Determinacy and measurable cardinals in HOD

MSc Thesis *(Afstudeerscriptie)*

written by

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Chapter 1

Introduction

The study of the axiom of choice, AC, and of the axiom of determinacy, AD, are often seen as complementary endeavours in set theory since these axioms are incompatible. However, the contemporary development of set theory has allowed the emergence of an intricate connection between determinacy axioms and large cardinal axioms. In particular the hierarchy of the consistency strength of ZFC with large cardinal axioms has been used to gauge with precision the consistency strength of determinacy axioms. This enterprise is twofold. On one hand large cardinal assumptions in ZFC have been used to derive various degrees of determinacy of projective pointclasses, as well as the consistency of AD. On the other hand, models of AD, where AC is absent, have been used to create inner models that satisfy AC and contain large cardinals notions, even those that may not provably exist in a model of AD.

The latter, developed especially in the 1980's, lead to a focal result by Woodin:

Theorem 1.1 ([KW10])**.** *Assume that $V = \mathbf{L}({}^{\omega}\omega) + \mathsf{AD}$. If $\Theta = \delta$ we have that*

$$\text{HOD} \models \delta \textit{ is a Woodin cardinal.}$$

In this thesis, we study the underlying technique with which the above result is achieved. Namely, taking a *combinatorial large cardinal property* created in $\mathbf{L}({}^{\omega}\omega)$ via the axiom of determinacy and then pulling it back into HOD, which satisfies ZFC, resulting a much stronger large cardinal property.

The phrase *combinatorial large cardinal property* is used to highlight a difference between large cardinal properties in models of ZFC and of $\mathsf{ZF} + \mathsf{AD}$. In ZFC, the existence of a κ-complete non-principal ultrafilter over κ is equivalent to the existence of a non-trivial elementary embedding with critical point κ. In ZF though, we cannot prove this equivalence: The existence of a non-trivial elementary embedding with critical point κ implies that κ is a large cardinal in a meaningful way even in models of $\mathsf{ZF} + \mathsf{AD}$ (Corollary 3.13) whereas it is consistent with ZF that $\aleph_1$ carries a non-principal ω_1-complete ultrafilter. In fact, in a model of $\mathsf{ZF} + \mathsf{AD}$ this is the case (Corollary 4.21). We refer to the first description of a large cardinal notion as a *combinatorial notion* and the second as an *embedding notion*. In ZFC, the combinatorial notions are generally equivalent to appropriate embedding notions. At the same time, in ZF without choice, the embedding notions can be considerably stronger than the combinatorial notions, as has been studied in [Kie06] for example.

Here, we will first present large cardinal notions, focusing on combinatorial and embedding formulations of measurable cardinals, and study the relations of these with and without AC. Then, working in a model of AD, we will show the existence of combinatorial large cardinals. Finally we will present the technique of pulling the combinatorial objects in HOD in order to obtain embedding large cardinals.

Our main goal is to isolate the technique of pulling back combinatorial properties from the models of AD to get embedding properties in inner models that satisfy AC. This technique is not new: it is the backbone of Woodin's Theorem 1.1 and has been used by other authors. However, the technique has never been presented in isolation, independent of a particular application. By focusing on large cardinal properties that are much weaker than Woodinness, we manage to present the technique in its purest form, allowing for easily accessible proofs.

The main result of the thesis is the transfer theorem, Corollary 5.11, allowing us to pull back measurable cardinals into inner models with the axiom of choice. The main application of this is Theorem 5.14:

Theorem. *Assume* $V = \mathbf{L}({}^{\omega}\omega) + \mathsf{AD}$ *and let* $\Theta = \delta$*. We have*

$$\mathrm{HOD} \models \delta \textit{ is a strong inaccessible limit of } 1\textit{-embedding cardinals.}$$

This result also highlights the difference between combinatorial notions and embedding notions and the power of moving in an inner model with choice, it is impossible that any $\kappa < \Theta$ is an 1-embedding cardinal in a model of AD (Corollary 3.10).

Inner models of ZFC with embedding cardinals of higher order were originally investigated in [Gre78]. Green produced inner models of the form $\mathbf{L}[U]$. The technique used was significantly different from what is presented in this thesis. In [Gre78], the embedding cardinals were countable ordinals in the base model and therefore carried no complex ultrafilters. The ultrafilters that were thus constructed using determinacy were only ultrafilters in the inner model. Contrary to this our technique first defines ultrafilters in the base models using AD and then pulls them back into HOD, to yield embedding cardinals.

The structure of the thesis is as follows: In Chapter 2 we present the basics of the axiom of choice (Section 2.1), filters and combinatorics (Section 2.2), descriptive set theory (Section 2.3), infinite games and determinacy (Section 2.4) and model theory (Section 2.5).

In Section 3.1 we discuss ZFC-equivalent formulations of measurable cardinals in the context of ZF. We carefully present established ZFC results in the ZF framework to show what are the exact assumptions that are required. Furthermore we focus on embedding cardinals, whose study in the ZF context has been scarce, and highlight the great discrepancy they have with measurable cardinals in terms of strength. In Section 3.2 we present established results by Kleinberg that show that combinatorial properties of a cardinal κ imply that κ has normal ultrafilters.

In Section 4.1 we define the notions of norms in pointclasses and of Spector pointclasses. We present various of their properties as well as the existence of such pointclasses. In Section 4.2 we use the axiom of determinacy to derive the existence of ultrafiters on cardinals related to Spector pointclasses. We present two different proofs for this, in which we have tried to show with as little assumptions as possible the existence of normal ultrafilters, by abstracting away from the methods used in [KKMW81]. Then we present some results from [KKMW81] that show that cardinals associated with specific Spector pointclasses have strong combinatorial properties and using the theorems from Section 3.2 we derive the under $\mathsf{AD} + \mathsf{DC}$ one can prove the existence many 1-measurable cardinals below Θ.

Finally in Chapter 5 we define HOD and present in abstraction the argument that allows us to move from ultrafilters in the AD model to ultrafilters and embeddings in HOD. Then we apply the results from Chapter 4 to establish our final results.

Chapter 2

Preliminaries

The ambient theory in which we will be working will be the Zermelo-Fraenkel set theory, ZF. Every time some other axiom is used in a proof this will be stated in a parenthesis before the statement of the theorem. In this chapter we present the basic definitions and properties that will be needed throughout this text. We assume some familiarity with basic set theory, model theory and recursion theory. Some references for these topics are [Jec02] for set theory and [Hod97] for model theory. Furthermore, a good reference for large cardinals is [Kan08], while for descriptive set theory is [Mos09].

A binary transitive and irreflexive relation R on a set X is called *well-founded* if every subset of X has an R-minimal element. If a relation is not well-founded it is called *ill-founded*. If the relation is also total it is called a *well-ordering*. A preordering $\preceq$ on a set X (that is a binary transitive reflexive relation) will be called a *prewellordering* if the order $\preceq$ induces on the equivalence class $\equiv = \preceq \cap \preceq^{-1}$ is a well-ordering.

It is well-known that well-orderings are unique up to isomorphism. We identify ordinals with the Von Neumann ordinals and hence use $<$ and $\in$ interchangeably with them. We use the first letters of the greek alphabet $\alpha, \beta, \gamma, \ldots$ to denote (infinite) ordinals, while finite ordinals are denoted with $k, l, m, n, \ldots$ and are identified with the natural numbers. We denote with ω the first infinite ordinal and identify it with the set of the natural numbers.

We reserve the word *cardinal* for ordinals that are not equinumerous with any of its elements (even though in the absence of the axiom of choice there may exist cardinalities incomparable with all cardinals). We denote cardinals with $\kappa, \lambda, \mu, \nu, \ldots$. We denote with ω_α the α-th cardinal. The cofinality of an ordinal α, denoted by $\mathrm{cf}(\delta)$ is the least ordinal β such that an unbounded function $f : \beta \to \alpha$ exists. The cofinality of an ordinal is always a cardinal. A cardinal κ is called *regular* if $\mathrm{cf}(\kappa) = \kappa$.

Considering ω equipped with the discrete topology, we define the real numbers as ${}^\omega\omega$ equipped with the product topology. This space is called the *Baire Space*. The basic open sets are denoted with $N_s = \{f \in {}^\omega\omega : s \subset f\}$. We fix once and for all a recursive bijection $\pi : \omega \times \omega \to \omega$. Using this we can define a recursive bijection $\chi : {}^{\omega}\omega^{\omega} \to {}^{\omega}\omega$. Given a real r we will (ambiguously) denote with r_n the real $\chi^{-1}(r)$. Likewise, if a real a codes a countable ordinal α, then with r_ξ ($\xi \in \alpha$) we denote r_n where n has order $\xi + 1$ as coded by a.

Given reals a, b we say that $a \leq_{\mathrm{T}} b$ (a is Turing reducible from b) if there exists an (oracle) Turing machine that given access to b can compute a. This relation is a preorder, and hence induce the Turing equivalence relation $\equiv_{\mathrm{T}} = \leq_{\mathrm{T}} \cap \leq_{\mathrm{T}}^{-1}$. The equivalence classes $[a]_{\mathrm{T}}$ are called Turing degrees and $\leq_{\mathrm{T}}$ is an order relation on the Turing degrees, where each element has countable many predecessors. We denote $\mathcal{D}_{\mathrm{T}} = \{[a]_{\mathrm{T}} : a \in {}^{\omega}\omega\}$ the set of Turing degrees. The cone of Turing degrees with base a is the set $C_a = \{[b]_{\mathrm{T}} \in \mathcal{D}_{\mathrm{T}} : a \leq_{\mathrm{T}} b\}$.

2.1 Choice principles

Given a set X such that all $x \in X$ are non-empty, we cal $f : X \to \bigcup X$ a *choice function* for X if $f(x) \in x$ for all $x \in X$. The *Axiom of Choice* (AC) is the statement that every set that does not contain the empty has a choice function. It is well known that the axiom of choice is equivalent to the well ordering principle and Zorn's lemma. Given a set X and a cardinal κ we call $\mathsf{AC}_\kappa(X)$ the statement that every set of non-empty subsets of X, equinumerous with κ has a choice function. We denote with AC_κ the statement that $\mathsf{AC}_\kappa(X)$ is true for all sets X.

Proposition 2.1. *The axiom $\mathsf{AC}_\omega({}^{\omega}\omega)$ implies that ω_1 is regular.*

Proof. Heading towards a contradiction assume that there is a function $f : \omega \to \omega_1$ unbounded in ω_1. For every $n \in \omega$ define $X_n = \{r \in {}^{\omega}\omega : r \text{ codes } f(n)\}$. By $\mathsf{AC}_\omega({}^{\omega}\omega)$ there is a choice function for $\{X_n : n \in \omega\}$, g. Define the order on $\omega \times \omega$:

$$(n, m) < (k, l) \iff n < k \vee (n = k \wedge f(n)(m, l) = 1)$$

It is clear that this is a well-ordering and since f is unbounded in ω_1 it has length ω_1. By the bijection between $\omega \times \omega$ and ω, we have a bijection $h : \omega_1 \to \omega$. $\square$

The *Axiom of Dependent Choice* (DC) states that if R is a binary relation on a non-empty set X such that for all $x \in X$ there is a $y \in X$ with xRy then, there is a function $f : \omega \to X$ such that for all $n \in \omega$ $f(n)Rf(n+1)$.

Proposition 2.2. *(DC) A binary relation R on X is ill-founded if and only if there exists an R-descending ω-sequence $\langle x_n : n \in \omega \rangle$.*

Proof. If an R-descending sequence $\langle x_n : n \in \omega \rangle$ exists then $\{x_n : n \in \omega\} \subseteq X$ is a subset of X with no minimal element, hence R is ill-founded.

The converse requires the axiom of dependent choice. If R is ill-founded, then there exists $Y \subseteq X$ such that Y has no minimal element. In particular for every $x \in Y$ there is a $y \in Y$ such that yRx. Hence by DC there is an R^{-1}-sequence of length ω, i.e., an R-descending ω-sequence. $\square$

Proposition 2.3. *The axiom of dependent choice implies that AC_ω.*

Proof. Assume that we have a countable set $X = \{x_n : n \in \omega\}$. Let $A = \{p \in [\bigcup X]^{<\omega} : (\forall k \in \mathrm{dom}(p))[p(k) \in x_k]\}$. Since every p in A can be extended to a q in A, by the axiom of dependent choice we have a sequence $p_1 \subsetneq \ldots \subsetneq p_n \subsetneq \ldots$. Then $\mathrm{dom}(\bigcup_{n\in\omega} p_n) = \omega$. We define $f : X \to \bigcup X$ as $f(x_k) = \bigcup_{n\in\omega} p_n(k)$. This f is a choice function for X. □

2.2 Filters and combinatorics

Given a set S, we say that $\mathcal{F} \subseteq \wp(S)$ is a *filter* over X (or on $\wp(S)$) if it has the following properties:

1. We have $S \in \mathcal{F}$ and $\varnothing \notin \mathcal{F}$;
2. if $X \in \mathcal{F}$ and $X \subseteq Y$ then $Y \in \mathcal{F}$;
3. if $X, Y \in \mathcal{F}$ then $X \cap Y \in \mathcal{F}$.

A filter $\mathcal{F}$ over S is called an *ultrafilter* if for every $X \subseteq S$, $X \in \mathcal{F}$ or $S \setminus X \in \mathcal{F}$. Notice that X and $S \setminus X$ cannot be both in $\mathcal{F}$.

Proposition 2.4. *A filter $\mathcal{F}$ is maximal under $\subseteq$ if and only if $\mathcal{F}$ is an ultrafilter.*

Proof. Assume that $\mathcal{F}$ is not maximal, i.e., it can be extended to $\mathcal{F}'$ such that $X \in \mathcal{F}'$ and $X \notin \mathcal{F}$. Then $S \setminus X \notin \mathcal{F}$ since otherwise $\varnothing = X \cap (S \setminus X) \in \mathcal{F}'$. Hence $\mathcal{F}$ is not an ultrafilter.

On the other hand assume that $\mathcal{F}$ is not an ultrafilter. Then there is some $X \subseteq S$ such that $X \notin \mathcal{F}$ and $S \setminus X \notin \mathcal{F}$. For every $Y \in \mathcal{F}$ $X \cap Y \neq \varnothing$, since otherwise $Y \subset S \setminus X$, which would imply that $S \setminus X \in \mathcal{F}$. Because of this it follows that

$$\{Y \subset S : (\exists W \in \mathcal{F})[W \cap X \subseteq Y]\}$$

is a filter that contains $\mathcal{F} \cup \{X\}$, i.e., $\mathcal{F}$ is not maximal. □

An ultrafilter $\mathcal{U}$ over S is called *principal* if there exists some $s \in S$ such that $\{s\} \in \mathcal{U}$. If $\mathcal{U}$ is not principal it will be called *non-principal*.

Given a cardinal κ a filter $\mathcal{F}$ over S will be called *κ-complete* if for every $\lambda < \kappa$ and $\{X_\xi : \xi \in \lambda\} \subseteq \mathcal{F}$ we have $\bigcap_{\xi\in\lambda} X_\xi \in \mathcal{F}$.

Proposition 2.5. *An ultrafilter $\mathcal{U}$ over S is κ-complete if and only if for all $\lambda < \kappa$, $\bigcup_{\xi\in\lambda} X_\xi = S$ implies that there exists some $\xi \in \lambda$ such that $X_\xi \in \mathcal{F}$.*

Proof. If $\mathcal{U}$ is not κ-complete and this is witnessed by $\{X_\xi : \xi \in \lambda\}$ then since $\mathcal{U}$ is an ultrafilter $(\bigcap_{\xi\in\lambda} X_\xi) \cup (\bigcup_{\xi\in\lambda}(S \setminus X_\xi)) = S$ but by definition none of these are in $\mathcal{U}$. On the other hand if $\bigcup_{\xi\in\lambda} X_\xi = S$ and $X_\xi \notin \mathcal{F}$ for all $\xi \in \lambda$, then $\mathcal{F}$ cannot be κ-complete. If it were then since $\mathcal{U}$ is an ultrafilter, we would have that $\xi \in \lambda$ $S \setminus X_\xi \in \mathcal{U}$, and by the κ-completeness of $\mathcal{U}$

$$\bigcap_{\xi\in\lambda}(S \setminus X_\xi) = S \setminus \bigcup_{\xi\in\lambda} X_\xi = \varnothing \in \mathcal{F}.$$

□

Proposition 2.6. *Let $f : S \to W$ and let $\mathcal{F}$ be a κ-complete filter over S. Then the set $f^*[\mathcal{F}] = \{Y \subseteq W : f^{-1}[Y] \in \mathcal{F}\}$ is a κ-complete filter over W. Furthermore if $\mathcal{F}$ is an ultrafilter so is $f^*[\mathcal{F}]$.*

Proof. Since $f^{-1}[\bigcap_{\xi\in\lambda} Y_\xi] = \bigcap_{\xi\in\lambda} f^{-1}[Y_\xi]$, if $Y_1 \subseteq Y_2$ we have $f^{-1}[Y_1] \subseteq f^{-1}[Y_2]$, $f^{-1}[W] = S$ and $f^{-1}[\varnothing] = \varnothing$, it follows that $f^*[\mathcal{F}]$ is a filter. If $\mathcal{F}$ is an ultrafilter, $f^{-1}[Y_1 \cup Y_2] = f^{-1}[Y_1] \cup f^{-1}[Y_2]$ implies that so is $f^*[\mathcal{F}]$. $\square$

The *cone filter* M_{T} over $\mathcal{D}_{\mathrm{T}}$ is defined as follows:

$$X \in M_{\mathrm{T}} \iff \exists a \in {}^{\omega}\omega : X \supseteq C_a.$$

We note that if a and b are reals and c codes them both recursively then $C_c \subseteq C_a \cap C_b$, granting that M_{T} is closed under intersections and hence it is a filter.

Let κ be a regular cardinal. If $X \subseteq \kappa$, we say that α is a *limit* of X if X is unbounded below α (that is for all $\beta \in \alpha$ there is a $\gamma \ni \beta$ with $\gamma \in X$). We denote the set of all limits of X with $\lim(X)$. We say that $C \subset \kappa$ is a *closed unbounded set* on κ (or a *club* on κ) if C is unbounded in κ and $\lim(C) \subseteq C$. Given $\{C_\xi : \xi \in \kappa\}$, where each $C_\xi \subset \kappa$, we call the *diagonal intersection* of $\{C_\xi : \xi \in \kappa\}$ the following set:

$$\triangle_{\xi\in\kappa} C_\xi = \{\alpha \in \kappa : \alpha \in \bigcap_{\beta\in\alpha} C_\beta\}.$$

The following proposition, given without proof, states two basic properties of closed unbounded sets:

Proposition 2.7. *Let κ be a regular cardinal. The following are true:*

1. *If $\lambda < \kappa$ and C_ξ is a club set for all $\xi \in \lambda$ then $\bigcap_{\xi\in\lambda} C_\xi$ is a club set.*
2. *If C_ξ is a club set for all $\xi \in \kappa$, then $\triangle_{\xi\in\kappa} C_\xi$ is a club set.*

A filter $\mathcal{F}$ over κ is called *normal* if it is closed under the diagonal intersection. That is for all $\xi \in \kappa$ $X_\xi \in \mathcal{F}$ then $\triangle_{\xi\in\kappa} X_\xi \in \mathcal{F}$. It is clear that closure under diagonal intersection implies κ-completeness.

Proposition 2.8. *If $\mathcal{F}$ is a filter over κ that contains the all final segments of κ and is normal, then it contains all closed unbounded sets.*

Proof. First of all we note that $\lim(\kappa) \in \mathcal{F}$ since $\lim(\kappa) = \triangle_{\xi\in\kappa}(\kappa \setminus (\xi + 2))$. Then if $C = \{\alpha_\xi : \xi \in \kappa\}$ is a club set $\triangle_{\xi\in\kappa}(\kappa \setminus (\alpha_\xi + 1)) \cap \lim(\kappa) \subseteq C$, i.e., $C \in \mathcal{F}$. $\square$

A function $f : S \to \kappa$ where $S \subseteq \kappa$ is called *regressive* if for all $\alpha \in S \setminus \omega$ we have that $f(\alpha) < \alpha$.

Proposition 2.9. *Let $\mathcal{F}$ be a filter over κ, where κ is a regular cardinal, that contains all final segments of κ. Then the following are equivalent:*

1. *The filter $\mathcal{F}$ is a κ-complete normal ultrafilter.*

2. *For every regressive function $f : \kappa \to \kappa$ there exists some $\alpha \in \kappa$ such that $\{\xi \in \kappa : f(\xi) = \alpha\} \in \mathcal{F}$.*

Proof. Let $f : \kappa \to \kappa$ be a regressive function and assume towards a contradiction that $X_\alpha = \{\xi \in \kappa : f(\xi) \neq \alpha\} \in \mathcal{F}$. Then $X = \triangle_{\alpha\in\kappa} X_\alpha \in \mathcal{F}$, and since $\mathcal{F}$ contains all final segments of κ, there exists some $\alpha \in X$ which is infinite. But because $\alpha \in X$, we have that $f(\alpha) \neq \beta$, for all $\beta < \alpha$, hence $f(\alpha) \geq \alpha$, a contradiction.

For the converse, let $X \subset \kappa$ and define the function $f : \kappa \to \kappa$ with $f(x) = 1$ if $x \in X$ and $f(x) = 0$ if $x \notin X$. This is regressive and hence either X or $\kappa \setminus X$ is in $\mathcal{F}$. To show normality, let $X_\xi \in \mathcal{F}$ and assume towards a contradiction that $\triangle_{\xi\in\kappa} X_\xi \notin \mathcal{F}$. Define the function $f(\alpha) = \bigcap\{\xi \in \alpha : \alpha \notin X_\xi\} + 1$ if $\alpha \notin \triangle_{\xi\in\kappa} X_\xi \notin \mathcal{F}$, and 0 otherwise. Then this is a regressive function and hence there is some α such that $\{\xi \in \kappa : f(\xi) = \alpha\} \in \mathcal{F}$. This α cannot be 0 thus $\kappa \setminus X_{\alpha-1} \in \mathcal{F}$. □

Given a regular cardinal κ we define $\mathcal{C}_\kappa \subseteq \wp(\kappa)$ as:

$$X \in \mathcal{C}_\kappa \iff \text{there is a closed unbounded set } C \text{ such that } X \supseteq C.$$

Since closed unbounded sets are closed under (finite) intersections it follows that $\mathcal{C}_\kappa$ is filter and it is called the *closed unbounded filter* (or club filter) over κ. Under AC_κ, by the properties of club sets, it follows that $\mathcal{C}_\kappa$ is a normal κ-complete filter.

Given a regular cardinal κ a set $S \subset \kappa$ is called *stationary* if for all closed unbounded set C on κ, $S \cap C \neq \varnothing$. Since if C_1, C_2 are club sets $C_1 \cap C_2$ is also a club set, it immediately follows that given a stationary set S and club set C on κ, $S \cap C$ is also stationary on κ. If $\lambda \in \kappa$ is a regular cardinal we define

$$E^\kappa_\lambda = \{\alpha \in \kappa : \mathrm{cf}(\alpha) = \lambda\}.$$

It is easy to see that the E^κ_λ are stationary sets on κ. We also notice that if $\mathcal{U}$ is a non-principle normal ultrafilter on κ then all its elements are stationary sets. This follows because by Proposition 2.8 the club filter is a subset of $\mathcal{U}$ and the intersection of any two elements of an ultrafilter are non-empty.

If S is a stationary set on κ, we can now define $\mathcal{C}^S_\kappa$, a refinement of the closed unbounded filter:

$$X \in \mathcal{C}^S_\kappa \iff \text{there is a closed unbounded set } C \text{ such that } X \supseteq C \cap S.$$

We notice that for all stationary sets S $\mathcal{C}^S_\kappa$ is a filter as well and that $\mathcal{C}^S_\kappa \supseteq \mathcal{C}_\kappa$. Again, under AC_κ it follows that C^S_κ is a normal κ-complete filter. In the case that $S = E^\kappa_\lambda$ we simply write $\mathcal{C}^\lambda_\kappa$ instead of $\mathcal{C}^{E^\kappa_\lambda}_\kappa$, and call this the λ-*closed unbounded filter*.

If $\lambda \in \kappa$ we define $[\kappa]^\lambda = \{X \subset \kappa : |X| = \lambda\}$ and $[\kappa]^\lambda = \wp_\lambda(\kappa) = \bigcup_{\mu\in\lambda}[\kappa]^\mu$. We note that $[\kappa]^\lambda$ can also be considered the set of strictly increasing functions from κ to λ. If $f : [\kappa]^\lambda \to \mu$ we say that $H \subset \kappa$ is a *homogeneous* set for f if $|f[[H]^\lambda]| = 1$. We define the relation

$$\kappa \to (\lambda)^\mu_\nu$$

to designate that all $f : [\kappa]^\mu \to \nu$ have a homogeneous set of size λ. It is hard not to notice that the relation remains true if we increase what is to the left of the arrow or decrease what is to the right of it. in the case that $\nu = 2$ we may omit the subscript.

We have the following results:

Lemma 2.10. *If $\kappa \to (\kappa)^2_2$ then κ is a regular cardinal.*

Proof. Assume that there is a $\lambda < \kappa$ and a partition of κ $\langle X_\xi : \xi \in \lambda\rangle$ such that $|X_\xi| < \kappa$. Then define a function $f : [\kappa]^2 \to 2$ as

$$f(\alpha, \beta) = 1 \iff \exists \xi \in \lambda(\alpha, \beta \in X_\xi).$$

If H is a homogeneous set for f, then if $f[[H]^2] = \{0\}$ then the elements of H pairwise belong to different elements of the partition, hence $|H| \leq \lambda$. If $f[[H]^2] = \{1\}$, then the elements of H belong in the same element of the partition, hence $|H| < \kappa$. $\square$

Lemma 2.11 ([Kle70])**.** *If λ is a regular cardinal and $\kappa \to (\kappa)^{\lambda+\lambda}_2$, then $\kappa \to (\kappa)^\lambda_\gamma$ for all $\gamma < \kappa$.*

Proof. Let $f : [\kappa]^\lambda \to \gamma$. We define the function $g : [\kappa]^{\lambda+\lambda} \to 2$ as

$$g(s_1 \cup s_2) = 0 \iff f(s_1) = f(s_2)$$

where if $s_1 \cup s_2$ is a $\lambda + \lambda$ sequence s_1 are its first λ elements and s_2 are the rest.

By Lemma 2.10 κ is regular and hence $\lambda \cdot \kappa = \kappa$. Thus if H is a homogeneous set for g of size κ we can write H as an increasing sequence $\langle s_\xi : \xi \in \kappa\rangle$ such that $s_\xi \in [\kappa]^\lambda$ and $\cup s_\xi < \cap s_\delta$ for $\xi < \delta$. Hence it cannot be the case that $g[[H]^{\lambda+\lambda}] = \{1\}$ because then $f(s_\xi) \neq f(s_\delta)$ for each $\xi \neq \delta$, a contradiction since f takes γ many values only. Therefore $g[[H]^{\lambda+\lambda}] = \{0\}$, which implies that $f(s_\xi) = f(s_\delta)$ for all $\xi, \delta \in \kappa$, i.e., H is a homogeneous set for f. $\square$

The axiom of choice restricts the possibilities of this relation:

Lemma 2.12 ([ER52])**.** *(AC) There exist no κ such that $\kappa \to (\omega)^\omega_2$.*

Proof. Let $s, t \in [\kappa]^\omega$. We say that $s \equiv t$ if and only if $\{n \in \omega : s(n) \neq t(n)\}$ is finite. Using the axiom of choice let's pick a representative from each equivalence class. Then we can define $f(s) = 1$ if s differs from the representative of its class at even many places and 0 otherwise. Let $H \in [\kappa]^\omega$. Let s be such that $s(n) = H(2n)$. There exists some m such that for all $k \geq m$ $s(k)$ coincides with its representative. Then let s' be equal to s everywhere but in m and let $s'(m) = H(2m + 1)$. It is clear that $f(s) \neq f(s')$. $\square$

2.3 Descriptive set theory

We call a *space* $\mathcal{X}$ the finite product of copies of ${}^{\omega}\omega$ (seen as the Baire Space) and ω (equipped with the discrete topology) and topologised by the product topology. A *pointset* will be a subset of a space while a set of pointsets will be called a *pointclass*. If $\mathcal{X}$ is a space and Γ is a pointclass we denote with $\Gamma \restriction \mathcal{X} = \{P \subseteq \mathcal{X} : P \in \Gamma\}$.

We can think of pointsets as sets or relations:

$$x \in P \iff P(x).$$

If $P \subseteq {}^{\omega}\omega \times \mathcal{X}$ and $e \in {}^{\omega}\omega$ we denote with $P_e = \{x \in \mathcal{X} : P(e, x)\}$. Given a pointset $P, Q \subset \mathcal{X}$ we define $\neg P(x) \iff x \in \mathcal{X} \setminus P$, $P \wedge Q(x) \iff x \in P \cap Q$, $P \vee Q(x) \iff x \in P \cup Q$. Furthermore, if $P \subseteq \mathcal{Y} \times \mathcal{X}$ we define

$$\exists^{\mathcal{Y}} P = \{x \in \mathcal{X} : \exists y P(y, x)\}$$

and likewise

$$\forall^{\mathcal{Y}} P = \{x \in \mathcal{X} : \forall y P(y, x)\} = \neg\exists^{\mathcal{Y}} \neg P.$$

For $P \subseteq \omega \times \mathcal{X}$ we also define

$$\exists^{\leq} P = \{(x, n) \in \mathcal{X} \times \omega : \exists m \leq n P(m, x)\}$$

and

$$\forall^{\leq} P = \{(x, n) \in \mathcal{X} \times \omega : \forall m \leq n P(m, x)\}.$$

We say that a pointclass is Γ is closed under an operation F on pointsets if whenever $P_1, \dots, P_n \in \Gamma$ we have that $F(P_1, \dots, P_n) \in \Gamma$ as well.

If Γ is a pointclass we denote with $\check{\Gamma}$ the *dual pointclass*, i.e., for $P \subseteq \mathcal{X}$ we have

$$P \in \check{\Gamma} \iff \neg P \in \Gamma.$$

We denote with $\Delta_\Gamma = \Gamma \cap \check{\Gamma}$. If Γ is clear from the context, we drop the subscript.

Given a pointclass Γ, for $e \in {}^{\omega}\omega$, we define $\Gamma(e)$ by

$$P \in \Gamma(e) \iff (\exists P^* \in \Gamma)[P = P^*_e].$$

Hence associated with Γ we can define the *boldface pointclass*

$$\boldsymbol{\Gamma} = \bigcup_{e \in {}^{\omega}\omega} \Gamma(e).$$

If $f : \mathcal{X} \rightharpoonup \mathcal{Y}$ is a partial function, $D \subseteq \mathrm{dom}(f)$ and $P \subseteq \mathcal{X} \times \omega$,we say that P *computes* f *on* D if

$$x \in D \implies (\forall s)[f(x) \in N_s \iff P(x, s)].$$

Given a pointclass Γ we say that a partial function f is Γ-*recursive on* D if $D \subseteq \mathrm{dom}(f)$, $D \in \Gamma$ and there is a $P \in \Gamma$ that computes f on D. We say that

f is Γ-*recursive* if $\mathrm{dom}(f) \in \Gamma$ and f is Γ-recursive on $\mathrm{dom}(f)$. We say that Γ has the *substitution property* if for every $Q \subseteq \mathcal{Y}$ and for every Γ-recursive $f : \mathcal{X} \rightharpoonup \mathcal{Y}$ there is a $Q' \in \Gamma$ such that

$$f(x)\downarrow \Longrightarrow [Q'(x) \iff Q(f(x))].$$

A pointclass Γ is called *adequate* if it contains all recursive pointsets and it is closed under recursive substitution, $\wedge, \vee, \exists^{\leq}$ and $\forall^{\leq}$. A pointclass Γ is called a Σ-*pointclass* if it contains all semirecursive pointsets and it is closed under trivial substitution, $\wedge$, $\vee$, $\exists^{\leq}$, $\forall^{\leq}$ and $\exists^{\omega}$. We say that a pointclass Γ is I-parametrised if for every space $\mathcal{X}$ there exists some surjective function $h : I \twoheadrightarrow \Gamma \restriction \mathcal{X}$.

Lemma 2.13 (Good parametrisation lemma)**.** *Suppose that Γ is a pointclass which is ω-parametrised and closed under recursive substitutions. Then for each space $\mathcal{X}$ there is a $G^{\mathcal{X}} \subseteq {}^{\omega}\omega \times \mathcal{X}$ which is universal for $\boldsymbol{\Gamma} \restriction \mathcal{X}$ and the following properties hold:*

1. *For $P \subseteq \mathcal{X}$ we have that*

$$P \in \Gamma^{\mathcal{X}} \iff P = G_e \textit{ with } e \textit{ recursive}$$

2. *For each space $\mathcal{X}$ of type 0 or 1 and each space $\mathcal{Y}$, there is a recursive function*

$$S^{\mathcal{X},\mathcal{Y}} : {}^{\omega}\omega \times \mathcal{X} \to {}^{\omega}\omega$$

such that

$$G^{\mathcal{X}\times\mathcal{Y}}(e, x, y) \iff G^{\mathcal{Y}}(S^{\mathcal{X},\mathcal{Y}}(e, x), y).$$

Proof. See [Mos09, 3H.1]. □

Such a G will be called a *good universal set*. Given such a G for Γ we will say that e is a $\boldsymbol{\Gamma}$-code for some $P \in \boldsymbol{\Gamma}$ if $P = G_e$. We see that it immediately follows that if Γ is such that it admits a good universal set then $\boldsymbol{\Gamma}$ is ${}^{\omega}\omega$-parametrised. We have the following important corollaries:

Corollary 2.14. *Supposed Γ is an ω-parametrised adequate pointclass. If Γ is closed under any of the operations $\wedge, \vee, \exists^{\leq}, \forall^{\leq}, \exists^{{}^{\omega}\omega}, \forall^{{}^{\omega}\omega}$ then $\boldsymbol{\Gamma}$ is uniformly closed under the same operations.*

Proof. We will only show the case for $\exists^{{}^{\omega}\omega}$; the others are similar. Assume that Γ is closed under $\exists^{{}^{\omega}\omega}$. We need to find a recursive function $u : {}^{\omega}\omega \to {}^{\omega}\omega$ such that if for $P \subseteq {}^{\omega}\omega \times \mathcal{X}$ and $a \in {}^{\omega}\omega$

$$P(x, y) \iff G(a, x, y)$$

then

$$(\exists x)P(x, y) \iff G(u(a), y).$$

By Lemma 2.13 we can assume that G is a good universal set. Then we let

$$Q(a, y) \iff (\exists x)G(a, x, y).$$

Then since G is good there is a recursive e such that

$$Q(a,y) \iff G(e,a,y) \iff G(S(e,a),y).$$

Hence we let $u(a) = S(e,a)$. □

Corollary 2.15 (Kleene's recursion theorem)**.** *Suppose Γ is an ω-parametrised pointclass, closed under recursive substitution and let $R \subset {}^{\omega}\omega \times \mathcal{X}$. Then there can be found a recursive e such that R_e has $\mathbf{\Gamma}$-code e, i.e.,*

$$R(e,x) \iff G(e,x),$$

where G is a fixed good universal set for $\mathbf{\Gamma} \restriction \mathcal{X}$.

Proof. Let

$$P(a,x) \iff R(S(a,a),x)$$

where S is as from Lemma 2.13, i.e.,

$$G(a,b,x) \iff G(S(a,b),x).$$

P is in Γ and hence there is a recursive e such that

$$G(e,a,x) \iff P(a,x).$$

Now let $e^* = S(e,e)$. We have

$$R(e^*,x) \iff P(e,x) \iff G(e,e,x) \iff G(e^*,x).$$

□

2.4 Games and determinacy

Given $A \subseteq X^\omega$ we define the *two-player perfect-information zero-sum game* $G_X(A)$ as follows: There are two players I and II. Player I plays first and players alternate turns. Each player plays an element of X during their turn. The game last ω many rounds, and hence player I plays during the even rounds while player II during the odd ones. Schematically this is described as follows:

$$\begin{array}{rcccccccc} I: & x(0) & & x(2) & & x(4) & & \ldots \\ II: & & x(1) & & x(3) & & x(5) & \ldots \end{array}$$

Whenever a player makes a move it is assumed that they are aware of all the previously played moves. The outcome of the game is the function $x \in X^\omega$. We say that player I wins the game if $x \in A$; otherwise player II wins (so there can be no draw). We denote with $(x)_I \in X^\omega$ the function $(x)_I(n) = x(2n)$ and with $(x)_{II} \in X^\omega$ the function $(x)_{II}(n) = x(2n+1)$. $(x)_I$ are the elements I played in order while $(x)_{II}$ are the elements II played in order. A *strategy* for I is a function $\sigma : \bigcup_{n\in\omega} X^{2_n} \to X$. For player II is a function $\tau : \bigcup_{n\in\omega} X^{2n+1} \to X$.

The purpose of the strategy is to dictate to the player what move to make, given the previous moves. Given $x, y \in X^\omega$ we denote with $x \star y = z$ such that $(z)_I = x$ and $(z)_{II} = y$, that is the outcome if I plays x and II plays y. If σ is a strategy for I we denote with $\sigma \star y$ the outcome of the game if II played y and I followed σ, and likewise we define $x \star \tau$.

We call a strategy σ *winning* for I if no matter what II plays, following σ leads to I winning the game. The definition for a winning strategy for II is symmetric. That is, σ is winning for I if $\{\sigma \star y : y \in X^\omega\} \subseteq A$ and τ is winning for II if $\{x \star \tau : x \in X^\omega\} \cap A = \varnothing$. We say that the game $G_X(A)$ is *determined* if one of the two players has a winning strategy. We say that a position $s \in X^k$, where k is even, is winning for I in $G_X(A)$ if there is a winning strategy σ such that $\{s^\frown \sigma \star y : y \in X^\omega\} \subseteq A$. Likewise a winning position for II can be defined.

Theorem 2.16 (Gale-Stewart)**.** *Assume that $(X, \wp(X))$ is a topological space equipped with the discrete topology, and $(X^\omega, \mathcal{T})$ is the product space equipped with the product topology. Then for all $A \in \mathcal{T}$ the game $G_X(A)$ is determined.*

Proof. Let $A \in \mathcal{T}$ and assume that there is no winning strategy for I. Then we define the strategy τ for II, in which he picks the element that leads to a position that is not winning for I. First of all we notice that this is well-defined. To see this we observe that if at a position s there is some $i \in X$ that I can play such for all $j \in X$ the position $s^\frown i^\frown j$ is winning for I (witnessed by σ_j), so is s. Indeed let $\sigma(\varnothing) = i$ and $\sigma(i^\frown j^\frown t) = \sigma_j(t)$.

We claim that this strategy is winning for II. Assume that it is not, i.e., we have some $x \in X^\omega$ such that $x \star \tau \in A$. This was witnessed at some finite point of the game. This is because A is a union of base open set. Hence x is in some base open set $A' = A_1 \times \cdots \times A_k \times X^\omega$, where each $A_i \subseteq X$. Hence the fact that the game was won by I, was already known (i.e., I was in a winning position) at the k-th move, contradicting the definition of τ. □

If $X = \omega$, then the players form a real number. In that case we omit the subscript and write the game as $G(A)$, for $A \subseteq {}^\omega\omega$. We also note that strategies are of the form $\sigma : \omega^{<\omega} \to \omega$ and hence they can be recursively coded into real numbers.

The statement "*For every $A \in {}^\omega\omega$ the game $G_\omega(A)$ is determined*" is called the *Axiom of Determinacy (*AD*).*

Lemma 2.17. *Assume that M and N are models of set theory such that $(V_{\omega+7})^M = (V_{\omega+7})^N$. Then $M \models \mathsf{AD}$ if and only if $N \models \mathsf{AD}$.*

Proof. The statement of the axiom of determinacy is about sets of real numbers and real numbers. Because all subsets of reals as well as the winning strategies exist in $V_{\omega+7}$, AD is decided in $V_{\omega+7}$. □

We note that 7 is a gross estimate of what we need. It is chosen because it is certain that all sets of reals and all strategies are in $V_{\omega+7}$. If one wants to be careful 7 can be decreased.

Because strategies can be coded by real numbers, the axiom of determinacy is not compatible with the axiom of choice:

Theorem 2.18. *(AC) The axiom of determinacy is false.*

Proof. Assume that we have enumerations of the strategies $\{\sigma_\xi : \xi \in 2^\omega\}$ and $\{\tau_\xi : \xi \in 2^\omega\}$ for I and II, respectively. We note that given a strategy there are 2^ω different outcomes from games according to that strategy.

We create sets A_ξ, B_ξ for $\xi \in 2^\omega$ such that $|A_\xi| = |B_\xi| = |\xi|$ by transitive recursion: Assume that A_ξ and B_ξ are defined. The sets $\{\sigma_\xi \star y : y \in {}^\omega\omega\} \setminus (A_\xi \cup B_\xi)$ and $\{x \star \tau_\xi : x \in {}^\omega\omega\} \setminus (A_\xi \cup B_\xi)$ are or size 2^ω and thus let us choose four distinct elements, $a^\sigma_\xi, b^\sigma_\xi$ from the first and a^τ_ξ, b^τ_ξ from the second and let $A_{\xi+1} = A_\xi \cup \{a^\sigma_\xi, a^\tau_\xi\}$ and $B_{\xi+1} = B_\xi \cup \{b^\sigma_\xi, b^\tau_\xi\}$. For the limit case we let $A_\xi = \bigcup_{\delta\in\xi} A_\delta$ and $B_\xi = \bigcup_{\delta\in\xi} B_\delta$.

Now we let $A = \bigcup_{\xi\in 2^\omega} A_\xi$ and $B = \bigcup_{\xi\in 2^\omega} B_\xi$. First we note that by construction $A \cap B = \varnothing$. Now we claim that there is no winning strategy in $G(A)$. Indeed every winning strategy for I has an outcome landing in B (and hence outside of A) and every winning strategy for II has an element that lands in A. $\square$

Lemma 2.19. *If there exists an non-principal ultrafilter over ω then the axiom of determinacy is false.*

Proof. Let $\mathcal{U}$ be a non-principle ultrafilter over ω. We define the following game: Each player play a natural number. The first player who plays a number smaller than or equal to the previous number played loses. If both players manage to follow the above restriction and the outcome of the game is x then I wins unless $\bigcup_{n\in\omega}[x(2n+1) \setminus x(2n)] \in \mathcal{U}$. That is, in this game players play consecutive intervals and the winner is the one whose union of intervals lands in $\mathcal{U}$.

We claim that the above game is not determined and this is because players can steal each other's strategy:

If I has a winning strategy σ then we define a winning strategy τ for II as follows: $\tau(\langle n, 0\rangle) = \max\{n+1, \sigma(\varnothing)\}$. If s is a position of the game then $\tau(s) = \sigma(s')$ where $s'(0) = \sigma(\varnothing)$ and $s'(n) = s(n+1)$. Then the difference between $\bigcup_{n\in\omega}[(x\star\tau)(2n+1)\setminus(x\star\tau)(2n)]$ and $\bigcup_{n\in\omega}[(\sigma\star y)(2n)\setminus(\sigma\star y)(2n-1)]$ (where $(\sigma\star y)(-1) = 0$ and $y(0) = \sigma(\varnothing)$ and $y(n+1) = x(n+2)$) is finite, hence one is in the ultrafilter if and only if the other is, i.e., τ is winning for II.

If II has a winning strategy τ then we define a winning strategy σ for I as follows: $\sigma(s) = \tau(0^\frown s)$. Then

$$\bigcup_{n\in\omega}[(\sigma \star y)(2n) \setminus (\sigma \star y)(2n-1)] = \bigcup_{n\in\omega}[(x \star \tau)(2n+1) \setminus (x \star \tau)(2n)],$$

where $x = 0^\frown y$ and thus one is in the ultrafilter if and only if the other is, i.e., σ is winning for I. $\square$

Corollary 2.20. *(AD) Every ultrafilter is ω_1-complete.*

Proof. Assume that there is an ultrafilter $\mathcal{U}$ that is not ω_1-complete, that is there are $X_n \notin \mathcal{U}$ but $\bigcup_{n\in\omega} X_n \in \mathcal{U}$. Then we can define an ultrafilter $\mathcal{F}$ over ω as follows

$$Y \in \mathcal{F} \iff \bigcup_{n\in Y} X_n \in \mathcal{U}.$$

It is clear that this is an non-principal ultrafilter over ω, contradicting –via Lemma 2.19– the Axiom of Determinacy. $\square$

On the other hand the axiom of determinacy implies some choice:

Lemma 2.21. *(AD) It is the case that $\mathsf{AC}_\omega({}^\omega\omega)$ holds.*

Proof. Assume that we have a countable set of nonempty sets of reals $A = \{X_n : n \in \omega\}$. We define the following game: If the outcome of a play is x then Player I wins if $x_{II} \notin X_{x(0)}$. It is impossible for I to have a winning strategy, since given any $n \in \omega$ X_n is non-empty. Hence any strategy σ with $\sigma(\varnothing) = n$ playing against an element of X_n will lose. Therefore, by the axiom of determinacy II has a winning strategy τ. If for $n \in \omega$ $c_n \in {}^\omega\omega$ is the constant function with value n, then a choice function for A is $f(n) = (c_n \star \tau)_{II}$. $\square$

Furthermore the following result by Kechris allows us to work with $\mathsf{AD} + \mathsf{DC}$:

Theorem 2.22 ([Kec84])**.** *(AD) $\mathbf{L}({}^\omega\omega)$ is a model of $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}$.*

Theorem 2.23. *(AD) The cone filter M_{T} on the Turing degrees is an ultrafilter.*

Proof. We need to show that given a set of reals A, closed under Turing equivalence, either A or ${}^\omega\omega \setminus A$ contains a Turing cone. Assume that σ is a winning strategy for I in $G(A)$. As we noted every strategy can be coded recursively into a real, hence let's assume that $\sigma \in {}^\omega\omega$.

We note that $y \leq_{\mathrm{T}} \sigma \star y$ since the star operation is recursive. On the other hand if $\sigma \leq_{\mathrm{T}} y$ then $\sigma \star y \leq_{\mathrm{T}} y$. Hence if $\sigma \leq_{\mathrm{T}} y$ then $y \equiv_{\mathrm{T}} \sigma \star y$. We have that $\sigma \star y \in A$ and since A is closed under Turing equivalence, $y \in A$, i.e., the Turing cone with base σ is a subset of A. In the case where II has a winning strategy, a symmetric argument yields a Turing cone in ${}^\omega\omega \setminus A$. $\square$

Theorem 2.24 (Wadge's Lemma)**.** *(AD) If A and B are pointclasses of type 1, then either A is the continuous preimage of B or B is the continuous preimage of $\mathcal{X} \setminus A$.*

Proof. Given A and B we define the following two player game: Player I plays a real a and player II plays a real b. I wins the game unless

$$a \in A \iff b \in B.$$

If II has a winning strategy τ then since the map $a \mapsto (a \star \tau)_{II}$ is continuous we have that A is the continuous preimage of B. If I has a winning strategy σ then again since the map $b \mapsto (\sigma \star b)_I$ is continuous B is the continuous preimage of $\mathcal{X} \setminus A$. $\square$

We define

$$\Theta = \bigcup\{\alpha \in \mathrm{Ord} : (\exists f)[f : {}^\omega\omega \twoheadrightarrow \alpha]\}.$$

That is Θ is the supremum of all ordinals that can be covered by ${}^\omega\omega$ via a surjective function. That Θ is an ordinal follows from replacement and the fact that every surjection from the reals to an ordinal can be coded by a prewellordering that can be seen as an element of $\wp({}^{\omega}\omega)$. In the context of the axiom of choice, since all sets are equinumerous with ordinals $\Theta = (2^{\aleph_0})^+$, because $2^{\aleph_0} = \aleph_0^{\aleph_0}$. The axiom of determinacy implies on the other hand that in some sense Θ is quite large:

Theorem 2.25 ([Mos70]). *(AD) Assume that $\alpha < \Theta$. Then there exists a surjection $F : {}^{\omega}\omega \twoheadrightarrow \wp(\alpha)$.*

Proof. Let $f : {}^{\omega}\omega \twoheadrightarrow \alpha$. We construct inductively surjections $f_\xi : {}^{\omega}\omega \twoheadrightarrow \wp(\xi)$, for $\xi \leq \lambda$. For $\xi \leq \omega$ it is trivial. For $\xi + 1$ this can be achieved using a bijection between ξ and $\xi + 1$, and f_ξ. Hence the only nontrivial step is the limit case.

Assume that for every $\xi < \gamma$ we have defined f_ξ. Given $X \subseteq \gamma$ we define the following game, $G(X)$: Player I plays x_{I}, player II plays x_{II}. If player I fails to make sure that $X \cap \xi_{\mathrm{I}} = f_{f((x_{\mathrm{I}})_0)}((x_{\mathrm{I}})_1)$ (for some ξ_{I}) then player II wins, and vice versa. If both players make sure of that, then I wins if and only if $\xi_{\mathrm{II}} < \xi_{\mathrm{I}}$.

It is clear that a winning strategy for some player for $G(X)$, cannot be a winning strategy for some $G(Y)$, with $Y \neq X$: If $\beta \in X \setminus Y$, then the opponent can play a real y such that $X \cap (\beta + 1) = f_{f((y)_0)}((y)_1)$.

Hence we can define a function

$$f_\gamma(x) = \begin{cases} X & \text{if } x \text{ is a winning strategy for } G(X), \\ \varnothing & \text{otherwise.} \end{cases}$$

That it is a surjection follows directly from AD. $\square$

We end this section by mentioning a result about Θ in $\mathbf{L}({}^{\omega}\omega)$:

Lemma 2.26 (Solovay). *Assume that $V = \mathbf{L}({}^{\omega}\omega)$. Then Θ is a regular cardinal.*

Proof. By the definition of $\mathbf{L}({}^{\omega}\omega)$ it immediately follows that there exists a surjection $\Phi : \mathrm{On} \times {}^{\omega}\omega \twoheadrightarrow \mathbf{L}({}^{\omega}\omega)$. Assume towards a contradiction that $\mathrm{cf}(\Theta) = \lambda < \Theta$, let this be witnessed by $f : \lambda \to \Theta$, and let $g : {}^{\omega}\omega \twoheadrightarrow \lambda$ be a surjection. Define a function $\rho : \Theta \to \mathrm{On}$ where

$$\rho(\xi) = \bigcap\{\delta : (\exists x \in {}^{\omega}\omega)[\Phi(\delta, x) \text{ is a surjection from } {}^{\omega}\omega \text{ onto } \xi]\}.$$

Then define the function

$$h(x) = \begin{cases} \Phi(\rho(f(g((x)_0))), (x)_1)((x)_2) & \text{if } \Phi(\rho(f(g((x)_0))), (x)_1) : {}^{\omega}\omega \twoheadrightarrow f(g((x)_0)), \\ \varnothing & \text{otherwise.} \end{cases}$$

A moment's reflection reveals that this is a surjection from ${}^{\omega}\omega$ onto Θ, a contradiction. $\square$

2.5 Model theory

If M, N are models of the language of set theory we say that $j : M \to N$ is an *embedding* if j is injective and for every $a, b \in M$ we have that

$$M \models a \in b \text{ if and only if } N \models j(a) \in j(b).$$

Furthermore, if whenever $a_1, \ldots, a_n \in M$ we have that for all formulas φ

$$M \models \varphi(a_1, \ldots, a_n) \text{ if and only if } N \models \varphi(j(a_1), \ldots, j(a_n))$$

then we say that j is an *elementary embedding*.

Let $\mathcal{M} = (M, \ldots)$ be a model of the language of set theory. Let I be a set and let $\mathcal{U}$ be an ultrafilter over I. We can define the ultrapower of $\mathcal{M}$ modulo $\mathcal{U}$, denoted by $\mathcal{M}^I/\mathcal{U}$, as follows: The domain of the ultrapower will be

$$M^I/\mathcal{U} = \{[f] : f \in M^I\}$$

where $[f]$ is the equivalence class induced by the equivalence relation

$$f \equiv g \iff \{i \in I : f(i) = g(i)\} \in \mathcal{U}.$$

Furthermore we define

$$[f_1] \in^{\mathcal{M}^I/\mathcal{U}} [f_2] \iff \{i \in I : f_1(i) \in^{\mathcal{M}} f_2(i)\} \in \mathcal{U}.$$

From the properties of filters it follows that this is well-defined. The existence of a choice function on $\wp(M)$ implies the following pivotal result:

Theorem 2.27 (Łoś)**.** *(*AC*) Assume that $\mathcal{M}$ is a model, I a set and $\mathcal{U}$ is an ultrafilter over I. Then*

$$\mathcal{M}^I/\mathcal{U} \models \varphi([f]_1, \ldots, [f]_n) \textit{ if and only if } \{i \in I : \mathcal{M} \models \varphi(f_1(i), \ldots, f_n(i))\} \in \mathcal{U}$$

for all $f_1, \ldots, f_n \in M^I$ and all formulas φ.

Corollary 2.28. *There exists an elementary embedding $j : \mathcal{M} \to \mathcal{M}^I/\mathcal{U}$.*

Proof. For every $x \in M$ let $c_x \in M^I$ be the constant function with value x. Then define $j(x) = [c_x]$. Since $I \in \mathcal{U}$, Theorem 2.27 implies that j is an elementary embedding. $\square$

Given the universe of set theory, V, and a set I we can define as V^I the class of functions with domain I. Even though the equivalence classes modulo $\mathcal{U}$ are not sets but proper classes we can turn them into sets by cutting them at the least level of the cumulative hierarchy in which they are non-empty (this trick was first applied by Dana Scott and it is known as Scott's Trick). Thus we can define $V^I/\mathcal{U}$ as above (which is extensional because if some $[f]$ is cut at height α all $[g] \in [f]$ will be cut at some size $\beta \in \alpha$ and hence the extension of $[f]$ will be a set), and prove Łoś' theorem, as a theorem scheme.

Chapter 3

Measurable cardinals

In this chapter we will study measurable cardinals in their various forms. In the first section we will introduce some large cardinal notions. The three main notions that we will consider wll be pre-measurable cardinals, measurable cardinals and embedding cardinals. The axiom of choice implies that these concepts are equivalent, but without it this is not the case. We will discuss the implications between the axioms as well as some of their consequences. It will turn out that in ZF embedding cardinals have much more structural consequences that measurable ones. We will also present large cardinal concepts that naturally arise from these cardinals, 1-measurable and 1-embedding.

In the second section we will present arguments that in the absence of choice combinatorial properties such as $\kappa \to (\kappa)^{\lambda+\lambda}_2$ yield that the filter $\mathcal{C}^\lambda_\kappa$ over κ is a normal ultrafilter. Stronger combinatorial properties will guarantee that any stationary set on κ is a member of a normal ultrafilter over κ.

3.1 Large cardinal notions

A cardinal κ is called *(weakly) inaccessible* if it is a limit regular cardinal. In the context of the axiom of choice we call a cardinal κ a *strong limit* if for all $\lambda \in \kappa$ $2^\lambda \in \kappa$.

Proposition 3.1. *(*AC*) If κ is an inaccessible strong limit cardinal, then we have that $V_\kappa \models \mathsf{ZFC}$.*

Proof. The only axiom in question is the axiom of replacement. That κ is a strong limit regular cardinal implies that $|V_\kappa| = \kappa$. Hence if $x \in V_\kappa$, we have that $|x| = \lambda < \kappa$, and let this be witnessed by a function h. If $f \subseteq V_\kappa$ is a function with domain x, then we define $g : \lambda \to \kappa$ by $g(\alpha) = \text{rank}(f(h(\alpha)))$. Since κ is regular we have that g is bounded on κ, hence $f[x] \in V_\alpha$ for some $\alpha \in \kappa$, granting the replacement axiom. $\square$

Proposition 3.2. *The existence of a weakly inaccessible cardinal cannot be* *proved in* ZF *and neither can the consistency (relative to* ZF*) of the existence of weakly inaccessible cardinals.*

Proof. It there exists a weakly inaccessible cardinal κ then it the constructible universe it will be strongly inaccessible and hence the constructible universe up the the height of κ would be a model of ZF, contradicting Gödel's second incompleteness theorem. If the consistency of ZF could show the consistency of ZF along with the existence of a weakly inaccessible cardinal then by the above ZF along with the existence of a weakly inaccessible cardinal would prove its own consistency, again contradicting Gödel's second incompleteness theorem. $\square$

A cardinal κ is called *Mahlo* if the set $\{\lambda \in \kappa : \lambda \text{ is regular}\}$ is a stationary subset of κ. Obviously every Mahlo cardinal is weakly inaccessible and furthermore because the set $\{\lambda \in \kappa : \lambda \text{ limit cardinal}\}$ is closed unbounded in κ, if fact there are stationary many inaccessible cardinals below a Mahlo cardinal.

A cardinal κ will be called *pre-measurable* cardinal if there exists a non-principle κ-complete ultrafilter over κ. If the ultrafilter is also normal then κ will be called *measurable*.

Proposition 3.3. *If κ is the least cardinal with a non-principal ω_1-complete ultrafilter, then κ is pre-measurable.*

Proof. Assume that $\mathcal{U}$ is an ω_1-complete non-principle ultrafilter over κ. We claim that $\mathcal{U}$ is κ-complete. Heading towards a contradiction assume that there is a $\lambda \in \kappa$ and X_ξ for $\xi \in \lambda$ such that $\bigcup_{\xi\in\lambda} X_\xi = \kappa$ while $X_\xi \notin \mathcal{U}$ for all $\xi \in \lambda$. Then define an ultrafilter $\mathcal{V}$ over λ as follows:

$$Y \in \mathcal{V} \iff \bigcup_{\xi\in Y} X_\xi \in \mathcal{U}.$$

It is routine to check that this is a non-principal ω_1-complete ultrafilter over λ, using the properties of $\mathcal{U}$; this contradicts the minimality of κ. $\square$

Proposition 3.4. *If κ is a pre-measurable cardinal then κ is regular.*

Proof. Let $\mathcal{U}$ be a non-principle κ-complete ultrafilter over κ and let $f : \lambda \to \kappa$ be a function and define $X_\xi = \kappa \setminus f(\xi)$. Since $\mathcal{U}$ is non-principle $X_\xi \in \mathcal{U}$. We notice that $\bigcap_{\xi\in\lambda} X_\xi = \kappa \setminus \bigcup f[\lambda]$. By the κ completeness $\bigcap_{\xi\in\lambda} X_\xi \in \mathcal{U}$ hence it is not empty, i.e., f is bounded in κ. $\square$

Under DC the notion of pre-measurable and measurable cardinals coincide:

Lemma 3.5. *(*DC*) Every pre-measurable cardinal is measurable.*

Proof. Let $\mathcal{U}$ be a non-principal κ-complete ultrafiter over κ. Then we define the equivalence relation on κ^κ

$$f \equiv g \iff \{\alpha \in \kappa : f(\alpha) = g(\alpha)\} \in \mathcal{U},$$

and an order relation on equivalence classes

$$[f] < [g] \iff \{\alpha \in \kappa : f(\alpha) \in g(\alpha)\} \in \mathcal{U}.$$

The properties of the ultrafilter imply that this is well defined. We claim that $<$ is a well-ordering. Heading towards a contradiction assume that it is ill-founded, and thus by Proposition 2.2 let $[f_0] > [f_1] > \dots$ be an ω-sequence, and by Proposition 2.3 we can pick a representative from each equivalence class, f_n. Now we define the sets $X_n = \{\alpha \in \kappa : f_{n+1}(\alpha) \in f_n(\alpha)\}$, and by the definition of the sequence we have that $X_n \in \mathcal{U}$. Since $\mathcal{U}$ is κ-complete $\bigcap_{n\in\omega} X_n \in \mathcal{U}$ and in particular there is some $\xi \in \bigcap_{n\in\omega} X_n$. Then we have that for all n $f_{n+1}(\xi) \in f_n(\xi)$, i.e., an $\in$-descending ω-sequence of ordinals, a contradiction.

Thus there exists a least equivalence class $[f]$ such that for all $g \in [f]$ and $\delta \in \kappa$ $\{\alpha \in \kappa : g(\alpha) \neq \delta\} \in \mathcal{U}$ (that this subset of the equivalence classes is non-empty is witnessed by [id]), and let f be a representative of it. We claim that $f^*[\mathcal{U}] = \{Y \subseteq \kappa : f^{-1}[Y] \in \mathcal{U}\}$ is a normal ultrafilter over κ. That it is an ultrafilter (in fact κ-complete) follows from Proposition 2.6. To show that it is normal assume that h is regressive on κ. Then $[h\circ f] < [f]$ and by the minimality of f there exists some $\delta \in \kappa$ such that $X = \{\alpha \in \kappa : h \circ f(\alpha) = \delta\} \in \mathcal{U}$. Then $f-1[f[X]] \supseteq X$. hence $f[X] \in \mathcal{V}$ and for all $\alpha \in f[X]$ we have that $h(\alpha) = \delta$. $\square$

Without DC on the other hand, these two notions are different as has been shown in [BG12].

We will now investigate elementary embeddings of the universe into standard transitive inner models. Henceforth in this section, when we say that $j : V \to M$ is an elementary embedding we will always mean that M is a standard transitive class unless otherwise explicitly stated. At first glance it is not clear that in ZF we can talk about the elementarity of a class function, since it is a statement about the equivalence between the truth of all formulas. The following lemma shows that equivalence for Σ_1 formulas (which is expressible in ZF) is enough to guarantee elementary equivalence:

Lemma 3.6. *Assume M is a transitive standard model of some finite fragment of* ZF *and $j : V \to M$ is an embedding such that for any Σ_1 formula $\varphi(x_1, \dots, x_n)$ we have*

$$\varphi(a_1, \dots, a_n) \iff M \models \varphi(j(a_1), \dots, j(a_n)).$$

Then for all $k \in \omega$ and for any Σ_k formula $\psi(x_1, \dots, x_n)$ we have

$$\psi(u_1, \dots, u_n) \iff M \models \psi(j(u_1), \dots, j(u_n)).$$

Proof. The proof is by induction on k. First of all we notice that if α is an ordinal, then $j(\alpha)$ an ordinal since the notion of ordinals is Δ_0 and furthermore, $j(\alpha) \geq \alpha$ (since j is an embedding and hence for all $\beta \in \alpha$ we have $j(\beta) \in j(\alpha)$).

Assume that for all Σ_k formulas the statement holds. Let $\psi(x, y)$ be a Π_k formula and consider $\exists x \psi(x, y)$. If for any set a we have that $\exists x \psi(x, a)$ then this is witnesses by a set b, i.e., $\psi(b,a)$ holds and by the induction hypothesis we have that $M \models \psi(j(b), j(a))$ hence $M \models \exists x \psi(x, j(a))$.

For the other direction assume that $M \models \psi(b, j(a))$. The formula $v = V_\alpha$ is Π_1 and hence $j(V_\alpha)$ is $(V_{j(\alpha)})^M$. Then there is some ordinal α such that

$$M \models \exists x \in V_{j(\alpha)} \psi(x, j(a)).$$

Since this is a Π_k statement we have that $\exists x \in V_\alpha \psi(x, a)$. $\square$

Therefore, when we say that $j : V \to M$ is an elementary embedding we will assume that this is formalised by saying that j is an embedding and that the two models satisfy the same Σ_1 formulas.

Lemma 3.7. *Assume that $j : V \to M$ is a non-trivial elementary embedding of the universe into a standard transitive class. Then there is an ordinal δ such that $j(\delta) \neq \delta$.*

Proof. Heading towards a contradiction, let's assume that $j \restriction \mathrm{On} = \mathrm{id}$. We will show by induction on γ that $j \restriction V_\gamma = \mathrm{id}$. The base case is trivial, as well as the limit case. Let's thus assume that $j \restriction V_\gamma = \mathrm{id}$. Let $x \subseteq V_\gamma$ such that $x \notin V_\gamma$, i.e., $\mathrm{rank}(x) = \gamma$. Then by the elementarity of j and the fact that $j(\gamma) = \gamma$

$$M \models \mathrm{rank}(j(x)) = \gamma$$

and thus since $M \cap V_\gamma = V_\gamma$ we have that $j(x) \subseteq V_\gamma$. Given $y \in V_\gamma$ since $j(y) = y$ and by the elementarity of j we have

$$y \in x \iff M \models y \in j(x)$$

i.e., $j(x) = x$. Thus $j \restriction V_{\gamma+1} = \mathrm{id}$. $\square$

Given an elementary embedding j, the least ordinal γ such that $j(\gamma) \neq \gamma$ will be called the *critical point* of j. Because ω is absolute, the elementarity of j implies that the critical point will be above ω.

Lemma 3.8. *Assume that $j : V \to M$ is an elementary embedding, with critical point κ. Then for every $\delta < \kappa$ every function $f : V_\delta \to \kappa$ we have that $j(f) = f$. In particular such an f is bounded in κ and hence κ is a regular cardinal.*

Proof. Let $\delta < \kappa$ and let $f : V_\delta \to \kappa$. Of course κ is a limit ordinal by the elementarity of j since for every ordinal γ we have $j(\gamma + 1) = j(\gamma) + 1$. Thus, since $j \restriction (\delta + 1) = \mathrm{id}$ we can show as in Lemma 3.7 that $j \restriction V_\delta = \mathrm{id}$ and $j(V_\delta) = V_\delta$. Hence

$$M \models \mathrm{dom}(j(f)) = V_\delta.$$

Furthermore for $x \in V_\delta$ because $j(x) = x$ and $j \restriction \kappa = \mathrm{id}$ we have

$$f(x) = \delta \iff M \models j(f)(x) = \delta.$$

From this it follows that $j(f) = f$. Now if f is unbounded in κ, then

$$M \models f \text{ is unbounded in } j(\kappa)$$

but $j(\kappa) > \kappa$, a contradiction. Therefore f is bounded. $\square$

A cardinal κ, that is the critical point of a non-trivial elementary embedding $j : V \to M$ will be called an *embedding cardinal*.

Corollary 3.9. *Assume that $j : V \to M$ is an elementary embedding with critical point κ. Then $V_{\kappa+1} \subset M$.*

Proof. From Lemma 3.8 it follows that $V_\kappa \subseteq M$. On the other hand if $X \subset V_\kappa$ then the above proof implies that $j(X) \cap V_\kappa = X$, hence $X \in M$. $\square$

Corollary 3.10. *Assume that $j : V \to M$ is an elementary embedding with critical point κ. Then $\Theta < \kappa$.*

Proof. We observe that there exists a surjection $g : \wp({}^\omega\omega) \twoheadrightarrow \Theta$; $g(X)$ sends X to its length if X codes a prewellordering, or to 0 if it does not. If $\kappa \leq \Theta$ then we can define $h : \wp({}^\omega\omega) \twoheadrightarrow \kappa$. Since $\wp({}^\omega\omega) \in V_{\omega+7}$ Lemma 3.8 yields that h is bounded, a contradiction. $\square$

Proposition 3.11. *Let κ be the critical point of a non-trivial elementary embedding $j : V \to M$. Then κ is a limit cardinal and thus it is inaccessible.*

Proof. Assume, heading towards a contradiction that κ is a successor cardinal, i.e., $\kappa = \lambda^+$. Since κ is the critical point of j, we have that $j(\lambda) = \lambda$. Hence $M \models j(\kappa) = \lambda^+$. Thus $M \models |\kappa| = \lambda$, a contradiction since $M \subseteq V$ and thus κ is a cardinal in M too. Since by Lemma 3.8 κ is regular we have that it is inaccessible. $\square$

Lemma 3.12. *Assume that $V \models \mathsf{ZF}$ and let $j : V \to M$ be an elementary embedding with critical point κ. Then $V_\kappa \models \mathsf{ZF}$. Furthermore if $V \models \mathsf{AD}$ we also have that $V_\kappa \models \mathsf{AD}$.*

Proof. For all limit ordinal γ it is clear that V_γ satisfies all axioms of ZF with the exception of the replacement scheme. Thus we need to show that V_κ satisfies the replacement axiom scheme. Let $f : V_\delta \to V_\kappa$ be a function. Then we define $g : V_\delta \to \kappa$ by $g(x) = \operatorname{rank}(f(x))$. By Lemma 3.8 we have that g is bounded. Hence $f[V_\delta] \subseteq V_\alpha$ for some $\alpha \in \kappa$ and hence $f[V_\delta] \in V_\kappa$. Therefore the axiom of replacement is true in V_κ. Now since κ is above $\omega + \omega$, we have by Lemma 2.17 that if $V \models \mathsf{AD}$ so will V_κ. $\square$

Corollary 3.13. $\mathsf{ZF} + \mathsf{AD}$ *cannot prove the existence of an embedding cardinal. Furthermore* $\mathsf{ZF} + \mathsf{AD}$ *cannot prove the consistency (relative to* $\mathsf{ZF} + \mathsf{AD}$*) of the existence of an embedding cardinal.*

Proof. If $\mathsf{ZF} + \mathsf{AD}$ could prove the existence of an embedding cardinal κ, then by Lemma 3.12, we have that V_κ is a model of $\mathsf{ZF} + \mathsf{AD}$. Thus we have shown the consistency of $\mathsf{ZF} + \mathsf{AD}$, contradicting Gödel's second incompleteness theorem.

Let I be the statement "*there exists an embedding cardinal*". Lemma 3.12 implies that $\mathsf{ZF} + \mathsf{AD} + \mathsf{I}$ can prove the consistency of $\mathsf{ZF} + \mathsf{AD}$. Hence if $\mathsf{ZF} + \mathsf{AD}$ could show that its own consistency implies the consistency of $\mathsf{ZF} + \mathsf{AD} + \mathsf{I}$, then $\mathsf{ZF} + \mathsf{AD} + \mathsf{I}$ would show its own consistency, again contradicting Gödel's theorem. $\square$

Lemma 3.14. *Let κ be the critical point of a non-trivial elementary emdedding $j : V \to M$. Then κ is a measurable cardinal. Furthermore κ is Mahlo and in fact there are stationary many Mahlo cardinals below κ.*

Proof. We define $\mathcal{U} \subseteq \wp(\kappa)$ as follows:

$$X \in \mathcal{U} \iff \kappa \in j(X).$$

We will show that $\mathcal{U}$ is a normal ultrafilter over κ. First of all it follows that $\kappa \in \mathcal{U}$ and $\varnothing = j(\varnothing) \notin \mathcal{U}$. If $X \subseteq Y$, the elementarity of j implies that $j(X) \subseteq j(Y)$ and thus if $\kappa \in j(X)$ we have $\kappa \in j(Y)$. Also if $X, Y \in \mathcal{U}$, again by the elementarity of j we have that $j(X \cap Y) = j(X) \cap j(Y)$. Finally let $f : \kappa \to \kappa$ be regressive. Then $j(f)$ is also regressive and thus $j(f)(\kappa) = \alpha < \kappa$. Let $X = \{\xi \in \kappa : f(\xi) = \alpha\}$. Then $j(X) = \{\xi \in j(\kappa) : j(f)(\xi) = \alpha\}$, hence $\kappa \in j(X)$ and thus $X \in \mathcal{U}$.

To show that κ is Mahlo we observe that κ is regular, hence κ is regular in M and thus $X = \{\lambda \in \kappa : \lambda \text{ regular cardinal}\} \in \mathcal{U}$ because $\kappa \in j(X) = \{\lambda \in j(\kappa) : M \models \text{“}\lambda \text{ is a regular cardinal”}\}$. Since all elements of a normal ultrafilter are stationary, if follows that κ is Mahlo. Furthermore because $V \subseteq M$, all clubs of M are also clubs of V, and thus κ is Mahlo in M. Thus $Y = \{\lambda \in \kappa : \lambda \text{ is Mahlo}\} \in \mathcal{U}$, because $j(Y) \ni \kappa$, and thus there are stationary Mahlo cardinals below κ. □

The axiom of choice implies, via Łoś' theorem, that the notions of pre-measurable cardinal, measurable cardinal and the critical point of an elementary embedding are in fact equivalent:

Theorem 3.15. *(AC) Assume that κ is a pre-measurable cardinal, i.e., there exists a κ-complete non-principal ultrafiter over κ. Then κ is the critical point of an elementary embedding $j : V \to M$.*

Proof. Let $\mathcal{U}$ be a κ-complete ultrafilter over κ. We can define $V^\kappa/\mathcal{U}$, as a class. We have that DC (a consequence of AC) implies that $V^\kappa/\mathcal{U}$ is well-founded. The proof is identical with the proof in Lemma 3.5 that the order of the functions is a well-order. Then because $V^\kappa/\mathcal{U}$ is well-founded and extensional we have from Mostowski's theorem that it can be collapsed to a standard transitive class M.

Due to the axiom of choice Łoś' theorem applies and hence we have an elementary embdedding $j : (V, \in) \to (V^\kappa/\mathcal{U}, E)$ (note here that $V^\kappa/\mathcal{U}$ is not standard and transitive). Taking the concatenation with the transitive collapse π we have an elementary embedding into a standard transitive class $\iota = \pi \circ j : M \to V$. We show by induction that $\iota \restriction \kappa = \text{id}$. Indeed by absoluteness $\iota \restriction \omega + 1 = \text{id}$. For $\alpha \in \kappa$ assume that $\iota \restriction \alpha = \text{id}$ and let $\pi([f]) \in \iota(\alpha)$. Then $A = \{\xi \in \kappa : f(\xi) \in \alpha\} \in \mathcal{U}$. Define $X_\delta = \{\xi \in \kappa : f(\xi) = \delta\}$ for $\delta \in \alpha$. Then $A = \bigcup_{\delta \in \alpha} X_\delta$ and by the κ-completeness of $\mathcal{U}$ we have that there is some $\delta \in \alpha$ such that $X_\delta \in \mathcal{U}$. But this implies that $\pi([f]) = \iota(\delta) = \delta$. Hence $\iota(\alpha) = \alpha$. To see that $\iota(\kappa) > \kappa$ we observe that for all $\alpha \in \kappa$ we have $\iota(\alpha) = \alpha \in \pi([\text{id}]) \in \iota(\kappa)$. □

Theorem 3.16 (Scott)**.** *If there exists a pre-measurable cardinal then $V \neq \mathbf{L}$.*

Proof. Let κ be the least pre-measurable cardinal. If $V = \mathbf{L}$ then the axiom of choice holds and thus there exists an elementary embedding $j : V \to M$ with critical point κ. Since $\mathbf{L}$ is the least inner model $M = \mathbf{L}$. Hence by the elementarity of j we have that $j(\kappa)$ is the least pre-measurable cardinal, a contradiction since $j(\kappa) > \kappa$. □

We have thus far observed a pattern in which a new large cardinal notion is introduced that implies the existence of many weaker large cardinal notions below it. If κ is a Mahlo cardinal then there are κ inaccessible cardinals below it, or if there exists an elementary embedding from $j : V \to M$ with critical point κ, then there are κ Mahlo cardinals below κ. In view of this we can iterate on this process on measurable cardinals:

We call κ a 1-*measurable cardinal* if there is a normal ultrafilter $\mathcal{U}$ over κ such that $\{\lambda \in \kappa : \lambda \text{ is measurable}\} \in \mathcal{U}$. We also say that κ is an 1-*embedding cardinal* if there exists an elementary embedding $j : V \to M$ with critical point κ such that $M \models$"κ is measurable".

Lemma 3.17. *(*AC*) A cardinal is* 1*-measurable if and only if it is* 1*-embedding.*

Proof. Assume that κ is 1-embedding and let $j : V \to M$ witness this. Then the ultrafilter

$$X \in \mathcal{U} \iff \kappa \in j(X)$$

over κ contains the set $A = \{\lambda \in \kappa : \lambda \text{ is measurable}\}$ since $j(A) = \{\lambda \in j(\kappa) : M \models \text{"}\lambda \text{ is measurable"}\}$ and $M \models$"κ is measurable".

On the other hand assume that κ is 1-measurable and let this be witnessed by the normal ultrafilter $\mathcal{U}$. We will show that if $j : (V, \in) \to (V^\kappa/\mathcal{U}, E)$ and $\pi : V^\kappa/\mathcal{U} \to M$ is the transitive collapse map as defined in Theorem 3.15 then $\pi([\mathrm{id}]) = \kappa$. This follows directly by the normality of the ultrafilter: if $[f]E[\mathrm{id}]$ then f is regressive on an element of $\mathcal{U}$ and hence it is constant with value $\delta \in \kappa$ on an element of $\mathcal{U}$, i.e., $[f] = j(\delta)$. Thus $\pi([\mathrm{id}])$ is the least cardinal above all $j(\delta)$ for $\delta \in \kappa$, i.e., $\pi([\mathrm{id}]) = \kappa$. Now because $\{\lambda \in \kappa : \lambda \text{ is measurable}\} \in \mathcal{U}$, we have by Łoś' Theorem that $(V^\kappa/\mathcal{U}, E) \models$"$[\mathrm{id}]$ is measurable". Hence $M \models$"κ is measurable". □

3.2 Ultrafilters from combinatorics

Theorem 3.18 ([Kle70])**.** *Suppose λ is a regular cardinal and $\kappa \to (\kappa)^{\lambda+\lambda}$. Then $\mathcal{C}^\lambda_\kappa$ is a normal κ-complete ultrafilter over κ.*

Proof. By Lemma 2.11 we have that $\kappa \to (\kappa)^\lambda_\gamma$ for all $\gamma < \kappa$. To show that $\mathcal{C}^\lambda_\kappa$ is a κ-complete ultrafilter we need to show that for $\gamma < \kappa$ if $\bigcup_{\xi \in \gamma} X_\xi = \kappa$ then for some $\xi \in \gamma$ $X_\xi \in \mathcal{C}^\lambda_\kappa$. Let $f : [\kappa]^\lambda \to \gamma$ be defined as $f(s) = \bigcap\{\xi \in \gamma : \cup s \in X_\xi\}$. Let H be a homogeneous set for f of size κ and let $f[[H]^\lambda] = \{\delta\}$. Then

$$C = \lim(H) \cap \{\xi \in \kappa : \mathrm{cf}(\xi) = \lambda\}$$

is an element of $\mathcal{C}_\kappa^\lambda$ and by the definition of f we have that $C \subseteq X_\delta$.

To show that $\mathcal{C}_\kappa^\lambda$ is normal, given κ-completeness, we need to show that given a regressive function $g : \kappa \to \kappa$ then there is some $C \in \mathcal{C}_\kappa^\lambda$ and some $\delta < \kappa$ such that $f[C] \subseteq \delta$. Let $G : [\kappa]^\lambda \to 2$ be defined as

$$G(s) = 0 \iff f(\cup s) < \cap s.$$

If H is a homogeneous set for f, then $G[[H]^\lambda] = \{0\}$ since for $s \in [H]^\lambda$, $s' = s \setminus (f(\cup s) + 1)$ is a λ sequence because λ is a cardinal, and thus $f(\cup s') < \cap s'$. This means that for every $s \in [H]^\lambda$ $f(\cup s) < \bigcap H$. Taking C as above, yields that $f[C] \subseteq \bigcap H$. $\square$

Lemma 3.19 ([Kle70]). *Let κ be a regular cardinal and suppose that κ is not Mahlo (i.e., $\{\alpha \in \kappa : \alpha$ regular cardinal$\}$ is non-stationary), and assume that for any regular cardinal $\lambda \in \kappa$, $\mathcal{C}_\kappa^\lambda$ is a normal ultrafilter over κ. Then these are the only normal ultrafilters over κ.*

Proof. Assume that $\mathcal{U}$ is a normal ultrafilter over κ. Then by Proposition 2.8 $\mathcal{C}_\kappa \subseteq \mathcal{U}$. Furthermore by the assumption we have that $A = \{\lambda \in \kappa : \lambda$ not regular$\} \in \mathcal{C}_\kappa$. We define the function on $f : A \to \kappa$ with $f(\lambda) = \mathrm{cf}(\lambda)$. Then f is regressive and since $\mathcal{U}$ is normal we have that there is some regular $\lambda \in \kappa$ such that $X = \{\alpha \in A : f(\alpha) = \lambda\} \in \mathcal{U}$. Then $X \subseteq E_\lambda^\kappa$, hence $E_\lambda^\kappa \in \mathcal{U}$. Thus, because $\mathcal{U}$ contains the club filter, $\mathcal{C}_\kappa^\lambda \subseteq \mathcal{U}$. The equality follows because $\mathcal{C}_\kappa^\lambda$ is an ultrafilter and thus maximal. $\square$

A cardinal κ is said to have the *strong partition property* if $\kappa \to (\kappa)_\mu^\kappa$ for all $\mu < \kappa$. If κ has the strong partition property and given a stationary subset S of κ, we can show that there exists a normal ultrafilter over κ that contains S. This is of particular importance if κ is Mahlo and hence contains very fine stationary sets.

Theorem 3.20 ([Kle82]). *Suppose κ has the strong partition property and let S be any stationary subset of κ that only contains ordinals with uncountable cofinality. Then there exists a κ-complete normal ultrafilter $\mathcal{U}$ over κ such that $S \in \mathcal{U}$.*

Proof. Let $\hat{S} = \{\xi \in S :$ there is a closed unbounded set C of ξ such that $C \cap S = \varnothing\}$. It is trivial to see that $\hat{S}$ is a stationary set: For every closed unbounded set C the least element of $\lim(C) \cap S$ is an element of $\hat{S}$.

We define the following ultrafilter $\mathcal{U}$ on κ:

$$X \in \mathcal{U} \iff \text{ for some closed unbounded set } C \text{ of } \kappa, C \cap \hat{S} \subseteq X.$$

By the definition and the basic properties of closed unbounded sets it follows that $\mathcal{U}$ is a filter. It also follows that immediately that $S \in \mathcal{U}$.

Let's assume that for $\alpha < \kappa$ $\bigcup_{\xi \in \alpha} X_\xi = \kappa$. We want to show that for some $\xi \in \alpha$ $X_\xi \in \mathcal{U}$. We define $F : [\kappa]^\kappa \to \alpha$ as

$$F(Y) = \bigcap\{\xi \in \alpha : \bigcap(\lim(Y) \cap S) \in X_\xi\}.$$

Let H be a homogeneous set for F, with $F[[H]^\kappa] = \{\delta\}$. We will show that $\lim(H) \cap \hat{S} \subseteq X_\delta$, i.e., that $X_\delta \in \mathcal{U}$.

Indeed, let $\beta \in \lim(H) \cap \hat{S}$. Then, there is some C_β, a closed unbounded set of β, such that $C_\beta \cap S = \varnothing$. We note that $\lim(H) \cap C_\beta$ is a closed unbounded set of β with the same property. Let $D \subset \beta \cap H$, such that $\lim(D) = \lim(H) \cap C_\beta$. Such a set is easy to construct, since removing limits can be achieved by removing some final segment of H below the limit. Then $D' = D \cup (H \setminus \beta) \subset H$, and hence $F(D') = \delta$. But the least limit of D' that intersects S is β because by construction $D \cap S \cap \beta = \varnothing$, and hence $\beta \in X_\delta$.

To show that $\mathcal{U}$ is normal, given the κ-completeness of the ultrafilter, we just need to show that for any regressive on κ function f there is some $\delta < \kappa$ such that for some $X \in \mathcal{U}$ it is the case that $f[X] \subseteq \delta$. Let us define $G : [\kappa]^\kappa \to 2$ as

$$G(Y) = 0 \iff f(\bigcap(\lim(Y) \cap S)) < \bigcap Y.$$

Let H be a homogeneous set for G. For $Y \in [H]^\kappa$, let $Y' = Y \setminus f(\bigcap(\lim(Y) \cap S))$. First of all it cannot be the case that $G[[H]^\kappa] = \{1\}$. Indeed, since f is regressive we have that $\bigcap(\lim(Y) \cap S) = \bigcap(\lim(Y') \cap S)$ and we have that $G(Y) = 0$.

It now follows that $f[\lim(H) \cap \hat{S}] \subseteq \bigcap H$. To see this let $\beta \in \lim(H) \cap \hat{S}$ and let C_β a closed unbounded set of β, subset of $\lim(H)$ such that $C_\beta \cap S = \varnothing$. Then let $D \subseteq \beta \cap H$, that contains the least element of H and $\lim(D) = C_\beta$. Then $G(D \cup (H \setminus \beta)) = 0$ and hence $f(\beta) < \bigcap H$, by the definition of G. $\square$

Chapter 4

Determinacy

In this chapter we will study how the axiom of determinacy can yield normal ultrafilters on cardinals below Θ. In the first section we introduce the concept of a Spector pointclass, an abstraction of Π^1_1 and discuss their properties.

In the second section, under the axiom of determinacy and with the aid of Moschovakis' Coding Lemma, we show that cardinals that naturally arise from Spector pointclasses have large cardinal properties. In particular we will show that for certain cardinals κ, $\mathcal{C}^{\omega}_{\kappa}$ is a normal ultrafilter over κ.

We will present two different proofs: One that generalises Solovay's well-known result that $\mathcal{C}_{\omega_1}$ is an ultrafilter over ω_1, and one that shows that κ has some combinatorial properties and then apply the results from the previous chapter. The first proof requires the axiom of dependence choice to show that the ultrafilter is normal but it's more natural. The second doesn't require any choice, but the game that is used is slightly more complex.

Finally by generalising the second proof and using a more complex Spector pointclass we will show that under $\mathsf{AD} + \mathsf{DC}$ there are $\kappa < \Theta$ that are 1-measurable cardinals.

4.1 Spector pointclasses

A *norm* on some pointset P of length α is a surjective function $f : P \twoheadrightarrow \alpha$. Given Γ a pointclass, $P \in \Gamma$ and $\varphi : P \twoheadrightarrow \lambda$ a regular norm. We say that φ is a Γ-*norm* if there are relations $\leq_{\varphi} \in \Gamma$ and $\leq_{\check{\varphi}} \in \check{\Gamma}$ that satisfy the following:

$$P(y) \implies (\forall x)[P(x) \wedge \varphi(x) \leq \varphi(y) \iff x \leq_{\varphi} y \iff x \leq_{\check{\varphi}} y].$$

Given the regular norm $\varphi : P \twoheadrightarrow \lambda$ we can also define two other relations:

$$x \leq^*_{\varphi} y \iff P(x) \wedge [\neg P(y) \vee \varphi(x) \leq \varphi(y)]$$

and

$$x <^*_{\varphi} y \iff P(x) \wedge [\neg P(y) \vee \varphi(x) < \varphi(y)].$$

We have the following result:

Proposition 4.1. *Given an adequate pointclass* Γ *and a regular norm* $\varphi : P \twoheadrightarrow \lambda$, φ *is a* Γ*-norm if and only if the relations* $\leq^*_\varphi$ *and* $<^*_\varphi$ *are in* Γ.

Proof. Let's assume that φ is a Γ-norm. Then

$$x \leq^*_\varphi y \iff P(x) \wedge [x \leq_\varphi y \vee \neg y \leq_{\check{\varphi}} x]$$

and

$$x <^*_\varphi y \iff P(x) \wedge \neg y \leq_{\check{\varphi}} x.$$

On the other hand assume that $\leq^*_\varphi$ and $<^*_\varphi$ are both in Γ. Then let

$$x \leq_\varphi y \iff x \leq^*_\varphi y$$

and

$$x \leq_{\check{\varphi}} y \iff \neg y <^*_\varphi x.$$

It is routine to check that these are indeed correct. □

Given a pointclass Γ we define

$$o(\mathbf{\Delta}) = \bigcup\{\xi : \text{ there is a prewellordering } \leq \in \mathbf{\Delta} \text{ of length } \xi\}.$$

For example $o(\mathbf{\Delta}^1_1) = \omega_1$ (see [Mos09, 2G.2]). We say that a pointclass Γ is *normed* or has the *prewellordering property* if every $P \in \Gamma$ admits a Γ-norm.

Proposition 4.2. *If* Γ *is adequate and normed then* $\mathbf{\Gamma}$ *is also normed.*

Proof. Assume that $P(z, x) \in \Gamma$ and let $e \in {}^\omega\omega$. Since Γ is normed P admits a Γ-norm φ, with $\leq^*_\varphi, <^*_\varphi \in \Gamma$. Then let ψ be the norm on P_e where $\psi(x) = \varphi(e, x)$. Then

$$x \leq^*_\psi y \iff (e, x) \leq^*_\varphi (e, y)$$

and likewise for $<^*_\psi$. □

We say that Γ has the *reduction property* if whenever $P, Q \in \Gamma$ there are $P^*, Q^* \in \Gamma$ such that $P^* \cup Q^* = P \cup Q$, $P^* \subseteq P$, $Q^* \subseteq Q$ and $P^* \cap Q^* = \varnothing$.

Proposition 4.3. *If* Γ *is adequate and normed then* Γ *has the reduction property.*

Proof. Let $P, Q \in \Gamma$. Let $R \in \Gamma$ be defined as

$$R(x, n) \iff [P(x) \wedge n = 0] \vee [Q(x) \wedge n = 1]$$

and let φ be a Γ-norm for R. Then take

$$P^*(x) \iff (x, 0) \leq^*_\varphi (x, 1)$$

and

$$Q^*(x) \iff (x, 1) <^*_\varphi (x, 0).$$

Obviously $P^* \cup Q^* = P \cup Q$. Also if $x \in Q^*$ then it cannot be the case that $x \in P^*$, hence $P^* \cap Q^* = \varnothing$. □

We say that Γ has the *separation property* if whenever $P, Q \in \Gamma$ such that $P \cap Q = \varnothing$, there some $R \in \Delta$ such that $P \subset R$ and $Q \cap R = \varnothing$.

Proposition 4.4. *If Γ is adequate and has the reduction property then $\check{\Gamma}$ has the separation property.*

Proof. Let $P, Q \in \check{\Gamma}$ such that $P \cap Q = \varnothing$. $\neg P, \neg Q \in \Gamma$ and $\neg P \cup \neg Q = {}^{\omega}\omega$. Then since Γ has the reduction property we have $P^*, Q^* \in \Gamma$ such that $P^* \cap Q^* = \varnothing$ and $P^* \cup Q^* = {}^{\omega}\omega$. That is $P^* \in \Delta$, and it separates P from Q. $\square$

Proposition 4.5. *If Γ is adequate, ω-parametrised or ${}^{\omega}\omega$-parametrised and has the reduction property then Γ cannot have the separation property.*

Proof. We prove this only for ${}^{\omega}\omega$-parametrised; the other case is analogous. Let $G : {}^{\omega}\omega \times {}^{\omega}\omega$ be a universal set of $\Gamma \restriction {}^{\omega}\omega$ and let's define

$$P(x) \iff G((x)_0, x), \qquad Q(x) \iff G((x)_1, x).$$

Chose P^* and Q^* that reduce P and Q and let's assume that $S \in \Delta$ separates P^* and Q^* and $P^* \subseteq S$. Choose reals a, b such that $S = G_b$ and $\neg S = G_a$. Then let $c = (a, b)$. Then $c \in S$ implies that $c \in Q$, which implies that $c \in P$, i.e., $c \notin S$. And likewise if $c \notin S$. $\square$

Corollary 4.6. *If Γ is adequate and ${}^{\omega}\omega$-parametrised it cannot be the case that both Γ and $\check{\Gamma}$ have the prewellordering property.*

Proof. If Γ and $\check{\Gamma}$ were both normed, Proposition 4.3 implies that both have the reduction property, hence by Proposition 4.4 both have the separation property, contradicting Proposition 4.5. $\square$

Definition 4.7. A *Spector pointclass* Γ is a pointclass that satisfies the following conditions:

1. Γ is a Σ-pointclass with the substitution property and closed under $\forall^{\omega}$.
2. Γ is ω-parametrised.
3. Γ has the prewellordering property.

We note that from the definition of the substitution property it immediately follows that a Spector pointclass is closed under recursive substitution. It is also the case that any Spector pointclass contains Π^1_1 ([Mos09, 4C.2]).

Proposition 4.8. *If Γ is a Spector pointclass then $\boldsymbol{\Gamma}$ contains $\boldsymbol{\Pi^1_1}$, it is closed under Borel substitutions, $\exists^{\omega}, \forall^{\omega}$, countable unions and intersections, it is ${}^{\omega}\omega$-parametrised and it is normed.*

Proof. That $\boldsymbol{\Gamma}$ contains $\boldsymbol{\Pi^1_1}$ follows from the fact that Γ contains Π^1_1 and that it is closed under Borel substitutions follows from the fact Γ is closed under recursive substitutions. The closure under $\exists^{\omega}$ and $\forall^{\omega}$ follows immediately from the definition of the boldface pointclass. Since Γ has a good universal set it follows that $\mathbf{\Gamma}$ is ${}^{\omega}\omega$-parametrised and that it is normed follows from Proposition 4.2. Now assume that we have $A_n \subset \mathcal{X}$ for $n \in \omega$, with $A_n \in \mathbf{\Gamma}$. If G is a good universal set for $\mathbf{\Gamma} \restriction \mathcal{X}$ and let a_n be the $\mathbf{\Gamma}$-code for A_n. Then if a codes all a_n we have that

$$x \in \bigcup_{n\in\omega} A_n \iff (\exists k \in \omega) G((a)_k, x)$$

and

$$x \in \bigcap_{n\in\omega} A_n \iff (\forall k \in \omega) G((a)_k, x)$$

which are in $\mathbf{\Gamma}$ since Γ is closed under recursive substitution. □

Proposition 4.9. *If Γ is a Spector pointclass then if $\varphi : P \twoheadrightarrow \lambda$ is a regular $\mathbf{\Gamma}$-norm then for every $\xi \in \lambda$ we have that $\{x \in P : \varphi(x) \leq \xi\} \in \mathbf{\Delta}$.*

Proof. Given $y \in P$ such that $\varphi(y) = \xi$ we have that

$$x \in P \wedge \varphi(x) \leq \xi \iff x \leq_{\varphi} y \iff x \leq_{\breve{\varphi}} y.$$

□

Corollary 4.10. *If Γ is a Spector pointclass and $\varphi : P \twoheadrightarrow \lambda$ is a regular $\mathbf{\Gamma}$-norm then $\lambda \leq o(\mathbf{\Delta})$.*

Proof. By Proposition 4.9 we have that P is the λ union of sets $P_\xi \in \mathbf{\Delta}$. Every P_ξ has a prewellordering of order ξ in $\mathbf{\Delta}$ defined by

$$x \leq_\xi y \iff P_\xi(y) \wedge x \leq_{\varphi} y \iff P_\xi(y) \wedge x \leq_{\breve{\varphi}} y.$$

Thus by the definition of $o(\mathbf{\Delta})$ we have that $\lambda \leq o(\mathbf{\Delta})$. □

Lemma 4.11. *Let Γ be a Spector pointclass closed under $\forall^{{}^{\omega}\omega}$, let $S \in \mathbf{\Gamma} \setminus \mathbf{\Delta}$, and let $\varphi : S \twoheadrightarrow \delta$ be a regular $\mathbf{\Gamma}$-norm. If $Q \subseteq S$ and $Q \in \breve{\mathbf{\Gamma}}$, then there is a $\xi \in \delta$ such that $Q \subseteq \{x \in S : \varphi(x) \leq \xi\}$.*

Proof. Heading towards a contradiction let's assume that for every $\xi \in \delta$, there is some $x \in Q$ such that $\varphi(x) \geq \xi$. Since φ is a $\mathbf{\Gamma}$-norm we have by definition that the relation $\leq_{\varphi} \in \mathbf{\Delta}$. Furthermore since $\mathbf{\Gamma}$ is closed under $\forall^{{}^{\omega}\omega}$, we have that $\breve{\mathbf{\Gamma}}$ is closed under $\exists^{{}^{\omega}\omega}$. We now observe that

$$x \in S \iff \exists y (y \in Q \wedge x \leq_{\varphi} y).$$

But the right hand side is in $\breve{\mathbf{\Gamma}}$. Hence $S \in \mathbf{\Delta}$. □

Lemma 4.12 ([Mos70])**.** *Let Γ be a Spector pointclass closed under $\forall^{{}^{\omega}\omega}$, let G be a good universal set in Γ and let φ be a regular Γ-norm on G. Then the length of φ is $o(\mathbf{\Delta})$.*

Proof. See for example [Mos09, 4C.14]. □

We will now present, without proof, a few results about the existence and construction of Spector pointclasses, along with a few concrete examples:

Lemma 4.13. Π^1_1 *is the smallest Spector pointclass.* Σ^1_2 *is the smallest Spector pointclass closed under* $\exists^{\omega^\omega}$.

Proof. See [Mos09, 4C.2]. □

A pointset R is said to be *inductive* if there exists a Σ^1_n relation $\varphi(a, x, A)$ such that $\Phi_x : \wp(\mathcal{X}) \to \wp(\mathcal{X})$ defined by $\Phi_x(A) = \{a : \varphi(a, x, A)\}$ is a monotone map, and R is the least fixed point of Φ_x. If x is recursive then we say that R is *absolute inductive*.

Lemma 4.14. *The class of absolute inductive pointsets is the smallest Spector pointclass that is closed under* $\forall^{\omega^\omega}$ *and* $\exists^{\omega^\omega}$. *Furthermore, for every pointset A the class of all pointsets absolute inductive in A is a Spector pointclass that contains A, $\neg A$ and is closed under* $\forall^{\omega^\omega}$ *and* $\exists^\omega\omega$.

Proof. See [Mos09, 7C.3]. □

For any spaces $\mathcal{X}$,$\mathcal{Y}$, we call any $U : \omega^{\mathcal{X}} \times \mathcal{Y} \to \omega$ a *type* 3 *objects*. The type 3 object ${}^3\mathrm{E}$ is defined as follows:

$$ {}^3\mathrm{E}(h) = \begin{cases} 0 & \text{if}(\exists^{\omega^\omega} a)[h(a) = 0], \\ 1 & \text{otherwise.} \end{cases} $$

If U is a type 3 object we say that a pointclass Γ is *closed under* U if for each Γ-recursive $h : \mathcal{Y} \times \mathcal{Y} \to \omega$ there is a $P \in \Gamma$ such that

$$ P(i, x, y) \iff (\forall z)[h(x, z)\downarrow] \wedge U(\lambda z.h(x, z), y) = i. $$

Given a type 3 object U we define the *envelope* of U, denoted by $\mathrm{Env}(U)$ to be the pointclass that contains all pointsets semirecursive in U. We say that a a type 3 object U is *normal* if ${}^3\mathrm{E}$ is recursive in U.

Lemma 4.15. *If U is a normal type* 3 *object then* $\mathrm{Env}(U) = \Gamma$ *is the least Spector pointclass closed under U and under* $\forall^{\omega^\omega}$. *Furthermore if for every $A \in \Gamma$ there is a $B \in \Gamma$ such that*

$$ x \notin A \iff \exists a(x, a) \in B. $$

Proof. See [Mos74] and [Mos67]. □

4.2 Games for large cardinals

A similar game the one used in Theorem 2.25 provides the following technical but significant lemma. Given a function $f : \lambda^n \to \wp(\mathcal{Y})$ and a norm ρ of length λ we say that C is a choice set for f if

$$ (y_1, \ldots, y_n, x) \in C \iff y \in f(\rho(y_1), \ldots, \rho(y_n)) $$

and if $f(\xi_1, \ldots, \xi_n) \neq \varnothing$ then there are an $x \in f(\xi_1, \ldots, \xi_n)$ and $y_1, \ldots, y_n$ with $\rho(y_1) = \xi_1, \ldots \rho(y_n) = \xi_n$, such that $(y_1, \ldots, y_n, x) \in C$.

Theorem 4.16 (Coding Lemma[Mos70]). *(AD) Let $\leq$ be a prewellordering on $S \subseteq \mathcal{X}$ with rank function $\rho : S \twoheadrightarrow \lambda$ and let Γ be a Spector pointclass closed under $\forall^{^\omega\omega}$ such that $\leq \in \mathbf{\Delta}$. Then every function $f : \lambda^n \to \wp(\mathcal{Y})$ has a choice set in $\check{\mathbf{\Gamma}}$.*

Proof. [Mos09, 7D.5]. $\square$

Theorem 4.17 (The Suslin theorem for the odd levels (Martin and [Mos09])). *(AD) Let Γ be a Spector pointclass closed under $\forall^{^\omega\omega}$ and let $\delta = o(\mathbf{\Delta})$. If $\lambda < \delta$ and for all $\xi < \lambda$ we have $A_\xi \in \mathbf{\Delta}$, then $\bigcup_{\xi\in\lambda} A_\xi \in \mathbf{\Delta}$.*

Proof. Heading towards a contradiction assume that λ is least such that $A_\xi \in \mathbf{\Delta}$ while $A = \bigcup_{\xi\in\lambda} A_\xi \notin \mathbf{\Delta}$. Since $\lambda < \delta$ we have that there is a prewellordering $\leq \in \mathbf{\Delta}$ of length λ, with rank function $\rho : {^\omega\omega} \twoheadrightarrow \lambda$.

Let G be a universal set of $\mathbf{\Gamma}$ and let us define

$$f(\xi) = \{x : A_\xi = \mathcal{X} \setminus G_x\}.$$

By the Coding Lemma 4.16 we have that there is a choice set $C \in \check{\mathbf{\Gamma}}$ for f. Then the relation

$$x \in A_{\rho(y)} \iff (\exists y')(\exists z)[y \leq y' \wedge y' \leq y \wedge C(y', z) \wedge \neg G(z, x)]$$

and therefore if $P(x, y) \iff x \in A_{\rho(y)}$ we have that $P \in \check{\mathbf{\Gamma}}$. From this and since $\check{\mathbf{\Gamma}}$ is closed under $\exists^{^\omega\omega}$ it follows that $\bigcup_{\xi\in\lambda} A_\xi \in \check{\mathbf{\Gamma}}$.

By the choice of λ, for $\zeta < \lambda$ we have that $\bigcup_{\xi<\zeta} A_\xi \in \mathbf{\Delta}$. Thus completely analogously to P we can define relations

$$Q(x, y) \iff x \notin \bigcup_{\xi<\zeta} A_{\rho(y)}$$

and

$$R(x, y) \iff x \notin \bigcup_{\xi\leq\zeta} A_{\rho(y)}$$

such that $Q, R \in \check{\mathbf{\Gamma}}$. Hence A is $\check{\mathbf{\Gamma}}$-normed from obvious prewellordering $\varphi(x) = \mu[\xi : x \in A_\xi]$:

$$x \leq^*_\varphi y \iff \exists a[P(x, a) \wedge Q(y, a)]$$

and

$$x <^*_\varphi y \iff \exists a[P(x, a) \wedge R(y, a)].$$

By the assumption that $A \notin \mathbf{\Delta}$ we have that $A \notin \mathbf{\Gamma}$. Hence A cannot be the the continuous preimage of B for $B \in \mathbf{\Gamma}$. Thus, by Wadge's Lemma 2.24 we have that $\neg B$ is the continuous preimage of A and hence using φ every element of $\check{\mathbf{\Gamma}}$ is normed, contradicting Corollary 4.6. $\square$

Corollary 4.18. *(AD) If Γ is a Spector pointclass closed under $\forall^{^\omega\omega}$ then $\kappa = o(\mathbf{\Delta})$ is a regular cardinal.*

Proof. Let $\varphi : G \twoheadrightarrow \kappa$ be a $\mathbf{\Gamma}$-norm. Assume that $f : \lambda \to \kappa$ is given, where $\lambda < \kappa$. Then for every $\xi \in \lambda$ we have $A_\xi = \{x \in G : \varphi(x) < f(x)\} \in \mathbf{\Delta}$. By Theorem 4.17 we have that $A = \bigcup_{\xi \in \lambda} A_\xi \in \mathbf{\Delta}$. Then by Lemma 4.11 there is some $\alpha \in \kappa$ such that $A \subseteq \{x \in G : \varphi(x) < \alpha\}$. Hence f is bounded in κ. $\square$

The following theorem generalises Solovay's argument that the closed unbounded filter over ω_1 is an ω_1-complete ultrafilter.

Theorem 4.19. *(AD) Let Γ be a Spector pointclass closed under $\forall^{\omega^\omega}$ and let $\kappa = o(\mathbf{\Delta})$. Then $\mathcal{C}^\omega_\kappa$ is a κ-complete ultrafilter over κ.*

Proof. Let $\varphi : G \twoheadrightarrow \kappa$ be a regular Γ-norm, where G is a good universal set for $\mathbf{\Gamma}$. Given $X \subseteq \kappa$ we define the following game:

Both players play ω many reals x_n and y_n. Player I loses unless the least n such that $y_n \notin G$ is strictly less than the least n such that $x_n \notin G$. If both players manage to play elements of G player I loses unless

$$\bigcup(\{\varphi(x_n) : n \in \omega\} \cup \{\varphi(y_n) : n \in \omega\}) \in X.$$

By the axiom of determinacy the above game is determined. We will show that if σ is a winning strategy for I then $X \in \mathcal{C}^\omega_\kappa$. If the game is won by II, then symmetrically one can show that $\kappa \setminus X \in \mathcal{C}^\omega_\kappa$.

By definition we have that $\leq_\varphi \in \mathbf{\Delta}$. For given $\nu \in \kappa$ and $b \in G$ such that $\phi(b) = \nu$ we have that the set B_ν defined as

$$x \in B_\nu \iff (\exists y)(\exists n)[x = ((\sigma \star y)_I)_n \wedge (\forall m < n)(y_m \leq_\varphi b)]$$

is an element of $\check{\mathbf{\Gamma}}$, since $\mathbf{\Gamma}$ is assumed to be a Spector pointclass closed under $\forall^{\omega^\omega}$.

From the definition of the game and the fact that σ is winning for I, we have that $B_\nu \subseteq G$. Thus, by Lemma 4.11 we have that there is a $\lambda \in \kappa$ such that $B_\nu \subseteq \{x \in G : \varphi(x) \leq \lambda\}$, and let the least such λ be called $\rho(\nu)$. Then the set

$$C = \{\alpha \in \kappa : (\forall \beta < \alpha)[\rho(\beta) < \alpha]\}$$

is a closed unbounded subset of κ. Furthermore, by the definition of C we have that every element of $C \cap E^\kappa_\omega$ cab be the outcome of the game played according to σ. Hence $X \supseteq C \cap E^\kappa_\omega$.

The argument for the κ-completeness generalises the above: Let $\lambda < \kappa$ and assume that $\bigcup_{\delta \in \lambda} A_\xi = \kappa$ and $A_\delta \notin \mathcal{C}^\omega_\kappa$, for all $\delta \in \lambda$. By the Coding Lemma (Theorem 4.16) there is some $S \in \check{\mathbf{\Gamma}}$ that contains a winning strategy for I for every $\kappa \setminus A_\xi$. Then

$$x \in B'_\nu \iff (\exists y)(\exists \sigma)(\exists n)[x = ((\sigma \star y)_I)_n \wedge \sigma \in S \wedge (\forall m < n)(\varphi(y_m) \leq \xi)]$$

is in $\check{\mathbf{\Gamma}}$ and thus by Lemma 4.11 there is some $\mu \in \kappa$ such that $B'_\nu \subseteq \{x \in G : \varphi(x) < \nu\}$. Hence we can define an increasing function and a closed unbounded set on κ whose every ω-limit in that set will be an outcome of all the strategies. But that would mean that such an element will be in $\bigcap_\xi \in \lambda\kappa \setminus A_\xi = \varnothing$, a contradiction. $\square$

To show normality using the above arguments we require the use of the axiom of dependent choice:

Lemma 4.20. *($\mathsf{AD}+\mathsf{DC}$) Let Γ be a Spector pointclass closed under $\forall^{\omega_\omega}$ and let $\kappa = o(\boldsymbol{\Delta})$. Then $\mathcal{C}^\omega_\kappa$ is a normal κ-complete ultrafilter over κ.*

Proof. We consider the game from Theorem 4.19. Since we have shown in Theorem 4.19 that $\mathcal{C}^\omega_\kappa$ is an ultrafilter, all that we now need to show is that if $f \in \kappa^\kappa$ is a regressive function, then it is constant on an element of $\mathcal{C}^\omega_\kappa$. Let's assume towards a contradiction that $A_\xi = \{\alpha \in \kappa : f(\alpha) \neq \xi\} \in \mathcal{C}^\omega_\kappa$. Given a fixed $\eta \in \kappa$, by the Coding Lemma we have that there is some $S_\xi \in \check{\boldsymbol{\Gamma}}$ that contains winning strategies for I for all A_ξ where $\xi \in \eta$. Furthermore the set

$$B_\eta = \{((\sigma \star y)_I)_n : \sigma \in S_\eta \wedge (\forall m < n)(\varphi(y_m) < \eta)\}$$

is in $\check{\boldsymbol{\Gamma}}$ and hence by Lemma 4.11 there is some $\nu \in \kappa$ such that $B_\eta \subseteq \{x \in G : \varphi(x) < \nu\}$. Thus we define the following relation on elements of $\check{\boldsymbol{\Gamma}}$, that are of the form S_η as described above: $S_\eta \sqsubseteq S_\delta$ if $S_\eta \subseteq S_\delta$, $\eta < \delta$ and the bound of B_η is less than or equal to δ.

Then by DC, there is a sequence of such sets (and of ordinals) S_{η_n}, for $n \in \omega$. Now by $\mathrm{AC}_\omega({}^\omega\omega)$ we can pick y such that $\varphi(y_0) = 0$ and $\varphi(y_{n+1}) = \eta_n$. Then $\bigcup_{n\in\omega} \eta_n$ will be the outcome of every strategy in $\bigcup_{n\in\omega} S_{\eta-n}$ played against y. hence $f(\eta) \neq \delta$ for all $\delta < \eta$, contradicting the fact that f is regressive. □

Corollary 4.21 (Solovay)**.** *($\mathsf{AD}+\mathsf{DC}$)The cardinal ω_1 is measurable. In particular, the club filter, $\mathcal{C}_{\omega_1}$, over ω_1 is the unique normal ω_1-complete ultrafilter over ω_1.*

Proof. We have that Π^1_1 is a Spector pointclass closed under $\forall^{\omega_\omega}$. On the other hand $o(\boldsymbol{\Delta}^1_1) = \omega_1$. Hence C_{ω_1} is a normal ultrafilter over ω_1, by Theorem 4.19 and Lemma 4.20. □

We can show directly that $\mathcal{C}^\omega_\kappa$ is a normal κ-complete ultrafilter over κ, without using DC by using a slightly more complex game and showing that $\kappa \to (\kappa)^{\omega+\omega}_2$:

Theorem 4.22. *(AD) Let Γ be a Spector pointclass closed under $\forall^{\omega_\omega}$ and let $\kappa = o(\boldsymbol{\Delta})$. Then $\kappa \to (\kappa)^{\omega+\omega}_2$. In particular κ is a measurable cardinal.*

Proof. By Corollary 4.18, κ is a regular cardinal. Let $\varphi : G \twoheadrightarrow \kappa$ be a regular Γ-norm, where G is a good universal set for $\boldsymbol{\Gamma}$. Let a code $\omega \cdot (\omega+\omega)$. Given $\{A_1, A_2\}$ a partition of $[\kappa]^{\omega+\omega}$ we define the following game:

Both players play $\omega \cdot (\omega+\omega)$ reals (coded by a) x_ξ and y_ξ. Player I loses unless the least ξ such that $y_\xi \notin G$ is strictly less than the least ξ such that $x_\xi \notin G$. If both players manage to play elements of G, then for $\delta \in \omega+\omega$ let us define

$$\alpha_\delta = \bigcup(\{\phi(x_{\omega\cdot\delta+n}) : n \in \omega\} \cup \{\phi(y_{\omega\cdot\delta+n}) : n \in \omega\}).$$

Then player I loses unless $\{\alpha_\delta : \delta \in \omega+\omega\} \in A_1$.

By the axiom of determinacy the above game is determined. Due to the symmetry of the game we can, without loss of generality, assume that there is a winning strategy σ for I. By definition we have that $\leq_\varphi \in \mathbf{\Delta}$. For given $\xi \in \omega \cdot (\omega + \omega)$, $\nu \in \kappa$ and $b \in G$ such that $\phi(b) = \nu$ we observe that the set $A_{\xi,\nu}$ defined as

$$y \in A_{\xi,\nu} \iff (\exists z)[y_\xi = z \wedge z \leq_\varphi b]$$

is in $\check{\mathbf{\Gamma}}$. Hence also the set $\bigcap_{\zeta<\xi} A_{\zeta,\nu} \in \check{\mathbf{\Gamma}}$, since $\check{\mathbf{\Gamma}}$ is closed under countable intersections. Therefore the set $B_{\xi,\nu}$ defined as

$$x \in B_{\xi,\nu} \iff (\exists y)[x = ((\sigma \star y)_I)_\xi \wedge y \in \bigcap_{\zeta<\xi} A_{\zeta,\nu}]$$

is an element of $\check{\mathbf{\Gamma}}$, since $\mathbf{\Gamma}$ is assumed to be closed under $\forall^{^\omega\omega}$. And since $\check{\mathbf{\Gamma}}$ is closed under countable unions we have that $B_\nu = \bigcup_{\xi\in\omega\cdot(\omega+\omega)} B_{\xi,\nu} \in \check{\mathbf{\Gamma}}$.

From the definition of the game and the fact that σ is winning for I, we have that $B_\nu \subseteq G$. Thus, by Lemma 4.11 we have that there is a $\lambda \in \kappa$ such that $B_\nu \subseteq \{x \in G : \varphi(x) \leq \lambda\}$, and let the least such λ be called $\rho(\nu)$. Then the set

$$C = \{\alpha \in \kappa : (\forall \beta < \alpha)[\rho(\beta) < \alpha]\}$$

is a closed unbounded subset of κ. By the definition of C we have that every element of $[C \cap E^\kappa_\omega]^{\omega+\omega}$ can be the outcome of the game played according to σ, i.e., $[C \cap E^\kappa_\omega]^{\omega+\omega} \subseteq A_1$. Thus $C \cap E^\omega_\kappa$ is a stationary homogeneous set for the partition.

By Theorem 3.18 if follows that $\mathcal{C}^\omega_\kappa$ is a normal non-principal ultrafilter over κ, i.e., κ is measurable. □

Now, suppose that we have a Spector pointclass closed under $\forall^{^\omega\omega}$ with $o(\mathbf{\Delta}) = \kappa$ and let $\varphi : G \twoheadrightarrow \kappa$ be a regular Γ-norm and suppose that for each $\epsilon \in {}^\omega\omega$ we have a (potentially) partial function $f_\epsilon : \kappa \to \kappa$. Then we will say that φ and $\{f_\epsilon : \epsilon \in {}^\omega\omega\}$ define a *good coding* in Γ of the functions from κ to κ the following conditions hold:

1. For fixed $\gamma, \delta \in \kappa$ the relation

$$\epsilon \in C_{\gamma,\delta} \iff (\forall \alpha \leq \gamma)[f_\epsilon(\alpha) \leq \delta]$$

   is in $\mathbf{\Delta}$;

2. There is a relation $V(\epsilon, a, b) \in \check{\mathbf{\Gamma}}$ which computes the values of each f_ϵ relative to φ in the following sense:

$$\begin{aligned} &a \in G \wedge f_\epsilon(\varphi(a))\downarrow \Longrightarrow \\ &\quad (\exists b) V(\epsilon, a, b) \wedge (\forall b)[V(\epsilon, a, b) \implies (b \in G \wedge f_\epsilon(\varphi(a)) = \varphi(b))]; \end{aligned}$$

3. For every total function in κ^κ there is some $\epsilon \in {}^\omega\omega$ such that $f = f_\epsilon$.

Lemma 4.23. *(*$\mathsf{AD}+\mathsf{DC}$*) Assume that* Γ *is a Spector pointclass closed under* $\forall^{\omega_\omega}$ *and* $\exists^{\omega_\omega}$*, if* $\kappa = o(\mathbf{\Delta})$*, then there is a good coding* $\{f_\epsilon : \epsilon \in {}^\omega\omega\}$ *in* $\mathbf{\Gamma}$ *of the functions from* κ *to* κ*.*

Proof. See [KKMW81, Lemma 1.6]. □

Theorem 4.24 ([KKMW81])**.** *(*$\mathsf{AD}+\mathsf{DC}$*) Assume that* Γ *is a Spector pointclass closed under* $\forall^{\omega_\omega}$ *and* $\exists^{\omega_\omega}$*, if* $\kappa = o(\mathbf{\Delta})$ *then for all* $\mu < \kappa$ *we have that* $\kappa \to (\kappa)^\kappa_\mu$*, i.e.,* κ *has the strong partition property.*

Proof. Let $\varphi : G \twoheadrightarrow \kappa$ be a regular Γ-norm, where G is a good universal set for $\mathbf{\Gamma}$. By Lemma 4.23 we have that there is a good coding $\{f_\epsilon : \epsilon \in {}^\omega\omega\}$. Assume now that we have a partition $\{A_\xi : \xi \in \mu\}$ of $[\kappa]^\kappa$. For $\xi \in \mu$ we define the game G_ξ similarly to the game in the proof of Theorem 4.22:

Players play one real number x and y. Player I loses unless the least δ such that $f_x(\delta)$ is not defined is strictly less that the least δ such that $f_y(\delta)$ is not defined. If both players play a code for a total function then we define for each $\delta \in \kappa$:

$$\alpha_\delta = \bigcup(\{f_x(\omega\cdot\delta + n) : n \in \omega\} \cup \{f_y(\omega\cdot\delta+n) : n \in \omega\}).$$

Then player I loses unless $\{\alpha_\delta : \delta \in \kappa\} \in A_\xi$.

If σ is a winning strategy for I in one of the G_ξ games then, because we have a good coding, for every $\gamma, \delta \in \kappa$ it is the case that

$$C_{\gamma,\delta} = \{\epsilon \in {}^\omega\omega : (\forall\alpha \le \gamma)[f_\epsilon(\alpha) \le \delta]\} \in \mathbf{\Delta}.$$

Hence the set $A_{\gamma,\delta} = \{(\sigma \star y)_I : y \in A_\delta\} \in \check{\mathbf{\Gamma}}$ and therefore

$$B_{\gamma,\delta} = \{x \in G : (\exists\epsilon \in A_{\gamma,\delta})(\exists y)[\varphi(y) \le \delta \wedge \varphi(\alpha) = f_\epsilon(\varphi(y))]\} \in \check{\mathbf{\Gamma}}.$$

Thus by Lemma 4.11 there is a $\nu \in \kappa$ such that $B_{\gamma,\delta} \subseteq \{x \in G : \varphi(x) < \nu\}$. Hence as in Theorem 4.22 we can define an increasing function $\rho : \kappa^2 \to \kappa$ and a closed unbounded set

$$C = \{\alpha \in \kappa : (\forall\beta,\gamma < \alpha)[\rho(\beta,\gamma) < \alpha]\},$$

whose ω-limit points provide the homogeneous set for the partition. What is left to show is that II cannot be the winner in G_ξ for all $\xi \in \mu$.

Assume towards a contradiction that II has a winning strategy in G_ξ for all $\xi \in \mu$. If S_ξ is the set of winning strategies of II for G_ξ and since $\mu \in \kappa$, by the Coding Lemma 4.16, there is a set $S \in \check{\mathbf{\Gamma}}$ such that for every $\xi \in \mu$ there is a winning strategy for II in G_ξ, $\tau_\xi \in S$. Then the set

$$A'_{\gamma,\delta} = \{(y \star \tau)_{II} : y \in C_{\gamma,\delta} \wedge \tau \in S\} \in \check{\mathbf{\Gamma}}$$

and therefore by Lemma 4.11 there is a $\nu \in \kappa$ such that if

$$B'_{\gamma,\delta} = \{x \in G : (\exists\epsilon \in A'_{\gamma,\delta})(\exists y)[\varphi(y) \le \delta \wedge \varphi(\alpha) = f_\epsilon(\varphi(y))]\}$$

then

$$B'_{\gamma,\delta} \subseteq \{x \in G : \varphi(x) \leq \nu\}.$$

Then as before we can define a closed unbounded set D. Any $h \in [D \cap E^{\kappa}_{\omega}]^{\kappa}$ is an outcome of a play according to every strategy in S, hence for all $\xi \in \mu$ $h \notin A_{\xi}$, a contradiction. $\square$

Lemma 4.25. *($\mathsf{AD} + \mathsf{DC}$) If Γ is a Spector pointclass closed under $\forall^{^\omega\omega}$ and $\exists^{^\omega\omega}$ then $o(\mathbf{\Delta}) = \kappa$ is a regular limit of measurable cardinals. In particular it is measurable and weakly inaccessible.*

Proof. By Theorem 4.22 we have that κ is a measurable cardinal. Given $\lambda < \kappa$ and a prewellordering $\leq \in \mathbf{\Delta}$ of length (at least) λ, let $\Gamma^{\star} = \mathrm{Env}({}^3\mathrm{E}, \leq)$. This is a Spector pointclass closed under $\forall^{^\omega\omega}$ that contains $\leq$, $\neg \leq$. Furthermore since Γ is closed under both $\exists^{^\omega\omega}$ and $\forall^{^\omega\omega}$, it immediately follows that Γ is closed under ${}^3\mathrm{E}$ and thus $\Gamma^{\star} \subseteq \Gamma$. Furthermore because Γ is closed under $\exists^{^\omega\omega}$ and by Lemma 4.15 if $\neg A \in \Gamma^{\star}$,

$$A(x) \iff \exists a B(a, x)$$

for some $B \in \Gamma$ we have that $\Gamma \subseteq \mathbf{\Delta}$. Hence $\lambda \leq o(\mathbf{\Delta}^{\star}) < \kappa$. But by Theorem 4.22 $o(\mathbf{\Delta}^{\star})$ is a measurable cardinal. $\square$

A more technical argument guarantees us that in such a case the set of measurable cardinals form a stationary set. We give only a sketch of the proof:

Theorem 4.26 ([KKMW81])**.** *($\mathsf{AD} + \mathsf{DC}$) If Γ is a Spector pointclass closed under $\forall^{^\omega\omega}$ and $\exists^{^\omega\omega}$ then the set of measurable cardinals below $o(\mathbf{\Delta}) = \kappa$ form a stationary set. In particular it is a measurable and Mahlo cardinal.*

Proof sketch. Let $\varphi : G \twoheadrightarrow \kappa$ be a regular Γ-norm, where G is a good universal set for $\mathbf{\Gamma}$. Given any normal function $f : \kappa \to \kappa$ we will define a Spector pointclass $\Gamma^{\star}$ closed under $\forall^{^\omega\omega}$ such that $o(\mathbf{\Delta}^{\star})$ is a fixed point of κ. Thus let's fix such a function f. For any function $h : {}^{\omega}\omega \to \omega$ we denote with $PW_{\kappa}(h)$ the fact that h is the characteristic function of a prewellordering with length less than κ. We define the following type 3 object:

$${}^3\mathrm{F}(h, a, b) = \begin{cases} 0 & \text{if } PW_{\kappa}(h) \wedge a, b \in G \wedge (\varphi(a) < \varphi(b) < f(|h|)), \\ 1 & \text{otherwise.} \end{cases}$$

where $|h|$ is the length of the prewellordering h defines.

We let $\Gamma^{\star} = \mathrm{Env}({}^3\mathrm{E}, {}^3\mathrm{F})$. Let $\kappa^{\star} = o(\mathbf{\Delta}^{\star})$. If h is the characteristic function of a prewellordering in $\mathbf{\Delta}^{\star}$, then $h \in \mathbf{\Delta}^{\star}$ and hence

$$\{(a, b) : {}^3F(h, a, b) = 0\} \in \mathbf{\Delta}^{\star}.$$

This is by definition a prewellordering of length $f(|h|)$, hence $f(|h|) < \kappa^{\star}$. That is we have that for all $\xi < \kappa^{\star}$ it is the case that $f(\xi) < \kappa^{\star}$. Since f is normal it follows that $f(\kappa^{\star}) = \kappa^{\star}$.

One can show that $\Gamma^\star \subseteq \mathbf{\Delta}$. Hence $\kappa^\star < \kappa$. Furthermore, since $\Gamma^\star$ is a Spector pointclass closed under $\forall^{\omega^\omega}$, $\kappa^\star$ is measurable by Theorem 4.22 and it meets the club set that consists of the fixed points of f.

Since every closed unbounded set is the set of fixed points of a normal function, we have that every closed unbounded set of κ contains a measurable cardinal, i.e., the set of measurable cardinals below κ is stationary. $\square$

Corollary 4.27. *($\mathsf{AD}+\mathsf{DC}$) If Γ is a Spector pointclass closed under $\forall^{\omega^\omega}$ and $\exists^{\omega^\omega}$ then $o(\mathbf{\Delta}) = \kappa$ is an 1-measurable cardinal.*

Proof. By Theorem 4.26 we have that $S = \{\alpha \in \kappa : \alpha \text{ is measurable}\}$ is stationary in κ. Theorem 4.24 implies that κ has the strong partition property. Then Theorem 3.20 implies that there is a normal κ-complete ultrafilter $\mathcal{U}$ over κ such that $S \in \mathcal{U}$. $\square$

Corollary 4.28. *($\mathsf{AD}+\mathsf{DC}$) Given any $\delta < \Theta$ there exists a 1-measurable cardinal κ such that $\delta < \kappa < \Theta$. In particular Θ is a limit of 1-measurable cardinals.*

Proof. Let $\leq$ be a prewellordering of length δ. Then by Lemma 4.14 the class of all inductive in $\leq$ pointsets, Γ, is a Spector pointclass closed under $\forall^{\omega^\omega}$ and $\exists^{\omega^\omega}$ and obviously $\delta < o(\mathbf{\Delta}) = \kappa$. Then Corollary 4.27 implies that κ is 1-measurable. $\square$

Chapter 5

Large cardinals in inner models

In this chapter we will try to wrap together everything that has been established thus far. We will define the class of hereditarily ordinal definable sets, HOD, and briefly discuss its properties. Specifically we will show that HOD is a model of ZFC. We will then proceed to show that under $\mathsf{AD} + \mathsf{DC}$ ultrafilters over cardinals $\kappa < \Theta$ yield analogue ultrafilters in HOD. We will try to present this method in an abstract fashion. Finally we will apply the results from previous chapters to this abstract method, showing some lower bounds of the consistency strength of the axiom of determinacy.

5.1 HOD

A set x is *ordinal definable* if there exists a formula φ such that

$$x = \{u : \varphi(u, \alpha_1, \ldots, \alpha_n)\}$$

where $\alpha_1, \ldots, \alpha_n$ are ordinal numbers. The following result shows that the statement "x is ordinal definable" is definable by a first-order formula $\mathrm{OD}(x)$:

Proposition 5.1. *A set x is ordinal definable if and only if there is some $\alpha \in \mathrm{Ord}$ such that x is ordinal definable in V_α.*

Proof. Assume that x is ordinal definable witnessed by $\varphi(u, \alpha_1, \ldots, \alpha_n)$. By the Reflection theorem, there is some β above all α_i, such that $x \in V_\beta$ and such that

$$(\forall u)\{[V_\beta \models \varphi(u, \alpha_1, \ldots, \alpha_n)] \iff \varphi(u, \alpha_1, \ldots, \alpha_n)\}.$$

Thus x is ordinal definable in V_β.

On the other hand if x is ordinal definable in some V_β from $\varphi(u, \alpha_1, \ldots, \alpha_n)$ and since V_β is definable from β, then x is definable from the formula:

$$\psi(u, \alpha_1, \ldots, \alpha_n) \iff V_\beta \models \varphi(u, \alpha_1, \ldots, \alpha_n),$$

i.e., x is ordinal definable. $\square$

The following is immediate:

Proposition 5.2. *If X is a class and there exists a definable function $F : \mathrm{Ord} \twoheadrightarrow X$ then every element of X is ordinal definable.*

Proof. The formula $(\forall x)[F(\alpha) = x \to u \in x]$ defines $F(\alpha)$. $\square$

We have that OD, the class of the ordinal definable sets, is the largest class with such a property:

Proposition 5.3. *There is a definable well ordering of* OD*. In fact there is a definable function from* Ord *onto* OD*.*

Proof. Using the standard well ordering of pairs of ordinals we can define a well ordering of finite sequences of ordinals, in a typical way. Thus given some V_α and some fixed enumeration of the formulas of the language of set theory, there is a definable well ordering $<_\alpha$ of its ordinal definable sets using a well ordering of $\omega \times \mathrm{Ord}$. Then we define for $x, y \in \mathrm{OD}$, $x <_{\mathrm{OD}} y$ if either the least ordinal α such that x is definable in V_α is less than the least ordinal β such that y is definable in V_β, or if these ordinals are equal to α then $x <_\alpha y$. It is clear that this is a well ordering. To show that there is a definable function with domain the ordinals that covers OD we note that $<_{\mathrm{OD}}$ is defined in such a way so that for any $x \in \mathrm{OD}$, $\{y : y <_{\mathrm{OD}} x\}$ is a set. Transitive recursion, thus yields the definable function. $\square$

We call HOD the class of *hereditarily ordinal definable sets*, that contains the ordinal definable sets whose transitive closure consists of ordinal definable sets:

$$x \in \mathrm{HOD} \iff \mathrm{TC}(\{x\}) \subset \mathrm{OD}.$$

It is clear that HOD is a transitive class that contains all the ordinals; HOD also has a definable well ordering:

$$x <_{\mathrm{HOD}} y \iff \mathrm{HOD}(x) \wedge \mathrm{HOD}(y) \wedge x <_{\mathrm{OD}} y.$$

Furthermore Proposition 5.2 implies that any transitive class that is the image of a definable function on Ord is a subclass of HOD.

Theorem 5.4. *The class* HOD *is a transitive model of* ZFC*.*

Proof. The axioms of extensionality, empty set, infinity, regularity are trivial. The axiom of pairing follows by noticing that if two sets are ordinal definable then the set that contains them exactly is ordinal definable. If x is definable by $\varphi(u, \alpha_1, \ldots, \alpha_n)$ then $\bigcup x$ is definable by $(\exists u)[\varphi(u, \alpha_1, \ldots, \alpha_n) \wedge v \in u]$ and $\wp(x)^{\mathrm{HOD}}$ is definable by $(\forall u)(u \in v \to \varphi(u, \alpha_1, \ldots, \alpha_n)) \wedge \mathrm{HOD}(v)$. It trivially follows that the transitive closure of these consists of ordinal definable sets. Hence the axioms of union and powerset are also true in HOD.

If $\psi(u,v)$ is a functional relation with domain x and codomain a subclass in HOD then the codomain is definable by $(\exists u)[\psi(u,v) \wedge \varphi(u,\alpha_1,\ldots,\alpha_n)]$ hence it is an ordinal definable set, by the axiom of replacement. Finally, a well-ordering of x is definable by $\varphi(u,\alpha_1,\ldots,\alpha_n) \wedge \varphi(v,\alpha_1,\ldots,\alpha_n) \wedge (u <_{\mathrm{OD}} v)$. $\square$

5.2 Large cardinals in HOD

Lemma 5.5. *Assume that $\mathcal{U}$ is a κ-complete normal ultrafilter over κ, and $\mathcal{U}$ is* OD*. Then $\mathcal{U} \cap \mathrm{HOD} \in \mathrm{HOD}$ is a κ-complete normal ultrafilter over κ in* HOD*.*

Proof. Let's assume that $\mathcal{U}$ is definable from $\varphi(x,\alpha_1,\ldots,\alpha_n)$. Then $\mathcal{U} \cap \mathrm{HOD}$ is definable by $\varphi(x,\alpha_1,\ldots,\alpha_n) \wedge \mathrm{HOD}(x)$. Hence, it is OD and as a subset of HOD, it is an element of HOD.

Since $\mathcal{U}$ is an ultrafilter, the same has to be the case for $\mathcal{U} \cap \mathrm{HOD}$. Now assume that $\langle X_\alpha : \alpha \in \lambda\rangle$ is an OD-sequence of less than κ elements of $\mathcal{U} \cap \mathrm{HOD}$. Then the intersection of this sequence is the intersection in HOD, hence by the fact that $\mathcal{U}$ is κ-complete, it is an element of $\mathcal{U}$, hence it is in $\mathcal{U} \cap \mathrm{HOD}$. Finally, let f be a regressive function on κ, $f \in \mathrm{HOD}$, and let $\psi(x,\beta_1,\ldots,\beta_m)$ be the formula that defines f. Since f is regressive and because $\mathcal{U}$ is normal we have that f is constant on some element of $\mathcal{U}$, so let $C = \{\xi \in \kappa : f(\xi) = \delta_0\} \in \mathcal{U}$. Then C is definable by the formula $\psi((x,\delta),\beta_1,\ldots,\beta_m)$, i.e., $C \in \mathcal{U} \cap \mathrm{HOD}$. $\square$

Corollary 5.6. *Let δ be an ordinal, assume that $\varphi(x)$ is a property such that*

$$(\forall x \in \delta)[\varphi(x) \implies \varphi^{HOD}(x)]$$

and assume that $\mathcal{U}$ is an OD *ultrafilter over δ such that $\{\alpha \in \delta : \varphi(\alpha)\} \in \mathcal{U}$. Then*

$$\mathrm{HOD} \models \{\alpha \in \delta : \varphi(\alpha)\} \in \mathcal{U} \cap \mathrm{HOD}.$$

Proof. Since $\mathrm{HOD} \models \mathsf{ZFC}$ it follows that $A = \{\alpha \in \delta : \varphi^{\mathrm{HOD}}(\alpha)\} \in \mathrm{HOD}$. Furthermore it follows from the assumption

$$(\forall x \in \delta)[\varphi(x) \implies \varphi^{HOD}(x)]$$

that $A \supseteq \{\alpha \in \delta : \varphi(\alpha)\} \in \mathcal{U}$, and thus $A \in \mathcal{U}$, which yields that $A \in \mathcal{U} \cap \mathrm{HOD}$. $\square$

Proposition 5.7. *(*AD*) Θ is a strong limit cardinal in* HOD*.*

Proof. Assume towards a contradiction that there is some $\alpha < \Theta$ and an OD-surjection $f : \wp(\alpha)^{\mathrm{HOD}} \twoheadrightarrow \Theta$. By Theorem 2.25, there is a surjection $g : {}^\omega\omega \twoheadrightarrow \wp(\alpha)$. Since $\wp(\alpha)^{\mathrm{HOD}} \subseteq \wp(\alpha)$, there is a surjection $h : \wp(\alpha) \twoheadrightarrow \wp(\alpha)^{\mathrm{HOD}}$. Then $f \circ h \circ g : {}^\omega\omega \twoheadrightarrow \Theta$. $\square$

Theorem 5.8 (Kunen)**.** *(*AD*) Assume that $\kappa < \Theta$ and $\mathcal{F}$ is an ω_1-complete ultrafilter over κ. Then $\mathcal{F}$ can be extended into a ω_1-complete ultrafilter over κ.*

Proof. By Theorem 2.25 there exists a surjection $g : {}^\omega\omega \twoheadrightarrow \wp(\kappa)$. For each $x \in {}^\omega\omega$ let us define
$$A_x = \bigcap\{g(y) \in \mathcal{F} :\ y \leq_T x\}.$$
Each A_x is non-emply because $\mathcal{F}$ is ω_1-complete, and if $x \equiv_T y$ then $A_x = A_y$. Hence we can define a function on the Turing degrees as $f(x) = \bigcap A_x$. We observe that if $A \in \mathcal{F}$ and $g(x) = A$ then if $x \leq_T y$, it is the case that $f(y) \in A$. This means that the image through f of the cone with root x is contained in A and therefore $f^{-1}[A] \in M_T$. Now, $f^*[M_T]$ is an ultrafilter, and by the observation above, it contains $\mathcal{F}$. □

Corollary 5.9 (Kunen)**.** *(*$\mathsf{AD} + \mathsf{DC}$*) Let* $\kappa < \Theta$*. Then every* ω_1*-complete ultrafilter* $\mathcal{U}$ *on* κ *is* OD*.*

Proof. Let f be the function defined in Theorem 5.8. We observe that $f^*[M_T]$ is $\mathcal{U}$, because it is an ultrafilter that contains $\mathcal{U}$. We take $V^{{}^\omega\omega}/M_T$. Because of DC and the fact that M_T is ω_1-complete, the ultrapower is well-founded. Hence we can collapse it to some standard model M, and let $j : V \to M$ be the canonical embedding and γ be the ordinal that $[f]$ is collapsed to. Since $\mathcal{D}_T$ is OD, so will be j, because the Mostowski's collapsing function is definable by transitive recursion. Now we have

$$X \in \mathcal{U} \iff f^{-1}[X] \in M_T \iff \{x \in {}^\omega\omega : f(x) \in X\} \in M_T \iff \gamma \in j(X).$$

Hence $\mathcal{U}$ is OD. □

Lemma 5.10. *(*DC*) Assume that* $\mathcal{U}$ *is an* OD ω_1*-complete ultrafilter on* κ*. Then if* M *is the transitive collapse of* $\mathrm{HOD}^\kappa/\mathcal{U}$*,* $j : \mathrm{HOD} \to M$ *and* $M \subseteq \mathrm{HOD}$*.*

Proof. That j is OD is shown as in Corollary 5.9. So it is left to show that $M \subseteq \mathrm{HOD}$. Let's assume that π is Mostowski's collapsing function.

SinceHOD has a definable well ordering $<_{\mathrm{HOD}}$, the relation $<^\star$ on $HOD^\kappa/\mathcal{U}$ defined as
$$[f] <^\star [g] \iff \{\xi \in \kappa : f(\xi) <_{\mathrm{HOD}} g(\xi)\} \in \mathcal{U}$$
is a well ordering by Łoś' Theorem. Then its collapse $<_\star$ defined by
$$\pi([f]) <_\star \pi([g]) \iff [f] <^\star [g]$$
is a well-ordering of M. It is also the case that for $x \in M$ $\{y : y <_\star x\}$ is a set. This is because the same is the case for $<_{\mathrm{HOD}}$ and thus also for $<^\star$. Therefore, by transfinite recursion we can define a class function from Ord onto M. By Proposition 5.2 $M \subseteq \mathrm{OD}$, and since M is transitive, $M \subseteq \mathrm{HOD}$. □

Corollary 5.11. *(*$\mathsf{AD} + \mathsf{DC}$*) Define the formula*
$$\varphi(x) \iff x \textit{ is a measurable cardinal.}$$
Then we have that
$$\forall(x \in \Theta)[\varphi(x) \implies \varphi^{HOD}(x)].$$

Proof. If $\kappa < \Theta$ is a measurable cardinal, this is witnessed by a normal ultrafilter $\mathcal{U}$ over κ. By Corollary 5.9 we have that $\mathcal{U}$ is OD. Then Lemma 5.5 implies that $\mathcal{U} \cap \mathrm{HOD}$ is a a normal ultrafilter over κin HOD. Thus κ is measurable in HOD. $\square$

Corollary 5.12. *($\mathsf{AD} + \mathsf{DC}$) If $\kappa < \Theta$ is a measurable cardinal, witnessed by $\mathcal{U}$, such that any one of the sets $\{\alpha \in \kappa : \alpha$ is regular$\}$, $\{\alpha \in \kappa : \alpha$ is inaccessible$\}$, $\{\alpha \in \kappa : \alpha$ is Mahlo$\}$, $\{\alpha \in \kappa : \alpha$ is measurable$\}$ are in $\mathcal{U}$, then the respective set $\{\alpha \in \kappa : \mathrm{HOD} \models \alpha$ is regular$\}$, $\{\alpha \in \kappa : \mathrm{HOD} \models \alpha$ is inaccessible$\}$, $\{\alpha \in \kappa : \mathrm{HOD} \models \alpha$ is Mahlo$\}$, $\{\alpha \in \kappa : \mathrm{HOD} \models \alpha$ is measurable$\}$ is in $\mathcal{U} \cap \mathrm{HOD}$.*

Proof. If a cardinal is regular, inaccessible or Mahlo, it is so in any inner model, since these are Π_1^{ZF} statements, and Corollary 5.11 implies that this is also the case for measurable cardinals below Θ. Thus Corollary 5.6 yields the desired result. $\square$

We can apply Corollary 5.12 to the results from Chapter 3:

Theorem 5.13. *(AD) Assume $V = \mathbf{L}({}^\omega\omega)$. Let Γ be a Spector pointclass closed under $\forall^{{}^\omega\omega}$ and $\exists^{{}^\omega\omega}$ and let $\kappa = o(\mathbf{\Delta})$. Then*

$$\mathrm{HOD} \models \kappa \textit{ is an } 1\textit{-measurable cardinal.}$$

Furthermore

$$\mathrm{HOD} \models \kappa \textit{ is an } 1\textit{-embedding cardinal.}$$

Proof. Corollary 4.27 implies that κ is 1-measurable, that is there is a normal ultrafilter $\mathcal{U}$ over κ such that $\{\alpha \in \kappa : \alpha$ is measurable$\} \in \mathcal{U}$. Now Corollary 5.12 implies that $\{\alpha \in \kappa : \mathrm{HOD} \models \alpha$ is measurable$\} \in \mathcal{U} \cap \mathrm{HOD}$. Since $\mathrm{HOD} \models \mathsf{ZFC}$, the second assertion follows from Lemma 3.17. $\square$

Theorem 5.14. *(AD) Assume $V = \mathbf{L}({}^\omega\omega)$ and let $\Theta = \delta$. Then*

$$\mathrm{HOD} \models \delta \textit{ is a strong inaccessible limit of } 1\textit{-measurable cardinals.}$$

Furthermore

$$\mathrm{HOD} \models \delta \textit{ is a strong inaccessible limit of } 1\textit{-embedding cardinals.}$$

Proof. That Θ is inaccessible in HOD follows from Proposition 5.7 and Lemma 2.26. Corollary 4.28 implies that Θ is a limit of 1-measurable cardinals. Theorem 5.13 then yields the first statement. The second assertion follows from Lemma 3.17, because $\mathrm{HOD} \models \mathsf{ZFC}$. $\square$

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