TOPOLOGICAL MODELS FOR BELIEF AND BELIEF REVISION

MSc Thesis *(Afstudeerscriptie)*

written by

Aybüke Özgün
(born January 28th, 1988 in Aydin, Turkey)

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Date of the public defense:	**Members of the Thesis Committee:**
August 21, 2013	Dr Alexandru Baltag
	Dr Nick Bezhanishvili
	Dr Dion Coumans
	Prof. Dr Dick de Jongh
	Dr Sonja Smets
	Dr Jakub Szymanik

Institute for Logic, Language and Computation

Abstract

We introduce a new topological semantics for belief logics in which the belief modality is interpreted as the closure of the interior operator. We show that our semantics validates the axioms of Stalnaker's combined system of knowledge and belief, in fact, that it constitutes the most general extensional (and compositional) semantics validating these axioms. We further prove that in this semantics the logic **KD45** is sound and complete with respect to the class of extremally disconnected spaces. We have a critical look at the topological interpretation of belief in terms of the derived set operator [45] and compare it with our proposal. We also provide two topological semantics for conditional beliefs of which especially the latter is quite successful in capturing the rationality postulates of AGM theory. We further investigate a topological analogue of dynamic belief change, namely, *update*. In addition, we provide a completeness result of the system **wKD45**, a weakened version of **KD45**.

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Contents

Chapter 1

Introduction

1.1 Logics for Knowledge and Belief

Ever since Hintikka [28] interpreted knowledge and belief in terms of standard Kripke semantics, the properties of knowledge and belief have come to be formulated as axioms in the language of basic modal logic. Hintikka evaluated knowledge and belief as modal operators K and B by providing possible world semantics similar to the one for the modal operator of necessity and assigning them the following readings:

$K\varphi$ reads *the agent knows* φ

$B\varphi$ reads *the agent believes* φ.[1]

More precisely, knowledge/belief is interpreted as truth in all possible worlds that are *epistemically/doxastically accessible* to the agent where epistemic/doxastic accessibility is defined by means of a binary relation that is a primitive component of a Kripke model. Although modeling knowledge and belief in the possible worlds framework boils down to imposing constraints on the accessibility relation of a Kripke frame[2], this simplification has a downside: since tautologies are true in all possible worlds, the agents know/believe all tautologies. Moreover, the K-axiom (also known as *Normality*)

$$K(\varphi \to \psi) \to (K\varphi \to K\psi)^3$$

is valid in all Kripke frames independent of the properties of the accessibility relation. This axiom states that *the agent knows all the consequences of her knowledge* [44, 51].

[1]In this thesis, we only consider the single agent case. In a multi agent case, knowledge and belief modalities of each agent are generally denoted together with a subscript specifying the corresponding agent. For example, the knowledge modality for agent a is denoted by K_a and reads *agent a knows* φ.

[2]Many well-known axioms of basic modal logic, including the ones concerning knowledge and belief that we will see below, correspond to a property of the accessibility relation of a Kripke frame. In this sense, by specifying the modal axioms which characterize the properties of knowledge/belief in a syntactic way, we can easily construct the Kripke model in which the respective axioms are valid. We will introduce some frame properties together with the corresponding modal axioms in the next chapter.

[3]We state the same axiom for belief by replacing K with B.

Our agents are therefore logically omniscient, highly idealized reasoners. In this thesis, we refrain from the issue of logical omniscience and work within this idealized framework[4].

1.1.1 Properties of Knowledge and Belief

There has been a great debate among philosophers about which axioms of basic modal logic characterize the notions of knowledge and belief and how they are related to each other (see, e.g., [35] for a review). We now give a brief summary of these issues.

One of the more uncontroversial properties of knowledge is its being factive, which is captured by the T-axiom

$$K\varphi \rightarrow \varphi$$

also known as *Truthfulness of Knowledge* [5] or *Veridicality* [27, 51] in the epistemic logic literature[5]. This property is considered to be the essential one that distinguishes knowledge from belief. Another commonly accepted, yet more controversial axiom, is the so-called axiom of *Positive Introspection for Knowledge*[6]

$$K\varphi \rightarrow KK\varphi$$

stating that *the agent knows what he knows.* Above principles together with the K-axiom and the inference rules *Necessitation* and *Modus Ponens* constitute the modal system **S4**. Some authors, such as e.g., Hintikka, consider this system to capture the right notion of knowledge. Lastly, we elaborate on the axiom

$$\neg K\varphi \rightarrow K\neg K\varphi$$

of *Negative Introspection for Knowledge*[7] saying that *the agent knows what she does not know.* From a philosophical point of view, this axiom is arguably the most controversial in the characterization of knowledge. It could be considered odd that negative introspection is more debatable than positive introspection, as one could think that if an agent knows what she knows, she should also know what she does not know; the introspection must work both ways. However, if not the axiom of Negative Introspection alone, its consequences when added to the other epistemic axioms contradict certain intuitively accepted principles, e.g., *it makes it impossible for a rational agent to believe that she knows something which is in fact false*[8] [44].

[4] We refer the reader to [22, 43] for a more detailed survey of the problem of logical omniscience and to [23, 26, 29] for some proposed solutions.

[5] Throughout this thesis, we refer to this axiom as *Truthfulness of Knowledge* in the context of a logic of knowledge.

[6] In formal epistemology, it is also named the KK-principle (see, e.g. [28]). In the modal logic literature, it is known as axiom 4.

[7] This axiom is also known as axiom 5 and yields the modal system **S5** when added to **S4**.

[8] Negative Introspection together with Truthfulness of Knowledge entails $(\neg\varphi \rightarrow K\neg K\varphi)$ in the system **S5**, i.e. $(\neg\varphi \rightarrow K\neg K\varphi)$ is a theorem of **S5**. Suppose that φ is not true, but the agent believes

On the other hand, *public announcement logics* have been studied assuming the **S5** properties of knowledge (see, e.g. [50, 58, 60]). Moreover, **S5**-type knowledge is widely accepted in the applications of epistemic logic to areas such as computer science and artificial intelligence.

Other than the above mentioned extremes of logics of knowledge, **S4** and **S5**, we present in Chapter 2 intermediate systems such as **S4.2** and **S4.3** which are also considered to be knowledge systems of different strengths. In fact, our main focus in this thesis will be the system **S4.2** as a logic of knowledge.

The axioms listed above, except for the T-axiom, can also be stated for belief by simply replacing K with B and assigned a doxastic interpretation in a similar way as was done in the epistemic case. More precisely, we obtain the (doxastic) axioms

$$B(\varphi \to \psi) \to (B\varphi \to B\psi)$$

of *Normality*

$$B\varphi \to BB\varphi$$

of *Positive Introspection for Belief* and

$$\neg B\varphi \to B\neg B\varphi$$

of *Negative Introspection for Belief*[9] by replacing K with B in the respective epistemic axioms. The T-axiom, however, does not apply to belief. Intuitively, it makes perfect sense to assume that an agent might believe things that are not true. For example, it is very likely that a master student, who is working really hard on her thesis and very close to the deadline, loses track of the days and believes it to be Wednesday today when in fact it is Thursday[10]. The T-axiom thus needs to be abandoned when characterizing belief in order to avoid truthfulness of belief. However, instead of the T-axiom, we add the D-axiom

$$B\varphi \to \neg B\neg\varphi$$

saying that *the agent cannot believe both* φ *and* $\neg\varphi$ *at the same time*. Therefore, we obtain the system **KD45** which is usually assumed to be the standard logic of belief (see, e.g., [40, 44, 58]). In this thesis, we choose to work with the system **KD45** as our main reference point of a logic of belief.

Besides the modal systems in which *only* knowledge or *only* belief modalities occur, it is also very interesting to study frameworks in which *both*, knowledge and belief modalities, as primitive operators, occur since these systems allow us to investigate the interaction between the two (see, e.g., [35, 44, 56, 59]).

that she knows φ, i.e. $\neg\varphi \wedge BK\varphi$ is true. If the agent has consistent beliefs, she does not believe that she does not know φ ($\neg B\neg K\varphi$ is the case), and therefore she does not know that she knows φ, i.e. $\neg K\neg K\varphi$ is the case, contradicting $\neg\varphi \to K\neg K\varphi$ [28, 44].

[9] From a philosophical point of view, as belief is a more subjective notion than knowledge, the Negative Introspection principle is less debatable for belief than it is for knowledge: "If we take knowledge to be a relation between an agent and an external reality, then it is as problematic to account for an agent's knowledge of their own knowledge as it is to account for any other type of knowledge. But to the extent that belief is an "internal" relation, it seems easier to say that fully-aware agents should have access to their own beliefs" [6, p. 490].

[10] This example is a slightly modified version of the example given in [58, p. 38].

1.1.2 Relation between Knowledge and Belief

Epistemologists assumed for a long time that the correct analysis of knowledge is the one concluded from Plato's dialogue *Theaetetus*, i.e. that 'knowledge is true, justified belief'. This analysis is often abbreviated as "JTB" for 'justified true belief' [30]. According to JTB, *an agent knows* φ *iff* φ *is true, she believes that it is true and she is justified in believing that* φ[11]. However, in his paper [25], Edmund Gettier presented two counterexamples to the JTB analysis of knowledge showing that having a justified true belief is not sufficient for having knowledge. A Gettier-type example could run as follows: Suppose that you see the driving license of your classmate Pablo and that it is a Spanish driving license, when Pablo is in fact, unbeknownst to you, Mexican. You have also heard him speaking perfect Spanish. You therefore have strong evidence for the proposition

(a) Pablo is Spanish.

and, on the basis of (a), you believe

(b) One of your classmates is Spanish.

You are thereby justified in believing (b). Also unbeknownst to you, your classmate David is Spanish. Therefore, your justified belief in (b) is true, but it does not constitute knowledge, since (b), from which you inferred (a), is in fact false. Your belief in (b) is true by mere coincidence. Gettier's arguments against JTB invited an interesting discussion among formal epistemologists and philosophers concerned with understanding the correct relation between knowledge and belief, and, in particular, with identifying the exact properties and conditions that distinguishes a piece of belief from a piece of knowledge and vice versa.

This question can be approached from two angles; on the one hand, one can assume that the JTB approach is on the right track and start with the weakest notion of true justified belief, then enhance it by adding new conditions X that render the enhanced analysis JTB+X immune to Gettier counterexamples. On the other hand, one can take a preferred notion of knowledge as primitive and weaken it to obtain a "good" (e.g. consistent, strong, introspective, possibly false) notion of belief. Most of the proposals found in the literature responding to this issue fall under the first approach. An example of this is the *defeasibility analysis of knowledge* proposed by Lehrer and Paxson [34] and later by Klein [32]. They analyzed the Gettier counterexamples and found a common underlying feature: In each case the justification for the agent's belief is *defeasible* by a piece of new true information. They therefore proposed to define knowledge as *true belief with undefeated justification* [34] in order to prevent it from being 'gettiered'. More precisely, according to this approach, knowledge must be resistant to new true information. Other responses to the Gettier challenge include, among others, the *sensitivity account* [39], the *contextualist account* [17] and the *safety account* [42][12].

[11] For a discussion on what justification consists in, we refer the reader to [48].

[12] For an overview of responses to the Gettier challenge and a detailed discussion, we refer the reader to [30, 41].

Moreover, the standard topological semantics for knowledge (based on the interior operator) can be considered to belong to the first category in which knowledge is defined as "correctly justified belief": according to the interior semantics, a proposition P is known iff there exists "true evidence" (i.e. an open set U which includes the real world) that entails P (i.e. $U \subseteq P$).

The second approach, on the other hand, has received much less attention from formal epistemologists than the first one. In fact, the only formal account following it that we are aware of (prior to our own work) is the one given by Stalnaker [44], using a relational semantics for knowledge based on *reflexive, transitive and directed* Kripke models. In his work, Stalnaker analyzes the relation between knowledge and belief and builds a combined modal system for these notions with the axioms extracted from his analysis. He intends to capture a strong notion of belief based on the conception of "subjective certainty"

$$B\varphi \to BK\varphi$$

meaning that *believing implies believing that one knows* [44, p. 179]. Stalnaker refers to this concept as "strong belief", but we prefer to call it *full belief* as in [3][13]. In fact, the above axiom holds biconditionally in his system and belief therefore becomes *subjectively indistinguishable from knowledge: an agent (fully) believes φ iff she (fully) believes that she knows φ* [3]. Moreover, Stalnaker argues that the 'true' logic of knowledge is **S4.2** and that (full) belief can be defined as *epistemic possibility of knowledge.* More precisely,

$$B\varphi = \neg K \neg K \varphi$$

meaning that *an agent believes φ iff she does't know that she does't know φ.*

1.2 Topology and Modal Logic

The first significant work on topological semantics for modal logic was conducted by McKinsey and Tarski [36]. They interpreted the modal operator $\Box$, which in this thesis is denoted by K in the context of an epistemic logic, as the interior operator on topological spaces and showed that **S4** is complete wrt the class of all topological spaces. They further improved this result and showed that **S4** is also the complete logic of any dense-in-itself separable metric space[14].

In general, the work of relating topology and modal logic can be approached from two directions: on the one hand, the primarily interest can lie in spatial structures, in particular, topological spaces and building modal logics as tools to reason about them. In this case, the respective modal logic can be seen as a formal machinery for topological

[13]This terminology was first used in [3] and it was acknowledged that the authors of [3] wanted to avoid clash with the very different notion of strong belief (due to Battigalli and Siniscalchi [9]) that is standard in epistemic game theory and emphasize the similarity between the intuitions behind Stalnaker's notion and ones behind Van Fraassen's probabilistic concept of full belief [24].

[14]In recent years, there has been increase in the work that provides topological completeness proofs of **S4** for some special class of spaces such as for the Cantor space [1,37], the rational line [53] and the real line [1,12]. For a general overview of these results, we refer the reader to [52].

reasoning (see, e.g., [1,11,12,37]). For instance, given that the modality $\Box$ is interpreted as the interior operator and **S4** is the complete logic of all topological spaces, the axiom

$$\Box\varphi \to \varphi$$

of the system **S4** admits a topological interpretation saying that *every subset of a topological space includes its interior*. In this approach, the success of the respective modal logic is judged by how much it helps in reasoning about topological spaces, how it facilitates topological reasoning [52] .

On the other hand, topological spaces can be seen as natural mathematical structures which provide new semantics for some already well-established and interesting modal systems, such as epistemic and doxastic logics (see, e.g. [38, 40, 46, 55]) [15]). Our work in this thesis is in line with the latter approach. In this section, we provide an overview of the features of topological semantics concerning interpretations of knowledge and belief.

Topological Semantics vs. Kripke Semantics for Knowledge. It is well-known that every reflexive and transitive Kripke frame[16] corresponds to an Alexandroff space[17] (see, e.g., [1, 13, 52]). This close connection remains at the level of models as well: the evaluation of a modal formula in a reflexive and transitive Kripke frame coincides with its evaluation in the corresponding Alexandroff space wrt the interior-based semantics. Thus, as topological semantics includes standard Kripke semantics as a particular case in the from of Alexandroff spaces, we can say that interior-based topological semantics generalizes the standard Kripke semantics.

One of the reasons why the interior-based topological semantics provides a nice interpretation for knowledge is that while all topological spaces validate the axioms of **S4**[18], in particular, the axioms

$$K\varphi \to \varphi$$

of Truthfulness of Knowledge and

$$K\varphi \to KK\varphi$$

of Positive Introspection, the philosophically debatable axiom (for reasons mentioned in the previous sections)

$$\neg K\varphi \to K\neg K\varphi$$

[15]In their work [38], Moss and Parikh pursue a slightly different direction: while interior-based topological semantics for modal logic is based on a unimodal language interpreted on *topological spaces*, Moss and Parikh consider a bimodal language interpreted on a larger class of spaces, namely, subset spaces. They interpret the two modalities in their system as *knowledge* and *effort*, respectively, and the subsets of a space as the *possible observations*. For the interested reader we refer to [38, 40].

[16]Reflexive and transitive Kripke frames are also called **S4**-frames, since **S4** is sound and complete wrt the class of reflexive and transitive Kripke frames.

[17]This connection will be further explained in later chapters

[18]Recall that **S4** can be assumed to be the weakest logic for knowledge.

of Negative Introspection is not valid on all topological spaces. Moreover, just as for the standard Kripke semantics, we can obtain completeness results for stronger logics of knowledge such as **S4.2** and **S5** wrt some interesting classes of topological spaces (see, e.g. Chapter 3 or [52, p. 253]). Another argument in favor of topological models for logics of knowledge is of a more 'semantic' nature. Dating back to the 1930's, topological models have been used for an epistemic interpretation in the context of intuitionistic logic [49] where open sets are interpreted as 'pieces of evidence', e.g. about the location of a point [13, 55]. We can employ this idea in the context of logics of knowledge and interpret $K\varphi$ in a topological model as *there exists a piece of evidence (i.e an open set in the corresponding space) which validates* φ.

Another important topological semantics for modal logic is given via the *derived set operator* which was originally suggested by Tarski and McKinsey [36] and mostly developed by Esakia (see, e.g., [10,11,20,21]). In this semantics, the modality $\Box$, which in this thesis is denoted by B in the context of a doxastic logic, is interpreted as the co-derived set operator t on topological spaces[19]. This semantics has been studied from a mathematical perspective for a long time. Several topological completeness results have been provided for the logic **K4** and for its normal extensions [20].

C. Steinsvold [45, 46] was the first to interpret the co-derived set operator as belief. He provides a topological completeness result for the belief logic **KD45** wrt the class of DSO-spaces.

1.3 In This Thesis

In this thesis, we aim to provide topological semantics for belief logics which can also account for the relation between knowledge and belief. To this end, we choose to work with Stalnaker's combined system for knowledge and belief presented in [44]. We think that this system provides an accurate analysis of the relation between knowledge and belief both because the first principles he starts with are very natural and uncontroversial and because it gives a concrete definition of belief in terms of knowledge. Moreover, we claim that our topological belief semantics provides a nice interpretation for Stalnaker's notion of belief as 'subjective certainty'.

We generalize Stalnaker's formalization to a topological setting, making it independent from the relational framework. We then provide *the most general extensional semantics*[21] validating the axioms of Stalnaker's system and prove that it is indeed *topological*. We do it so by extending the interior-based semantics for knowledge with a semantic clause for belief in terms of the closure of interior operator. Furthermore,

[19] Given the derived set operator d on a topological space (X, τ) and a subset $A \subseteq X$, the co-derived set operator t is defined as $t(A) := X \setminus d(X \setminus A)$.

[20] For instance, it has been proven in [20] that the logic **K4** sound and complete for T_d spaces wrt the co-derived set-based semantics. See, e.g., [52] for an overview of the topological completeness results for some normal extensions of **K4**.

[21] An extensional semantics is a semantics that assigns the same meaning to the sentences having the same extension. It takes a meaning of a sentence to be given by U.C.L.A propositions, i.e., a set of possible worlds [3].

we focus on the unimodal fragments of Stalnaker's system having knowledge and belief, respectively, as only modalities and prove that while the complete logic of knowledge in this setting is **S4.2**, the complete logic of belief is **KD45**. We further aim at providing a topological account for static and dynamic belief revision, in particular, for conditional beliefs and update modalities. To this end, we supplement Stalnaker's system with conditional belief and update modalities and further extend our proposed topological semantics with semantic clauses for these modalities in a standard way: we obtain the semantic clause for conditional belief modalities from the semantic clause for the simple belief modality by relativizing the interior and the closure operators and the semantic clause for the update modalities by restricting topological spaces to the proposition representing the new information. We further improve the conditional belief semantics obtained the aforementioned way by using an advantage of working with extremally disconnected spaces. Extremally disconnected spaces allow for an alternative semantic definition of conditional beliefs which is more successful in capturing the rationality postulates of AGM theory in a modal framework.

As mentioned, it is well-known that the interior-based topological semantics and the standard Kripke models are strongly connected. In our topological completeness proofs, we benefit from this connection and use Kripke semantics and relational completeness results of the respective logics. This thesis is structured as follows:

In Chapter 2, we introduce the standard Kripke semantics for logics of knowledge and logics of belief and mention some important frame properties. Moreover, we list relational completeness results relevant to our work.

Chapter 3 aims to provide sufficient topological background for understanding this thesis and introduces the two aforementioned topological approaches to basic modal logic: *the interior semantics* and *the co-derived semantics* together with topological completeness results for the logics of knowledge, **S4** and **S4.2**, wrt the interior semantics and for the logic of belief **KD45** wrt the co-derived semantics.

Chapter 4 and Chapter 5 constitute the original part of this thesis. In Chapter 4 we introduce Stalnaker's combined logic and briefly outline his analysis regarding the relation between knowledge and belief. We then propose a topological semantics for the system, carry it to an extensional framework and show that our proposed topological semantics is the most general extensional semantics validating the axioms of Stalnaker's system. Furthermore, we prove that his system in fact forms a complete axiomatization of extremally disconnected spaces in our setting. We then continue with investigating the unimodal fragments **S4.2** for knowledge and **KD45** for belief of Stalnaker's system, and give topological completeness results for these logics, again wrt the class of extremally disconnected spaces. We also compare our topological belief semantics with Steinsvold's co-derived set semantics.

Chapter 5 focuses on topological semantics for belief revision, assuming the distinction between *static* and *dynamic belief revision* made in, e.g., [4, 5, 50, 57]. It includes two proposals: the *basic* and the *refined* topological semantics for conditional beliefs. The former is obtained from the semantic clause of the (simple) belief modality by relativization, yet, it does not give a 'good' semantics for conditional beliefs in the sense of capturing the rationality postulates of AGM theory. However, by using the properties of extremally disconnected spaces, we improve the first proposal and obtain a 'better' (the refined) semantics for conditional beliefs. We further investigate the natural topological analogue of updates. Our last original result in Chapter 5 is of a more technical nature. The *refined* semantics that we introduce for conditional beliefs invites us to further explore the setting without restricting our models to extremally disconnected spaces. In the last section of this chapter, we define a more complex semantics for the belief modality and provide a completeness result for a weaker logic, **wKD45**, wrt a larger class of spaces, namely, wrt the class of all topological spaces.

Finally we conclude with Chapter 6 by giving a brief summary of this thesis and pointing out number of directions for future research.

NOTE: Work on this thesis resulted in the joint paper *The Topology of Belief, Belief Revision and Defeasible Knowledge* by A. Baltag, N. Bezhanishvili, A. Özgün and S. Smets to appear in *Proceedings of the Fourth International Workshop Logic, Rationality and Interaction (LORI 2013)*. Chapter 4 and parts of Chapter 5, here in particular the sections on the basic topological semantics for conditional beliefs and dynamic belief revision, are based on the work presented in this paper. The proofs of the theorems are missing from the paper and are presented in this thesis.

Chapter 2

A Brief Introduction to Logics of Knowledge and Belief

2.1 Kripke Semantics for Logics of Knowledge and Belief

In the previous chapter, we briefly mentioned some of the important properties of knowledge and belief and the modal systems used in reasoning about them. In this section, we introduce the formal setting properly and supply the most standard semantics, i.e. Kripke semantics, for those logics. Here and throughout the thesis we focus on the single agent case. The reader familiar with the topic can skip this chapter.

We start with the logics of knowledge by recalling the standard unimodal language $\mathcal{L}_K$. The language $\mathcal{L}_K$ has a countable set of propositional letters Prop, Boolean operators $\neg, \wedge$ and a modal operator K. The language $\mathcal{L}_K$ is then given by following grammar

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi$$

where $p \in \mathrm{Prop}$. Abbreviations for the connectives $\vee$, $\rightarrow$ and $\leftrightarrow$ are standard. Moreover, the existential modal operator $\langle K \rangle$ is defined as $\neg K \neg$ and $\bot := p \wedge \neg p$.

Definition 1 (Kripke Frame/Model)**.** *A Kripke frame $\mathcal{F} = (X, R)$ is a pair where X is a non-empty set and R is a binary relation on X. A Kripke model $\mathcal{M} = (X, R, \nu)$ is a tuple where (X, R) is a Kripke frame and ν is a valuation, i.e. a map $\nu : \mathrm{Prop} \rightarrow \mathcal{P}(X)$.*

Elements of X are called *states* or *possible worlds* and R is known as the *accessibility relation.*

Definition 2 (Standard Kripke Semantics)**.** *Let $\mathcal{M} = (X, R, \nu)$ be a Kripke model and x be a state in X. The truth of modal formulas at a world x in $\mathcal{M}$ is defined recursively as:*

$$\begin{array}{lll}
\mathcal{M}, x \models p & \textit{iff} & x \in \nu(p) \\
\mathcal{M}, x \models \neg\varphi & \textit{iff} & \textit{not } \mathcal{M}, x \models \varphi \\
\mathcal{M}, x \models \varphi \wedge \psi & \textit{iff} & \mathcal{M}, x \models \varphi \textit{ and } \mathcal{M}, x \models \psi \\
\mathcal{M}, x \models K\varphi & \textit{iff} & (\forall y \in X)(xRy \rightarrow \mathcal{M}, y \models \varphi)
\end{array}$$

It is useful to note that

$$\mathcal{M}, x \models \langle K \rangle \varphi \quad \text{iff} \quad (\exists y \in X)(xRy \wedge \mathcal{M}, y \models \varphi).$$

We say that φ is *true* in $\mathcal{M}$ if it is true in all the states of $\mathcal{M}$. We say that φ is *valid* in a Kripke frame $\mathcal{F}$ if it is true in every model based on $\mathcal{F}$. Finally, we say that φ is valid in a class of Kripke frames if it is valid in every member of the class.

We let $\|\varphi\|^{\mathcal{M}} = \{x \in X : \mathcal{M}, x \models \varphi\}$ and call $\|\varphi\|^{\mathcal{M}}$ the *extension* of the modal formula φ in $\mathcal{M}$. In other words, we write $x \in \|\varphi\|^{\mathcal{M}}$ for $\mathcal{M}, x \models \varphi$[1]. We skip the index when it is clear from the context. It is useful to give the equivalent definitions of truth and validity on Kripke models in terms of extensional notation:

- φ is true in $\mathcal{M} = (X, R, \nu)$ if $\|\varphi\|^{\mathcal{M}} = X$,
- φ is valid in $\mathcal{F} = (X, R)$ if $\|\varphi\|^{\mathcal{M}} = X$ for all Kripke models $\mathcal{M}$ based on $\mathcal{F}$, and
- φ is valid in a class of Kripke frames if φ is valid in every member of the class.

As mentioned in the introduction, some well-known modal axioms characterize some properties of the accessibility relation of a Kripke frame. Here, we only introduce the relational properties that are relevant to our work. They are given in the following Table[2]:

Name of the Property	Condition on R
Reflexivity	$\forall x(xRx)$
Symmetry	$(\forall x, y)(xRy \to yRx)$
Transitivity	$(\forall x, y, z)(xRy \wedge yRz \to xRz)$
Seriality	$(\forall x)(\exists y)(xRy)$
Euclidean-ness	$(\forall x, y, z)(xRy \wedge xRz \to yRz)$
Directedness	$(\forall x, y, z)(xRy \wedge xRz) \to (\exists w)(yRw \wedge zRw)$
Connectedness	$(\forall x, y)(xRy \vee yRz)$

Table 1: Frame Conditions

We discussed some of the modal axioms together with their interpretation regarding knowledge and belief in the introduction. The following table provides a complete list of the knowledge axioms that we will use to in the remainder of the thesis together with their traditional names.

[1] For the most parts we will use the extensional notation in the proofs, since a similar notation for topological semantics renders it more intuitive.

[2] For a more extensive presentation of relational properties, we refer to [14–16].

Name	Axiom
K	$K(\varphi \to \psi) \to (K\varphi \to K\psi)$
T	$K\varphi \to \varphi$
4	$K\varphi \to KK\varphi$
5	$\langle K\rangle\varphi \to K\langle K\rangle\varphi$
.2	$\langle K\rangle K\varphi \to K\langle K\rangle\varphi$
.3	$K(K\varphi \to K\psi) \vee K(K\psi \to K\varphi)$

Table 2: Knowledge Axioms

As mentioned before, the axioms T, 4, 5 are known as *Truthfulness of Knowledge*, *Positive Introspection* and *Negative Introspection* respectively in the context of an epistemic logic.

A very important property of Kripke semantics is that each of the above axioms corresponds to a constraint on the accessibility relation of a Kripke frame such as the ones listed in Table 1. For instance, the T-axiom is valid on a Kripke frame $\mathcal{F}$ if and only if its accessibility relation is reflexive. The logics of knowledge we have considered and the corresponding frame conditions of Kripke frames wrt which they are sound and complete are listed in the following table. The logics are listed in increasing order of strength.

Logic of Knowledge	Axioms	Frame Condition
S4	K + T + 4	Reflexivity + Transitivity
S4.2	K + T + 4 + .2	Reflexivity + Transitivity +Directedness
S4.3	K + T + 4 + .3	Reflexivity + Transitivity + Connectedness
S5	K + T + 4 + 5	Reflexivity + Transitivity + Symmetric
Inference Rules		
Modus Ponens	From φ and $\varphi \to \psi$ infer ψ	
Necessitation	From φ infer $K\varphi$	

Table 3: Some Soundness and Completeness Results

We read the table in the following manner: **S4.3** *is sound and complete wrt the class of reflexive, transitive and connected frames.* The reader can find a complete presentation of the soundness and completeness results mentioned above in, e.g., [14–16].

The standard Kripke semantics for logics of belief can be presented the same way as for logics of knowledge. For belief logics, we work with the unimodal language $\mathcal{L}_B$ obtained by replacing B for K in $\mathcal{L}_K$. The Kripke semantics for the language $\mathcal{L}_B$ is exactly the same as it is for $\mathcal{L}_K$.

As mentioned in the introduction, the standard logic of belief, and the one we assume in this thesis, is the system **KD45**, whose axioms and inference rules are given in the following table:

Name	Axiom
K	$B(\varphi \to \psi) \to (B\varphi \to B\psi)$
D	$B\varphi \to \langle B \rangle \varphi$
4	$B\varphi \to BB\varphi$
5	$\langle B \rangle \varphi \to B\langle B \rangle \varphi$
Inference Rules	
Modus Ponens	From φ and $\varphi \to \psi$ infer ψ
Necessitation	From φ infer $B\varphi$

Table 4: The system **KD45**

In the presence the other axioms, the D-axiom is equivalent to $\neg B\bot$[3] and called *Consistency of Belief*.

Since the Kripke semantics for the language $\mathcal{L}_B$ is defined the same way as for $\mathcal{L}_K$, we have a similar correspondence between the axioms of **KD45** and properties of the accessibility relation on Kripke frames. The frame conditions corresponding to the axiom 4 and 5 are already given in Table 4. In addition to this, the D-axiom is valid in a Kripke frame $\mathcal{F}$ iff its accessibility relation satisfies seriality. Moreover, the belief system **KD45** is sound and complete wrt the class of all serial, transitive and Euclidean Kripke frames (see, e.g [14]).

2.2 Belief Revision Theory

2.2.1 Static Belief Revision: AGM Theory vs. Conditional Beliefs

AGM Theory

In their seminal work [2], Alchourrón, Gärdenfors and Makinson proposed a syntactic approach, known as *AGM theory*, to the theory of belief change. In this system, the beliefs of an agent are represented as a logically closed, consistent set of sentences (her belief set) from a given propositional language $\mathcal{L}_0$. Belief revision is defined as an operation $*$ that associates with every belief set T and formula $\varphi \in \mathcal{L}_0$ a new belief set $T * \varphi$ representing the agent's revised beliefs[4]. The authors of [2] then imposed certain constraints on this operation, known as AGM Postulates for revision, in order to capture the belief change of a *rational* agent. Since they take the belief set of an agent to be a consistent set of sentences, they already capture the idea that a rational

[3] $(B\varphi \to \neg B\neg\varphi) \leftrightarrow (\neg B\varphi \vee \neg B\neg\varphi) \leftrightarrow \neg(B\varphi \wedge B\neg\varphi) \leftrightarrow \neg B(\varphi \wedge \neg\varphi)$.

[4] In fact, AGM theory captures three different kinds of belief change: revision, expansion and contraction. However, only revision is of interest to us in the context of this thesis.

agent should have consistent beliefs. They further postulate the following conditions:

(1)	Closure :	$T * \varphi$ is a belief set
(2)	Success :	$\varphi \in T * \varphi$
(3)	Inclusion :	$T * \varphi \subseteq T + \varphi$
(4)	Preservation :	If $\neg\varphi \notin T$ then $T + \varphi \subseteq T * \varphi$
(5)	Vacuity :	$T * \varphi$ is inconsistent iff $\vdash \neg\varphi$
(6)	Extensionality :	If $\vdash \varphi \leftrightarrow \psi$ then $T * \varphi = T * \psi$
(7)	Subexpansion :	$T * (\varphi \wedge \psi) \subseteq (T * \varphi) + \psi$
(8)	Superexpansion :	If $\neg\psi \notin T * \varphi$ then $(T * \varphi) + \psi \subseteq T * (\varphi \wedge \psi)$

where the *expansion* $T + \varphi$ of a belief set T with a sentence φ is defined as $T + \varphi = \{\psi : T \cup \{\varphi\} \vdash \psi\}$.

The syntactic approach of AGM turns out to be too limited to explicitly model the beliefs and higher-order beliefs of an introspective agent. To be able to capture the properties of belief introspection, AGM should be able to talk about 'beliefs about beliefs', i.e. higher-order beliefs. However, while $\varphi \in T$ for some belief set T and formula $\varphi \in \mathcal{L}_0$, we can express neither $B\varphi \in T$ nor $\neg B\varphi \in T$ (since the belief sets by design include only sentences from a propositional language $\mathcal{L}_0$). Moreover, even if we allow for higher-order beliefs in belief sets, the AGM postulates are too weak to retain the consistency of the revised belief set of an introspective agent. This is illustrated by the Paradox of Serious Possibility: *suppose φ is a consistent sentence and the agent believes neither φ nor $\neg\varphi$. By negative introspection, we then have $\neg B\varphi \in T$ and $\neg B\neg\varphi \in T$, where T is the agent's initial belief set. Hence, by preservation, we have $\neg B\varphi \in T * \varphi$. Now let the agent learn φ. Then, positive introspection and the success postulate yield $B\varphi \in T * \varphi$, contradicting the consistency of $T * \varphi$, i.e. the closure postulate*[5] [58, p. 59].

One way of defending AGM theory is by simply accepting that it can only model the beliefs about *ontic facts* and cannot deal with higher-order beliefs. On the other hand, it can be saved as a coherent theory for belief revision by using the distinction between *static* and *dynamic belief revision* made in [4, 5, 50, 57]. Against this background, we can interpret $*$ as static belief revision operator: $T * \varphi$ captures the beliefs of an agent after learning φ about what was the case before the learning, i.e. AGM theory is static in the sense that it captures *the agent's changing beliefs about an unchanging world* [5]. This *static* interpretation of AGM theory is captured by *conditional beliefs* in a modal framework, in the style of dynamic epistemic logic.

Conditional Beliefs

In Dynamic Epistemic Logic, static belief revision captures the agent's revised beliefs about how the world was before learning new information and is implemented by conditional belief operators $B^{\varphi}\psi$. Using van Benthem's terminology, "[c]onditional beliefs

[5] Another example concerning the limitations of AGM theory with regard to belief introspection is the Moore Paradox which results from the conflict between the Success Postulate and positive introspection (see, e.g., [58, p. 60]).

pre-encode beliefs that we would have if we learnt certain things." [50, p. 139]. The statement $B^{\varphi}\psi$ says that *if the agent would learn φ, then she would come to believe that ψ was the case before the learning* [5, p. 12]. That means conditional beliefs are hypothetical by nature, hinting at possible future belief changes of the agent. The semantics for conditional beliefs is generally given in terms of plausibility models (or equivalently, in terms of sphere models) [5, 50, 54].

Plausibility Models. A *plausibility model* $\mathcal{M} = (X, \leq, \nu)$ is simply a Kripke model where $\leq$ is a well-founded[6], reflexive and transitive accessibility relation. For any non-empty subset $P \subseteq X$, the set of minimal elements belonging to P is defined as follows:

$$min_{\leq}P := \{y \in P : y \leq x \text{ for all } x \in P\}.$$

In plausibility models, the accessibility relation is called the *plausibility relation* and for any $x, y \in X$, $x \leq y$ is read as "the state x is at least as plausible as y". $min_{\leq}P$ therefore is the set of the most plausible worlds in P.

We can now introduce semantics for knowledge, belief and conditional beliefs on plausibility models. Let $\mathcal{L}$ be the modal language obtained by adding to $\mathcal{L}_K$ the modalities B and B^{φ} for belief and conditional beliefs. The clauses for the propositional variables and Boolean connectives are defined the same way as in standard Kripke semantics. For the knowledge K and the belief B modalities we put

$$\begin{array}{lll} x \in \|K\varphi\|^{\mathcal{M}} & \text{iff} & \|\varphi\|^{\mathcal{M}} = X \\ x \in \|B\varphi\|^{\mathcal{M}} & \text{iff} & min_{\leq}X \subseteq \|\varphi\|^{\mathcal{M}}. \end{array}$$

The interpretation of knowledge and belief on plausibility models is thus different from the one on standard Kripke models: while *knowledge* is interpreted as "truth in all possible worlds", *belief* is interpreted as "truth in the most plausible worlds". Moreover, the semantic clause for conditional beliefs $B^{\varphi}\psi$ is obtained from the semantic clause of the belief modality in a natural way by *relativizing* the minimal states (the most plausible worlds) of the model to the set of worlds in which the new information φ is true (i.e. to the $\|\varphi\|^{\mathcal{M}}$-worlds):

$$x \in \|B^{\varphi}\psi\|^{\mathcal{M}} \quad \text{iff} \quad min_{\leq}\|\varphi\|^{\mathcal{M}} \subseteq \|\psi\|^{\mathcal{M}}$$

where $min_{\leq}\|\varphi\|^{\mathcal{M}} = min_{\leq}X \cap \|\varphi\|^{\mathcal{M}}$. We again emphasize that the semantics for none of the modalities K, B and B^{φ} on plausibility models is given by the standard Kripke semantics introduced in the previous section.

Given a plausibility model $\mathcal{M} = (X, \leq, \nu)$ and a state $x \in X$, we can define the current belief set T of an agent at state x as

$$T = \{\psi \in \mathcal{L} : x \in \|B\psi\|^{\mathcal{M}}\}$$

and the revised belief set $T * \varphi$ as

$$T * \varphi = \{\psi \in \mathcal{L} : x \in \|B^{\varphi}\psi\|^{\mathcal{M}}\}.$$

[6] A binary relation R is well-founded iff every non-empty subset of X has a minimal element wrt R.

Hence, saying that ψ *belongs to the agent's revised theory* $T * \varphi$ *at state* x boils down to saying that *the agent believes* ψ *conditional on* φ *at state* x [5]. This way, i.e by interpreting AGM revision as *static conditioning*, we can in fact embed AGM theory into the following complete modal system **CDL**.

The logic of conditional beliefs (CDL) [4, 5][7]. The syntax of **CDL** is given by

$$\varphi := \ p \mid \neg\varphi \mid \varphi \wedge \varphi \mid B^{\varphi}\varphi$$

and the semantics is given on plausibility models as above. In this system, knowledge and belief are defined as $K\varphi := B^{\neg\varphi}\varphi$ and $B\varphi := B^{\top}\varphi$, where $\top := \neg(p \wedge \neg p)$ is some tautological sentence. A sound and complete system of **CDL** (wrt plausibility models) is given as follows:

The inference rules and axioms of propositional logic	
Necessitation Rule:	From $\vdash \varphi$ infer $\vdash B^{\psi}\varphi$
Normality:	$B^{\theta}(\varphi \to \psi) \to (B^{\theta}\varphi \to B^{\theta}\psi)$
Truthfulness of Knowledge:	$K\varphi \to \varphi$
Persistence of Knowledge:	$K\varphi \to B^{\theta}\varphi$
Strong Positive Introspection:	$B^{\theta}\varphi \to KB^{\theta}\varphi$
Strong Negative Introspection:	$\neg B^{\theta}\varphi \to K\neg B^{\theta}\varphi$
Success of Belief Revision:	$B^{\varphi}\varphi$
Consistency of Revision:	$\neg K\neg\varphi \to \neg B^{\varphi}\bot$
Inclusion:	$B^{\varphi\wedge\psi}\theta \to B^{\varphi}(\psi \to \theta)$
Rational Monotonicity:	$B^{\varphi}(\psi \to \theta) \wedge \neg B^{\varphi}\neg\psi \to B^{\varphi\wedge\psi}\theta$

In later chapters, we propose topological semantics for conditional beliefs and judge its success in representing AGM theory by checking the validity of the **CDL** axioms within the new semantics.

2.2.2 Dynamic Belief Revision: Updates

Unlike conditional beliefs, a belief update captures the agent's beliefs about a world as it is after the update. In Dynamic Epistemic Logic, update is represented by dynamic modalities such as $[!\varphi]B\psi$, meaning that *the agent would come to believe* ψ *is the case (in the world after the learning) after learning* φ. An agent *updates* her beliefs when she receives "hard information", i.e. when the information comes from an infallible and truthful source.

We now add *dynamic modalities* $[!\varphi]\psi$ associated with updates to the language, where $[!\varphi]\psi$ means that *if* φ *is true then after the agent learns it* ψ *becomes true.* $\langle!\varphi\rangle$ is defined in the usual way as $\neg[!\varphi]\neg$ and $\langle!\varphi\rangle\psi$ is read as φ *is true and after the agent learns it* ψ *becomes true.*

[7] This system was first introduced in [4] with common knowledge and common belief operators. We work with the simplified version introduced in [5].

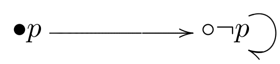
An update with a piece of *true* information φ is modeled by simply deleting the worlds in which φ is not true from the current model $\mathcal{M}$, i.e. by *restricting* $\mathcal{M}$ to the $\|\varphi\|^{\mathcal{M}}$-worlds. Given a plausibility model $\mathcal{M} = (X, \leq, \nu)$ and a formula φ, we let

$$\mathcal{M}_{\varphi} = (\|\varphi\|, \leq_{\|\varphi\|}, \nu_{\|\varphi\|})$$

be the restricted model where $\|\varphi\| = \|\varphi\|^{\mathcal{M}}$, $\leq_{\|\varphi\|} = \leq \cap \|\varphi\| \times \|\varphi\|$ and $\nu_{\|\varphi\|}(p) = \nu(p) \cap \|\varphi\|$ for each propositional variable p. The semantics for an update modality $[!\varphi]\psi$ is given in the following way on a plausibility model $\mathcal{M} = (X, \leq, \nu)$ at a world $x \in X$:

$$x \in \|[!\varphi]\psi\|^{\mathcal{M}} \text{ iff } x \in \|\varphi\|^{\mathcal{M}} \text{ implies } x \in \|\psi\|^{\mathcal{M}_{\varphi}}.$$

Why plausibility models are preferred over Kripke models for belief update: Consider the following two-state **KD45** Kripke model $\mathcal{M}$ [50, pp. 137-138]:



modeling the case where p is true and the agent believes $\neg p$[8]. Suppose the agent learns p from an infallible source, i.e. the agent receives the hard information p. Then, after the update, we obtain the relativized model $\mathcal{M}_p$

$$\bullet p$$

with an empty accessibility relation. The agent therefore comes to believe *everything*, in particular $\perp$, contradicting the principle Consistency of Belief. On the other hand, on a plausibility model $\mathcal{M}'$ we have $min_{\leq}\|\varphi\|^{\mathcal{M}'} \neq \emptyset$ for any new *true* information φ, since plausibility models are well-founded. More precisely, it is never the case that $min_{\leq}\|\varphi\|^{\mathcal{M}'} \subseteq \|\perp\|^{\mathcal{M}'_{\varphi}}$ on a plausibility model $\mathcal{M}'$. Hence, the agent never comes to believe $\perp$ after learning true information φ[9].

[8] $\bullet$ in the figure represents the actual world.

[9] $\|\perp\|^{\mathcal{M}} = \emptyset$ for all plausibility and Kripke models $\mathcal{M}$.

Chapter 3

Topological Semantics for Modal Logic

3.1 Topological Preliminaries

We start by introducing the basic topological concepts that will be used throughout this thesis. For a more detailed discussion of general topology we refer the reader to [18,19].

Definition 3 (Topological Space)**.** *A topological space $\mathcal{X} = (X, \tau)$ is a pair consisting of a set X and a family τ of subsets of X satisfying the following conditions:*

(O1) $\emptyset \in \tau$ *and* $X \in \tau$.

(O2) *If* $U_1 \in \tau$ *and* $U_2 \in \tau$, *then* $U_1 \cap U_2 \in \tau$.

(O3) *If* $\mathcal{A} \subseteq \tau$, *then* $\bigcup \mathcal{A} \in \tau$.

The set X is called *space*, the elements of X are called *points* of the space. The subsets of X belonging to τ are called *open sets* (or *opens*) in the space; the family τ of open subsets of X is also called a *topology* on X. If for some $x \in X$ and an open $U \subseteq X$ we have $x \in U$, we say that U is an *open neighborhood* of x. The complements of opens are called *closed* in the space.

A point x is called an *interior point* of a set $A \subseteq X$ if there is an open neighborhood U of x such that $U \subseteq A$. The set of all interior points of A is called the *interior* of A and denoted by $\mathrm{Int}(A)$. We can then easily observe that for any $A \subseteq X$, $\mathrm{Int}(A)$ is the largest open subset of A. Dually, for any $x \in X$, x belongs to $\mathrm{Cl}(A)$ if and only if $U \cap A \neq \emptyset$ for each open neighborhood U of x. $\mathrm{Cl}(A)$ is called the *closure* of A. It is not hard to see that $\mathrm{Cl}(A)$ is the smallest closed set containing A. We call Int and Cl the *interior operator* and the *closure operator* of $\mathcal{X}$, respectively.

A point $x \in X$ is called a *limit point* (or accumulation point) of a set $A \subseteq X$ if for each open neighborhood U of x, we have $A \cap (U \setminus \{x\}) \neq \emptyset$. The set of all limit points of A is called the *derived set* of A and denoted by $d(A)$. For any $A \subseteq X$, we also let $t(A) = X \setminus d(X \setminus A)$ and call $t(A)$ the *co-derived set* of A. It is easy to see that $x \in t(A)$

if and only if there exists an open neighborhood U of x such that $U \subseteq A \setminus \{x\}$. We call d the *derived set operator* and t the *co-derived set operator* of $\mathcal{X}$. Moreover, a set $A \subseteq X$ is called *dense-in-itself* if $A \subseteq d(A)$ and a space $\mathcal{X}$ is called *dense-in-itself* if $X = d(X)$.

3.2 The Interior Semantics for Modal Logic

In this section, we aim at providing the formal background for the aforementioned interior-based topological semantics that started with the work of McKinsey and Tarski [36]. In this semantics, the universal modal operator is interpreted as the interior operator on topological spaces. Referring to this fact, we call the semantics the *interior semantics*. While presenting some important completeness results (concerning logics of knowledge) of previous works, we also explain the connection between the interior semantics and standard Kripke semantics and focus on the topological (evidence-based) interpretation of knowledge.

Definition 4 (Topological Model)**.** *A topological model (or simply a topo-model)* $\mathcal{M} = (X, \tau, \nu)$ *is a triple where* (X, τ) *is a topological space and* $\nu : \mathrm{Prop} \to \mathcal{P}(X)$ *is a valuation function.*

Since our focus is the topological interpretation of knowledge, we work with the uni-modal epistemic language $\mathcal{L}_K$ introduced in Chapter 2.

Definition 5 (Interior Semantics)**.** *Let* $\mathcal{M} = (X, \tau, \nu)$ *be a topo-model and* x *be a point in* X*. The truth of modal formulas at a world* x *in* $\mathcal{M}$ *is defined recursively as:*

$$
\begin{array}{lll}
\mathcal{M}, x \models p & \textit{iff} & x \in \nu(p) \\
\mathcal{M}, x \models \neg\varphi & \textit{iff} & \textit{not } \mathcal{M}, x \models \varphi \\
\mathcal{M}, x \models \varphi \wedge \psi & \textit{iff} & \mathcal{M}, x \models \varphi \textit{ and } \mathcal{M}, x \models \psi \\
\mathcal{M}, x \models K\varphi & \textit{iff} & (\exists U \in \tau)(x \in U \wedge \forall y \in U,\ \mathcal{M}, y \models \varphi)
\end{array}
$$

It is useful to note a pointwise definition for the semantics of $\langle K \rangle$:

$$\mathcal{M}, x \models \langle K \rangle\varphi \quad \text{iff} \quad (\forall U \in \tau)(x \in U \to \exists y \in U,\ \mathcal{M}, y \models \varphi)$$

Truth and *validity* of a modal formula φ are defined the same way as for standard Kripke semantics. We let $[\![\varphi]\!]^{\mathcal{M}} = \{x \in X \mid \mathcal{M}, x \models \varphi\}$ denote the *extension* of a modal formula φ in $\mathcal{M}$ (where $\mathcal{M}$ is a topo-model and $\models$ refers to the interior semantics)[1]. We skip the index when it is clear from context. It is now easy to see that $[\![K\varphi]\!] = \mathrm{Int}([\![\varphi]\!])$ and $[\![\langle K \rangle\varphi]\!] = \mathrm{Cl}([\![\varphi]\!])$. We use this extensional notation throughout the thesis as it makes clear the fact that we interpret the modalities in terms of a specific topological operator. In particular, as stated above, in the case of the interior semantics we interpret K as the interior operator and, dually, $\langle K \rangle$ as the closure operator.

[1] Recall that we denote the extension of a modal formula φ in a *Kripke model* $\mathcal{M}$ by $\|\varphi\|^{\mathcal{M}}$. The reader should be aware of this distinction, especially for the proofs in which we use both the interior and the Kripke semantics.

We are now ready to introduce the topological soundness and completeness results of the epistemic logics **S4** and **S4.2** which are of particular interest to us.

It is well-known that the interior (Int) and the closure (Cl) operators of a topological space $\mathcal{X} = (X, \tau)$ satisfy the following properties (the so-called Kuratowski axioms) for any $A, B \subseteq X$ (see, e.g., [19, pp. 14-15]):

(I1) $\mathrm{Int}(X) = X$	(C1) $\mathrm{Cl}(\emptyset) = \emptyset$
(I2) $\mathrm{Int}(A) \subseteq A$	(C2) $A \subseteq \mathrm{Cl}(A)$
(I3) $\mathrm{Int}(A \cap B) = \mathrm{Int}(A) \cap \mathrm{Int}(B)$	(C3) $\mathrm{Cl}(A \cup B) = \mathrm{Cl}(A) \cup \mathrm{Cl}(B)$
(I4) $\mathrm{Int}(\mathrm{Int}(A)) = \mathrm{Int}(A)$	(C4) $\mathrm{Cl}(\mathrm{Cl}(A)) = \mathrm{Cl}(A)$

Moreover, as in the case for modal operators K and $\langle K \rangle$, Int and Cl are duals of each other:

$$\mathrm{Int}(A) = X \setminus \mathrm{Cl}(X \setminus A).$$

Given the interior semantics, it is not hard to see that above properties (Kuratowski axioms) of the interior operator are the axioms of the system **S4** written in topological terms. This implies the soundness of **S4** with respect to the class of all topological spaces in the interior semantics[2]. In order to give the topological completeness result for **S4** which was first proven in [36], we further investigate the connection between standard Kripke semantics and the interior semantics.

Connection between Kripke models and topo-models.

Definition 6 (Alexandroff space)**.** *A topological space $\mathcal{X} = (X, \tau)$ is called Alexandroff if τ is closed under arbitrary intersections, i.e., $\bigcap \mathcal{A} \in \tau$ for any $\mathcal{A} \subseteq \tau$.*

A very important feature of an Alexandroff space $\mathcal{X}$ is that every point $x \in X$ has a smallest open neighborhood around it. It is well-known that there is a one-to-one correspondence between reflexive and transitive Kripke frames and Alexandroff spaces [13, 40, 52]. Given a reflexive and transitive Kripke frame $\mathcal{F} = (X, R)$, we can construct a topological space, indeed an Alexandroff space, $\mathcal{X} = (X, \tau_R)$ by defining τ_R to be the set of all upsets[3] of $\mathcal{F}$. $R(x) = \{y \in X \mid xRy\}$ forms the smallest open neighborhood containing the point x. Conversely, for every topological space (X, τ), the relation R_τ defined by

$$xR_\tau y \text{ iff } x \in \mathrm{Cl}(\{y\})$$

is reflexive and transitive. (X, R_τ) hence constitutes a reflexive and transitive Kripke frame. We also have that $R = R_{\tau_R}$ if and only if $\mathcal{X}$ is Alexandroff [7, 52].

The very same connection also exists between reflexive and transitive Kripke models and Alexandroff topo-models: the extension of a modal formula $\varphi \in \mathcal{L}_K$ in a reflexive and transitive Kripke frame coincides with its extension in the corresponding Alexandroff space.

[2] See [13, 40, 52] for a more detailed discussion of the topological soundness of **S4**.

[3] A set $A \subseteq X$ is called an *upset* of (X, R) if for each $x, y \in X$, xRy and $x \in A$ imply $y \in A$.

Proposition 1 (Parikh et al., 2007). *For all reflexive and transitive Kripke models* $\mathcal{M} = (X, R, \nu)$ *and all* $\varphi \in \mathcal{L}_K$,

$$\|\varphi\|^{\mathcal{M}} = [\![\varphi]\!]^{\mathcal{M}_{\tau_R}}$$

where $\mathcal{M}_{\tau_R} = (X, \tau_R, \nu)$.

Proof. By induction on the complexity of φ, see [40, p. 306]. □

Theorem 1. S4 *is sound and complete wrt the class of all topological spaces in the interior semantics.*

Proof. The soundness proof is a routine check and in fact immediately follows from the Kuratowski axioms for the interior operator (see, e.g., [52, p. 237] for a detailed proof). For completeness, let $\varphi \in \mathcal{L}_K$ such that φ is not a theorem of **S4**, i.e., $\mathbf{S4} \nvdash \varphi$. Then, by the relational completeness of **S4**, there exists a reflexive and transitive Kripke model $\mathcal{M} = (X, R, \nu)$ such that $\|\varphi\|^{\mathcal{M}} \neq X$. Hence, by Proposition 1, we have that $[\![\varphi]\!]^{\mathcal{M}_{\tau_R}} \neq X$ where $\mathcal{M}_{\tau_R} = (X, \tau_R, \nu)$ is the corresponding topo-model[4].

□

Since we will mainly work with the logic **S4.2** in later chapters, we also elaborate on the soundness and completeness of **S4.2** in the interior semantics.

Definition 7 (Extremally Disconnected Space). *A topological space* $\mathcal{X} = (X, \tau)$ *is called extremally disconnected if the closure of each open subset of* X *is open.*

Recall that **S4.2** is a *strengthening* of **S4** defined as

$$\mathbf{S4.2} = \mathbf{S4} + (\langle K\rangle K\varphi \rightarrow K\langle K\rangle\varphi)$$

where $L + \varphi$ is the smallest logic including L and φ.

Proposition 2. *For any topological space* $\mathcal{X}$,

$$\langle K\rangle K\varphi \rightarrow K\langle K\rangle\varphi \text{ is valid in } \mathcal{X} \text{ iff } \mathcal{X} \text{ is extremally disconnected}$$

Proof. [52, p. 253]

Let $\mathcal{X} = (X, \tau)$ be a topological space and $\mathcal{M} = (X, \tau, \nu)$ be a topo-model on $\mathcal{X}$. Then,

$$\begin{aligned} [\![\langle K\rangle K\varphi \rightarrow K\langle K\rangle\varphi]\!]^{\mathcal{M}} = X \quad & \text{iff} \quad \mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]^{\mathcal{M}})) \subseteq \mathrm{Int}(\mathrm{Cl}([\![\varphi]\!]^{\mathcal{M}})) \\ & \text{iff} \quad \mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]^{\mathcal{M}})) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]^{\mathcal{M}}))) \\ & \text{iff} \quad (X, \tau) \text{ is extremally disconnected.} \end{aligned}$$

□

Just as in the case for **S4**, we can prove the topological completeness of **S4.2** by using its relational completeness wrt the reflexive, transitive and directed Kripke models.

[4] A more elegant proof via *topo-canonical models* can be found in [52].

Proposition 3. *For any reflexive, transitive and directed Kripke frame (X, R), the corresponding Alexandroff space (X, τ_R) is extremally disconnected.*

Proof. Let (X, R) be a reflexive, transitive and directed Kripke frame and $U \in \tau_R$. We want to show that $\mathrm{Cl}(U) \in \tau_R$. As stated before, the elements of τ_R are the upsets of (X, R). Hence, it suffices to show that $\mathrm{Cl}(U)$ is an upset. Let $x, y \in X$ such that $x \in \mathrm{Cl}(U)$ and xRy. $x \in \mathrm{Cl}(U)$ implies that $V \cap U \neq \emptyset$ for every open neighborhood V of x. Recall that (X, τ_R) is an Alexandroff space. Hence, $R(x)$ is the smallest open neighborhood of x. Thus, $x \in \mathrm{Cl}(U)$ implies, in particular, that $R(x) \cap U \neq \emptyset$. Then, there exists a $z \in X$ such that $z \in R(x)$, i.e. xRz, and $z \in U$. Since R is directed, xRy and xRz, there exists a $w \in X$ such that yRw and zRw. yRw means that $w \in R(y)$. Moreover, as R being transitive and $xRyRw$, we have xRw. Then, $w \in U$ (since $x \in U$ and U is an upset). Thus, $w \in R(y) \cap U$ implying $R(y) \cap U \neq \emptyset$. Since $R(y)$ is the smallest open neighborhood of y, we have that $V' \cap U \neq \emptyset$ for every open neighborhood V' of y. Therefore, $y \in \mathrm{Cl}(U)$. □

Theorem 2. S4.2 *is sound and complete wrt the class of extremally disconnected spaces in the interior semantics.*

Proof. The soundness result is obtained from soundness of **S4** and Proposition 2. The completeness proof follows from Proposition 3 and Proposition 1 in a similar way as in the proof of Theorem 1. □

In fact, the following more general result about the completeness of the normal extensions of **S4** concerning the connection between topo-models and Kripke models has been proven:

Proposition 4 (van Benthem et al., 2007)**.** *Every normal extension of* **S4** *that is complete with respect to the standard Kripke semantics is also complete with respect to the interior semantics.*

As implied by Proposition 4, the standard Kripke semantics is a particular case of the interior semantics. A relational completeness together with a soundness result in the interior semantics for **S4**, more generally, for any normal extension of **S4**, directly yields the topological completeness result in the interior semantics.

3.2.1 Evidential-Based Interpretation of Knowledge

As very briefly mentioned in the introduction, the interior semantics provides a deeper insight into the evidence-based interpretation of knowledge. We can interpret opens in a topological model as 'pieces of evidence' and, in particular, open neighborhoods of a state x as the pieces of *true* (*sound, correct*) evidence that are observable by the agent at state x. Opens being closed under finite intersection captures the ability of an agent to combine finitely many pieces of evidence into a single piece. If an open set U is included in the extension of a proposition φ in a topo-model $\mathcal{M}$, i.e. if $U \subseteq [\![\varphi]\!]^{\mathcal{M}}$, we say that *the piece of evidence U entails (supports, justifies) the proposition φ.* Recall that, for any topo-model $\mathcal{M} = (X, \tau, \nu)$, any $x \in X$ and any $\varphi \in \mathcal{L}_K$, we have

$$x \in [\![K\varphi]\!]^{\mathcal{M}} \text{ iff } (\exists U \in \tau)(x \in U \wedge U \subseteq [\![\varphi]\!]^{\mathcal{M}}).$$

Thus, taking open sets as pieces of evidence and in fact open neighborhoods of a point x as *true* pieces of evidence (that the agent can observe at x), we obtain the following evidence-based interpretation for knowledge: *the agent knows φ iff she has a true piece of evidence U that justifies φ.* In other words, *knowing* φ is the same as *having a correct justification for φ.* The necessary and sufficient conditions for one's belief to qualify as knowledge consist in it being not only truthful, but also in having a correct (evidential) justification. Therefore, the interior semantics implements the widespread intuitive response to Gettier's challenge: knowledge is *correctly* justified belief (rather than being simply true justified belief) [3].

3.3 The Co-derived Semantics for Belief

In this section, we introduce another topological semantics in which the existential modality (denoted by $\langle B \rangle$) of the language of basic modal logic is interpreted as the derived set operator d and, dually, B as the co-derived set operator t[5]. Since the universal modal operator B is interpreted as the co-derived set operator on topo-models, we call this semantics the *co-derived semantics*, analogous to the case for the interior semantics and the knowledge modality K. In his recent work [45], Steinsvold proposed a doxastic interpretation for the *co-derived semantics* and proved that the belief logic **KD45** is sound and complete wrt the class of DSO-spaces[6].
As our focus is the topological interpretation of belief, we work with the unimodal doxastic language $\mathcal{L}_B$ introduced in Chapter 2.

Definition 8 (Co-derived Semantics)**.** *Let $\mathcal{M} = (X, \tau, \nu)$ be a topo-model and x be a point in X. The truth of modal formulas at a world x in $\mathcal{M}$ is defined recursively as:*

$\mathcal{M}, x \models_d p$	*iff*	$x \in \nu(p)$
$\mathcal{M}, x \models_d \neg\varphi$	*iff*	*not* $\mathcal{M}, x \models_d \varphi$
$\mathcal{M}, x \models_d \varphi \wedge \psi$	*iff*	$\mathcal{M}, x \models_d \varphi$ *and* $\mathcal{M}, x \models_d \psi$
$\mathcal{M}, x \models_d B\varphi$	*iff*	$(\exists U \in \tau)(x \in U \wedge \forall y \in U \setminus \{x\},\ \mathcal{M}, y \models_d \varphi)$

A pointwise definition for the semantics of $\langle B \rangle$ is then given as:

$$\mathcal{M}, x \models_d \langle B \rangle\varphi \quad \text{iff} \quad (\forall U \in \tau)(x \in U \to \exists y \in U \setminus \{x\},\ \mathcal{M}, y \models_d \varphi)$$

In order to clarify in which topological semantics we work, we say *d-true, d-valid, d-sound, d-complete*, etc. (following the notation in [11] and [52]) when we work in the co-derived semantics.

[5]This semantics was also first suggested by McKinsey and Tarski in [36], and then mainly developed by Esakia and his colleagues (see, e.g., [11, 20, 21]).

[6]For the interested reader, Steinsvold, in his work [47], also gives completeness proofs for extensions of the logic **K4** via canonical models.

d-truth and *d-validity* of modal formulas in the co-derived semantics are defined the same way as in the interior semantics.

We again let $[\![\varphi]\!]_d^{\mathcal{M}} = \{x \in X \mid \mathcal{M}, x \models_d \varphi\}$ for the *d-extension* of a modal formula $\varphi \in \mathcal{L}_B$ and skip the indices if they are clear from context. It is easy to see that $[\![B\varphi]\!] = t([\![\varphi]\!])$ and $[\![\langle B\rangle\varphi]\!] = d([\![\varphi]\!])$.

We can now focus on the main issue: soundness and completeness. It is well-known that the co-derived set and the derived set operators of a topological space $\mathcal{X}$ satisfy the following properties for each $A, B \subseteq X$:

(t1) $t(X) = X$	(d1) $d(\emptyset) = \emptyset$
(t2) $t(A \cap B) = t(A) \cap t(B)$	(d2) $d(A \cup B) = d(A) \cup d(B)$
(t3) $A \cap t(A) \subseteq t(t(A))$	(d3) $d(d(A)) \subseteq A \cup d(A)$

Recall that the logic **wK4** (weak **K4**) is defined as

$$\mathbf{wK4} = \mathbf{K} + ((\varphi \wedge B\varphi) \to BB\varphi).$$

Just as in the case of **S4**, the properties of the co-derived set operator match exactly the axioms of **wK4**. This proves the d-soundness of **wK4** wrt the class of all topological spaces. Moreover, it has been proven in [20] that **wK4** is d-complete wrt the class of all topological spaces.

We finish this section by introducing a recently established d-completeness result of the system **KD45** wrt the class of *DSO*-spaces [45].

d-Completeness of KD45 wrt the class of *DSO*-spaces.

Definition 9 (T_d-space[7]). *A topological space $\mathcal{X} = (X, \tau)$ is called a T_d-space if every singleton is the intersection of an open and a closed set. Equivalently, $\mathcal{X}$ is a T_d-space if and only if $d(d(A)) \subseteq d(A)$ for each $A \subseteq X$.*

We call a space $\mathcal{X}$ a *DSO-space* if it is dense-in itself and $d(A)$ is open for every $A \subseteq X$ [13, 40, 45]. A nice example for a *DSO*-space is the topology $(\mathbb{N}, \tau)$ where $\mathbb{N}$ is the set of natural numbers and $\tau = \{\emptyset, \text{all cofinite sets}\}$ [13, p. 32]. Then, for each $A \subseteq \mathbb{N}$, we have

$$d(A) = \begin{cases} \emptyset & \text{if } A \text{ is finite} \\ \mathbb{N} & \text{otherwise} \end{cases}$$

As in Kripke semantics, each of the axioms D, 4 and 5 corresponds to a property of a topological space within the co-derived semantics:

Proposition 5 (Parikh et al., 2007). *For any topological space $\mathcal{X}$,*

1. *$B\varphi \to \langle B\rangle\varphi$ is d-valid in $\mathcal{X}$ iff $\mathcal{X}$ is dense-in-itself,*

2. *$B\varphi \to BB\varphi$ is d-valid in $\mathcal{X}$ iff $\mathcal{X}$ is a T_d-space,*

[7] T_d-spaces are also known as $T_{\frac{1}{2}}$-spaces (see, e.g., [19]).

3. $\neg B\varphi \to B\neg B\varphi$ *is d-valid in* $\mathcal{X}$ *iff for all* $A \subseteq X, d(A)$ *is open.*

Proof. See [40, p. 335], Theorem 6.27. □

Proposition 6 (Parikh et al., 2007)**.** *Every DSO-space is a* T_d*-space.*

Proof. See [40, p. 333], Proposition 6.24. □

Theorem 3. KD45 *is d-sound and d-complete wrt the class of DSO-spaces.*

Proof. The d-soundness follows from Proposition 5 and Proposition 6. The proof of the d-completeness of **KD45** is rather intricate and, for the purpose of this thesis, its details are not relevant. We refer the reader to [40, 45, 47] for the proof. □

Chapter 4

The Topology of Full Belief and Knowledge

4.1 Stalnaker's Combined Logic of Knowledge and Belief

In his paper [44], Stalnaker focuses on the properties of belief and its relation with knowledge and proposes an interesting analysis. Most research in the formal epistemology literature concerning the relation between knowledge and belief, in particular, dealing with the attempt to provide a definition of the one in terms of the other, takes belief as a primitive notion and tries to determine additional properties which render a piece of belief knowledge (see, e.g., [17, 32, 34, 39, 41]). In contrast, Stalnaker chooses to start with a notion of knowledge and weakens it to have a "good" notion of belief. He *initially* considers knowledge to be an **S4**-type modality and analyzes belief based on the conception of "subjective certainty": *if the agent believes φ she believes that she knows it.* To remind the reader, in this thesis we refer to Stalnaker's notion of belief as "full belief"[1].

The bimodal language $\mathcal{L}_{KB}$ of knowledge and full belief is given by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid B\varphi$$

where $p \in \text{Prop}$. Abbreviations for the connectives $\rightarrow$ and $\leftrightarrow$ are standard. The existential modalities $\langle K \rangle$ and $\langle B \rangle$ are defined as $\neg K \neg$ and $\neg B \neg$ respectively. We call Stalnaker's system, given in the following table, **KB**:

[1] In [44] he calls it "strong belief".

	Stalnaker's Epistemic-Doxastic Axioms	
(K)	$K(\varphi \to \psi) \to (K\varphi \to K\psi)$	Knowledge is additive
(T)	$K\varphi \to \varphi$	Knowledge implies truth
(KK)	$K\varphi \to KK\varphi$	Positive introspection for K
(CB)	$B\varphi \to \neg B \neg\varphi$	Consistency of belief
(PI)	$B\varphi \to KB\varphi$	(Strong) positive introspection of B
(NI)	$\neg B\varphi \to K\neg B\varphi$	(Strong) negative introspection of B
(KB)	$K\varphi \to B\varphi$	Knowledge implies Belief
(FB)	$B\varphi \to BK\varphi$	Full Belief
	Inference Rules	
(MP)	From φ and $\varphi \to \psi$ infer ψ.	Modus Ponens
(K-Nec)	From φ infer $K\varphi$.	Necessitation

Table 5: Stalnaker's System **KB**

Stalnaker takes the axioms of the logic **S4** as his axioms for knowledge and, based on his analysis of the relationship between knowledge and belief, adds above combined principles to his logic. He argues that our agents, as they are idealized, should have introspective access to their own beliefs: *the agents know what they believe and what not.* In a sense, "...beliefs are *conscious*, in the sense of *known*" [56, p. 272]. This principle is captured by (PI) and (NI). Recall that the analogous principle for knowledge, namely Negative Introspection for Knowledge, is considered to be unreasonable because of its consequences together with the principle Truth of Knowledge [28, 35, 44]. Since belief is assumed to be non-factive, the corresponding principle $\neg B\varphi \to K\neg B\varphi$ does not encounter the same problem. Moreover, it is reasonable to assume and commonly accepted that our logically omniscient agents have consistent beliefs, which is captured by the axiom (CB). Given that knowledge is a stronger notion than belief and belief is more subjective than knowledge, it is quite uncontroversial (at least for our idealized agents) that *the agents believe everything that they know*, as the axiom (KB) says. Lastly, (FB) captures the intended meaning of full belief as *subjective certainty.*

Proposition 7 (Stalnaker)**.** *All axioms of the standard belief system* **KD45** *are provable in the system* **KB***. More precisely, the axioms*

(K) $B(\varphi \to \psi) \to (B\varphi \to B\psi)$

(D) $B\varphi \to \langle B\rangle\varphi$

(4) $B\varphi \to BB\varphi$

(5) $\neg B\varphi \to B\neg B\varphi$

are provable in **KB***.*

Proposition 7 thus says that Stalnaker's logic **KB** of knowledge and full belief yields the belief logic **KD45**. Moreover, one can define belief in terms of knowledge in this system.

Proposition 8. *The following equivalence is provable in the system* **KB***:*

$$B\varphi \leftrightarrow \langle K\rangle K\varphi$$

Proof.

($\Rightarrow$) $B\varphi \rightarrow \langle K\rangle K\varphi$

1.	$K\neg K\varphi \rightarrow B\neg K\varphi$	Ax.(KB)
2.	$B\neg K\varphi \rightarrow \neg BK\varphi$	Ax.(CB)
3.	$\neg BK\varphi \rightarrow \neg B\varphi$	Ax.(FB)
4.	$K\neg K\varphi \rightarrow \neg B\varphi$	Propositional tautology and MP
5.	$B\varphi \rightarrow \langle K\rangle K\varphi$	Contraposition, 4

($\Leftarrow$) $\langle K\rangle K\varphi \rightarrow B\varphi$

1.	$\neg B\varphi \rightarrow K\neg B\varphi$	Ax.(NI)
2.	$\neg B\varphi \rightarrow \neg K\varphi$	Ax.(KB)
3.	$K(\neg B\varphi \rightarrow \neg K\varphi)$	K-Nec, 2
4.	$K(\neg B\varphi \rightarrow \neg K\varphi) \rightarrow (K\neg B\varphi \rightarrow K\neg K\varphi)$	Ax.(K)
5.	$K\neg B\varphi \rightarrow K\neg K\varphi$	MP, 3, 4
6.	$\neg B\varphi \rightarrow K\neg K\varphi$	Propositional tautology and MP
7.	$\langle K\rangle K\varphi \rightarrow B\varphi$	Contraposition, 6

□

Proposition 8 in fact constitutes one of the most important features of Stalnaker's combined system **KB**. This equivalence allows us to have a combined logic of knowledge and belief in which the only modality is K and the belief modality B is defined in terms of it. We therefore obtain "...a more economical formulation of the combined belief-knowledge logic..." [44, p. 179]. Moreover, substituting $\langle K\rangle K\varphi$ for $B\varphi$ in the axiom (CB) results in the modal axiom

$$\langle K\rangle K\varphi \rightarrow K\langle K\rangle\varphi$$

also known as the (.2)-axiom in the modal logic literature [14]. Recall that we obtain the logic of knowledge **S4.2** by adding the (.2)-axiom to the system **S4**. If we substitute $\langle K\rangle K$ for B in all the other axioms of **KB**, they turn out to be theorems of **S4.2** [44]. Therefore, given the equivalence $B\varphi \leftrightarrow \langle K\rangle K\varphi$, we can obtain the unimodal logic of knowledge **S4.2** by substituting $\langle K\rangle K$ for B in all the axioms of **KB** implying that the logic **S4.2** by itself forms a *unimodal* combined logic of knowledge and belief[2]. Stalnaker then argues that his analysis of the relation between knowledge and belief suggests that the "true" logic of knowledge should be **S4.2** and that belief can be defined as the *epistemic possibility of knowledge: the agent believes* φ *if and only if it is possible, for all that she knows, that she knows* φ.

We will elaborate on the unimodal fragment $\mathcal{L}_B$ for belief of $\mathcal{L}_{KB}$ later in more detail. We now introduce a topological semantics for **KB**.

[2]We already pointed out in Chapter 2 that the logic **S4.2** is sound and complete wrt the class of extremally disconnected spaces in the interior semantics [52] (also see Chapter 2).

4.2 Topological Semantics for KB

In this section, we introduce a new topological semantics for the language $\mathcal{L}_{KB}$, which is an extension of the interior semantics for knowledge with a new topological semantics for belief given by the *closure of the interior operator.*

Definition 10 (Topological Semantics for Full Belief and Knowledge). *Let $\mathcal{M} = (X, \tau, \nu)$ be a topological model. The semantics for the formulas in $\mathcal{L}_{KB}$ is defined for Boolean cases and $K\varphi$ the same way as in the interior semantics. The semantics for $B\varphi$ is defined as*

$$[\![B\varphi]\!]^{\mathcal{M}} = \mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]^{\mathcal{M}})).$$

Truth and validity of a formula is defined the same way as in the interior semantics.

We now generalize the above semantics given on topological spaces to an extensional framework independent from topologies and show that the most general extensional (and compositional) semantics validating the axioms of the system **KB** is again topological and based on extremally disconnected spaces.

Definition 11 (Extensional Semantics for $\mathcal{L}_{KB}$). *An extensional (and compositional) semantics for the language $\mathcal{L}_{KB}$ of knowledge and full belief is a triple $(X, \mathrm{K}, \mathrm{B})$, where X is a set of possible worlds and $\mathrm{K} : \mathcal{P}(X) \to \mathcal{P}(X)$ and $\mathrm{B} : \mathcal{P}(X) \to \mathcal{P}(X)$ are unary operations on (sub)sets of worlds.*

Any extensional semantics $(X, \mathrm{K}, \mathrm{B})$, together with a valuation $\nu : Prop \to \mathcal{P}(X)$, gives us an extensional model $\mathcal{M} = (X, \mathrm{K}, \mathrm{B}, \nu)$, in which we can interpret the formulas φ of $\mathcal{L}_{KB}$ in the obvious way: the clauses for propositional connectives are the same as in the topological semantics above, and the remaining cases are given by

$$\begin{array}{lcl} [\![K\varphi]\!]^{\mathcal{M}} & = & \mathrm{K}[\![\varphi]\!]^{\mathcal{M}} \\ [\![B\varphi]\!]^{\mathcal{M}} & = & \mathrm{B}[\![\varphi]\!]^{\mathcal{M}}. \end{array}$$

As usual, a formula $\varphi \in \mathcal{L}_{KB}$ is valid in an extensional semantics $(X, \mathrm{K}, \mathrm{B})$ if $[\![\varphi]\!]^{\mathcal{M}} = X$ for all extensional models $\mathcal{M}$ based on $(X, \mathrm{K}, \mathrm{B})$.

Our proposed topological semantics given by Definition 10 is in fact a special case of the extensional semantics for the language $\mathcal{L}_{KB}$:

Definition 12 (Topological Extensional Semantics). *A topological extensional semantics for the language $\mathcal{L}_{KB}$ is an extensional semantics $(X, \mathrm{K}^{\tau}, \mathrm{B}^{\tau})$, where (X, τ) is a topological space, $\mathrm{K}^{\tau} = \mathrm{Int}^{\tau}$ is the interior operator and $\mathrm{B}^{\tau} = \mathrm{Cl}^{\tau}(\mathrm{Int}^{\tau})$ is the closure of the interior operator with respect to the topology τ.*

We can now state one of the main results of this section; a topological representation theorem for extensional models of **KB**:

Theorem 4. *An extensional semantics $(X, \mathrm{K}, \mathrm{B})$ validates all the axioms and rules of Stalnaker's system* **KB** *iff it is a topological extensional semantics given by an extremally disconnected topology τ on X, such that $\mathrm{K} = \mathrm{K}^{\tau} = \mathrm{Int}^{\tau}$ and $\mathrm{B} = \mathrm{B}^{\tau} = \mathrm{Cl}^{\tau}(\mathrm{Int}^{\tau})$.*

Proof.
($\Leftarrow$) Observe that for any extensional semantics $(X, \mathrm{K}, \mathrm{B})$ and for any $\varphi,\ \psi \in \mathcal{L}_{KB}$,

$$[\![\varphi \to \psi]\!] = X \text{ iff } [\![\varphi]\!] \subseteq [\![\psi]\!].$$

Let $(X, \mathrm{K}, \mathrm{B})$ be a topological extensional semantics given by an extremally disconnected topology τ on X. The validity of the axioms (K), (T), (KK) and the inference rules follows from the soundness of **S4** wrt the interior semantics.

(CB):

$$\begin{aligned} X = [\![B\varphi \to \langle B\rangle\varphi]\!] \quad &\text{iff} \quad [\![B\varphi]\!] \subseteq [\![\langle B\rangle\varphi]\!] \\ &\text{iff} \quad \mathrm{Cl}^\tau(\mathrm{Int}^\tau([\![\varphi]\!])) \subseteq \mathrm{Int}^\tau(\mathrm{Cl}^\tau([\![\varphi]\!])) \\ &\text{iff} \quad \mathrm{Cl}^\tau(\mathrm{Int}^\tau([\![\varphi]\!])) = \mathrm{Int}^\tau(\mathrm{Cl}^\tau(\mathrm{Int}^\tau([\![\varphi]\!]))) \\ &\text{iff} \quad (X, \tau) \text{ is extremally disconnected.} \end{aligned}$$

(PI):

$$\begin{aligned} X = [\![B\varphi \to KB\varphi]\!] \quad &\text{iff} \quad [\![B\varphi]\!] \subseteq [\![KB\varphi]\!] \\ &\text{iff} \quad \mathrm{Cl}^\tau(\mathrm{Int}^\tau([\![\varphi]\!])) \subseteq \mathrm{Int}^\tau(\mathrm{Cl}^\tau(\mathrm{Int}^\tau([\![\varphi]\!]))) \\ &\text{iff} \quad (X, \tau) \text{ is extremally disconnected.} \end{aligned}$$

(NI):

$$\begin{aligned} X = [\![\neg B\varphi \to K\neg B\varphi]\!] \quad &\text{iff} \quad [\![\neg B\varphi]\!] \subseteq [\![K\neg B\varphi]\!] \\ &\text{iff} \quad X \setminus (\mathrm{Cl}^\tau(\mathrm{Int}^\tau([\![\varphi]\!]))) \subseteq \mathrm{Int}^\tau(X \setminus (\mathrm{Cl}^\tau(\mathrm{Int}^\tau([\![\varphi]\!])))) \\ &\text{iff} \quad \mathrm{Int}^\tau(\mathrm{Cl}^\tau(X \setminus [\![\varphi]\!])) \subseteq \mathrm{Int}^\tau(\mathrm{Int}^\tau(\mathrm{Cl}^\tau(X \setminus [\![\varphi]\!]))) \end{aligned}$$

Since $\mathrm{Int}^\tau(\mathrm{Cl}^\tau(X \setminus [\![\varphi]\!])) = \mathrm{Int}^\tau(\mathrm{Int}^\tau(\mathrm{Cl}^\tau(X \setminus [\![\varphi]\!])))$ is true in all topological spaces (by (I4) in Chapter 3), (NI) is valid. The proof for the validity of the axioms (KB) and (FB) are very similar to the ones above and they both are valid in all topological extensional semantics.

($\Rightarrow$) Let $(X, \mathrm{K}, \mathrm{B})$ be an extensional semantics which validates all the axioms of **KB**. As we stated before, the axioms (K), (T), (KK) together with the inference rules of **KB** yields the knowledge logic **S4**. Hence, K satisfies the properties of an interior operator on X, thus, generates a topology τ on X in which $\mathrm{K} = \mathrm{Int}^\tau$ by the Theorem 5.3 in [18, p. 74] (see also Proposition 1.2.9 in [19, p. 23]). Then, since $(X, \mathrm{K}, \mathrm{B})$ validates all the axioms of **KB**, we have $[\![B\varphi \leftrightarrow \langle K\rangle K\varphi]\!]^{\mathcal{M}} = X$ for any model $\mathcal{M} = (X, \mathrm{K}, \mathrm{B}, \nu)$ and for all $\varphi \in \mathcal{L}_{KB}$ by Proposition 8. Hence, $[\![B\varphi]\!]^{\mathcal{M}} = \mathrm{B}[\![\varphi]\!]^{\mathcal{M}} = \mathrm{Cl}^\tau(\mathrm{Int}^\tau([\![\varphi]\!]^{\mathcal{M}})$, i.e., $\mathrm{B} = \mathrm{Cl}^\tau(\mathrm{Int}^\tau)$. Thus, $(X, \mathrm{K}, \mathrm{B})$ is a topological extensional semantics. Finally, the validity of the axiom (CB) proves that (X, τ) is extremally disconnected.

□

Theorem 4 shows that **KB** is just another axiomatization of extremally disconnected spaces in which both the interior (Int) and the closure of the interior (Cl(Int)) are taken to be primitive operators of a topological space (corresponding to the primitive modalities K and B in **KB**, respectively). We also conclude that our topological semantics for full belief and knowledge is the most general extensional semantics validating the axioms of **KB**.

Theorem 5. *The sound and complete logic of knowledge and belief on extremally disconnected spaces is given by Stalnaker's system* **KB**.

Proof. Since axioms of **KB** are Sahlqvist formulas, **KB** is canonical, hence, complete wrt its canonical model. However, the canonical model of **KB** is in fact an extensional model validating all of its axioms. Thus, by Theorem 4, we have that **KB** is sound and complete wrt the class of extremally disconnected spaces. □

4.2.1 Unimodal Logic for Belief: KD45

As mentioned in the previous section, Stalnaker's system **KB** yields the logic of belief **KD45**. In this section, we introduce a new semantics for the unimodal language $\mathcal{L}_B$ in which the *closure of interior* operator is taken to be the only primitive operator. This approach can be seen as the unimodal component of the topological semantics for full belief and knowledge, capturing only the notion of belief. We name our proposed semantics in this section *topological belief semantics*. The main result of this section is the topological soundness and completeness for **KD45** wrt the class of extremally disconnected spaces in the topological belief semantics.

The language $\mathcal{L}_B$ of **KD45** is given by

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid B\varphi$$

and we again denote $\neg B \neg$ with $\langle B \rangle$. Let $\mathcal{M} = (X, \tau, \nu)$ be a topo-model. The semantic clauses for the propositional variables and the Boolean connectives are the same as in the interior semantics. For the modal operator B, we put

$$[\![B\varphi]\!]^{\mathcal{M}} \quad = \quad \mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]^{\mathcal{M}}))$$

and the semantic clause for $\langle B \rangle$ is easily obtained as

$$[\![\langle B \rangle\varphi]\!]^{\mathcal{M}} \; = \; \mathrm{Int}(\mathrm{Cl}([\![\varphi]\!]^{\mathcal{M}})).$$

Throughout this section, we use the notation $[\varphi]^{\mathcal{M}}$ for the extension of a formula $\varphi \in \mathcal{L}_K$ wrt the *interior semantics* in order to make clear in which semantics we work. We reserve the notation $[\![\varphi]\!]^{\mathcal{M}}$ for the extension of the formula $\varphi \in \mathcal{L}_B$ wrt the *topological belief semantics*. We skip the index when confusion is unlikely to occur.

Definition 13 (Translation $(.)^* : \mathcal{L}_B \to \mathcal{L}_K$). *For any $\varphi \in \mathcal{L}_B$, the translation $(\varphi)^*$ of φ into $\mathcal{L}_K$ is defined recursively as follows:*

1. $(\bot)^* = \bot$
2. $(p)^* = p$, *where* $p \in \mathrm{Prop}$
3. $(\neg\varphi)^* = \neg\varphi^*$
4. $(\varphi \wedge \psi)^* = \varphi^* \wedge \psi^*$
5. $(B\varphi)^* = \langle K \rangle K\varphi^*$
6. $(\langle B \rangle\varphi)^* = K\langle K \rangle\varphi^*$

Proposition 9. *For any topo-model* $\mathcal{M} = (X, \tau, \nu)$ *and for any formula* $\varphi \in \mathcal{L}_B$ *we have*

$$[\![\varphi]\!]^{\mathcal{M}} = [\varphi^*]^{\mathcal{M}}.$$

Proof. We prove the lemma by induction on the complexity of φ. The cases for

1. $\varphi = \bot$,
2. $\varphi = p$,
3. $\varphi = \neg\psi$, and
4. $\varphi = \psi \wedge \chi$

are straightforward. Now let $\varphi = B\psi$, then

$$\begin{aligned}
[\![\varphi]\!]^{\mathcal{M}} &= [\![B\psi]\!]^{\mathcal{M}} \\
&= \mathrm{Cl}(\mathrm{Int}([\![\psi]\!]^{\mathcal{M}})) && \text{(by the topological belief semantics for } \mathcal{L}_B\text{)} \\
&= \mathrm{Cl}(\mathrm{Int}([\psi^*]^{\mathcal{M}})) && \text{(by I.H.)} \\
&= [\langle K \rangle K\psi^*]^{\mathcal{M}} && \text{(by the interior semantics for } \mathcal{L}_K\text{.)} \\
&= [(B\psi)^*]^{\mathcal{M}} && \text{(by the translation.)} \\
&= [\varphi^*]^{\mathcal{M}}.
\end{aligned}$$

□

Soundness of KD45

The proof of soundness follows as usual. Recall that

$$\mathbf{KD45} = \mathbf{K} + (B\varphi \to \langle B \rangle\varphi) + (B\varphi \to BB\varphi) + (\langle B \rangle\varphi \to B\langle B \rangle\varphi).$$

Proposition 10. *For any topo-model* $\mathcal{M} = (X, \tau, \nu)$ *and any* $\varphi \in \mathcal{L}_B$ *we have*

1. $[\![B\varphi \to BB\varphi]\!] = X$,
2. $[\![\langle B \rangle\varphi \to B\langle B \rangle\varphi]\!] = X$.

Proof. Let $\mathcal{M} = (X, \tau, \nu)$ be a topo-model and $\varphi \in \mathcal{L}_B$. Note that for any $\varphi, \psi \in \mathcal{L}_B$ we have

$$[\![\varphi \to \psi]\!] = X \text{ iff } [\![\varphi]\!] \subseteq [\![\psi]\!]. \quad (4.1)$$

1. By equation (1), it suffices to show that $[\![B\varphi]\!] \subseteq [\![BB\varphi]\!]$. By our semantics, we have $[\![B\varphi]\!] = \mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]))$ and $[\![BB\varphi]\!] = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]))))$.

   As known, the closure of an open set is a closed domain[3] [19, p. 20]. We then have
   $$\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!])) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!])))).$$
   as $\mathrm{Int}([\![\varphi]\!])$ is open in (X, τ). Therefore, we obtain $[\![B\varphi]\!] = [\![BB\varphi]\!]$ which implies $[\![B\varphi \to BB\varphi]\!] = X$.

2. Similar to part-(1), it suffices to show that $[\![\langle B\rangle\varphi]\!] \subseteq [\![B\langle B\rangle\varphi]\!]$ and the proof follows:
   $$\begin{array}{rcll} [\![\langle B\rangle\varphi]\!] & = & \mathrm{Int}(\mathrm{Cl}([\![\varphi]\!])) & \\ & \subseteq & \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}([\![\varphi]\!]))) & \text{(by (C2))} \\ & = & \mathrm{Cl}(\mathrm{Int}(\mathrm{Int}(\mathrm{Cl}([\![\varphi]\!])))) & \text{(by (I4))} \\ & = & [\![B\langle B\rangle\varphi]\!]. & \end{array}$$
   Therefore, by (1), we have $[\![\langle B\rangle\varphi \to B\langle B\rangle\varphi]\!] = X$.

□

It follows from Proposition 10 that all topological spaces validate the axioms 4 and 5 wrt the topological belief semantics. However, the K-axiom $B\varphi \wedge B\psi \to B(\varphi \wedge \psi)$ and the D-axiom $B\varphi \to \langle B\rangle\psi$ are not valid on all topological spaces. We thus have to restrict the class of topological spaces in order to obtain a topological soundness result for the belief logic **KD45**. Moreover, we use the translation defined above in order to prove the validity of the K-axiom. Recall that for any Kripke model $\mathcal{M}$, we denote the extension of the formula φ in $\mathcal{M}$ wrt the standard Kripke semantics by $\|\varphi\|^{\mathcal{M}}$.

Lemma 1. *For any* $\varphi \in \mathcal{L}_K$*,* $\mathbf{S4.2} \vdash \langle K\rangle K\varphi \wedge \langle K\rangle K\psi \to \langle K\rangle K(\varphi \wedge \psi)$.

Proof. We know that **S4.2** is complete wrt the class of reflexive, transitive and directed Kripke frames. So, it suffices to show that $\langle K\rangle K\varphi \wedge \langle K\rangle K\psi \to \langle K\rangle K(\varphi \wedge \psi)$ is valid on all reflexive, transitive and directed Kripke frames.

Let $\mathcal{F} = (X, R)$ be a reflexive, transitive and directed Kripke frame, $\mathcal{M} = (X, R, \nu)$ be a model on $\mathcal{F}$ and $x \in X$. Suppose $x \in \|\langle K\rangle K\varphi \wedge \langle K\rangle K\psi\|$. Hence, $x \in \|\langle K\rangle K\varphi\|$ and $x \in \|\langle K\rangle K\psi\|$. Then, there exist $y, z \in X$ with xRy and xRz such that $y \in \|K\varphi\|$ and $z \in \|K\psi\|$. By directedness of R, there exists a $w \in X$ such that yRw and zRw. Since $y \in \|K\varphi\|$, $z \in \|K\psi\|$ and R is transitive, $w \in \|K\varphi\|$ and $w \in \|K\psi\|$, i.e., $w \in \|K\varphi \wedge K\psi\|$. So $w \in \|K(\varphi \wedge \psi)\|$ and by reflexivity of R, $x \in \|\langle K\rangle K(\varphi \wedge \psi)\|$. Hence, $\langle K\rangle K\varphi \wedge \langle K\rangle K\psi \to \langle K\rangle K(\varphi \wedge \psi)$ is valid in all reflexive, transitive and directed Kripke frames. Therefore, by the completeness of **S4.2**, we have $\mathbf{S4.2} \vdash \langle K\rangle K\varphi \wedge \langle K\rangle K\psi \to \langle K\rangle K(\varphi \wedge \psi)$. □

[3] A subset A of a topological space is called *closed domain* if $A = \mathrm{Cl}(\mathrm{Int}(A))$ [19, p. 20]. In the literature, a closed domain is also called *regular closed*.

Proposition 11. *A topological space validates the* K-*axiom if it is extremally disconnected.*

Proof. Let (X, τ) be an extremally disconnected space and $\mathcal{M} = (X, \tau, \nu)$ be a topo-model on it. Also suppose $\varphi, \psi \in \mathcal{L}_B$. Then,

$$\begin{aligned} [\![B\varphi \wedge B\psi \rightarrow B(\varphi \wedge \psi)]\!]^{\mathcal{M}} &= [(B\varphi \wedge B\psi \rightarrow B(\varphi \wedge \psi))^*]^{\mathcal{M}} && \text{(by Proposition 9)} \\ &= [\langle K \rangle K\varphi^* \wedge \langle K \rangle K\psi^* \rightarrow \langle K \rangle K(\varphi^* \wedge \psi^*)]^{\mathcal{M}} && \text{(by the translation)} \\ &= X && \text{(by Lemma 1)} \end{aligned}$$

Hence, (X, τ) validates $B\varphi \wedge B\psi \rightarrow B(\varphi \wedge \psi)$.[4] □

Proposition 12. *A topological space (X, τ) validates the* D-*axiom iff (X, τ) is extremally disconnected.*

Proof. See the proof of Theorem 4-(CB). □

It follows from Proposition 12 that the D-axiom is not only valid on extremally disconnected spaces, it also characterizes them wrt the topological belief semantics. Hence, the class of extremally disconnected spaces is the largest class of topological spaces which validates the D-axiom in topological belief semantics. In fact, this result together with the previous ones yields the soundness of **KD45**:

Theorem 6. *The belief logic* **KD45** *is sound wrt the class of extremally disconnected spaces in topological belief semantics. In fact, a topological space (X, τ) validates all the axioms and rules of the system* **KD45** *in the topological belief semantics iff (X, τ) is extremally disconnected.*

Proof. The validity of the axioms of **KD45** follows from Propositions 10, 24 and 12. We only need to show that the inference rules *Necessitation* and *Modus Ponens* preserve validity.

> *Necessitation:* Let $\varphi \in \mathcal{L}_B$ such that $B\varphi$ is not valid. Then, there exists a topo-model $\mathcal{M} = (X, \tau, \nu)$ such that $[\![B\varphi]\!]^{\mathcal{M}} \neq X$, i.e., $\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]^{\mathcal{M}})) \neq X$. Hence, $[\![\varphi]\!]^{\mathcal{M}} \neq X$ implying that φ is not valid.
>
> *MP:* Let $\varphi, \psi \in \mathcal{L}_B$ such that φ and $\varphi \rightarrow \psi$ are valid in all topological spaces. Let (X, τ) be a topological space and $\mathcal{M} = (X, \tau, \nu)$ be a topo-model on (X, τ). By assumption, $[\![\varphi]\!]^{\mathcal{M}} = X$ and $[\![\varphi \rightarrow \psi]\!]^{\mathcal{M}} = X$. Note that $[\![\varphi \rightarrow \psi]\!]^{\mathcal{M}} = (X \setminus [\![\varphi]\!]^{\mathcal{M}}) \cup [\![\psi]\!]^{\mathcal{M}}$. As $[\![\varphi]\!]^{\mathcal{M}} = X$, we obtain $X \setminus [\![\varphi]\!]^{\mathcal{M}} = \emptyset$ implying that $[\![\psi]\!]^{\mathcal{M}} = X$. As (X, τ) has been chosen arbitrarily, ψ is valid in all topological spaces as well.

□

[4] In his paper [31], D.S. Janković provided a series of equivalent conditions characterizing extremally disconnected spaces. One of these conditions is that a topological space (X, R) is extremally disconnected iff for every $U, V \in \tau$ we have $\mathrm{Cl}(U) \cap \mathrm{Cl}(V) = \mathrm{Cl}(U \cap V)$. By using this fact, Proposition 24 can also be proven directly.

Completeness of KD45

We prove the completeness of **KD45** by using the translation of the language $\mathcal{L}_B$ into the language $\mathcal{L}_K$ defined above and the completeness of **S4.2** wrt the class of extremally disconnected spaces in the interior semantics.

For the topological completeness proof of **KD45** we also make use of the completeness of **KD45** and **S4.2** in the standard Kripke semantics. We first recall some frame conditions concerning the relational completeness of the corresponding systems.

Let (X, R) be a transitive Kripke frame. A non-empty subset $C \subseteq X$ is a *cluster* if

(1) for each $x, y \in C$ we have xRy, and

(2) there is no $D \subseteq X$ such that $C \subset D$ and D satisfies (1).

A point $x \in X$ is called a *maximal point* if there is no $y \in X$ such that xRy and $\neg(yRx)$. We call a cluster a *final cluster* if all its points are maximal. It is not hard to see that for any final cluster C of (X, R) and any $x \in C$, we have $R(x) = C$. A transitive Kripke frame (X, R) is called *cofinal* if it has a unique final cluster C such that for each $x \in X$ and $y \in C$ we have xRy. We call a cofinal frame a *brush* if $X \setminus C$ is an irreflexive antichain, i.e., for each $x, y \in X \setminus C$ we have $\neg(xRy)$ where C is the final cluster. A brush with a singleton $X \setminus C$ is called a *pin*. By definition, every brush and every pin is transitive. Finally, a transitive frame (X, R) is called *rooted*, if there is an $x \in X$, called a *root*, such that for each $y \in X$ with $x \neq y$ we have xRy. Hence, every rooted brush is in fact a pin. The following figures illustrate brushes and pins:

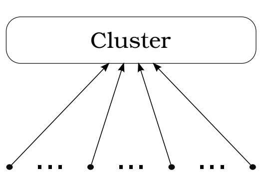


Figure 4.1: Brush

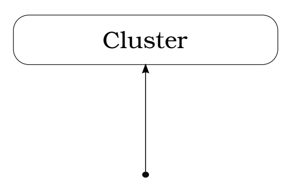


Figure 4.2: Pin

Lemma 2.

1. *Each reflexive and transitive cofinal frame is an* **S4.2***-frame. Moreover,* **S4.2** *is sound and complete wrt the class of finite rooted reflexive and transitive cofinal frames.*

2. *Each brush is a* **KD45***-frame. Moreover,* **KD45** *is sound and complete wrt the class of finite brushes, indeed, wrt the class of finite pins.*

Proof. See, e.g., [15, Chapter 5]. □

For any reflexive and transitive cofinal frame (X, R) we define R_B on X by

$$xR_By \text{ if } y \in C$$

for each $x, y \in X$ where C is the final cluster of (X, R). It is easy to see that for each $x \in X$, we have $R_B(x) = C$.

Lemma 3. *For any reflexive and transitive cofinal frame* (X, R),

1. (X, R_B) *is a brush.*

2. *For any valuation* ν *on* X *and for each formula* $\varphi \in \mathcal{L}_B$ *we have*

$$\|\varphi^*\|^{\mathcal{M}} = \|\varphi\|^{\mathcal{M}_B}$$

where $\mathcal{M} = (X, R, \nu)$ *and* $\mathcal{M}_B = (X, R_B, \nu)$.

Proof. Let (X, R) be a reflexive and transitive cofinal frame.

1. By its definition, R_B is transitive. We can also show that the final cluster C of (X, R) is also a cluster (X, R_B). For each $x, y \in C$, xR_By by definition of R_B. Moreover, suppose for a contradiction that there is a $D \subseteq X$ such that $C \subset D$ and for each $x, y \in D$ we have xR_By. As $C \subset D$, there is an $x_0 \in D$ such that $x_0 \notin C$ contradicting that xR_Bx_0 for all $x \in D$. Hence, C is a cluster of (X, R_B) too. By definition of R_B, we also have that for any $x \in X$, $R_B(x) = C$, i.e., for any $x \in X$ and $y \in C$ we have xR_By. Hence, (X, R_B) is a cofinal frame with the final cluster C. Now consider $X \setminus C$. Suppose there is an $x \in X \setminus C$ such that xR_Bx. This implies, by definition of R_B, that $x \in C$ contradicting our assumption. Hence, each point $x \in X \setminus C$ is irreflexive. Suppose also that $X \setminus C$ is not an antichain, i.e., there exist $x, y \in X \setminus C$ such that either xR_By or yR_Bx. W.l.o.g, assume xR_By. This also implies, by definition of R_B, that $y \in C$ contradicting $y \in X \setminus C$. Hence, (X, R_B) is a brush.

2. We prove this item by induction on the complexity of φ. Let $\mathcal{M} = (X, \tau, \nu)$ be a model on (X, R). The cases for $\varphi = \bot$, $\varphi = p$, $\varphi = \neg\psi$, $\varphi = \psi \wedge \chi$ are straightforward. Let $\varphi = B\psi$.

   ($\subseteq$) Let $x \in \|(B\psi)^*\|^{\mathcal{M}} = \|\langle K\rangle K\psi^*\|^{\mathcal{M}}$. Then, by the standard Kripke semantics, there is a $y \in X$ with xRy such that $R(y) \subseteq \|\psi^*\|^{\mathcal{M}}$. Since (X, R) is a cofinal frame, we have $C \subseteq R(y)$, hence, $C \subseteq \|\psi^*\|^{\mathcal{M}}$. Then, by induction hypothesis, $C \subseteq \|\psi\|^{\mathcal{M}_B}$. Since $R_B(x) = C$ in the brush (X, R_B), we have $R_B(x) \subseteq \|\psi\|^{\mathcal{M}_B}$ implying that $x \in \|B\psi\|^{\mathcal{M}_B}$.

   ($\supseteq$) Let $x \in \|B\psi\|^{\mathcal{M}_B}$. Then, by the standard Kripke semantics, for all $y \in X$ with xR_By we have $y \in \|\psi\|^{\mathcal{M}_B}$, i.e., $R_B(x) \subseteq \|\psi\|^{\mathcal{M}_B}$. Then, $C \subseteq \|\psi\|^{\mathcal{M}_B}$, since $R_B(x) = C$. Hence, by induction hypothesis, $C \subseteq \|\psi^*\|^{\mathcal{M}}$. By definition of a final cluster, we have $R(y) = C$ for any $y \in C$. Hence, $y \in \|K\psi^*\|^{\mathcal{M}}$ for any $y \in C$. Since (X, R) is a cofinal frame, xRy for any $y \in C$. Thus, $x \in \|\langle K\rangle K\psi^*\|^{\mathcal{M}}$, i.e., $x \in \|(B\psi)^*\|^{\mathcal{M}}$.

□

For each Kripke frame (X, R) we let R^+ be the reflexive closure of R.

Lemma 4. *For any brush* (X, R),

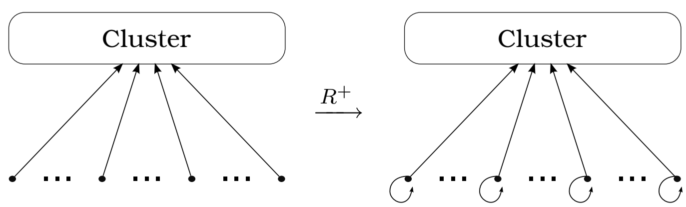
1. (X, R^+) *is a reflexive and transitive cofinal frame.*
2. *For any valuation* ν *on* X *and for each formula* $\varphi \in \mathcal{L}_B$ *we have*

$$\|\varphi\|^{\mathcal{M}} = \|\varphi^*\|^{\mathcal{M}^+}$$

where $\mathcal{M} = (X, R, \nu)$ *and* $\mathcal{M}^+ = (X, R^+, \nu)$.

Proof. Let (X, R) be a brush.

1. Since a brush is also a transitive cofinal frame, (X, R^+) is also transitive and cofinal. Moreover, R^+ is reflexive by definition. Therefore, (X, R^+) is a reflexive and transitive cofinal frame.



2. We prove (2) by induction on the complexity of φ. Let $\mathcal{M} = (X, \tau, \nu)$ be a model on (X, R). The cases for $\varphi = \bot$, $\varphi = p$, $\varphi = \neg\psi$, $\varphi = \psi \wedge \chi$ are straightforward. Let $\varphi = B\psi$.

   ($\subseteq$) Let $x \in \|B\psi\|^{\mathcal{M}}$. Then, by the standard Kripke semantics, for all $y \in X$ with xRy we have $y \in \|\psi\|^{\mathcal{M}}$, i.e., $R(x) \subseteq \|\psi\|^{\mathcal{M}}$. This implies, since $\mathcal{M}$ is a model based on a brush, $C \subseteq \|\psi\|^{\mathcal{M}}$. By I.H., $C \subseteq \|\psi^*\|^{\mathcal{M}^+}$. Since (X, R^+) is in fact just a reflexive brush, $C \subseteq R^+(x)$. Hence there is a $z \in C$ such that xRz and, since $R^+(z) = C$ and $C \subseteq \|\psi^*\|^{\mathcal{M}^+}$, $z \in \|K\psi^*\|^{\mathcal{M}^+}$. Therefore, $x \in \|\langle K\rangle K\psi^*\|^{\mathcal{M}^+} = \|(B\psi)^*\|^{\mathcal{M}^+}$.

   ($\supseteq$) Let $x \in \|(B\psi)^*\|^{\mathcal{M}^+} = \|\langle K\rangle K\psi^*\|^{\mathcal{M}^+}$. Then, by the standard Kripke semantics, there is a $y \in X$ with xR^+y such that $R^+(y) \subseteq \|\psi^*\|^{\mathcal{M}^+}$. Observe that either $y = x$ or xRy (equivalently, $y \in C$).

   If $x = y$, $R^+(y) \subseteq \|\psi^*\|^{\mathcal{M}^+}$ means that $R^+(x) \subseteq \|\psi^*\|^{\mathcal{M}^+}$. Then, since $R(x) \subseteq R^+(x)$, we have $R(x) \subseteq \|\psi\|^{\mathcal{M}}$ by induction hypothesis. Therefore, $x \in \|B\psi\|^{\mathcal{M}}$.

   If xRy, i.e., $y \in C$, we have $R(x) = R^+(y)$. Hence, by induction hypothesis, $R(x) \subseteq \|\psi\|^{\mathcal{M}}$. Therefore, $x \in \|B\psi\|^{\mathcal{M}}$.

□

Theorem 7. *For each formula* $\varphi \in \mathcal{L}_B$,

$$\mathbf{S4.2} \vdash \varphi^* \textit{ iff } \mathbf{KD45} \vdash \varphi$$

.

Proof. Let $\varphi \in \mathcal{L}_B$.

> ($\Rightarrow$) Suppose **KD45** $\nvdash \varphi$. By Lemma 2(2), there exists a Kripke model $\mathcal{M} = (X, R, \nu)$ where (X, R) is a finite pin such that $\|\varphi\|^{\mathcal{M}} \neq X$. Then, by Lemma 4, $\mathcal{M}^+$ is a model based on the finite reflexive and transitive cofinal frame (X, R^+) and $\|\varphi^*\|^{\mathcal{M}^+} \neq X$. Hence, by Lemma 2(1), we have **S4.2** $\nvdash \varphi^*$.
>
> ($\Leftarrow$) Suppose **S4.2** $\nvdash \varphi^*$. By Lemma 2(1), there exists a Kripke model $\mathcal{M} = (X, R, \nu)$ where (X, R) is a finite reflexive and transitive cofinal frame such that $\|\varphi^*\|^{\mathcal{M}} \neq X$. Then, by Lemma 3, $\mathcal{M}_B$ is a model based on the brush (X, R_B) and $\|\varphi\|^{\mathcal{M}_B} \neq X$. Hence, by Lemma 2(2), we have **KD45** $\nvdash \varphi$.

□

Theorem 8. KD45 *is complete wrt the class of extremally disconnected spaces in the topological belief semantics.*

Proof. Let $\varphi \in \mathcal{L}_B$ such that **KD45** $\nvdash \varphi$. By Theorem 7, **S4.2** $\nvdash \varphi^*$. Hence, by topological completeness of **S4.2** wrt the class of extremally disconnected spaces in the interior semantics, there exists a topo-model $\mathcal{M} = (X, \tau, \nu)$ where (X, τ) is an extremally disconnected space such that $[\varphi^*]^{\mathcal{M}} \neq X$. Then, by Proposition 9, $[\![\varphi]\!]^{\mathcal{M}} \neq X$. Thus, we found an extremally disconnected space (X, τ) which refutes φ in the topological belief semantics. Hence, **KD45** is complete wrt the class of extremally disconnected spaces in the topological belief semantics. □

Comparison with Related Work

Although the coincidence between the interior operator and **S4**-type knowledge was realized quite early and has been studied together with its extensions to multi-agent cases [53,55], to common knowledge [53], to logics of learning known as topo-logic [38,40] extensively, Steinsvold was the first to acknowledge the match between the co-derived set operator and belief, and propose a doxastic interpretation for the co-derived semantics in his work [45]. We will now have a critical look at Steinsvold's proposal and compare it with our topological semantics for belief.

As emphasized in the introduction, one of the desirable properties of belief is a negative one; namely the property of its being non-factive. This implies that a right notion of belief should hold the *possibility of error*: it must be possible for an agents to have false beliefs. A good semantics for belief should therefore allow for models and worlds at which some beliefs are false. However, the co-derived semantics entails that *there is at least one false belief in all worlds of every topo-model*, i.e., it demands not only the possibility, but also the *necessity of error*. We take this to be a defect of the co-derived semantics which results from the definition of the derived set operator. Recall that for any topological space (X, τ), any subset $A \subseteq X$ and any $x \in X$, we have

$$x \in d(A) \text{ iff for all open neighborhoods } U \text{ of } x,\ x \in U \cap (A \setminus \{x\}).$$

This implies, in particular that $x \notin d(\{x\})$ for any $x \in X$, i.e., no $x \in X$ is in the derived set of its singleton set $\{x\}$. As is standard, we take the set of states where the proposition P is true to represent this proposition (see, e.g., [5, 22]). More precisely, we say that P is true at x iff $x \in P$. We now consider an arbitrary topological space (X, τ) and the singleton proposition $\{x\}$[5] where $x \in X$. By definition of d, we have $x \notin d(\{x\})$, i.e., $x \notin \langle B \rangle(\{x\})$. Hence, $x \in B(X \setminus \{x\})$ meaning that *the agent believes the proposition* $X \setminus \{x\}$ *at the world* x. However, $X \setminus \{x\}$ is in fact false at x since $x \notin X \setminus \{x\}$. This argument holds for any topological space (X, τ) and any $x \in X$ implying that the co-derived semantics entails the *necessity of error*: *"the actual world is always dis-believed"* [3].

Another objection against the co-derived semantics concerns in the relation between knowledge and belief: *it is vulnerable to Gettier counterexamples.* For any topological space (X, τ) and any $A \subseteq X$, we have

$$\mathrm{Int}(A) = t(A) \cap A.$$

Assuming that the interior operator corresponds to the knowledge modality, we have

$$KP = BP \wedge P$$

for any proposition P. Therefore, the co-derived set interpretation of belief together with the interior-based interpretation of knowledge yields that *knowledge is true belief.* Even if true belief comes with a canonical justification, it can easily be 'gettiered'.

The last argument concerning the advantages of our proposal over the co-derived set semantics is of a technical nature. While the belief logic **KD45** is sound and complete wrt the class of extremally disconnected spaces in the topological belief semantics, it is sound and complete wrt only the class of *DSO*-spaces in the co-derived semantics. Therefore, as the following proposition shows, our topological interpretation of belief works on a larger class of frames than the co-derived semantics does:

Proposition 13. *Every DSO-space is extremally disconnected. However, not every extremally disconnected space is a DSO-space.*

Proof. Let (X, τ) be a *DSO*-space and $U \in \tau$. Recall that for any $A \subseteq X$, $\mathrm{Cl}(A) = d(A) \cup A$. So $\mathrm{Cl}(U) = d(U) \cup U$. Since (X, τ) is a *DSO*-space, $d(U)$ is an open subset of X. Thus, since U is open as well, $d(U) \cup U = \mathrm{Cl}(U)$ is open. Therefore, (X, τ) is an extremally disconnected spaces.

Now consider the topological space (X, τ) where $X = \{1, 2, 3\}$ and $\tau = \{X, \emptyset, \{2\}, \{1, 2\}\}$. It is easy to check that for all $U \in \tau$, $\mathrm{Cl}(U)$ is open (in fact, for each $U \in \tau$, $\mathrm{Cl}(U) = X$). Hence, (X, τ) is an extremally disconnected space. However, as $\mathrm{Cl}(X \setminus \{2\}) = \{1, 3\}$, we have $2 \notin d(X)$. Thus, (X, τ) is not dense in itself and thus not a *DSO*-space. □

[5] We can consider the singleton proposition $\{x\}$ as the complete description of the world x.

Chapter 5

Topological Semantics for Belief Revision

5.1 Static Belief Revision: Conditional Beliefs

As discussed in Chapter 2, AGM theory is not successful in dealing with dynamic belief change and it is in fact assumed to be static in the sense that *it captures the agent's changing beliefs about an unchanging world* [5]. Following [4, 5, 50, 57], we capture "static" belief revision, i.e. AGM theory, in a modal framework by adding *conditional belief* operators $B^{\varphi}\psi$ -which generalize the notion of belief- to our language $\mathcal{L}_{KB}$[1].

As conditional beliefs capture *hypothetical* belief changes of an agent in case she would learn certain things, we can obtain the semantics for a conditional belief modality $B^{\varphi}\psi$ in a natural and standard way by relativizing the semantics for the simple belief modality to the extension of the learnt formula φ. By *relativization* we mean a local change in the sense that it only affects one occurrence of the belief modality $B\varphi$. It does not cause a change in the model (i.e., a global change) due to its static nature. In this section, we investigate the natural topological analogue of modeling static belief revision.

The basic topological semantics for conditional beliefs. For any subset P of a topological space (X, τ), we can generalize the belief operator B on the topological extensional frames given in the previous section by relativizing the closure and the interior operators to the set P. More precisely, we define the *conditional belief operator* $\mathrm{B}^P : \mathcal{P}(X) \to \mathcal{P}(X)$ as

$$\mathrm{B}^{\mathrm{P}}(A) = \mathrm{Cl}(P \cap \mathrm{Int}(P \to A))$$

for any $A \subseteq X$ where $P \to A := (X \setminus P) \cup A$. This immediately gives us a topological semantics for the language $\mathcal{L}_{KCB}$ obtained by adding the conditional belief modalities $B^{\varphi}\psi$ to $\mathcal{L}_{KB}$. Given a topological model $\mathcal{M} = (X, \tau, \nu)$, the additional semantic clause

[1] Conditional beliefs generalize simple beliefs in the sense that the latter can be defined in terms of the former.

reads

$$[\![B^{\varphi}\psi]\!]^{\mathcal{M}} = \mathrm{Cl}([\![\varphi]\!]^{\mathcal{M}} \cap \mathrm{Int}([\![\varphi]\!]^{\mathcal{M}} \rightarrow [\![\psi]\!]^{\mathcal{M}}))$$

where $[\![\varphi]\!]^{\mathcal{M}} \rightarrow [\![\psi]\!]^{\mathcal{M}} := (X \setminus [\![\varphi]\!]^{\mathcal{M}}) \cup [\![\psi]\!]^{\mathcal{M}}$. We refer to this semantics as *the basic topological semantics for conditional beliefs and knowledge.*

Proposition 14. *The following equivalences are valid in all topological spaces wrt the basic topological semantics for conditional beliefs and knowledge*

$$B^{\varphi}\psi \;\leftrightarrow\; \neg K(\varphi \rightarrow \neg K(\varphi \rightarrow \psi)) \;\leftrightarrow\; \langle K\rangle(\varphi \wedge K(\varphi \rightarrow \psi)).$$

Proof. Follows immediately from the semantics of conditional belief modalities and of the knowledge modality. □

Proposition 14 shows that, just like simple beliefs, conditional beliefs can be defined in terms of knowledge. It also constitutes another justification for the semantic clause of the conditional belief operator: the topological semantics for conditional beliefs and knowledge validates above equivalence which generalizes the one[2] for simple beliefs in a natural way. More precisely, as a corollary of above proposition, we have

$$B^{\top}\psi \;\leftrightarrow\; \langle K\rangle(\top \wedge K(\top \rightarrow \psi)) \;\leftrightarrow\; \langle K\rangle K\psi \;\leftrightarrow\; B\psi.$$

Therefore, the logic **KCB** of knowledge and conditional beliefs, **KB**, and even the unimodal fragment of **KB** having K as the only modality, have the same expressive power, since we can define conditional beliefs and simple beliefs in terms of knowledge. We thereby obtain the completeness for **KCB** trivially:

Theorem 9. *The logic* **KCB** *of knowledge and conditional beliefs is axiomatized completely by the system* **S4.2** *for the knowledge modality* K *together with the following equivalences:*

1. $B^{\varphi}\psi \leftrightarrow \langle K\rangle(\varphi \wedge K(\varphi \rightarrow \psi))$
2. $B\varphi \leftrightarrow B^{\top}\varphi$

Proof. The validity of (1) and (2) is given by Proposition 14. While the latter reduces belief to conditional belief, the former reduces conditional beliefs to knowledge. Hence, the proof follows from the topological completeness of **S4.2**. □

As a last observation on definability of the modalities in $\mathcal{L}_{KCB}$ in terms of each other we note that, unlike in the case of simple beliefs, knowledge can be defined in terms of conditional beliefs.

Proposition 15. *The following equivalences are valid in all topological spaces wrt the basic topological semantics for conditional beliefs and knowledge*

$$K\varphi \;\leftrightarrow\; \neg B^{\neg\varphi}\top \;\leftrightarrow\; \neg B^{\neg\varphi}\neg\varphi.$$

[2]The corresponding equivalence for simple beliefs is stated in Proposition 8.

Proof. Follows immediately from the semantics of conditional belief modalities and of the knowledge modality. □

Proposition 15 also implies that the conditional belief operator is not a normal modality, i.e. it does not obey the necessitation rule. In particular, $B^{\varphi}\top$ is not always valid.

Observations on AGM theory. As mentioned in the beginning of the section and explained in Chapter 2, we try to embody AGM theory as static belief revision by adding conditional belief modalities to our language and providing a semantics for them. Now it is time to see how successful our topological analogue of static conditioning is in representing AGM theory and its 8 postulates presented in Chapter 2.

The second equivalence in Proposition 15 implies that the modal formula $B^{\varphi}\varphi$ corresponding to the AGM *Success Postulate* is not valid and that it has to be restricted to new information that is consistent with the agent's knowledge[3]:

Proposition 16. *The formula $\neg K\neg\varphi \rightarrow B^{\varphi}\varphi$ is valid in all topological spaces wrt the basic topological semantics for conditional beliefs and knowledge.*

Proof. Follows immediately from the semantics of conditional belief modalities and of the knowledge modality. □

This actually means that *as long as the new information φ is consistent with the agent's knowledge, the agent could come to believe φ was the case (before the learning) if she would learn φ.*

Another valid axiom in our system is the so-called *Consistency of Revision*:

Proposition 17. *The formula $\neg B^{\varphi}\bot$ is valid in all topological spaces wrt the basic topological semantics for conditional beliefs and knowledge.*

Proof. Follows immediately from the semantics of conditional belief modalities. □

This axiom corresponds to the AGM postulate *Non-Vacuity*. Although this postulate is criticized for being too liberal in case the agent's knowledge is taken into account, above validity successfully captures said postulate of AGM theory.

Ideally, we would like to have all the AGM postulates in the appropriate form stated in terms of conditional beliefs to be valid in our semantics. Our semantics validates some of those, such as Non-vacuity and a restricted version of Success as stated above, however, it does not validate Sub- and Super-expansion[4]. The basic topological semantics for conditional beliefs is thus not optimal in capturing all of the AGM postulates for static belief revision. This motivates the search for an alternative semantics for conditional beliefs which captures more of the AGM postulates and is compatible with the

[3]Proposition 15 implies that $B^{\neg\varphi}\top \leftrightarrow B^{\neg\varphi}\neg\varphi$ is valid. Hence, since $B^{\varphi}\top$ is not always valid, $B^{\varphi}\varphi$ is not valid either. However, we have $\neg K\neg\varphi \leftrightarrow \neg\neg B^{\neg\neg\varphi}\neg\neg\varphi \leftrightarrow B^{\varphi}\varphi$ by Proposition 15.

[4]In modal terms, these postulates correspond to the axioms Inclusion and Rational Monotonicity, respectively, given in Chapter 2.

notion of belief in Stalnaker's system. Fortunately, the definition of extremally disconnected spaces suggests an alternative semantics for conditional beliefs. We now present this *refined* semantics for conditional beliefs and see how well it does in representing the rationality postulates of AGM theory by checking the validity of the axioms of **CDL**.

5.1.1 A 'Refined' Topological Semantics for Conditional Beliefs

We start by recalling some properties of extremally disconnected spaces and topological belief semantics. A topological space (X, τ) is *extremally disconnected* if the closure of every open set in it is open. Equivalently, (X, τ) is extremally disconnected if for any $A \subseteq X$ we have $\mathrm{Cl}(\mathrm{Int}(A)) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$. Hence, given a topological extensional frame $(X, \mathrm{K}^\tau, \mathrm{B}^\tau)$ based on an extremally disconnected topology τ, we obtain

$$\mathrm{B}^\tau(A) = \mathrm{Cl}^\tau(\mathrm{Int}^\tau(A)) = \mathrm{Int}^\tau(\mathrm{Cl}^\tau(\mathrm{Int}^\tau(A)))$$

for any $A \subseteq X$. Therefore, for any subset P of an *extremally disconnected space* (X, τ) we can generalize the belief operator B on topological extensional frames (*based on extremally disconnected topologies*) the same way as we did for topological semantics for conditional beliefs and knowledge, i.e. by relativizing the closure and the interior operators to the set P. However, this time we use the alternative definition Int(Cl(Int)) for the belief operator which is supplied by extremally disconnected spaces. More precisely, we can define the *new conditional belief operator* $\mathrm{B}^P : \mathcal{P}(X) \to \mathcal{P}(X)$ as

$$\mathrm{B}^P(A) = \mathrm{Int}(P \to \mathrm{Cl}(P \cap \mathrm{Int}(P \to A)))$$

for any $A \subseteq X$. This again immediately gives us a topological semantics for the language $\mathcal{L}_{KCB}$. Given a topological model $\mathcal{M} = (X, \tau, \nu)$, the additional semantic clause reads

$$[\![B^\varphi \psi]\!]^\mathcal{M} = \mathrm{Int}([\![\varphi]\!]^\mathcal{M} \to \mathrm{Cl}([\![\varphi]\!]^\mathcal{M} \cap \mathrm{Int}([\![\varphi]\!]^\mathcal{M} \to [\![\psi]\!]^\mathcal{M}))).$$

We consider this semantics an improvement of the basic topological semantics of conditional beliefs and knowledge, since, as we will see in Theorem 11, it is more successful in capturing the rationality postulates of AGM theory. We refer to this semantics as *the refined topological semantics for conditional beliefs and knowledge.* We denote the logic of conditional beliefs and knowledge as **KCB′** in this section, since its axiomatization, which will be given in Theorem 10, is slightly different than the one previously presented.

Proposition 18. *The following equivalence is valid in all topological spaces wrt the refined topological semantics for conditional beliefs and knowledge*

$$B^\varphi \psi \;\leftrightarrow\; K(\varphi \to \langle K \rangle(\varphi \wedge K(\varphi \to \psi))).$$

Proof. Follows immediately from the semantics of conditional belief modalities and of the knowledge modality. □

Proposition 18 is indeed an analogue to Proposition 14 in the refined semantics; it says that conditional beliefs can be defined in terms of knowledge. Moreover, the refined topological semantics for conditional beliefs and knowledge also generalizes the equivalence for simple beliefs given in Proposition 8:

Corollary 1. *The following equivalences are valid in all topological spaces wrt the refined topological semantics for conditional beliefs and knowledge*

$$B^{\top}\psi \;\leftrightarrow\; K(\top \to \langle K\rangle(\top \wedge K(\top \to \psi))) \;\leftrightarrow\; K\langle K\rangle K\psi.$$

Proof. Follows from Proposition 18 and the semantics of belief modality. □

Proposition 19. *The following equivalence is valid in all extremally disconnected spaces wrt the refined topological semantics for conditional beliefs and knowledge*

$$B^{\top}\psi \;\leftrightarrow\; B\psi.$$

Proof. Let (X, τ) be an extremally disconnected space and $\mathcal{M} = (X, \tau, \nu)$ be a model on it. By the semantic clause for K, we have $[\![K\langle K\rangle K\psi]\!]^{\mathcal{M}} = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]^{\mathcal{M}})))$. Then, since (X, τ) is extremally disconnected, $[\![K\langle K\rangle K\psi]\!]^{\mathcal{M}} = \mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]^{\mathcal{M}}))$. Hence, by the semantic clause for B, we have

$$[\![K\langle K\rangle K\psi]\!]^{\mathcal{M}} = \mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]^{\mathcal{M}})) = [\![B\varphi]\!]^{\mathcal{M}}$$

implying that $K\langle K\rangle K\psi \leftrightarrow B\varphi$ is valid on all extremally disconnected spaces. Then, by Corollary 1, we have that $B^{\top}\psi \leftrightarrow B\varphi$ is valid on all extremally disconnected spaces. □

Therefore, we again trivially obtain the completeness result for $\mathbf{KCB'}$:

Theorem 10. *The logic* $\mathbf{KCB'}$ *of knowledge and conditional beliefs is axiomatized completely by the system* **S4.2** *for the knowledge modality* K *together with the following equivalences:*

1. $B^{\varphi}\psi \leftrightarrow K(\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to \psi)))$
2. $B\varphi \leftrightarrow B^{\top}\varphi$

Proof. The validity of (1) and (2) is given by Proposition 18 and 19, respectively. While the latter reduces belief to conditional belief, the former reduces conditional beliefs to knowledge. Hence, the proof follows from the topological completeness of **S4.2**. □

Once again, the logic $\mathbf{KCB'}$ of knowledge and conditional beliefs, Stalnaker's system **KB** of knowledge and full belief, and the unimodal fragment of **KB** having K as the only modality are co-expressive. Furthermore, just as in the previous case, we can define knowledge in terms of conditional beliefs:

Proposition 20. *The following equivalences are valid in all topological spaces wrt the refined topological semantics for conditional beliefs and knowledge*

$$K\varphi \;\leftrightarrow\; B^{\neg\varphi}\bot \;\leftrightarrow B^{\neg\varphi}\varphi.$$

Proof. Follows immediately from the semantics of conditional belief modalities and of the knowledge modality. □

As a corollary of Proposition 20, we can also say that even the unimodal fragment **KCB** having only conditional belief modalities is as expressive as the logics stated above.

AGM theory vs. the refined topological semantics for conditional beliefs

Now it is time to demonstrate what we promised at the beginning of this section, i.e. that the refined semantics provides a better semantics for conditional beliefs in the sense that most of the AGM postulates in the appropriate form are valid in all topological spaces wrt the refined topological semantics for conditional beliefs and knowledge or, in other words, that most of the axioms of the system **CDL** are valid wrt this semantics.

Theorem 11. *The following formulas are valid in all topological spaces wrt the refined topological semantics for conditional beliefs and knowledge*

Normality:	$B^\theta(\varphi \to \psi) \to (B^\theta\varphi \to B^\theta\psi)$
Truthfulness of Knowledge:	$K\varphi \to \varphi$
Persistence of Knowledge:	$K\varphi \to B^\theta\varphi$
Strong Positive Introspection:	$B^\theta\varphi \to KB^\theta\varphi$
Success of Belief Revision:	$B^\varphi\varphi$
Consistency of Revision:	$\neg K\neg\varphi \to \neg B^\varphi\bot$
Inclusion:	$B^{\varphi\wedge\psi}\theta \to B^\varphi(\psi \to \theta)$
Cautious Monotonicity:	$B^\varphi\psi \wedge B^\varphi\theta \to B^{\varphi\wedge\psi}\theta$

Moreover, the Necessitation rule for conditional beliefs:

$$\textit{From } \vdash \varphi \textit{ infer } \vdash B^\psi\varphi$$

preserves validity.

Proof. See proof in the Appendix. □

The validity of the Normality principle and the Necessitation rule shows that, unlike in case of the basic topological semantics for conditional beliefs, the conditional belief modality is a normal modal operator with respect to the refined semantics. Moreover, the refined semantics also validates the Success Postulate without any restriction[5]. However, in this case, we have to restrict the principle of Consistency of Belief Revision to the formulas that are consistent with the agent's knowledge. This is in fact a desirable

[5]Recall that the basic semantics demands to restrict the Success Postulate to the formulas that are consistent with the agent's knowledge.

restriction, as mentioned before, taking into account the agent's knowledge. If the agent knows $\neg\varphi$ with some degree of certainty, she should not revise her beliefs with φ. As conditional beliefs *pre-encode* possible future belief changes of an agent and the future belief changes must be based on the new information consistent with the agent's knowledge, her consistent conditional beliefs must pre-encode the possibilities that are in fact consistent with her knowledge.

More generally, all the axioms of the system **CDL** except for Strong Negative Introspection and Rational Monotonicity are valid on all topological spaces wrt the refined topological semantics for conditional beliefs and knowledge. In fact, the failure of Strong Negative Introspection is an expected result for the following reasons. First of all, observe that Theorem 10 and Theorem 11 imply that all the formulas stated in Theorem 11 are theorems of the system $\mathbf{KCB'}$. Recall that

$$\neg B^{\theta}\varphi \rightarrow K\neg B^{\theta}\varphi$$

is the principle of Strong Negative Introspection. If this principle were a theorem of $\mathbf{KCB'}$, then in particular $\neg B^{\neg\varphi}\varphi \rightarrow K\neg B^{\neg\varphi}\varphi$ would be a theorem of $\mathbf{KCB'}$. Then, by Proposition 20, we would obtain

$$\neg K\varphi \rightarrow K\neg K\varphi$$

as a theorem of $\mathbf{KCB'}$. However, Theorem 10 says that the knowledge modality of $\mathbf{KCB'}$ is an **S4.2**-type modality implying that $\neg K\varphi \rightarrow K\neg K\varphi$ is not a theorem of the system.

Moreover, even the extremally disconnected spaces fail to validate Rational Monotonicity, which captures the AGM postulate of Superexpansion, wrt the refined topological semantics for conditional beliefs and knowledge. However, a weaker principle, namely, the principle of Cautious Monotonicity is valid in all topological spaces. This principle says that *if the agent would come to believe ψ and would also come to believe θ if she would learn φ, her learning ψ should not defeat her belief in θ and vice versa.* In [33], the authors state that D. Gabbay also gives a convincing argument to accept Cautious Monotonicity: "if φ is an enough reason to believe ψ and also to believe θ, then φ and ψ should also be enough to make us believe θ, since φ was enough anyway and, on this basis, ψ was accepted" [33, p. 178].

The refined conditional belief semantics therefore captures the AGM postulates 1-7 together with a weaker version of 8. It is thus more successful than the basic one in modeling static belief change of a rational agent.

5.2 Dynamic Belief Revision: Updates

In Dynamic Epistemic Logic, update (dynamic conditioning) corresponds to change of beliefs through learning hard information. It is the operation of taking the restriction of the model to the set of worlds in which the new information is true [50]. Unlike the case for conditional beliefs, update induces a global change in the model.

The topological analogue of this corresponds to taking the restriction of a topology τ on X to a subset $P \subseteq X$ (see, e.g. [7, 8, 61]). This way, we obtain a *subspace* of a given topological space.

Definition 14 (Subspace)**.** *Given a topological space (X, τ) and a set $P \subseteq X$, a space (P, τ_P) is called a subspace of (X, τ) where $\tau_P = \{U \cap P : U \in \tau\}$.*

We can define the closure operator Cl_{τ_P} and the interior operator Int_{τ_P} of the subspace (P, τ_P) in terms of the closure and the interior operators of the space (X, τ) as follows[6]:

$$\mathrm{Cl}_{\tau_P}(A) = \mathrm{Cl}(A) \cap P$$

$$\mathrm{Int}_{\tau_P}(A) = \mathrm{Int}(P \to A) \cap P.$$

Topological semantics for update modalities. We now add to the language $\mathcal{L}_{KCB}$ new modalities $\langle !\varphi \rangle$ to be thought of as (existential) dynamic modalities associated with updates. $\langle !\varphi \rangle \psi$ means that φ *is true and after the agent learns the new information* φ*,* ψ *becomes true.* The dual $[!\varphi]$ is defined as $\neg \langle !\varphi \rangle \neg$ as usual and $[!\varphi]\varphi$ means that *if* φ *is true then after the agent learns the new information* φ*,* ψ *becomes true.*

Let (X, τ, ν) be a topological model and $P \subseteq X$. We let $\mathcal{M}_P = (P, \tau_P, \nu_P)$ denote a new model where (P, τ_P) is a subspace of (X, τ) and $\nu_P(p) = \nu(p) \cap P$ for each $p \in \mathrm{Prop}$. Given a formula φ, we denote the *restricted model* as

$$\mathcal{M}_\varphi = (\llbracket \varphi \rrbracket, \tau_{\llbracket \varphi \rrbracket}, \nu_{\llbracket \varphi \rrbracket})$$

where $\llbracket \varphi \rrbracket = \llbracket \varphi \rrbracket^{\mathcal{M}}$. Then, the semantics for $\langle !\varphi \rangle \psi$ on a model $\mathcal{M} = (X, \tau, \nu)$ is given as follows for any $x \in X$:

$$x \in \llbracket \langle !\varphi \rangle \psi \rrbracket^{\mathcal{M}} \text{ iff } x \in \llbracket \varphi \rrbracket^{\mathcal{M}} \text{ and } x \in \llbracket \psi \rrbracket^{\mathcal{M}_\varphi}.$$

Hence,

$$x \in \llbracket \langle !\varphi \rangle \psi \rrbracket^{\mathcal{M}} \text{ iff } x \in \llbracket \varphi \rrbracket^{\mathcal{M}} \cap \llbracket \psi \rrbracket^{\mathcal{M}_\varphi}.$$

As $\llbracket \psi \rrbracket^{\mathcal{M}_\varphi} \subseteq \llbracket \varphi \rrbracket$, the semantics for $\langle !\varphi \rangle \psi$ boils down to

$$\llbracket \langle !\varphi \rangle \psi \rrbracket^{\mathcal{M}} = \llbracket \psi \rrbracket^{\mathcal{M}_\varphi} \tag{5.1}$$

Proposition 21. *Let $\mathcal{M} = (X, \tau, \nu)$ be a topo-model, φ be a formula and p be an atomic sentence. Then we have*

$$\llbracket Bp \rrbracket^{\mathcal{M}_\varphi} = \llbracket \varphi \wedge B^\varphi p \rrbracket^{\mathcal{M}}$$

As a consequence, the following formula is valid:

$$\langle !\varphi \rangle Bp \;\leftrightarrow\; (\varphi \wedge B^\varphi p).$$

More generally, the following Reduction Law for belief is valid, for arbitrary formulas φ, ψ:

$$\langle !\varphi \rangle B\psi \;\leftrightarrow\; (\varphi \wedge B^\varphi \langle !\varphi \rangle \psi).$$

[6] See [19, pp. 65-74].

Proof. We will only prove that $\langle !\varphi\rangle B\psi \leftrightarrow (\varphi \wedge B^{\varphi}\langle !\varphi\rangle\psi)$ is valid on any topo-model, the previous equivalences follow from this one.

Let $\mathcal{M} = (X, \tau, \nu)$ be a topo-model and φ, ψ be any two formulas. Then,

$$\begin{array}{rcl}
[\![\langle !\varphi\rangle B\psi]\!]^{\mathcal{M}} & = & [\![B\psi]\!]^{\mathcal{M}_\varphi} \\
& = & \mathrm{Int}_{\tau_{[\![\varphi]\!]}}(\mathrm{Cl}_{\tau_{[\![\varphi]\!]}}(\mathrm{Int}_{\tau_{[\![\varphi]\!]}}([\![\psi]\!]^{\mathcal{M}_\varphi}))) \\
& = & \mathrm{Int}_{\tau_{[\![\varphi]\!]}}(\mathrm{Cl}_{\tau_{[\![\varphi]\!]}}(\mathrm{Int}_{\tau_{[\![\varphi]\!]}}([\![\langle !\varphi\rangle\psi]\!]^{\mathcal{M}}))) \\
& = & \mathrm{Int}_{\tau_{[\![\varphi]\!]}}(\mathrm{Cl}_{\tau_{[\![\varphi]\!]}}(\mathrm{Int}([\![\varphi]\!]^{\mathcal{M}} \rightarrow [\![\langle !\varphi\rangle\psi]\!]^{\mathcal{M}}) \cap [\![\varphi]\!]^{\mathcal{M}})) \\
& = & \mathrm{Int}_{\tau_{[\![\varphi]\!]}}(\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]^{\mathcal{M}} \rightarrow [\![\langle !\varphi\rangle\psi]\!]^{\mathcal{M}}) \cap [\![\varphi]\!]^{\mathcal{M}}) \cap [\![\varphi]\!]^{\mathcal{M}}) \\
& = & \mathrm{Int}([\![\varphi]\!]^{\mathcal{M}} \rightarrow (\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]^{\mathcal{M}} \rightarrow [\![\langle !\varphi\rangle\psi]\!]^{\mathcal{M}}) \cap [\![\varphi]\!]^{\mathcal{M}}) \cap [\![\varphi]\!]^{\mathcal{M}})) \cap [\![\varphi]\!]^{\mathcal{M}} \\
& = & \mathrm{Int}([\![\varphi]\!]^{\mathcal{M}} \rightarrow \mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]^{\mathcal{M}} \rightarrow [\![\langle !\varphi\rangle\psi]\!]^{\mathcal{M}}) \cap [\![\varphi]\!]^{\mathcal{M}})) \cap [\![\varphi]\!]^{\mathcal{M}} \\
& = & \mathrm{B}^{[\![\varphi]\!]^{\mathcal{M}}}[\![\langle !\varphi\rangle\psi]\!]^{\mathcal{M}} \cap [\![\varphi]\!]^{\mathcal{M}} \\
& = & [\![B^{\varphi}\langle !\varphi\rangle\psi]\!]^{\mathcal{M}} \cap [\![\varphi]\!]^{\mathcal{M}} \\
& = & [\![B^{\varphi}\langle !\varphi\rangle\psi \wedge \varphi]\!]^{\mathcal{M}}
\end{array}$$

Therefore, as $\mathcal{M}$ has been chosen arbitrarily, $\langle !\varphi\rangle B\psi \leftrightarrow (\varphi \wedge B^{\varphi}\langle !\varphi\rangle\psi)$ is valid on any topo-model. □

Theorem 12. *The logic obtained by adding update modalities to the language $\mathcal{L}_{KCB}$ is axiomatized completely by adding the following reduction axioms to any complete axiomatization of the logic* **KCB′***:*

1. $\langle !\varphi\rangle p \leftrightarrow (\varphi \wedge p)$
2. $\langle !\varphi\rangle\neg\psi \leftrightarrow (\varphi \wedge \neg\langle !\varphi\rangle\psi)$
3. $\langle !\varphi\rangle(\psi \wedge \theta) \leftrightarrow (\langle !\varphi\rangle\psi \wedge \langle !\varphi\rangle\theta)$
4. $\langle !\varphi\rangle K\psi \leftrightarrow (\varphi \wedge K(\varphi \rightarrow \langle !\varphi\rangle\psi))$
5. $\langle !\varphi\rangle B^{\theta}\psi \leftrightarrow (\varphi \wedge B^{\langle !\varphi\rangle\theta}\langle !\varphi\rangle\psi)$

Proof. See proof in the Appendix. □

5.3 A Unimodal Case: wKD45

The result in this section is of a more technical nature. We propose another topological semantics for the language $\mathcal{L}_B$ and present a soundness and completeness result for the system **wKD45**, a weakening of **KD45** which will be presented below, wrt the class of all topological semantics.

Recall that given an extremally disconnected space (X, τ), we have

$$\mathrm{Cl}(\mathrm{Int}(A)) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$$

for any $A \subseteq X$. Hence, given a topo-model $\mathcal{M} = (X, \tau, \nu)$, the semantic clause for the belief modality can be written in the following equivalent forms

$$[\![B\varphi]\!]^{\mathcal{M}} = \mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]^{\mathcal{M}})) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]^{\mathcal{M}})))$$

if (X, τ) is an extremally disconnected space. However, $\mathrm{Cl}(\mathrm{Int}(A)) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$ is not always the case for all topological spaces and all $A \subseteq X$; the equation demands the restriction to extremally disconnected spaces. In this section, we investigate the topological semantics based on the *interior of the closure of the interior operator* on the class of all topological spaces, i.e. without the restriction to extremally disconnected topologies.

5.3.1 The system wKD45 and its topological semantics

We define the logic **wKD45** as

$$\mathbf{wKD45} = \mathbf{K} + (B\varphi \to \langle B\rangle\varphi) + (B\varphi \to BB\varphi) + (B\langle B\rangle B\varphi \to B\varphi).$$

and call it *weak* **KD45**. This logic is weaker than **KD45** since it is obtained by replacing the 5-axiom with the axiom $B\langle B\rangle B\varphi \to B\varphi$ and while $B\langle B\rangle B\varphi \to B\varphi$ is a theorem of **KD45**, the 5-axiom is not a theorem of **wKD45**. More precisely, $\mathbf{KD45} \vdash B\langle B\rangle B\varphi \to B\varphi$ but $\mathbf{wKD45} \nvdash \langle B\rangle\varphi \to B\langle B\rangle\varphi$. We find it hard to give a direct and clear interpretation for this axiom as is given for the axiom of Negative Introspection, since it is too complex in the sense that it includes three consecutive modalities. However, we can interpret it on the basis of the axioms that we have already given an interpretation, in particular, based on the interpretation of Negative Introspection. It is easier to see the correspondence if we state the weak axiom in the following equivalent form:

$$\neg B\varphi \to \langle B\rangle B\neg B\varphi.$$

Recall that the principle of Negative Introspection says that *if an agent does not believe φ, then she believes that she does not believe φ*. On the other hand, taking the reading of Negative Introspection as the reference point, one possible doxastic reading for this axiom can be given as *if the agent does not believe φ, then it is doxastically possible to her that she believes that she does not believe φ*.

Semantics. Let $\mathcal{M} = (X, \tau, \nu)$ be a topo-model. The semantic clauses for the propositional variables and the Boolean connectives are the same as in the interior semantics. For the modal operator B, we put

$$[\![B\varphi]\!]^{\mathcal{M}} \quad = \quad \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]^{\mathcal{M}})))$$

and the semantic clause for $\langle B\rangle$ is easily obtained as

$$[\![\langle B\rangle\varphi]\!]^{\mathcal{M}} \ = \ \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}([\![\varphi]\!]^{\mathcal{M}}))).$$

Validity of a formula is defined as usual. We call this semantics **w**-*topological belief semantics* referring to the system **wKD45** for which we will prove soundness and completeness. This way we distinguish it from the *topological belief semantics* presented in Chapter 4 wrt to which we proved the soundness and completeness of the system **KD45**. Throughout this section, we again use the notation $[\varphi]^{\mathcal{M}}$ for the extension of a formula $\varphi \in \mathcal{L}_K$ wrt the *interior semantics* in order to make clear in which semantics we work. We reserve the notation $[\![\varphi]\!]^{\mathcal{M}}$ for the extensions of the formulas $\varphi \in \mathcal{L}_B$ wrt the **w**-*topological belief semantics*. We skip the index when confusion is unlikely to occur.

Definition 15 (Translation $(.)^{\circledast} : \mathcal{L}_B \to \mathcal{L}_K$). *For any $\varphi \in \mathcal{L}_B$, the translation $(\varphi)^{\circledast}$ of φ into $\mathcal{L}_K$ is defined recursively as follows:*

1. $(\bot)^{\circledast} = \bot$
2. $(p)^{\circledast} = p$, *where* $p \in \text{Prop}$
3. $(\neg\varphi)^{\circledast} = \neg\varphi^{\circledast}$
4. $(\varphi \wedge \psi)^{\circledast} = \varphi^{\circledast} \wedge \psi^{\circledast}$
5. $(B\varphi)^{\circledast} = K\langle K\rangle K\varphi^{\circledast}$
6. $(\langle B\rangle\varphi)^{\circledast} = \langle K\rangle K\langle K\rangle\varphi^{\circledast}$

Proposition 22. *For any topo-model $\mathcal{M} = (X, \tau, \nu)$ and for any formula $\varphi \in \mathcal{L}_B$ we have*

$$[\![\varphi]\!]^{\mathcal{M}} = [\varphi^{\circledast}]^{\mathcal{M}}.$$

Proof. We prove the lemma by induction on the complexity of φ. The cases for

1. $\varphi = \bot$,
2. $\varphi = p$,
3. $\varphi = \neg\psi$, and
4. $\varphi = \psi \wedge \chi$

are straightforward. Now let $\varphi = B\psi$, then

$$\begin{aligned}
[\![\varphi]\!]^{\mathcal{M}} &= [\![B\psi]\!]^{\mathcal{M}} \\
&= \text{Int}(\text{Cl}(\text{Int}([\![\psi]\!]^{\mathcal{M}}))) && \text{(by the } \mathbf{w}\text{-topological belief semantics for } \mathcal{L}_B) \\
&= \text{Int}(\text{Cl}(\text{Int}([\psi^{\circledast}]^{\mathcal{M}}))) && \text{(by I.H.)} \\
&= [K\langle K\rangle K\psi^{*}]^{\mathcal{M}} && \text{(by the interior semantics for } \mathcal{L}_K.) \\
&= [(B\psi)^{\circledast}]^{\mathcal{M}} && \text{(by the translation } ^{\circledast}.) \\
&= [\varphi^{\circledast}]^{\mathcal{M}}.
\end{aligned}$$

□

5.3.2 Soundness of wKD45

The proof of soundness follows as usual.

Proposition 23. *For any topo-model $\mathcal{M} = (X, \tau, \nu)$ and any $\varphi \in \mathcal{L}_B$ we have*

1. $[\![B\varphi \to \langle B\rangle\varphi]\!] = X$
2. $[\![B\varphi \to BB\varphi]\!] = X$,
3. $[\![B\langle B\rangle B\varphi \to B\varphi]\!] = X$.

Proof. Let $\mathcal{M} = (X, \tau, \nu)$ be a topo-model and $\varphi \in \mathcal{L}_B$. Note that for any $\varphi, \psi \in \mathcal{L}_B$ we have

$$[\![\varphi \to \psi]\!] = X \text{ iff } [\![\varphi]\!] \subseteq [\![\psi]\!]. \tag{5.2}$$

1. By equation 5.2, it suffices to show that $[\![B\varphi]\!] \subseteq [\![\langle B\rangle B\varphi]\!]$ and the proof follows:

$$\begin{array}{rcll} [\![B\varphi]\!] & = & \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]))) & \\ & \subseteq & \mathrm{Int}(\mathrm{Cl}(([\![\varphi]\!]))) & \text{(by (I2) and (C2))} \\ & \subseteq & \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}([\![\varphi]\!]))) & \text{(by (C2))} \\ & = & [\![\langle B\rangle\varphi]\!]. & \end{array}$$

2. Similar to part-(1), it suffices to show that $[\![B\varphi]\!] \subseteq [\![BB\varphi]\!]$. As known, the interior of a closed set is an open domain[7] [19, p. 20]. We then have

$$[\![B\varphi]\!] = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]))) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!])))))$$

as $\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]))$ is closed in (X, τ). Then, we have

$$\begin{array}{rcll} \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]))))) & = & \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!])))))) & \text{(by (I4))} \\ & = & [\![BB\varphi]\!] & \end{array}$$

Therefore, we obtain $[\![B\varphi]\!] = [\![BB\varphi]\!]$ which implies $[\![B\varphi \to BB\varphi]\!] = X$.

3. The proof proceeds in a similar way as in above cases:

$$\begin{array}{rcll} [\![B\langle B\rangle B\varphi]\!] & = & \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(([\![\varphi]\!]))))))))) & \\ & \subseteq & \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]))) & \text{(by the argument on open domains in (2))} \\ & = & [\![B\varphi]\!]. & \end{array}$$

Therefore, by equation 5.2, we have $[\![B\langle B\rangle B\varphi \to B\varphi]\!] = X$.

□

[7] A subset A of a topological space is called an *open domain* if $A = \mathrm{Int}(\mathrm{Cl}(A))$ [19, p. 20]. In the literature, an open domain is also called *regular open*.

Proposition 22 therefore shows that all the modal axioms of **wKD45** except for the K-axiom are valid on all topological spaces wrt the **w**-topological belief semantics. However, just as in the case for **KD45** and the topological belief semantics, we will use the Kripke semantics and the defined translation in order to prove validity of the K-axiom.

Lemma 5. *For any* $\varphi \in \mathcal{L}_K$, $\mathbf{S4} \vdash K\langle K\rangle K\varphi \wedge K\langle K\rangle K\psi \rightarrow K\langle K\rangle K(\varphi \wedge \psi)$.

Proof. We know that **S4** is complete wrt the class of reflexive and transitive Kripke frames. So, it suffices to show that $K\langle K\rangle K\varphi \wedge K\langle K\rangle K\psi \rightarrow K\langle K\rangle K(\varphi \wedge \psi)$ is valid on all reflexive and transitive Kripke frames.
Let $\mathcal{F} = (X, R)$ be a reflexive and transitive Kripke frame, $\mathcal{M} = (X, R, \nu)$ a model on $\mathcal{F}$ and $x \in X$. Suppose $x \in \|K\langle K\rangle K\varphi \wedge K\langle K\rangle K\psi\|$. Hence, $x \in \|K\langle K\rangle K\varphi\|$ and $x \in \|K\langle K\rangle K\psi\|$. This implies

$$R(x) \subseteq \|\langle K\rangle K\varphi\| \text{ and } R(x) \subseteq \|\langle K\rangle K\psi\| \tag{5.3}$$

Note that, since R is reflexive, $R(x) \neq \emptyset$. Now let $y \in R(x)$. Then, by 5.3, $y \in \|\langle K\rangle K\varphi\|$ and $y \in \|\langle K\rangle K\psi\|$. Hence, there exists a $z_1 \in X$ with yRz_1 such that $z_1 \in \|K\varphi\|$. Since R is transitive and $xRyRz_1$, we have xRz_1. Then, by 5.3, $z_1 \in \|\langle K\rangle K\psi\|$. Thus, there exists $z_2 \in X$ with z_1Rz_2 such that $z_2 \in \|K\psi\|$. Since R is transitive and z_1Rz_2, $z_1 \in \|K\varphi\|$ implies that $z_2 \in \|K\varphi\|$ as well. Hence, $z_2 \in \|K\varphi \wedge K\psi\|$ implying $z_2 \in \|K(\varphi \wedge \psi)\|$. Since R is transitive and yRz_1Rz_2, $y \in \|\langle K\rangle K(\varphi \wedge \psi)\|$. Since y has been chosen arbitrarily from $R(x)$, it holds for all $y \in R(x)$. Therefore, $x \in \|K\langle K\rangle K(\varphi \wedge \psi)\|$. Therefore, by the completeness of **S4**, we have $\mathbf{S4} \vdash K\langle K\rangle K\varphi \wedge K\langle K\rangle K\psi \rightarrow K\langle K\rangle K(\varphi \wedge \psi)$. □

Proposition 24. *The* K-*axiom* $B\varphi \wedge B\psi \rightarrow B(\varphi \wedge \psi)$ *is valid in all topological spaces wrt the* **w***-topological belief semantics.*

Proof. Let (X, τ) be a topological space and $\mathcal{M} = (X, \tau, \nu)$ be a topo-model on it. Also suppose $\varphi, \psi \in \mathcal{L}_B$. Then,

$$\begin{aligned}
[\![B\varphi \wedge B\psi \rightarrow B(\varphi \wedge \psi)]\!]^{\mathcal{M}} &= [(B\varphi \wedge B\psi \rightarrow B(\varphi \wedge \psi))^{\circledast}]^{\mathcal{M}} && \text{(by Proposition 22)}\\
&= [K\langle K\rangle K\varphi^{\circledast} \wedge K\langle K\rangle K\psi^{\circledast} \rightarrow K\langle K\rangle K(\varphi^{\circledast} \wedge \psi^{\circledast})]^{\mathcal{M}} &&\\
&= X && \text{(by Lemma 5)}
\end{aligned}$$

Hence, (X, τ) validates $B\varphi \wedge B\psi \rightarrow B(\varphi \wedge \psi)$. □

Theorem 13. *The logic* **wKD45** *is sound wrt the class all topological spaces in* **w***-topological belief semantics.*

Proof. The validity of the axioms of **wKD45** follows from Propositions 23 and 24. We only need to show that the inference rules *Necessitation* and *Modus Ponens* preserve validity.

Necessitation: Let $\varphi \in \mathcal{L}_B$ such that $B\varphi$ is not valid. Then, there exists a topo-model $\mathcal{M} = (X, \tau, \nu)$ such that $[\![B\varphi]\!]^{\mathcal{M}} \neq X$, i.e., $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]^{\mathcal{M}}))) \neq X$. Hence, $[\![\varphi]\!]^{\mathcal{M}} \neq X$ implying that φ is not valid.

MP: See the proof of Theorem 6.

□

Completeness of wKD45

We prove the completeness of **wKD45** by using the translation $^{\circledast}$ from the language $\mathcal{L}_B$ into the language $\mathcal{L}_K$ and the topological completeness of **S4**. In other words, we follow the same strategy as in the proof of the completeness for **KD45** in the topological belief semantics presented in Chapter 4.

For the topological completeness proof of **wKD45** we also make use of the completeness of **wKD45** and **S4** in the standard Kripke semantics. We first recall some frame conditions concerning the relational completeness of the respective systems.

We denote the set of final clusters of a transitive Kripke frame (X, R) by $\mathfrak{C}_R$. A transitive Kripke frame (X, R) having at least one final cluster is called *weak cofinal* if for each $x \in X$ there is a $C \in \mathfrak{C}_R$ such that for all $y \in C$ we have xRy. In fact, every finite reflexive and transitive frame is weak cofinal. We call a weak cofinal frame a *weak brush* if $X \setminus \bigcup \mathfrak{C}_R$ is an irreflexive anti-chain, i.e., for each $x, y \in X \setminus \bigcup \mathfrak{C}_R$ we have $\neg(xRy)$. A weak brush with a singleton $X \setminus \bigcup \mathfrak{C}_R$ is called a *weak pin*. By definition, every weak brush and every weak pin is transitive and also serial. Note that a brush (defined in Chapter 4) is a weak brush with a singleton $\mathfrak{C}_R$. Finally, we say that a transitive frame (X, R) is of depth n if there is a chain of points $x_1 R x_2 R \ldots R x_n$ such that $\neg(x_{i+1} R x_i)$ for any $i \leq n$ and there is no chain of greater length satisfying this condition. It is hard to draw a generic picture of a weak brush, but the following figures illustrate weak pins and how a weak brush could look like (where top squares correspond to final clusters).

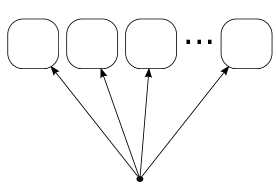


Figure 5.1: Weak pin

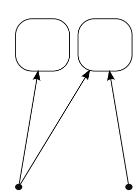


Figure 5.2: An example of a weak brush

Lemma 6. *If $\mathcal{F} = (X, R)$ is a rooted* **wKD45***-frame with depth at least 2 then it is a weak pin.*

Proof. Let $\mathcal{F} = (X, R)$ be a rooted **wKD45**-frame with depth of at least 2 and x be the root. $\mathcal{F}$ is both transitive and serial since it validates the axioms D and 5. Moreover, as it is a frame of depth 2, there exists a $y_0 \in X$ such that xRy_0 and $\neg(y_0 Rx)$. As

$\mathcal{F}$ is serial, every maximal point of it is in a final cluster. Hence, for any $x \in X$, x is maximal point iff there is a final cluster C of $\mathcal{F}$ such that $x \in C$, i.e. the set of maximal points of $\mathcal{F}$ is $\bigcup \mathfrak{C}_R$. Recall that a weak pin is a weak cofinal frame with a singleton irreflexive $X \setminus \bigcup \mathfrak{C}_R$. We hence need to show that x is an irreflexive point and every successor of x is a maximal point. Suppose for a contradiction that xRx or there is a $t_0 \in X$ such that xRt_0 and t_0 is not a maximal point of $\mathcal{F}$.

- *Case 1*: xRx

  Consider the valuation ν on (X, R) such that $\nu(p) = X \setminus \{x\}$ for some $p \in$ Prop. We want to show that $x \in \|B\langle B\rangle Bp\|$ but $x \notin \|Bp\|$. Let $y \in X$ such that xRy.

  Case 1.1: $x = y$
  Since $\neg(y_0 Rx)$ and xRx, we have that $y_0 \neq x$ and $x \notin R(y_0)$. Hence, $R(y_0) \subseteq (X \setminus \{x\})$. Then, as $\nu(p) = X \setminus \{x\}$, $R(y_0) \subseteq \nu(p)$ implying that $y_0 \in \|Bp\|$. Therefore, since yRy_0, $y \in \|\langle B\rangle Bp\|$.

  Case 1.2: $x \neq y$
  If yRx, then by transitivity of R we have yRy_0. Since $y_0 \in \|Bp\|$, we obtain that $y \in \|\langle B\rangle Bp\|$.

  If $\neg(yRx)$ then for all $z \in R(y)$ we have $\neg(zRx)$ by transitivity of R. Hence, for all $z \in R(y)$, $x \notin R(z)$ implying that $R(z) \subseteq (X \setminus \{x\})$ (since R is serial, $R(y) \neq \emptyset$). Therefore, $R(z) \subseteq \nu(p)$. Hence, as yRz, $y \in \|\langle B\rangle Bp\|$.

  Therefore $x \in \|B\langle B\rangle Bp\|$. On the other hand, as xRx and $x \notin \nu(p)$, $x \notin \|Bp\|$. We then have that (X, R) refutes $B\langle B\rangle Bp \to Bp$ implying that $\mathcal{F}$ cannot be a **wKD45** frame. Therefore, x is an irreflexive point.

- *Case 2*:

  There is a $t_0 \in X$ such that xRt_0 and t_0 is not a maximal point of $\mathcal{F}$. Since t_0 is not a maximal point, there exists a $z_0 \in X$ such that $t_0 Rz_0$ but $\neg(z_0 Rt_0)$. Consider the valuation ν on (X, R) such that $\nu(p) = X \setminus \{t_0\}$ for some $p \in$ Prop. Observe that, as $t_0 \notin R(z_0)$, $R(z_0) \subseteq (X \setminus \{t_0\})$, thus, $z_0 \in \|Bp\|$. We want to show that $x \in \|B\langle B\rangle Bp\|$ but $x \notin \|Bp\|$. Let $y \in X$ such that xRy. Then, since x is an irreflexive point, $y \neq x$.

  Case 2.1: yRz_0
  Then, as $z_0 \in \|Bp\|$, we have $y \in \|\langle B\rangle Bp\|$.

  Case 2.2: $\neg(yRz_0)$
  Then, $\neg(yRt_0)$ by transitivity of R. This implies $t_0 \notin R(y)$. Therefore, $R(y) \subseteq X \setminus \{t_0\}$ meaning that $R(y) \subseteq \nu(p)$. Hence, $y \in \|Bp\|$. Then, by seriality and transitivity of R, we have $y \in \|\langle B\rangle Bp\|$.

  Therefore $x \in \|B\langle B\rangle Bp\|$. On the other hand, as xRt_0 and $t_0 \notin \nu(p)$, $x \notin \|Bp\|$. We then have that (X, R) refutes $B\langle B\rangle Bp \to Bp$ implying that every successor of x is a maximal point.

Therefore, every rooted **wKD45** frame which is of depth at least 2 is a weak pin. This implies that every rooted **wKD45** is of at most depth 2.

□

Lemma 7.

1. *Each reflexive and transitive weak cofinal frame is an* **S4***-frame. Moreover,* **S4** *is sound and complete wrt the class of finite rooted reflexive and transitive weak cofinal frames.*

2. *Each weak brush is a* **wKD45***-frame. Moreover,* **wKD45** *is sound and complete wrt the class of finite weak brushes, indeed, wrt the class of finite weak pins.*

Proof. (1) is a very well-known and we refer to [14,15]. For (2), we proved in Lemma 6 that the **wKD45**-frames are of finite depth. It is well known that every logic over K4 that has finite depth has the finite model property (e.g., [15, Chapter 12 (tabularity)]). This implies that wKD45 has the finite model property. □

For any reflexive and transitive weak cofinal frame (X, R) we define R_B on X by

$$xR_By \text{ if } y \in \bigcup \mathfrak{C}_{R(x)}$$

for each $x, y \in X$, where $\bigcup \mathfrak{C}_{R(x)} = R(x) \cap \bigcup \mathfrak{C}_R$. In other words, $R_B(x) = \bigcup \mathfrak{C}_{R(x)}$ for each $x \in X$. Moreover, we have the following equivalence:

Lemma 8. *For any reflexive and transitive weak cofinal frame* (X, R)*,*

$$\bigcup \mathfrak{C}_{R_B} = \bigcup \mathfrak{C}_R.$$

Proof. Let (X, R) be a reflexive and transitive weak cofinal frame and $x \in X$.

($\subseteq$) Suppose $x \in \bigcup \mathfrak{C}_{R_B}$ and $x \notin \bigcup \mathfrak{C}_R$. $x \in \bigcup \mathfrak{C}_{R_B}$ means that $x \in C$ for some $C \in \mathfrak{C}_{R_B}$. As C is a final cluster, there is no $y \in X$ such that xR_By and $\neg(yR_Bx)$. On the other hand, since (X, R) is a weak cofinal frame, there is a $C' \in \mathfrak{C}_R$ such that xRz for all $z \in C'$. Hence, $C' \subseteq \bigcup \mathfrak{C}_{R(x)}$. Thus, by definition of R_B, we have $C' \subseteq R_B(x)$. However, as $x \notin \bigcup \mathfrak{C}_R$, we have that $\neg(zRx)$ and thus $\neg(zR_Bx)$ for any $z \in C'$ contradicting $x \in C$ for a final cluster C of (X, R_B).in fact, there is a unique $C \in \mathfrak{C}_{R_B}$ such that $R_B(x) = C$ since C is a final cluster.

($\supseteq$) Suppose $x \in \bigcup \mathfrak{C}_R$. Then, there is a (unique) $C \in \mathfrak{C}_R$ such that $x \in C$ and in fact $R(x) = C$. Also suppose that $x \notin \bigcup \mathfrak{C}_{R_B}$. Hence, there is a $y_0 \in X$ such that xR_By_0 and $\neg(y_0R_Bx)$. Then, $y_0 \in \bigcup \mathfrak{C}_{R(x)}$ but $x \notin \bigcup \mathfrak{C}_{R(y_0)}$ by definition of R_B. By definition of R_B, xR_By_0 implies xRy_0. Hence, as $y_0 \in R(x)$, we also have $R(y_0) = R(x) = C$. Thus, $\bigcup \mathfrak{C}_{R(y_0)} = \bigcup \mathfrak{C}_{R(x)}$. As R is reflexive, $x \in \bigcup \mathfrak{C}_{R(x)}$ and hence $x \in \bigcup \mathfrak{C}_{R(y_0)}$ contradicting $\neg(y_0R_Bx)$.

□

Lemma 9. *For any reflexive and transitive weak cofinal Kripke model $\mathcal{M} = (X, R, \nu)$, any $\varphi \in \mathcal{L}_K$ and any $x \in X$, we have*

$$\bigcup \mathfrak{C}_{R(x)} \subseteq \|\varphi\|^{\mathcal{M}} \text{ iff } x \in \|K\langle K\rangle K\varphi\|^{\mathcal{M}}.$$

Proof. Let $\mathcal{M} = (X, R, \nu)$ be a reflexive and transitive weak cofinal model, $\varphi \in \mathcal{L}_K$ and $x \in X$.

($\Rightarrow$) Suppose $\bigcup \mathfrak{C}_{R(x)} \subseteq \|\varphi\|^{\mathcal{M}}$. Let $y \in X$ such that xRy. As R is transitive and xRy, $R(y) \subseteq R(x)$ implying that $\bigcup \mathfrak{C}_{R(y)} \subseteq \bigcup \mathfrak{C}_{R(x)}$. Hence, by assumption, $\bigcup \mathfrak{C}_{R(y)} \subseteq \|\varphi\|^{\mathcal{M}}$. Thus, there is a $C \in \mathfrak{C}_R$ such that $C \subseteq R(y)$ and $C \subseteq \|\varphi\|^{\mathcal{M}}$. Since for all $z \in C$, we have $R(z) = C$ and $C \subseteq \|\varphi\|^{\mathcal{M}}$, we have $C \subseteq \|K\varphi\|^{\mathcal{M}}$. As $C \subseteq R(y)$, we have $y \in \|\langle K\rangle K\varphi\|^{\mathcal{M}}$. Therefore, since y has been chosen arbitrarily from $R(x)$, $x \in \|K\langle K\rangle K\varphi\|^{\mathcal{M}}$.

($\Leftarrow$) Suppose $\bigcup \mathfrak{C}_{R(x)} \not\subseteq \|\varphi\|^{\mathcal{M}}$. This implies that there exists a $y \in \bigcup \mathfrak{C}_{R(x)}$ such that $y \notin \|\varphi\|^{\mathcal{M}}$. $y \in \bigcup \mathfrak{C}_{R(x)}$ implies that there is a $C \in \mathfrak{C}_R$ such that $R(y) = C$ and $R(y) \subseteq R(x)$. As zRy for all $z \in C$ and $y \notin \|\varphi\|^{\mathcal{M}}$, we have $z \notin \|K\varphi\|^{\mathcal{M}}$ for all $z \in C$. Then, as $R(y) = C$, $y \notin \|\langle K\rangle K\varphi\|^{\mathcal{M}}$. Then, since xRy, $x \notin \|K\langle K\rangle K\varphi\|^{\mathcal{M}}$.

□

Lemma 10. *For any reflexive and transitive weak cofinal frame (X, R),*

1. *(X, R_B) is a weak brush.*
2. *For any valuation ν on X and for each formula $\varphi \in \mathcal{L}_B$ we have*

$$\|\varphi^{\circledast}\|^{\mathcal{M}} = \|\varphi\|^{\mathcal{M}_B}$$

where $\mathcal{M} = (X, R, \nu)$ and $\mathcal{M}_B = (X, R_B, \nu)$.

Proof. Let (X, R) be a reflexive and transitive weak cofinal frame.

1.
   - *Transitivity:* Let $x, y, z \in X$ such that xR_By and yR_Bz. This means that $y \in \bigcup \mathfrak{C}_{R(x)}$ and $z \in \bigcup \mathfrak{C}_{R(y)}$. As R being transitive and xRy, $\bigcup \mathfrak{C}_{R(y)} \subseteq \bigcup \mathfrak{C}_{R(y)}$. Hence, $z \in \bigcup \mathfrak{C}_{R(x)}$, i.e., xR_Bz.
   - *Seriality:* Let $x \in X$. Since (X, R) is weak cofinal, there is a $y \in \bigcup \mathfrak{C}_{R(x)}$, i.e., xR_By.
   - *Irreflexive, antichain:* Suppose there is an $x \in X \setminus \bigcup \mathfrak{C}_{R_B}$ such that xR_Bx. This implies, $x \in \bigcup \mathfrak{C}_{R(x)}$, thus, $x \in \bigcup \mathfrak{C}_R$ Then, by Lemma 8, $x \in \bigcup \mathfrak{C}_{R_B}$ which contradicts our assumption. Moreover, suppose that $X \setminus \bigcup \mathfrak{C}_{R_B}$ is not an antichain, i.e., there are $x, y \in X \setminus \bigcup \mathfrak{C}_{R_B}$ such that either xR_By or yR_Bx. W.l.o.g., suppose xR_By. Hence, by definition of R_B, $y \in \bigcup \mathfrak{C}_{R(x)}$. Thus, $y \in \bigcup \mathfrak{C}_R$ and, by Lemma 8, $y \in \bigcup \mathfrak{C}_{R_B}$ contradicting $y \in X \setminus \bigcup \mathfrak{C}_{R_B}$.
2. We prove this item by induction on the complexity of φ. Let $\mathcal{M} = (X, R, \nu)$ be a model on (X, R). The cases for $\varphi = \bot$, $\varphi = p$, $\varphi = \neg\psi$, $\varphi = \psi \wedge \chi$ are straightforward. Let $\varphi = B\psi$.

($\subseteq$) Let $x \in \|(B\psi)^{\circledast}\|^{\mathcal{M}} = \|K\langle K\rangle K\psi^{\circledast}\|^{\mathcal{M}}$. Then, by Lemma 9, $\bigcup \mathfrak{C}_{R(x)} \subseteq \|\psi^{\circledast}\|^{\mathcal{M}}$. By I.H, we obtain $\bigcup \mathfrak{C}_{R(x)} \subseteq \|\psi\|^{\mathcal{M}_B}$. Since $\bigcup \mathfrak{C}_{R(x)} = R_B(x)$, we have $R_B(x) \subseteq \|\psi\|^{\mathcal{M}_B}$ implying that $x \in \|B\psi\|^{\mathcal{M}_B}$.

($\supseteq$) Let $x \in \|B\psi\|^{\mathcal{M}_B}$. Then, by the standard Kripke semantics, we have $R_B(x) \subseteq \|\psi\|^{\mathcal{M}_B}$. By I.H, we obtain $R_B(x) \subseteq \|\psi^{\circledast}\|^{\mathcal{M}}$. Since $\bigcup \mathfrak{C}_{R(x)} = R_B(x)$, we have $\bigcup \mathfrak{C}_{R(x)} \subseteq \|\psi^{\circledast}\|^{\mathcal{M}}$. Thus, by Lemma 9, $x \in \|K\langle K\rangle K\psi^{\circledast}\|^{\mathcal{M}} = \|(B\psi)^{\circledast}\|^{\mathcal{M}}$.

□

Lemma 11. *For any weak brush* (X, R)*,*

1. (X, R^+) *is a reflexive and transitive weak cofinal frame.*
2. *For any valuation* ν *on* X *and for each formula* $\varphi \in \mathcal{L}_B$ *we have*

$$\|\varphi\|^{\mathcal{M}} = \|\varphi^{\circledast}\|^{\mathcal{M}^+}$$

where $\mathcal{M} = (X, R, \nu)$ *and* $\mathcal{M}^+ = (X, R^+, \nu)$*.*

Proof. Let (X, R) be a serial weak brush.

1. Since R is transitive, R^+ is also transitive and it is reflexive by definition. Moreover, (X, R^+) is weak cofinal since (X, R) is a weak brush.
2. We prove (2) by induction on the complexity of φ. Let $\mathcal{M} = (X, \tau, \nu)$ be a model on (X, R). The cases for $\varphi = \bot$, $\varphi = p$, $\varphi = \neg\psi$, $\varphi = \psi \wedge \chi$ are straightforward. Let $\varphi = B\psi$.

   ($\subseteq$) Let $x \in \|B\psi\|^{\mathcal{M}}$. Then, by the standard Kripke semantics, we have $R(x) \subseteq \|\psi\|^{\mathcal{M}}$. Hence, by I.H., $R(x) \subseteq \|\psi^{\circledast}\|^{\mathcal{M}^+}$. Since (X, R) is a weak brush, $R(x) = \bigcup \mathfrak{C}_{R(x)} \subseteq \bigcup \mathfrak{C}_{R^+(x)}$. Hence, $x \in \bigcup \mathfrak{C}_{R^+(x)}$. Then, by Lemma 9, $x \in \|K\langle K\rangle K\psi^{\circledast}\|^{\mathcal{M}^+}$.

   ($\supseteq$) Let $x \in \|K\langle K\rangle K\psi^{\circledast}\|^{\mathcal{M}^+}$. Then, by Lemma 9, $\bigcup \mathfrak{C}_{R^+(x)} \|\psi^{\circledast}\|^{\mathcal{M}^+}$. Thus, by I.H., $\bigcup \mathfrak{C}_{R^+(x)} \|\psi\|^{\mathcal{M}}$. Then, by a similar argument above, $R(x) \subseteq \|\psi\|^{\mathcal{M}}$ implying that $x \in \|B\psi\|^{\mathcal{M}}$.

□

Theorem 14. *For each formula* $\varphi \in \mathcal{L}_B$*,*

$$\mathbf{S4} \vdash \varphi^{\circledast} \textit{ iff } \mathbf{wKD45} \vdash \varphi$$

.

Proof. Let $\varphi \in \mathcal{L}_B$.

($\Rightarrow$) Suppose $\mathbf{wKD45} \nvdash \varphi$. By Lemma 7(2), there exists a Kripke model $\mathcal{M} = (X, R, \nu)$ where (X, R) is a finite weak pin such that $\|\varphi\|^{\mathcal{M}} \neq X$. Then, by Lemma 11, $\mathcal{M}^+$ is a model based on the finite reflexive and transitive weak cofinal frame (X, R^+) and $\|\varphi^{\circledast}\|^{\mathcal{M}^+} \neq X$. Hence, by Lemma 7(1), we have $\mathbf{S4} \nvdash \varphi^{\circledast}$.

($\Leftarrow$) Suppose **S4** $\nvdash \varphi^{\circledast}$. By Lemma 7(1), there exists a Kripke model $\mathcal{M} = (X, R, \nu)$ where (X, R) is a finite reflexive and transitive weak cofinal frame such that $\|\varphi^{\circledast}\|^{\mathcal{M}} \neq X$. Then, by Lemma 10, $\mathcal{M}_B$ is a model based on the (finite) weak brush (X, R_B) and $\|\varphi\|^{\mathcal{M}_B} \neq X$. Hence, by Lemma 7(2), we have **wKD45** $\nvdash \varphi$.

□

Theorem 15. **wKD45** *is complete wrt the class of all topological spaces in the* **w**-*topological belief semantics.*

Proof. Let $\varphi \in \mathcal{L}_B$ such that **wKD45** $\nvdash \varphi$. By Theorem 14, **S4** $\nvdash \varphi^{\circledast}$. Hence, by topological completeness of **S4** wrt the class of all topological spaces in the interior semantics, there exists a topo-model $\mathcal{M} = (X, \tau, \nu)$ such that $[\varphi^{\circledast}]^{\mathcal{M}} \neq X$. Then, by Proposition 22, $[\![\varphi]\!]^{\mathcal{M}} \neq X$. Thus, we found a topological space (X, τ) which refutes φ in the **w**-topological belief semantics. Hence, **wKD45** is complete wrt the class of all topological spaces in the **w**-topological belief semantics. □

Chapter 6

Conclusion

6.1 Summary

In this thesis, we proposed a new topological semantics for belief in terms of the *closure of the interior operator*. Combining it with the interior semantics for knowledge, our topological semantics for (full) belief constitutes *the most general extensional semantics for Stalnaker's system of full belief and knowledge*. Moreover, our proposal provides an intuitive interpretation of Stalnaker's conception of (full) belief as *subjective certainty* due to the nature of topological spaces, in particular, through the definitions of interior and closure operators. Recall that for any subset P of a topological space (X, τ) and any $x \in X$,

$$x \in \mathrm{Int}(P) \text{ iff } (\exists U \in \tau)(x \in U \wedge U \subseteq P).$$

In other words, a state x is in the interior of P iff there is an open neighborhood U of x such that $U \cap (X \setminus P) = \emptyset$, i.e., x can be sharply distinguished from all non-P states by an open neighborhood U. Therefore, under this interpretation, we can say that an agent knows P at a world x iff she can sharply distinguish it from all the non-P worlds. Dually,

$$x \in \mathrm{Cl}(P) \text{ iff } (\forall U \in \tau)(x \in U \rightarrow U \cap P \neq \emptyset)$$

meaning that a state x is in the closure of P iff it is very close to P, i.e., it cannot be sharply distinguished from P states. Thus, according to our topological belief semantics, an agent (fully) believes P at a state x iff she cannot sharply distinguish x from the worlds in which she has knowledge of P, i.e., the agent cannot sharply distinguish the states in which she has belief of P from the states in which she has knowledge of P. Belief, under this semantics, therefore becomes subjectively indistinguishable from knowledge, implying that our topological semantics perfectly captures the conception of belief as 'subjective certainty'.

Furthermore, we explore topological analogues of static and dynamic conditioning by providing a topological semantics for *conditional belief* and *update modalities*. We proposed two, *basic* and *refined*, topological semantics for conditional beliefs the latter of which is an improvement of the former. We demonstrated that the refined semantics for conditional beliefs quite successfully captures the rationality postulates of AGM theory: *it validates the appropriate versions of the AGM postulates 1-7 and a weaker version of postulate 8.* We moreover gave a complete axiomatization of the logic of conditional beliefs, as well as a complete axiomatization of the corresponding dynamic logic. Finally, we concluded the thesis with a result of a more technical nature by providing a more complex topological semantics for the language $\mathcal{L}_B$ and proved completeness for the logic **wKD45** obtained from **KD45** by weakening the 5-axiom.

The majority of approaches to knowledge and belief take belief – as the weaker notion, – as basic and then strengthen it to obtain a 'good' concept of knowledge. Our work provides a semantics for Stalnaker's system which approaches the issue from the other direction, i.e. taking knowledge as primitive. The formal setting developed in our studies therefore adds a precise semantic framework to a rather non-standard approach to knowledge and belief, providing a novel semantics to Stalnaker's system and imparting if not additional momentum at least an additional interpretation of it.

Moreover, on a purely formal level this thesis follows an only recently initiated approach [45, 46] to doxastic notions by interpreting them in a topological setting; these having traditionally been modeled in relational or probabilistic frameworks. This in itself provides interesting new insights, but also allows for an intuitive interpretation.

6.2 Future Work

There are several directions in which this work can be extended. In this thesis, we focused on providing nice topological semantics for *single agent* logics for knowledge, belief, conditional beliefs and updates. However, reasoning about knowledge, belief and especially about information change becomes especially interesting when applied to multi-agent cases. One natural continuation of this work therefore consists in extending our framework to a multi-agent setting and providing topological semantics for operators, such as *common knowledge* and *common belief*, in line with, e.g., [45, 55].

Here, we focused on only one type of dynamic belief change, namely, *updates*. However, there are many ways to change the beliefs of an agent depending on the information she receives. While receiving 'hard' information (information coming from an infallible source) corresponds to *updates*, receiving 'soft' information (information coming from a less reliable source) corresponds to *lexicographic upgrade* or *conservative upgrade*, as e.g. presented in [50] based on the plausibility models. On a plausibility model, while lexicographic upgrade with φ makes all φ-worlds more plausible than all $\neg\varphi$-worlds, conservative upgrade with φ changes the model by making the most plausible φ-worlds the most plausible worlds of the model. It would be worth trying to capture these belief revision policies in our topological setting.

We are not entirely satisfied with the doxastic interpretation given to the new axiom of the system **wKD45** presented in Chapter 5. A better understanding of this axiom and its doxastic implications are instrumental in providing a satisfactory doxastic interpretation of **wKD45**.

Moreover, the topological nature of the effort modality as introduced in the language of ‘topo-logic’, proposed by Parikh and Moss [38], indicates a close connection to our framework. This suggests the attempt of embedding said modality into our setting and investigating its epistemic/doxastic interpretation within this new context.

Last but not least, the interaction between topology and learning theory, and learning theory and dynamic epistemic logic, suggests learning theory to be a fruitful direction in which to extend our framework.

Appendix A

Proofs for Chapter 5

Theorem 11.

Proof. By Proposition 18, we know that each of the axioms can be rewritten by using only the knowledge modality K. We also know that the logic of knowledge **S4** is complete wrt to the class of reflexive and transitive Kripke frames. In this proof, we will first show that each of the axioms is a theorem of **S4** by using Kripke frames and the relational completeness of **S4**. Then, we can conclude that these axioms are also valid on all topological spaces, since **S4** is sound wrt the class of all topological spaces in the interior semantics. Recall that the semantic clauses for knowledge in the interior semantics and in the refined topological semantics for conditional beliefs and knowledge coincide.

Let (X, R) be a reflexive and transitive Kripke frame, $\mathcal{M} = (X, R, \nu)$ a model on (X, R) and x any element of X.

1. *Normality:* $B^\theta(\varphi \to \psi) \to (B^\theta\varphi \to B^\theta\psi)$
   By Proposition 18, we can rewrite the Normality principle as

   $K(\theta \to \langle K\rangle(\theta \wedge K(\theta \to (\varphi \to \psi)))) \to$
   $(K(\theta \to \langle K\rangle(\theta \wedge K(\theta \to \varphi))) \to K(\theta \to \langle K\rangle(\theta \wedge K(\theta \to \psi)))).$

   Suppose $x \in \|K(\theta \to \langle K\rangle(\theta \wedge K(\theta \to (\varphi \to \psi))))\|$ and $x \in \|K(\theta \to \langle K\rangle(\theta \wedge K(\theta \to \varphi)))\|$. This implies,

   $$R(x) \subseteq \|\theta \to \langle K\rangle(\theta \wedge K(\theta \to (\varphi \to \psi)))\| \quad \text{(A.1)}$$
   $$R(x) \subseteq \|\theta \to \langle K\rangle(\theta \wedge K(\theta \to \varphi))\| \quad \text{(A.2)}$$

   We want to show that $R(x) \subseteq \|\theta \to \langle K\rangle(\theta \wedge K(\theta \to \psi))\|$

Let $y \in X$ such that xRy, i.e. $y \in R(x)$. Suppose $y \in \|\theta\|$. Then,

$$\begin{array}{ll} y \in \|\langle K\rangle(\theta \wedge K(\theta \to (\varphi \to \psi))\| & \text{by (A.1)} \\ y \in \|\langle K\rangle(\theta \wedge K(\theta \to \varphi))\| & \text{by (A.2)} \end{array}$$

These imply that there exists a $y_1 \in X$ with yRy_1 such that $y_1 \in \|\theta \wedge K(\theta \to (\varphi \to \psi))\|$, and there exists a $y_2 \in X$ with yRy_2 such that $y_2 \in \|\theta \wedge K(\theta \to \varphi)\|$. Since R is transitive and $xRyRy_2$, we also have $y_2 \in \|\theta \to \langle K\rangle(\theta \wedge K(\theta \to (\varphi \to \psi))\|$, by (A.1).

Similar as above, there exists $y_2' \in X$ with y_2Ry_2' such that $y_2' \in \|\theta \wedge K(\theta \to (\varphi \to \psi))\|$. Hence, we have

$$y_2' \in \|\theta\|, \text{ and} \tag{A.3}$$

$$R(y_2') \subseteq \|\theta \to (\varphi \to \psi)\|. \tag{A.4}$$

As $y_2 \in \|K(\theta \to \varphi)\|$, y_2Ry_2' and R is transitive, $y_2' \in \|K(\theta \to \varphi)\|$ as well. Hence,

$$R(y_2') \subseteq \|\theta \to \varphi\|. \tag{A.5}$$

Thus, since $((\theta \to (\varphi \to \psi)) \wedge (\theta \to \varphi)) \to (\theta \to \psi)$ is a tautology, $R(y_2') \subseteq \|\theta \to \psi\|$ by (A.4) and (A.5).

Hence, $y_2' \in \|K(\theta \to \psi)\|$. Then, by (A.3), $y_2' \in \|\theta \wedge K(\theta \to \psi)\|$.

Thus, as yRy_2Ry_2' and R is transitive, we have $y \in \|\langle K\rangle(\theta \wedge K(\theta \to \psi))\|$. Therefore, $y \in \|\theta \to \langle K\rangle(\theta \wedge K(\theta \to \psi))\|$. Since we have chosen y arbitrarily from $R(x)$,

$$R(x) \subseteq \|\theta \to \langle K\rangle(\theta \wedge K(\theta \to \psi))\|, \text{ implying that}$$

$$x \in \|K(\theta \to \langle K\rangle(\theta \wedge K(\theta \to \psi)))\|.$$

2. *Success of Belief Revision:* $B^{\varphi}\varphi$

   By Proposition 18, we can rewrite this axiom as

$$K(\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to \varphi))).$$

We want to show that $x \in \|K(\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to \varphi)))\|$, i.e., that $R(x) \subseteq \|\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to \varphi))\|$.

Let $y \in X$ such that $y \in R(x)$ and $y \in \|\varphi\|$. As R is reflexive,

$$y \in \|\langle K\rangle\varphi\| \tag{A.6}$$

Observe that $(\varphi \wedge K(\varphi \to \varphi)) \leftrightarrow \varphi$. Thus, (A.6) implies $y \in \|\langle K\rangle(\varphi \wedge K(\varphi \to \varphi))\|$. Therefore, $y \in \|\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to \varphi))\|$. Since we have chosen y arbitrarily from $R(x)$,

$$R(x) \subseteq \|\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to \varphi))\|, \text{ implying that}$$

$$x \in \|K(\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to \varphi)))\|.$$

3. *Truthfulness of Knowledge:* $K\varphi \to \varphi$
   This is the T axiom of **S4**, hence its validity immediately follows from the soundeness of **S4** wrt the class of reflexive and transitive frames.

4. *Persistence of Knowledge:* $K\varphi \to B^{\psi}\varphi$
   By Proposition 18, we can rewrite this axiom as

$$K\varphi \to K(\psi \to \langle K\rangle(\psi \wedge K(\psi \to \varphi))).$$

   Suppose $x \in \|K\varphi\|$ and let $y \in X$ such that xRy and $y \in \|\psi\|$. By the first assumption, $y \in \|\varphi\|$ as well. As $x \in \|K\varphi\|$, $x \in \|K(\psi \to \varphi)\|$. Then, since xRy and R is tarnsitive, $y \in \|K(\psi \to \varphi)\|$ too. Thus, $y \in \|\psi \wedge K(\psi \to \varphi)\|$ and by reflexivity of R, $y \in \|\langle K\rangle(\psi \wedge K(\psi \to \varphi))\|$. Hence, $y \in \|\psi \to \langle K\rangle(\psi \wedge K(\psi \to \varphi))\|$. As y has been chosen arbitrarily from $R(x)$, $x \in \|K(\psi \to \langle K\rangle(\psi \wedge K(\psi \to \varphi)))\|$.

5. *Strong Positive Introspection:* $B^{\psi}\varphi \to KB^{\psi}\varphi$
   By Proposition 18, we can rewrite this axiom as

$$K(\psi \to \langle K\rangle(\psi \wedge K(\psi \to \varphi))) \to KK(\psi \to \langle K\rangle(\psi \wedge K(\psi \to \varphi))).$$

   Obviously, it is an instance of the 4-axiom. Hence, it is valid.

6. *Inclusin:* $B^{\varphi\wedge\psi}\theta \to B^{\varphi}(\psi \to \theta)$
   By Proposition 18, we can rewrite this axiom as

$$K((\varphi\wedge\psi) \to \langle K\rangle(\varphi\wedge\psi\wedge K(\varphi\wedge\psi \to \theta))) \to K(\varphi \to \langle K\rangle(\varphi\wedge K(\varphi \to (\psi \to \theta))))$$

   Suppose $x \in \|K((\varphi \wedge \psi) \to \langle K\rangle(\varphi \wedge \psi \wedge K(\varphi \wedge \psi \to \theta)))\|$. This implies, $R(x) \subseteq \|(\varphi \wedge \psi) \to \langle K\rangle(\varphi \wedge \psi \wedge K(\varphi \wedge \psi \to \theta))\|$, i.e.,
   $R(x) \subseteq \|\varphi \to (\psi \to \langle K\rangle(\varphi \wedge \psi \wedge K(\varphi \wedge \psi \to \theta))\|$.

   We want to show that $R(x) \subseteq \|\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to (\psi \to \theta)))\|$.
   Let $y \in X$ with $y \in R(x)$ such that $y \in \|\varphi\|$. Then, by assumption,

$$y \in \|\psi \to \langle K\rangle(\varphi \wedge \psi \wedge K(\varphi \wedge \psi \to \theta))\|.$$

Case 1: $y \notin \|\psi\|$
   Suppose for contradiction that $y \notin \|\langle K\rangle(\varphi \wedge K(\varphi \to (\psi \to \theta)))\|$. This implies, for every $z \in X$ with yRz, $z \notin \|\varphi \wedge K(\varphi \to (\psi \to \theta))\|$. Hence, for every $z \in X$ with yRz, $z \notin \|\varphi\|$ or $z \notin \|K(\varphi \to (\psi \to \theta))\|$. Then, since $y \in \|\varphi\|$ and R is reflexive, $y \notin \|K(\varphi \to (\psi \to \theta))\|$. Thus, there is a $z_0 \in X$ with yRz_0 such that $z_0 \notin \|\varphi \to (\psi \to \theta)\|$, i.e., $z_0 \in \|\varphi\|$, $z_0 \in \|\psi\|$ but $z_0 \notin \|\theta\|$.

On the other hand, as $xRyRz_0$ and R being transitive,
$z_0 \in \|\varphi \to (\psi \to \langle K\rangle(\varphi \wedge \psi \wedge K(\varphi \wedge \psi \to \theta)))\|$ by the first assumption.
Thus, $z_0 \in \|\langle K\rangle(\varphi\wedge\psi\wedge K(\varphi\wedge\psi \to \theta))\|$. This implies, there is a $z_1 \in X$ with z_0Rz_1 such that $z_1 \in \|\varphi\wedge\psi\wedge K(\varphi\wedge\psi \to \theta)\|$. Hence, $z_1 \in \|K(\varphi\wedge\psi \to \theta)\|$.

Then, as yRz_0Rz_1, we have by the first asuumption of this case that $z_1 \notin \|\varphi\|$ or $z_1 \notin \|K(\varphi \to (\psi \to \theta))\|$, which contradictions above fact. Hence,

$$y \in \|\langle K\rangle(\varphi \wedge K(\varphi \to (\psi \to \theta)))\|.$$

Case 2: $y \in \|\langle K\rangle(\varphi \wedge \psi \wedge K(\varphi \wedge \psi \to \theta))\|$
This implies that $\exists z_0 \in X$ with yRz_0 such that $z_0 \in \|\varphi\wedge\psi\wedge K(\varphi\wedge\psi \to \theta)\|$. Hencce, $z_0 \in \|\varphi \wedge K(\varphi \wedge \psi \to \theta)\|$ as well. Thus,

$$y \in \|\langle K\rangle(\varphi \wedge K(\varphi \to (\psi \to \theta)))\|.$$

Therefore, $y \in \|\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to (\psi \to \theta)))\|$. Sinca y has been choosen arbitrarily,

$$x \in \|K(\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to (\psi \to \theta))))\|.$$

7. *Cautious Monotonicity:* $B^{\varphi}\psi \wedge B^{\varphi}\theta \to B^{\varphi\wedge\psi}\theta$
By Proposition 18, we can rewrite this axiom as

$$K(\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to \psi))) \wedge K(\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to \theta))) \to$$
$$K((\varphi \wedge \psi) \to \langle K\rangle((\varphi \wedge \psi) \wedge K((\varphi \wedge \psi) \to \theta))).$$

Suppose $x \in \|K(\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to \psi))) \wedge K(\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to \theta)))\|$. Then,

$$R(x) \subseteq \|\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to \psi))\|, \text{ and} \quad (A.7)$$
$$R(x) \subseteq \|\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to \theta))\| \quad (A.8)$$

We want to show that $R(x) \subseteq \|(\varphi \wedge \psi) \to \langle K\rangle((\varphi \wedge \psi) \wedge K((\varphi \wedge \psi) \to \theta))\|$

Let $y \in R(x)$ such that $y \in \|\varphi \wedge \psi\|$. Then, by (A.7) and (A.8), we have $y \in \|\langle K\rangle(\varphi \wedge K(\varphi \to \psi))\|$ and $y \in \|\langle K\rangle(\varphi \wedge K(\varphi \to \theta))\|$, respectively. These imply there exists a $z_0 \in X$ with yRz_0 such that

$$z_0 \in \|\varphi \wedge K(\varphi \to \psi)\| \quad (A.9)$$

and there exists a $z_1 \in X$ with z_1Ry such that

$$z_1 \in \|\varphi \wedge K(\varphi \to \theta)\|. \quad (A.10)$$

Hence, as R is reflexive, we have $z_0 \in \|\psi\|$ and thus $z_0 \in \|\varphi \wedge \psi\|$ by (A.9). Then, since $xRyRz_0$ and R is transitive, we have $z_0 \in \|\varphi \to \langle K\rangle(\varphi \wedge K(\varphi \to \theta))\|$, by

(A.8). Thus, as $z_0 \in \|\varphi\|$, we obtain $z_0 \in \|\langle K\rangle(\varphi \wedge K(\varphi \to \theta))\|$. This implies that these is a $z_2 \in X$ with z_0Rz_2 such that $z_2 \in \|\varphi \wedge K(\varphi \to \theta)\|$. Then, since R is transitive and $z_0 \in \|K(\varphi \to \psi)\|$, we have $z_2 \in \|K(\varphi \to \psi)\|$. Hence, since R is reflexive and $z_2 \in \|\varphi\|$, we get $z_2 \in \|\psi\|$ implying that $z_2 \in \|\varphi \wedge \psi\|$. Moreover, $z_2 \in \|K(\varphi \to \psi)\|$ and $z_2 \in \|K(\varphi \to \theta)\|$ imply that $z_2 \in \|K((\varphi \wedge \psi) \to \theta)\|$. Therefore, $z_2 \in \|(\varphi \wedge \psi) \wedge K((\varphi \wedge \psi) \to \theta)\|$. Hence, as yRz_0Rz_2 and R is transitive, $y \in \|\langle K\rangle((\varphi \wedge \psi) \wedge K((\varphi \wedge \psi) \to \theta))\|$. Hence, $y \in \|(\varphi \wedge \psi) \to \langle K\rangle((\varphi \wedge \psi) \wedge K((\varphi \wedge \psi) \to \theta))\|$. Sinca y has been choosen arbitrarily from $R(x)$, we have $R(x) \subseteq \|(\varphi \wedge \psi) \to \langle K\rangle((\varphi \wedge \psi) \wedge K((\varphi \wedge \psi) \to \theta))\|$, i.e.

$$x \in \|K((\varphi \wedge \psi) \to \langle K\rangle((\varphi \wedge \psi) \wedge K((\varphi \wedge \psi) \to \theta)))\|.$$

Therefore, each of the above axioms is valid on all reflexive and transitive Kripke frames. Thus, they are theorems of **S4**, since **S4** is complete wrt the class of all reflexive and transitive Kripke frames. Then, by Theorem 1, we obtain that they are valid on all topological spaces in the interior semantics. As the semantic clause of knowledge in the interior semantics and the semantic clause of knowledge in the refined topological semantics for conditional beliefs and knowledge are the same, the above axioms are also valid in all topological spaces wrt the refined semantics.

We finally prove that the Necessitation Rule for conditional beliefs preserves validity:

Let $\varphi, \psi \in \mathcal{L}_{KBC}$ such that $B^{\psi}\varphi$ is not valid. Then, there exists a topo-model $\mathcal{M} = (X, \tau, \nu)$ such that $[\![B^{\psi}\varphi]\!]^{\mathcal{M}} \neq X$, i.e., $\mathrm{Int}([\![\psi]\!] \to \mathrm{Cl}([\![\psi]\!] \wedge \mathrm{Int}([\![\psi]\!] \to [\![\varphi]\!]))) \neq X$. Now suppose $[\![\varphi]\!] = X$. Then,

$$\begin{aligned}
\mathrm{Int}([\![\psi]\!] \to \mathrm{Cl}([\![\psi]\!] \cap \mathrm{Int}([\![\psi]\!] \to [\![\varphi]\!]))) &= \mathrm{Int}((X \setminus [\![\psi]\!]) \cup \mathrm{Cl}([\![\psi]\!] \cap \mathrm{Int}((X \setminus [\![\psi]\!]) \cup X))) \\
&= \mathrm{Int}((X \setminus [\![\psi]\!]) \cup \mathrm{Cl}([\![\psi]\!])) \\
&= \mathrm{Int}(X) \\
&= X
\end{aligned}$$

contradicting $[\![B^{\psi}\varphi]\!] = X$. Hence, $[\![\varphi]\!] \neq X$

□

Theorem 12.

Proof. The result follows from the validity of the new axioms. The cases for (1-3) are straightforward. We only prove the validity of (4) and (5). Let $\mathcal{M} = (X, \tau, \nu)$ be a topo-model and φ, ψ two formulas in the language. Then, 4.

$$\begin{aligned}
[\![\langle !\varphi\rangle K\psi]\!]^{\mathcal{M}} &= [\![K\psi]\!]^{\mathcal{M}_\varphi} \\
&= \mathrm{Int}_{\tau_{[\![\varphi]\!]}}([\![\psi]\!]^{\mathcal{M}_\varphi}) \\
&= \mathrm{Int}_{\tau_{[\![\varphi]\!]}}([\![\langle !\varphi\rangle\psi]\!]^{\mathcal{M}}) \\
&= \mathrm{Int}([\![\varphi]\!]^{\mathcal{M}} \to [\![\langle !\varphi\rangle\psi]\!]^{\mathcal{M}}) \cap [\![\varphi]\!]^{\mathcal{M}} \\
&= [\![K(\varphi \to \langle !\varphi\rangle\psi)]\!]^{\mathcal{M}} \cap [\![\varphi]\!]^{\mathcal{M}} \\
&= [\![K(\varphi \to \langle !\varphi\rangle\psi) \wedge \varphi]\!]^{\mathcal{M}}
\end{aligned}$$

5.

$$\begin{aligned}
[\![\langle !\varphi\rangle B^\theta\psi]\!]^{\mathcal{M}} &= [\![B^\theta\psi]\!]^{\mathcal{M}_\varphi} \\
&= \mathrm{Int}_{\tau_{[\![\varphi]\!]}}([\![\theta]\!]^{\mathcal{M}_\varphi} \to \mathrm{Cl}_{\tau_{[\![\varphi]\!]}}([\![\theta]\!]^{\mathcal{M}_\varphi} \cap \mathrm{Int}_{\tau_{[\![\varphi]\!]}}([\![\theta]\!]^{\mathcal{M}_\varphi} \to [\![\psi]\!]^{\mathcal{M}_\varphi}))) \\
&= \mathrm{Int}([\![\varphi]\!] \to ([\![\theta]\!]^{\mathcal{M}_\varphi} \to \mathrm{Cl}_{\tau_{[\![\varphi]\!]}}([\![\theta]\!]^{\mathcal{M}_\varphi} \cap \mathrm{Int}_{\tau_{[\![\varphi]\!]}}([\![\theta]\!]^{\mathcal{M}_\varphi} \to [\![\psi]\!]^{\mathcal{M}_\varphi})))) \cap [\![\varphi]\!] \\
&= \mathrm{Int}(([\![\varphi]\!] \wedge [\![\theta]\!]^{\mathcal{M}_\varphi}) \to (\mathrm{Cl}_{\tau_{[\![\varphi]\!]}}([\![\theta]\!]^{\mathcal{M}_\varphi} \cap \mathrm{Int}_{\tau_{[\![\varphi]\!]}}([\![\theta]\!]^{\mathcal{M}_\varphi} \to [\![\psi]\!]^{\mathcal{M}_\varphi})))) \cap [\![\varphi]\!] \\
&= \mathrm{Int}(([\![\varphi]\!] \wedge [\![\theta]\!]^{\mathcal{M}_\varphi}) \to (\mathrm{Cl}([\![\theta]\!]^{\mathcal{M}_\varphi} \cap \mathrm{Int}_{\tau_{[\![\varphi]\!]}}([\![\theta]\!]^{\mathcal{M}_\varphi} \to [\![\psi]\!]^{\mathcal{M}_\varphi})) \cap [\![\varphi]\!])) \cap [\![\varphi]\!] \\
&= \mathrm{Int}([\![\theta]\!]^{\mathcal{M}_\varphi} \to (\mathrm{Cl}([\![\theta]\!]^{\mathcal{M}_\varphi} \cap \mathrm{Int}_{\tau_{[\![\varphi]\!]}}([\![\theta]\!]^{\mathcal{M}_\varphi} \to [\![\psi]\!]^{\mathcal{M}_\varphi})) \cap [\![\varphi]\!])) \cap [\![\varphi]\!] \\
&= \mathrm{Int}([\![\theta]\!]^{\mathcal{M}_\varphi} \to (\mathrm{Cl}([\![\theta]\!]^{\mathcal{M}_\varphi} \cap (\mathrm{Int}([\![\varphi]\!] \cap [\![\theta]\!]^{\mathcal{M}_\varphi}) \to [\![\psi]\!]^{\mathcal{M}_\varphi}) \cap [\![\varphi]\!]))) \cap [\![\varphi]\!] \\
&= \mathrm{Int}([\![\langle !\varphi\rangle\theta]\!]^{\mathcal{M}} \to (\mathrm{Cl}([\![\langle !\varphi\rangle\theta]\!]^{\mathcal{M}} \cap (\mathrm{Int}([\![\varphi]\!] \cap [\![\langle !\varphi\rangle\theta]\!]^{\mathcal{M}}) \to [\![\langle !\varphi\rangle\psi]\!]^{\mathcal{M}}) \cap [\![\varphi]\!]))) \cap [\![\varphi]\!] \\
&= [\![B^{\langle !\varphi\rangle\theta}\langle !\varphi\rangle\psi]\!]^{\mathcal{M}} \cap [\![\varphi]\!] \\
&= [\![\varphi \wedge B^{\langle !\varphi\rangle\theta}\langle !\varphi\rangle\psi]\!]^{\mathcal{M}}
\end{aligned}$$

□

Bibliography

[1] Aiello, M., van Benthem, J., and Bezhanishvili, G. (2003) Reasoning about space: The modal way. *J. Log. Comput.*, **13**, 889–920.

[2] Alchourrón, C. E., Gärdenfors, P., and Makinson, D. (1985) On the logic of theory change: Partial meet contraction and revision functions. *J. Symb. Log.*, **50**, 510–530.

[3] Baltag, A., Bezhanishvili, N., Özgün, A., and Smets, S. The topology of belief, belief revision and defeasible knowledge. Huang, H., Grossi, D., and Roy, O. (eds.), *In Proceedings of the Fourth International Workshop, Logic, Rationality and Interaction (LORI 2013)*, Springer, to appear.

[4] Baltag, A. and Smets, S. (2006) Conditional doxastic models: A qualitative approach to dynamic belief revision. *In Proceedings of the 13th Workshop on Logic, Language, Information and Computation (WOLLIC 2006)*, pp. 5–21.

[5] Baltag, A. and Smets, S. (2008) A qualitative theory of dynamic interactive belief revision. *Logic and the Foundations of Game and Decision Theory (LOFT7)*, pp. 11–58.

[6] Baltag, A., van Ditmarsh, H., and Moss, L. (2008) Epistemic logic and information update. *Handbook of Philosophy of Information*, pp. 217–298.

[7] Baskent, C. (2007) *Topics in Subset Space Logic*. Master's thesis, University of Amsterdam, Amsterdam, The Netherlands.

[8] Baskent, C. (2011) Geometric public announcement logics. Murray, R. C. and McCarthy, P. M. (eds.), *Preceedings of the 24th Florida Artificial Intelligence Research Society Conference (FLAIRS-24)*, pp. 87–88.

[9] Battigalli, P. and Siniscalchi, M. (2002) Strong belief and forward induction reasoning. *Journal of Economic Theory*, **105**, 356–391.

[10] Bezhanishvili, G., Esakia, L., and Gabelaia, D. (2005) Some results on modal axiomatization and definability for topological spaces. *Studia Logica: An International Journal for Symbolic Logic*, **81**, pp. 325–355.

[11] Bezhanishvili, G., Esakia, L., and Gabelaia, D. (2011) Spectral and T_0-spaces in d-semantics. *Logic, Language, and Computation - 8th International Tbilisi Symposium on Logic, Language, and Computation. Revised Selected Papers*, vol. 6618 of *LNAI*, pp. 16–29.

[12] Bezhanishvili, G. and Gehrke, M. (2005) Completeness of S4 with respect to the real line: revisited. *Ann. Pure Appl. Logic*, **131**, 287–301.

[13] Bezhanishvili, N. and van der Hoek, W. Structures for epistemic logic. Baltag, A. and Smets, S. (eds.), *Logical and Informational Dynamics. A volume in honour of Johan van Benthem*, Trends in Logic, Springer, to appear.

[14] Blackburn, P., de Rijke, M., and Venema, Y. (2001) *Modal Logic*, vol. 53 of *Cambridge Tracts in Theoretical Computer Scie*. Cambridge University Press, Cambridge.

[15] Chagrov, A. V. and Zakharyaschev, M. (1997) *Modal Logic*, vol. 35 of *Oxford logic guides*. Oxford University Press.

[16] Chellas, B. F. (1980) *Modal logic*. Cambridge University Press, Cambridge.

[17] DeRose, K. (2009) *The Case for Contextualism*. New York: Oxford University Press, 1st edn.

[18] Dugundji, J. (1965) *Topology*. Allyn and Bacon Series in Advanced Mathematics, Prentice Hall.

[19] Engelking, R. (1989) *General Topology*. Sigma series in Pure Mathematics, Berlin: Heldermann, revised and completed edn.

[20] Esakia, L. (2001) Weak transitivity-restitution. *Study in logic*, **8**, 244–254, in Russian.

[21] Esakia, L. (2004) Intuitionistic logic and modality via topology. *Ann. Pure Appl. Logic*, **127**, 155–170.

[22] Fagin, R., Halpern, J. Y., Moses, Y., and Vardi, M. Y. (1995) *Reasoning about Knowledge*. MIT Press.

[23] Fagin, R., Halpern, J. Y., and Vardi, M. Y. (1990) A nonstandard approach to the logical omniscience problem. *Artificial Intelligence*, pp. 41–55, Morgan Kaufmann.

[24] Fraassen, B. v. (1995) Fine-grained opinion, probability, and the logic of full belief. *Journal of Philosophical logic*, **24**, 349–377.

[25] Gettier, E. (1963) Is justified true belief knowledge? *Analysis*, **23**, 121–123.

[26] Halpern, J. Y. and Pucella, R. (2007) Dealing with logical omniscience. Samet, D. (ed.), *TARK*, pp. 169–176.

[27] Hendricks, V. and Symons, J. (2009) Epistemic logic. Zalta, E. N. (ed.), *The Stanford Encyclopedia of Philosophy*, spring 2009 edn.

[28] Hintikka, J. (1962) *Knowledge and Belief*. Cornell University Press.

[29] Hintikka, J. (1975) Impossible possible worlds vindicated. *Journal of Philosophical Logic*, **4**, 475–484.

[30] Ichikawa, J. J. and Steup, M. (2013) The analysis of knowledge. Zalta, E. N. (ed.), *The Stanford Encyclopedia of Philosophy*, fall 2013 edn.

[31] Janković, D. S. (1983) On locally irreducible spaces. *Annales de la Société scientifique de Bruxelles*, **66(2)**, 59–72.

[32] Klein, P. (1971) A proposed definition of propositional knowledge. *Journal of Philosophy*, **68**, 471–482.

[33] Kraus, S., Lehmann, D., and Magidor, M. (1990) Nonmonotonic reasoning, preferential models and cumulative logics. *Artif. Intell.*, **44**, 167–207.

[34] Lehrer, K. and Paxson, T. J. (1969) Knowledge: Undefeated justified true belief. *Journal of Philosophy*, **66**, 225–237.

[35] Lenzen, W. (1978) *Recent Work in Epistemic Logic*, vol. 30. North Holland: Acta Philosophica Fennica.

[36] McKinsey, J. C. C. and Tarski, A. (1944) The algebra of topology. *Ann. of Math. (2)*, **45**, 141–191.

[37] Mints, G. (1999) A completeness proof for propositional S4 in Cantor space. *Logic at Work*.

[38] Moss, L. S. and Parikh, R. (1992) Topological reasoning and the logic of knowledge. Moses, Y. (ed.), *TARK*, pp. 95–105, Morgan Kaufmann.

[39] Nozick, R. (1981) *Philosophical Explanations*. Harvard University Press, Cambridge, MA.

[40] Parikh, R., Moss, L. S., and Steinsvold, C. (2007) Topology and epistemic logic. *Handbook of Spatial Logics*, pp. 299–341.

[41] Rott, H. (2004) Stability, strength and sensitivity:converting belief into knowledge. *Erkenntnis*, **61**, 469–493.

[42] Sosa, E. (1999) How to defeat opposition to moore. *Noûs*, **33**, pp. 141–153.

[43] Stalnaker, R. (1991) The problem of logical omniscience, i. *Synthese*, **89**, 425–440.

[44] Stalnaker, R. (2006) On logics of knowledge and belief. *Philosophical Studies*, **128**, 169–199.

[45] Steinsvold, C. (2007) *Topological models of belief logics*. Ph.D. thesis, New York, NY, USA.

[46] Steinsvold, C. (2008) A grim semantics for logics of belief. *Journal of Philosophical Logic*, **37**, 45–56.

[47] Steinsvold, C. (2010) A canonical topological model for extensions of K4. *Studia Logica*, **94**, 433–441.

[48] Steup, M. (2012) Epistemology. Zalta, E. N. (ed.), *The Stanford Encyclopedia of Philosophy*, winter 2012 edn.

[49] Troelstra, A. S. and van Dalen, D. (1988) *Constructivism in mathematics : an introduction. Volume I*. Studies in logic and the foundations of mathematics, North-Holland, Amsterdam, New-York, Oxford.

[50] van Benthem, J. (2007) Dynamic logic for belief revision. *Journal of Applied Non-Classical Logics*, **17**, 129–155.

[51] van Benthem, J. (2010) *Modal Logic for Open Minds*. CSLI Publications.

[52] van Benthem, J. and Bezhanishvili, G. (2007) Modal logics of space. *Handbook of Spatial Logics*, pp. 217–298.

[53] van Benthem, J., Bezhanishvilli, G., ten Cate, B., and Sarenac, D. (2005) Modal logics for products of topologies. *Studia Logica*, **84(3)**.

[54] van Benthem, J. and Pacuit, E. (2011) Dynamic logics of evidence-based beliefs. *Studia Logica*, **99(1)**, 61–92.

[55] van Benthem, J. and Sarenac, D. (2005) The geometry of knowledge. *Aspects of universal Logic*, **17**.

[56] van der Hoek, W. (1991) Systems for knowledge and beliefs. van Eijck, J. (ed.), *Logics in AI: Proc. of the European Workshop JELIA'90*, pp. 267–281, Springer, Springer Berlin Heidelberg.

[57] van Ditmarsch, H. (2005) Prolegomena to dynamic logic for belief revision. *Synthese*, **147**.

[58] van Ditmarsch, H., van der Hoek, W., and Kooi, B. (2007) *Dynamic Epistemic Logic*. Springer Publishing Company, Incorporated, 1st edn.

[59] Voorbraak, F. (1993) *As Far as I Know*. Ph.D. thesis, Utrecht University.

[60] Wáng, Y. N. and Ågotnes, T. (2011) Public announcement logic with distributed knowledge. van Ditmarsch, H., Lang, J., and Ju, S. (eds.), *Logic, Rationality, and Interaction*, vol. 6953 of *Lecture Notes in Computer Science*, pp. 328–341, Springer Berlin Heidelberg.

[61] Zvesper, J. (2010) *Playing with Information*. Ph.D. thesis, ILLC, University of Amsterdam.