \rhd i_2 \vee \Box(\rhd q \vee r), i_3 \leq q, i_4 \leq r, \rhd i_2 \leq m_0, \rhd i_3 \leq m_0, \rhd i_4 \leq m_0, r \leq m_0\}$;

then by residuation rule and right-handed Ackermann rule we can eliminate q:

$\{i_0 \leq \Diamond i_1, \blacklozenge(\blacklozenge i_1 - \rhd i_2) \leq \rhd i_3 \vee r, i_4 \leq r, \rhd i_2 \leq m_0, \rhd i_3 \leq m_0, \rhd i_4 \leq m_0, r \leq m_0\}$;

finally by residuation rule and right-handed Ackermann rule we can eliminate r:

$\{i_0 \leq \Diamond i_1, (\blacklozenge(\blacklozenge i_1 - \rhd i_2) - \rhd i_3) \vee i_4 \leq m_0, \rhd i_2 \leq m_0, \rhd i_3 \leq m_0, \rhd i_4 \leq m_0\}$, which is pure.

Stage 3:

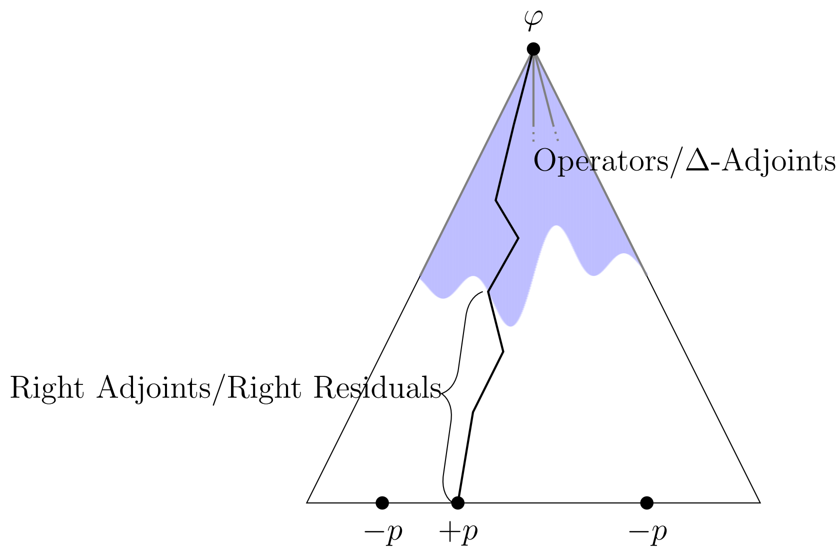
Output $\{\&\{i_0 \leq \Diamond(\Box\top \wedge \top), \rhd\top \leq m_0, \rhd m_0 \leq m_0\} \Rightarrow i_0 \leq m_0, \&\{i_0 \leq \Diamond i_1, (\blacklozenge(\blacklozenge i_1 - \rhd i_2) - \rhd i_3) \vee i_4 \leq m_0, \rhd i_2 \leq m_0, \rhd i_3 \leq m_0, \rhd i_4 \leq m_0\} \Rightarrow i_0 \leq m_0\}$.

In fact, we have the following result (for a proof, see [10]):

Theorem 5.2. *ALBA succeeds on all inductive inequalities.*

Intuitively, each formula on the left-hand side of an inductive inequality looks as follows, and the right-hand side formula can be seen order-dually[1]:

[1]This picture has been created by Willem Conradie.



This specific shape guarantees that every inductive inequality can be processed by ALBA in the following way:

By applying the approximation and splitting rules to the nodes of the upper part, we can always obtain a transformation in which the "lower part is in display". Namely, all the subformulas whose main node does not belong anymore to the upper part, which is shaded in the picture, and which contain critical occurrences of proposition variables, occur now as the main formulas either on the left or on the right of inequalities, i.e. they occur "in display". The syntactic restrictions on inductive and Sahlqvist inequalities guarantee that if such a formula is in display on the left (resp. right), then it is a left (resp. right) adjoint or a left (resp. right) residual in the critical coordinate; hence, by applying the appropriate adjunction or residuation rules, the given inequality can be equivalently rewritten in such a way that the critical occurrence is now in display, either on the left or on the right. This provides us with a "minimal valuation", namely with inequalities of the form $\alpha \leq p$ if the order-type of p is 1, and of the form $p \leq \alpha$ if the order-type of p is ∂. The non-critical occurrences of the proposition variable p will have the appropriate polarity to receive the minimal valuation, hence inductive and Sahlqvist inequalities make sure that the Ackermann rule can always be applied.

Finally, referring back to the discussion in Chapter 3, we noticed that the compositional structure of Sahlqvist formulas guarantees that at least one of the two routes which constitutes the core of the Jónsson-style strategy. Here, the same order-theoretic properties guarantee that the decompositional strategy of the ALBA is always successful. In Chapter 7, we will continue this discussion.

5.2 ALBA-Style Canonicity

Suppose ALBA succeeds on $\varphi \leq \psi$. Let ALBA($\varphi \leq \psi$) be the set of quasi-inequalities &Reduce($\varphi_i \leq \psi_i$) $\Rightarrow i_0 \leq m_0$ for $\varphi_i \leq \psi_i \in$Preprocess($\varphi \leq \psi$). It is easy to see that all quasi-inequalities in ALBA($\varphi \leq \psi$) are pure. In the following we will call the inequalities $\varphi \leq \psi$ on which ALBA succeeds *ALBA inequalities*. It is stated in Theorem 5.2 that the inductive inequalities are included in ALBA inequalities.

The following theorem is the main result of this chapter:

Theorem 5.3. *All inequalities on which ALBA succeeds are canonical, therefore by Theorem 5.2, inductive inequalities are canonical.*

Proof. For the proof, consider the inequality $\varphi \leq \psi$ and the resulting quasi-inequalities ALBA($\varphi \leq \psi$) after executing ALBA. We can see that the quasi-inequalities in ALBA($\varphi \leq \psi$) are pure, therefore $\mathbb{A}^\delta \vDash$ALBA($\varphi \leq \psi$) holds iff $\mathbb{A}^\delta \vDash_{\mathbb{A}}$ALBA($\varphi \leq \psi$) holds, hence we can use the U-shaped argument represented below to show that from $\mathbb{A}^\delta \vDash_{\mathbb{A}} \varphi \leq \psi$ (i.e. $\mathbb{A} \vDash \varphi \leq \psi$) we can get $\mathbb{A}^\delta \vDash \varphi \leq \psi$:

$$\begin{array}{ccc} \mathbb{A}^\delta \vDash_{\mathbb{A}} \varphi \leq \psi & & \mathbb{A}^\delta \vDash \varphi \leq \psi \\ \Updownarrow & & \Updownarrow \\ \mathbb{A}^\delta \vDash_{\mathbb{A}} \text{ALBA}(\varphi \leq \psi) & \Leftrightarrow & \mathbb{A}^\delta \vDash \text{ALBA}(\varphi \leq \psi). \end{array}$$

The right-hand equivalence is proved in Theorem 5.4 below, the bottom-line equivalence holds by the fact that the value of ALBA($\varphi \leq \psi$) depends only on the assignment for nominals and co-nominals, so there is no difference between admissible assignment and arbitrary assignment. The proof strategy for the right-hand equivalence is similar to that of the left-hand equivalence. In fact, almost for all steps, the proofs are similar (by the admissible assignment versions of Lemma 5.5, 5.6, 5.7), except for the Ackermann rule, which, as discussed in the previous chapter, is sound thanks to the topological Ackermann lemmas only when additional syntactic constraints are satisfied. By Lemma 5.8 (for a proof, see [10]), these syntactic constraints are satisfied, therefore the Ackermann rule preserves the equivalence for admissible assignments. Hence, we get the canonicity of all ALBA inequalities. □

In the following we list the theorems and lemmas needed in the proof of Theorem 5.3.

Theorem 5.4. *If ALBA succeeds on $\varphi \leq \psi$ and yields ALBA($\varphi \leq \psi$), then $\mathbb{A}^\delta \vDash \varphi \leq \psi$ iff $\mathbb{A}^\delta \vDash$ALBA($\varphi \leq \psi$).*

Proof. The correctness of the algorithm on the $\mathbb{A}^\delta$ side follows from the following chain of equivalences:

1. $\mathbb{A}^\delta \vDash \varphi \leq \psi$;
2. $\mathbb{A}^\delta \vDash$Preprocess($\varphi \leq \psi$);
3. $\mathbb{A}^\delta \vDash j_0 \leq \varphi' \ \&\ \psi' \leq m_0 \Rightarrow j_0 \leq m_0$ for all $\varphi' \leq \psi' \in$Preprocess($\varphi \leq \psi$);
4. $\mathbb{A}^\delta \vDash$ &Reduce($\varphi' \leq \psi'$) $\Rightarrow j_0 \leq m_0$ for all $\varphi' \leq \psi' \in$Preprocess($\varphi \leq \psi$);
5. $\mathbb{A}^\delta \vDash$ALBA($\varphi \leq \psi$).

The equivalence between (1) and (2) is given in Lemma 5.5, the equivalence between (2) and (3) is given in Lemma 5.6, the equivalence between (3) and (4) is given in Lemma 5.7, the equivalence between (4) and (5) is trivial. For the proofs of these lemmas (in frame-theoretic setting), see [10]. □

Lemma 5.5. *If a set S' of inequalities is obtained from a set S by the application of one step of preprocessing rule, then $\mathbb{A}^\delta \vDash S$ iff $\mathbb{A}^\delta \vDash S'$.*

Lemma 5.6. $\mathbb{A}^\delta \vDash \varphi' \leq \psi'$ *iff* $\mathbb{A}^\delta \vDash j_0 \leq \varphi'$ & $\psi' \leq m_0 \Rightarrow j_0 \leq m_0$.

Lemma 5.7. *Let S be a finite system of inequalities and S' be a system of inequalities obtained from S by applying one of the reduction rules, and h be any assignment. Then*

1. *If $\mathbb{A}^\delta, h \vDash S$, then $\mathbb{A}^\delta, h' \vDash S'$ for some h' such that $h'(j_0) = h(j_0)$, $h'(m_0) = h(m_0)$;*
2. *If $\mathbb{A}^\delta, h \vDash S'$, then $\mathbb{A}^\delta, h' \vDash S$ for some h' such that $h'(j_0) = h(j_0)$, $h'(m_0) = h(m_0)$.*

Lemma 5.8 (cf. Lemma 9.5 in [10])**.** *In every non-pure inequality $\varphi' \leq \psi'$ obtained when ALBA is run on an inequality $\varphi \leq \psi$, the left-hand side φ' is always syntactically closed while the right-hand side ψ' is always syntactically open.*

Chapter 6

Generalized Canonical Extensions for Maps

The success in the use of the expanded language to prove the canonicity result via the ALBA algorithm inspires us to consider the magic power of the expanded language and the possibility of proving Jónsson-style canonicity using the expanded language. However, for the expanded language, the subalgebra $\mathbb{A}$ might not be closed under the new operations, therefore the standard definition for canonical extension needs to be revised, and we need a theory of generalized canonical extension for functions $f^{\mathbb{A}} : \mathbb{A} \to \mathbb{B}^{\delta}$ such that the value of $f^{\mathbb{A}}$ is not restricted to clopen elements in $\mathbb{B}$.

In this chapter, we define a notion of generalized canonical extension $(f^{\mathbb{A}})^{\lambda} : \mathbb{A}^{\delta} \to \mathbb{B}^{\delta}$ for functions $f^{\mathbb{A}} : \mathbb{A} \to \mathbb{B}^{\delta}$ which are restrictions of $f^{\mathbb{A}^{\delta}} : \mathbb{A}^{\delta} \to \mathbb{B}^{\delta}$ to $\mathbb{A}$, in such a way that the value of $f^{\mathbb{A}^{\delta}}(u)$ for $u \in \mathbb{A}^{\delta}$ can be approximated by the set of values $f^{\mathbb{A}}(a)$ for $a \in \mathbb{A}$. The spirit of generalized canonical extensions of maps is similar to the fact that the value of a continuous real variable function on an irrational real number can be taken as a limit of its values on rational numbers. Although the definition we are going to give looks the same as in the standard case, the generalized version here is intended for getting approximations of functions $f^{\mathbb{A}^{\delta}} : \mathbb{A}^{\delta} \to \mathbb{B}^{\delta}$ based on their values on clopen elements in $\mathbb{A}$ and then comparing $f^{\mathbb{A}^{\delta}}$ and $(f^{\mathbb{A}})^{\lambda}$ so as to show whether $f^{\mathbb{A}^{\delta}}$ is contracting or expanding. In fact, the idea of this chapter elaborates on Section 2.3 of [16] on the continuous extensions of maps; however, in [16], canonical extensions are fully discussed only for functions $f^{\mathbb{A}} : \mathbb{A} \to \mathbb{B}$, not for functions $f^{\mathbb{A}} : \mathbb{A} \to \mathbb{B}^{\delta}$, and here we fill in the gap.

In the following we use superscripts to indicate the domain of the function, therefore $f^{\mathbb{A}}$ is the restriction of $f^{\mathbb{A}^{\delta}}$ to $\mathbb{A}$ or $\mathbb{A}^{\epsilon}$, depending on the context. Here we abuse the notation: $\blacklozenge^{\mathbb{A}}$ is not an operation on $\mathbb{A}$, but a map from $\mathbb{A}$ to $\mathbb{A}^{\delta}$, which is the restriction of $\blacklozenge^{\mathbb{A}^{\delta}}$ to $\mathbb{A}$. In this chapter, $\mathbb{A}$, $\mathbb{B}, \mathbb{C}$ are always BDLs.

6.1 Generalized Canonical Extensions for Maps

First of all, we have the following observation:

Proposition 6.1. *For any monotone function $f^{\mathbb{A}} : \mathbb{A} \to \mathbb{B}$ and its extension $f^{\mathbb{A}^{\delta}} : \mathbb{A}^{\delta} \to \mathbb{B}^{\delta}$,*

1. $(f^{\mathbb{A}})^\sigma \leq f^{\mathbb{A}^\delta}$ *iff the following* closed Esakia condition *holds:* $f^{\mathbb{A}^\delta}(\bigwedge\{c_i : i \in I\}) = \bigwedge\{f^{\mathbb{A}^\delta}(c_i) : i \in I\}$ *for any downward-directed collection* $\{c_i : i \in I\} \subseteq K(\mathbb{A}^\delta)$*;*

2. $f^{\mathbb{A}^\delta} \leq (f^{\mathbb{A}})^\pi$ *iff the following* open Esakia condition *holds:* $f^{\mathbb{A}^\delta}(\bigvee\{o_i : i \in I\}) = \bigvee\{f^{\mathbb{A}^\delta}(o_i) : i \in I\}$ *for any upward-directed collection* $\{o_i : i \in I\} \subseteq O(\mathbb{A}^\delta)$*.*

Proof. We prove 1, statement 2 being an order-variant of 1.

$\Leftarrow$: Assume the closed Esakia condition holds for $f^{\mathbb{A}^\delta}$. To show that $(f^{\mathbb{A}})^\sigma \leq f^{\mathbb{A}^\delta}$, we have to show that for all $u \in \mathbb{A}^\delta$, $\bigvee\{\bigwedge\{f^{\mathbb{A}}(a) : x \leq a \in \mathbb{A}\} : u \geq x \in K(\mathbb{A}^\delta)\} \leq f^{\mathbb{A}^\delta}(u)$, i.e., for all $u \in \mathbb{A}^\delta$, all $x \in K(\mathbb{A}^\delta)$ such that $x \leq u$, $\bigwedge\{f^{\mathbb{A}}(a) : x \leq a \in \mathbb{A}\} \leq f^{\mathbb{A}^\delta}(u)$. By the monotonicity of $f^{\mathbb{A}^\delta}$ and that $f^{\mathbb{A}}$ and $f^{\mathbb{A}^\delta}$ coincide on $\mathbb{A}$, it suffices to show that for all $x \in K(\mathbb{A}^\delta)$, $\bigwedge\{f^{\mathbb{A}^\delta}(a) : x \leq a \in \mathbb{A}\} \leq f^{\mathbb{A}^\delta}(x)$, which follows easily from the closed Esakia condition.

$\Rightarrow$: Suppose we have $(f^{\mathbb{A}})^\sigma \leq f^{\mathbb{A}^\delta}$, i.e., for all $u \in \mathbb{A}^\delta$, $\bigvee\{\bigwedge\{f^{\mathbb{A}}(a) : x \leq a \in \mathbb{A}\} : u \geq x \in K(\mathbb{A}^\delta)\} \leq f^{\mathbb{A}^\delta}(u)$. Then let us preliminarily show that $\bigwedge\{f^{\mathbb{A}^\delta}(a) : x \leq a \in \mathbb{A}\} = f^{\mathbb{A}^\delta}(x)$ for all $x \in K(\mathbb{A}^\delta)$:

For any $x \in K(\mathbb{A}^\delta)$, we have $\bigwedge\{f^{\mathbb{A}}(a) : x \leq a \in \mathbb{A}\} = \bigvee\{\bigwedge\{f^{\mathbb{A}}(a) : x' \leq a \in \mathbb{A}\} : x \geq x' \in K(\mathbb{A}^\delta)\} \leq f^{\mathbb{A}^\delta}(x)$. Since $f^{\mathbb{A}}$ and $f^{\mathbb{A}^\delta}$ coincide on $\mathbb{A}$, we have $\bigwedge\{f^{\mathbb{A}^\delta}(a) : x \leq a \in \mathbb{A}\} \leq f^{\mathbb{A}^\delta}(x)$. The converse inequality follows from the monotonicity of $f^{\mathbb{A}^\delta}$.

Let us prove that $f^{\mathbb{A}^\delta}$ satisfies the closed Esakia condition, i.e., for any downward-directed collection $\{c_i : i \in I\} \subseteq K(\mathbb{A}^\delta)$, $f^{\mathbb{A}^\delta}(\bigwedge\{c_i : i \in I\}) = \bigwedge\{f^{\mathbb{A}^\delta}(c_i) : i \in I\}$:

The inequality $f^{\mathbb{A}^\delta}(\bigwedge\{c_i : i \in I\}) \leq \bigwedge\{f^{\mathbb{A}^\delta}(c_i) : i \in I\}$ straightforwardly follows from the monotonicity of $f^{\mathbb{A}^\delta}$. For the converse direction $\bigwedge\{f^{\mathbb{A}^\delta}(c_i) : i \in I\} \leq f^{\mathbb{A}^\delta}(\bigwedge\{c_i : i \in I\})$, let $x = \bigwedge\{c_i : i \in I\} \in K(\mathbb{A}^\delta)$. By the preliminary fact shown above, we have that $f^{\mathbb{A}^\delta}(x) = \bigwedge\{f^{\mathbb{A}^\delta}(a) : a \in \mathbb{A}$ and $x \leq a\}$. Hence, it is enough to show that $\bigwedge\{f^{\mathbb{A}^\delta}(c_i) : i \in I\} \leq \bigwedge\{f^{\mathbb{A}^\delta}(a) : a \in \mathbb{A}$ and $x \leq a\}$, i.e., we need to show that for each $a \in \mathbb{A}$, if $x \leq a$, then there exists an $i_0 \in I$ such that $f^{\mathbb{A}^\delta}(c_{i_0}) \leq f^{\mathbb{A}^\delta}(a)$. By compactness, from $\bigwedge\{c_i : i \in I\} = x \leq a$ we get that $x' = \bigwedge\{c_i : i \in I'\} \leq a$ for some finite $I' \subseteq I$. By the downward-directedness of $\{c_i : i \in I\}$, we get $c_{i_0} \leq x' \leq a$ for some $i_0 \in I$. Therefore, $f^{\mathbb{A}^\delta}(c_{i_0}) \leq f^{\mathbb{A}^\delta}(a)$. ☐

From the proposition above, we can see that the essential property for a term to be σ-contracting (resp. π-expanding) is the closed (resp. open) Esakia condition. By checking the proof details, we can see that the fact that $f^{\mathbb{A}}$ maps elements in $\mathbb{A}$ to clopen elements in $\mathbb{B}$ plays no role. Therefore the result still holds if we allow $f^{\mathbb{A}}$ to map elements in $\mathbb{A}$ to non-clopen elements in $\mathbb{B}^\delta$, so we can work in a setting in which terms in the expanded signature $\mathcal{L}^{++}$ can also be accounted for. Therefore, we can define the generalized canonical extension $f^{\mathbb{A}^\delta} : \mathbb{A}^\delta \to \mathbb{B}^\delta$ for functions $f^{\mathbb{A}} : \mathbb{A} \to \mathbb{B}^\delta$, which we will give below.

The following definitions are very similar to those in Chapter 2, the only difference being the codomain of the functions, but for the sake of clarity we repeat them in full.

Definition 6.2 (Generalized Canonical Extension for Maps). For any map $f^{\mathbb{A}} : \mathbb{A} \to \mathbb{B}^\delta$, for all $u \in \mathbb{A}^\delta$, we define

$$(f^{\mathbb{A}})^{\sigma}(u) = \bigvee\{\bigwedge\{f(a) : a \in \mathbb{A}, x \leq a \leq y\} : K(\mathbb{A}^{\delta}) \ni x \leq u \leq y \in O(\mathbb{A}^{\delta})\}$$
$$(f^{\mathbb{A}})^{\pi}(u) = \bigwedge\{\bigvee\{f(a) : a \in \mathbb{A}, x \leq a \leq y\} : K(\mathbb{A}^{\delta}) \ni x \leq u \leq y \in O(\mathbb{A}^{\delta})\}.$$

For example, we can consider $\blacklozenge^{\mathbb{A}^{\delta}}$; let $\blacklozenge^{\mathbb{A}} : \mathbb{A} \to \mathbb{A}^{\delta}$ be its restriction to $\mathbb{A}$. Then the generalized canonical extension of $\blacklozenge^{\mathbb{A}}$ is $(\blacklozenge^{\mathbb{A}})^{\lambda}$, where $\lambda \in \{\sigma, \pi\}$. Then we will show that $(\blacklozenge^{\mathbb{A}})^{\sigma} = (\blacklozenge^{\mathbb{A}})^{\pi} = \blacklozenge^{\mathbb{A}^{\delta}}$ (see Section 7.1.2).

For order-preserving maps, we also have the following:

Theorem 6.3. *If $f^{\mathbb{A}} : \mathbb{A} \to \mathbb{B}^{\delta}$ is order-preserving, then for all $u \in \mathbb{A}^{\delta}$,*

$$(f^{\mathbb{A}})^{\sigma}(u) = \bigvee\{\bigwedge\{f^{\mathbb{A}}(a) : x \leq a \in \mathbb{A}\} : u \geq x \in K(\mathbb{A}^{\delta})\}$$
$$(f^{\mathbb{A}})^{\pi}(u) = \bigwedge\{\bigvee\{f^{\mathbb{A}}(a) : y \geq a \in \mathbb{A}\} : u \leq y \in O(\mathbb{A}^{\delta})\}.$$

In the following, we are mainly working with order-preserving maps.

Many properties about σ- and π-extensions still hold in the generalized setting. Here we state some of them without proof.

Proposition 6.4. *For any order-preserving map $f^{\mathbb{A}} : \mathbb{A} \to \mathbb{B}^{\delta}$,*

1. *Both $(f^{\mathbb{A}})^{\sigma}$ and $(f^{\mathbb{A}})^{\pi}$ are order-preserving;*
2. *$(f^{\mathbb{A}})^{\sigma}(x) = \bigwedge\{f^{\mathbb{A}}(a) : x \leq a \in \mathbb{A}\}$ for $x \in K(\mathbb{A}^{\delta})$;*
3. *$(f^{\mathbb{A}})^{\pi}(y) = \bigvee\{f^{\mathbb{A}}(a) : y \geq a \in \mathbb{A}\}$ for $y \in O(\mathbb{A}^{\delta})$;*
4. *$(f^{\mathbb{A}})^{\sigma}(a) = (f^{\mathbb{A}})^{\pi}(a) = f^{\mathbb{A}}(a)$ for all $a \in \mathbb{A}$;*
5. *$(f^{\mathbb{A}})^{\sigma}(u) \leq (f^{\mathbb{A}})^{\pi}(u)$ for all $u \in \mathbb{A}^{\delta}$, and "$=$" holds for $u \in K(\mathbb{A}^{\delta}) \cup O(\mathbb{A}^{\delta})$.*

Proof. The items 1.-4. follows straightforwardly from the definition. The proof of item 5. is very similar to the proof of Theorem 3.1 in [15]. □

We can also generalize the definition of stable, expanding and contracting maps to this new setting:

Definition 6.5 (Generalized Stable, Expanding and Contracting Maps). For any order-preserving maps $f^{\mathbb{A}} : \mathbb{A} \to \mathbb{B}^{\delta}$ and $f^{\mathbb{A}^{\delta}} : \mathbb{A}^{\delta} \to \mathbb{B}^{\delta}$ where $f^{\mathbb{A}^{\delta}}$ extends $f^{\mathbb{A}}$, $\lambda \in \{\sigma, \pi\}$, we say that

- $f^{\mathbb{A}^{\delta}}$ is λ-*stable* if $f^{\mathbb{A}^{\delta}} = (f^{\mathbb{A}})^{\lambda}$;
- $f^{\mathbb{A}^{\delta}}$ is λ-*expanding* if $f^{\mathbb{A}^{\delta}} \leq (f^{\mathbb{A}})^{\lambda}$;
- $f^{\mathbb{A}^{\delta}}$ is λ-*contracting* if $f^{\mathbb{A}^{\delta}} \geq (f^{\mathbb{A}})^{\lambda}$.

6.2 Generalized σ-Contracting and π-Expanding Maps

In this section we give a characterization of σ-contracting and π-expanding maps in terms of the Esakia conditions and the intersection conditions.

We first give the generalized definition of the Esakia conditions for order-preserving maps from $\mathbb{A}^\delta$ to $\mathbb{B}^\delta$, and also generalize Proposition 6.1.

Definition 6.6 (Generalized Esakia Conditions). For any order-preserving map $f^{\mathbb{A}^\delta} : \mathbb{A}^\delta \to \mathbb{B}^\delta$,

1. $f^{\mathbb{A}^\delta}$ satisfies the *closed Esakia condition*, if for any downward-directed collection $\mathcal{C} = \{c_i : i \in I\} \subseteq K(\mathbb{A}^\delta)$, $f^{\mathbb{A}^\delta}(\bigwedge\{c_i : i \in I\}) = \bigwedge f^{\mathbb{A}^\delta}(\{c_i : i \in I\})$;
2. $f^{\mathbb{A}^\delta}$ satisfies the *open Esakia condition*, if for any upward-directed collection $\mathcal{O} = \{o_i : i \in I\} \subseteq O(\mathbb{A}^\delta)$, $f^{\mathbb{A}^\delta}(\bigvee\{o_i : i \in I\}) = \bigvee f^{\mathbb{A}^\delta}(\{o_i : i \in I\})$.

Theorem 6.7. *For any order-preserving map $f^{\mathbb{A}^\delta} : \mathbb{A}^\delta \to \mathbb{B}^\delta$, both $(f^{\mathbb{A}})^\sigma$ and $(f^{\mathbb{A}})^\pi$ satisfy both the open and the closed Esakia condition.*

Proof. We prove only the open Esakia condition, the closed Esakia condition being similar.

For $(f^{\mathbb{A}})^\pi$, for any upward-directed collection $\mathcal{O} = \{o_i : i \in I\} \subseteq O(\mathbb{A}^\delta)$, we have that $(f^{\mathbb{A}})^\pi(\bigvee \mathcal{O}) = \bigvee\{f^{\mathbb{A}}(a) : \mathbb{A} \ni a \leq \bigvee \mathcal{O}\}$ and $\bigvee\{(f^{\mathbb{A}})^\pi(o) : o \in \mathcal{O}\} = \bigvee\{f^{\mathbb{A}}(a) : \mathbb{A} \ni a \leq o$ for some $o \in \mathcal{O}\}$, therefore it suffices to show that for all $a \in \mathbb{A}$, $a \leq \bigvee \mathcal{O}$ iff $a \leq o$ for some $o \in \mathcal{O}$. By compactness and upward-directedness of $\mathcal{O}$, it is easy to check. Therefore $(f^{\mathbb{A}})^\pi$ satisfies the open Esakia condition.

For $(f^{\mathbb{A}})^\sigma$, for any upward-directed collection $\mathcal{O} \subseteq O(\mathbb{A}^\delta)$, we have that

$$
\begin{array}{cll}
 & (f^{\mathbb{A}})^\sigma(\bigvee \mathcal{O}) & \\
= & (f^{\mathbb{A}})^\pi(\bigvee \mathcal{O}) & \text{(by Proposition 6.4(5))} \\
= & \bigvee\{(f^{\mathbb{A}})^\pi(o) : o \in \mathcal{O}\} & \text{(by } (f^{\mathbb{A}})^\pi \text{ satisfying the open Esakia condition)} \\
= & \bigvee\{(f^{\mathbb{A}})^\sigma(o) : o \in \mathcal{O}\} & \text{(by Proposition 6.4(5)),}
\end{array}
$$

therefore $(f^{\mathbb{A}})^\sigma$ satisfies the open Esakia condition. □

Theorem 6.8. *Given any order-preserving map $f^{\mathbb{A}^\delta} : \mathbb{A}^\delta \to \mathbb{B}^\delta$,*

1. *$f^{\mathbb{A}^\delta}$ is σ-contracting iff $f^{\mathbb{A}^\delta}$ satisfies the closed Esakia condition;*
2. *$f^{\mathbb{A}^\delta}$ is π-expanding iff $f^{\mathbb{A}^\delta}$ satisfies the open Esakia condition.*

Proof. Straightforward generalization the proof of Proposition 6.1. □

In fact, for n-ary order-preserving maps, we can also show that the Esakia conditions of Definition 6.6 are equivalent to the following much more manageable intersection conditions, which will be used in the discussion about the comparison between the two approaches to canonicity:

Definition 6.9 (Intersection Conditions). For any order-preserving map $f^{\mathbb{A}^\delta}(\vec{u}, \vec{v}) : (\mathbb{A}^\delta)^\epsilon \to \mathbb{B}^\delta$ where the order type is 1 for coordinates in $\vec{u}$ and ∂ for coordinates in $\vec{v}$,

1. $f^{\mathbb{A}^\delta}$ satisfies the *closed intersection condition*, if for any coordinate w picked up from $\vec{u}, \vec{v}$ (however, we abuse notation, and we denote the function $f^{\mathbb{A}^\delta}$ as $f^{\mathbb{A}^\delta}(\vec{u}, \vec{v}, w)$), for any $\vec{c} \in K(\mathbb{A}^\delta), \vec{o} \in O(\mathbb{A}^\delta)$, any downward-directed collection $\{c_i : i \in I\} \subseteq K(\mathbb{A}^\delta)$, any upward-directed collection $\{o_i : i \in I\} \subseteq O(\mathbb{A}^\delta)$, the following holds:
    - if the order type of w is 1, then $f^{\mathbb{A}^\delta}(\vec{c}, \vec{o}, \bigwedge\{c_i : i \in I\}) = \bigwedge\{f^{\mathbb{A}^\delta}(\vec{c}, \vec{o}, c_i) : i \in I\}$;
    - if the order type of w is ∂, then $f^{\mathbb{A}^\delta}(\vec{c}, \vec{o}, \bigvee\{o_i : i \in I\}) = \bigwedge\{f^{\mathbb{A}^\delta}(\vec{c}, \vec{o}, o_i) : i \in I\}$.
2. $f^{\mathbb{A}^\delta}$ satisfies the *open intersection condition*, if for any coordinate w picked up from $\vec{u}, \vec{v}$ (we use the same abuse of notation, and we denote the function $f^{\mathbb{A}^\delta}$ as $f^{\mathbb{A}^\delta}(\vec{u}, \vec{v}, w)$), for any $\vec{c} \in K(\mathbb{A}^\delta), \vec{o} \in O(\mathbb{A}^\delta)$, any downward-directed collection $\{c_i : i \in I\} \subseteq K(\mathbb{A}^\delta)$, any upward-directed collection $\{o_i : i \in I\} \subseteq O(\mathbb{A}^\delta)$, the following holds:
    - if the order type of w is 1, then $f^{\mathbb{A}^\delta}(\vec{o}, \vec{c}, \bigvee\{o_i : i \in I\}) = \bigvee\{f^{\mathbb{A}^\delta}(\vec{o}, \vec{c}, o_i) : i \in I\}$;
    - if the order type of w is ∂, then $f^{\mathbb{A}^\delta}(\vec{o}, \vec{c}, \bigwedge\{c_i : i \in I\}) = \bigvee\{f^{\mathbb{A}^\delta}(\vec{o}, \vec{c}, c_i) : i \in I\}$.

In this definition, we can see that the intersection conditions are the coordinatewise version of the Esakia conditions, and we can prove the following equivalence:

Lemma 6.10. *Given any order-preserving map $f^{\mathbb{A}^\delta}(\vec{u}, \vec{v}) : (\mathbb{A}^\delta)^\epsilon \to \mathbb{B}^\delta$ where the order type is 1 for coordinates in $\vec{u}$ and ∂ for coordinates in $\vec{v}$,*

1. *$f^{\mathbb{A}^\delta}$ satisfies the closed Esakia condition iff it satisfies the closed intersection condition;*
2. *$f^{\mathbb{A}^\delta}$ satisfies the open Esakia condition iff it satisfies the open intersection condition;*

Proof. We only give the proof of item 2, item 1. being order-dual.

$\Rightarrow$: It is easy to see that the open intersection condition is a special case of the open Esakia condition. Indeed, for any $\vec{o}, \vec{c}$, $\{o_i : i \in I\}$ and $\{c_i : i \in I\}$ as above, the set $\{(\vec{o}, \vec{c}, o_i) : i \in I\}$ is an upward-directed collection in $O((\mathbb{A}^\delta)^\epsilon)$ if the order-type of w is 1, and the set $\{(\vec{o}, \vec{c}, c_i) : i \in I\}$ is an upward-directed collection in $O((\mathbb{A}^\delta)^\epsilon)$ if the order-type of w is ∂.

$\Leftarrow$: To keep notation manageable, here we only give the proof in the special case in which both $\vec{u}$ and $\vec{v}$ are of length 2. Let us denote $\mathbb{A}^\delta \times \mathbb{A}^\delta \times (\mathbb{A}^\delta)^\partial \times (\mathbb{A}^\delta)^\partial$ by $(\mathbb{A}^\delta)^\epsilon$. Then $f^{\mathbb{A}^\delta}(u_1, u_2, v_1, v_2) : (\mathbb{A}^\delta)^\epsilon \to \mathbb{B}^\delta$, and $O((\mathbb{A}^\delta)^\epsilon) = O(\mathbb{A}^\delta) \times O(\mathbb{A}^\delta) \times K(\mathbb{A}^\delta) \times K(\mathbb{A}^\delta)$. This can be easily generalized to an arbitrary number of coordinates.
We are going to show that, for any upward-directed collection $\{(o_i, o'_i, c_i, c'_i) : i \in I\} \subseteq O((\mathbb{A}^\delta)^\epsilon)$,

$$f^{\mathbb{A}^\delta}(o, o', c, c') = \bigvee\{f^{\mathbb{A}^\delta}(o_i, o'_i, c_i, c'_i) : i \in I\},$$

where $o = \bigvee\{o_i : i \in I\}$, $o' = \bigvee\{o'_i : i \in I\}$, $c = \bigwedge\{c_i : i \in I\}$, $c' = \bigwedge\{c'_i : i \in I\}$.
The inequality $\geq$ follows by the monotonicity properties of $f^{\mathbb{A}^\delta}$. As to the converse inequality, since by assumption, $f^{\mathbb{A}^\delta}$ satisfies the open intersection condition, we have that

$$f^{\mathbb{A}^\delta}(o, o', c, c') = \bigvee\{f^{\mathbb{A}^\delta}(o_{i_1}, o'_{i_2}, c_{i_3}, c'_{i_4}) : i_1, i_2, i_3, i_4 \in I\},$$

it suffices to show that for any $i_1, i_2, i_3, i_4 \in I$, there is an $i \in I$ such that

$$f^{\mathbb{A}^\delta}(o_{i_1}, o'_{i_2}, c_{i_3}, c'_{i_4}) \leq f^{\mathbb{A}^\delta}(o_i, o'_i, c_i, c'_i).$$

Fix $i_1, i_2, i_3, i_4 \in I$, and consider the finite subset

$$\{(o_{i_k}, o'_{i_k}, c_{i_k}, c'_{i_k}) : 1 \leq k \leq 4\} \subseteq \{(o_i, o'_i, c_i, c'_i) : i \in I\};$$

by the upward-directedness of $\{(o_i, o'_i, c_i, c'_i) : i \in I\}$ in $(\mathbb{A}^\delta)^\epsilon$, this finite subset has a common upper bound in $\{(o_i, o'_i, c_i, c'_i) : i \in I\}$. That is, there exists some $i \in I$ such that $(o_{i_k}, o'_{i_k}, c_{i_k}, c'_{i_k}) \leq_\epsilon (o_i, o'_i, c_i, c'_i)$ for $1 \leq k \leq 4$. This implies that $(o_{i_1}, o'_{i_2}, c_{i_3}, c'_{i_4}) \leq_\epsilon (o_i, o'_i, c_i, c'_i)$, and thus by the ϵ-monotonicity of $f^{\mathbb{A}^\delta}$, we get $f^{\mathbb{A}^\delta}(o_{i_1}, o'_{i_2}, c_{i_3}, c'_{i_4}) \leq f^{\mathbb{A}^\delta}(o_i, o'_i, c_i, c'_i)$.

□

6.3 Generalized σ-Expanding and π-Contracting Maps

In this section we give sufficient conditions for maps to be σ-expanding or π-contracting. In the following we only give the proofs for the property of being σ-expanding, and the proofs for the property of being π-contracting can be obtained order dually.

6.3.1 Sufficient Conditions

First of all, we can generalize the result stating that f^σ (resp. f^π) is the largest UC (resp. smallest LC) extension of f to the setting of generalized canonical extensions of maps. The proof strategy is the same as Proposition 117 in Chapter 6 in [4].

Theorem 6.11. *Given any order-preserving map $f^{\mathbb{A}} : \mathbb{A} \to \mathbb{B}^\delta$,*

1. *$(f^{\mathbb{A}})^\sigma$ is the largest order-preserving UC extension of $f^{\mathbb{A}}$ to $\mathbb{A}^\delta$.*
2. *$(f^{\mathbb{A}})^\pi$ is the smallest order-preserving LC extension of $f^{\mathbb{A}}$ to $\mathbb{A}^\delta$.*

Proof. Here we only give the proof of 1, and the proof of 2. can be obtained order dually. The upper continuity of $(f^{\mathbb{A}})^\sigma$ easily follows from the definition of generalized σ-extension. To show that $(f^{\mathbb{A}})^\sigma$ is the largest such map, suppose we have another order-preserving map $g^{\mathbb{A}^\delta} : \mathbb{A}^\delta \to \mathbb{B}^\delta$ extending $f^{\mathbb{A}}$ satisfying UC, then it suffices to show that $g^{\mathbb{A}^\delta} \leq (f^{\mathbb{A}})^\sigma$:

Suppose otherwise, there is a $u \in \mathbb{A}^\delta$ such that $g^{\mathbb{A}^\delta}(u) \nleq (f^{\mathbb{A}})^\sigma(u)$, then there is an $i \in J^\infty(\mathbb{B}^\delta)$ such that $i \leq g^{\mathbb{A}^\delta}(u)$ but $i \nleq (f^{\mathbb{A}})^\sigma(u)$, then since $g^{\mathbb{A}^\delta}$ satisfies UC, we have that there is an x such that $K(\mathbb{A}^\delta) \ni x \leq u$ such that $i \leq g^{\mathbb{A}^\delta}(x)$. Then by the monotonicity of $g^{\mathbb{A}^\delta}$ and the fact that both $(f^{\mathbb{A}})^\sigma$ and $g^{\mathbb{A}^\delta}$ extend $f^{\mathbb{A}}$, we have that $i \leq g^{\mathbb{A}^\delta}(x) \leq g^{\mathbb{A}^\delta}(a) = f^{\mathbb{A}}(a)$ for all a such that $\mathbb{A} \ni a \geq x$. Therefore $i \leq \bigwedge\{f^{\mathbb{A}}(a) : \mathbb{A} \ni a \geq x\} = (f^{\mathbb{A}})^\sigma(x) \leq (f^{\mathbb{A}})^\sigma(u)$, a contradiction. □

The following facts then immediately follow as a corollary of the theorem above:

Corollary 6.12. *Given any order-preserving map $g^{\mathbb{A}^\delta} : \mathbb{A}^\delta \to \mathbb{B}^\delta$,*

1. *if $g^{\mathbb{A}^\delta}$ is UC, then $g^{\mathbb{A}^\delta} \leq (g^{\mathbb{A}})^\sigma$, i.e. g is σ-expanding;*
2. *if $g^{\mathbb{A}^\delta}$ is LC, then $g^{\mathbb{A}^\delta} \geq (g^{\mathbb{A}})^\pi$, i.e. g is π-contracting.*

6.3.2 Scott and Dual Scott Continuity

In this subsection we consider *operators*, i.e., order-preserving maps $f^{\mathbb{A}} : \mathbb{A}^{\epsilon} \to \mathbb{B}^{\delta}$ that preserve finite joins in each coordinate. Recall that *complete operators* are order-preserving maps $f^{\mathbb{A}} : \mathbb{A}^{\epsilon} \to \mathbb{B}^{\delta}$ that preserve arbitrary joins in each coordinate. *Dual operators* and *complete dual operators* are defined dually.

Lemma 6.13. *Given any order-preserving map $f^{\mathbb{A}} : \mathbb{A}^{\epsilon} \to \mathbb{B}^{\delta}$,*

1. *if $f^{\mathbb{A}}$ is an operator, then $(f^{\mathbb{A}})^{\sigma} : (\mathbb{A}^{\epsilon})^{\delta} \to \mathbb{B}^{\delta}$ is a complete operator;*
2. *if $f^{\mathbb{A}}$ is a dual operator, then $(f^{\mathbb{A}})^{\pi} : (\mathbb{A}^{\epsilon})^{\delta} \to \mathbb{B}^{\delta}$ is a complete dual operator.*

Proof. Similar to Lemma 2.22 in [16]. □

Then we have the following theorem:

Theorem 6.14. *Given any order-preserving map $f^{\mathbb{A}^{\delta}} : (\mathbb{A}^{\epsilon})^{\delta} \to \mathbb{B}^{\delta}$,*

1. *if $f^{\mathbb{A}^{\delta}}$ is a complete operator, then $f^{\mathbb{A}^{\delta}}$ is Scott continuous, hence σ-expanding;*
2. *if $f^{\mathbb{A}^{\delta}}$ is a complete dual operator, then $f^{\mathbb{A}^{\delta}}$ is dual Scott continuous, hence π-contracting.*

Proof. Here we only prove 1, and 2. can be obtained order dually. For the first half about Scott continuity, the proof is similar to Theorem 5.6 in [17]. For the second half, since Scott continuity is a special case of UC, for complete operator $f^{\mathbb{A}^{\delta}} : (\mathbb{A}^{\epsilon})^{\delta} \to \mathbb{B}^{\delta}$, $f^{\mathbb{A}^{\delta}}$ is UC, by Corollary 6.12, $f^{\mathbb{A}^{\delta}}$ is σ-expanding. □

Then we can generalize Corollary 5.7 in [17] as follows:

Lemma 6.15. *Given any order-preserving maps $f^{\mathbb{A}^{\delta}} : \mathbb{A}^{\delta} \to (\mathbb{B}^{\delta})^{\epsilon}$ and $g^{\mathbb{B}^{\delta}} : (\mathbb{B}^{\delta})^{\epsilon} \to \mathbb{C}^{\delta}$ and $(g \circ f)^{\mathbb{A}}$ obtained by restricting $g^{\mathbb{B}^{\delta}} f^{\mathbb{A}^{\delta}}$ to $\mathbb{A}$,*

1. *if $g^{\mathbb{B}^{\delta}}$ is a complete operator, then $g^{\mathbb{B}^{\delta}}(f^{\mathbb{A}})^{\sigma} \leq ((g \circ f)^{\mathbb{A}})^{\sigma}$;*
2. *if $g^{\mathbb{B}^{\delta}}$ is a complete dual operator, then $g^{\mathbb{B}^{\delta}}(f^{\mathbb{A}})^{\pi} \geq ((g \circ f)^{\mathbb{A}})^{\pi}$.*

Proof. Here we only prove 1, and 2. can be obtained order dually. By Theorem 6.11, $(f^{\mathbb{A}})^{\sigma}$ is UC. By Theorem 6.14, $g^{\mathbb{B}^{\delta}}$ is Scott continuous, hence $g^{\mathbb{B}^{\delta}}(f^{\mathbb{A}})^{\sigma}$ is UC. By Theorem 6.11 again, $((g \circ f)^{\mathbb{A}})^{\sigma}$ is the largest order-preserving UC extension of $(g \circ f)^{\mathbb{A}}$ to $\mathbb{A}^{\delta}$. It is also easy to see that $g^{\mathbb{B}^{\delta}}(f^{\mathbb{A}})^{\sigma}$ coincode with $(g \circ f)^{\mathbb{A}}$ on $\mathbb{A}$, hence $g^{\mathbb{B}^{\delta}}(f^{\mathbb{A}})^{\sigma}$ is also an order-preserving UC extension of $(g \circ f)^{\mathbb{A}}$ to $\mathbb{A}^{\delta}$. Therefore we have $g^{\mathbb{B}^{\delta}}(f^{\mathbb{A}})^{\sigma} \leq ((g \circ f)^{\mathbb{A}})^{\sigma}$. □

6.3.3 Strong Upper and Lower Continuity

In [17], it is proved that for meet-preserving maps $f^{\mathbb{A}} : \mathbb{A} \to \mathbb{B}$, $(f^{\mathbb{A}})^\sigma$ is SUC. Here we generalize this result as the following theorem:

Theorem 6.16. *1. Let $f^{\mathbb{A}^\delta} : \mathbb{A}^\delta \to \mathbb{B}^\delta$ be completely meet-preserving, satisfying the open Esakia condition (cf. Definition 6.6), and such that $f^{\mathbb{A}^\delta}(a) \in O(\mathbb{B}^\delta)$ for all $a \in \mathbb{A}$. Then it is SUC. Hence, $f^{\mathbb{A}^\delta}$ is also σ-expanding.*

2. Let $f^{\mathbb{A}^\delta} : \mathbb{A}^\delta \to \mathbb{B}^\delta$ be completely join-preserving, satisfying the closed Esakia condition (cf. Definition 6.6), and such that $f^{\mathbb{A}^\delta}(a) \in K(\mathbb{B}^\delta)$ for all $a \in \mathbb{A}$. Then it is SLC. Hence, $f^{\mathbb{A}^\delta}$ is also π-contracting.

Proof. Here we only prove 1, and 2. can be obtained order dually. Let $u \in \mathbb{A}^\delta$ and $q \in K(\mathbb{B}^\delta)$ such that $q \leq f^{\mathbb{A}^\delta}(u)$. Then $u = \bigwedge\{\bigvee\{a : y \geq a \in \mathbb{A}\} : u \leq y \in O(\mathbb{A}^\delta)\}$. Since $f^{\mathbb{A}^\delta}$ is completely meet-preserving, we have that $q \leq \bigwedge\{f^{\mathbb{A}^\delta}(\bigvee\{a : y \geq a \in \mathbb{A}\}) : u \leq y \in O(\mathbb{A}^\delta)\}$. Hence for all y such that $O(\mathbb{A}^\delta) \ni y \geq u$, we have that $q \leq f^{\mathbb{A}^\delta}(\bigvee\{a : y \geq a \in \mathbb{A}\})$. Since by the open Esakia condition, we have that $f^{\mathbb{A}^\delta}(\bigvee\{a : y \geq a \in \mathbb{A}\}) = \bigvee\{f^{\mathbb{A}^\delta}(a) : y \geq a \in \mathbb{A}\}$, by compactness we have that there is a finite subset $\mathcal{A}_0 \subseteq \{a : y \geq a \in \mathbb{A}\}$ such that $q \leq \bigvee\{f^{\mathbb{A}^\delta}(a) : a \in \mathcal{A}_0\}$, therefore by taking $a_y = \bigvee \mathcal{A}_0 \in \mathbb{A}$, we have that $q \leq f^{\mathbb{A}^\delta}(a_y)$ by the monotonicity of $f^{\mathbb{A}^\delta}$. Hence, we have that $q \leq \bigwedge\{f^{\mathbb{A}^\delta}(a_y) : O(\mathbb{A}^\delta) \ni y \geq u\}$, and since $f^{\mathbb{A}^\delta}$ is completely meet preserving, we have $q \leq f^{\mathbb{A}^\delta}(\bigwedge\{a_y : O(\mathbb{A}^\delta) \ni y \geq u\})$. Since $a_y = \bigvee \mathcal{A}_0 \leq \bigvee\{a : y \geq a \in \mathbb{A}\} = y$ for all y such that $O(\mathbb{A}^\delta) \ni y \geq u$, we have $\bigwedge\{a_y : O(\mathbb{A}^\delta) \ni y \geq u\} \leq u$. $\bigwedge\{a_y : O(\mathbb{A}^\delta) \ni y \geq u\} \in K(\mathbb{A}^\delta)$ follows from that $a_y \in \mathbb{A}$ for all y such that $O(\mathbb{A}^\delta) \ni y \geq u$.

As to the second part of the statement, since SUC is a special case of UC, we have that $f^{\mathbb{A}^\delta}$ satisfies UC, since $(f^{\mathbb{A}})^\sigma$ is the largest order-preserving UC map extending $f^{\mathbb{A}}$, we have that $f^{\mathbb{A}^\delta} \leq (f^{\mathbb{A}})^\sigma$, therefore $f^{\mathbb{A}^\delta}$ is σ-expanding. ∎

We can also generalize Corollary 5.9 in [17] as follows:

Lemma 6.17. *Given any order-preserving maps $f^{\mathbb{A}^\delta} : \mathbb{A}^\delta \to (\mathbb{B}^\delta)^\epsilon$ and $g^{\mathbb{B}^\delta} : (\mathbb{B}^\delta)^\epsilon \to \mathbb{C}^\delta$ and $(g \circ f)^{\mathbb{A}}$ obtained by restricting $g^{\mathbb{B}^\delta} f^{\mathbb{A}^\delta}$ to $\mathbb{A}$,*

1. *if $f^{\mathbb{A}^\delta}$ is completely meet-preserving, satisfies the open Esakia condition and is such that $f^{\mathbb{A}^\delta}(a) \in O(\mathbb{B}^\delta)$ for all $a \in \mathbb{A}$, and $g^{\mathbb{B}^\delta}$ satisfies the open Esakia condition, then $(g^{\mathbb{B}})^\sigma f^{\mathbb{A}^\delta} \leq ((g \circ f)^{\mathbb{A}})^\sigma$;*

2. *if $f^{\mathbb{A}^\delta}$ is completely join-preserving, satisfies the closed Esakia condition and is such that $f^{\mathbb{A}^\delta}(a) \in K(\mathbb{B}^\delta)$ for all $a \in \mathbb{A}$, and $g^{\mathbb{B}^\delta}$ satisfies the closed Esakia condition, then $(g^{\mathbb{B}})^\pi f^{\mathbb{A}^\delta} \geq ((g \circ f)^{\mathbb{A}})^\pi$.*

Proof. Here we only prove 1, and 2. can be obtained order dually. By Theorem 6.11, $(g^{\mathbb{B}})^\sigma$ is UC. By Theorem 6.16, $f^{\mathbb{A}^\delta}$ is SUC, hence $(g^{\mathbb{B}})^\sigma f^{\mathbb{A}^\delta}$ is UC. By Theorem 6.11 again, $((g \circ f)^{\mathbb{A}})^\sigma$ is the largest order-preserving UC extension of $(g \circ f)^{\mathbb{A}}$ to $\mathbb{A}^\delta$. Since $f^{\mathbb{A}^\delta}(a) \in O(\mathbb{B}^\delta)$ for all $a \in \mathbb{A}$, by Theorem 6.7 and that $g^{\mathbb{B}^\delta}$ satisfies the open Esakia condition, $(g^{\mathbb{B}})^\sigma f^{\mathbb{A}^\delta}$ coincode with $g^{\mathbb{B}^\delta} f^{\mathbb{A}^\delta}$ on $\mathbb{A}$, hence $(g^{\mathbb{B}})^\sigma f^{\mathbb{A}^\delta}$ is also an order-preserving UC extension of $(g \circ f)^{\mathbb{A}}$ to $\mathbb{A}^\delta$. Therefore we have $(g^{\mathbb{B}})^\sigma f^{\mathbb{A}^\delta} \leq ((g \circ f)^{\mathbb{A}})^\sigma$. ∎

Chapter 7

ALBA-Aided Jónsson-Style Canonicity

In this chapter, we prove the canonicity of ALBA inequalities in the language of distributive modal logic using the method of generalized canonical extensions of maps. As we have already seen in Section 3.2, when trying to prove the canonicity of inductive inequalities within the original signature DML, the term functions for α_1 and β_1 do not have the good properties (namely, the property of being σ-expanding for α_1 and the property of being π-contracting for β_1) which would be sufficient to guarantee the canonicity according to the strategy in [17]. Therefore, in this chapter we generalize the proof scheme (in fact, the proof scheme after the refinement in Section 3.1.3) to the expanded signature $\mathcal{L}^{++}$, by making manipulations aimed at reformulating the inequality $\alpha \leq \beta$ into inequalities of the form $\alpha_1 \leq \gamma$, where α_1 is of order type ϵ and σ-expanding, i.e., it is generalized ϵ-left Sahlqvist, and γ is of order type ϵ and σ-contracting, i.e., it is a generalized **n**-part (see Section 3.1.1, discussion before Lemma 3.7). However, we need to use more complex manipulations than the simple **n**-trick in order to make full use of the expanded signature. After the proof, a comparison between the Jónsson-style canonicity and the ALBA-style canonicity is given.

Before getting into the details, we first define canonical extensions for terms in $\mathcal{L}^{++}$, and prove some results about the corresponding contracting or expanding term functions.

7.1 Generalized Canonical Extensions for $\mathcal{L}^{++}$

In this section, we define generalized canonical extensions for the expanded language $\mathcal{L}^{++}$, and show that certain terms in $\mathcal{L}^{++}$ are contracting or expanding, which will be used in the canonicity proof for ALBA inequalities in distributive modal logic in Section 7.2.

7.1.1 Generalized Canonical Extensions for Term Functions in $\mathcal{L}^{++}$

Since, in $\mathcal{L}^{++}$, not only new logical symbols are introduced, but also nominals and co-nominals (see Section 2.5), we need to generalize the definitions of term functions, canonical extensions and contracting and expanding terms to this setting. Here we give two different kinds of definitions of term functions for convenience of work: the first one takes nominals and co-nominals as already assigned constants, the second one takes them as arguments.

Definition 7.1 (Term Functions in $\mathcal{L}^{++}$). Let $\mathbb{A}$ be a DMA and $t(\vec{p}, \vec{j}, \vec{m})$ be an $\mathcal{L}^{++}$-term, where the proposition variables, nominals and co-nominals actually occurring in t are in $\vec{p}, \vec{j}, \vec{m}$, respectively. For any arrays $\vec{x}, \vec{y}$ of suitable lengths, and such that $x \in J^\infty(\mathbb{A}^\delta)$ for each coordinate x in $\vec{x}$, and $y \in M^\infty(\mathbb{A}^\delta)$ for each coordinate y in $\vec{y}$, we use $t^{\mathbb{A}^\delta}_{\vec{x},\vec{y}}$ to denote the term function

$$(\mathbb{A}^\delta)^n \to \mathbb{A}^\delta$$

such that the variables in $\vec{j}, \vec{m}$ are orderly interpreted as elements in $\vec{x}, \vec{y}$, respectively. In the following, we abuse notation and write expressions such as $\vec{x} \in J^\infty(\mathbb{A}^\delta)$, even if they are not set theoretically proper. We use the symbol $t^{\mathbb{A}^\delta}$ to denote as usual the function

$$(\mathbb{A}^\delta)^n \times (J^\infty(\mathbb{A}^\delta))^k \times (M^\infty(\mathbb{A}^\delta))^l \to \mathbb{A}^\delta.$$

Finally, we use the symbols $t^{\mathbb{A}}_{\vec{x},\vec{y}}$, $t^{\mathbb{A}}$ to respectively denote the restrictions of the term functions $t^{\mathbb{A}^\delta}_{\vec{x},\vec{y}}$ and $t^{\mathbb{A}^\delta}$ to $\mathbb{A}^n$. In other words, we have:

$$t^{\mathbb{A}}_{\vec{x},\vec{y}} : (\mathbb{A})^n \to \mathbb{A}^\delta$$

and

$$t^{\mathbb{A}} : (\mathbb{A})^n \times (J^\infty(\mathbb{A}^\delta))^k \times (M^\infty(\mathbb{A}^\delta))^l \to \mathbb{A}^\delta,$$

respectively.

The theory developed in Chapter 6 now can be applied to $\mathcal{L}^{++}$-term functions where nominals and co-nominals have been fixed. So we start by giving the following definition which is nothing else but the Definition 6.2 applied to $t^{\mathbb{A}}_{\vec{x},\vec{y}}$:

Definition 7.2 (Generalized Canonical Extensions for Term Functions in $\mathcal{L}^{++}$). Let the term $t(\vec{p}, \vec{q}, \vec{j}, \vec{m})$ be a uniform $\mathcal{L}^{++}$-term, such that the nominals and co-nominals actually occurring in t are in $\vec{j}, \vec{m}$, and let $\vec{p}, \vec{q}$ be the arrays of positive and negative variables, respectively. For any DMA $\mathbb{A}$ and all $\vec{u}, \vec{v} \in \mathbb{A}^\delta$, $\vec{x} \in J^\infty(\mathbb{A}^\delta)$, $\vec{y} \in M^\infty(\mathbb{A}^\delta)$, let

$$(t^{\mathbb{A}}_{\vec{x},\vec{y}})^\sigma : (\mathbb{A}^\delta)^{n_1} \times ((\mathbb{A}^\delta)^\partial)^{n_2} \to \mathbb{A}^\delta \quad \text{and} \quad (t^{\mathbb{A}}_{\vec{x},\vec{y}})^\pi : (\mathbb{A}^\delta)^{n_1} \times ((\mathbb{A}^\delta)^\partial)^{n_2} \to \mathbb{A}^\delta$$

be respectively defined by the assignments $(\vec{u}, \vec{v}) \mapsto (t^{\mathbb{A}})^\sigma(\vec{u}, \vec{v}, \vec{x}, \vec{y})$, and $(\vec{u}, \vec{v}) \mapsto (t^{\mathbb{A}})^\pi(\vec{u}, \vec{v}, \vec{x}, \vec{y})$, where:

$$(t^{\mathbb{A}})^\sigma(\vec{u}, \vec{v}, \vec{x}, \vec{y}) = \bigvee\{\bigwedge\{t^{\mathbb{A}}(\vec{a}, \vec{b}, \vec{x}, \vec{y}) : \vec{c} \leq \vec{a} \in \mathbb{A}, \vec{o} \geq \vec{b} \in \mathbb{A}\} : \vec{u} \geq \vec{c} \in K(\mathbb{A}^\delta), \vec{v} \leq \vec{o} \in O(\mathbb{A}^\delta)\}$$

$$(t^{\mathbb{A}})^\pi(\vec{u}, \vec{v}, \vec{x}, \vec{y}) = \bigwedge\{\bigvee\{t^{\mathbb{A}}(\vec{a}, \vec{b}, \vec{x}, \vec{y}) : \vec{o} \geq \vec{a} \in \mathbb{A}, \vec{c} \leq \vec{b} \in \mathbb{A}\} : \vec{u} \leq \vec{o} \in O(\mathbb{A}^\delta), \vec{v} \geq \vec{c} \in K(\mathbb{A}^\delta)\}.$$

Next, we give the corresponding definition of contracting and expanding terms:

Definition 7.3 (Generalized Contracting and Expanding Terms in $\mathcal{L}^{++}$). For a uniform term $t(\vec{p}, \vec{j}, \vec{m})$ in $\mathcal{L}^{++}$ where $\vec{j}, \vec{m}$ are all nominals and co-nominals in t, and $\lambda \in \{\sigma, \pi\}$,

- t is λ-*stable*, if for all DMAs $\mathbb{A}$, $t^{\mathbb{A}^\delta} = (t^{\mathbb{A}})^\lambda$, i.e. $t^{\mathbb{A}^\delta}_{\vec{x},\vec{y}} = (t^{\mathbb{A}}_{\vec{x},\vec{y}})^\lambda$ for all $\vec{x} \in J^\infty(\mathbb{A}^\delta)$, $\vec{y} \in M^\infty(\mathbb{A}^\delta)$;

- t is λ-*expanding*, if for all DMAs $\mathbb{A}$, $t^{\mathbb{A}^\delta} \leq (t^{\mathbb{A}})^\lambda$, i.e. $t^{\mathbb{A}^\delta}_{\vec{x},\vec{y}} \leq (t^{\mathbb{A}}_{\vec{x},\vec{y}})^\lambda$ for all $\vec{x} \in J^\infty(\mathbb{A}^\delta)$, $\vec{y} \in M^\infty(\mathbb{A}^\delta)$;
- t is λ-*contracting*, if for all DMAs $\mathbb{A}$, $t^{\mathbb{A}^\delta} \geq (t^{\mathbb{A}})^\lambda$, i.e. $t^{\mathbb{A}^\delta}_{\vec{x},\vec{y}} \geq (t^{\mathbb{A}}_{\vec{x},\vec{y}})^\lambda$ for all $\vec{x} \in J^\infty(\mathbb{A}^\delta)$, $\vec{y} \in M^\infty(\mathbb{A}^\delta)$.

The new connectives turn out to be very well behaved, as we are going to discuss in the next subsection.

7.1.2 σ- and π-Stability of Connectives in the Expanded Signature

The Esakia properties of $\blacksquare, \blacklozenge, \blacktriangleright, \blacktriangleleft, -, \rightarrow$ were fundamental to prove the topological Ackermann lemmas in Chapter 4. In this part, we will use them again to establish the stability properties of these connectives, as well as those of $\mathbf{n}, \mathbf{l}$. Throughout this section, we omit superscripts of term functions when they are clear from the context, since their domain is typically $\mathbb{A}^\delta$.

In the lemma below, we list all the preservation properties which follow from the fact, well known from the general order theory (see [12]), that each of the new operators is either a right or left residual or adjoint.

Lemma 7.4. *For any $\mathcal{A} \subseteq \mathbb{A}^\delta, u \in \mathbb{A}^\delta$,*

1. $\blacksquare \bigwedge \mathcal{A} = \bigwedge\{\blacksquare v : v \in \mathcal{A}\}$;
2. $\blacklozenge \bigvee \mathcal{A} = \bigvee\{\blacklozenge v : v \in \mathcal{A}\}$;
3. $\blacktriangleright \bigvee \mathcal{A} = \bigwedge\{\blacktriangleright v : v \in \mathcal{A}\}$;
4. $\blacktriangleleft \bigwedge \mathcal{A} = \bigvee\{\blacktriangleleft v : v \in \mathcal{A}\}$;
5. $u - \bigwedge \mathcal{A} = \bigvee\{u - v : v \in \mathcal{A}\}$, $\bigvee \mathcal{A} - u = \bigvee\{v - u : v \in \mathcal{A}\}$;
6. $u \rightarrow \bigwedge \mathcal{A} = \bigwedge\{u \rightarrow v : v \in \mathcal{A}\}$, $\bigvee \mathcal{A} \rightarrow u = \bigwedge\{v \rightarrow u : v \in \mathcal{A}\}$;
7. $\mathbf{n}(u, \bigwedge \mathcal{A}) = \bigvee\{\mathbf{n}(u, v) : v \in \mathcal{A}\}$, $\mathbf{n}(\bigvee \mathcal{A}, u) = \bigvee\{\mathbf{n}(v, u) : v \in \mathcal{A}\}$;
8. $\mathbf{l}(u, \bigwedge \mathcal{A}) = \bigwedge\{\mathbf{l}(u, v) : v \in \mathcal{A}\}$, $\mathbf{l}(\bigvee \mathcal{A}, u) = \bigwedge\{\mathbf{l}(v, u) : v \in \mathcal{A}\}$.

This implies in particular that $\blacksquare, \blacktriangleright, \rightarrow, \mathbf{l}$ satisfy the closed Esakia condition, and $\blacklozenge, \blacktriangleleft, -, \mathbf{n}$ satisfy the open Esakia condition (see Definition 6.6).

The next step is to prove the following proposition:

Proposition 7.5. $\blacksquare, \blacklozenge, \blacktriangleright, \blacktriangleleft, -, \rightarrow, \mathbf{n}, \mathbf{l}$ *are both σ-contracting and π-expanding.*

Proof. By Theorem 6.8, it is enough to show that all the connectives mentioned above satisfy both the open and the closed Esakia condition (see Definition 6.6). By the conclusion of Lemma 7.4, it remains to be shown that $\blacksquare, \blacktriangleright, \rightarrow, \mathbf{l}$ satisfy the open Esakia condition, and $\blacklozenge, \blacktriangleleft, -, \mathbf{n}$ satisfy the closed Esakia condition. Lemma 4.10 equivalently says that $\blacksquare, \blacktriangleright, \rightarrow$ enjoy the open Esakia condition, and that $\blacklozenge, \blacktriangleleft, -$ enjoy the closed Esakia condition. Actually, for the binary connectives $-$ and $\rightarrow$, Lemma 4.10 states that they enjoy the intersection condition. However, by Lemma 6.10, this condition is equivalent to the Esakia condition we require. Finally, Lemma 7.6 below takes care of the remaining intersection conditions for $\mathbf{n}$ and $\mathbf{l}$, which again by Lemma 6.10 imply the required Esakia conditions. □

Lemma 7.6. *Let $\mathcal{C} = \{c_i : i \in I\} \subseteq K(\mathbb{A}^\delta)$ be downward-directed, $\mathcal{O} = \{o_i : i \in I\} \subseteq O(\mathbb{A}^\delta)$ be upward-directed. Then:*

1. $\mathbf{n}(\bigwedge \mathcal{C}, \bigvee \mathcal{O}) = \bigwedge\{\mathbf{n}(c, o) : c \in \mathcal{C}, o \in \mathcal{O}\}$;
2. $\mathbf{l}(\bigwedge \mathcal{C}, \bigvee \mathcal{O}) = \bigvee\{\mathbf{l}(c, o) : c \in \mathcal{C}, o \in \mathcal{O}\}$.

Proof. We prove 2., since 1. is an order-variant of 2. To show 2, notice that the inequality from right to left straightforwardly follows by the $(\partial, 1)$-monotonicity of $\mathbf{l}$. For the converse, since the value of $\mathbf{l}$ is either $\top$ or $\bot$, we only consider the case in which $\mathbf{l}(\bigwedge \mathcal{C}, \bigvee \mathcal{O}) = \top$, and show that there exist some $c \in \mathcal{C}$ and $o \in \mathcal{O}$ such that $\mathbf{l}(c, o) = \top$. From $\mathbf{l}(\bigwedge \mathcal{C}, \bigvee \mathcal{O}) = \top$ we have that $\bigwedge \mathcal{C} \leq \bigvee \mathcal{O}$, hence by compactness, $\bigwedge \mathcal{C}_0 \leq \bigvee \mathcal{O}_0$ for some finite $\mathcal{C}_0 \subseteq \mathcal{C}$ and $\mathcal{O}_0 \subseteq \mathcal{O}$. By the directedness of $\mathcal{C}$ and $\mathcal{O}$, there exist some $c \in \mathcal{C}$ and $o \in \mathcal{O}$ such that $c \leq \bigwedge \mathcal{C}_0 \leq \bigvee \mathcal{O}_0 \leq o$, therefore $\mathbf{l}(c, o) = \top$. □

Next, we are in a position to prove the following:

Proposition 7.7. $\blacksquare, \blacklozenge, \blacktriangleright, \blacktriangleleft$ *are both σ-stable and π-stable.*

Proof. By Theorem 6.16.1, in order to prove that $\blacksquare : \mathbb{A}^\delta \rightarrow \mathbb{A}^\delta$ and $\blacktriangleright : (\mathbb{A}^\delta)^\partial \rightarrow \mathbb{A}^\delta$ are σ-expanding, it is enough to show that they are completely meet-preserving, satisfying the open Esakia condition, and such that their value on clopen input is always open. These conditions are respectively satisfied by Lemmas 7.4, 4.10, and 4.7. Likewise, in order to prove that $\blacklozenge : \mathbb{A}^\delta \rightarrow \mathbb{A}^\delta$ and $\blacktriangleleft : (\mathbb{A}^\delta)^\partial \rightarrow \mathbb{A}^\delta$ are π-contracting, we use Theorem 6.16.2 and Lemmas 7.4, 4.10, and 4.7. Together with Proposition 7.5, this proves that $\blacksquare$ and $\blacktriangleright$ are σ-stable, and that $\blacklozenge$ and $\blacktriangleleft$ are π-stable.

For the remaining parts of the statement, let us prove that $\blacksquare$ is π-stable, the others being order-variants of it. In the following proof, to make the domain of the functions clear, we use superscripts. Indeed, for any $u \in \mathbb{A}^\delta$, since $\blacksquare^{\mathbb{A}^\delta}$ is completely meet-preserving and satisfies the open Esakia condition, the following chain of identities holds:

$$
\begin{aligned}
\blacksquare^{\mathbb{A}^\delta} u &= \blacksquare^{\mathbb{A}^\delta}(\bigwedge\{\bigvee\{a : o \geq a \in \mathbb{A}\} : u \leq o \in O(\mathbb{A}^\delta)\}) \\
&= \bigwedge\{\blacksquare^{\mathbb{A}^\delta}(\bigvee\{a : o \geq a \in \mathbb{A}\}) : u \leq o \in O(\mathbb{A}^\delta)\} \\
&= \bigwedge\{\bigvee\{\blacksquare^{\mathbb{A}^\delta} a : o \geq a \in \mathbb{A}\} : u \leq o \in O(\mathbb{A}^\delta)\} \\
&= \bigwedge\{\bigvee\{\blacksquare^{\mathbb{A}} a : o \geq a \in \mathbb{A}\} : u \leq o \in O(\mathbb{A}^\delta)\} \\
&= (\blacksquare^{\mathbb{A}})^\pi(u).
\end{aligned}
$$

□

Finally, we state the stability properties enjoyed by the remaining binary connectives.

Proposition 7.8. *$\mathbf{n}, -$ are σ-stable and $\mathbf{l}, \rightarrow$ are π-stable.*

Proof. By Proposition 7.5, it remains to be shown that $\mathbf{n}, -$ are σ-expanding and $\mathbf{l}, \rightarrow$ are π-contracting. By Theorem 6.14.1, to show that $\mathbf{n}, - : \mathbb{A}^\delta \times (\mathbb{A}^\delta)^\partial \rightarrow \mathbb{A}^\delta$ are σ-expanding, it is enough to show that they are complete operators, which follows from Lemma 7.4. Likewise, in order to prove that $\mathbf{l}, \rightarrow: (\mathbb{A}^\delta)^\partial \times \mathbb{A}^\delta \rightarrow \mathbb{A}^\delta$ are π-contracting, we use Theorem 6.14.2, Proposition 7.5, and Lemma 7.4. $\square$

7.1.3 Generalized σ-Contracting and π-Expanding Terms in $\mathcal{L}^{++}$

The aim of this subsection is to show that all uniform syntactically closed (resp. open) terms are σ-contracting (resp. π-expanding) (Theorem 7.11). This will be useful to treat the ϵ^∂-part in the Jónsson-style canonicity strategy.

Lemma 7.9 (Intersection Lemma)**.** *Let $\varphi(\vec{p}, \vec{q}, r, \vec{j}, \vec{m})$ and $\psi(\vec{p}, \vec{q}, r, \vec{j}, \vec{m})$ be uniform syntactically closed and open terms in $\mathcal{L}^{++}$, respectively, where $\vec{p}$ and $\vec{q}$ respectively occur positively and negatively, and $\vec{j}, \vec{m}$ are all nominals and co-nominals occuring in φ and ψ. Let $\vec{x} \in J^\infty(\mathbb{A}^\delta), \vec{y} \in M^\infty(\mathbb{A}^\delta)$, $\vec{c} \in K(\mathbb{A}^\delta), \vec{o} \in O(\mathbb{A}^\delta)$, $\{c_i : i \in I\} \subseteq K(\mathbb{A}^\delta)$ be downward-directed, $\{o_i : i \in I\} \subseteq O(\mathbb{A}^\delta)$ be upward-directed.*

1. *If $\varphi(\vec{p}, \vec{q}, r, \vec{j}, \vec{m})$ is positive in r, then $\varphi^{\mathbb{A}^\delta}(\vec{c}, \vec{o}, \bigwedge\{c_i : i \in I\}, \vec{x}, \vec{y}) = \bigwedge\limits_{i\in I} \varphi^{\mathbb{A}^\delta}(\vec{c}, \vec{o}, c_i, \vec{x}, \vec{y})$.*

2. *If $\psi(\vec{p}, \vec{q}, r, \vec{j}, \vec{m})$ is negative in r, then $\psi^{\mathbb{A}^\delta}(\vec{o}, \vec{c}, \bigwedge\{c_i : i \in I\}, \vec{x}, \vec{y}) = \bigvee\limits_{i\in I} \psi^{\mathbb{A}^\delta}(\vec{o}, \vec{c}, c_i, \vec{x}, \vec{y})$.*

3. *If $\varphi(\vec{p}, \vec{q}, r, \vec{j}, \vec{m})$ is negative in r, then $\varphi^{\mathbb{A}^\delta}(\vec{c}, \vec{o}, \bigvee\{o_i : i \in I\}, \vec{x}, \vec{y}) = \bigwedge\limits_{i\in I} \varphi^{\mathbb{A}^\delta}(\vec{c}, \vec{o}, o_i, \vec{x}, \vec{y})$.*

4. *If $\psi(\vec{p}, \vec{q}, r, \vec{j}, \vec{m})$ is positive in r, then $\psi^{\mathbb{A}^\delta}(\vec{o}, \vec{c}, \bigvee\{o_i : i \in I\}, \vec{x}, \vec{y}) = \bigvee\limits_{i\in I} \psi^{\mathbb{A}^\delta}(\vec{o}, \vec{c}, o_i, \vec{x}, \vec{y})$.*

Proof. The proof goes by simultaneous induction on φ and ψ, which is similar to the proof of Lemma 4.12. $\square$

Corollary 7.10. *Let $\varphi(\vec{p}, \vec{j}, \vec{m})$ and $\psi(\vec{p}, \vec{j}, \vec{m})$ be uniform syntactically closed and open terms in $\mathcal{L}^{++}$, respectively, where $\vec{p}, \vec{j}, \vec{m}$ are all proposition variables, nominals and co-nominals occuring in φ and ψ, respectively. Let $\vec{x} \in J^\infty(\mathbb{A}^\delta), \vec{y} \in M^\infty(\mathbb{A}^\delta)$. Then,*

1. *$\varphi^{\mathbb{A}^\delta}_{\vec{x},\vec{y}}(\vec{p})$ satisfies the closed Esakia condition;*

2. *$\psi^{\mathbb{A}^\delta}_{\vec{x},\vec{y}}(\vec{p})$ satisfies the open Esakia condition.*

Proof. By Lemma 6.10, it is enough to show that the intersection conditions hold, which have been verified in Lemma 7.9. $\square$

Theorem 7.11. *In $\mathcal{L}^{++}$, every uniform syntactically closed term is σ-contracting, and every uniform syntactically open term is π-expanding.*

Proof. By Corollary 7.10, for a uniform syntactically closed term $\varphi(\vec{p}, \vec{j}, \vec{m})$ and arbitrary DMA $\mathbb{A}$, $\vec{x} \in J^{\infty}(\mathbb{A}^{\delta}), \vec{y} \in M^{\infty}(\mathbb{A}^{\delta})$, $\varphi^{\mathbb{A}^{\delta}}_{\vec{x},\vec{y}}(\vec{p})$ satisfies the closed Esakia condition, therefore by Theorem 6.8, $\varphi^{\mathbb{A}^{\delta}}_{\vec{x},\vec{y}}(\vec{p})$ is σ-contracting, hence by Definition 7.3, φ is σ-contracting. The second part of the statement is proven similarly. □

7.1.4 Generalized σ-Expanding and π-Contracting Terms in $\mathcal{L}^{++}$

In this subsection, we consider generalized σ-expanding and π-contracting terms in $\mathcal{L}^{++}$, give the definition of generalized Sahlqvist terms (i.e., pseudo inductive terms) in $\mathcal{L}^{++}$ (Definition 7.14), and show that all left (resp. right) pseudo inductive terms are σ-expanding (resp. π-contracting) (Theorem 7.17).

We first give the following lemma, which is proved, analogously to Lemma 4.11, by simultaneous induction on the shape of φ and ψ:

Lemma 7.12. *Let $\varphi(\vec{p}, \vec{q}, \vec{j}, \vec{m})$ and $\psi(\vec{p}, \vec{q}, \vec{j}, \vec{m})$ be uniform syntactically closed and open terms in $\mathcal{L}^{++}$, respectively, where $\vec{p}$ and $\vec{q}$ respectively occur positively and negatively, and $\vec{j}, \vec{m}$ are all nominals and co-nominals occuring in φ and ψ. Let $\vec{x} \in J^{\infty}(\mathbb{A}^{\delta}), \vec{y} \in M^{\infty}(\mathbb{A}^{\delta})$, $\vec{c} \in K(\mathbb{A}^{\delta}), \vec{o} \in O(\mathbb{A}^{\delta})$, then*

1. $\varphi^{\mathbb{A}^{\delta}}(\vec{c}, \vec{o}, \vec{x}, \vec{y}) \in K(\mathbb{A}^{\delta})$;
2. $\psi^{\mathbb{A}^{\delta}}(\vec{o}, \vec{c}, \vec{x}, \vec{y}) \in O(\mathbb{A}^{\delta})$.

By Corollary 7.10, given the valuation for nominals and co-nominals, for any uniform syntactically open term t, the term function $t^{\mathbb{A}^{\delta}}_{\vec{x},\vec{y}}$ satisfies the open Esakia condition. Hence we have the following:

Corollary 7.13. *For any DMA $\mathbb{A}$, $\vec{x} \in J^{\infty}(\mathbb{A}^{\delta})$ and $\vec{y} \in M^{\infty}(\mathbb{A}^{\delta})$,*

1. *for any uniform syntactically open term t in $\mathcal{L}^{++}$, if $t^{\mathbb{A}^{\delta}}_{\vec{x},\vec{y}}$ is completely meet-preserving, then $t^{\mathbb{A}^{\delta}}_{\vec{x},\vec{y}}$ is SUC;*
2. *if a term t in $\mathcal{L}^{++}$ is uniform syntactically open and completely meet-preserving in proposition variables, then it is σ-expanding.*

Proof. By Corollary 7.10, $t^{\mathbb{A}^{\delta}}_{\vec{x},\vec{y}}$ satisfies the open Esakia condition; by Lemma 7.12, for all $\vec{a}, \vec{a}' \in \mathbb{A}$, $t^{\mathbb{A}^{\delta}}_{\vec{x},\vec{y}}(\vec{a}, \vec{a}') \in O(\mathbb{A}^{\delta})$; since $t^{\mathbb{A}^{\delta}}_{\vec{x},\vec{y}}$ is completely meet-preserving, by Theorem 6.16, $t^{\mathbb{A}^{\delta}}_{\vec{x},\vec{y}}$ is SUC. The second part follows easily from the first part and Theorem 6.16. □

The corollary above is all we need to apply in the proof of canonicity for the ALBA inequalities in Section 7.2. In fact, we can use the results in Section 6.3 about sufficient conditions for terms to be σ-expanding and π-contracting to define pseudo inductive terms, and show that left (resp. right) pseudo inductive terms in $\mathcal{L}^{++}$ are σ-expanding (resp. π-contracting).

In Chapter 6, we showed that in order to guarantee that a function $f^{\mathbb{A}^{\delta}}$ is σ-expanding (resp. π-contracting), it suffices to show that $f^{\mathbb{A}^{\delta}}$ satisfies UC (resp. LC). These results make it possible to reproduce the Jónsson-style core strategy which was discussed in Subsection 3.1.2 in the context

Polarity	Outer Skeleton							
$+$	$\wedge$	$\vee$	$\Diamond$	$\lhd$	$\blacklozenge$	$\blacktriangleleft$	$-$	$\mathbf{n}$
$-$	$\vee$	$\wedge$	$\Box$	$\rhd$	$\blacksquare$	$\blacktriangleright$	$\rightarrow$	$\mathbf{l}$
	Inner Structure							
$+$	$\wedge$	$\Box$	$\rhd$	$\blacksquare$	$\blacktriangleright$	$\rightarrow$	$\mathbf{l}$	$\vee$
$-$	$\vee$	$\Diamond$	$\lhd$	$\blacklozenge$	$\blacktriangleleft$	$-$	$\mathbf{n}$	$\wedge$

Table 7.1: Classification of Nodes for Pseudo Inductive Terms

of generalized canonical extensions. Here we take UC as an example: as discussed in Subsection 3.1.2, UC is of the form $(\forall J \exists K)$, and hence this property can be guaranteed in two ways: either by requiring the inner function to satisfy UC $(\forall J \exists K)$ and the outer function to satisfy Scott continuity $(\forall J \exists J)$, or by requiring the inner function to satisfy SUC $(\forall K \exists K)$ and the outer function to satisfy UC $(\forall J \exists K)$. In addition, the differences between this generalized setting and the standard one require that we put extra constraints on the terms in order to make them satisfy the assumptions of Lemma 6.15 and Lemma 6.17, e.g. the positive (resp. negative) inner component need to be syntactically open (resp. closed). These extra constraints are given in the following definition in the form of Sahlqvist-style syntactic conditions on $\mathcal{L}^{++}$-terms. Instead of giving a choice/universal classification of nodes, we give a classification in "positive configuration" style as in [6].

Definition 7.14 (**Pseudo Inductive Terms in $\mathcal{L}^{++}$**)**.** Given an order type ϵ, the signed generation tree of an $\mathcal{L}^{++}$-term $+s(p_1, \ldots, p_n)$ (resp. $-s(p_1, \ldots, p_n)$) is *ϵ-left pseudo inductive*, abbreviated as ϵ-LPI (resp. *ϵ-right pseudo inductive*, abbreviated as ϵ-RPI), if:

- it agrees with ϵ;
- on every branch with some p_i as its leaf, going from the root to the leaf, there are two groups of nodes, as given in Table 7.1. The first group (outer skeleton) occurs before the second group (inner structure); in fact, either group can be empty;
- the binary nodes in the inner structure part of branches ending with some p_i are in either of the following forms:

Polarity									
$+$	$\alpha \rightarrow \beta$	$\gamma \rightarrow \delta$	$\mathbf{l}(\alpha, \beta)$	$\mathbf{l}(\gamma, \delta)$	$\delta \vee \gamma$	$\gamma \vee \delta$	$\delta \wedge \gamma$	$\gamma \wedge \delta$	$\beta \wedge \gamma$
$-$	$\gamma - \delta$	$\alpha - \beta$	$\mathbf{n}(\gamma, \delta)$	$\mathbf{n}(\alpha, \beta)$	$\alpha \wedge \beta$	$\beta \wedge \alpha$	$\alpha \vee \beta$	$\beta \vee \alpha$	$\beta \vee \gamma$

  where α is a syntactically closed pure term (see Definition 4.3), δ is a syntactically open pure term, β and γ are not pure.

Let us use the abbreviation PI for pseudo inductive terms.

Remark 7.15. The following facts are worth noticing about the definition above:

1. The definition above ensures that if t is a left-LPI term and $\mathbb{A}$ is a DMA, then, regarding $t^{\mathbb{A}^\delta}$ as a composition of order-preserving maps with respect to an appropriate order-type, $t^{\mathbb{A}^\delta}$ is composed of two parts: the interpretation of the outer skeleton, which is the composition of complete operators, and the interpretation of the inner structure, which is the composition of completely meet-preserving maps such that each subterm in the inner structure is an open (resp. closed) term if it is labelled $+$ (resp. $-$). Proposition 7.16 below describes the inner structure more in detail.

2. Pure terms are trivially both left and right ϵ-LPI for any ϵ, since none of their branches end with a proposition variable.

3. All uniform Sahlqvist terms in the basic DML signature are pseudo inductive terms.

4. This definition reminds of inductive formulas, in a sense which can be made more precise as follows. Consider a non-pure PI term t in which only symbols in $\mathcal{L}$ occur on branches ending with some p_i; suppose that on such branches, any choice node occurring in the scope of a universal node is of the form $+(\delta \vee \gamma)$ or $-(\alpha \wedge \beta)$, where δ is a syntactically open pure term and α is a syntactically closed pure term. Suppose that $\mathbf{n}, \mathbf{l}$ do not occur in t, and $i, m, \blacklozenge, \blacktriangleleft, -, \blacksquare, \blacktriangleright, \rightarrow$ only occur in terms δ or α mentioned above, and nowhere else. If we replace each δ with a new proposition variable $p_{+\delta}$ of order type ∂ and α with a new proposition variable $p_{-\alpha}$ of order type 1, and put all the new variables below all existing variables in the dependence order, then the term we obtain is inductive. Since the minimal valuation for a variable of order type 1 (resp. ∂) is a syntactically closed (resp. open) term, PI terms can be regarded as inductive terms after substituting some minimal valuations into its order dual part. This also explains why the expanded language is a crucial tool when dealing with the canonicity of inductive inequalities.

Proposition 7.16. *For any non-pure PI term $*s$, if the outer skeleton of $*s$ is empty, then for every subterm $\star t$ of $*s$, t is syntactically open (resp. closed) if $\star$ is $+$ (resp. $-$).*

Proof. The proof is simple but very lengthy, and hence we omit it. $\square$

Theorem 7.17. *If α is an ϵ-LPI (resp. ϵ-RPI) term, then α is σ-expanding (resp. π-contracting).*

Proof. The proof is similar to Lemma 5.10 in [17], by simultaneous induction on the syntactic structure of LPI and RPI terms. Here we consider ϵ-LPI terms, ϵ-RPI terms being order dual. For basic cases, the proof is trivial. For pure terms, it is also easy to check that they are both σ-stable and π-stable. Hence, without loss of generality, we only consider non-pure terms; for the inductive step, suppose that for the ϵ-LPI term α, the outmost operation is f. Now the discussion breaks into cases.

If f is a node in the outer skeleton, then it is among $\wedge$, $\vee$, $\Diamond$, $\lhd$, $\blacklozenge$, $\blacktriangleleft$, $-$, $\mathbf{n}$. We consider $\alpha = \beta - \gamma$, since other cases are easier. It can be readily checked that $+\beta$ is ϵ-LPI and $-\gamma$ is ϵ-RPI; indeed, if not, then their flaw would make $+\alpha$ non-LPI. Hence, by induction hypothesis, β is σ-expanding and γ is π-contracting, i.e.

$$\beta^{\mathbb{A}^\delta}_{\vec{x},\vec{y}} \leq (\beta^{\mathbb{A}}_{\vec{x},\vec{y}})^\sigma \quad \text{and} \quad (\gamma^{\mathbb{A}}_{\vec{x},\vec{y}})^\pi \leq \gamma^{\mathbb{A}^\delta}_{\vec{x},\vec{y}}, \qquad \text{i.e. } (\gamma^{\mathbb{A}^\delta}_{\vec{x},\vec{y}})^\partial \leq ((\gamma^{\mathbb{A}}_{\vec{x},\vec{y}})^\partial)^\sigma$$

for any DMA $\mathbb{A}$, any $\vec{x} \in J^\infty(\mathbb{A}^\delta)$, any $\vec{y} \in M^\infty(\mathbb{A}^\delta)$, and where $\beta^{\mathbb{A}^\delta}_{\vec{x},\vec{y}}$ and $\gamma^{\mathbb{A}^\delta}_{\vec{x},\vec{y}}$ are regarded as follows:

$$\beta^{\mathbb{A}^\delta}_{\vec{x},\vec{y}} : (\mathbb{A}^\delta)^\epsilon \to \mathbb{A}^\delta \text{ and } \gamma^{\mathbb{A}^\delta}_{\vec{x},\vec{y}} : ((\mathbb{A}^\delta)^\epsilon)^\partial \to \mathbb{A}^\delta.$$

Since $-^{\mathbb{A}^\delta} : \mathbb{A}^\delta \times (\mathbb{A}^\delta)^\partial \to \mathbb{A}^\delta$ is a complete operator and $\alpha^{\mathbb{A}^\delta}_{\vec{x},\vec{y}} = -^{\mathbb{A}^\delta}(\beta^{\mathbb{A}^\delta}_{\vec{x},\vec{y}}, (\gamma^{\mathbb{A}^\delta}_{\vec{x},\vec{y}})^\partial)$ is a composition of the order-preserving maps $(\beta^{\mathbb{A}^\delta}_{\vec{x},\vec{y}}, (\gamma^{\mathbb{A}^\delta}_{\vec{x},\vec{y}})^\partial) : (\mathbb{A}^\delta)^\epsilon \to \mathbb{A}^\delta \times (\mathbb{A}^\delta)^\partial$ and $-^{\mathbb{A}^\delta} : \mathbb{A}^\delta \times (\mathbb{A}^\delta)^\partial \to \mathbb{A}^\delta$, by Lemma 6.15, we have

$$\begin{array}{rcll}
(\beta - \gamma)^{\mathbb{A}^\delta}_{\vec{x},\vec{y}} & = & -^{\mathbb{A}^\delta}(\beta^{\mathbb{A}^\delta}_{\vec{x},\vec{y}}, (\gamma^{\mathbb{A}^\delta}_{\vec{x},\vec{y}})^\partial) & \\
& \leq & -^{\mathbb{A}^\delta}((\beta^{\mathbb{A}}_{\vec{x},\vec{y}})^\sigma, ((\gamma^{\mathbb{A}}_{\vec{x},\vec{y}})^\partial)^\sigma) & \text{(by induction hypothesis)} \\
& = & -^{\mathbb{A}^\delta}(\beta^{\mathbb{A}}_{\vec{x},\vec{y}}, (\gamma^{\mathbb{A}}_{\vec{x},\vec{y}})^\partial)^\sigma & \\
& \leq & ((\beta - \gamma)^{\mathbb{A}}_{\vec{x},\vec{y}})^\sigma & \text{(by Lemma 6.15).}
\end{array}$$

If f is a node in the inner structure, then by Proposition 7.16, α is syntactically open, and by appropriately flipping the polarity of coordinates and functions, $\alpha^{\mathbb{A}^\delta}_{\vec{x},\vec{y}}$ is a composition of completely meet-preserving maps, therefore by Corollary 7.13, α is σ-expanding. □

7.2 ALBA-Aided Jónsson-Style Canonicity for ALBA Inequalities

The aim of the present section is to prove the following:

Theorem 7.18. *ALBA inequalities in distributive modal logic are canonical.*

Proof. Our proof strategy is a combination of ALBA-style canonicity and Jónsson-style canonicity. The main structure of the proof is similar to the U-shaped argument in ALBA-style canonicity on Page 44, while for the bottom line, the proof strategy is similar to the Jónsson-style canonicity.

Let $\varphi \leq \psi$ be an ALBA inequality, on which ALBA succeeds. Let us run ALBA up to the point (in the second stage, therefore we are dealing with systems of inequalities) in which for every system, only one proposition variable remains, and the systems are in Ackermann shape, i.e., each system is either in the *right-handed Ackermann form*:

$$\begin{cases}
\alpha_1 \leq p \\
\vdots \\
\alpha_m \leq p \\
\beta_1(p) \leq \gamma_1(p) \\
\vdots \\
\beta_n(p) \leq \gamma_n(p) \\
\delta_1 \leq \theta_1 \\
\vdots \\
\delta_k \leq \theta_k
\end{cases}$$

where the α_ls are pure and syntactically closed, $\mathsf{Prop}(\beta_j(p)) \cup \mathsf{Prop}(\gamma_j(p)) = \{p\}$, the $\beta_j(p)$s are syntactically closed and positive in p, the $\gamma_j(p)$s are syntactically open and negative in p, the δ_hs and θ_hs are pure; or in the *left-handed Ackermann form*, which is the analogous but symmetric situation.

Since we treat each system independently of the others, we can assume for simplicity that all the systems of inequalities are of the *right-handed Ackermann form* $\{\alpha_{i,1} \leq p_i, \ldots, \alpha_{i,m} \leq p_i, \beta_{i,1}(p_i) \leq \gamma_{i,1}(p_i), \ldots, \beta_{i,n}(p_i) \leq \gamma_{i,n}(p_i), \delta_{i,1} \leq \theta_{i,1}, \ldots, \delta_{i,k} \leq \theta_{i,k}\}$, which we denote as $\mathrm{RAF}(\varphi_i \leq \psi_i)$.

Our goal now is to show that the Jónsson-style strategy can be applied to prove the equivalence between items 2. and 3. in the following list:

1. $\mathbb{A}^\delta \models_{\mathbb{A}} \varphi \leq \psi$;
2. $\mathbb{A}^\delta \models_{\mathbb{A}} \&\mathrm{RAF}(\varphi_i \leq \psi_i) \Rightarrow i_0 \leq m_0$ for all $\varphi_i \leq \psi_i$;
3. $\mathbb{A}^\delta \models \&\mathrm{RAF}(\varphi_i \leq \psi_i) \Rightarrow i_0 \leq m_0$ for all $\varphi_i \leq \psi_i$;
4. $\mathbb{A}^\delta \models \varphi \leq \psi$.

The equivalence between 3. and 4. follows from the soundness of ALBA, and the equivalence between 1. and 2. similarly follows from the soundness of ALBA when the valuations are admissible[1]. So the crucial step is the equivalence between 2. and 3.

To show that 2. implies 3, we adapt the chain of entailments discussed in the proof of Theorem 3.1. Below, we use the following abbreviations:

$$
\begin{array}{lcl}
\mathrm{A}(p_i) & = & \alpha_{i,1} \vee \ldots \vee \alpha_{i,m} \leq p_i, \\
\mathrm{BG}(p_i) & = & \beta_{i,1}(p_i) \leq \gamma_{i,1}(p_i) \& \ldots \& \beta_{i,n}(p_i) \leq \gamma_{i,n}(p_i), \\
\mathrm{DT} & = & \delta_{i,1} \leq \theta_{i,1} \& \ldots \& \delta_{i,k} \leq \theta_{i,k}, \\
\mathbf{l}(\alpha, p_i) & = & \mathbf{l}(\alpha_{i,1} \vee \ldots \vee \alpha_{i,m}, p_i), \\
\mathrm{PURE} & = & \mathbf{n}(i_0, m_0) \wedge \bigwedge_{1 \leq j \leq k} \mathbf{l}(\delta_{i,j}, \theta_{i,j}), \\
\gamma(p_i) & = & \bigvee_{1 \leq j \leq n} \mathbf{n}(\beta_{i,j}(p_i), \gamma_{i,j}(p_i)).
\end{array}
$$

In what follows, the arrays $\vec{x}$ and $\vec{y}$ are taken in $J^\infty(\mathbb{A}^\delta)$ and $M^\infty(\mathbb{A}^\delta)$, respectively.

[1]Notice that the fact that we are using the ALBA manipulations does not mean that this proof is a canonicity-via-correspondence one. More details on this are given in Remark 7.21 and Section 7.3.

$$
\begin{array}{lll}
& \mathbb{A}^\delta \models_{\mathbb{A}} \& \text{ RAF } (\varphi_i \leq \psi_i) \Rightarrow i_0 \leq m_0 & \\
\iff & \mathbb{A}^\delta \models_{\mathbb{A}} (\mathrm{A}(p_i) \& \text{ BG } (p_i) \& \text{ DT}) \Rightarrow i_0 \leq m_0 & \text{(trivial)} \\
\iff & \mathbb{A}^\delta \models_{\mathbb{A}} \mathbf{l}(\alpha, p_i) \wedge \bigwedge_{1 \leq j \leq n} \mathbf{l}(\beta_{i,j}(p_i), \gamma_{i,j}(p_i)) \wedge \bigwedge_{1 \leq j \leq k} \mathbf{l}(\delta_{i,j}, \theta_{i,j}) \leq \mathbf{l}(i_0, m_0) & (*) \\
\iff & \mathbb{A}^\delta \models_{\mathbb{A}} \mathbf{l}(\alpha, p_i) \wedge \mathbf{n}(i_0, m_0) \wedge \bigwedge_{1 \leq j \leq k} \mathbf{l}(\delta_{i,j}, \theta_{i,j}) \leq \bigvee_{1 \leq j \leq n} \mathbf{n}(\beta_{i,j}(p_i), \gamma_{i,j}(p_i)) & (*) \\
\iff & (\mathbf{l}(\alpha, p_i) \wedge \text{ PURE})^{\mathbb{A}}_{\vec{x},\vec{y}} \leq (\gamma(p_i))^{\mathbb{A}}_{\vec{x},\vec{y}} \text{ for all } \vec{x}, \vec{y} & \text{(trivial)} \\
\iff & ((\mathbf{l}(\alpha, p_i) \wedge \text{ PURE})^{\mathbb{A}}_{\vec{x},\vec{y}})^\sigma \leq ((\gamma(p_i))^{\mathbb{A}}_{\vec{x},\vec{y}})^\sigma \text{ for all } \vec{x}, \vec{y} & \text{(Def. } \sigma\text{-extension)} \\
\Longrightarrow & ((\mathbf{l}(\alpha, p_i) \wedge \text{ PURE})^{\mathbb{A}}_{\vec{x},\vec{y}})^\sigma \leq (\gamma(p_i))^{\mathbb{A}^\delta}_{\vec{x},\vec{y}} \text{ for all } \vec{x}, \vec{y} & \text{(Lemma 7.19)} \\
\Longrightarrow & (\mathbf{l}(\alpha, p_i) \wedge \text{ PURE})^{\mathbb{A}^\delta}_{\vec{x},\vec{y}} \leq (\gamma(p_i))^{\mathbb{A}^\delta}_{\vec{x},\vec{y}} \text{ for all } \vec{x}, \vec{y} & \text{(Lemma 7.20)} \\
\iff & \mathbb{A}^\delta \models \mathbf{l}(\alpha, p_i) \wedge \text{ PURE} \leq \gamma(p_i) & \text{(trivial)} \\
\iff & \mathbb{A}^\delta \models \& \text{ RAF } (\varphi_i \leq \psi_i) \Rightarrow i_0 \leq m_0 & (*).
\end{array}
$$

The equivalences marked with $(*)$ can be shown by straightforward algebraic manipulations by using the definitions of $\mathbf{n}$ and $\mathbf{l}$. Finally, the proof is complete with the lemmas below. ☐

Lemma 7.19. *$\gamma(p_i)$ is σ-contracting.*

Proof. Recall that $\gamma(p_i) = \bigvee_{1 \leq j \leq n} \mathbf{n}(\beta_{i,j}(p_i), \gamma_{i,j}(p_i))$ where the βs are syntactically closed and positive in p_i, and the γs are syntactically open and negative in p_i. Hence $\gamma(p_i)$ is syntactically closed and positive in p_i. Then the statement follows from Theorem 7.11. ☐

Lemma 7.20. *$\mathbf{l}(\alpha, p_i) \wedge PURE$ is σ-expanding.*

Proof. By Theorem 7.17, it is enough to show that $\mathbf{l}(\alpha, p_i) \wedge$PURE is LPI with order type (1). Let us recall that PURE is of the form $\mathbf{n}(i_0, m_0) \wedge \bigwedge_{1 \leq j \leq k} \mathbf{l}(\delta_{i,j}, \theta_{i,j})$, and $\mathbf{l}(\alpha, p_i)$ is of the form $\mathbf{l}(\alpha_{i,1} \vee \ldots \vee \alpha_{i,m}, p_i)$. The outer skeleton of $\mathbf{l}(\alpha, p_i) \wedge$PURE only consists of the node $+\wedge$, and the inner structure only consists of the subterm $+\mathbf{l}(\alpha_{i,1} \vee \ldots \vee \alpha_{i,m}, p_i)$, where $\alpha_{i,1} \vee \ldots \vee \alpha_{i,m}$ is syntactically closed (cf. Lemma 5.8) and pure. ☐

Remark 7.21.

- The tool of ALBA has been used for two different purposes: in [10], it was used to calculate a first-order correspondent for formulas or inequalities; in this chapter, ALBA is used to manipulate the initial inequality into a shape to which the Jónsson strategy can be applied.
- Since the inductive inequalities are ALBA inequalities, their Jónsson-style canonicity for them follows straightforwardly from the theorem above.
- Some aspects of the proof might still look to be a bit ad hoc, e.g. the fact that we run ALBA up to the point immediately before the last application of the Ackermann rule. We will discuss this aspect in Section 7.3.1.

7.3 Discussion: Comparing the Two Approaches

Certainly, the main motivation for the proof above is methodological. Namely, we were looking for a better understanding of the relationship between the two proof strategies for canonicity. Indeed, we are able to recognize that some ingredients of the two approaches are two faces of the same coin. In this part, we collect our findings in this respect.

7.3.1 General Strategy

We now summarize the two proof strategies restricted to Sahlqvist inequalities (in fact, the situation complicates a bit when dealing with formulas or inequalities outside this class, but we will come back to those later on).

- As discussed below Lemma 3.8, in the Jónsson-style canonicity of Sahlqvist inequalities, the $\mathbf{n}$-trick is used to transform Sahlqvist inequalities of the form $\alpha \leq \beta$ into inequalities of the form $\alpha_1 \leq \beta_1 \vee \gamma$; in this new shape, there is a clear separation between the $\mathbf{n}$-*part* γ and the *Sahlqvist part* α_1, β_1; it is then shown that the Sahlqvist part α_1 (resp. β_1) is σ-expanding (resp. π-contracting), and the $\mathbf{n}$-part γ, which is uniform, is σ-contracting.
- In the ALBA-style canonicity-via-correspondence, all the essentially algebraic and order-topological manipulation steps aim at achieving pure quasi-inequalities. Immediately before the last application of the Ackermann rule, the inequalities in a system can be classified into two types: the first type, which we call *minimal valuation part*, contains inequalities which are used to compute the minimal valuation; the second type, which we will call *receiving part*, contains the inequalities where the minimal valuation is going to be substituted into.

The most notable difference between the two approaches is that the Jónsson-style transformations preserve the integrity of the initial inequality and proceed from the bottom of the generation tree by taking out subformulas and moving them to the γ-part, whereas the ALBA-style approach systematically decomposes the initial inequality into systems of inequalities and proceeds from the top of the generation tree. However, the two approaches are similar in that they both aim at achieving a state in which the ingredients are neatly separated into two types: in the Jónsson-style approach, the two types are the Sahlqvist part and the $\mathbf{n}$-part, and in the ALBA-style approach, they are the minimal valuation part and the receiving part.

The core of the ALBA-aided Jónsson-style canonicity is the recognition that, modulo some manipulations, the minimal valuation part is σ-expanding and the receiving part is σ-contracting[2]. Namely, the minimal valuation part and the receiving part, generated by ALBA, respectively enjoy the same properties enjoyed by the Sahlqvist part and the $\mathbf{n}$-part respectively, and which are crucial in the Jónsson-style canonicity. In the following subsections, we will expand on this. Finally, the ad hoc aspect of the proof, namely the fact that ALBA runs up to the step immediately before the last application of the Ackermann rule, is in fact motivated by the need to highlight this recognition in a simple way; however, this is not in general the only step in which this recognition is possible.

[2]Of course, in order to be able to formally express these facts, we had to introduce the extra connectives $\mathbf{n}$ and $\mathbf{l}$.

7.3.2 The Sahlqvist Part and the Minimal Valuation Part

In this subsection, we expand on the roles of the Sahlqvist part in the Jónsson-style canonicity, and of the minimal valuation part in the ALBA-style canonicity. We first consider the case of Sahlqvist inequalities, then briefly discuss the situation in the case of inductive inequalities. Without loss of generality, we consider left Sahlqvist and left inductive terms.

The Sahlqvist Case

Let us sum up the observations so far, relative to the treatment of Sahlqvist inequalities in the two approaches:

- In the Jónsson-style canonicity, the *compositional* structure of uniform left Sahlqvist terms crucially guarantees that they are σ-expanding. Indeed, uniform left Sahlqvist terms consist of operators, which are Scott continuous ($\forall J \exists J$), as the outer skeleton, and meet-preserving maps, which are SUC ($\forall K \exists K$), as the inner structure. This guarantees that the resulting composition is UC ($\forall J \exists K$), which guarantees their being σ-expanding.

- In the ALBA-style canonicity-via-correspondence, the uniform left Sahlqvist terms are *decomposed* to "get out" the critical occurrences of proposition variables, which are the ones to be minimally valued. After the first approximation, we get inequalities e.g. of the form $i \leq \alpha$, where α is of certain restricted left Sahlqvist form. The algorithm proceeds deconstructing this inequality, and this deconstruction is made possible by the structure of α (without loss of generality we only consider critical occurrences of proposition variables of order type 1).

  Indeed, we recall that the structure of α consists of connectives in the outer skeleton which are interpreted in $\mathbb{A}^\delta$ as maps which are completely join-preserving in each coordinate, and of connectives in the inner structure which are interpreted in $\mathbb{A}^\delta$ as maps which are right adjoints. Hence, the inequality $i \leq \alpha$ is decomposed in two stages:

  1. The first stage transforms the inequality $i \leq f(\beta_1, \ldots, \beta_n)$ into inequalities $i \leq f(i_1, \ldots, i_n)$, $i_1 \leq \beta_1$, …, $i_n \leq \beta_n$, which keep nominals (interpreted as completely join-irreducibles) in the left-hand side of the inequalities. These transformations are sound thanks to the properties of the outer skeleton, namely that complete operators are completely join-preserving in each coordinate.

     In fact, in the algorithm, the rule above is not applied to $\wedge$ and $\vee$, since they have even better properties: the outer $\vee$ has been eliminated in the first stage due to its being the left adjoint of Δ, and $\wedge$ is the right adjoint of Δ, which makes it possible to apply the corresponding splitting rule, which preserves the nominals on the left-hand side.

  2. The second stage transforms inequalities $\beta \leq g(\alpha)$ where g is completely meet-preserving, into inequalities $g'(\beta) \leq \alpha$, where $g' \dashv g$. These transformations are sound thanks to the properties of the inner structure, namely that completely meet-preserving maps are right adjoints.

Finally, after having performed the transformations in the second stage exhaustively (recall that we are only considering critical occurrences of proposition variables of order type 1), all the critical occurrences of p have been dragged out and we are reduced to a set of inequalities of the form $\beta \leq p$, which gives the minimal valuation part of a set of inequalities in Ackermann form.

This account is intended to stress the fact that one and the same syntactic structure, namely that of Sahlqvist terms, is used in two different ways to achieve canonicity. Namely, it is used in a compositional way in the Jónsson strategy, and in a decompositional way for the ALBA strategy. The order-theoretic properties relevant to Jónsson-style canonicity, and guaranteed by the Sahlqvist shape, are: complete join-preservation in each coordinate for the outer skeleton and complete meet-preservation for the inner part, whereas, the order-theoretic properties relevant to ALBA, and guaranteed by the Sahlqvist shape, are: complete join-preservation in each coordinate for the outer skeleton and being a right adjoint for the inner part. The equivalence between completely meet-preserving maps and right adjoints is another aspect of this two-faces-of-same-coin situation.

The Inductive Case

In [30], it has been observed that the Jónsson-style canonicity proof cannot be straightforwardly extended to inductive inequalities. In Section 3.2, we have seen that the minimal collapse algorithm, which is based on the plain **n**-trick, does not work on inductive inequalities, because it transforms inductive inequalities into inequalities which do not enjoy the required order-theoretic properties in general. However, from this we can only conclude that this kind of manipulation is too rough, since the order-theoretic properties guaranteed by the inductive shape are enough for the order-topological ALBA-style analysis. For instance, by Lemma 5.8, we know that the ALBA manipulations preserve the syntactic closedness and openness of terms.

Therefore, in order to achieve Jónsson-style canonicity for inductive/ALBA inequalities, we realized the need to supplement the **n**-trick with additional syntactic manipulations coming from ALBA. Hence, we have combined the two approaches, and used ALBA to transform the inductive inequalities into some suitable form where the Jónsson-style strategy for canonicity can be applied. Analogously to the original proof in [17], this suitable form exhibits a separation between a part corresponding to the minimal valuation part, which in its turn plays the role of what we called the "Sahlqvist part" in [17], and a part corresponding to the receiving part, which in its turn plays the role of what we called the "**n**-part" in [17]. In particular, it is shown in Lemma 7.20 that the term corresponding to the minimal valuation part is LPI. In Remark 7.15, we mention that pseudo inductive terms can be recognized as inductive terms with minimal valuations substituted into. The fact that pseudo inductive terms have good enough compositional structure to guarantee that they are σ-expanding or π-contracting makes it possible for the Jónsson-style strategy to go through.

7.3.3 The n-Part and the Receiving Part

The similarity between the **n**-part and the receiving part is even more straightforward:

- In the Jónsson-style canonicity, the **n**-part is σ-contracting. However, in [17], the proof goes through only on the basis of the fact that it is uniform. This is due to the fact that the setting of that proof only treats the original signature. When moving to the expanded signature, the hidden mechanism of σ-contracting and π-expanding terms starts to appear: namely, the Esakia conditions, or equivalently, the intersection conditions (cf. Section 6.2).

- In the ALBA-style canonicity, at the stage before the last application of the Ackermann rule, the left-hand side of the receiving part is syntactically closed and uniform in p, its right-hand side is syntactically open and uniform in p. In Lemma 7.10 and the discussion in Section 6.2, we highlighted the connections between these properties and σ-contracting or π-expanding maps. In the ALBA-aided Jónsson-style canonicity, these two aspects, which provide another two-faces-of-same-coin instance, come together. Namely, the receiving part is shown to be σ-contracting.

In the next chapter, we will turn to the signature of distributive modal μ-calculus. In this setting, we will explore further the connection between σ-contracting and π-expanding terms and the intersection property.

Chapter 8

Jónsson-Style Canonicity in Distributive Modal μ-Calculus

Recently, Sahlqvist theory for modal μ-calculus has been explored, concerning both correspondence and canonicity, e.g. in [27, 28, 29, 5, 2]. In [29], Sahlqvist correspondence has been extended to modal μ-formulas on a classical base, and these results have been extended to the modal μ-calculus on an intuitionistic base in [5]. In [2], some completeness results for μ-formulas with respect to clopen semantics in the classical case have been achieved for a fragment of μ-formulas which is much more restricted than the class of formulas defined in [29]. In particular, fixed points are not allowed on critical branches.

In [2], the completeness proof has some canonicity flavor: indeed, Theorem 5.3 in [2], viewed algebraically, is a canonicity result. The proof is frame-theoretic, and its crucial tools are the Esakia lemma and the intersection lemma, following the route of [23]. In particular, it is proven that the positive μ-formulas, which play an analogous role to the **n**-part, satisfy the closed intersection condition, which, as we have proved, is equivalent to being σ-contracting. Therefore, it is imaginable that this proof can be done also in Jónsson-style.

In this chapter, we generalize the result in [2] to the distributive setting. Our proof is in Jónsson-style, and comes as an application of the connection between the two canonicity approaches. Particularly important for the proof is the connection between being σ-contracting and satisfying the closed intersection condition.

8.1 Distributive Modal μ-Calculus

In this section we consider the distributive modal μ-calculus (μ-DML), which is an extension of distributive modal logic with fixed point operators. Given a set Prop of proposition variables and a set FVar of fixed point variables, the formulas of μ-DML are given by the following inductive definition:

$$\alpha ::= p \mid x \mid \top \mid \bot \mid \alpha \wedge \beta \mid \alpha \vee \beta \mid \Diamond\alpha \mid \Box\alpha \mid \rhd\alpha \mid \lhd\alpha \mid \mu x.\alpha \mid \nu x.\alpha,$$

where $p \in \mathsf{Prop}$, $x \in \mathsf{FVar}$ and x occurs positively in α, i.e. in the scope of an even number of triangles. We denote the language as $\mathcal{L}^\mu$, and $\mathcal{L}^\mu_{term}$, $\mathcal{L}^\mu_{\leq}$, $\mathcal{L}^\mu_{quasi}$ are defined similarly to Section 2.1. We interpret the formulas over distributive modal algebras (DMA). In fact, many definitions concerning the syntax and semantics of distributive modal logic can be carried over to distributive modal μ-calculus, therefore we only mention the ones which are new or need revision.

Definition 8.1 (**Free and Bound Variables**). For any formula $\alpha \in \mathcal{L}^\mu$ and fixed point variable x occurring in α, we say

- an occurrence of x in α is *free*, if it is not in the scope of μx or νx;
- an occurrence of x in α is *bound*, if it is in the scope of μx or νx;
- x is a *free variable*, if there is a free occurrence of x in α;
- x is a *bound variable*, if all occurences of x in α are bound;
- α is a *sentence*, if there is no free occurence of any fixed point variables.

We denote $\mathsf{Var}(\alpha)$ as the set of all free fixed point variables in α.

Therefore, in any DMA $\mathbb{A}$, α induces a term function $\alpha^{\mathbb{A}} : \mathsf{Prop}(\alpha) \cup \mathsf{Var}(\alpha) \to \mathbb{A}$. Let $\mathbb{A}$ be a DMA and $h : \mathsf{Prop} \cup \mathsf{FVar} \to \mathbb{A}$ be an assignment. We use the symbol $[\alpha]^{\mathbb{A}}_h$ to denote the interpretation of α in $\mathbb{A}$ under the assignment h, as defined in Chapter 2. The interpretation for $\mu x.\alpha$ (resp. $\nu x.\alpha$) is defined as the least fixed point (resp. greatest fixed point) of the term function α with respect to x, namely,

$$[\mu x.\alpha]^{\mathbb{A}}_h = \bigwedge\{a \in \mathbb{A} : [\alpha]^{\mathbb{A}}_{h^a_x} \leq a\}$$
$$[\nu x.\alpha]^{\mathbb{A}}_h = \bigvee\{a \in \mathbb{A} : a \leq [\alpha]^{\mathbb{A}}_{h^a_x}\},$$

where h^a_x is the assignment which is the same as h except that it maps x to $a \in \mathbb{A}$. It is easy to see that the value of $[\mu x.\alpha]^{\mathbb{A}}_h$ (resp. $[\nu x.\alpha]^{\mathbb{A}}_h$) might not exist, since $\mathbb{A}$ might not be closed under taking arbitrary meets (resp. joins), therefore we only consider DMAs $\mathbb{A}$ such that all the required meets and joins exist:

Definition 8.2 (**Distributive Modal μ-Algebra**). A DMA $\mathbb{A}$ is a *distributive modal μ-algebra* (μ-DMA), if the interpretations $[\mu x.\alpha]^{\mathbb{A}}_h$ and $[\nu x.\alpha]^{\mathbb{A}}_h$ of $\mu x.\alpha$ and $\nu x.\alpha$, respectively, exist for all formulas $\alpha \in \mathcal{L}^\mu_{term}$ (where x occurs positively in α) and all assignments h on $\mathbb{A}$.

Definition 8.3 (**$\mathcal{F}$-Canonical Extension of μ-DMA**). Let $\mathbb{A}$ be a μ-DMA, and let $\mathcal{F}$ be a subalgebra of $\mathbb{A}^\delta$ considering $\mathbb{A}$ is a DMA. The *$\mathcal{F}$-canonical extension* $\mathbb{A}^\delta_{\mathcal{F}}$ of $\mathbb{A}$ is defined as the canonical extension $\mathbb{A}^\delta$ of the underlying DMA, while for the interpretation of fixed point formulas, the least and greatest fixed points on $\mathbb{A}^\delta_{\mathcal{F}}$ are computed relatively to $\mathcal{F}$. That is,

$$[\mu x.\alpha]^{\mathbb{A}^\delta_{\mathcal{F}}}_h = \bigwedge\{a \in \mathcal{F} : [\alpha]^{\mathbb{A}^\delta_{\mathcal{F}}}_{h^a_x} \leq a\}$$
$$[\nu x.\alpha]^{\mathbb{A}^\delta_{\mathcal{F}}}_h = \bigvee\{a \in \mathcal{F} : a \leq [\alpha]^{\mathbb{A}^\delta_{\mathcal{F}}}_{h^a_x}\}.$$

When interpreting μ-terms on canonical extensions, we call *clopen semantics* the interpretation of terms on the $\mathbb{A}$-canonical extensions, and we call *set-theoretic semantics* the interpretation of terms on the $\mathbb{A}^\delta$-canonical extensions. Admissible assignments and admissible validity are defined similarly to Section 2.6.

As is discussed in Chapter 7 in [2], the clopen semantics looks unnatural, since, in this semantics, the interpretation of μ- and ν-formulas is not necessarily given by fixed points as the name suggests. However, when proving completeness (and also canonicity), this semantics is more manageable, especially when it comes to the Esakia lemma and the intersection lemma. This will be clear in the next section.

8.2 Jónsson-Style Canonicity for μ-Sahlqvist Inequalities

In this section we define the notion of μ-Sahlqvist inequalities. This definition generalizes the μ-Sahlqvist formulas defined in [2]. We prove that μ-Sahlqvist inequalities $\alpha \leq \beta$ are $\mathbb{A}$-*canonical* in the following sense:

$$\mathbb{A} \models \alpha \leq \beta \quad \Rightarrow \quad \mathbb{A}^{\delta}_{\mathbb{A}} \models \alpha \leq \beta.$$

where $\mathbb{A}$ is an arbitrary μ-DMA.

Before defining the μ-Sahlqvist inequalities, we first generalize the definition of signed generation tree, by adding the clause that when a node is labelled by μx or νx, its child node inherits the same sign. We erase the sign of any bound occurrence of a fixed point variable, and we consider such an occurence non-critical by default. Hence, every ϵ-critical branch ends with a proposition variable or with a free occurrence of a fixed point variable. Extending the choice/universal classification to fixed point nodes is irrelevant in this setting, since these nodes never occur on critical branches. Since we are going to use the **n**-trick in the proof, we will also consider the language $\mathcal{L}^{n\mu}$ which is $\mathcal{L}^{\mu}$ expanded by **n**.

Definition 8.4 (**μ-Terms and ν-Terms**)**.** Let $+s$ be the positive generation tree of a term $s(p_1, \ldots, p_n) \in \mathcal{L}^{n\mu}_{term}$.

1. s is a *μ-term* if every μ (resp. ν) node on $+s$ is labelled $+$ (resp. $-$);

2. s is a *ν-term* if every ν (resp. μ) node on $+s$ is labelled $+$ (resp. $-$).

Definition 8.5 (**μ-Sahlqvist Inequalities**)**.** Given an order type ϵ and $* \in \{-, +\}$, the term $*s(p_1, \ldots, p_n) \in \mathcal{L}^{\mu}_{term}$ is *(ϵ, μ)-Sahlqvist* if

1. no μ or ν node appears on an ϵ-critical branch;

2. on every ϵ-critical branch with p_i as a leaf, for $1 \leq i \leq n$, no choice node has a universal node as ancestor;

3. every μ node is labelled $-$ and every ν node is labelled $+$.

An inequality $s \leq t \in \mathcal{L}^{\mu}_{\leq}$ is *(ϵ, μ)-Sahlqvist* if both $+s$ and $-t$ are (ϵ, μ)-Sahlqvist. An inequality $s \leq t$ is *μ-Sahlqvist* if it is (ϵ, μ)-Sahlqvist for some ϵ.

According to the definition above, in μ-Sahlqvist inequalities, the skeleton part on the ϵ-critical branches is the same as in the case of Sahlqvist inequalities in the basic DML signature. The difference between the two definitions is that we allow fixed point operators to appear on non ϵ-critical branches. Note, however, that the occurrences of fixed point nodes are restricted by condition 3 in the definition above.

Example 8.6. The inequality $\Diamond p \leq \Box \mu x.(p \vee \Diamond x)$ is (ϵ, μ)-Sahlqvist for $\epsilon = (1)$.

Theorem 8.7. *Every μ-Sahlqvist inequality $s \leq t$ is $\mathbb{A}$-canonical, i.e., for every μ-DMA $\mathbb{A}$, if $\mathbb{A} \models s \leq t$, then $\mathbb{A}^{\delta}_{\mathbb{A}} \models s \leq t$.*

Proof. The proof strategy is an adaptation of the Jónsson-style canonicity of Sahlqvist inequalities in the basic DML signature. Namely, we use the **n**-trick to separate the maximal non ϵ-critical parts (i.e. both the ϵ^{∂}-parts and the parts under the scope of fixed points) from the Sahlqvist "skeleton". Note that μ and ν nodes can occur neither on ϵ-critical branches nor on ϵ^{∂}-critical branches, where all the leaf nodes are bound occurences of fixed point variables. The essential facts we use are that $\mathbf{n}(\alpha_1, \beta_1)$ is σ-expanding (Lemma 3.18), and the **n**-part is σ-contracting.

The proof strategy goes as follows:

$$
\begin{array}{lll}
 & \mathbb{A} \models \alpha \leq \beta & \\
\Longleftrightarrow & \mathbb{A} \models \alpha_1 \leq \beta_1 \vee \gamma & (\mathbf{n}\text{-trick in } \mu\text{-DML, see Lemma 8.8}) \\
\Longleftrightarrow & \alpha_1^{\mathbb{A}} \leq (\beta_1 \vee \gamma)^{\mathbb{A}} & (\text{trivial}) \\
\Longleftrightarrow & (\mathbf{n}(\alpha_1, \beta_1))^{\mathbb{A}} \leq \gamma^{\mathbb{A}} & (\text{Lemma 3.16}) \\
\Longleftrightarrow & \left((\mathbf{n}(\alpha_1, \beta_1))^{\mathbb{A}}\right)^{\sigma} \leq \left(\gamma^{\mathbb{A}}\right)^{\sigma} & (\text{Def. of } \sigma\text{-extension}) \\
\Longrightarrow & \left((\mathbf{n}(\alpha_1, \beta_1))^{\mathbb{A}}\right)^{\sigma} \leq \gamma^{\mathbb{A}^{\delta}_{\mathbb{A}}} & (\text{Lemma 8.8 and Lemma 8.11}) \\
\Longrightarrow & (\mathbf{n}(\alpha_1, \beta_1))^{\mathbb{A}^{\delta}_{\mathbb{A}}} \leq \gamma^{\mathbb{A}^{\delta}_{\mathbb{A}}} & (\text{Lemma 3.18}) \\
\Longleftrightarrow & \alpha_1^{\mathbb{A}^{\delta}_{\mathbb{A}}} \leq \beta_1^{\mathbb{A}^{\delta}_{\mathbb{A}}} \vee \gamma^{\mathbb{A}^{\delta}_{\mathbb{A}}} & (\text{see Lemma 3.16}) \\
\Longleftrightarrow & \mathbb{A}^{\delta}_{\mathbb{A}} \models \alpha_1 \leq \beta_1 \vee \gamma & (\text{trivial}) \\
\Longleftrightarrow & \mathbb{A}^{\delta}_{\mathbb{A}} \models \alpha \leq \beta & (\mathbf{n}\text{-trick in } \mu\text{-DML, see Lemma 8.8}).
\end{array}
$$

Most of the steps are the same as in the case of distributive modal logic, for instance, the **n**-trick and the second **n**-trick can be applied as in the basic DML case. A bit of extra complication arises to show that the uniform μ-terms in the γ-parts are σ-contracting. The rest of this chapter focuses mainly on this issue. $\square$

Lemma 8.8 (μ-Analogue of Lemma 5.14 in [17]). *Every (ϵ, μ)-Sahlqvist inequality $\alpha \leq \beta$ is equivalent to an inequality $\alpha_1 \leq \beta_1 \vee \gamma$ where $+\alpha_1$ is ϵ-uniform left Sahlqvist in the basic DML sense, $-\beta_1$ is ϵ-uniform right Sahlqvist in the basic DML sense, and $+\gamma$ is an ϵ-uniform μ-term.*

Proof. Since every fixed point node only occurs in the non ϵ-critical part, the **n**-trick can be applied so as to separate all the maximal non ϵ-critical parts, therefore the statement follows straightforwardly from the results in Subsection 3.1.1. By the definition of (ϵ, μ)-Sahlqvist inequalities, it is easy to check that $+\alpha_1$ is ϵ-uniform left Sahlqvist in the basic DML sense, $-\beta_1$ is ϵ-uniform right Sahlqvist in the basic DML sense, and $+\gamma$ is an ϵ-uniform μ-term. □

Lemma 8.9. *Let $\varphi(\vec{p}, \vec{q})$ be a μ-term and $\psi(\vec{p}, \vec{q})$ be a ν-term where $\vec{p}$ occur positively and $\vec{q}$ occur negatively, and let $\vec{c} \in K(\mathbb{A}^\delta_\mathbb{A}), \vec{o} \in O(\mathbb{A}^\delta_\mathbb{A})$. Then*

1. $\varphi^{\mathbb{A}^\delta_\mathbb{A}}(\vec{c}, \vec{o}) \in K(\mathbb{A}^\delta_\mathbb{A})$.
2. $\psi^{\mathbb{A}^\delta_\mathbb{A}}(\vec{o}, \vec{c}) \in O(\mathbb{A}^\delta_\mathbb{A})$.

Proof. We prove 1. and 2. by simultaneous induction on φ and ψ. The base case and connective cases are easily obtained by Lemma 4.6 and 4.9. For the case where $\varphi(\vec{p}, \vec{q}) = \mu x.\theta(\vec{p}, \vec{q}, x)$, since $\varphi^{\mathbb{A}^\delta_\mathbb{A}}(\vec{c}, \vec{o}) = (\mu x.\theta)^{\mathbb{A}^\delta_\mathbb{A}}(\vec{c}, \vec{o}, x) = \bigwedge\{a \in \mathbb{A} : \theta^{\mathbb{A}^\delta_\mathbb{A}}(\vec{c}, \vec{o}, a) \leq a\}$, we have that $\varphi^{\mathbb{A}^\delta_\mathbb{A}}(\vec{c}, \vec{o}) \in K(\mathbb{A}^\delta_\mathbb{A})$. The case of $\psi(\vec{p}, \vec{q})$ is similar. □

Lemma 8.10 (Intersection Lemma)**.** *Let $\varphi(\vec{p}, \vec{q}, r)$ be a μ-term and $\psi(\vec{p}, \vec{q}, r)$ be a ν-term where $\vec{p}$ occur positively and $\vec{q}$ occur negatively, and let $\vec{c} \in K(\mathbb{A}^\delta_\mathbb{A}), \vec{o} \in O(\mathbb{A}^\delta_\mathbb{A})$, $\{c_i : i \in I\} \subseteq K(\mathbb{A}^\delta_\mathbb{A})$ be downward-directed, $\{o_i : i \in I\} \subseteq O(\mathbb{A}^\delta_\mathbb{A})$ be upward-directed. Then*

1. *if $\varphi(\vec{p}, \vec{q}, r)$ is positive in r, then $\varphi^{\mathbb{A}^\delta_\mathbb{A}}(\vec{c}, \vec{o}, \bigwedge\{c_i : i \in I\}) = \bigwedge\{\varphi^{\mathbb{A}^\delta_\mathbb{A}}(\vec{c}, \vec{o}, c_i) : i \in I\}$;*
2. *if $\psi(\vec{p}, \vec{q}, r)$ is negative in r, then $\psi^{\mathbb{A}^\delta_\mathbb{A}}(\vec{o}, \vec{c}, \bigwedge\{c_i : i \in I\}) = \bigvee\{\psi^{\mathbb{A}^\delta_\mathbb{A}}(\vec{o}, \vec{c}, c_i) : i \in I\}$;*
3. *if $\varphi(\vec{p}, \vec{q}, r)$ is negative in r, then $\varphi^{\mathbb{A}^\delta_\mathbb{A}}(\vec{c}, \vec{o}, \bigvee\{o_i : i \in I\}) = \bigwedge\{\varphi^{\mathbb{A}^\delta_\mathbb{A}}(\vec{c}, \vec{o}, o_i) : i \in I\}$;*
4. *if $\psi(\vec{p}, \vec{q}, r)$ is positive in r, then $\psi^{\mathbb{A}^\delta_\mathbb{A}}(\vec{o}, \vec{c}, \bigvee\{o_i : i \in I\}) = \bigvee\{\psi^{\mathbb{A}^\delta_\mathbb{A}}(\vec{o}, \vec{c}, o_i) : i \in I\}$.*

Proof. We prove 1.-4. by simultaneous induction on φ and ψ. The base case and connective cases are similar to the proof of Lemma 4.12. We only consider the case where $\varphi(\vec{p}, \vec{q}, r) = \mu x.\theta(\vec{p}, \vec{q}, r, x)$ is positive in r, the remaining cases being order-variants.
Spelling out the relevant definitions, we obtain:

$$
\begin{aligned}
& \varphi^{\mathbb{A}^\delta_\mathbb{A}}(\vec{c}, \vec{o}, \textstyle\bigwedge\{c_i : i \in I\}) \\
= \ & (\mu x.\theta)^{\mathbb{A}^\delta_\mathbb{A}}(\vec{c}, \vec{o}, \textstyle\bigwedge\{c_i : i \in I\}, x) \\
= \ & \textstyle\bigwedge\{a \in \mathbb{A} : \theta^{\mathbb{A}^\delta_\mathbb{A}}(\vec{c}, \vec{o}, \bigwedge\{c_i : i \in I\}, a) \leq a\} \quad \text{(by Definition 8.3)}
\end{aligned}
$$

$$
\begin{aligned}
& \textstyle\bigwedge\{\varphi^{\mathbb{A}^\delta_\mathbb{A}}(\vec{c}, \vec{o}, c_i) : i \in I\} \\
= \ & \textstyle\bigwedge\{(\mu x.\theta)^{\mathbb{A}^\delta_\mathbb{A}}(\vec{c}, \vec{o}, c_i, x) : i \in I\} \\
= \ & \textstyle\bigwedge\{\bigwedge\{a \in \mathbb{A} : \theta^{\mathbb{A}^\delta_\mathbb{A}}(\vec{c}, \vec{o}, c_i, a) \leq a\} : i \in I\} \quad \text{(by Definition 8.3)} \\
= \ & \textstyle\bigwedge\{a \in \mathbb{A} : (\exists i \in I)\theta^{\mathbb{A}^\delta_\mathbb{A}}(\vec{c}, \vec{o}, c_i, a) \leq a\}.
\end{aligned}
$$

Hence we need to prove the equality between the last members of the two chains of equalities above.

The inequality $\bigwedge\{a \in \mathbb{A} : \theta^{\mathbb{A}^\delta_\mathbb{A}}(\vec{c}, \vec{o}, \bigwedge\{c_i : i \in I\}, a) \leq a\} \leq \bigwedge\{a \in \mathbb{A} : (\exists i \in I)\theta^{\mathbb{A}^\delta_\mathbb{A}}(\vec{c}, \vec{o}, c_i, a) \leq a\}$ immediately follows from the hypothesis that φ, and hence θ, is positive in r.

Conversely, let us show that if $a \in A$, and $\theta^{\mathbb{A}^\delta_\mathbb{A}}(\vec{c}, \vec{o}, \bigwedge\{c_i : i \in I\}, a) \leq a$, then $\theta^{\mathbb{A}^\delta_\mathbb{A}}(\vec{c}, \vec{o}, c_i, a) \leq a$ for some $i \in I$. By induction hypothesis,

$$\bigwedge\{\theta^{\mathbb{A}^\delta_\mathbb{A}}(\vec{c}, \vec{o}, c_i, a) : i \in I\} = \theta^{\mathbb{A}^\delta_\mathbb{A}}(\vec{c}, \vec{o}, \bigwedge\{c_i : i \in I\}, a) \leq a.$$

Since φ is a μ-term and is positive in r, so is θ. By Lemma 8.9, this implies that $\theta^{\mathbb{A}^\delta_\mathbb{A}}(\vec{c}, \vec{o}, c_i, a) \in K(\mathbb{A}^\delta_A)$. Hence, by compactness, $\bigwedge\{\theta^{\mathbb{A}^\delta_\mathbb{A}}(\vec{c}, \vec{o}, c_i, a) : i \in I'\} \leq a$ for some finite $I' \subseteq I$, and by the downward-directedness of $\{c_i : i \in I\}$, there exists some $i_0 \in I$ such that

$$\theta^{\mathbb{A}^\delta_\mathbb{A}}(\vec{c}, \vec{o}, c_{i_0}, a) \leq \bigwedge\{\theta^{\mathbb{A}^\delta_\mathbb{A}}(\vec{c}, \vec{o}, c_i, a) : i \in I'\} \leq a.$$

□

The proof above shows that the clopen semantics plays an important role in applying compactness.

Lemma 8.11. *Every uniform μ-term $s(\vec{p}, \vec{q})$ is σ-contracting, and every uniform ν-term $t(\vec{p}, \vec{q})$ is π-expanding, where $\vec{p}$ occur positively and $\vec{q}$ occur negatively.*

Proof. Straightforwardly follows from Theorem 6.8, Lemma 6.10 and Lemma 8.10. □

Chapter 9

Conclusion and Future Work

9.1 Conclusion

Summing up, the main results of this thesis are:

- a treatment of terms in the expanded signature $\mathcal{L}^{++}$ within a generalized definition of canonical extensions of maps (Chapter 6 and 7), and conditions on their compositional structure which guarantee them to be contracting or expanding.
- a new proof of canonicity of ALBA inequalities in the language of distributive modal logic (Chapter 7). This proof is Jónsson-style, and ALBA-aided.
- a Jónsson-style canonicity proof for the Sahlqvist fragment of distributive modal μ-calculus (Chapter 8), generalizing the Sahlqvist formulas of modal μ-calculus in the classical setting defined in [2].

9.2 Future Work

9.2.1 Extending Algebraic Canonicity to μ-Inductive Fragment

In Chapter 8, we generalize the result in [2] to the distributive setting. However, compared to the correspondence results proved in e.g. [29] and [5], the fragment identified by [2] is very restricted, especially in the sense that fixed point nodes are not allowed on critical branches; in addition, the more interesting set-theoretic semantics is unexplored regarding completeness and canonicity results.

In the future, along this line of research, the following directions are of interest:

- to generalize the μ-Sahlqvist inequalities to μ-inductive inequalities, using the inductive skeleton instead of the Sahlqvist skeleton, while still restricting fixed point nodes on non ϵ-critical branches.
- to allow fixed point nodes to appear on critical branches. This requires additional properties on the terms in the scope of those fixed points.

- to consider the set-theoretic semantics. The main difficulties here are due to the lack of good topological properties of μ and ν formulas.

9.2.2 Algebraic Canonicity Using Additive Terms

Although the Jónsson-style canonicity and the order-topological ALBA-style canonicity have many similarities on Sahlqvist and inductive inequalities as discussed in Section 7.3, the power of the combination of the two methods is still unexplored, and is certainly not limited to the previous chapters.

As is shown in [21] and discussed in [30], the Jónsson-style canonicity has the advantage that canonicity is independent from correspondence. This is particularly important for formulas (e.g. Fine formula in [14]) which do not have a first-order correspondent. In this section, we briefly discuss one example given in [21], which can serve as a case study, towards of improving both Jónsson-style canonicity (by proving that larger classes of terms enjoy sufficient conditions for being contracting or expanding), and the order-topological ALBA-style canonicity (by proving that compositions of certain connectives have adjoints). In the following we will only briefly discuss this issue, its systematic study being a future direction.

Example in Jónsson's Paper

Example 9.1 (cf. Theorem 6.1 in [21], with revision)**.** For a uniform σ-stable term $t(p)$ in $\mathcal{L}_{term}$ with order type (1), $t(p \vee q) \leq t(p) \vee t(q)$ is canonical.

Proof. We sketch the idea behind the proof, as is discussed in [21]. The aim is to prove that for any DMA $\mathbb{A}$,

$$\mathbb{A} \models t(p \vee q) \leq t(p) \vee t(q) \quad \Rightarrow \quad \mathbb{A}^\delta \models t(p \vee q) \leq t(p) \vee t(q).$$

Consider $t^{\mathbb{A}}$. Since $t(p)$ is uniform and positive in p, from $\mathbb{A} \models t(p \vee q) \leq t(p) \vee t(q)$ we have that $t^{\mathbb{A}}(a \vee b) = t^{\mathbb{A}}(a) \vee t^{\mathbb{A}}(b)$, therefore $t^{\mathbb{A}}$ is an operator. By Lemma 6.13, $(t^{\mathbb{A}})^\sigma$ is a complete operator. Since by assumption, $t(p)$ is a uniform σ-stable term, this implies $(t^{\mathbb{A}})^\sigma = t^{\mathbb{A}^\delta}$. Hence, $t^{\mathbb{A}^\delta}$ is also a complete operator, and so $\mathbb{A}^\delta \models t(p \vee q) \leq t(p) \vee t(q)$. $\square$

The example above shows that the power of Jónsson-style canonicity method is not confined to what we discussed in the previous chapters. In fact, this example can inspire further enhancements of both the Jónsson-style canonicity proofs and the ALBA-style canonicity proofs when considering the canonicity of the set $\{t(p \vee q) \leq t(p) \vee t(q), \varphi \leq \psi\}$, as we will briefly mention in the following.

Enhancement of Jónsson-Style Canonicity Proofs

In the Jónsson-style canonicity, the compositional structure of the uniform Sahlqvist terms is key, and, as we know, it consists of the outer skeleton, made with operators, and the inner structure, made of meet-preserving operations. Therefore, when we restrict ourselves to the class of all DMAs validating $t(p \vee q) \leq t(p) \vee t(q)$ where t is σ-stable, more terms will have the order-theoretic properties guaranteed by the Sahlqvist structure when interpreted on such algebras. For instance, we can take $t(p)$ as a whole to act as a generalized diamond with respect to that class; therefore, in the definition of Sahlqvist terms, the outer skeleton part can contain terms like t. Possible future work includes generalizing this idea to generalized boxes, triangles and n-ary operators, as well as incorporating the results in [31] stating that equations like $t(p \vee q) \leftrightarrow t(p) \vee t(q)$ are canonical for all positive terms $t(p)$.

Enhancement of ALBA-Style Canonicity Proofs

In the proof of Example 9.1, one essential property is that the σ-extension of an operator is a complete operator. Since being completely join-preserving and being a left adjoint are two faces of the same coin, we can make use of this fact in the design of strengthenings of ALBA. In the presence of an inequality such as $t(p \vee q) \leq t(p) \vee t(q)$ where $t(p)$ is a σ-stable term, to show the canonicity of $\{t(p \vee q) \leq t(p) \vee t(q), \varphi \leq \psi\}$, we can add the following rules on manipulating $\varphi \leq \psi$:

$$\frac{i \leq t(\alpha)}{j \leq \alpha \quad i \leq t(j)} \qquad \frac{t(\alpha) \leq m}{\alpha \leq \blacksquare_t m}$$

Here $t^{\mathbb{A}^\delta}$ is completely join-preserving and $\blacksquare_t^{\mathbb{A}^\delta}$ is the right adjoint of $t^{\mathbb{A}^\delta}$. The resulting "Jónsson-aided version" of ALBA can be used in its turn to prove more canonicity results.

Pseudo Correspondence

As we mentioned above, in [31], canonicity results are given, generalizing Jónsson's example to equations such as $t(p \vee q) \leftrightarrow t(p) \vee t(q)$ for any positive terms $t(p)$. It is also proved that these equations have pseudo first-order correspondence. Therefore, a natural question is whether these and other pseudo correspondence results can be obtained from the Jónsson-aided ALBA enhancement.

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