Completeness proofs via canonical models on increasingly generalized settings

MSc Thesis *(Afstudeerscriptie)*

written by

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Contents

Introduction

In the present thesis, we will expand Restall's completeness proof [**Restall 2005**] and present it on a wider context. He proposes an adaptation of the completeness proof for constant domains predicate modal logic found in [**Garson 2001**] to the wider case of a distributive setting expanded with unary modal operators and enriched with constant domains quantification. The overall motivation stems from the pending problem of finding a clearer semantics for quantified relevance logics. First, unlike Restall's paper, soundness and the truth lemma are explicitly proved, in fact the overall proof is presented in a more clarified and structured way, in line with classic literature on modal completeness. Moreover, a flaw in the original proof is repaired.

Another modest contribution of this thesis will be to provide an overview of some of the most central concepts and methods of modal logic on a wide array of settings, starting with quantified distributive modal logic and arriving at propositional non-distributive modal-substructural logic. In both cases we start with preliminaries that provide the general understanding of the subject and then we present completeness proofs that illustrate the different methods and their powerful insights. Thus Chapter 1 will present the syntax and semantics (and associated discrete duality) of quantified substructural modal logic for the distributive setting. Chapters 2 (soundness) and 3 (completeness) will portray the canonical model method for relational semantics completeness, giving a first glimpse at how the classical methods look under a first jump in algebraic generality (from Boolean to distributive context). On Chapter 4 we now take a step further in this line of generalization and expose the non-distributive setting, seen as a natural descendant of classical and distributive settings. Here the algebraic and dualization methods show their true potential in guiding the far more obscure relational enterprise. On Chapter 5, the last part of the thesis, we will make explicit the methods underlying a proof of completeness for non-distributive modal logic, as presented in [**Gehrke 2006**], this time restricting ourselves to the propositional case only. While Restall's completeness proof proceeds via the canonical model, thus working on the frame side, the present proof will rather work on the algebraic side, or more precisely with the complex algebra of the canonical model better known as the (canonical extension of) the Lindenbaum Tarski algebra of the logic (we can identify both up to isomorphism).

In this way, not only we provide a wider context to Restall's initial problem, but we present an array of methods and techniques along with their natural habitats in a hopefully more unified -and thus more transparent- view than usually found in the literature, showing the interplay between these techniques and the different logical settings. Throughout the entire presentation we emphasize the modularity of each component (classical vs distributive vs non-distributive settings, propositional vs 1st order logic, unary vs binary modal operators, etc.)

CHAPTER 1

Preliminaries

In this chapter we introduce the preliminaries on both the syntax and the relational semantics of substructural modal logic with constant domain quantification. The relational semantics for the propositional reduct is presented in *Section 1.2.* along with the discrete duality for the substructural operators (viewed as modal operators). The discrete duality will mostly help to show the path when jumping from the distributive setting -and its frame-based canonical model strategy- (Chapter 3) to the non-distributive one, worked from the algebraic side with the canonical extension method (Chapter 5). Such presentation, which is an adaptation of the material found in [**Conradie & Palmigiano 2012**], is rounded up with an overview of the residuation laws linking the three main substructural operators. Finally, on *Section 1.3.* the models for (constant domain) quantified substructural logic are defined. Our initial intention was to extend Restall's proof to encompass the substructural connectives, fusion, implication and co-implication. The associated truth lemma for them turned harder than expected and we decided to leave them for later research. For uniformity's sake, we keep the general presentation as originally intended, with the full substructural logic as main player, restricting to distributive modal logic when detailing Restall's proof.

1.1. The constant domains quantified substructural modal logic

In order to make transparent the link between the quantified distributive case and the big picture schematized in the introduction, we should present propositional distributive modal logic on its own (*DML* henceforth). However, as this comes out straightforwardly by considering only the propositional and modal reduct of Constant Domains Quantified Substructural Modal logic (*CQSML* from now on), which is a lattice-based logic, we prefer to get straight into the syntax of the latter while clearly highlighting the different modules of this logic.

1.1.1. Syntax and axioms. The propositional and modal reduct of *CQSML* is the propositional distributive modal logic as introduced for the first time in [**Gehrke, Nagahashi & Venema-2005**]. Since its main feature consists in removing Boolean negation, it is closely related with Dunn's Positive Modal Logic (PML) [**Dunn 1995**]. In classical modal logic, propositional logic is expanded with modalities. Here we are to expand the *positive fragment* of propositional logic with four normal modalities $\Box, \Diamond, \lhd, \rhd$ where the last two represent weak forms of negation. Moreover, we will handle quantifiers within the restricted arena of constant domains and the substructural operators, fusion $\circ$ and its residuals $\to$ and $\leftarrow$.

In the classical setting, any of the four modal operators can be used to syntactically define the other three with the help of negation, likewise the quantifiers are also inter-definable via negation. But here, since we have no negation, the modalities $\Box, \Diamond, \rhd$ and $\lhd$ are no longer inter-definable in this way and need to be explicitly introduced as primitives [**Gehrke, Nagahashi & Venema-2005**], the same happens with the existential quantifier now in need of independent treatment. Thus we define our distributive modal language of type $\{\bot, \top, \wedge, \vee, \Box, \Diamond, \lhd, \rhd, \circ, \to, \leftarrow, \exists, \forall\}$ as the set $\mathcal{L}_{SQ}$ of formulas given by the rule

$$\varphi ::= P(x_1,\ldots,x_n) \in \mathsf{AtProp} \mid \top \mid \bot \mid x_i = x_j \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \Diamond\varphi \mid \Box\varphi \mid \lhd \varphi \mid \rhd \varphi \mid \varphi \circ \psi \mid \\ \varphi \to \psi \mid \varphi \leftarrow \psi \mid \exists y\varphi \mid \forall y\varphi$$

where P is an n-ary predicate and $\{x_1, x_2, \ldots, x_n\} \subseteq Var$ the set of denumerably many individual variables and AtProp a denumerably infinite set of propositions. The notion of free and bound occurrences of variables are defined as expected. Equality is a designated primitive binary predicate symbol that is treated separately from the rest of predicates as it will be given a fixed interpretation as the identity relation on the domain of quantification.

If $\bar{x}$ is a list of distinct variables and $\bar{y}$ a list of variables of the same length as $\bar{x}$, then $\psi\,[\bar{y}/\bar{x}]$ is the formula ψ where the variables $\bar{y}$ have been simultaneously substituted for all free occurrences of the variables $\bar{x}$. Thus we assume that no variable x_i in $\bar{x}$ occurs free in ψ within the scope of a quantifier Qy_i -we thus follow the conventions in [**Braüner & Ghilardi 2007**]-.[1]

The following *normal* properties still hold, however, as in classical modal logic:

- diamond preserves disjunction and bottom: $\Diamond(\varphi \vee \psi) = \Diamond\varphi \vee \Diamond\psi$ and $\Diamond\bot = \bot$ (note that $\bot = \bigvee \varnothing$ so $\Diamond \bigvee \varnothing = \bigvee \Diamond\varnothing = \bigvee \varnothing$).
- the box preserves conjunction and top: $\Box(\varphi \wedge \psi) = \Box\varphi \wedge \Box\psi$ and $\Box\top = \top$ (note that $\top = \bigwedge \varnothing$ so $\Box \bigwedge \varnothing = \bigwedge \Box\varnothing = \bigwedge \varnothing$).
- the modal operator $\rhd$ reverses disjunction and bottom: $\rhd(\varphi \vee \psi) = \rhd\varphi \wedge \rhd\psi$ and $\rhd\bot = \top$.
- finally, the modal operator $\lhd$ reverses conjunction and top: $\lhd(\varphi \wedge \psi) = \lhd\varphi \vee \lhd\psi$ and $\lhd\top = \bot$.

Thus, the four modalities can be seen as weak forms of negation or, regarding this *normal* behaviour, as combinations of a classical modality with negation ($\rhd\varphi \equiv \Box\neg\varphi$ and $\lhd\varphi \equiv \neg\Box\varphi$).[2] Thus the intuitive meanings of $\Box\varphi$, $\Diamond\varphi$,$\lhd\varphi$ and $\rhd\varphi$ are as expected: *φ is necessary*, *φ is possible*, *φ might not be the case*, *φ is impossible.* For sake of generality, they will be interpreted by four different accessibility relations, as will be clear in the relational semantics presented in the next section (1.2.1.).

Coming back to syntax proper, the logics for lattices lack in general the expressive power required to define an implication connective in terms of the primitive connectives. More precisely, whenever $\neg$ is not present in the language of a logic Λ, we can no longer define $\neg(A \wedge \neg B) \dashv\vdash_\Lambda A \to B \dashv\vdash_\Lambda \neg A \vee B$. Thereby entailment relations cannot be recovered from the set of tautologies alone, for entailment can no longer be represented as a tautological formula of conditional shape -or to state it syntactically, a theorem-. This amounts to a failure of the deduction theorem in these non-classical settings. To compensate this deficiency we must take sequents into account.[3]

[1]In full detail, letting $Q \in \{\exists, \forall\}$, this amounts to define $\psi\,[y/x]$ as the result of taking a *bound alphabetic variant* of ψ in which there is no quantifier Qy and then replacing every x free in such variant of ψ by y. Two well-formed formulas ψ and ψ' are bound alphabetic variants of each other iff ψ has a well formed part $Qx\delta$ where ψ' has $Qx\gamma$, and γ, δ differ only in that γ has free instances of x where and only where δ has free instances of y [**Hughes & Cresswell 1996**]

[2]As noted in [**Gehrke, Nagahashi & Venema-2005**] this may be seen as a generalization of classical modal logics: when $\rhd$ and $\lhd$ are taken to be the classical negation $\neg$ then the classical normal modal logics are those DMLs containing the sequents: $\rhd\alpha \Rightarrow \lhd\alpha$, $\lhd\alpha \Rightarrow \rhd\alpha$, $\Diamond\alpha \Rightarrow \rhd\Box\rhd\alpha$, $\rhd\Box\rhd\alpha \Rightarrow \Diamond\alpha$, $\top \Rightarrow \alpha \vee \rhd\alpha$ and $\alpha \wedge \rhd\alpha \Rightarrow \bot$.

[3]The *deduction theorem* is one of the main ingredients -the other one being *compactness*- needed to reduce entailments from any kind of assumptions to tautologies (which can be seen as entailments from empty assumptions). When both properties hold, and given an entailment $\Gamma \vdash_\Lambda \varphi$ in a logic Λ with Γ a set of formulas, we can assume by compactness that there is a finite set $\Gamma' \subseteq \Gamma$ such that $\bigwedge \Gamma' \vdash_\Lambda \varphi$. But now the deduction theorem states that $\bigwedge \Gamma' \vdash_\Lambda \varphi$ iff $\vdash_\Lambda \bigwedge \Gamma' \to \varphi$. So we have captured or reduced an arbitrary inference (with nontrivial assumptions) to an entailment with empty assumptions, i.e. a tautology. That is, *compactness* and the *deduction theorem* together imply that *entailment* in a logic Λ can be reduced to *theoremhood* in Λ. Hence, the set of all tautologies (theorems) in the logic Λ captures all the inferential power of Λ. But when the deduction theorem fails, tautologies are no longer sufficient to capture the inferential properties of our logic and deducibility has to be captured directly in terms of sequents.

A sequent is a pair of formulas from $\mathcal{L}$ such that the second follows from the first. A pair (φ, ψ) which forms a Λ-sequent is written $\varphi \vdash_{\Lambda} \psi$, as expected.[4] Sequents allow us to express any entailment. Thus, now a logic will be a set of sequents -rather than a set of formulas- containing certain axioms and closed under certain inference rules.

Definition 1. A Constant Domains Quantified Substructural Modal logic ($CQSML$) Λ is the smallest set of sequents $\varphi \Rightarrow \psi$ with $\varphi, \psi \in \mathcal{L}_{SQ}$ such that

- Λ contains all instances of the following axiom schemes:

(1) Distributive lattice axioms :
  (a) $\varphi \Rightarrow \varphi$
  (b) $\bot \Rightarrow \varphi$ and $\varphi \Rightarrow \top$
  (c) $\varphi \Rightarrow \varphi \vee \psi$ and $\psi \Rightarrow \varphi \vee \psi$
  (d) $\varphi \wedge \psi \Rightarrow \varphi$ and $\varphi \wedge \psi \Rightarrow \psi$
  (e) $\varphi \wedge (\psi \vee \chi) \Rightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$

(2) Unary modalities axioms:
  (a) $\Box\varphi \wedge \Box\psi \Rightarrow \Box(\varphi \wedge \psi)$ and $\top \Rightarrow \Box\top$
  (b) $\Diamond(\varphi \vee \psi) \Rightarrow \Diamond\varphi \vee \Diamond\psi$ and $\Diamond\bot \Rightarrow \bot$
  (c) $\lhd(\varphi \wedge \psi) \Rightarrow \lhd\varphi \vee \lhd\psi$ and $\lhd\top \Rightarrow \bot$
  (d) $\rhd\varphi \wedge \rhd\psi \Rightarrow \rhd(\varphi \vee \psi)$ and $\top \Rightarrow \rhd\bot$

(3) Binary modalities axioms:
  (a) $(\varphi \vee \psi) \circ \chi \Rightarrow (\varphi \circ \chi) \vee (\psi \circ \chi)$ and $\chi \circ (\varphi \vee \psi) \Rightarrow (\chi \circ \varphi) \vee (\chi \circ \psi)$
  (b) $\bot \circ \varphi \Rightarrow \bot$ and $\varphi \circ \bot \Rightarrow \bot$

(4) Quantifiers laws
  (a) $\forall x\varphi \Rightarrow \varphi[y/x]$, with y being free for x in φ [5]
    (i) As corollary of a, $[\forall\ left]$: $\frac{\varphi(y)\Rightarrow\psi}{\forall x\varphi(x)\Rightarrow\psi}$ (immediately follows by cut from $\forall x\varphi(x) \Rightarrow_{by\ 4(a)} \varphi(y) \Rightarrow_{by\ assumption} \psi$)
  (b) $\varphi[y/x] \Rightarrow \exists x\varphi$, with y being free for x in φ.
    (i) As corollary of b, $[\exists\ right]$: $\frac{\psi\Rightarrow\varphi(y)}{\psi\Rightarrow\exists x\varphi(x)}$ (immediately follows by cut from $\psi \Rightarrow_{by\ assumption} \varphi(y) \Rightarrow_{by\ 4(b)} \exists x\varphi(x)$)

(5) Barcan law's
  (a) $\Box\forall x\varphi \Leftrightarrow \forall x\Box\varphi$ and $\Diamond\exists x\varphi \Leftrightarrow \exists x\Diamond\varphi$
  (b) $\lhd\forall x\varphi \Leftrightarrow \exists x \lhd \varphi$ and $\rhd\exists x\varphi \Leftrightarrow \forall x \rhd \varphi$
  (c) $\exists x(\varphi \circ \psi) \Leftrightarrow \exists x\varphi \circ \psi$ provided x does not occur free in ψ
  (d) $\varphi \circ \exists x\psi \Leftrightarrow \exists x(\varphi \circ \psi)$ provided x does not occur free in φ

[4]However, when sequents are taken themselves as basic syntactic objects, they are often written $\varphi \Rightarrow \psi$ to reserve the turnstile $\vdash$ for deductions *between* sequents. Note as well that the symbols $\Sigma, \Pi, \Delta, \Gamma, \ldots$ are sometimes taken as lists -rather than sets- of finite formulas and accordingly φ, ψ are taken to be lists of at most one formula.

[5]"y being free for x" means that we exclude the cases where y becomes bounded when replacing x. For more details, see [**van Dalen 2004**] p.66, and for quantifier rules p.98.

- Λ is closed under the following *inference rules* (where x, y, z and α, β, γ are arbitrary variables and arbitrary terms respectively) :

Structural rules

$$(1a)\frac{\alpha \Rightarrow \beta \quad \beta \Rightarrow \gamma}{\alpha \Rightarrow \gamma}\ \mathit{cut}$$

$$\mathit{weakening\ to\ the\ left}\quad (1b)\frac{\alpha \Rightarrow \gamma}{\alpha \wedge \beta \Rightarrow \gamma} \qquad \frac{\beta \Rightarrow \gamma}{\alpha \wedge \beta \Rightarrow \gamma}$$

$$\mathit{weakening\ to\ the\ right}\quad (1c)\frac{\gamma \Rightarrow \alpha}{\gamma \Rightarrow \alpha \vee \beta} \qquad \frac{\gamma \Rightarrow \beta}{\gamma \Rightarrow \alpha \vee \beta}$$

Lattice rules

$$(2a)\frac{\alpha \Rightarrow \gamma \quad \beta \Rightarrow \gamma}{\alpha \vee \beta \Rightarrow \gamma}$$

$$(2b)\frac{\gamma \Rightarrow \alpha \quad \gamma \Rightarrow \beta}{\gamma \Rightarrow \alpha \wedge \beta}$$

$$(2c)\frac{\alpha_1 \Rightarrow \beta_1 \quad \alpha_2 \Rightarrow \beta_2}{\alpha_1 \vee \alpha_2 \Rightarrow \beta_1 \vee \beta_2}$$

$$(2d)\frac{\alpha_1 \Rightarrow \beta_1 \quad \alpha_2 \Rightarrow \beta_2}{\alpha_1 \wedge \alpha_2 \Rightarrow \beta_1 \wedge \beta_2}$$

unary modalities rules

$$(3a)\frac{\alpha \Rightarrow \beta}{\Diamond\alpha \Rightarrow \Diamond\beta} \qquad (3b)\frac{\alpha \Rightarrow \beta}{\Box\alpha \Rightarrow \Box\beta} \qquad (3c)\frac{\alpha \Rightarrow \beta}{\lhd\beta \Rightarrow \lhd\alpha} \qquad (3d)\frac{\alpha \Rightarrow \beta}{\rhd\beta \Rightarrow \rhd\alpha}$$

binary modalities rules

$$(4a)\frac{\alpha \Rightarrow \gamma \leftarrow \beta}{\dfrac{}{(4b)\dfrac{\beta \Rightarrow \alpha \rightarrow \gamma}{\dfrac{}{\alpha \circ \beta \Rightarrow \gamma}}}}$$

$$(4c)\frac{\alpha_1 \Rightarrow \beta_1 \quad \alpha_2 \Rightarrow \beta_2}{\alpha_1 \circ \alpha_2 \Rightarrow \beta_1 \circ \beta_2}$$

first order rules

$$(5a)\quad \frac{\alpha \Rightarrow \beta\,[y/x]}{\alpha \Rightarrow \forall x \beta\,(x)}\ \text{provided } x \text{does not occur free in } \alpha$$

$$(5b)\quad \frac{\alpha\,[y/x] \Rightarrow \beta}{\exists x \alpha\,(x) \Rightarrow \beta}\ \text{provided } x \text{does not occur free in } \beta$$

$$\mathit{substitution}\quad (5c)\frac{\alpha \Rightarrow \beta}{\alpha\,(\gamma/x) \Rightarrow \beta\,(\gamma/x)}$$

Notice that (2a) amounts to state that $\vee$ is the least upper bound for $\Rightarrow$, namely, $\alpha \vee \beta \Rightarrow \gamma$ iff $\alpha \Rightarrow \gamma$ and $\beta \Rightarrow \gamma$. The left to right direction trivially follows from order theoretic properties and thus is usually not explicitly stated as part of the inference rule. A similar observation applies to (2b) and $\wedge$ as greatest lower bound for $\Rightarrow$.

In what follows we will often make reference to the axioms and inference rules stated above. While making reference to them we will use the following notation: for an *axiom* as $\varphi \Rightarrow \varphi$ we will use (A.1a) as reference and for an *inference rule* as cut we will use (IR.1a) as reference.

On chapters 2 and 3 in which we detail Restall's completeness proof, we will work with a smaller language $\mathcal{L}_Q \subset \mathcal{L}_{SQ}$ and an accordingly reduced set of axioms and rules, namely, all the ones concerned with substructural operators $\circ, \rightarrow, \leftarrow$ will be dropped. As expected, $\mathcal{L}_Q \subseteq \mathcal{L}_{SQ}$ is given by the rule:

$$\varphi ::= P(x_1, \ldots, x_n) \in \mathsf{AtProp} \mid \top \mid \bot \mid x_i = x_j \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \Diamond\varphi \mid \Box\varphi \mid \lhd \varphi \mid \rhd \varphi \mid \exists y \varphi \mid \forall y \varphi$$

Similarly, $\mathcal{L}_S \subset \mathcal{L}_{SQ}$ is given by the rule:

$$\varphi ::= p \in \mathsf{AtProp} \mid \top \mid \bot \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \Diamond\varphi \mid \Box\varphi \mid \lhd \varphi \mid \rhd \varphi \mid \varphi \circ \psi \mid \varphi \rightarrow \psi \mid \varphi \leftarrow \psi$$

1.2. Substructural discrete duality

1.2.1. Relational semantics for the propositional distributive modal reduct. A relational semantics interprets a (modal) logic on a Kripke frame. A Kripke frame for a distributive modal logic is based on a partially ordered set $(W, \leq)$ and equipped with four binary relations, one for each modal operator of a

DML. Given such poset, we use $\wp^{\uparrow}(W)$ to denote the set of all its upsets (upward closed sets). The language associated to this propositional modal reduct is given by the set $\mathcal{L} \subseteq \mathcal{L}_Q \subseteq \mathcal{L}_{SQ}$ of formulas given by the rule:

$$\varphi ::= p \in \mathsf{AtProp} \mid \top \mid \bot \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \Diamond\varphi \mid \Box\varphi \mid \ \lhd \varphi \mid \ \rhd \varphi$$

Definition 2. (DML-frame or Kripke frame for distributive modal logic). An (ordered) Kripke frame for DML is a structure $\mathbb{F} = ((W, \leq), R_\Box, R_\Diamond, R_\rhd, R_\lhd)$ where W is a non-empty set, $\leq$ is a partial order on W, and $R_\Box, R_\Diamond, R_\rhd, R_\lhd$ are binary relations on W satisfying the following weak set of inclusion conditions (WIC):

(1) $\geq \circ R_\Diamond \subseteq R_\Diamond \circ \geq$ that is $\forall t, u, v\, [(t \geq u \ \wedge\ R_\Diamond uv) \to \exists w\, (R_\Diamond tw \ \wedge\ w \geq v)]$
(2) $\leq \circ R_\Box \subseteq R_\Box \circ \leq$ that is $\forall t, u, v\, [(t \leq u \ \wedge\ R_\Box uv) \to \exists w (R_\Box tw \ \wedge\ w \leq v]$
(3) $\leq \circ R_\rhd \subseteq R_\rhd \circ \geq$ that is $\forall t, u, v, w\, [(t \leq u \ \wedge\ R_\rhd uv) \to \exists w\, (R_\rhd tw \ \wedge\ w \geq v)]$
(4) $\geq \circ R_\lhd \subseteq R_\lhd \circ \leq$ that is $\forall t, u, v\, [(t \geq u \ \wedge\ R_\lhd uv) \to \exists w\, (R_\lhd tw \ \wedge\ w \leq v)]$

The symbol $\circ$ denotes relation composition, i.e. given relations R and S, $R \circ S = \{\langle x, z\rangle \mid \exists y\, (\langle x, y\rangle \in R\, \&\ \langle y, z\rangle \in S)\}$. Then, for instance, $\geq \circ R_\Diamond = \{\langle x, z\rangle \mid \exists y\, (x \geq y\, \&\, yR_\Diamond z)\}$.[6]

Remark 3. Classic Kripke frames are special cases of DML-frames with an implicit ordering relation $a \leq b$ iff $a = b$ and the relations $R_\Box$, $R_\rhd$ and $R_\lhd$ are subsumed by $R_\Diamond$.

Remark 4. Observe that the conditions imposed on the relations have no other purpose than to guarantee a well defined complex algebra $\mathbb{F}^+$ (the dual of $\mathbb{F}$). They constitute the first-order properties of $R_\Diamond$, $R_\Box$, $R_\rhd$ and $R_\lhd$ needed to obtain modal operations $\langle R_\Diamond\rangle$, $[R_\Box]$, $[R_\rhd\rangle$ and $\langle R_\lhd]$ such that they map upsets onto upsets and thus well-defined on $\wp^{\uparrow}(W)$, the carrier of the complex algebra of $\mathbb{F}$. In other words, they are the first order properties of $R_\Diamond$, $R_\Box$, $R_\rhd$ and $R_\lhd$ that are equivalent via correspondence to the closure of $\wp^{\uparrow}(W)$ under $\langle R_\Diamond\rangle$, $[R_\Box]$, $[R_\rhd\rangle$ and $\langle R_\lhd]$. [7]

Thus, these conditions on the frames lead to a nice behaviour of the modal (algebraic) operations -to preserve the property of being an upset- which, in turn, allows us to define a lattice of upsets *à la Birkhoff*, extended with these operations. That is, we can set the dual or complex algebra of $\mathbb{F}$ as in definition 9 further down.

Let us now come back to the frame-based semantics by introducing the definitions of model, valuations and validity.

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[6] I will depart from standard notation, which would consist in reading $\circ$ in the same way as functional composition, with $\geq \circ R_\Diamond$ read as "$\geq$ after $R_\Diamond$" just as $f \circ g$ reads "f after g" (cf [**Meulen, Partee & Wall 1990**]). Instead I follow the same notation as in [**Conradie & Palmigiano 2012**], with $\geq \circ R_\Diamond$ read as "$\geq$ and then $R_\Diamond$", which is more convenient for readability.

[7] We should warn the reader that some papers (see for instance [**Gehrke, Nagahashi & Venema-2005**]) choose a slightly different -and stronger- set of inclusion conditions (IC):

(1) $\geq \circ R_\Diamond \circ \geq \subseteq R_\Diamond$ that is $\forall t, u, v, w\, [(t \geq u \ \wedge\ R_\Diamond uv \ \wedge\ v \geq w) \to R_\Diamond tw]$
(2) $\leq \circ R_\Box \circ \leq \subseteq R_\Box$ that is $\forall t, u, v, w\, [(t \leq u \ \wedge\ R_\Box uv \ \wedge\ v \leq w) \to R_\Box tw]$
(3) $\leq \circ R_\rhd \circ \geq \subseteq R_\rhd$ that is $\forall t, u, v, w\, [(t \leq u \ \wedge\ R_\rhd uv \ \wedge\ v \geq w) \to R_\rhd tw]$
(4) $\geq \circ R_\lhd \circ \leq \subseteq R_\lhd$ that is $\forall t, u, v, w\, [(t \geq u \ \wedge\ R_\lhd uv \ \wedge\ v \leq w) \to R_\lhd tw]$

We have adopted, however, the weaker set of inclusion conditions shown above (WIC) because the fact that we are working with upsets is all we need to get the same result (namely that $\wp^{\uparrow}(W)$ is closed under modal operations). The weaker conditions (WIC) allow us to use the assumption that $X \subseteq W$ is an upset, for if we had (IC) this is not needed, i.e. these conditions are really stronger than necessary here: for instance, (IC) implies that $\langle R_\lhd] : \wp(W) \longrightarrow \wp^{\uparrow}(W)$ is well defined, whereas (WIC) by itself does not. Rather, we need to consider $\langle R_\lhd] : \wp^{\uparrow}(W) \longrightarrow \wp^{\uparrow}(W)$ instead and use the fact that the inputs are upsets.

DEFINITION 5. Valuations, Models and Validity

- A valuation on a frame $\mathbb{F}$ is a map $V : \mathsf{AtProp} \longrightarrow \wp(W)$ from the set AtProp of propositional variables to the power set of the domain W of $\mathbb{F}$. We say that such a valuation is *persistent* if it assigns an *upset* for each variable x, as a result, if $v \in V(x)$ and $v \leq w$ then $w \in V(x)$.
- A model based on a DML-frame $\mathbb{F}$ is a tuple $\mathbb{M} = (\mathbb{F}, V)$ where $V : \mathsf{AtProp} \longrightarrow \wp^{\uparrow}(W)$ is a *persistent* valuation on $\mathbb{F}$.
- A model $\mathbb{M}$ validates a sequent $\alpha \Rightarrow \beta$ (notation: $\mathbb{M} \Vdash \alpha \Rightarrow \beta$) if for each $w \in W$ such that $\mathbb{M}, w \Vdash \alpha$ we have $\mathbb{M}, v \Vdash \beta$ for all $v \in W$ with $w \leq v$.
- A frame $\mathbb{F}$ validates a sequent $\alpha \Rightarrow \beta$ at a point w (notation: $\mathbb{F}, w \Vdash \alpha \Rightarrow \beta$) if for each model $(\mathbb{F}, V)$ -with V a persistent valuation- we have $(\mathbb{F}, V), w \Vdash \alpha \Rightarrow \beta$.
- A frame $\mathbb{F}$ validates a sequent $\alpha \Rightarrow \beta$ (notation: $\mathbb{F} \Vdash \alpha \Rightarrow \beta$) if for each model $(\mathbb{F}, V)$ -with V a persistent valuation- we have $(\mathbb{F}, V), w \Vdash \alpha \Rightarrow \beta$ for any point $w \in W$.
- A frame $\mathbb{F}$ validates a set of sequents Γ (notation: $\mathbb{F} \Vdash \Gamma$) if for each sequent $\alpha \Rightarrow \beta \in \Gamma$ we have $\mathbb{F} \Vdash \alpha \Rightarrow \beta$.

Now fix a model $\mathbb{M} = (\mathbb{F}, V)$ with $\mathbb{F} = \langle (W, \leq), R_{\Box}, R_{\Diamond}, R_{\lhd}, R_{\rhd} \rangle$ and a point $w \in W$. Then the semantics of our propositional language $\mathcal{L}$ is given by:

- $\mathbb{M}, w \Vdash \top$, is always the case.
- $\mathbb{M}, w \nVdash \bot$, is never the case.
- For $p \in \mathsf{AtProp}$, $\mathbb{M}, w \Vdash p$ iff $w \in V(p)$.
- $\mathbb{M}, w \Vdash \alpha \vee \beta$ iff $\mathbb{M}, w \Vdash \alpha$ or $\mathbb{M}, w \Vdash \beta$.
- $\mathbb{M}, w \Vdash \alpha \wedge \beta$ iff $\mathbb{M}, w \Vdash \alpha$ and $\mathbb{M}, w \Vdash \beta$.
- $\mathbb{M}, w \Vdash \Diamond\alpha$ iff there exists a $v \in W$ such that $R_{\Diamond}wv$ and $\mathbb{M}, v \Vdash \alpha$.
- $\mathbb{M}, w \Vdash \Box\alpha$ iff for all $v \in W$ such that $R_{\Box}wv$ we have $\mathbb{M}, v \Vdash \alpha$.
- $\mathbb{M}, w \Vdash \rhd\alpha$ iff for all $v \in W$ such that $R_{\rhd}wv$ we have $\mathbb{M}, v \nVdash \alpha$.
- $\mathbb{M}, w \Vdash \lhd\alpha$ iff there exists a $v \in W$ such that $R_{\lhd}wv$ and $\mathbb{M}, v \nVdash \alpha$.

REMARK 6. Notice that when V is a persistent valuation, then $\Vdash$ is an hereditary satisfaction relation.

1.2.2. Perfect distributive modal algebras and their correspondence with frames. Now we introduce the algebraic semantics for distributive modal logics. By considering each sequent $\alpha \Rightarrow \beta$ as an algebraic inequality $\alpha \leq \beta$ (which by lattice theoretic laws amounts to an equality of shape $\alpha \wedge \beta = \alpha$ or $\alpha \vee \beta = \beta$), the algebraic face of distributive modal logics is immediately apparent. Seen like this, DMLs are equational theories corresponding to varieties of algebras. In fact, just as modal algebras are obtained by adding modal operators to Boolean algebras, distributive modal algebras (DMA) are obtained by adding modal operators to distributive lattices presented as algebras.

DEFINITION 7. A *distributive modal algebra* (DMA) is an algebra $\mathbb{A} = (A, \vee, \wedge, \bot, \top, \Diamond, \Box, \lhd, \rhd)$ where $(A, \vee, \wedge, \bot, \top)$ is a bounded distributive lattice (DL) and the additional modal operations satisfy the following conditions:

$$\Box(x \wedge y) = \Box x \wedge \Box y \qquad \Box\top = \top$$
$$\Diamond(x \vee y) = \Diamond x \vee \Diamond y \qquad \Diamond\bot = \bot$$
$$\rhd(x \vee y) = \rhd x \wedge \rhd y \qquad \rhd\bot = \top$$
$$\lhd(x \wedge y) = \lhd x \vee \lhd y \qquad \lhd\top = \bot$$

REMARK 8. In the Boolean setting it was furthermore possible to obtain again a frame from a given modal algebra by using the ultrafilter frame construction, where ultrafilters -generalizing the notion of atom- are taken as points of the frame. In the case of distributive modal algebras it is no longer possible to use the exact same construction: in general, there are not enough ultrafilters to ensure that every proper filter is an intersection of all the ultrafilters extending it. Fortunately for us, it is still the case that every proper filter of a distributive lattice is the intersection of all *prime* filters extending it [**Conradie & Palmigiano 2012**]. This crucial fact allows for the extension of classical constructions and results like the stone duality to accommodate the present setting. Thus, just as any BAO $\mathbb{A}$ can give rise to an ultrafilter frame whose points are the ultrafilters of $\mathbb{A}$, every DMA $\mathbb{D}$ can be associated with its prime filter frame whose points are the prime filters of $\mathbb{D}$. Ultrafilters form an anti-chain by definition, (since they are maximal, if an ultrafilter is included in another then they are the same, so they are only order-related to themselves) while the inclusion ordering between prime filters is non trivial and thereby needs to be recorded in the associated frame which is henceforth based on a poset rather than on a set. We introduce these prime filter frames later (definition 18), but first we present the opposite construction, namely, how the obtain a DMA from a given DML-frame.

DEFINITION 9. *Complex algebra of a DML-frame.* Given a DML-frame $\mathbb{F} = ((W, \leq), R_\Diamond, R_\Box, R_\triangleright, R_\triangleleft)$, let $\wp^\uparrow(W)$ be the collection of all upward closed sets (that is, the upsets) of W. For every binary accessibility relation $R_\circledcirc \subseteq W \times W$ with $R_\circledcirc \in \{R_\Diamond, R_\Box, R_\triangleright, R_\triangleleft\}$ and for every $S \subseteq W$ we define the following operations on subsets of W:[8]

- $\langle R_\Diamond \rangle S := \{x \in W \mid \exists v (R_\Diamond xv \wedge v \in S)\} = \{x \in W \mid R_\Diamond [x] \cap S \neq \varnothing\} = R_\Diamond^{-1}[S]$
- $[R_\Box] S := \{x \in W \mid \forall v (R_\Box xv \to v \in S)\} = \{x \in W \mid R_\Box [x] \subseteq S\} = \left(R_\Box^{-1}[S^c]\right)^c$
- $[R_\triangleright \rangle S := \{x \in W \mid \forall v (R_\triangleright xv \to v \notin S)\} = \{x \in W \mid R_\triangleright [x] \subseteq S^c\} = \left(R_\triangleright^{-1}[S]\right)^c$
- $\langle R_\triangleleft] S := \{x \in W \mid \exists v (R_\triangleleft xv \wedge v \notin S)\} = \{x \in W \mid R_\triangleleft [x] \cap S^c \neq \varnothing\} = R_\triangleleft^{-1}[S^c]$

Then the complex algebra of $\mathbb{F}$ is $\mathbb{F}^+ = \left(\wp^\uparrow(W), \cap, \cup, \varnothing, W, \langle R_\Diamond \rangle, [R_\Box], [R_\triangleright \rangle, \langle R_\triangleleft]\right)$

CLAIM 10. :$\wp^\uparrow(W)$ is closed under the operations $[R_\Box]$, $\langle R_\Diamond \rangle$, $[R_\triangleright \rangle$ and $\langle R_\triangleleft]$.

PROOF. The claim will follow from the weaker inclusion conditions (WIC) that any distributive modal frame satisfies by definition and the fact that the carrier of $\mathbb{F}^+$ only contains upsets. □

In the modal classical setting the complex algebras of Kripke frames can be characterized abstractly (that is, independently of the discrete duality linking them to frames) as follows: in pure algebraic terms they are precisely (up to isomorphism) the complete and atomic Boolean algebras with operators (also known as *Perfect BAO*s). This extends to the distributive case where the complex algebras of DML-frames are characterized as *perfect DMA*s with properties that constitute generalizations of the ones held by perfect BAOs. In particular, a more relaxed notion than atomicity is needed. The following definitions generalize the notion of atom to the non-Boolean cases (since the set of completely join-prime elements of a complete Boolean algebra is precisely the set of its atoms).

[8]For any $S \subseteq W$, S^c is the complement of S relative to W. For every relation $R \subseteq W \times W$ and every $S \subseteq W$ let:

- $R[Y] := \{x \in W \mid \exists y (y \in Y \wedge yRx)\}$ = the set of R-successors of points in Y, i.e the R-closure of Y.
- $R^{-1}[Y] := \{x \in W \mid \exists y (y \in Y \wedge xRy)\}$ = the set of R-predecessors of points in Y.

We will adopt the convenient abbreviations of $R[\{x\}]$ and $R^{-1}[\{x\}]$ as $R[x]$ and $R^{-1}[x]$. Now for any set S, let $\uparrow S$ be the upward closure of S under $\leq$. Applying the previous abbreviation, we will write: $\uparrow p = \leq [p] = \leq [\{p\}] = \{a \in W \mid a \geq p\}$ instead of $\uparrow \{p\}$. Likewise, $\downarrow p = \{a \in W \mid a \leq p\}$. A subset $S \subseteq W$ in a partial order $(W, \leq)$ is an upset if $u \geq v \in S$ implies $u \in S$, that is $S = \uparrow S = \leq [S]$. Order closures generated by singletons are called *principal* upsets/down-sets respectively. Observe that the principal upset of p is the smallest upset that contains p and thus is included in all upsets of p.

DEFINITION 11. *(Join prime element/Meet prime element)* Let $\mathbb{L}$ be a complete lattice, then

- $a \in L$ is said to be *join-prime* if and only if for any $b, c \in L$ if $a \leq b \vee c$ then $a \leq b$ or $a \leq c$.
- $a \in L$ is said to be *completely join-prime* if and only if for any $b_i \in L$ $(i \in I)$ if $a \leq \bigvee_{i \in I} b_i$ then $a \leq b_i$ for some $i \in I$.
- $a \in L$ is said to be *meet-prime* if and only if for any $b, c \in L$ if $a \geq b \wedge c$ then $a \geq b$ or $a \geq c$.
- $a \in L$ is said to be *completely meet-prime* if and only if for any $b_i \in L$ $(i \in I)$ if $a \geq \bigwedge_{i \in I} b_i$ then $a \geq b_i$ for some $i \in I$.
- The set of completely join-primes of $\mathbb{L}$ is denoted $J_P^{\infty}(\mathbb{L})$, the set of completely meet-primes of $\mathbb{L}$ is denoted $M_P^{\infty}(\mathbb{L})$

DEFINITION 12. *(Join irreducible element/Meet irreducible element)* Let $\mathbb{L}$ be a complete lattice, then

- $a \in L$ is said to be *join-irreducible* if and only if for any $b, c \in L$ if $a = b \vee c$ then $a = b$ or $a = c$ (it is not the finite join of strictly smaller elements).
- $a \in L$ is said to be *completely join-irreducible* if and only if for any $b_i \in L$ $(i \in I)$ if $a = \bigvee_{i \in I} b_i$ then $a = b_i$ for some $i \in I$ (it is not the supremum of all elements strictly below it).
- $a \in L$ is said to be *meet-irreducible* if and only if for any $b, c \in L$ if $a = b \wedge c$ then $a = b$ or $a = c$ (it is not the finite meet of strictly greater elements).
- $a \in L$ is said to be *completely meet-irreducible* if and only if for any $b_i \in L$ $(i \in I)$ if $a = \bigwedge_{i \in I} b_i$ then $a = b_i$ for some $i \in I$ (it is not the infimum of all elements strictly above it).
- The set of completely join-irreducibles of $\mathbb{L}$ is denoted $J^{\infty}(\mathbb{L})$, the set of completely meet-irreducibles of $\mathbb{L}$ is denoted $M^{\infty}(\mathbb{L})$

Clearly, any (completely) join-prime element is (completely) join-irreducible as well. The converse is not true in general, but it is a nice property of distributive lattices which allows us to set $J^{\infty}(\mathbb{A}) = J_P^{\infty}(\mathbb{A})$ and $M^{\infty}(\mathbb{A}) = M_P^{\infty}(\mathbb{A})$ in definition (15) below. That is, the set of all *completely join irreducible* elements of $\mathbb{A}$ is the set of all *completely join prime* elements, and the set of all *completely meet irreducible* elements of $\mathbb{A}$ is the set of all *completely meet prime* elements, respectively.

DEFINITION 13. A complete lattice $\mathbb{L}$ is called *perfect* if it is join-generated by its completely join-irreducibles and meet-generated by its completely meet-irreducibles, that is, if for any $x \in L$ we have $\bigvee \{j \in J^{\infty}(\mathbb{L}) \mid j \leq x\} = x = \bigwedge \{m \in M^{\infty}(\mathbb{L}) \mid m \geq x\}$. It is then said that $J^{\infty}(\mathbb{L})$ and $M^{\infty}(\mathbb{L})$ are join-dense and meet-dense (respectively) in $\mathbb{L}$.

Notice that this definition encompasses both distributive and non-distributive complete lattices, but for a lattice to qualify as a *perfect distributive lattice* it also needs to be completely distributive (arbitrary meets distribute over arbitrary joins). Perfect distributive lattices can also be pinned down as those lattices that are isomorphic to $\wp^{\uparrow}(P)$ for some poset P, just as the complete and atomic Boolean algebras are precisely the Boolean algebras that happen to be isomorphic to $\wp(X)$ for some set X [**Conradie & Palmigiano 2012**]. Furthermore, for any perfect distributive lattice $\mathbb{L}$, when $J^{\infty}(\mathbb{L})$ and $M^{\infty}(\mathbb{L})$ are seen as subposets of $\mathbb{L}$, the following proposition holds:

PROPOSITION 14. *The maps* $\kappa : \begin{cases} J^{\infty}(\mathbb{L}) \longrightarrow M^{\infty}(\mathbb{L}) \\ j \longmapsto \bigvee \{u \in L \mid u \ngeq j\} \end{cases}$ *and* $\lambda : \begin{cases} M^{\infty}(\mathbb{L}) \longrightarrow J^{\infty}(\mathbb{L}) \\ m \longmapsto \bigwedge \{u \in L \mid u \nleq m\} \end{cases}$ *are order isomorphisms*

DEFINITION 15. A DMA $\mathbb{A}$ is a *perfect* distributive modal algebra (DMA^{+}) if, in addition, $(A, \vee, \wedge, \perp, \top)$ is complete, completely distributive, join generated by $J_P^{\infty}(\mathbb{A})$ (as well as meet generated by $M_P^{\infty}(\mathbb{A})$) and such that:

- $\Diamond\left(\bigvee X\right)=\bigvee\left(\Diamond X\right)=\bigvee\left\{\Diamond x\,|\,x\in X\right\}$
- $\Box\left(\bigwedge X\right)=\bigwedge\left(\Box X\right)=\bigwedge\left\{\Box x\,|\,x\in X\right\}$
- $\rhd\left(\bigvee X\right)=\bigwedge\left(\rhd X\right)=\bigwedge\left\{\rhd x\,|\,x\in X\right\}$
- $\lhd\left(\bigwedge X\right)=\bigvee\left(\lhd X\right)=\bigvee\left\{\lhd x\,|\,x\in X\right\}$

REMARK 16. Via the discrete duality, Perfect DMAs can be seen as frames in algebraic disguise (cf.[**Gehrke, Nagahashi & Venema-2005**])

PROPOSITION 17. *Given any DML-frame $\mathbb{F}$, $\mathbb{F}^{+}$ is a* perfect *distributive modal algebra.*

In the classical case, by following the discrete duality, any complete and atomic Boolean algebra with operators can retrieve back a Kripke frame based on the set of its atoms, it then comes as no surprise that a similar fact holds regarding the perfect DMAs and their sets of join-prime irreducibles.

DEFINITION 18. For every perfect DMA $\mathbb{A}$, the associated *prime structure*-frame is defined as $\mathbb{A}_{+}=\left(\left(J^{\infty}\left(\mathbb{A}\right),\geq\right),R_{\Diamond},R_{\Box},R_{\rhd},R_{\lhd}\right)$ where $\left(J^{\infty}\left(\mathbb{A}\right),\geq\right)$ is the dualized[9] subposet of the completely join-prime elements of $\mathbb{A}$, and for every $j,\,j'\in J^{\infty}\left(\mathbb{A}\right)$

- $jR_{\Diamond}j'$iff $j\leq\Diamond j'$
- $jR_{\lhd}j'$iff $j\leq\lhd\kappa\left(j'\right)$
- $jR_{\Box}j'$iff $\Box\kappa\left(j'\right)\leq\kappa\left(j\right)$
- $jR_{\rhd}j'$iff $\rhd j'\leq\kappa\left(j\right)$

The following captures the discrete duality in the distributive setting (details in [**Conradie & Palmigiano 2012**]).

PROPOSITION 19. *For every perfect DMA $\mathbb{A}$ and every DML-frame $\mathbb{F}$, the following holds:* $\left(\mathbb{A}_{+}\right)^{+}\cong\mathbb{A}$ *and* $\left(\mathbb{F}^{+}\right)_{+}\cong\mathbb{F}$

1.2.3. The expanded language with substructural connectives.

We now add the extra connectives of $\mathcal{L}_{S}$ to the discrete duality picture.

DEFINITION 20. ($SDML$-frame or Substructural Distributive Modal logic frame). A structure $\mathbb{F}=\left(\left(W,\leq\right),\,R_{\Box},\,R_{\Diamond},\,R_{\rhd},\,R_{\lhd},\,R_{\circ},\,R_{\rightarrow},\,R_{\leftarrow},\,D\right)$ is a $SDML$-frame if $\left(\left(W,\leq\right),R_{\Box},R_{\Diamond},R_{\rhd},R_{\lhd}\right)$ is a DML-frame, D is a non-empty set of objects which remains the same for all $w\in W$, and $R_{\circ},R_{\rightarrow},R_{\leftarrow}$ are binary relations on W satisfying the following weak set of inclusion conditions (WIC):[10]

1. $\left(\geq\right)\circ R_{\circ}\subseteq R_{\circ}\circ\left(\geq,\geq\right)$ that is: $\forall t,u,v,z\left[\left(t\geq u\,\wedge\,R_{\circ}uvz\right)\rightarrow\exists w,w'\left(R_{\circ}tww'\,\wedge\,w\geq v\,\wedge\,w'\geq z\right)\right]$
2. $\left(\geq\right)\circ R_{\rightarrow}\subseteq R_{\rightarrow}\circ\left(\leq,\geq\right)$ that is: $\forall t,u,v,z\left[\left(t\geq u\,\wedge\,R_{\rightarrow}uvz\right)\rightarrow\exists w,w'\left(R_{\rightarrow}tww'\,\wedge\,w\leq v\,\wedge\,w'\geq z\right)\right]$
3. $\left(\geq\right)\circ R_{\leftarrow}\subseteq R_{\leftarrow}\circ\left(\geq,\leq\right)$ that is: $\forall t,u,v,z\left[\left(t\geq u\,\wedge\,R_{\leftarrow}uvz\right)\rightarrow\exists w,w'\left(R_{\leftarrow}tww'\,\wedge\,w\geq v\,\wedge\,w'\leq z\right)\right]$

As expected, these conditions are there to ensure that the corresponding operations in the complex algebra of a $SDML$-frame are well behaved (sending upsets to upsets and thus well-defined on $\wp^{\uparrow}\left(W\right)$, the carrier of the complex algebra of $\mathbb{F}$).

REMARK 21. Notice that just as the frames for classical first order modal logic -which are simply Kripke-frames as used for classical propositional modal logic (cf. p.244 [**Hughes & Cresswell 1996**] and p.272 [**Garson 2001**])-, here we will be using $SDML$-frames for $CQSML$ logic. The added complexity of the

[9]We mean order-dual, since $\left(J^{\infty}\left(\mathbb{A}\right),\geq\right)$ has inverse order to the one inherited from $\mathbb{A}$

[10]As the first order formula makes explicit, the relation composition is to be read component-wise

quantification over objects is built into the definition of the *models* for these logics, keeping the frames untouched. This means that we need to delay our frame-semantics to the last subsection of this chapter where we present such model definitions.

DEFINITION 22. A *Substructural distributive modal algebra* ($SDMA$) is an algebra $\mathbb{A} = (A, \vee, \wedge, \bot, \top, \Diamond, \Box, \triangleleft, \triangleright, \circ, \rightarrow, \leftarrow)$ where $(A, \vee, \wedge, \bot, \top, \Diamond, \Box, \triangleleft, \triangleright)$ is a DMA and the additional (binary modal) operations satisfy the following conditions:

(1) $(\varphi \vee \psi) \circ \chi = (\varphi \circ \chi) \vee (\psi \circ \chi)$ and $\chi \circ (\varphi \vee \psi) = (\chi \circ \varphi) \vee (\chi \circ \psi)$
(2) $\bot \circ \varphi = \bot$ and $\varphi \circ \bot = \bot$
(3) (the *residuation* law) For all $a, b, c \in A$ we have $a \circ b \leq c \iff b \leq a \rightarrow c \iff a \leq c \leftarrow b$

DEFINITION 23. Complex algebra of a $SDML$-frame. Given a $SDML$-frame $\mathbb{F} = ((W, \leq), R_\Box, R_\Diamond, R_\triangleright, R_\triangleleft, R_\circ, R_\rightarrow, R_\leftarrow)$, let $\wp^\uparrow(W)$ be the collection of all upward closed sets (that is, the upsets) of W. For every binary accessibility relation $R_\circledcirc \subseteq W \times W$ with $R_\circledcirc \in \{R_\Diamond, R_\Box, R_\triangleright, R_\triangleleft\}$ and for every $S \subseteq W$ we have the same operations as before (definition 9). But for every ternary accessibility relation $R_@ \subseteq W \times W \times W$ with $R_@ \in \{R_\circ, R_\rightarrow, R_\leftarrow\}$ and for every $T, S \subseteq W$ we define new operations on subsets of W:[11]

$$\begin{aligned}
\langle R_\circ \rangle (T, S) : &= \{x \in W \mid \exists t, s\, (R_\circ xts \wedge t \in T \wedge s \in S)\} = \{x \in W \mid R_\circ [x] \cap T \times S \neq \varnothing\} \\
&= R_\circ^{-1} [(T, S)]
\end{aligned}$$

$$\begin{aligned}
\langle R_\rightarrow \rangle (T, S) : &= \{x \in W \mid \forall v, u\, (R_\rightarrow xvu \Rightarrow (v \in T^c) \vee (u \in S))\} \\
&= \{x \in W \mid R_\rightarrow [x] \subseteq ((T^c \times W) \cup (W \times S))\} = R_\rightarrow^{-1} [((T \times W) \cap (W \times S^c)) \downarrow^c]
\end{aligned}$$

$$\begin{aligned}
\langle R_\leftarrow \rangle (T, S) : &= \{x \in W \mid \forall v, u\, (R_\leftarrow xvu \Rightarrow (u \in T^c) \vee (v \in S))\} \\
&= \{x \in W \mid R_\leftarrow [x] \subseteq ((W \times T^c) \cup (S \times W))\} = R_\leftarrow^{-1} [((W \times T) \cap (S^c \times W)) \downarrow^c]
\end{aligned}$$

Then the complex algebra of $\mathbb{F}$ is $\mathbb{F}^+ = \left(\wp^\uparrow(W), \cap, \cup, \varnothing, W, \langle R_\Diamond \rangle, [R_\Box], [R_\triangleright\rangle, \langle R_\triangleleft], \langle R_\circ \rangle, (R_\rightarrow), (R_\leftarrow)\right)$[12]

REMARK 24. Observe the standard procedure to define the function $\langle R_\odot \rangle$ associated as dual -in the complex algebra- to a given n-ary relation $R_\odot$ in the frame. While we usually consider a point w in the frame and then look for its $R_\odot$-successors (where a *successor* is a sequence of points of length $n-1$), the corresponding operation in the complex algebra works in the opposite direction: it takes $n-1$ sets of points and retrieves back some operation on the set of $R_\odot$-antecedents of the elements in the product of such sets. So if the relation is n-ary, then the corresponding operation is $(n-1)$-ary. This general scheme comes handy when we know how the operations look like in the algebra (cf. fusion, implication and co-implication in proposition 26 below) and we wish to define relations in the frame that emulate or mirror such behaviour.

The conditions on the ternary relations in definition 20 should guarantee now that the corresponding operations in the complex algebra above are well behaved. We show the fusion case to illustrate, all others being similar.

[11]For ease of notation we treat here $R_\circ$ as a binary relation between points and *pairs* of points. So $R_\circ[x]$ has pairs of points as elements and $R_\circ[x] \cap T \times S$ is to be read accordingly.
[12]Notice that we use $\langle - \rangle$ for a diamond-like operator and $[-]$ for a box-like operator, similarly, $\langle -]$ is an operator that is diamond-like from the outside and box-like on the inside. However, in the case of binary operators corresponding to $\rightarrow$ and $\leftarrow$ the notation no longer works quite right as each coordinate is treated differently (the behaviour on the first coordinate is anti-tone with $(R_\rightarrow)$ and monotone with $(R_\leftarrow)$, but the inverse happens on the second coordinate of these same operators).

PROOF. Closure under $\langle R_\circ \rangle$: Assume $X, Z \subseteq W$ are upsets. We have to show that: $\langle R_\circ \rangle (X, Z) := \{x \in W \mid R_\circ [x] \cap X \times Z \neq \varnothing\} = R_\circ^{-1} [(X, S)]$ is an upset. Fix an $a \in \langle R_\circ \rangle (X, Z)$ and suppose $a \leq a'$ for some $a' \in W$. We have to show that $a' \in \langle R_\circ \rangle (X, Z)$. Since $a \in \langle R_\circ \rangle (X, Z)$ then there are some b, b' such that $aR_\circ bb'$ and $b \in X$, $b' \in Z$. Hence we have fulfilled the antecedent of WIC5. Therefore there are c, c' with $c \geq b$ and $c' \geq b'$ and , $a'R_\circ cc'$. Since $b \in X$, $b' \in Z$ with X, Z upsets, then $c \in X$ and $c' \in Z$. Therefore $a' \in \langle R_\circ \rangle (X, Z)$ □

1.2.3.1. *Fusion and its residuals as a generalization of meet and implication.* There is hardly any connective as prominent for logicians as $\rightarrow$ is (the role it plays in the deduction theorem is one key reason). It is worth noticing that the two connectives $\rightarrow$ and $\leftarrow$ as residuals of fusion are a generalization of the intuitionistic implication $\rightarrow$ as residual of $\wedge$ (which again is an order-theoretical generalization of classical implication $\rightarrow$ and its residual $\wedge$). But within the growing field of substructural logic, the focus has somewhat shifted to the study on how the properties of *fusion* determine those of the *implication* as its residual. The reason for this shift of focus is undoubtedly due to the fact that fusion is simpler to study than implication. In fact, the same relation $R_\circ$ can be used -with relevant coordinate permutations- to interpret all three operators $\circ, \rightarrow, \leftarrow$, since the corresponding relations are just systematic swappings of one of them.

$$(b, a, c) \in R_\rightarrow \text{ iff } (a, b, c) \in R_\circ \text{ iff } (a, c, b) \in R_\leftarrow$$

Fusion allows us to consider the three properties of meet (associativity, commutativity and absorption) modularly and constitutes a generalization of meet (conjunction) in two directions:

(1) it is not commutative in general and thus has two different residuals/adjoints $\rightarrow$ and $\leftarrow$, associated each with one of its coordinates (when commutative, fusion has the same adjoint for both coordinates)
(2) the relation $R_\circ$ that interprets $\circ$ is non-trivial, i.e. fusion has a clear modal flavor

Fusion has a modal flavor in that its interpretation is linked to an accessibility relation $R_\circ$, while $\wedge$ is a special case of fusion where $R_\wedge$ is the diagonal relation along all three of its coordinates, that is, $R_\wedge := \{(x, x, x) \mid x \in W\}$. It follows immediately that $R_\rightarrow := \{(x, x, x) \mid x \in W\}$ as well, since swapping the coordinates does not yield any difference. This amounts to both $\wedge$ and $\rightarrow$ being interpreted locally regarding the underlying accessibility relations $R_\wedge$ and $R_\rightarrow$ which can be ignored -in other words, these connectives don't have any modal flavor anymore. Regarding its modal properties, fusion behaves like a binary diamond, it is join preserving in both coordinates[13] and sends closed sets to closed sets.

In this more general setting, where $\wedge$ is not in general commutative (and thus written '$\circ$') we have still have the residuation laws

$$\begin{cases} A \circ B \vdash C \text{ iff } B \vdash A \rightarrow C \\ A \circ B \vdash C \text{ iff } A \vdash C \leftarrow B \end{cases}$$

Since $\circ$ is not commutative, $A \circ B$ is not equivalent to $B \circ A$ with

$$\begin{cases} B \circ A \vdash C \text{ iff } A \vdash B \rightarrow C \\ B \circ A \vdash C \text{ iff } B \vdash C \leftarrow A \end{cases}$$

[13]In fact it is a complete operator, it preserves meets on both sides as well.

But in the particular case where $\circ$ is commutative, in which case we write it as $\wedge$, we have

$$\begin{cases} A \wedge B \vdash C \text{ iff } A \vdash C \leftarrow B & (1) \\ A \wedge B \vdash C \text{ iff } B \vdash A \rightarrow C & (2) \\ \Updownarrow \textit{ by commutativity of } \wedge & \\ B \wedge A \vdash C \text{ iff } A \vdash B \rightarrow C & (3) \\ B \wedge A \vdash C \text{ iff } B \vdash C \leftarrow A & (4) \end{cases}$$

where $A \wedge B$ iff $B \wedge A$, and thus the residuals $\rightarrow$ and $\leftarrow$ collapse into a single one, as they are indistinguishable due to the commutativity bridge which leads to

$$\begin{cases} (1)\, A \vdash C \leftarrow B \; iff \; (3)\, A \vdash B \rightarrow C \\ (2)\, B \vdash A \rightarrow C \; iff \; (4)\, B \vdash C \leftarrow A \end{cases}$$

Both intuitionistically and classically, the following residuation law holds: $A \wedge B \vdash C$ iff $B \vdash A \rightarrow C$ a simple replacement of symbols gives: $B \wedge A \vdash C$ iff $A \vdash B \rightarrow C$ as well.

Now let us take a closer look to residuation.

1.2.3.2. *Preliminaries on residuation.* Let $\mathbb{P}$, $\mathbb{Q}$ and $\mathbb{R}$ be partial orders and let

$$f : \mathbb{P} \times \mathbb{Q} \rightarrow \mathbb{R}$$
$$g : \mathbb{P} \times \mathbb{R} \rightarrow \mathbb{Q}$$
$$h : \mathbb{R} \times \mathbb{Q} \rightarrow \mathbb{P}$$

We say that g is the **right residual** (or right adjoint) of f if and only if for every $p \in P$, $q \in Q$ and $r \in R$, and once p is fixed:

$$f(\boldsymbol{p}, q) \leq r \text{ iff } q \leq g(\boldsymbol{p}, r).$$

We say that h is the **left residual** (or left adjoint) of f iff for every $p \in P$, $q \in Q$ and $r \in R$, and once q is fixed:[14]

$$f(p, \boldsymbol{q}) \leq r \text{ iff } p \leq h(r, \boldsymbol{q}).$$

Graphically:

$$\begin{tikzcd} \mathbb{P} & \mathbb{R} & \mathbb{Q} \\ h(r,q) & r \arrow[l, "h(-,q)"'] \arrow[r, "g(p,-)"'] & g(p,r) \\ p \arrow[u, dashed, "\leq"'] \arrow[r, "f(-,q)"] & f(p,q) \arrow[u, dashed, "\leq"'] & q \arrow[u, dashed, "\leq"'] \arrow[l, "f(p,-)"'] \end{tikzcd}$$

The following lemma states some useful facts about residual maps.

[14] Note that both h and g are *upper* adjoints while f is their *lower* adjoint.

LEMMA 25. *If g and h are the right and left residual of f respectively, then*

(1) $f : \mathbb{P} \times \mathbb{Q} \to \mathbb{R}$, $g : \mathbb{P}^\partial \times \mathbb{R} \to \mathbb{Q}$ *and* $h : \mathbb{R} \times \mathbb{Q}^\partial \to \mathbb{P}$ *are order preserving (where* $\mathbb{X}^\partial$ denotes the order-dual of the poset $\mathbb{X}$)
(2) *If* $\mathbb{P}$, $\mathbb{Q}$ *and* $\mathbb{R}$ *are complete lattices, then* $f : \mathbb{P} \times \mathbb{Q} \to \mathbb{R}$ *preserves arbitrary joins in each coordinate, and* $g : \mathbb{P}^\partial \times \mathbb{R} \to \mathbb{Q}$ *and* $h : \mathbb{R} \times \mathbb{Q}^\partial \to \mathbb{P}$ *preserve arbitrary meets in each coordinate.*

PROOF. (1) The fact that g is right residual of f implies that both f and g are order preserving in the second coordinate, indeed let $p \in P$, $q_1, q_2 \in Q$, and assume that $q_1 \leq q_2$. As $f(p, q_2) \leq f(p, q_2)$ then by applying right residuation we get $q_1 \leq q_2 \leq g(p, f(p, q_2))$, and so by applying right residuation again $f(p, q_1) \leq f(p, q_2)$. Analogously, one can show that for every $p \in P$, $r_1, r_2 \in R$, if $r_1 \leq r_2$ then $g(p, r_1) \leq g(p, r_2)$. Similarly, one can show that h being left residual of f implies that both f and h are order preserving in the first coordinate.

Let us show that g is order reversing in the first coordinate: indeed let $r \in R$, $p_1, p_2 \in P$, and assume that $p_1 \leq p_2$. As $g(p_2, r) \leq g(p_2, r)$ then by applying right residuation we get $f(p_2, g(p_2, r)) \leq r$, and as g is order preserving in the second coordinate, then $g(p_1, f(p_2, g(p_2, r))) \leq g(p_1, r)$. As $p_1 \leq p_2$ and f is order preserving in the first coordinate, then $f(p_1, g(p_2, r)) \leq f(p_2, g(p_2, r))$, hence $g(p_1, f(p_1, g(p_2, r))) \leq g(p_1, f(p_2, g(p_2, r))) \leq g(p_1, r)$. So the proof is complete if we show that $g(p_2, r) \leq g(p_1, f(p_1, g(p_2, r)))$. By right residuation, this is true iff $f(p_1, g(p_2, r)) \leq f(p_1, g(p_2, r))$, which is indeed the case. Similarly, one can show that h is order reversing in the second coordinate.

(2) The fact that h is left residual of f implies that f preserves complete joins in the first coordinate and h preserves complete meets in the first coordinate. Let $X \subseteq P$, $y \in Q$ and let us show that $f(\bigvee X, y) = \bigvee\{f(x, y) \mid x \in X\}$.

As h is left residual of f, then, by item 1 of this lemma, f is order preserving in the first coordinate, hence $f(x, y) \leq f(\bigvee X, y)$ for every $x \in X$, and so $\bigvee\{f(x, y) \mid x \in X\} \leq f(\bigvee X, y)$.

Let us now show that $f(\bigvee X, y) \leq \bigvee\{f(x, y) \mid x \in X\}$. By applying left residuation, this is equivalent to show that $\bigvee X \leq h(\bigvee\{f(x, y) \mid x \in X\}, y)$. As $f(x, y) \leq f(x, y)$ for every $x \in X$, then by left residuation $x \leq h(f(x, y), y)$, hence $\bigvee X \leq \bigvee\{h(f(x, y), y) \mid x \in X\}$. To show that $\bigvee\{h(f(x, y), y) \mid x \in X\} \leq h(\bigvee\{f(x, y) \mid x \in X\}, y)$ it is enough to verify that for every $x \in X$, $h(f(x, y), y) \leq h(\bigvee\{f(x, y) \mid x \in X\}, y)$. As h is left residual of f, then, by item 1 of this lemma, h is order preserving in the first coordinate, hence this last inequality follows from the fact that $f(x, y) \leq \bigvee\{f(x, y) \mid x \in X\}$ for every $x \in X$.

The proof that h preserves complete meets in the first coordinate is similar, and analogously one can show that the fact that g is right residual of f implies that both f preserves complete joins in second coordinate and g preserve complete meets in second coordinate.

Let us show that g being right residual of f and f preserving complete joins in the first coordinate imply together that g reverses complete joins in first coordinate: Let $X \subseteq P$ and $z \in R$ to show that $g(\bigvee X, z) = \bigwedge\{f(x, z) \mid x \in X\}$. As g is right residual of f and f is order preserving in first coordinate, then by item 1 of this lemma, g is order reversing in the first coordinate, hence $g(\bigvee X, z) \leq g(x, z)$ for every $x \in X$, and so $g(\bigvee X, z) \leq \bigwedge\{g(x, z) \mid x \in X\}$. Let us show that $\bigwedge\{g(x, z) \mid x \in X\} \leq g(\bigvee X, z)$. By right residuation, this is equivalent to show that $f(\bigvee X, \bigwedge\{g(x, z) \mid x \in X\}) \leq z$. As $g(x, z) \leq g(x, z)$ for every $x \in X$, then, by applying right residuation, $f(x, g(x, z)) \leq z$ for every $x \in X$, hence $\bigvee\{f(x, g(x, z)) \mid x \in X\} \leq z$. As f is order preserving in the second coordinate and preserves arbitrary joins in the first coordinate, then for every $x \in X$, $f(\bigvee X, \bigwedge\{g(x, z) \mid x \in X\}) \leq f(\bigvee X, g(x, z)) = \bigvee\{f(x, g(x, z)) \mid x \in X\} \leq z$.

Similarly, one can show that h being left residual of f and f preserving complete joins in second coordinate imply together that h reverses complete joins in the second coordinate. $\square$

By applying the above lemma to our present connectives, we get:

PROPOSITION 26. Fusion $\circ$ is **monotone** in both coordinates. Its right residual, implication $\rightarrow$, is anti-tone in the first coordinate and monotone in the second coordinate. Its left residual, co-implication $\leftarrow$, is monotone in the first coordinate and anti-tone in the second coordinate. On perfect lattices, fusion $\circ$ **preserves** arbitrary joins in each coordinate, *implication* $\rightarrow$ sends joins to meets in the first coordinate and preserves meets in the second coordinate, and finally, co-implication $\leftarrow$ preserves meets in the first coordinate but sends joins to meets on the second coordinate.

We illustrate with a similar diagram as above:

$$\begin{tikzcd}
\mathbb{C} & \mathbb{C} & \mathbb{C} \\
r \leftarrow q & r \arrow[l, "(-\leftarrow q)"] \arrow[r, "(p\rightarrow -)"'] & p \rightarrow r \\
p \arrow[u, dashed, "\leq"'] \arrow[r, "(-\circ q)"] & p \circ q \arrow[u, dashed, "\leq"'] & q \arrow[l, "(p\circ -)"'] \arrow[u, dashed, "\leq"']
\end{tikzcd}$$

1.3. Models for quantifiers

Once we have frames for substructural distributive modal logics, models for *propositional SDML* would amount to add valuations. But here we are interested in models to interpret (constant domain) quantified *SDML*. Thus we need to handle more detailed models.

In general, first order modal logic semantics requires to split the meaning of an expression between the two notions of *extension* and *intension*. Given a set W of points, the intension of an expression is a function that takes each such point and associates it with an extension for that expression. Thus we reconcile the fact the a given expression has the same intensional meaning (embodies the same concept) across worlds in the frame with the fact that such expression does not have the same extensional meaning in each world. In such general setting, a model is defined as follows:[15]

[15] We follow [**Garson 2001**] presentation but for one detail, we keep variables-assignments separate from the models themselves as done in [**Braüner & Ghilardi 2007**], the convenience of this separation is argued for in page 238 of [**Hughes & Cresswell 1996**]

DEFINITION 27. A model for a first order modal logic of similarity type τ is a tuple $\left\langle (W, \leq), \{R_{\triangledown}\}_{\triangledown \in \tau}, D, Q, \{V_w\}_{w \in W} \right\rangle$ where

- W is a non-empty set of points
- $\leq$ is a partial order over W (in the classical case this boils down to a trivial order in which $x \leq y$ iff $x = y$, and hence it is usually omitted)
- $\{R_{\triangledown}\}_{\triangledown \in \tau}$ is the family of accessibility relations that interpret each one of the modal operators $\triangledown$ contained in the modal similarity type τ.
- D -for *Domain of quantification*- is a non-empty set of all possible objects
- $Q : \begin{cases} W \longrightarrow \wp(D) \\ w \longrightarrow D_w \end{cases}$ is a function that determines the (possibly different) domain of quantification $D_w \subseteq D$ associated to each point $w \in W$.
- V_w is valuation at w that to each n-place predicate symbol assigns a subset of D_w^n (both the resulting set and D_w^n maybe different depending on w)

A particular instance of such general definition consists of the following simplification: We consider only a single fixed domain D shared by all the points $w \in W$ and independent of them. In such case, it is said that we are in a *constant domain* system and $Q : W \longrightarrow \wp(D)$ is the constant function $\lambda x.D$ that assigns the same domain of quantification $Q(w) = D_w = D$ to all $w \in W$. Such simplification allows us to add the usual satisfaction conditions for $\forall x$ to the semantics of a modal logic without further complications. Moreover, we will later need the Barcan law ($\forall x \Box \varphi \vdash \Box \forall x \varphi$) and its converse, which together do enforce constant domains.

REMARK 28. Since in a propositional environment the addition of a *valuation* is enough to have a *model*, [**Braüner & Ghilardi 2007**] introduce an intermediate concept between the concept of a *frame* -which only consists of $\mathbb{F} := \left\langle (W, \leq), \{R_{\triangledown}\}_{\triangledown \in \tau} \right\rangle$ - and the concept of a (first-order) *model* -as in definition 27-. The intermediate concept is that of a *skeleton*, which consists of $\langle \mathbb{F}, D, Q \rangle$, that is: a *frame* enriched with an *object domain system*. In this way, we recover again the idea that a *model* is essentially given by valuations. Then such models can be *based* on a *frame* (which is enough to interpret a propositional language) or on a *skeleton* (required to interpret languages with non-trivial predicates). Validity is then understood as expected, relative to frames or relative to frame-skeleton pairs, respectively.

Thus our models can be greatly simplified and in fact we end up with a distributive version of a *fixed domain objectual model with rigid terms*, called $Q1$-model in [**Garson 2001**], where the domain of quantification is D for all points[16]

DEFINITION 29. In general, a model for *Constant Domains Quantified Substructural Modal logic* (*CQSML*) of similarity type τ is a tuple $\left\langle (W, \leq), \{R_{\triangledown}\}_{\triangledown \in \tau}, D, \{V_w\}_{w \in W} \right\rangle$ where

- W is a non-empty set of points
- $\leq$ is a partial order over W
- $\{R_{\triangledown}\}_{\triangledown \in \tau}$ is the family of accessibility relations that interpret each of the modal operators contained in the modal similarity type τ.
- D is a non-empty set of all possible objects, which acts as the domain of quantification for all $w \in W$ (that is, the function Q from the previous definition assigns D to every point as domain of quantification) .

[16] In fact, the satisfaction conditions (i.e. the semantics) also get much simpler. In particular, g -the function assigning denotation to terms in Garson's presentation- meets the rigidity condition that turns term's intensions into constant functions (Rigidity condition: $g(t)(w) = g(t)(w')$ for all w, w' in W). Thus, terms are rigid designators which are assigned constant functions as intensions (or equivalently, they are assigned extensions directly).

- V_w is valuation at w that to each n-place predicate symbol assigns a subset of D^n (such subset maybe different depending on w), and moreover, since we are constrained to persistent valuations: $w \leq w'$ implies $V_w(P) \subseteq V_{w'}(P)$ for all points $w, w' \in W$ and for all predicates $P \in \mathcal{L}_{SQ}^{Pred}$ (with $\mathcal{L}_{SQ}^{Pred}$ the set of predicates of $\mathcal{L}_{SQ}$) .

REMARK 30. Notice that we could use a different way of defining the interpretation of predicates to make it more similar to the propositional semantics presentation: for each n-place predicate symbol $P \in \mathcal{L}_{SQ}^{Pred}$ there is valuation $V_P : D^n \longrightarrow \mathbb{F}^+$ which to each $n-$tuple of objects assigns the (up)set of worlds $w \in W$ in which the property P applies to them.[17] Then we could state that V is a map such that for every predicate symbol P of arity n, $V(P) = V_P$. In such case, we would define the model as $\left\langle (W, \leq), \{R_\triangledown\}_{\triangledown \in \tau}, D, \{V_P\}_{P \in \mathcal{L}_{SQ}^{Pred}} \right\rangle$

To capture our satisfaction conditions for formulas in the language $\mathcal{L}_{SQ}$, besides a model based on a *constant domain skeleton*, we will need to use assignments. An assignment is a map $g : Var \longrightarrow D$. If g, g' agree on all variables but (possibly) y we write $g' \equiv_y g$. Now to define the satisfaction relation $\Vdash \subseteq W \times Form$, fix a model $\mathbb{M} = \left(\mathbb{F}, D, \{V_w\}_{w \in W}\right)$ with $\mathbb{F}$ a $SDML$-frame and fix a point $w \in W$ and an an assignment g. Then the Kripke-semantics of our first order modal language $\mathcal{L}_{SQ}$ is given by the rules:

- $\mathbb{M}, w, g \Vdash \top$, is always the case.
- $\mathbb{M}, w, g \nVdash \bot$, is never the case.
- $\mathbb{M}, w, g \Vdash P(x_1 \dots x_n)$ iff $\langle g(x_1) \dots g(x_n) \rangle \in V_w(P)$
- $\mathbb{M}, w, g \Vdash x_1 = x_2$ iff $g(x_1) = g(x_2)$
- $\mathbb{M}, w, g \Vdash \alpha \vee \beta$ iff $\mathbb{M}, w, g \Vdash \alpha$ or $\mathbb{M}, w, g \Vdash \beta$.
- $\mathbb{M}, w, g \Vdash \alpha \wedge \beta$ iff $\mathbb{M}, w, g \Vdash \alpha$ and $\mathbb{M}, w, g \Vdash \beta$.
- $\mathbb{M}, w, g \Vdash \Diamond\alpha$ iff there exists a $v \in W$ such that $R_\Diamond wv$ and $\mathbb{M}, v, g \Vdash \alpha$.
- $\mathbb{M}, w, g \Vdash \Box\alpha$ iff for all $v \in W$ such that $R_\Box wv$ we have $\mathbb{M}, v, g \Vdash \alpha$.
- $\mathbb{M}, w, g \Vdash \rhd\alpha$ iff for all $v \in W$ such that $R_\rhd wv$ we have $\mathbb{M}, v, g \nVdash \alpha$.
- $\mathbb{M}, w, g \Vdash \lhd\alpha$ iff there exists a $v \in W$ such that $R_\lhd wv$ and $\mathbb{M}, v, g \nVdash \alpha$
- $\mathbb{M}, w, g \Vdash \forall xA$ iff for each g' such that $g' \equiv_x g$ we have $\mathbb{M}, w, g' \Vdash A$
- $\mathbb{M}, w, g \Vdash \exists xA$ iff for some g' such that $g' \equiv_x g$ we have $\mathbb{M}, w, g' \Vdash A$
- $\mathbb{M}, w, g \Vdash \varphi \circ \psi$ iff $\exists a, b \in W\,(wR_\circ ab \;\&\; \mathbb{M}, a, g \Vdash \varphi \;\&\; \mathbb{M}, b, g \Vdash \psi)$
- $\mathbb{M}, w, g \Vdash \varphi \to \psi$ iff $\forall a, b \in W\,(bR_\to aw \;\Rightarrow if\; \mathbb{M}, b, g \Vdash \varphi \; then\; \mathbb{M}, a, g \Vdash \psi)$[18]
- $\mathbb{M}, w, g \Vdash \psi \leftarrow \varphi$ iff $\forall a, b \in W\,(aR_\leftarrow bw \;\Rightarrow if\; \mathbb{M}, b, g \Vdash \varphi \; then\; \mathbb{M}, a, g \Vdash \psi)$ [19]

A formula φ is true at a point $w \in W$ if $\mathbb{M}, w, g \Vdash \varphi$ for an assignment g and false at w otherwise. $\mathbb{M}, w \Vdash \varphi$ is to be read as: $\mathbb{M}, w, g \Vdash \varphi$ for all assignments g, and $\mathbb{M}, w \nVdash \varphi$ is to be read as $\mathbb{M}, w, g \nVdash \varphi$ for all assignments g.

REMARK 31. The Kripke-semantics of the first order modal language $\mathcal{L}_Q$ is obtained by simply deleting the semantic rules for $\circ. \to, \leftarrow$.

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[17] Since we are interested in persistent valuations, once a predicate applies to a given sequence of objects at a point w, it will hold for such sequence in all $\leq -$successive worlds. This allows a simplification. Instead of having to consider a separate valuation for each world w, we simply have a global valuation from pairs of predicates and n-tuples into upsets of worlds.

[18] This can also be rewritten as $\forall a, b \in W\,(aR_\circ bw \;\Rightarrow if\; \mathbb{M}, b, g \Vdash \varphi \; then\; \mathbb{M}, a, g \Vdash \psi)$ and the condition for $\mathbb{M}, w, g \Vdash \psi \leftarrow \varphi$ below can likewise be rewritten as $\forall a, b \in W\,(aR_\circ wb \;\Rightarrow if\; \mathbb{M}, a, g \Vdash \varphi \; then\; \mathbb{M}, b, g \Vdash \psi)$.

[19] The alternative notation of $\psi \to \varphi$ as $\psi\backslash\varphi$ and $\psi \leftarrow \varphi$ as ψ/φ is sometimes used.

CHAPTER 2

Soundness

In this chapter we briefly verify the soundness of the axioms and inference rules put forward in definition 1 which amounts to show that no theorem derived from them is falsified anywhere in the class of $SDML$-frames. The proof of soundness for the corresponding set of axioms and rules in the reduced language $\mathcal{L}_Q$ with respect to DML-frames is entirely skipped in [**Restall 2005**] except for IR.5a and the second equivalence of A.5b. Here we will dedicate a short space to it.

2.1. Soundness of the axiom schemes.

We will first show that the axioms are valid in any constant domain skeleton based on a $SDML$-frame (or a DML-frame, when dealing with the corresponding logic deprived of substructural operators $\circ, \to, \leftarrow$).

Proof. let us fix an $CQSML$-model $\mathbb{M} = \left(\mathbb{F}, D, \{V_w\}_{w \in W}\right)$ based on a $SDML$-frame $\mathbb{F}$. If dealing with the corresponding logic deprived of substructural operators $\circ, \to, \leftarrow$ then we fix an $CQML$-model $\mathbb{M} = \left(\mathbb{F}, D, \{V_w\}_{w \in W}\right)$ based on a DML-frame $\mathbb{F}$. In such case, all axioms and rules involving $\circ, \to, \leftarrow$ are obviously ignored.

Now, the sequents (A1b) $\bot \Rightarrow \varphi$ and $\varphi \Rightarrow \top$ are trivially valid in $\mathbb{M}$. For $\bot \Rightarrow \varphi$, notice that by definition of the semantics, the antecedent is never satisfied at any point which makes the sequent vacuously true at all points. Also by definition, $\top$ is satisfied everywhere and thus can be deduced at any point. To verify the 2nd part of axioms of (A2a-b-c-d) is straightforward, since for similar reasons , $\Diamond \bot \Rightarrow \bot$ and $\top \Rightarrow \Box \top$ are valid ($\Diamond \bot$ is not satisfied anywhere and $\Box \top$ is satisfied everywhere, including terminal points). The validity of $\lhd \top \Rightarrow \bot$ and $\top \Rightarrow \rhd \bot$ and finally of (A3b) $\bot \circ \varphi \Rightarrow \bot$ and $\varphi \circ \bot \Rightarrow \bot$ are shown likewise. The sequent (A1a) $\varphi \Rightarrow \varphi$ is trivially valid as well because if a point $w \in \mathbb{F}$ is such that $w, g \Vdash \varphi$, then we can deduce φ at this point under the given assignment. If on the other hand $w \nVdash \varphi$, then the antecedent of the sequent is not satisfied and thus $\varphi \Rightarrow \varphi$ will not be falsified at w.

For the rest, let us fix a point $w \in \mathbb{F}$ and an assignment g. Then:

(1) Suppose that $\mathbb{M}, w, g \Vdash \varphi$. Then by definition $\mathbb{M}, w, g \Vdash \varphi \vee \psi$ with ψ any formula. Thus given φ we can semantically deduce $\varphi \vee \psi$. Since w is arbitrary, the (A1c) sequent $\varphi \Rightarrow \varphi \vee \psi$ is valid in $\mathbb{M}$. Similar reasoning shows that sequents (A1c-d) $\psi \Rightarrow \varphi \vee \psi$, $\varphi \wedge \psi \Rightarrow \varphi$ and $\varphi \wedge \psi \Rightarrow \psi$ are valid in $\mathbb{M}$.

(2) Now suppose that $\mathbb{M}, w, g \Vdash \chi \wedge (\varphi \vee \psi)$. By definition, this means that $\mathbb{M}, w, g \Vdash \chi$ and $\mathbb{M}, w, g \Vdash \varphi \vee \psi$, which then gives either (a) $\mathbb{M}, w, g \Vdash \chi$ and $\mathbb{M}, w, g \Vdash \varphi$ or (b) $\mathbb{M}, w, g \Vdash \chi$ and $\mathbb{M}, w, g \Vdash \psi$. Again by definition, we have $\mathbb{M}, w, g \Vdash (\chi \wedge \varphi) \vee (\chi \wedge \psi)$. Since w is arbitrary, the sequent (A1e) $\varphi \wedge (\psi \vee \chi) \Rightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$ is valid in $\mathbb{M}$

(3) Suppose that $\mathbb{M}, w, g \Vdash \Diamond (\varphi \vee \psi)$. By definition there exists a point v such that $R_{\Diamond} wv$ and $\mathbb{M}, v, g \Vdash \varphi \vee \psi$, which again semantically implies that $\mathbb{M}, v, g \Vdash \varphi$ or $\mathbb{M}, v, g \Vdash \psi$. Hence, either $\mathbb{M}, w, g \Vdash \Diamond\varphi$ or $\mathbb{M}, w, g \Vdash \Diamond\psi$, that is $\mathbb{M}, w, g \Vdash \Diamond\varphi \vee \Diamond\psi$. Since w is arbitrary the sequent (A2b) $\Diamond (\varphi \vee \psi) \Rightarrow \Diamond\varphi \vee \Diamond\psi$ is valid in $\mathbb{M}$.

(4) Suppose that $\mathbb{M}, w, g \Vdash \Box\varphi \wedge \Box\psi$. Hence $\mathbb{M}, w, g \Vdash \Box\varphi$ and $\mathbb{M}, w, g \Vdash \Box\psi$. Then, by definition, for all points v such that $R_{\Box}wv$ we have $\mathbb{M}, v, g \Vdash \varphi \wedge \psi$, which again semantically implies that for all points v such that $R_{\Box}wv$: $\mathbb{M}, w, g \Vdash \Box(\varphi \wedge \psi)$. Since w is arbitrary, the sequent (A2a) $\Box\varphi \wedge \Box\psi \Rightarrow \Box(\varphi \wedge \psi)$ is valid in $\mathbb{M}$.

(5) Suppose that $\mathbb{M}, w, g \Vdash \rhd\varphi \wedge \rhd\psi$. Then $\mathbb{M}, w, g \Vdash \rhd\varphi$ and $\mathbb{M}, w, g \Vdash \rhd\psi$ which semantically imply that for all points v such that $R_{\rhd}wv$ we have $\mathbb{M}, v, g \nVdash \varphi$ and $\mathbb{M}, v, g \nVdash \psi$. Hence for all points v such that $R_{\rhd}wv$ we have $\mathbb{M}, v, g \nVdash \varphi \vee \psi$, which again semantically implies that $\mathbb{M}, w, g \Vdash \rhd(\varphi \vee \psi)$. Since w is arbitrary, the sequent (A2d) $\rhd\varphi \wedge \rhd\psi \Rightarrow \rhd(\varphi \vee \psi)$ is valid in $\mathbb{M}$.

(6) Suppose that $\mathbb{M}, w, g \Vdash \lhd(\varphi \wedge \psi)$. Then there exists a point v such that $R_{\lhd}wv$ and $\mathbb{M}, v, g \nVdash \varphi \wedge \psi$, which by definition implies that either $\mathbb{M}, v, g \nVdash \varphi$ or $\mathbb{M}, v, g \nVdash \psi$. This semantically implies that, either $\mathbb{M}, w, g \Vdash \lhd\varphi$ or $\mathbb{M}, w, g \Vdash \lhd\psi$ (respectively) and thus $\mathbb{M}, w, g \Vdash \lhd\varphi \vee \lhd\psi$. Since w is arbitrary, the sequent (A2c) $\lhd(\varphi \wedge \psi) \Rightarrow \lhd\varphi \vee \lhd\psi$ is valid in $\mathbb{M}$.

(7) Now suppose that $\mathbb{M}, w, g \Vdash \chi \circ (\varphi \vee \psi)$. Then there exist $s, t \in W$ such that $wR_{\circ}st$ with $\mathbb{M}, s, g \Vdash \chi$ and $\mathbb{M}, t, g \Vdash \varphi \vee \psi$, that is $\mathbb{M}, t, g \Vdash \varphi$ or $\mathbb{M}, t, g \Vdash \psi$. This means that there exist $s, t \in W$ such that (a) $\mathbb{M}, s, g \Vdash \chi$ and $\mathbb{M}, t, g \Vdash \varphi$ or (b) $\mathbb{M}, s, g \Vdash \chi$ and $\mathbb{M}, t, g \Vdash \psi$. So, by definition we have $\mathbb{M}, w, g \Vdash (\chi \circ \varphi) \vee (\chi \circ \psi)$. Since w is arbitrary, the sequent $\chi \circ (\varphi \vee \psi) \Rightarrow (\chi \circ \varphi) \vee (\chi \circ \psi)$ in (A3a) is valid in $\mathbb{M}$ and a similar reasoning proves the same for $(\varphi \vee \psi) \circ \chi \Rightarrow (\varphi \circ \chi) \vee (\psi \circ \chi)$.

(8) Let $\mathbb{M}, w, g \Vdash \Box\forall x\varphi$. Then for each $v \in W$ such that $R_{\Box}wv$ we have $\mathbb{M}, v, g \Vdash \forall x\varphi$. Let us fix v, we have for each $g' \equiv_x g$, $\mathbb{M}, v, g' \Vdash \varphi$. Since we have a constant domain (which makes assignments point-independent) and v was an arbitrary point (such that $R_{\Box}wv$) this means that $\mathbb{M}, w, g' \Vdash \Box\varphi$ for all $g' \equiv_x g$ and thus $\mathbb{M}, w, g \Vdash \forall x\Box\varphi$. Now for the opposite direction, let $\mathbb{M}, w, g \Vdash \forall x\Box\varphi$. Then we have $\mathbb{M}, w, g' \Vdash \Box\varphi$ for each $g' \equiv_x g$. Fix g', then we have -given the point-independence of assignments- $\mathbb{M}, v, g' \Vdash \varphi$ for all $v \in W$ such that $R_{\Box}wv$, and since $g' \in [g]_{\equiv_x}$is arbitrary, we get $\mathbb{M}, v \Vdash \forall x\varphi$ for all $v \in W$ such that $R_{\Box}wv$. Hence $\mathbb{M}, w, g \Vdash \Box\forall x\varphi$. Since w is arbitrary, the sequent $\Box\forall x\varphi \dashv\Vdash \forall x\Box\varphi$ in (A5a) is valid in $\mathbb{M}$.

(9) Let $\mathbb{M}, w, g \Vdash \Diamond\exists x\varphi$. By constant domain, the assignments of each $R_{\Diamond}-$successor of w are still the same assignments of w and thus there is a point $v \in W$ such that $R_{\Diamond}wv$ and $\mathbb{M}, v, g \Vdash \exists x\varphi$. Hence there is an assignment $g' \equiv_x g$ with $\mathbb{M}, v, g' \Vdash \varphi$ and then $\mathbb{M}, w, g' \Vdash \Diamond\varphi$. Therefore $\mathbb{M}, w, g \Vdash \exists x\Diamond\varphi$. Now for the opposite direction, let $\mathbb{M}, w, g \Vdash \exists x\Diamond\varphi$. Then there is an assignment $g' \equiv_x g$ with $\mathbb{M}, w, g' \Vdash \Diamond\varphi$. Again, by the point-independence of assignments, we get $\mathbb{M}, v, g' \Vdash \varphi$ for the witness v of $\Diamond\varphi$ with $R_{\Diamond}wv$. But then $\mathbb{M}, v, g \Vdash \exists x\varphi$ and therefore $\mathbb{M}, w, g \Vdash \Diamond\exists x\varphi$. Since w is arbitrary, the sequent $\Diamond\exists x\varphi \dashv\Vdash \exists x\Diamond\varphi$ in (A5a) is valid in $\mathbb{M}$.

(10) Let $\mathbb{M}, w, g \Vdash \rhd\exists x\varphi$. Then for each $v \in W$ such that $R_{\rhd}wv$ we have $\mathbb{M}, v, g \nVdash \exists x\varphi$. Fix such a v. This means that $\mathbb{M}, v, g' \nVdash \varphi$ for all assignments $g' \equiv_x g$. Since assignments are point-independent and v was an arbitrary state (such that $R_{\rhd}wv$) this means that $\mathbb{M}, w, g' \Vdash \rhd\varphi$ for all assignments $g' \equiv_x g$ and thus $\mathbb{M}, w, g \Vdash \forall x \rhd \varphi$. For the opposite direction, let $\mathbb{M}, w, g \Vdash \forall x \rhd \varphi$. Then for all $g' \equiv_x g$ we have $\mathbb{M}, w, g' \Vdash \rhd\varphi$, or in different words, for all $v \in W$ such that $R_{\rhd}wv$ we get $\mathbb{M}, v, g' \nVdash \varphi$ (again, because assignments are point-independent). But then $\mathbb{M}, v, g \nVdash \exists x\varphi$ for all $v \in W$ such that $R_{\rhd}wv$, and hence $\mathbb{M}, w, g \Vdash \rhd\exists x\varphi$. Since w is arbitrary, the sequent $\rhd\exists x\varphi \dashv\Vdash \forall x\rhd\varphi$ in (A5b) is valid in $\mathbb{M}$.

(11) Let $\mathbb{M}, w, g \Vdash \lhd\forall x\varphi$. Then there is a $v \in W$ such that $R_{\lhd}wv$ and $\mathbb{M}, v, g \nVdash \forall x\varphi$. Therefore $\mathbb{M}, v, g' \nVdash \varphi$ for some assignment $g' \equiv_x g$. By point-independence of assignments we obtain $\mathbb{M}, w, g \Vdash \exists x \lhd \varphi$ with g' itself being the $g' \equiv_x g$ witness assignment. For the opposite direction, let $\mathbb{M}, w, g \Vdash \exists x \lhd \varphi$. Then there is an assignment $g' \equiv_x g$ such that $\mathbb{M}, w, g' \Vdash \lhd\varphi$. Hence there is a point $v \in W$ such that $R_{\lhd}wv$ and $\mathbb{M}, v, g' \nVdash \varphi$ (we used again the fact that assignments are point-independent).

Therefore $\mathbb{M}, v, g \nVdash \forall x\varphi$ and thus $\mathbb{M}, w, g \Vdash \lhd\forall x\varphi$. Since w is arbitrary, the sequent $\lhd\forall x\varphi \dashv\Vdash \exists x\lhd\varphi$ in (A5b) is valid in $\mathbb{M}$.

(12) Let $\mathbb{M}, w, g \Vdash \exists x\varphi \circ \psi$ with x not free in ψ. Then there are points $v, v' \in W$ such that $R_\circ wvv'$, $\mathbb{M}, v, g \Vdash \exists x\varphi$ and $\mathbb{M}, v', g \Vdash \psi$. Then there is an assignment $g' \equiv_x g$ with $\mathbb{M}, v, g' \Vdash \varphi$. Since assignments are point independent, we obtain $\mathbb{M}, w, g \Vdash \exists x\,(\varphi \circ \psi)$ via the assignment $g' \equiv_x g$ since $\mathbb{M}, w, g' \Vdash \varphi\circ\psi$. For the opposite direction, let $\mathbb{M}, w, g \Vdash \exists x\,(\varphi \circ \psi)$ with x not free in ψ. Then there is an assignment $g' \equiv_x g$ with $\mathbb{M}, w, g' \Vdash \varphi \circ \psi$. Again, by point-independence of assignments, we can write for the witness points $v, v' \in W$ with $R_\circ wvv'$: $\mathbb{M}, v, g' \Vdash \varphi$ and $\mathbb{M}, v', g' \Vdash \psi$. Since x not free in ψ, the assignment g' which varies at most on x is irrelevant and we simplify as $\mathbb{M}, v', g \Vdash \psi$. From $\mathbb{M}, v, g' \Vdash \varphi$ we obtain $\mathbb{M}, v, g \Vdash \exists x\varphi$, and thus $\mathbb{M}, w, g \Vdash \exists x\varphi \circ \psi$. Since w is arbitrary, the sequent (A5c) $\exists x\varphi \circ \psi \dashv\Vdash \exists x(\varphi \circ \psi)$ is valid in $\mathbb{M}$ provided x does not occur free in ψ, and a similar reasoning proves the same for (A5d) $\varphi \circ \exists x\psi \dashv\Vdash \exists x(\varphi \circ \psi)$ provided x does not occur free in φ.

(13) Let $\mathbb{M}, w, g \Vdash \forall x\varphi$. Then we have $\mathbb{M}, w, g' \Vdash \varphi$ for each $g' \equiv_x g$. One class of such assignments is $g_{i\in Var}^{y=x}$ with y not occurring in φ, which assign the same value i (whichever this might be) to both x and y. Hence $\mathbb{M}, w, g_i^{y=x} \Vdash \varphi$ for all $i \in Var$, with x being free in φ, and thus $\mathbb{M}, w, g \Vdash \varphi\,[y/x]$. Since w is arbitrary, the sequent (A4a) $\forall x\varphi \vdash \varphi\,[y/x]$, with x being free in φ, is valid in $\mathbb{M}$.

(14) Let $\mathbb{M}, w, g \Vdash \varphi\,[y/x]$ with x being free in φ. Then there is an assignment $g' \equiv_x g$ such that $g'\,(x) = g'\,(y)$ and $\mathbb{M}, w, g' \Vdash \varphi$, therefore $\mathbb{M}, w, g \Vdash \exists x\varphi$. Since w is arbitrary, the sequent (A4b) $\varphi\,[y/x] \vdash \exists x\varphi$, with x being free in φ , is valid in $\mathbb{M}$.

Since the model $\mathbb{M}$ and $SDML$-frame $\mathbb{F}$ were arbitrary then the axioms are valid in any $SDML$-frame based constant domain skeleton. □

2.2. Soundness of the inference rules

We now verify that the inference rules preserve validity on any $SDML$-frame based constant domain skeleton (or a DML-frame, when dealing with the corresponding logic deprived of substructural operators $\circ, \rightarrow, \leftarrow$).

(IR.1a) Fix a model $\mathbb{M}$, a point w and a trio of formulas φ, ψ, χ . Assume that $\mathbb{M}, w, g \Vdash \varphi$ and that both $\varphi \vdash \psi$ and $\psi \vdash \chi$ are valid. By such validity it will follow that $\mathbb{M}, w, g \Vdash \psi$ and consequently $\mathbb{M}, w, g \Vdash \chi$. Since the model, the formulas and the point were arbitrary, this gives us our result. In fact all structural rules (IR1a-b-c) and all lattice rules (IR2a-b-c-d) are trivially sound as we interpret the sequent symbol $\Rightarrow$ by the partial order $\leq$ in the poset $(W, \leq)$ and given $\wedge$ and $\vee$ interpreted as lattice meet and join.

Soundness of (IR.5a)

Proof. Fix a model $\mathbb{M}$, a point w and:

(1) assume that $\varphi \Rightarrow \psi\,[y/x]$ is valid.
(2) suppose that $\mathbb{M}, w, g \Vdash \varphi$, with x not occurring free in it,

Observe that $\mathbb{M}, w, g \Vdash \forall x\psi\,(x)$ iff $\mathbb{M}, w, g' \Vdash \psi\,(x)$ for all $g' \equiv_x g$, but since x does not occur free in φ then any variation on the assignment of x does not destroy the satisfaction relation assumed in (2). In other words, since we assumed $\mathbb{M}, w, g \Vdash \varphi$, with x not occurring free, then we have $\mathbb{M}, w, g' \Vdash \varphi$ for all $g' \equiv_x g$. But then by applying (1) to all these satisfaction instances, we obtain: $\mathbb{M}, w, g' \Vdash \psi\,[y/x]$ for all $g' \equiv_x g$, and therefore $\mathbb{M}, w, g \Vdash \forall x\psi\,(x)$ as desired. □

Soundness of (IR.5b)

Proof. Fix a model $\mathbb{M}$, a point w and :

(1) assume that $\varphi[y/x] \Rightarrow \psi$ (with x not occurring free in ψ) is valid
(2) suppose that $\mathbb{M}, w, g \Vdash \exists x \varphi(x)$

Then by (2) we have $\mathbb{M}, w, g' \Vdash \varphi(x)$ for some $g' \equiv_x g$, and by applying (1) we get $\mathbb{M}, w, g' \Vdash \psi$ (notice that $\varphi[y/x]$ means that y is free for x -it will not get bound- and replaces only free instances of x, so we can simply take w.l.o.g. the special case where $y = x$, i.e. $\varphi[x/x]$). But since x does not occur free in ψ then any variation on the assignment of x does not destroy the satisfaction relation just obtained. Therefore by changing the assignment of x in order to reconvert g' into g, we still have $\mathbb{M}, w, g \Vdash \psi$, as desired. $\square$

Soundness of (IR.3c) -verification of IR.3a/3b/3d are all similar-

Proof. Fix a model $\mathbb{M}$, a point w and a pair of formulas φ, ψ . Assume that $\mathbb{M}, w, g \Vdash \triangleleft\psi$ and that $\varphi \vdash \psi$ is valid. By the satisfaction conditions on $\triangleleft$ we know that there is an $v \in W$ such that $R_{\triangleleft}wv$ and $\mathbb{M}, v, g \nVdash \psi$. By the validity of $\varphi \vdash \psi$ it must be the case that $\mathbb{M}, v, g \nVdash \varphi$ and therefore the conditions are given for $\mathbb{M}, w, g \Vdash \triangleleft\varphi$. Since the model, the formulas and the point were arbitrary, this gives us our result. $\square$

Soundness of inference rules IR.4a IR.4b IR.4c *:*

We will verify that $\varphi \circ \psi \Rightarrow \chi$ semantically implies $\psi \Rightarrow \varphi \to \chi$.

Proof. Fix a model $\mathbb{M}$, a point $c \in W$ and a let φ, ψ, χ be formulas. Assume that $\mathbb{M}, c, g \Vdash \psi$ and that $\varphi \circ \psi \Rightarrow \chi$ is valid. We aim to show $\mathbb{M}, c, g \Vdash \varphi \to \chi$, so suppose there exists $a, b \in W$ such that $bR_{\to}ac$ and $\mathbb{M}, b, g \Vdash \varphi$ to try and reach $\mathbb{M}, c, g \Vdash \chi$. Since $bR_{\to}ac$ then $aR_{\circ}bc$, and since $\mathbb{M}, b, g \Vdash \varphi$ and $\mathbb{M}, c, g \Vdash \psi$ then $\mathbb{M}, a, g \Vdash \varphi \circ \psi$ and thus $\mathbb{M}, a, g \Vdash \chi$ because $\varphi \circ \psi \Rightarrow \chi$ is valid. Thus if $bR_{\to}ac$ and $\mathbb{M}, b, g \Vdash \varphi$, we get $\mathbb{M}, a, g \Vdash \chi$ Therefore $\mathbb{M}, c, g \Vdash \varphi \to \chi$ as desired. $\square$

We will verify that $\varphi \circ \psi \Rightarrow \chi$ semantically implies $\varphi \Rightarrow \chi \leftarrow \psi$.

Fix a model $\mathbb{M}$, a point $b \in W$ and a let φ, ψ, χ be formulas.

Proof. Assume that $\mathbb{M}, b, g \Vdash \varphi$ and that $\varphi \circ \psi \Rightarrow \chi$ is valid. We aim to show $\mathbb{M}, b, g \Vdash \chi \leftarrow \psi$, so suppose there exists $a, c \in W$ such that $aR_{\leftarrow}cb$ and $\mathbb{M}, c, g \Vdash \psi$ to try and reach $\mathbb{M}, a, g \Vdash \chi$. Since $aR_{\leftarrow}cb$ then $aR_{\circ}bc$, and since $\mathbb{M}, b, g \Vdash \varphi$ and $\mathbb{M}, c, g \Vdash \psi$ then $\mathbb{M}, a, g \Vdash \varphi \circ \psi$ and thus $\mathbb{M}, a, g \Vdash \chi$ because $\varphi \circ \psi \Rightarrow \chi$ is valid. Thus if $aR_{\leftarrow}cb$ and $\mathbb{M}, c, g \Vdash \psi$, we get $\mathbb{M}, a, g \Vdash \chi$ Therefore $\mathbb{M}, b, g \Vdash \chi \leftarrow \psi$ as desired. $\square$

We will verify that $\psi \Rightarrow \varphi \to \chi$ semantically implies $\varphi \circ \psi \Rightarrow \chi$.

Proof. Fix a model $\mathbb{M}$, a point $c \in W$ and a let φ, ψ, χ be formulas. Assume that $\mathbb{M}, a, g \Vdash \varphi \circ \psi$ and that $\psi \Rightarrow \varphi \to \chi$ is valid. We aim to show $\mathbb{M}, a, g \Vdash \chi$. Since $\mathbb{M}, a, g \Vdash \varphi \circ \psi$ then there are $b, c \in W$ such that $aR_{\circ}bc$, and $\mathbb{M}, b, g \Vdash \varphi$ and $\mathbb{M}, c, g \Vdash \psi$. From the latter we obtain $\mathbb{M}, c, g \Vdash \varphi \to \chi$. From $aR_{\circ}bc$ and $\mathbb{M}, b, g \Vdash \varphi$ we get $\mathbb{M}, a, g \Vdash \varphi \circ (\varphi \to \chi)$, namely $\mathbb{M}, a, g \Vdash \chi$, as desired. $\square$

That $\varphi \Rightarrow \chi \leftarrow \psi$ semantically implies $\varphi \circ \psi \Rightarrow \chi$ is proven similarly.

CHAPTER 3

Completeness theorem for constant domains quantified modal logic (*CQML*)

The main goal of this part is to prove a completeness theorem for (constant domains) quantified distributive modal logic. This is not an original contribution, but rather a detailed exposition of [**Restall 2005**]'s proof. This will not be entirely redundant, though, because Restall's exposition is quite schematic, skipping or condensing many parts of the proof. In particular, the Truth lemma is omitted altogether. No verification is made as to the question whether the canonical frame is indeed member of the class of DML-frames. But moreover, the sketchy nature of some key points makes enough room for potential mistakes. A crucial step on the proof requires a Lindenbaum lemma analogue for pairs of sets of formulas (lemma 46 below). That is, we need to prove that given any pair (of sets of formulas), such pair can be extended to a *full* quantified-suited pair. At this stage, Restall's presentation of the proof seems to fail. He makes use of a single enumeration of formulas of a language which is not clearly specified: is it the original language L or the language L^+ extended with countably many new constants? (In our presentation of the proof, named as $\mathcal{L}_Q$ and $\mathcal{L}_Q^{+'}$ see definition 44) while in fact two enumerations are needed: the enumeration of all formulas from the extended language L^+ and the enumeration of all constants in L^+.

REMARK 32. Notice that the language we are using here is such that all terms are variables (while Restall uses also constants). We will follow [**Hughes & Cresswell 1996**] p256 in using the set of variables as the domain of quantification ($Var = D$), thus taking advantage of the language to build up the canonical model.

Regarding our initial goal to extend Restall's proof with the addition of substructural connectives to the proof, we have encountered difficulties to provide witness points for when another point has $\varphi \circ \psi$ within its members or fails to have $\varphi \to \psi$ as one of them. We will briefly comment on these after the presentation of the proof for (constant domain) quantified distributive modal logic.

Before advancing any further, let us first recall the definition of completeness as stated for the non-Boolean case:

DEFINITION 33. *(completeness)* A distributive modal logic Λ is strongly complete w.r.t. a class S of structures iff for every pair $\Sigma \nvdash_\Lambda \Delta$ there is a model $\mathbb{M}$ based on some $\mathbb{F} \in S$ such that $\Sigma \nVdash_\mathbb{M} \Delta$ (i.e. there exists a point $w \in \mathbb{F}$ such that $\mathbb{M}, w \Vdash \Sigma$ and $\mathbb{M}, w \nVdash \Delta$).

REMARK 34. We purposively leave some vagueness on the definition above as to the structures involved. These can be algebraic structures (and then $w \in \mathbb{F}$ is to be read as "w belongs to the carrier of the algebra $\mathbb{F}$") or relational ones (and then $w \in \mathbb{F}$ is to be read as "w belongs to the universe of the frame $\mathbb{F}$").

As it can be guessed from the definition above, completeness theorems are model-existence theorems, and the corresponding proof shows (in the classical setting) that for every consistent set of formulas there is a point, in a model suitable for the logic, that makes them simultaneously true. In fact a big conceptual part of the proof consists precisely in building such model out of the formulas from the logic's language. If we are seeking an algebraic completeness, the standard construction of a suitable model for the logic is known as the Lindenbaum-Tarski Algebra. But here we are concerned with a completeness result w.r.t. a relational semantics. We need to build a frame and then a canonical model based on a constant domain skeleton over it.

Three crucial properties are required from this model to work as a universal counterexample-source for our logic:

- any pair of formula sets such that the first one does not deduce any formula of the second (denoted $\Sigma \nvdash \Delta$) has to be represented by at least one point in the canonical model's frame. That is, we need to show that if $\Sigma \nvdash \Delta$ then there is a point $\langle \Sigma', \Gamma' \rangle$ in the canonical model with $\Sigma \subseteq \Sigma'$ and $\Delta \subseteq \Delta'$. In the classical case this is assured by the Lindenbaum lemma, while here we will use a version of it -the *pair extension in a new language* lemma 46-.
- From the fact that $\Sigma \subseteq \Sigma'$ and $\Delta \subseteq \Delta'$ we still need to be able to deduce that, by the canonical valuation, $\langle \Sigma', \Gamma' \rangle$ is a point in which every element of Σ is true but every element of Δ is false (we may refer to Σ' as the *positive* side and to Δ' as the *negative* side of the point $\langle \Sigma', \Gamma' \rangle$). It must be the case for any formula ψ of our logic and any point $\langle \Sigma, \Gamma \rangle$ in our model that $V_{\langle \Sigma, \Gamma \rangle}(\psi) = 1$ iff $\psi \in \Sigma$, i.e. that under the canonical valuation any formula is true at a point iff it is contained in the positive side of the point. While the canonical valuation definition (on page 32) already tells us that the truth at point w of any atomic sentence on the canonical model amounts to membership in the positive side of w, this only constitutes the first step. The lemma that lifts this property to all formulas is classically known as the *truth-lemma*. The truth lemma has the shape: $\Sigma \Vdash \psi \iff \psi \in \Sigma$ for an arbitrary formula ψ.
- the canonical model must suitable for the logic Λ, i.e. we need to verify that the underlying frame belongs to the class of frames that satisfy the logic Λ.

As a guide reference we show here a dependence diagrams on the several lemmas we will be proving in next sections:

$$
\begin{array}{cccccccc}
 & & & & \begin{array}{c} Pair\,extension \\ on\,a\,new\,language \\ lemma\ (46) \end{array} & \searrow & \\
 & & & & & & \begin{array}{c} Completeness \\ Proof \end{array} \\
 & & \begin{array}{c} Prime\,Theory \\ Claim\,(40) \end{array} & \longrightarrow & \begin{array}{c} Truth\,lemma \\ (\wedge,\vee)-cases \end{array} & \nearrow & \\
 & & \begin{array}{c} Well-behaved \\ quantifiers \\ lemma\,(43) \end{array} & \longrightarrow & (\forall,\exists)-cases & & \\
 & & & & | & & \\
 & & \begin{array}{c} witnessing-pair \\ existence \end{array} & & Truth\,lemma & & \\
 & \nearrow & lemma\,(55) & \longrightarrow & (\Box,\Diamond,\rhd,\lhd,\circ,\rightarrow,\leftarrow) & & \\
\begin{array}{c} Finite\,addition \\ lemma\ (53) \end{array} & \longrightarrow & \begin{array}{c} Pair\,extension \\ in\,the \\ same\,language \\ lemma\ (54) \end{array} & \nearrow & -cases & &
\end{array}
$$

3.1. The canonical frame

The whole intuition behind canonical models is to try to capture the logic in a single model by using the language itself as building material, thus in classical modal logic the points of the frame are maximal consistent sets of formulas, while here we need to have pairs of sets such that the first does not entail the second (they are still "maximal" in the sense that they constitute a partition of all formulas). Both the accessibility relations and the canonical valuation are defined in terms of formulas membership to one of the sides of the points concerned.

Remark 35. Even though we are talking of the pairs as the points of the canonical frame, it should be clear that since these pairs $\langle\Gamma,\Omega\rangle$ are *full* ($\Gamma\cup\Omega=\mathcal{L}_Q^{+'}$), each $\langle\Gamma,\Omega\rangle$ is fully determined by the first coordinate Γ (this property will disappear when we drop distributivity). In this sense, the points of the canonical frame are rather the first coordinates Γ of each pair $\langle\Gamma,\Omega\rangle$, which are Prime (filter) theories by the claim 40. Order dually, then, each second coordinate Ω is an Prime Ideal theory. Together they form a perfect "cut" of the logical space. Notice also that each point $\langle\Gamma,\Omega\rangle$ specifies explicitly *both* the set Γ of formulas that are contained in the *positive* side of the point (and thus will be true at that point by the canonical valuation and the truth lemma) and the set Ω of formulas that are contained in the *negative* side of the point (and thus will be false at that point by the canonical valuation and the truth lemma). In the classical modal case, each point of the canonical model only specifies explicitly the formulas that are true in it, the negative side is omitted altogether because the false propositions are directly expressed in the language and end up as (negative) formulas being true in the point. Since here we have not Boolean negation, we need to explicitly list the false propositions in Ω. The maximal consistent sets of classical modal logic are special cases of pairs: all point of the model are of the shape $\langle\Sigma,[\bot]_{\equiv}\rangle$, in other words, they do not derive bottom, they are consistent.

Let us first take a brief view at the ingredients of a canonical model approach to completeness for a classical quantified modal logic Λ with Barcan formulas (i.e. with constant domains).

3.1.1. The main properties required from the frame points in the classical setting.

In the classical setting, as said previously, the points are maximal $\Lambda-$consistent sets of formulas (in fact, maximal filters in $\wp(Form)$). The task of showing that any consistent set of formulas can be satisfied in the canonical model requires first the guarantee that any such set will be represented by at least one point in the canonical frame (which more concretely means it will be contained in such point). That is what the well-known Lindenbaum lemma provides:

Lemma 36. *(Lindenbaum lemma) Suppose Γ is an Λ-consistent set of well-formed formulas, then there is a maximally Λ-consistent set of well-formed formulas Γ' such that $\Gamma\subseteq\Gamma'$.*

The shape of this lemma will change accordingly to the setting we are in. When Boolean complementation is dropped we require a reformulation like lemma 46 in the sense that incomplete points to be extended have now both an explicit *positive* assertion side and an explicit *negative* assertion side (while Lemma 34 treats points that only have explicitly a *positive* assertion side). To see more clearly the continuity with the classical case, it is worth noticing that what we call the *positive* side of a point is in fact a prime filter in $\wp(Form)$, while the negative side is its complement: a prime ideal.

The lemma 46 is a bit more complex since in fact it also deals with an extra component: the quantifiers. To fully grasp what role these play in the extra requirements that the points need to comply, it is worth to pick up the matter right from the start, in the classical setting.

3.1.1.1. *Omega-saturation.* Given $\mathcal{L}_{Q\cup\{\neg\}}$ the language of quantified classical modal logic, we need to take care of the quantifiers, and in particular we want the model to satisfy the natural condition that if $\forall x\varphi$ does not hold at a point, there must be some object in the domain of quantification for which φ does not hold at that point. Since our domain D is in fact the set Var of individual variables of $\mathcal{L}_{Q\cup\{\neg\}}$, we can rephrase this property as follows: for any given point of the model, if $\neg\forall x\varphi$ holds then there must be some $y \in Var$ such that $\neg\varphi\,[y/x]$ holds as well.

Clearly it is not enough to have maximal consistency for such a property to arise. Let $\Omega = \{\neg\forall x\varphi x, \varphi y_1, \varphi y_2 \ldots\}$ be the set that consists of $\neg\forall x\varphi$ together with all formulas $\varphi\,[y/x]$ for $y \in Var$. Every finite subset of Ω being consistent (hence, having a model), the set Ω will be consistent as well by the compactness theorem of first order logic (a set of sentences has a model iff every finite subset has a model) which thereby has a maximal consistent extension Ω'. But we cannot hope for Ω' to have a witness $\neg\varphi\,[y/x]$ supporting the falsehood of $\neg\forall x\varphi$, because $y = y_i$ for some $i \in \mathbb{N}$ and thus $\varphi\,[y/x] \in \Omega \subseteq \Omega'$. So having both $\neg\varphi\,[y/x]$ and $\varphi\,[y/x]$ in Ω' would lead to inconsistency. This is so for any $y \in Var$ so not only Ω' has not such witness but we cannot hope to add it either. Hence the points of our (classical) canonical model need to independently satisfy the following additional property besides being maximally consistent:

DEFINITION 37. *(ω-completeness)* A set Γ of well-formed formulas is ω-complete iff for every formula φ and every individual variable x there is some individual variable y such that $\varphi\,[y/x] \to \forall x\varphi \in \Gamma$.

This property ensures that if Γ is a point in our (canonical) model and $\forall x\varphi \notin \Gamma$ then there is an individual variable $y \in Var$ such that $\varphi\,[y/x] \notin \Gamma$. For if $\varphi\,[z/x] \in \Gamma$ for all variables z, then certainly $\varphi\,[y/x] \in \Gamma$ and thus by $\varphi\,[y/x] \to \forall x\varphi \in \Gamma$ we get $\forall x\varphi \in \Gamma$. This property is precisely what will make the quantifiers case of the Truth-lemma work: the ω-completeness encoded as definition 37 is sufficient to obtain $\Sigma \vdash \forall x\varphi\,(x)$ iff $\forall x\varphi\,(x) \in \Sigma$. [1] Since $\exists$ and $\forall$ are inter-definable via Boolean negation, we get $\Sigma \vdash \exists x\varphi\,(x)$ iff $\exists x\varphi\,(x) \in \Sigma$ in the same shot.

Clearly, the maximal consistent set Ω' exemplified above is not ω-complete, but we need to include it somehow in the model otherwise a consistent set of formulas will not be satisfied by the model. The solution turns out to be the extension of $\mathcal{L}_{Q\cup\{\neg\}}$ with countably many new variables so that Ω' can find a proper witness $y' \in Var'$ such that $\varphi\,[y'/x] \to \forall x\varphi \in \Omega$ in the new language $\mathcal{L}'_{Q\cup\{\neg\}} \supset \mathcal{L}_{Q\cup\{\neg\}}$. This is possible because predicate logic (and its modal extension) has the property that if Γ is a consistent set of formulas from $\mathcal{L}$, it remains consistent in $\mathcal{L}'_{Q\cup\{\neg\}}$ [**Hughes & Cresswell 1996**]. On lemma 46, instead of extending the (non-Boolean analogues of) consistent sets to maximally consistent ones first, and then ensuring that all such sets acquire the ω-completeness property, we do both simultaneously.

In the distributive case, we don't have Boolean negation anymore and thus the ω-completeness property won't give us the existential case which depended on quantifier interdefinability. Now the statement of this property as definition 37 is no longer sufficient and turns out as too weak. So the ω-completeness property has to be strengthened with what we may call a "super-primeness" condition as can be seen on definition 41.

3.1.2. The points of the canonical frame in a non-Boolean setting. It is a standard practice to extend the relation $\vdash$ to relate sets of sentences. For every $\Sigma, \Delta \subseteq Form$ we set $\Sigma \vdash \Delta$ iff $\bigwedge \Sigma_0 \vdash \bigvee \Delta_0$ for some $\Sigma_0 \subseteq_\omega \Sigma$ and some $\Delta_0 \subseteq_\omega \Delta$, where $\subseteq_\omega$ denotes finitary inclusion. Note that Σ and Δ may not

[1] The property ensures (via contrapositive) the right-to-left direction of this key sentence: $\forall x\varphi\,(x) \in \Sigma$ iff $\varphi\,(y) \in \Sigma$ for all $y \in Var$. Then by IH $\varphi\,(y) \in \Sigma$ for all $y \in Var$ iff $\Sigma \vdash \varphi\,(y)$ for all $y \in Var$, and in the logic $\Sigma \vdash \varphi\,(y)$ for all $y \in Var$ iff $\Sigma \vdash \forall x\varphi\,(x)$ (where the left-to-right direction follows from order theoretic properties of lattices: since Σ is an upset and $\forall x\varphi\,(x)$ is a meet, namely $\forall x\varphi\,(x) \equiv \bigwedge_{g' \equiv_x g} \|\varphi\,(x)\|^{g'}$, then if $\forall x\varphi\,(x) \in \Sigma$ we must have $\varphi\,(y) \in \Sigma$ for all $y \in Var$ as well, since the meet is below all of is members).

be finite, but by $\Sigma \vdash \Delta$ we always mean that some finite conjunction of elements in Σ entail some finite disjunction of elements in Δ, as formally stated. Let $C(\Sigma) = \{\psi \mid \Sigma \vdash \psi\}$ and $C^d(\Delta) = \{\phi \mid \phi \vdash \Delta\}$, then

PROPOSITION 38. *Both C and C^d are closure operators on $\wp(Form)$.*

Given a pair of formula-sets $\langle \Sigma, \Delta \rangle$, it is said to be a $\vdash$-pair if $\Sigma \nvdash \Delta$. From now on we will refer to a *pair* (no extra properties) as an *ordered pair* and to a *$\vdash$-pair* simply as a *pair*. Moreover, a pair $\langle \Sigma, \Delta \rangle$ is said to be *full* in a language $\mathcal{L}$ iff $\Sigma \cup \Delta = Form(\mathcal{L})$, with $Form(\mathcal{L})$ the set of all formulas in $\mathcal{L}$.

Since we are in a non-Boolean setting, the dual frame of the Lindenbaum-Tarski algebra will have prime-filters rather than ultrafilters as points. Accordingly, the points of the canonical model will be prime theories, rather than maximally consistent ones.

DEFINITION 39. A set T of sentences is a *prime theory* if:

- T is a theory, i.e. it is closed under derivability[2], and
- $\phi \vee \psi \in T$ iff $\phi \in T$ or $\psi \in T$.

All points in our canonical frame will be full pairs, and thereby also prime theories:

CLAIM 40. If $\langle \Sigma, \Delta \rangle$ is a full pair, then Σ is a prime theory.

PROOF. Suppose $\langle \Sigma, \Delta \rangle$ is a full pair. Then $\Sigma \nvdash \Delta$ and $\Sigma \cup \Delta = Form(\mathcal{L})$. From this we can show a number of facts:

- Σ is closed under derivability. Observe that everything that is derivable from Σ is contained in either Σ or Δ by fullness. Only the first is possible, because otherwise $\Sigma \vdash \Delta$ which contradicts our assumption that$\Sigma \nvdash \Delta$. So the claim holds. This brings trivially that:
  - $\varphi \wedge \psi \in \Sigma$ iff $\varphi \in \Sigma$ and $\psi \in \Sigma$. For left to right direction, if $\varphi \wedge \psi \in \Sigma$ then both φ and ψ must be in Σ as well. To see why, suppose that at least one of φ and ψ is not in Σ. Then it is in Δ since $\Sigma \cup \Delta = Form(\mathcal{L})$. But then $\Sigma \vdash \Delta$ because $\varphi \wedge \psi \vdash \ldots \vee \varphi \vee \ldots$ and $\varphi \wedge \psi \vdash \ldots \vee \psi \vee \ldots$, which contradicts $\Sigma \nvdash \Delta$. Thus φ and ψ are in Σ. A similar argument shows that if $\varphi \in \Sigma$ and $\psi \in \Sigma$ then $\varphi \wedge \psi \in \Sigma$. Thus $\varphi \wedge \psi \in \Sigma$ iff $\varphi \in \Sigma$ and $\psi \in \Sigma$.
  - $\bot \notin \Sigma$. For everything can be derived from bottom. Now since $\Sigma \nvdash \Delta$ then $\bot \notin \Sigma$ because otherwise everything could be derived from Σ, in particular Δ.
  - $\top \in \Sigma$. $\top$ is derivable from anything. So if $\top \in \Delta$ then $\Sigma \vdash \Delta$. But $\Sigma \nvdash \Delta$ so $\top \notin \Delta$. It follows immediately that $\top \in \Sigma$ because $\Sigma \cup \Delta = Form(\mathcal{L})$.
- Suppose $\varphi \vee \psi \in \Sigma$ then either φ or ψ must be in Σ as well, otherwise they are both in Δ since $\Sigma \cup \Delta = Form(\mathcal{L})$. But then $\Sigma \vdash \Delta$ because $\varphi \vee \psi \vdash \ldots \vee \varphi \vee \psi \vee \ldots$ (note that $\varphi \vee \psi \nvdash \varphi$ and $\varphi \vee \psi \nvdash \psi$ so it is enough for one of them to be in Σ). A similar argument shows that if $\varphi \in \Sigma$ or $\psi \in \Sigma$ then $\varphi \vee \psi \in \Sigma$. Thus $\varphi \vee \psi \in \Sigma$ iff $\varphi \in \Sigma$ or $\psi \in \Sigma$.

Thus, Σ is a prime theory. □

The non-Boolean setting also means we have to pay special attention to both quantifiers instead of only focusing on the universal one. The ω-completeness property -in a distributive setting- has to be formulated for each quantifier separately, as follows:

[2]Observe that this, stated algebraically as $\forall x\,((x \in T \,\&\, x \leq y) \rightarrow y \in T)$ and $\forall x, y\,((x \in T \,\&\, y \in T) \rightarrow x \wedge y \in T)$, means that T is a filter.

DEFINITION 41. Quantifier suited pairs.

A pair $\langle \Sigma, \Delta \rangle$ is quantifier suited iff the following two conditions hold:

- (QS1) If $\Sigma \vdash \Delta \cup \{A(x)\}$ for all $x \in Var$ then $\Sigma \vdash \Delta \cup \{\forall v A[v/x]\}$.
- (QS2) If $\Sigma \cup \{A(x)\} \vdash \Delta$ for each $x \in Var$ then $\Sigma \cup \{\exists v A[v/x]\} \vdash \Delta$.

REMARK 42. Observe that, to bring it closer to the classical case formulation (definition 37), (QS1) can be stated contrapositively: $\Sigma \nvdash \Delta \cup \{\forall x A(x)\}$ implies $\Sigma \nvdash \Delta \cup \{A(x)\}$ for some $x \in Var$. The contrapositive of (QS2) would be $\Sigma \cup \{\exists x A(x)\} \nvdash \Delta$ implies $\Sigma \cup \{A(x)\} \nvdash \Delta$ for some $x \in Var$.[3]

Observe that the conditions (QS1) -a sort of "super-filter" condition- and (QS2) -a sort of "super-prime" condition- are non-trivial since they implicitly appeal to compactness. [4]Namely for (QS1), if $\Sigma \vdash \Delta \cup \{A(x)\}$ for each $x \in Var$ then we get $\Sigma \vdash \Delta \cup \{A(x_0) \wedge A(x_1) \wedge \ldots \wedge A(x_n)\}$ for any $n \in \mathbb{N}$ (we assume the language has countably many variables). This means that models satisfying Σ do satisfy the union of delta with any finite set of formulas of shape $A(v)$ with $v \in Var$, and thus by compactness they also satisfy the union of delta with the entire set of such formulas.

By ensuring that a pair if both quantifier suited and full, we obtain the expected behaviour of quantifiers:

LEMMA 43. *If $\langle \Sigma, \Delta \rangle$ is a full and quantifier suited pair, then*

- $\exists v A \in \Sigma$ *iff* $A(x) \in \Sigma$ *for some* $x \in Var$
- $\forall v A \in \Sigma$ *iff* $A(x) \in \Sigma$ *for all* $x \in Var$.

PROOF. Assume $\langle \Sigma, \Delta \rangle$ is a full and quantifier suited pair. Then $\Sigma \nvdash \Delta$ and $\Sigma \cup \Delta = Form(\mathcal{L}_Q)$.

First claim

($\Rightarrow$) Suppose $\exists v A \in \Sigma$. If $A(x) \in \Delta$ for some $x \in Var$ then $\Sigma \cup \{A(x)\} \vdash \Delta$. By the (QS2), $\Sigma \cup \{\exists v A[v/x]\} \vdash \Delta$. Thus, since $\exists v A \in \Sigma$ we have in fact $\Sigma \vdash \Delta$. But by assumption $\Sigma \nvdash \Delta$, contradiction! Hence $A(x) \notin \Delta$. Since $\Sigma \cup \Delta = Form(\mathcal{L}_Q)$, it follows that $A(x) \in \Sigma$.

($\Leftarrow$) Suppose that $A(x) \in \Sigma$ for some $x \in Var$. Clearly $\Sigma \vdash A(x)$. Then $\Sigma \vdash \exists v A$ by (A.4bi) from definition 1 (we can assume that a finite subset $\Gamma \subseteq \Sigma$ suffices, namely $\Gamma = \{A(x)\}$, so we apply the rule with $\bigwedge \Gamma \vdash A(x)$). Since $\Sigma \nvdash \Delta$ then $\exists v A \notin \Delta$. So $\exists v A \in \Sigma$ because $\Sigma \cup \Delta = Form(\mathcal{L}_Q)$.

Thus, $\exists v A \in \Sigma$ iff $A(x) \in \Sigma$ for some $x \in Var$.

Second claim follows in an analogous way:

($\Rightarrow$) Suppose $\forall v A \in \Sigma$. Then $\Sigma \vdash A(x)$ for each $x \in Var$, by application of universal instantiation (A.4a) (we have non-empty domain)[5]. If $A(x) \in \Delta$ for some x then we have $\Sigma \vdash \Delta$. But by assumption we have

[3]If Σ is maximal and $\Sigma \cup \{\exists x A(x)\} \nvdash \Delta$ we must have $\exists x A(x) \in \Sigma$, otherwise Σ is not *maximally* consistent (from $\Sigma \cup \{\exists x A(x)\} \nvdash \Delta$ we can see that $\Sigma \cup \{\exists x A(x)\}$ is consistent, since bottom would derive anything, in particular Δ). Same reasoning goes for the consequent part of (2), it implies $A(x) \in \Sigma$ for all $x \in Var$, by maximality. This is what we could call Σ-super primeness: $\exists x A(x) \in \Sigma \Rightarrow A(y) \in \Sigma$ for some $y \in Var$.

[4]In contrast, the analogous conditions 1 and 2 below, are completely trivial as they follow from usual inference rules for quantifiers, or algebraically from order theory.

(1) If $\Sigma \cup \{A(x)\} \vdash \Delta$ for all $x \in Var$, then $\Sigma \cup \{\forall v A[v/x]\} \vdash \Delta$. This trivially holds since $\Sigma \cup \{\forall v A v\}$ can be instantiated into $\Sigma \cup \{A(x)\}$ for any x in the domain which clearly is assumed non-empty. The domain is non-empty since the assumption contains a variable and $Var = D$. Algebraically, this implication follows on order-theoretic grounds: $\forall v A v$ is the meet of the set $\{A(x) \mid x \in Var\}$, thus if every component of the meet is below Δ, so is the meet (we interpret $\vdash$ as $\leq$).
(2) If $\Sigma \vdash \Delta \cup \{A(x)\}$ for each $x \in Var$ then $\Sigma \vdash \Delta \cup \{\exists v A[v/x]\}$. This trivially holds by existential generalization. Algebraically, this implication follows on order-theoretic grounds: $\exists v A$ is the join of the set $\{A(x) \mid x \in Var\}$, thus if every component of the join is above Σ, so is the join.

[5]Again, we can assume that a finite subset $\Gamma \subseteq \Sigma$ suffices, namely $\Gamma = \{\forall v A\}$

$\Sigma \nvdash \Delta$. Hence for all $x \in Var$ it holds that $A(x) \notin \Delta$. Since $\Sigma \cup \Delta = Form(\mathcal{L}_Q)$, it follows that $A(x) \in \Sigma$ for all variables x.

($\Leftarrow$) Suppose that $A(x) \in \Sigma$ for all $x \in Var$. Then $\Sigma \vdash \Delta \cup \{A(x)\}$ for each $x \in Var$. By the quantifier suited condition QS1, $\Sigma \vdash \Delta \cup \{\forall v A\}$. Since $\Sigma \nvdash \Delta$ then $\forall v A \notin \Delta$. So $\forall v A \in \Sigma$ because $\Sigma \cup \Delta = Form(\mathcal{L}_Q)$.

Thus, $\forall v A \in \Sigma$ iff $A(x) \in \Sigma$ for all $x \in Var$. □

To built our canonical frame, the strategy is as follows. If $\Sigma \nvdash \Delta$ we extend $\langle \Sigma, \Delta \rangle$ to a full quantifier-suited pair $\langle \Sigma', \Delta' \rangle$ using the lemma 46. Then we use the class of full quantifier-suited pairs in this language $\mathcal{L}_Q^{+'}$ as the universe of points W of the canonical frame. Because they are full and quantifier-suited, it is guaranteed that these points, as prime theories, will interpret the extensional part of the language adequately. But we still must guarantee that the modal part of the language is also adequately interpreted. A first step towards this goal is given by the definition of the accessibility relations in the canonical frame, which are as expected.

Definition 44. The Canonical model $\mathbb{M}^c$ for a *CDML*-logic on the language $\mathcal{L}_Q^{+'}$ is tuple $\left\langle (W, \subseteq^*), R_\Box^c, R_\Diamond^c, R_\triangleright^c, R_\triangleleft^c, Var, V^c \right\rangle$, where:

- The language $\mathcal{L}_Q^{+'}$ is the extension of $\mathcal{L}_Q^{+}$ with (intuitionistic) implication and substraction connectives, and $\mathcal{L}_Q^{+}$ is $\mathcal{L}_Q$ extended with countably many new variables.
- The domain (of quantification) of this frame is the set Var of variables in the language $\mathcal{L}_Q$.
- The universe of the frame is $(W, \subseteq^*)$ with W as the set of all full quantifier-suited pairs in the language $\mathcal{L}_Q^{+'}$ and $\subseteq^*$ the ordering relation between them. We set $\langle \Sigma, \Delta \rangle \subseteq^* \langle \Gamma, \Pi \rangle$ iff $\Sigma \subseteq \Gamma$.
- The canonical relations $R_\Box^c$, $R_\Diamond^c$, $R_\triangleright^c$,$R_\triangleleft^c$ are defined as follows:
  - $\langle \Sigma, \Delta \rangle R_\Box^c \langle \Gamma, \Pi \rangle$ iff for each $\varphi \in \mathcal{L}_Q^{+'}$ we have $\Box\varphi \in \Sigma \Rightarrow \varphi \in \Gamma$
  - $\langle \Sigma, \Delta \rangle R_\Diamond^c \langle \Gamma, \Pi \rangle$ iff for each $\varphi \in \mathcal{L}_Q^{+'}$ we have $\varphi \in \Gamma \Rightarrow \Diamond\varphi \in \Sigma$
  - $\langle \Sigma, \Delta \rangle R_\triangleright^c \langle \Gamma, \Pi \rangle$ iff for each $\varphi \in \mathcal{L}_Q^{+'}$ we have $\triangleright\varphi \in \Sigma \Rightarrow \varphi \notin \Gamma$
  - $\langle \Sigma, \Delta \rangle R_\triangleleft^c \langle \Gamma, \Pi \rangle$ iff for each $\varphi \in \mathcal{L}_Q^{+'}$ we have $\varphi \notin \Gamma \Rightarrow \triangleleft\varphi \in \Sigma$
- The canonical valuation V^c is defined such that the extension $V^c_{\langle \Gamma, \Pi \rangle}(F)$ of a predicate F at a point $\langle \Gamma, \Pi \rangle$ of the frame is the set of n-tuples $\langle x_1, \ldots, x_n \rangle$ such that $F(x_1, \ldots, x_n) \in \Gamma$.

We will see below that these conditions guarantee that the accessibility relations in the canonical frame maintain the right interactions with the ordering on the canonical frame universe. Such ordering is the inclusion relation $\subseteq$ between prime theories. As for the evaluation conditions of modal operators, however, these definitions only do half the job. Take $\Box$ as example. The half that is achieved is that when $\Box\varphi \in \Sigma$ then for each Γ with $R_\Box \Sigma\Gamma$ we have $\varphi \in \Gamma$. But we need to complement this with the property that if $\Box\varphi \notin \Sigma$ then there is some Γ such that $R_\Box \Sigma\Gamma$ and $\varphi \notin \Gamma$. In other words, we need to ensure that there are appropriate witnessing points when a universal quantification fails (box-like operators) or when an existential one occurs (diamond-like operators). This is more difficult than in the propositional case, for we need to not only construct the theory but also ensure that it is quantifier-suited in the same language. To achieve this we will use lemma 55.

Claim 45. the canonical frame defined above is of the appropriate kind, namely, it is an DML-frame.

We verify that the canonical frame is an DML-frame (definition 2) by checking that the conditions on the accessibility relations are fulfilled.

(1) $\geq \circ R_\Diamond \subseteq R_\Diamond \circ \geq$ that is $\forall t, u, v\,[(t \geq u \wedge R_\Diamond uv) \rightarrow \exists w\,(R_\Diamond tw \wedge w \geq v)]$. In the canonical frame, such condition takes the following shape (we assume universal closure): $\left(\langle \Sigma, \Delta \rangle \subseteq^* \langle \Sigma', \Delta' \rangle \;\&\; \langle \Sigma, \Delta \rangle R_\Diamond^c \langle \Gamma, \Pi \rangle\right) \Rightarrow \exists \langle \Gamma', \Pi' \rangle \left(\langle \Sigma', \Delta' \rangle R_\Diamond^c \langle \Gamma', \Pi' \rangle \;\&\; \langle \Gamma, \Pi \rangle \subseteq^* \langle \Gamma', \Pi' \rangle\right)$.

PROOF. Let $\langle\Sigma,\Delta\rangle, \langle\Gamma,\Pi\rangle$ and $\langle\Sigma',\Delta'\rangle$ be full quantified-suited pairs from $\mathcal{L}_Q^{+'}$ such that $\langle\Sigma,\Delta\rangle R^c_\Diamond \langle\Gamma,\Pi\rangle$ and $\langle\Sigma,\Delta\rangle \subseteq^* \langle\Sigma',\Delta'\rangle$. Then for each $\varphi \in \mathcal{L}_Q^{+'}$ we have $\varphi\in\Gamma \Rightarrow \Diamond\varphi\in\Sigma$, and since $\Sigma\subseteq\Sigma'$ then $\varphi\in\Gamma \Rightarrow \Diamond\varphi\in\Sigma \Rightarrow \Diamond\varphi\in\Sigma'$. Therefore $\langle\Sigma',\Delta'\rangle R^c_\Diamond \langle\Gamma,\Pi\rangle$. So just take $\langle\Gamma',\Pi'\rangle = \langle\Gamma,\Pi\rangle$ as the witness point. □

(2) $\leq\circ R_\Box \subseteq R_\Box\circ\leq$ that is $\forall t,u,v\,[(t\leq u \,\wedge\, R_\Box uv)\to\exists w(R_\Box tw \,\wedge\, w\leq v]$. In the canonical frame, such condition takes the following shape (we assume universal closure): $\left(\langle\Sigma,\Delta\rangle \subseteq^* \langle\Sigma',\Delta'\rangle \;\&\; \langle\Sigma',\Delta'\rangle R^c_\Box \langle\Gamma',\Pi'\rangle\right) \Rightarrow \exists\langle\Gamma,\Pi\rangle \left(\langle\Sigma,\Delta\rangle R^c_\Box \langle\Gamma,\Pi\rangle \;\&\; \langle\Gamma,\Pi\rangle \subseteq^* \langle\Gamma',\Pi'\rangle\right)$.

PROOF. Let $\langle\Sigma,\Delta\rangle, \langle\Gamma',\Pi'\rangle$ and $\langle\Sigma',\Delta'\rangle$ be full quantified-suited pairs from $\mathcal{L}_Q^{+'}$ such that $\langle\Sigma',\Delta'\rangle R^c_\Box \langle\Gamma',\Pi'\rangle$ and $\langle\Sigma,\Delta\rangle \subseteq^* \langle\Sigma',\Delta'\rangle$. By definition of the canonical relation $R^c_\Box$, we have, for each $\varphi\in\mathcal{L}_Q^{+'}$, $\Box\varphi\in\Sigma' \Rightarrow \varphi\in\Gamma'$, and since $\Sigma\subseteq\Sigma'$, then $\Box\varphi\in\Sigma \Rightarrow \Box\varphi\in\Sigma' \Rightarrow \varphi\in\Gamma'$. Hence $\langle\Sigma,\Delta\rangle R^c_\Box \langle\Gamma',\Pi'\rangle$. But then, there exists a point $\langle\Gamma,\Pi\rangle$ such that $\langle\Gamma,\Pi\rangle \subseteq^* \langle\Gamma',\Pi'\rangle$ and $\langle\Sigma,\Delta\rangle R^c_\Box \langle\Gamma,\Pi\rangle$, for we can just take $\langle\Gamma,\Pi\rangle = \langle\Gamma',\Pi'\rangle$. □

(3) $\leq\circ R_\triangleright \subseteq R_\triangleright\circ\geq$ that is $\forall t,u,v,w\,[(t\leq u \,\wedge\, R_\triangleright uv)\to\exists w\,(R_\triangleright tw \,\wedge\, w\geq v)]$. In the present case, such condition is rather written like this (we assume universal closure): $(\langle\Sigma,\Delta\rangle \subseteq^* \langle\Sigma',\Delta'\rangle \;\&\; \langle\Sigma',\Delta'\rangle R^c_\triangleright \langle\Gamma,\Pi\rangle) \Rightarrow \exists\langle\Gamma',\Pi'\rangle\,(\langle\Sigma,\Delta\rangle R^c_\triangleright \langle\Gamma',\Pi'\rangle \;\&\; \langle\Gamma,\Pi\rangle \subseteq^* \langle\Gamma',\Pi'\rangle)$.

PROOF. Let $\langle\Sigma,\Delta\rangle, \langle\Gamma,\Pi\rangle$ and $\langle\Sigma',\Delta'\rangle$ be full quantified-suited pairs from $\mathcal{L}_Q^{+'}$ such that $\langle\Sigma',\Delta'\rangle R^c_\triangleright \langle\Gamma,\Pi\rangle$ and $\langle\Sigma,\Delta\rangle \subseteq^* \langle\Sigma',\Delta'\rangle$. Then for each $\varphi\in\mathcal{L}_Q^{+'}$ we have $\triangleright\varphi\in\Sigma' \Rightarrow \varphi\notin\Gamma$, by definition of the relation $R^c_\triangleright$. By $\Sigma\subseteq\Sigma'$ we get $\triangleright\varphi\in\Sigma \Rightarrow \triangleright\varphi\in\Sigma' \Rightarrow \varphi\notin\Gamma$ for each $\varphi\in\mathcal{L}_Q^{+'}$, and thus $\langle\Sigma,\Delta\rangle R^c_\triangleright \langle\Gamma,\Pi\rangle$. But then, there exists a point $\langle\Gamma',\Pi'\rangle$ such that $\langle\Gamma,\Pi\rangle \subseteq^* \langle\Gamma',\Pi'\rangle$ and $\langle\Sigma,\Delta\rangle R^c_\triangleright \langle\Gamma',\Pi'\rangle$, as we can just take $\langle\Gamma',\Pi'\rangle = \langle\Gamma,\Pi\rangle$. □

(4) $\geq\circ R_\triangleleft \subseteq R_\triangleleft\circ\leq$ that is $\forall t,u,v\,[(t\geq u \,\wedge\, R_\triangleleft uv)\to\exists w\,(R_\triangleleft tw \,\wedge\, w\leq v)]$. In the present case, such condition is rather written like this (we assume universal closure): $(\langle\Sigma,\Delta\rangle \subseteq^* \langle\Sigma',\Delta'\rangle \;\&\; \langle\Sigma,\Delta\rangle R^c_\triangleleft \langle\Gamma',\Pi'\rangle) \Rightarrow \exists\langle\Gamma,\Pi\rangle\,(\langle\Sigma',\Delta'\rangle R^c_\triangleleft \langle\Gamma,\Pi\rangle \;\&\; \langle\Gamma,\Pi\rangle \subseteq^* \langle\Gamma',\Pi'\rangle)$.

PROOF. Let $\langle\Sigma,\Delta\rangle, \langle\Gamma',\Pi'\rangle$ and $\langle\Sigma',\Delta'\rangle$ be full quantified-suited pairs from $\mathcal{L}_Q^{+'}$ such that $\langle\Sigma,\Delta\rangle R^c_\triangleleft \langle\Gamma',\Pi'\rangle$ and $\langle\Sigma,\Delta\rangle \subseteq^* \langle\Sigma',\Delta'\rangle$. Then for each $\varphi\in\mathcal{L}_Q^{+'}$ we have $\varphi\notin\Gamma' \Rightarrow \triangleleft\varphi\in\Sigma$, and since $\Sigma\subseteq\Sigma'$ then $\varphi\notin\Gamma' \Rightarrow \triangleleft\varphi\in\Sigma \Rightarrow \triangleleft\varphi\in\Sigma'$. Hence $\langle\Sigma',\Delta'\rangle R^c_\triangleleft \langle\Gamma',\Pi'\rangle$. So just take $\langle\Gamma,\Pi\rangle = \langle\Gamma',\Pi'\rangle$. □

3.1.3. Lindenbaum lemma analogue to extend theories to prime theories (pairs to full pairs). Here we present the Lindenbaum lemma analogue for the distributive setting, which also takes care to enforce ω-completeness (quantifier suitedness).

LEMMA 46. *(Pair Extension on a new language). If $\langle\Sigma,\Delta\rangle$ is a pair then there is also a full quantifier-suited pair $\langle\Sigma',\Delta'\rangle$ extending $\langle\Sigma,\Delta\rangle$, in a new language $\mathcal{L}_Q^{+'}$ extending the original language $\mathcal{L}'_Q$ by at most countably many new variables.*

PROOF. We will prove the result by construction.

CONSTRUCTION:

Let $\mathcal{L}_Q^{+'}$ be the language $\mathcal{L}'_Q$ extended with countably many new variables. Let $Enum\left(\mathcal{L}_Q^{+'}\right) = \{A_n \mid n\in\omega\}$ be an enumeration of all the formulas $A_0, A_1, \ldots$ of $\mathcal{L}_Q^{+'}$, and $Var^+ = \{x_n \mid n\in\omega\}$ an enumeration of all the variables in $\mathcal{L}_Q^{+'}$. [6] Now define $\langle\Sigma_0,\Delta_0\rangle := \langle\Sigma,\Delta\rangle$, and $\langle\Sigma_{n+1},\Delta_{n+1}\rangle$ is defined as follows:

[6] The original proof on [**Restall 2005**] seems to only use an enumeration on $\mathcal{L}'_Q$ which only labels formulas from the old language, and thus, the verification stage of the proof fails, since the construction does not take care of new formulas made from the added variables.

Case 1. If $\Sigma_n \cup \{A_n\} \nvdash \Delta_n$ and A_n is not of the form $\exists vB$ then $\langle \Sigma_{n+1}, \Delta_{n+1} \rangle := \langle \Sigma_n \cup \{A_n\}, \Delta_n \rangle$. [7]

Case 2. If $\Sigma_n \cup \{A_n\} \nvdash \Delta_n$ and A_n is of the form $\exists vB$ then $\langle \Sigma_{n+1}, \Delta_{n+1} \rangle := \langle \Sigma_n \cup \{A_n, B(x)\}, \Delta_n \rangle$, where x is the first new variable in the enumeration Var^+ not appearing in Σ_n, A_n or Δ_n (The well-foundedness of $\mathbb{N}$ guarantees that every non-empty set of indices has a least element and thus we can choose the first such variable).[8]

Case 3. If $\Sigma_n \cup \{A_n\} \vdash \Delta_n$ and A_n is not of the form $\forall vB$ then $\langle \Sigma_{n+1}, \Delta_{n+1} \rangle := \langle \Sigma_n, \Delta_n \cup \{A_n\} \rangle$ (suppose it is of form $\exists vB$, then any witness will be added by construction to Δ as well because any witness derives $\exists vB$, by (A.4b) and thus derives Δ_n, so no extra precaution is needed).

Case 4. If $\Sigma_n \cup \{A_n\} \vdash \Delta_n$ and A_n is of the form $\forall vB$ then $\langle \Sigma_{n+1}, \Delta_{n+1} \rangle := \langle \Sigma_n, \Delta_n \cup \{A_n, B(x)\} \rangle$, where x is a new variable not appearing in Σ_n, A_n or Δ_n.[9] .

Then we set $\langle \Sigma', \Delta' \rangle := \langle \bigcup_n \Sigma_n, \bigcup_n \Delta_n \rangle$, which by construction is a partition of the formulas: every formula $A_n \in Form\left(\mathcal{L}_Q^{+'}\right)$ is either in Σ' or in Δ', thus $\langle \Sigma', \Delta' \rangle$ is full. □

VERIFICATION:

We just remarked that all formulas in the enumeration of $\mathcal{L}_Q^{+'}$ are put either in Δ' or Σ', so fullness is guaranteed. Remains to verify the rest of the properties.

CLAIM 47. $\langle \Sigma', \Delta' \rangle := \langle \bigcup_n \Sigma_n, \bigcup_n \Delta_n \rangle$ is a pair.

PROOF. First, we show by induction on n and m that $\Sigma_n \nvdash \Delta_m$ for all n, m. Notice that we can assume w.l.o.g. that $n = m$ for if $n \neq m$, it suffices to use weakening and add premises to the conjunction of Σ_n in case $n < m$ (those added premises are exactly the formulas that are added to Σ' on stages i with $n < i \leq m$) or add disjuncts to the consequent Δ_m in case $n > m$. So in fact this reduces to show by induction on n that $\Sigma_n \nvdash \Delta_n$ for all n.

(1) Base case: $n = 0$. Since $\langle \Sigma_0, \Delta_0 \rangle := \langle \Sigma, \Delta \rangle$ and $\langle \Sigma, \Delta \rangle$ is a pair, we have $\Sigma_0 \nvdash \Delta_0$.
(2) Inductive step. $n + 1$.

Suppose towards a contradiction that $\Sigma_n \nvdash \Delta_n$ but $\Sigma_{n+1} \vdash \Delta_{n+1}$.

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[7] Suppose A_n is of form $\forall vB$, then there is no problem, because we are sure that:
(1) since $\Sigma_n \cup \{\forall vB\} \nvdash \Delta_n$ then Δ_n contains no instantiation $B[x/v]$ and it is safe to add it to Σ_n,
(2) the universal quantifier is properly interpreted: the instantiations $B[x/v]$, for $x \in Var^+$, and not already in Σ will be added later by construction because all such formulas are in the enumeration and they are such that $\Sigma_n \cup \{B(x)\} \nvdash \Delta_n$ by (a).

[8]
(1) we need a variable not appearing in Δ_n for obvious reasons: to be sure we avoid deducibility of Δ_{n+1} from Σ_{n+1}, just in case that formulas of shape $B(x)$, $B(x) \vee \varphi$, etc. are in Δ_n
(2) we need a variable not appearing in Σ_n because then we could get deducibility of Δ_{n+1} from Σ_{n+1}, with for instance, $\varphi(x)$ in Σ and something like $\varphi(x) \wedge B(x)$ in Δ (which would be deducible once we have added $B(x)$ to Σ),
(3) we need a variable not appearing in A_n because otherwise we would get something stronger. Observe that with x in A_n we would have something like $A_n = \exists v\varphi(v, x)$. Then the instantiation $\varphi(x, x)$ added to Σ would make both $\exists v\varphi(v, x)$ and $\exists v\varphi(v, v)$ deducible from Σ. If the second happens to be in Δ_n, we get deducibility of Δ_n from Σ.
(4) But more importantly: note that since Σ and Δ contain only formulas from $\mathcal{L}_Q$ and for any $n \in \omega$, Σ_n and Δ_n are composed of finite formulas and contain only finitely many from $\mathcal{L}_Q^{+'}$, then we can always find an x new to Σ_n, A_n and Δ_n. Notice also that "x new to Σ_n, A_n and Δ_n" entails that $B(x)$ has not appeared earlier in the enumeration. Otherwise $B(x) = A_m$ would have been checked at stage $m < n$ and thrown into Σ_m or Δ_m. It cannot be checked at stage n either, otherwise $B(x) = A_n$. In both cases, x would not be new to Σ_n, A_n and Δ_n.

[9] Observe that you add the witness $B(x)$ to Δ so that Σ will no longer contain all witnesses, and thus, cannot derive $\forall vB = A_n$; if x appears in Σ it might be that $B(x)$ is in Σ and thus you would get derivability of Δ. So we need x not appearing in Σ_n. Moreover, observe that with x in A_n we would have something like $\forall v\varphi(v, x)$ then the instantiation $\varphi(x, x)$ added to Δ would make both $\forall v\varphi(v, x)$ and $\forall v\varphi(v, v)$ unsuitable for Σ (Only the first is desired). So we also reject x from appearing in A_n. The main reason to get x new to Δ_n, is again the guarantee that $B(x)$ has not occurred previously in the enumeration.

Case 1. Consider the case 1 of the construction. By hypothesis $\Sigma_n \cup \{A_n\} \nvdash \Delta_n$, hence $\Sigma_{n+1} \nvdash \Delta_{n+1}$ because $\Sigma_{n+1} = \Sigma_n \cup \{A_n\}$ and $\Delta_{n+1} = \Delta_n$. Clearly we get a contradiction with our assumption that $\Sigma_{n+1} \vdash \Delta_{n+1}$.$\blacksquare$

Case 2. Then $\Sigma_n \cup \{A_n\} \nvdash \triangle_n$ with A_n of the form $\exists vB$. We assumed that $\Sigma_{n+1} \vdash \Delta_{n+1}$ and in this case $\Sigma_{n+1} = \Sigma_n \cup \{A_n, B(x)\}$ (where x does not appear in Σ_n,Δ_n or A_n) and $\Delta_{n+1} = \Delta_n$. Hence, there is a conjunction C of formulas in Σ_n and a disjunction D of formulas in Δ_n such that $C \wedge A_n \wedge B(x) \vdash D$. By existential generalization we can set $\exists v\chi \vdash \chi(x)$ with $\chi := C \wedge A_n \wedge B(x)$ because x has not previously appeared and thus carries no assumptions along with it. Hence we have $\exists x(C \wedge A_n \wedge B(x)) \vdash C \wedge A_n \wedge B(x) \vdash D$ and then $C \wedge A_n \wedge \exists x B(x) \vdash \exists x(C \wedge A_n \wedge B(x)) \vdash C \wedge A_n \wedge B(x) \vdash D$ by existential distribution.[10] Finally $C \wedge A_n \wedge \exists x B(x) \vdash D$ by cut, and since A_n is of the form $\exists vB$ we have $C \wedge A_n \vdash D$, which contradicts the fact that $\Sigma_n \cup \{A_n\} \nvdash \Delta_n$.$\blacksquare$

Case 3. Then $\Sigma_n \cup \{A_n\} \vdash \Delta_n$. Now suppose that $\Sigma_n \vdash \Delta_n \cup \{A_n\}$ (that is $\Sigma_{n+1} \vdash \Delta_{n+1}$). By the first assumption, for some conjunction C of members of Σ_n and some disjunction D from Δ_n we have $C \wedge A_n \vdash D$ and, by the second assumption, we get $C \vdash A_n \vee D$ (observe that we can assume w.l.o.g. that the disjunction D from Δ_n is the same in both cases -if they are different just take the disjunction of both and replace D by this new disjunction-). Now notice that:

(a) By disjointing D to the first antecedent we have $(C \wedge A_n) \vee D \vdash D$ (by instantiation of rule $\frac{(C \wedge A_n) \vdash D \quad D \vdash D}{(C \wedge A_n) \vee D \vdash D}$ since both disjuncts derive D)

(b) By conjoining C to the second consequent we have $C \vdash C \wedge (A_n \vee D)$ (trivially, from $C \vdash C$ and $C \vdash A_n \vee D$ by instantiation of rule $\frac{C \vdash C \quad C \vdash A_n \vee D}{C \vdash C \wedge (A_n \vee D)}$).

(c) By an instance of the distribution law we have $C \wedge (A_n \vee D) \vdash (C \wedge A_n) \vee D$.

Thus $C \vdash (C \wedge A_n) \vee D$ by using (b)-(c) and cut. Then by applying (a) to the previous result, we get $C \vdash (C \wedge A_n) \vee D \vdash D$, that is $C \vdash D$. But this contradicts our starting point, namely $\Sigma_n \nvdash \Delta_n$.$\blacksquare$

Case 4. Assume that $\Sigma_n \cup \{A_n\} \vdash \Delta_n$ and A_n is of the form $\forall vB$. Now suppose that $\Sigma_n \vdash \triangle_n \cup \{A_n, B(x)\}$ where x is a new variable not appearing in Σ_n, A_n or Δ_n (that is, we assume $\Sigma_{n+1} \vdash \Delta_{n+1}$). Then for some conjunction C of elements in Σ_n and a disjunction D of elements in Δ_n we have $C \wedge A_n \vdash D$, by the first assumption, and $C \vdash A_n \vee B(x) \vee D$, by the second one (again, w.l.o.g. we can use the same C and D in both cases). Now notice that:

(a) By disjointing D to the first antecedent we have $(C \wedge A_n) \vee D \vdash D$ (as before)

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[10] $C \wedge A_n \wedge \exists x B(x) \vdash \exists x(C \wedge A_n \wedge B(x))$ can be seen algebraically as $\|C \wedge A_n\|_g \wedge \left(\bigvee_{g' \equiv_x g} \|B(x)\|_{g'}\right) \leq \bigvee_{g' \equiv_x g} \left\{\|C \wedge A_n\|_{g'} \wedge \|B(x)\|_{g'}\right\}$ which involves a strong form of distributivity (complete lattices that satisfy it are known as *Heyting algebras*). It is worth noticing that since x does not occur in $C \wedge A_n$, then $\|C \wedge A_n\|_{g'} = \|C \wedge A_n\|_g$. The opposite direction (with $\alpha = C \wedge A_n$) $\|\alpha\|_g \wedge \left(\bigvee_{g' \equiv_x g} \|B(x)\|_{g'}\right) \geq \bigvee_{g' \equiv_x g} \left\{\|\alpha\|_{g'} \wedge \|B(x)\|_{g'}\right\}$ always holds in any complete lattice, which allows binary meets to distribute over *arbitrary* joins. To see why, consider that for any g" such that $g" \equiv_x g$ it is clear that $\begin{cases} \left(\bigvee_{g' \equiv_x g} \|B(x)\|_{g'}\right) \geq \|B(x)\|_{g"} \geq \left(\|B(x)\|_{g"} \wedge \|\alpha\|_{g"}\right) \\ \textit{and} \qquad \|\alpha\|_{g"} \geq \left(\|B(x)\|_{g"} \wedge \|\alpha\|_{g"}\right) \end{cases}$, where $\|\alpha\|_{g"} = \|\alpha\|_g$. Hence $\left(\left(\bigvee_{g' \equiv_x g} \|B(x)\|_{g'}\right) \wedge \|\alpha\|_g\right) \geq \left(\|B(x)\|_{g"} \wedge \|\alpha\|_{g"}\right)$, in other words, since $\left(\|B(x)\|_{g"} \wedge \|\alpha\|_{g"}\right)$ is below each element of the meet, then it is below the meet as well. Since this holds for any g" such that $g" \equiv_x g$ it is clear that $\left(\left(\bigvee_{g' \equiv_x g} \|B(x)\|_{g'}\right) \wedge \|\alpha\|_g\right) \geq \|B(x)\|_{g"} \wedge \|\alpha\|_{g"}$ for all members of the join $\bigvee_{g' \equiv_x g} \left\{\|B(x)\|_{g'} \wedge \|\alpha\|_{g'}\right\}$. Therefore $\left(\left(\bigvee_{g' \equiv_x g} \|B(x)\|_{g'}\right) \wedge \|\alpha\|_g\right) \geq \bigvee_{g' \equiv_x g} \left\{\|B(x)\|_{g'} \wedge \|\alpha\|_{g'}\right\}$, in different words, since $\left(\left(\bigvee_{g' \equiv_x g} \|B(x)\|_{g'}\right) \wedge \|\alpha\|_g\right)$ is above each element of the join, then it is above the join as well.

(b) By conjoining C to the second consequent we have $C \vdash C \wedge (A_n \vee B(x) \vee D)$ with (*) x absent from C, A_n and D.

We obtain $C \vdash \forall v (C \wedge (A_n \vee B(v) \vee D)) \vdash C \wedge (A_n \vee \forall v B(v) \vee D)$, the first entailment by rule (5a) applied to (b), and then -on second entailment- the quantifier is pushed inside by (*). So $C \vdash C \wedge (A_n \vee \forall v B \vee D)$ by cut. Now $C \wedge (A_n \vee \forall v B \vee D)$ is the same as $C \wedge (A_n \vee D)$ because $A_n = \forall v B$, so $C \vdash C \wedge (A_n \vee D)$. By an instance of the distribution law we have $C \wedge (A_n \vee D) \vdash (C \wedge A_n) \vee D$. Thus $C \vdash (C \wedge A_n) \vee D$, and through (a) we get $C \vdash (C \wedge A_n) \vee D \vdash D$, that is $C \vdash D$. But this contradicts $\Sigma_n \nvdash \Delta_n$.■

We have a contradiction in all cases, which proves $\Sigma_{n+1} \nvdash \Delta_{n+1}$. Hence for all n we have $\Sigma_n \nvdash \Delta_n$, which implies $\Sigma' \nvdash \Delta'$. For suppose $\Sigma' \vdash \Delta'$. Then by compactness, there exists a finite $\Gamma \subseteq \Sigma'$ such that $\Gamma \vdash D$ for some disjunction of elements in Δ. Since $\{\Sigma_i\}_{i\in\omega}$ and $\{\Delta_i\}_{i\in\omega}$ form increasing chains, Γ is finite and D has an index in the enumeration of $\mathcal{L}_Q^{+'}$, we know there exists $z \in \omega$ such that $\Gamma \subseteq \Sigma_z$ and $D \in \Delta_z$, i.e. such that $\Sigma_z \vdash \Delta_z$. This contradicts the fact that for all n we have $\Sigma_n \nvdash \Delta_n$, and henceforth $\Sigma' \nvdash \Delta'$ and the claim is proved. □

REMARK 48. It is quite clear that $\langle \Sigma', \Delta' \rangle$ is a pair given the proof above. It is instructive, however, to discuss what happens with the formulas A_m that are inserted as companion witnesses of an existential formula A_t with $t < m$ (recall that we do know for sure that $t < m$ because the witness uses a new variable not present on any formula added at a stage $s \leq t$). Suppose we are at stage m and we are to examine the formula A_m which was previously entered in Σ_t at stage $t < m$ as witness companion of a formula $A_t = \exists v B$ (i.e., we assume for stage t that we are in case 2 with $\Sigma_t \cup \{\exists v B\} \nvdash \Delta_t$ and $A_m = B(y)$, y a fresh variable, is the witness). These are the cases when A_m is checked.

Case 1. A_m is put into Σ, but then it is all fine since A_m was already in Σ since stage t.

Case 2. As previous case, but now we add the witness corresponding to A_m.

Case 3. (and *Case 4*) If we were indeed in these cases we would be in trouble because $\Sigma_m \cup \{A_m\} \vdash \Delta_m$ and $A_m \in \Sigma_{t<m}$ by assumption. But note that we have $A_m = B(y)$ and $\Sigma_t \cup \{\exists v B\} \nvdash \Delta_t$. By the latter, certainly $\exists v B \nvdash \Delta_t$. Since, algebraically speaking, $\exists v B$ is the join of all witnesses $B(x)$, and $\leq$ interprets $\vdash$, then clearly $B(x) \nvdash \Delta_t$ for all $x \in Var^+$. In particular $A_m \nvdash \Delta_t$ so it is fine that $A_m \in \Sigma_{t<m}$. Moreover, since A_m enters Σ at stage t, then the construction takes care to preserve pairness afterwards, so $\Sigma_m \cup \{A_m\} \vdash \Delta_m$ (that is $\Sigma_m \vdash \Delta_m$) cannot be the case.

CLAIM 49. The pair $\langle \Sigma', \Delta' \rangle$ is *quantifier-suited* by construction.

PROOF. (QS1) Universal quantifier. We just proved $\Sigma' \nvdash \Delta'$ so if $\Sigma' \vdash \Delta' \cup \{A(x)\}$ then we must have $A(x) \in \Sigma'$, for all $x \in Var^+$. But then $\forall v A[v/x] \in \Sigma'$ for otherwise $\forall v A[v/x] \in \Delta'$ since $\langle \Sigma', \Delta' \rangle$ is full, and this would lead to $\Sigma' \vdash \Delta'$ through universal generalization (A.5a) applied to $\Sigma' \vdash \Delta' \cup \{A(x)\}$. Therefore $\Sigma' \vdash \Delta' \cup \{\forall v A\}$, as desired.

(QS2) Existential quantifier. We just proved $\Sigma' \nvdash \Delta'$ so, for all $x \in Var^+$, if $\Sigma' \cup \{A(x)\} \vdash \Delta'$ then we must have $A(x) \in \Delta'$. But then $\exists v A \in \Delta'$ (otherwise $\exists v A \in \Sigma'$ since $\langle \Sigma', \Delta' \rangle$ is full, and then $\Sigma' \vdash \Delta'$ through existential instantiation (A.4bi)), therefore $\Sigma' \cup \{\exists v A\} \vdash \Delta'$, as desired. □

3.2. Witnessing-pair existence lemma

In this section we will prove three lemmas, amongst which the *witnessing-pair existence* lemma is the key result. The other two lemmas are preparatory steps. The *pair extension in the same language* lemma will be used within the truth-lemma to turn each witnessing-information nucleus provided by lemma 55 into an actual point of the frame. The *finite addition* lemma, for which the language needs an extension, is used in both the witnessing-pair lemma proof and in the truth lemma.

EXTENDING THE LANGUAGE CONSERVATIVELY

We will conservatively add two connectives $\supset$ and $-$ to our language $\mathcal{L}_Q$ of distributive lattice logic as they will be required to push the proof of the finite addition lemma 53 through. The resulting language will be denoted by $\mathcal{L}'_Q$. By *conservatively* we mean that the set of tautologies of our logic Λ will not be affected in any way.

DEFINITION 50. Conservative extension of a language

- The logic Λ' based on $\mathcal{L}'_Q \supseteq \mathcal{L}_Q$ is an extension of the logic Λ if $\Lambda \subseteq \Lambda'$,
- it is a *conservative extension* if moreover $\Lambda' \cap \mathcal{L}_Q = \Lambda$, that is $\Lambda' \vdash \varphi \Rightarrow \Lambda \vdash \varphi$ for any φ not containing the new connectives.

To keep the extension conservative we define the new connectives by linking them to what we already have:

DEFINITION 51. new connectives (NC):

(1) $\varphi \wedge \psi \vdash \chi$ iff $\varphi \vdash \psi \supset \chi$ (thus $\supset$ is intuitionistic implication)
(2) $\varphi \vdash \psi \vee \chi$ iff $\varphi - \psi \vdash \chi$

The satisfaction relation $\Vdash$ needs to be extended accordingly:

(1) $\mathbb{M}, w, g \Vdash \varphi \supset \psi$ iff for all v with $w \leq v$, $\mathbb{M}, v, g \Vdash \varphi \Rightarrow \mathbb{M}, v, g \Vdash \psi$
(2) $\mathbb{M}, w, g \Vdash \varphi - \psi$ iff $\exists v \leq w\, (\mathbb{M}, v, g \Vdash \varphi \,\&\, \mathbb{M}, v, g \nVdash \psi)$,

REMARK 52. Defined algebraically, intuitionistic $\supset$ is the following operation: $A \supset B := \bigvee \{C \mid C \wedge A = B\}$, that is, the biggest C such that $C \wedge A \leq B$. When interpreted on an algebra of upsets, then $A \supset B := (A \cap B^c) \downarrow^c$. If we delete the arrow, we get the classical interpretation of $\supset$.

From the axioms we already have and the definitions of the new connectives, it can be shown that the following rules characterize the interaction between the new connectives and the quantifiers (we refer the reader to [**Restall 2005**] for details).

(QN) Quantifiers and the new connectives (φ, ψ stand for any formula)

(1) $\forall v\, (\varphi \supset \psi) \vdash \forall v \varphi \supset \forall v \psi$
(2) $\forall v\, (\varphi \supset \psi) \vdash \exists v \varphi \supset \psi$, with v not free in ψ
(3) $\exists v \psi - \exists v \varphi \vdash \exists v\, (\psi - \varphi)$
(4) $\psi - \forall v \varphi \vdash \exists v\, (\psi - \varphi)$, with v not free in ψ

We can now proceed with the proof of the finite addition lemma.

LEMMA 53. *Finite addition lemma.*

If $\langle \Sigma, \Delta \rangle$ is a quantifier-suited pair and X and Y are finite sets of formulas in the same language, and one of Σ and Δ is finite, then $\langle \Sigma \cup X, \Delta \cup Y \rangle$ is also quantifier-suited.[11]

[11] Observe that the resulting sequence $\langle \Sigma \cup X, \Delta \cup Y \rangle$ might not be -in general- a pair, we only claim that *if* it is a pair, it will be quantifier-suited. Accordingly, the proof does not take care of tracking the deducibility of $\Delta \cup Y$ from $\Sigma \cup X$.

PROOF. Assume that $\langle \Sigma, \Delta \rangle$ is a quantifier-suited pair.

Case 1. Σ is finite. X and Y are finite, so w.l.o.g. we can assume they are singletons $X = \{\varphi\}$ and $Y = \{\psi\}$.

Case i. (QS2) subcase. Fix an $x \in Var^+$. If $\Sigma \cup X \cup \{A(x)\} \vdash \Delta \cup Y$, we have $\bigwedge \Sigma \wedge \varphi \wedge A(x) \vdash D \vee \psi$, with D a finite disjunction in Δ ($\bigwedge \Sigma \wedge \varphi \wedge A(x)$ is a formula since Σ is finite). Equivalently $(\bigwedge \Sigma \wedge \varphi \wedge A(x)) - \psi \vdash D$ by substraction. Now assume that $\Sigma \cup X \cup \{A(x)\} \vdash \Delta \cup Y$ for all $x \in Var^+$. Therefore, by lattice rule (IR.2a) and for any $n \in \omega$ we have:

$$\left[\left(\bigwedge \Sigma \wedge \varphi \wedge A(x_0)\right) - \psi\right] \vee \left[\left(\bigwedge \Sigma \wedge \varphi \wedge A(x_1)\right) - \psi\right] \vee \ldots \vee \left[\left(\bigwedge \Sigma \wedge \varphi \wedge A(x_n)\right) - \psi\right] \vdash D$$

Since $(\ -\ )$ is order-preserving on the left argument (proposition 26), then $(\ - \psi)$ is order-preserving and thus we can push the disjunction inside the left argument of substraction, to obtain

$$\left[\left(\bigwedge \Sigma \wedge \varphi \wedge A(x_0)\right) \vee \left(\bigwedge \Sigma \wedge \varphi \wedge A(x_1)\right) \vee \ldots \vee \left(\bigwedge \Sigma \wedge \varphi \wedge A(x_n)\right)\right] - \psi \vdash D$$

(Alternatively, we can take $f(x) = (x - \bigvee Y)$, and use the well known proposition in lattice theory which states that in a lattice L, f being order preserving is a property equivalent to $f(a) \vee f(b) \leq f(a \vee b)$ -and order-dually $f(a) \wedge f(b) \geq f(a \wedge b)$- for all $a, b \in L$ ([**Davey & Priestley 2002**] p.44 proposition 2.19).[12] Then we take deduction as the ordering relation). By compactness[13] we finally have: $\exists v\,(\bigwedge \Sigma \wedge \varphi \wedge A[v/x]) - \psi \vdash D$ -we can choose v not free in $\bigwedge \Sigma \wedge \varphi$ since it is a finite formula- and then $\exists v\,(\bigwedge \Sigma \wedge \varphi \wedge A[v/x]) \vdash D \vee \psi$ and $(\bigwedge \Sigma \wedge \varphi \wedge \exists v A[v/x]) \vdash D \vee \psi$. So $\Sigma \cup X \cup \{\exists v A\} \vdash \Delta \cup Y$ as desired and $\langle \Sigma \cup X, \Delta \cup Y \rangle$ fulfills (QS2).■

Case ii. (QS1) subcase. Fix an $x \in Var^+$. If $\Sigma \cup X \vdash \Delta \cup Y \cup \{A(x)\}$, we have $\bigwedge \Sigma \wedge \varphi \vdash D \vee \psi \vee A(x)$, with D a finite disjunction in Δ ($\bigwedge \Sigma \wedge \varphi$ is a formula since Σ is finite). Equivalently $(\bigwedge \Sigma \wedge \varphi) - (\psi \vee A(x)) \vdash D$, by substraction. Now assume that $\Sigma \cup X \vdash \Delta \cup Y \cup \{A(x)\}$ for all $x \in Var^+$. Therefore, by lattice rule (IR.2a) and for any $n \in \omega$ we have:

$$\left[\left(\bigwedge \Sigma \wedge \varphi\right) - (\psi \vee A(x_0))\right] \vee \left[\left(\bigwedge \Sigma \wedge \varphi\right) - (\psi \vee A(x_1))\right] \vee \ldots \vee \left[\left(\bigwedge \Sigma \wedge \varphi\right) - (\psi \vee A(x_n))\right] \vdash D$$

Since $(\ -\ )$ is order-reversing on the right argument, then $((\bigwedge \Sigma \wedge \varphi) -\ )$ is order-reversing and thus we can push the disjunction inside the right argument of substraction, to obtain

$$\left(\bigwedge \Sigma \wedge \varphi\right) - [(\psi \vee A(x_0)) \wedge (\psi \vee A(x_1)) \wedge \ldots \wedge (\psi \vee A(x_n))] \vdash D$$

By compactness we finally have: $(\bigwedge \Sigma \wedge \varphi) - \forall v\,(\psi \vee A[v/x]) \vdash D$, -w.l.o.g. we can choose v not free in ψ - and then $(\bigwedge \Sigma \wedge \varphi) \vdash D \vee \forall v\,(\psi \vee A[v/x])$. Hence $\Sigma \cup X \vdash \Delta \cup Y \cup \{\forall v A\}$ as desired. Thus $\langle \Sigma \cup X, \Delta \cup Y \rangle$ fulfills (QS1).■

Case 2. Δ is finite. This case is dual with implication instead of substraction.

Case i. (QS1) subcase. Fix an $x \in Var^+$. If $\Sigma \cup X \cup \{A(x)\} \vdash \Delta \cup Y$, we have $\bigwedge C \wedge \varphi \wedge A(x) \vdash \bigvee \Delta \vee \psi$, with C a finite conjunction in Σ ($\bigvee \Delta \vee \psi$ is a formula since Δ is finite)

[12] Observe that $\begin{cases} a \leq a \vee b \\ b \leq a \vee b \end{cases} \Rightarrow_{by\ monotonicity\ of\ f} \begin{cases} f(a) \leq f(a \vee b) \\ f(b) \leq f(a \vee b) \end{cases}$ and since $f(a \vee b)$ is above both $f(a)$ and $f(b)$ then by lattice-theoretic principles, it is above its join $f(a) \vee f(b)$ as well.

[13] The model of any finite set $\bigcup_{0 \leq i \leq n} \{[(\bigwedge \Sigma \wedge \varphi \wedge A(x_0)) \vee (\bigwedge \Sigma \wedge \varphi \wedge A(x_1)) \vee \ldots \vee (\bigwedge \Sigma \wedge \varphi \wedge A(x_i))] - \psi\}$ with $n \in \omega$ is a model of D, so the infinite set $\bigcup_{i \in \omega} \{[(\bigwedge \Sigma \wedge \varphi \wedge A(x_0)) \vee (\bigwedge \Sigma \wedge \varphi \wedge A(x_1)) \vee \ldots \vee (\bigwedge \Sigma \wedge \varphi \wedge A(x_i))] - \psi\}$ is a model of D as well.

and equivalently $\bigwedge \Sigma \vdash (\varphi \wedge A(x)) \supset (\bigvee \Delta \vee \psi)$ by implication. Now assume that $\Sigma \cup X \cup \{A(x)\} \vdash \Delta \cup Y$ for all $x \in Var^{+}$. Then, by lattice rule (2b) and for any $n \in \omega$ we have:

$$\bigwedge \Sigma \vdash \left[(\varphi \wedge A(x_0)) \supset \left(\bigvee \Delta \vee \psi\right)\right] \wedge \left[(\varphi \wedge A(x_1)) \supset \left(\bigvee \Delta \vee \psi\right)\right] \wedge \ldots \wedge \left[(\varphi \wedge A(x_n)) \supset \left(\bigvee \Delta \vee \psi\right)\right]$$

Since implication is order-reversing on the left argument, by pushing the conjunction inside the implication we obtain:

$$\bigwedge \Sigma \vdash [(\varphi \wedge A(x_0)) \vee (\varphi \wedge A(x_1)) \vee \ldots \vee (\varphi \wedge A(x_n))] \supset \left(\bigvee \Delta \vee \psi\right)$$

By compactness we finally have: $\bigwedge \Sigma \vdash \exists v (\varphi \wedge A[v/x]) \supset (\bigvee \Delta \vee \psi)$, -w.l.o.g. we can choose v not free in φ - and then $\bigwedge \Sigma \vdash (\varphi \wedge \exists v A[v/x]) \supset (\bigvee \Delta \vee \psi)$. Hence $\bigwedge \Sigma \wedge \varphi \wedge \exists v A[v/x] \vdash \bigvee \Delta \vee \psi$ and $\Sigma \cup X \cup \{\exists v A(v)\} \vdash \Delta \cup Y$, as desired. Thus $\langle \Sigma \cup X, \Delta \cup Y\rangle$ fulfills (QS1).■

Case ii. (QS2) subcase. Fix an $x \in Var^{+}$. If $\Sigma \cup X \vdash \Delta \cup Y \cup \{A(x)\}$, we have $\bigwedge C \wedge \varphi \vdash \bigvee \Delta \vee \psi \vee A(x)$, with C a finite conjunction in Σ ($\bigvee \Delta \vee \psi$ is a formula since Δ is finite) and equivalently $\bigwedge C \vdash \varphi \supset (\bigvee \Delta \vee \psi \vee A(x))$ by implication. Now assume that $\Sigma \cup X \vdash \Delta \cup Y \cup \{A(x)\}$ for all $x \in Var^{+}$. Then, by lattice rule (2b) and for any $n \in \omega$ we have:

$$\bigwedge C \vdash \left[\varphi \supset \left(\bigvee \Delta \vee \psi \vee A(x_0)\right)\right] \wedge \left[\varphi \supset \left(\bigvee \Delta \vee \psi \vee A(x_1)\right)\right] \wedge \ldots \wedge \left[\varphi \supset \left(\bigvee \Delta \vee \psi \vee A(x_n)\right)\right]$$

Since implication is order-preserving on the second argument, by pushing the conjunction inside the implication we obtain:

$$\bigwedge C \vdash \varphi \supset \left[\left(\bigvee \Delta \vee \psi \vee A(x_0)\right) \wedge \left(\bigvee \Delta \vee \psi \vee A(x_1)\right) \wedge \ldots \wedge \left(\bigvee \Delta \vee \psi \vee A(x_n)\right)\right]$$

By compactness we finally have: $\bigwedge C \vdash \varphi \supset \forall v (\bigvee \Delta \vee \psi \vee A[v/x])$, -w.l.o.g. we can choose v not free in ψ nor Δ- and then $\bigwedge C \vdash \varphi \supset (\bigvee \Delta \vee \psi \vee \forall v A[v/x])$ to finalize with $\bigwedge C \wedge \varphi \vdash \bigvee \Delta \vee \psi \vee \forall v A[v/x]$. So $\Sigma \cup X \vdash \Delta \cup Y \cup \{\forall v A\}$ as desired. Thus $\langle \Sigma \cup X, \Delta \cup Y\rangle$ fulfills (QS2).■

□

LEMMA 54. *(pair extension in the same language).*

If $\langle \Sigma, \Delta\rangle$ is a quantifier-suited pair, and if one of Σ and Δ is finite, then there is a full quantifier-suited pair $\langle \Sigma', \Delta'\rangle$ extending $\langle \Sigma, \Delta\rangle$ in the same language.

PROOF. The procedure is similar to the one used in the proof of the pair extension lemma on an extended language (lemma 46). But now, instead of adding a new witness for each existential quantifier we show that an old one suffices.

Let $Enum\left(\mathcal{L}_Q^{+'}\right) = \{A_n \mid n \in \omega\}$ be an enumeration of all the formulas $A_0, A_1, \ldots$ of $\mathcal{L}_Q^{+'}$, and $Var^{+} = \{x_n \mid n \in \omega\}$ an enumeration of all the variables in $\mathcal{L}_Q^{+'}$. Now define $\langle \Sigma_0, \Delta_0\rangle := \langle \Sigma, \Delta\rangle$, and we let $\langle \Sigma_{n+1}, \Delta_{n+1}\rangle$ to be defined as follows:

Case 1. (as in lemma 46) If $\Sigma_n \cup \{A_n\} \nvdash \Delta_n$ and A_n is not of the form $\exists v B$ then $\langle \Sigma_{n+1}, \Delta_{n+1}\rangle := \langle \Sigma_n \cup \{A_n\}, \Delta_n\rangle$.

Case 2. If $\Sigma_n \cup \{A_n\} \nvdash \Delta_n$ and A_n is of the form $\exists v B$ then $\langle \Sigma_{n+1}, \Delta_{n+1}\rangle := \langle \Sigma_n \cup \{A_n, B(x)\}, \Delta_n\rangle$, for some $x \in Var^{+}$ where $\Sigma_n \cup \left\{A_n, B_{g'(v)=x}\right\} \nvdash \Delta_n$ with $g' \equiv_v g$.

Case 3. (as in lemma 46) If $\Sigma_n \cup \{A_n\} \vdash \Delta_n$ and A_n is not of the form $\forall vB$ then $\langle \Sigma_{n+1}, \Delta_{n+1} \rangle := \langle \Sigma_n, \Delta_n \cup \{A_n\} \rangle$.

Case 4. If $\Sigma_n \cup \{A_n\} \vdash \Delta_n$ and A_n is of the form $\forall vB$ then $\langle \Sigma_{n+1}, \Delta_{n+1} \rangle := \langle \Sigma_n, \Delta_n \cup \{A_n, B(x)\} \rangle$, for some $x \in Var^+$ where $\Sigma_n \nvdash \Delta_n \cup \{A_n, B(x)\}$

Then we set $\langle \Sigma', \Delta' \rangle := \langle \bigcup_n \Sigma_n, \bigcup_n \Delta_n \rangle$, which by construction is a partition of the formulas, as before: every formula A_n is either in Σ' or in Δ', thus $\langle \Sigma', \Delta' \rangle$ is full.

Verification. Assume $\langle \Sigma, \Delta \rangle$ is a quantifier-suited pair, and one of Σ and Δ is finite. Cases 1 and 3 are as before, so we are done with them. The crucial cases are 2 and 4, which previously added lots of new constants to the initial language. We need to show that at each stage n, if we use the 2nd or 4th cases of the construction, an appropriate x can be found in the *same* language $\mathcal{L}_Q^{+'}$. This is where the finite addition lemma (lemma 53) is used. Each $\langle \Sigma_n, \Delta_n \rangle$ is quantifier-suited, as it is a finite extension to the quantifier-suited $\langle \Sigma, \Delta \rangle$ and one of Σ or Δ is finite (note that an extension might be finite even if the original Σ or Δ -and thus the resulting extended set- is infinite).

Case 1. Case where Σ is finite.

Case i. Case 2 of construction: $\Sigma_n \cup \{A_n\} \nvdash \Delta_n$ and A_n is of the form $\exists vB$. Assume there is no suitable $x \in Var^+$ in the language such that $\Sigma_n \cup \left\{A_n, B(v)_{g'(v)=x}\right\} \nvdash \Delta_n$ with $g' \equiv_v g$.[14] Then it follows that for all $y \in Var^+$ we have $\Sigma_n \cup \{A_n, B(y)\} \vdash \Delta_n$. Then there is a finite $D \in \Delta_n$ such that $\bigwedge \Sigma_n \wedge A_n \wedge B(y) \vdash \bigvee D$, which leads us to $\Sigma_n \cup \{A_n \wedge B(y)\} \vdash \Delta_n$. But $\langle \Sigma_n, \Delta_n \rangle$ is quantifier-suited, thus by (QS2) we have $\Sigma_n \cup \{\exists v (\exists vB \wedge B(v))\} \vdash \Delta_n$. Since v is not free in the first occurrence of B, we get: $\Sigma_n \cup \{(\exists vB \wedge \exists vB)\} \vdash \Delta_n$, that is $\Sigma_n \cup \{\exists vB\} \vdash \Delta_n$. But by assumption $\Sigma_n \cup \{A_n\} \nvdash \Delta_n$ with A_n of the form $\exists vB$, contradiction! Hence there is such suitable $x \in Var^+$.

Case ii. Case 4 of construction (dual): $\Sigma_n \cup \{A_n\} \vdash \Delta_n$ with A_n the form $\forall vB$. Assume there is no suitable $x \in Var^+$ in the language such that $\Sigma_n \nvdash \Delta_n \cup \left\{A_n, B(v)_{g'(v)=x}\right\}$ with $g' \equiv_v g$. Then it follows that for all $y \in Var^+$ we have $\Sigma_n \vdash \Delta_n \cup \{A_n, B(y)\}$. Then there is a finite $D \in \Delta_n$ such that $\bigwedge \Sigma_n \vdash \bigvee D \vee A_n \vee B(y)$ which leads us to $\Sigma_n \vdash \Delta_n \cup \{A_n \vee B(y)\}$. By quantifier suitedness of $\langle \Sigma_n, \Delta_n \rangle$ we can apply (QS1) and get $\Sigma_n \vdash \Delta_n \cup \{\forall v (\forall vB \vee B(v))\}$. Since v is not free in the first occurrence of B, we get: $\Sigma_n \vdash \Delta_n \cup \{(\forall vB \vee \forall vB)\}$ that is $\Sigma_n \vdash \Delta_n \cup \{\forall vB\}$. But then we have $\Sigma_n \vdash \Delta_n \cup \{A_n\}$ and $\Sigma_n \nvdash \Delta_n \cup \{A_n, B[v := c]\}$, contradiction! Hence there is such suitable $x \in Var^+$.

Case 2. Case where Δ is finite.

Case i. Case 2 of construction: exactly the same proof as above, just change "Then there is a finite $D \in \Delta_n$ such that $\bigwedge \Sigma_n \wedge A_n \wedge B(y) \vdash \bigvee D$ " by "Then there is a finite $C \in \Sigma_n$ such that $\bigwedge C \wedge A_n \wedge B(y) \vdash \Delta_n$ ".

Case ii. Case 4: same proof as above with similar change as previous case.

Proof is complete. □

Now we are almost ready. The following lemma will ultimately provide witness-points for the hard direction of the truth lemma. Since we extended the language conservatively, the additional connectives $\supset$ and - need to be treated as well.

[14] Observe that $g'(v) = x$ implies that $x \in Var \subset Var^+$since the universe of objects is Var

LEMMA 55. *witnessing-pair existence lemma*[15]

If $\langle \Sigma, \Delta \rangle$ is full and quantifier-suited, then

(1) If $\Box\psi \notin \Sigma$ then $\left\langle \Box^{-1}\Sigma, \{\psi\} \right\rangle$ is a quantifier-suited pair.
(2) If $\rhd\psi \notin \Sigma$ then $\left\langle \{\psi\}, \rhd^{-1}\Sigma \right\rangle$ is a quantifier-suited pair.
(3) If $\Diamond\psi \in \Sigma$ then $\left\langle \{\psi\}, \Diamond^{-1}\Delta \right\rangle$ is a quantifier-suited pair.
(4) If $\lhd\psi \in \Sigma$ then $\left\langle \lhd^{-1}\Delta, \{\psi\} \right\rangle$ is a quantifier-suited pair.
(5) If $\varphi \supset \psi \notin \Sigma$ then $\langle \Sigma \cup \{\varphi\}, \{\psi\} \rangle$ is a quantifier-suited pair.
(6) If $\varphi - \psi \in \Sigma$ then $\langle \{\varphi\}, \{\psi\} \cup \Delta \rangle$is a quantifier-suited pair.

In all the claims that follow assume that $\langle \Sigma, \Delta \rangle$ is a full and quantifier-suited pair.

CLAIM 56. If $\Box\psi \notin \Sigma$ then $\left\langle \Box^{-1}\Sigma, \{\psi\} \right\rangle$ is a quantifier-suited pair.

PROOF. Assume that $\Box\psi \notin \Sigma$.

We first show that $\left\langle \Box^{-1}\Sigma, \{\psi\} \right\rangle$ is a pair. We know that $\left\langle \Box^{-1}\Sigma, \{\psi\} \right\rangle$ is a pair iff $\bigwedge S_n \nvdash \psi$ for any finite subset $S_n \subseteq_\omega \Box^{-1}\Sigma$ (each s a formula, and $\bigwedge S_n = s_0 \wedge s_1 \wedge \ldots \wedge s_{n-1}$). Assume towards a contradiction that $\bigwedge S_n \vdash \psi$ for some $n \in \omega$, then $\Box \bigwedge S_n \vdash \Box\psi$ (by IR.3b). But we have $\bigwedge_{0 \leq i \leq n-1} \Box s_i \vdash \Box \bigwedge S_n \vdash \Box\psi$ (first entailment is an instance of A.2a) and hence $\bigwedge_{0 \leq i \leq n-1} \Box s_i \vdash \Box\psi$ by cut. Since $\Box^{-1}\Sigma := \{\varphi \mid \Box\varphi \in \Sigma\}$ and each $s_i \in \Box^{-1}\Sigma$ then each $\Box s_i \in \Sigma$ and therefore $\Sigma \vdash \{\Box\psi\}$. So $\Box\psi \notin \Delta$ by the assumption of $\langle \Sigma, \Delta \rangle$ being a pair and then $\Box\psi \in \Sigma$ by fullness. But by assumption $\Box\psi \notin \Sigma$, contradiction! Thus $\bigwedge S_n \vdash \psi$ is not the case for any $n \in \omega$. We have $\bigwedge S_n \nvdash \psi$ for any finite subset $S_n \subseteq_\omega \Box^{-1}\Sigma$, as desired, so $\left\langle \Box^{-1}\Sigma, \{\psi\} \right\rangle$ is a pair .■

We now show that $\left\langle \Box^{-1}\Sigma, \{\psi\} \right\rangle$ is quantifier-suited. By the finite addition lemma (lemma 53), $\left\langle \Box^{-1}\Sigma, \{\psi\} \right\rangle$ is quantifier-suited if $\left\langle \Box^{-1}\Sigma, \varnothing \right\rangle$ is. So let us show the second.

So, for QS1, fix an $x \in Var^+$ and assume $\Box^{-1}\Sigma \vdash \varnothing \cup \{\zeta(x)\}$. Then there is a finite set of φ_is in $\Box^{-1}\Sigma$ such that $\bigwedge_i \varphi_i \vdash \zeta(x)$. Then $\Box \bigwedge_i \varphi_i \vdash \Box\zeta(x)$, by (IR.3b), and thus $\bigwedge_i \Box\varphi_i \vdash \Box\zeta(x)$ by (A.2a) and cut. Then, because $\Box^{-1}\Sigma := \{\varphi \mid \Box\varphi \in \Sigma\}$ and each $\varphi_i \in \Box^{-1}\Sigma$, we get each $\Box\varphi_i \in \Sigma$. Hence $\Sigma \vdash \Box\zeta(x)$. Now since x is arbitrary this holds for every $x \in Var^+$ and since $\langle \Sigma, \Delta \rangle$ is quantifier-suited, from $\Sigma \vdash \varnothing \cup \{\Box\zeta(x)\}$ for each $x \in Var^+$, we get $\Sigma \vdash \varnothing \cup \{\forall v \Box\zeta\}$, that is $C \vdash \forall v \Box\zeta$ for a finite conjunction C from Σ. The Barcan formula (A.5a) gives $C \vdash \forall v \Box\zeta \vdash \Box\forall v\zeta$ and thus $\Sigma \vdash \{\Box\forall v\zeta\}$ by cut. This leads to $\Box\forall v\zeta \notin \Delta$ by the pair condition and then to $\Box\forall v\zeta \in \Sigma$ by fullness. So $\forall v\zeta \in \Box^{-1}\Sigma$ because $\forall v\zeta$ is boxed in Σ. This means that $\Box^{-1}\Sigma \vdash \varnothing \cup \{\forall v\zeta\}$, as desired.■

For QS2, fix an $x \in Var^+$ and assume that $\Box^{-1}\Sigma \cup \{\zeta(x)\} \vdash \varnothing$. Then $C \wedge \zeta(x) \vdash \bigvee \varnothing$, with $\bigvee \varnothing = \bot$ and C a finite conjunction of formulas in $\Box^{-1}\Sigma$. Then $C \vdash \zeta(x) \supset \bot$ by implication. Hence $\Box^{-1}\Sigma \vdash \{\zeta(x) \supset \bot\}$ and the previous result (QS1 holds) can be used to get $\Box^{-1}\Sigma \vdash \{\forall v (\zeta(v) \supset \bot)\}$. So $C \vdash \forall v (\zeta(v) \supset \bot)$. Now we have $C \vdash \forall v (\zeta(v) \supset \bot) \vdash (\exists v \zeta(v) \supset \bot)$ by QN2, since obviously v does not occur in $\bot$. Then $C \vdash (\exists v\zeta \supset \bot)$ by cut and $C \wedge \exists v\zeta \vdash \bot$ follows immediately. This leads to our final result: $\Box^{-1}\Sigma \cup \{\exists v\zeta\} \vdash \varnothing$ as desired.■

Since $\left\langle \Box^{-1}\Sigma, \varnothing \right\rangle$ is quantifier suited then $\left\langle \Box^{-1}\Sigma, \{\psi\} \right\rangle$ is quantifier suited as well. □

[15]We will use the following notation:

- $\Box^{-1}\Sigma := \{\varphi \mid \Box\varphi \in \Sigma\}$, i.e. the set of formulas that are boxed in Σ.
- $\rhd^{-1}\Sigma := \{\varphi \mid \rhd\varphi \in \Sigma\}$,
- $\Diamond^{-1}\Delta := \{\varphi \mid \Diamond\varphi \in \Delta\}$,
- $\lhd^{-1}\Delta := \{\varphi \mid \lhd\varphi \in \Delta\}$,

CLAIM 57. If $\rhd\psi \notin \Sigma$ then $\left\langle \{\psi\}, \rhd^{-1}\Sigma \right\rangle$ is a quantifier-suited pair.

PROOF. Assume that $\rhd\psi \notin \Sigma$.

We first show that $\left\langle \{\psi\}, \rhd^{-1}\Sigma \right\rangle$ is a pair. This is the case iff $\psi \nvdash \rhd^{-1}\Sigma$, that is, $\psi \nvdash \bigwedge S_n$ for any finite subset $S_n \subseteq_\omega \rhd^{-1}\Sigma$ (each s a formula, and $\bigwedge S_n = s_0 \wedge s_1 \wedge \ldots \wedge s_{n-1}$). Assume towards a contradiction that $\psi \vdash \bigwedge S_n$ for some $n \in \omega$, then $\rhd \bigwedge S_n \vdash \rhd\psi$ by (IR.3d). But we have $\bigwedge_{0\leq i\leq n-1} \rhd s_i \vdash \rhd \bigwedge S_n \vdash \rhd\psi$ (first entailment is an instance of A.2d) and hence $\bigwedge_{0\leq i\leq n-1} \rhd s_i \vdash \rhd\psi$ by cut. Since $\rhd^{-1}\Sigma := \{\varphi \mid \rhd\varphi \in \Sigma\}$ and each $s_i \in \rhd^{-1}\Sigma$ then each $\rhd s_i \in \Sigma$ and therefore $\Sigma \vdash \{\rhd\psi\}$. So $\rhd\psi \notin \Delta$ by the assumption of $\langle \Sigma, \Delta \rangle$ being a pair and then $\rhd\psi \in \Sigma$ by fullness. But by assumption $\rhd\psi \notin \Sigma$, contradiction! Thus $\psi \vdash \bigwedge S_n$ is not the case for any $n \in \omega$. We have $\psi \nvdash \bigwedge S_n$ for any finite subset $S_n \subseteq_\omega \rhd^{-1}\Sigma$, as desired, so $\left\langle \{\psi\}, \rhd^{-1}\Sigma \right\rangle$ is a pair .■

We now show that $\left\langle \{\psi\}, \rhd^{-1}\Sigma \right\rangle$ is quantifier-suited. By the finite addition lemma (lemma 53), $\left\langle \{\psi\}, \rhd^{-1}\Sigma \right\rangle$ is quantifier-suited if $\left\langle \varnothing, \rhd^{-1}\Sigma \right\rangle$ is. So let us show the second.

So, for QS2, fix an $x \in Var^+$ and assume $\varnothing \cup \{\zeta(x)\} \vdash \rhd^{-1}\Sigma$. Then there is a finite set of φ_is in $\rhd^{-1}\Sigma$ such that $\top \wedge \zeta(x) \vdash \bigvee_i \varphi_i$. Since for any formula δ we have $\top \wedge \delta \dashv\vdash \delta$, then $\zeta(x) \vdash \bigvee_i \varphi_i$ and finally $\rhd \bigvee_i \varphi_i \vdash \rhd\zeta(x)$ by (IR.3d). Therefore $\bigwedge_i \rhd\varphi_i \vdash \rhd \bigvee_i \varphi_i \vdash \rhd\zeta(x)$ by (A.2d) and $\bigwedge_i \rhd\varphi_i \vdash \rhd\zeta(x)$ by cut. Then, because $\rhd^{-1}\Sigma := \{\varphi \mid \rhd\varphi \in \Sigma\}$ and each $\varphi_i \in \rhd^{-1}\Sigma$, we get each $\rhd\varphi_i \in \Sigma$. Hence $\Sigma \vdash \{\rhd\zeta(x)\}$ and by weakening (IR.1c) the consequent with Δ we get $\Sigma \vdash \Delta \cup \{\rhd\zeta(x)\}$. Now since x is arbitrary this holds for every $x \in Var^+$ and since $\langle \Sigma, \Delta \rangle$ is quantifier-suited,, we get $\Sigma \vdash \Delta \cup \{\forall v \rhd \zeta(v)\}$, that is $C \vdash D \vee \forall v \rhd \zeta(v)$ for a finite conjunction C from Σ and a finite disjunction D from Δ. The Barcan formula (A.5b) gives $C \vdash D \vee \forall v \rhd \zeta(v) \vdash D \vee \rhd\exists v\zeta(v)$ and thus $C \vdash D \vee \rhd\exists v\zeta(v)$ by cut. This leads to $\rhd\exists v\zeta(v) \notin \Delta$ by the assumption that $\langle \Sigma, \Delta \rangle$ is a pair. But then to $\rhd\exists v\zeta(v) \in \Sigma$ by fullness. So $\exists v\zeta(v) \in \rhd^{-1}\Sigma$. This means that $\varnothing \cup \{\exists v\zeta(v)\} \vdash \rhd^{-1}\Sigma$ as desired.■

Now, for QS1, fix an $x \in Var^+$ and assume $\varnothing \vdash \rhd^{-1}\Sigma \cup \{\zeta(x)\}$. Then there is a finite set of φ_is in $\rhd^{-1}\Sigma$ such that $\top \vdash \bigvee_i \varphi_i \vee \zeta(x)$ and then $\top - \zeta(x) \vdash \bigvee_i \varphi_i$ by substraction. Since (QS2) holds, we obtain $\exists v(\top - \zeta(v)) \vdash \bigvee_i \varphi_i$ and by QN4 we have $\top - \forall v\zeta(v) \vdash \exists v(\top - \zeta(v)) \vdash \bigvee_i \varphi_i$, that is $\top - \forall v\zeta(v) \vdash \bigvee_i \varphi_i$ by cut. Finally, $\top \vdash \bigvee_i \varphi_i \vee \forall v\zeta(v)$ which leads to $\varnothing \vdash \rhd^{-1}\Sigma \cup \{\forall v\zeta(v)\}$ as desired.■ □

CLAIM 58. If $\Diamond\psi \in \Sigma$ then $\left\langle \{\psi\}, \Diamond^{-1}\Delta \right\rangle$ is a quantifier-suited pair.

PROOF. Assume that $\Diamond\psi \in \Sigma$.

We first show that $\left\langle \{\psi\}, \Diamond^{-1}\Delta \right\rangle$ is a pair. Suppose towards a contradiction that $\{\psi\} \vdash \Diamond^{-1}\Delta$, then there exists a finite subset of formulas $\{\varphi_i\}_{i\in I} \subseteq_\omega \Diamond^{-1}\Delta$ such that $\psi \vdash \bigvee_{i\in I} \psi_i$. But then by (IR.3a) we have $\Diamond\psi \vdash \Diamond \bigvee_{i\in I} \varphi_i$ and then $\Diamond\psi \vdash \Diamond \bigvee_{i\in I} \varphi_i \vdash \bigvee_{i\in I} \Diamond\varphi_i$ by (A.2b). So $\Diamond\psi \vdash \bigvee_{i\in I} \Diamond\varphi_i$ by cut. Since $\varphi_i \in \Diamond^{-1}\Delta$ for all $i \in I$, then $\Diamond\varphi_i \in \Delta$ for all $i \in I$. Thus $\{\Diamond\psi\} \vdash \Delta$. Now since by assumption $\Diamond\psi \in \Sigma$, then $\Sigma \vdash \Delta$. This contradicts the assumption of $\langle \Sigma, \Delta \rangle$ being a pair, hence for any finite subset of formulas $\{\varphi_i\}_{i\in I} \subseteq \Diamond^{-1}\Delta$ we have $\psi \nvdash \bigvee_{i\in I} \varphi_i$ and $\left\langle \{\psi\}, \Diamond^{-1}\Delta \right\rangle$ is a pair.■

Now we show that $\left\langle \{\psi\}, \Diamond^{-1}\Delta \right\rangle$ is quantifier-suited. By the finite addition lemma (lemma 53), for this it suffices to show that $\left\langle \varnothing, \Diamond^{-1}\Delta \right\rangle$ is quantifier-suited.

For (QS2) fix an $x \in Var^+$. If $\varnothing \cup \{\zeta(x)\} \vdash \Diamond^{-1}\Delta$, there is a finite set $\{\varphi_i\}_{i\in I} \subseteq \Diamond^{-1}\Delta$ such that $\zeta(x) \vdash \bigvee_{i\in I} \varphi_i$. Thus by (IR.3a) we get $\Diamond\zeta(x) \vdash \Diamond \bigvee_{i\in I} \varphi_i$. But then, by (A.2b): $\Diamond\zeta(x) \vdash \Diamond \bigvee_{i\in I} \varphi_i \vdash \bigvee_{i\in I} \Diamond\varphi_i$. Now since $\{\varphi_i\}_{i\in I} \subseteq_\omega \Diamond^{-1}\Delta$, then $\{\Diamond\varphi_i\}_{i\in I} \subseteq_\omega \Delta$, and therefore $\{\Diamond\zeta(x)\} \vdash \Delta$. Now assume that $\varnothing \cup \{\zeta(x)\} \vdash \Diamond^{-1}\Delta$ for all $x \in Var^+$, by weakening the antecedent with Σ, we get $\Sigma \cup \{\Diamond\zeta(x)\} \vdash \Delta$ and then $\Sigma \cup \{\exists v\Diamond\zeta(v)\} \vdash \Delta$ by quantifier-suitedness of $\langle \Sigma, \Delta \rangle$. Now by Barcan laws (A.5a), $\Sigma \cup \{\Diamond\exists v\zeta(v)\} \vdash \Sigma \cup \{\exists v\Diamond\zeta(v)\} \vdash \Delta$ and then $\Sigma \cup \{\Diamond\exists v\zeta(v)\} \vdash \Delta$ by cut. Now because $\langle \Sigma, \Delta \rangle$ is a pair, $\{\Diamond\exists v\zeta(v)\} \vdash \Delta$ and

$\Diamond\exists v\zeta(v) \notin \Sigma$. So by fullness $\Diamond\exists v\zeta(v) \in \Delta$ and then $\exists v\zeta(v) \in \Diamond^{-1}\Delta$. So clearly $\varnothing \cup \{\exists v\zeta(v)\} \vdash \Diamond^{-1}\Delta$, as desired. This proves that (QS2) holds for $\langle \varnothing, \Diamond^{-1}\Delta \rangle$.■

For (QS1) fix an $x \in Var^{+}$. If $\varnothing \vdash \Diamond^{-1}\Delta \cup \{\zeta(x)\}$ then there is a finite set $\{\varphi_i\}_{i\in I} \subseteq_\omega \Diamond^{-1}\Delta$ such that $\top \vdash \zeta(x) \vee (\bigvee_{i\in I}\varphi_i)$. Then $\top - \zeta(x) \vdash \bigvee_{i\in I}\varphi_i$ by substraction. Now by (IR.3a) we get $\Diamond(\top - \zeta(x)) \vdash \Diamond\bigvee_{i\in I}\varphi_i$ and then, by (A.2b): $\Diamond(\top - \zeta(x)) \vdash \Diamond\bigvee_{i\in I}\varphi_i \vdash \bigvee_{i\in I}\Diamond\varphi_i$. Since $\{\varphi_i\}_{i\in I} \subseteq_\omega \Diamond^{-1}\Delta$ then $\{\Diamond\varphi_i\}_{i\in I} \subseteq_\omega \Delta$, and therefore $\{\Diamond(\top - \zeta(x))\} \vdash \Delta$. Now assume that $\varnothing \vdash \Diamond^{-1}\Delta \cup \{\zeta(x)\}$ for all $x \in Var^{+}$. By weakening the antecedent with Σ, we get $\Sigma \cup \{\Diamond(\top - \zeta(x))\} \vdash \Delta$ for all $x \in Var^{+}$ and then $\Sigma \cup \{\exists v\Diamond(\top - \zeta(v))\} \vdash \Delta$ by the quantifier-suitedness of $\langle\Sigma,\Delta\rangle$. Now by Barcan laws (5a), $\Sigma \cup \{\Diamond\exists v(\top - \zeta(v))\} \vdash \Sigma \cup \{\exists v\Diamond(\top - \zeta(v))\} \vdash \Delta$ and then $\Sigma \cup \{\Diamond\exists v(\top - \zeta(v))\} \vdash \Delta$ by cut. Now because $\langle\Sigma,\Delta\rangle$ is a pair, $\{\Diamond\exists v(\top - \zeta(v))\} \vdash \Delta$ and $\Diamond\exists v(\top - \zeta(v)) \notin \Sigma$. So by fullness $\Diamond\exists v(\top - \zeta(v)) \in \Delta$ and then $\exists v(\top - \zeta(v)) \in \Diamond^{-1}\Delta$. So clearly $\{\exists v(\top - \zeta(v))\} \vdash \Diamond^{-1}\Delta$. Now observe that $\top - \forall v\zeta(v) \vdash \exists v(\top - \zeta(v)) \vdash D$ for some finite disjunction D of formulas in $\Diamond^{-1}\Delta$. Hence $\top - \forall v\zeta(v) \vdash D$ and then $\top \vdash D \vee \forall v\zeta(v)$. So $\varnothing \vdash \Diamond^{-1}\Delta \cup \{\forall v\zeta(v)\}$ which proves that (QS1) holds for $\langle \varnothing, \Diamond^{-1}\Delta \rangle$.■ □

Claim 59. If $\lhd\psi \in \Sigma$ then $\langle \lhd^{-1}\Delta, \{\psi\} \rangle$ is a quantifier-suited pair.

Proof. Assume that $\lhd\psi \in \Sigma$.

First we show that $\langle \lhd^{-1}\Delta, \{\psi\} \rangle$ is a pair. Assume towards a contradiction that $\lhd^{-1}\Delta \vdash \{\psi\}$, then there exists a finite subset of formulas $\{\varphi_i\}_{i\in I} \subseteq_\omega \lhd^{-1}\Delta$ such that $\bigwedge_i \varphi_i \vdash \psi$. Then $\lhd\psi \vdash \lhd\bigwedge_i\varphi_i$ by (IR.3c), and $\lhd\bigwedge_i\varphi_i \vdash \bigvee_i \lhd\varphi_i$ by (A.2c). This gives us $\lhd\psi \vdash \bigvee_i\lhd\varphi_i$ by cut, and since by assumption $\lhd\psi \in \Sigma$ and $\varphi_i \in \lhd^{-1}\Delta$, this implies that $\Sigma \vdash \Delta$. But we assumed $\langle\Sigma,\Delta\rangle$ to be a pair, so this is a contradiction! So $\bigwedge_i\varphi_i \vdash \psi$ cannot be the case and we have $\bigwedge_i \varphi_i \nvdash \psi$ for any finite subset of formulas $\{\varphi_i\}_{i\in I} \subseteq_\omega \lhd^{-1}\Delta$ and thus $\lhd^{-1}\Delta \nvdash \{\psi\}$ as desired.■

We now show that $\langle \lhd^{-1}\Delta, \{\psi\} \rangle$ is quantifier-suited. By the finite addition lemma (lemma 53), $\langle \lhd^{-1}\Delta, \{\psi\} \rangle$ is a quantifier-suited if $\langle \lhd^{-1}\Delta, \varnothing \rangle$ is. So let us prove the second.

For (QS1) fix an $x \in Var^{+}$. If $\lhd^{-1}\Delta \vdash \varnothing \cup \zeta(x)$ then there is a finite subset of formulas $\{\varphi_i\}_{i\in I} \subseteq_\omega \lhd^{-1}\Delta$ such that such that $\bigwedge_i\varphi_i \vdash \zeta(x)$. Then $\lhd\zeta(x) \vdash \lhd\bigwedge_i\varphi_i$ by (IR.3c), and thus $\lhd\zeta(x) \vdash \bigvee_i\lhd\varphi_i$ by (A.2c) and cut. Then, because $\lhd^{-1}\Delta := \{\varphi \mid \lhd\varphi \in \Delta\}$ and $\varphi_i \in \lhd^{-1}\Delta$, we get each $\lhd\varphi_i \in \Delta$. Hence $\{\lhd\zeta(x)\} \vdash \Delta$. By weakening the antecedent we obtain $\Sigma \cup \{\lhd\zeta(x)\} \vdash \Delta$. Now assume that $\lhd^{-1}\Delta \vdash \varnothing \cup \zeta(x)$ for each $x \in Var^{+}$. Since $\langle\Sigma,\Delta\rangle$ is quantifier-suited, we get $\Sigma \cup \{\exists v \lhd \zeta(v)\} \vdash \Delta$ by (QS2). By the pairness of $\langle\Sigma,\Delta\rangle$ we know that $\Sigma \nvdash \Delta$, so it must be the case that $\exists v\lhd\zeta(v) \vdash D$ for some finite disjunction D in Δ (otherwise $\Sigma \cup \{\exists v\lhd\zeta(v)\} \nvdash \Delta$). The Barcan formula (A.5b) gives $\lhd\forall v\zeta(v) \vdash \exists v\lhd\zeta(v) \vdash D$, and by cut, $\lhd\forall v\zeta(v) \vdash D$. Therefore $\{\lhd\forall v\zeta(v)\} \vdash \Delta$. By pairness $\lhd\forall v\zeta(v) \notin \Sigma$ and then by fullness $\lhd\forall v\zeta(v) \in \Delta$. Consequently $\forall v\zeta(v) \in \lhd^{-1}\Delta$. Clearly then, $\lhd^{-1}\Delta \vdash \varnothing \cup \{\forall v\zeta(v)\}$, as desired.■

Now for QS2, fix an $x \in Var^{+}$. If $\lhd^{-1}\Delta \cup \{\zeta(x)\} \vdash \varnothing$ then $C \wedge \zeta(x) \vdash \bot$ with C a finite conjunction of formulas in $\lhd^{-1}\Delta$. We obtain $C \vdash \zeta(x) \supset \bot$ by implication. Assume that $\lhd^{-1}\Delta \cup \{\zeta(x)\} \vdash \varnothing$ for all $x \in Var^{+}$. Then the previous result (QS1 holds) can be applied to $\lhd^{-1}\Delta \vdash \varnothing \cup \{\zeta(x) \supset \bot\}$ to obtain $\lhd^{-1}\Delta \vdash \varnothing \cup \{\forall v(\zeta(v) \supset \bot)\}$. Then $C \vdash \forall v(\zeta(v) \supset \bot)$. By (QN2), and since v is not free in $\bot$, $C \vdash \forall v(\zeta(v) \supset \bot) \vdash \exists v\zeta(v) \supset \bot$. Thus $C \vdash \exists v\zeta(v) \supset \bot$ by cut and then $C \wedge \exists v\zeta(v) \vdash \bot$. Finally $\lhd^{-1}\Delta \cup \{\exists v\zeta(v)\} \vdash \varnothing$ as desired.■ □

Claim 60. If $\varphi \supset \psi \notin \Sigma$ then $\langle \Sigma \cup \{\varphi\}, \{\psi\} \rangle$ is a quantifier-suited pair.

PROOF. Assume that $\varphi \supset \psi \notin \Sigma$.

We first show that $\langle \Sigma \cup \{\varphi\}, \{\psi\} \rangle$ is a pair. Assume towards a contradiction that $\Sigma \cup \{\varphi\} \vdash \{\psi\}$. Then there exists a finite set of formulas $\{\chi_i\}_{i \in I} \subseteq_\omega \Sigma$ such that $\left(\bigwedge_{i \in I} \chi_i\right) \wedge \varphi \vdash \psi$. Then by implication rule, $\bigwedge_{i \in I} \chi_i \vdash \varphi \supset \psi$ which gives us $\Sigma \vdash \{\varphi \supset \psi\}$. Since $\langle \Sigma, \Delta \rangle$ is a pair by assumption, then $\varphi \supset \psi \notin \Delta$, which by fullness gives us $\varphi \supset \psi \in \Sigma$. Contradiction! Hence for any finite set of formulas $\{\chi_i\}_{i \in I} \subseteq_\omega \Sigma$ we have $\left(\bigwedge_{i \in I} \chi_i\right) \wedge \varphi \nvdash \psi$ and thus $\langle \Sigma \cup \{\varphi\}, \{\psi\} \rangle$ is a pair.■

By the finite addition lemma (lemma 53), $\langle \Sigma \cup \{\varphi\}, \{\psi\} \rangle$ is quantifier suited if $\langle \Sigma \cup \{\varphi\}, \varnothing \rangle$ is. So let us prove the second.

For QS1 , fix an $x \in Var^+$. If $\Sigma \cup \{\varphi\} \vdash \{\gamma(x)\} \cup \varnothing$. Then $C \wedge \varphi \vdash \gamma(x) \vee \perp$ (with $\bigvee \varnothing = \perp$ and C a finite conjunction composed of elements from Σ). A disjunction with bottom always simplifies into a single member, so $C \wedge \varphi \vdash \gamma(x)$ and then by implication rule $C \vdash \varphi \supset \gamma(x)$ .Given $\Sigma \vdash \{\varphi \supset \gamma(x)\}$ for all $x \in Var^+$, and by weakening the consequent with Δ we get $\Sigma \vdash \{\varphi \supset \gamma(x)\} \cup \Delta$ and finally $\Sigma \vdash \{\forall v (\varphi \supset \gamma(v))\} \cup \Delta$ by quantifier-suitedness of $\langle \Sigma, \Delta \rangle$. We can choose w.l.o.g. a v not free in φ and then $\Sigma \vdash \{\varphi \supset \forall v \gamma(v)\} \cup \Delta$. Since $\langle \Sigma, \Delta \rangle$ is a pair, then $\varphi \supset \forall v \gamma(v) \notin \Delta$ and then by fullness $\{\varphi \supset \forall v \gamma(v)\} \in \Sigma$. But then $\Sigma \vdash \{\varphi \supset \forall v \gamma(v)\}$, that is, $C \vdash \varphi \supset \forall v \gamma(v)$ and then $C \wedge \varphi \vdash \forall v \gamma(v)$ which finally leads us to $\Sigma \cup \{\varphi\} \vdash \{\forall v \gamma(v)\} \cup \varnothing$ as desired.■

For QS2, fix an $x \in Var^+$. If $\Sigma \cup \{\varphi\} \cup \{\gamma(x)\} \vdash \varnothing$ we have $C \wedge \varphi \wedge \gamma(x) \vdash \perp$, with C a finite conjunction composed of elements from Σ. Then $C \wedge \varphi \vdash \gamma(x) \supset \perp$ by implication rule. Assume that $\Sigma \cup \{\varphi\} \cup \{\gamma(x)\} \vdash \varnothing$ for all $x \in Var^+$. Then the previous result (QS1 holds) can be applied to $\Sigma \cup \{\varphi\} \vdash \{\gamma(x) \supset \perp\} \cup \varnothing$ to obtain $\Sigma \cup \{\varphi\} \vdash \{\forall v (\gamma(v) \supset \perp)\} \cup \varnothing$ and by QN2 from $C \wedge \varphi \vdash \forall v (\gamma(v) \supset \perp)$ we get $C \wedge \varphi \vdash \exists v \gamma(v) \supset \perp$ and then $C \wedge \varphi \wedge \exists v \gamma(v) \vdash \perp$ which means that $\Sigma \cup \{\varphi\} \cup \{\exists v \gamma(v)\} \vdash \varnothing$ as desired.■ □

CLAIM 61. If $\varphi - \psi \in \Sigma$ then $\langle \{\varphi\}, \{\psi\} \cup \Delta \rangle$ is a quantifier-suited pair.

PROOF. Assume that $\varphi - \psi \in \Sigma$.

We first show that $\langle \{\varphi\}, \{\psi\} \cup \Delta \rangle$ is a pair. Assume towards a contradiction that $\{\varphi\} \vdash \{\psi\} \cup \Delta$ so $\varphi \vdash \psi \vee D$ with D a finite disjunction composed of elements from Δ. By substraction we get $\varphi - \psi \vdash D$, that is $\{\varphi - \psi\} \vdash \Delta$. Since $\langle \Sigma, \Delta \rangle$ is a pair then $\Sigma \nvdash \Delta$ and $\varphi - \psi$ cannot be in Σ, contradiction! Hence $\langle \{\varphi\}, \{\psi\} \cup \Delta \rangle$ is a pair.■

By the finite addition lemma (lemma 53),$\langle \{\varphi\}, \{\psi\} \cup \Delta \rangle$ is a quantifier-suited if $\langle \varnothing, \{\psi\} \cup \Delta \rangle$ is. So let us prove the second.

For QS2, fix an $x \in Var^+$. If $\varnothing \cup \{\zeta(x)\} \vdash \{\psi\} \cup \Delta$ then $\zeta(x) \vdash \psi \vee D$ with D a finite disjunction composed of elements from Δ. Then $\zeta(x) - \psi \vdash D$ by substraction and by weakening the antecedent with $\bigwedge C$ and $C \subseteq \Sigma$ we get $\bigwedge C \wedge (\zeta(x) - \psi) \vdash D$. Now assume that $\varnothing \cup \{\zeta(x)\} \vdash \{\psi\} \cup \Delta$ for all $x \in Var^+$. Then $\Sigma \cup \{\zeta(x) - \psi\} \vdash \Delta$ for all $x \in Var^+$ and since $\langle \Sigma, \Delta \rangle$ is quantifier-suited then $\Sigma \cup \{\exists v (\zeta(v) - \psi)\} \vdash \Delta$ by (QS2). Since $\langle \Sigma, \Delta \rangle$ is a pair, then $\exists v (\zeta(v) - \psi) \notin \Sigma$ and then by fullness $\exists v (\zeta(v) - \psi) \in \Delta$, so $\{\exists v (\zeta(v) - \psi)\} \vdash \Delta$ which means that $\exists v (\zeta(v) - \psi) \vdash D$. By (QN3) we obtain $\exists v \zeta(v) - \exists v \psi \vdash \exists v (\zeta(v) - \psi) \vdash D$ and then $\exists v \zeta(v) - \exists v \psi \vdash D$ by cut. Since we can choose v not free in ψ, this amounts to $\exists v \zeta(v) - \psi \vdash D$ and then $\exists v \zeta(v) \vdash \psi \vee D$ which finally leads us to $\varnothing \cup \{\exists v \zeta(v)\} \vdash \{\psi\} \cup \Delta$, as desired.■

For QS1, fix an $x \in Var^+$. If $\varnothing \vdash \{\psi\} \cup \Delta \cup \{\zeta(x)\}$ then $\top \vdash \psi \vee D \vee \zeta(x)$ with D a finite disjunction composed of elements from Δ, and thus $\top \vdash \zeta(x) \vee \psi \vee D$ and then $\top - \zeta(x) \vdash \psi \vee D$ by substraction. This can be stated as $\varnothing \cup \{\top - \zeta(x)\} \vdash \{\psi\} \cup \Delta$. Now assume that $\varnothing \vdash \{\psi\} \cup \Delta \cup \{\zeta(x)\}$ for all $x \in Var^+$ and we can apply the previous result (QS2 holds) to obtain $\varnothing \cup \{\exists v (\top - \zeta(v))\} \vdash \{\psi\} \cup \Delta$ from $\varnothing \cup \{\top - \zeta(x)\} \vdash \{\psi\} \cup \Delta$. Thus

$\exists v (\top - \zeta(v)) \vdash \psi \vee D$ which by (QN4) gives us $\top - \forall v\zeta(v) \vdash \exists v (\top - \zeta(v)) \vdash \psi \vee D$, then $\top - \forall v\zeta(v) \vdash \psi \vee D$ by cut, and finally $\top \vdash \psi \vee D \vee \forall v\zeta(v)$ which stated as $\varnothing \vdash \{\psi\} \cup \Delta \cup \{\forall v\zeta(v)\}$ is our result.■ □

3.3. The truth lemma

The canonical valuation stipulates that truth at a point corresponds to membership to (the first coordinate of) such point. The Truth lemma lifts this property to arbitrary formulas (recall that any such point is a full quantifier suited pair in $\mathcal{L}_Q^{+'}$).

LEMMA 62. *(Truth lemma): for any formula ψ of our logic and any point $\langle\Sigma,\Gamma\rangle$ in the canonical model, $V^c_{\langle\Sigma,\Gamma\rangle}(\psi) = 1$ iff $\psi \in \Sigma$.*

PROOF. By induction on the complexity of ψ.

BASE CASE:

$V^c_{\langle\Sigma,\Delta\rangle}(P(x_1,\ldots,x_n)) = 1$ iff $P(x_1,\ldots,x_n) \in \Sigma$ for any atomic sentence $P(x_1,\ldots,x_n)$. This follows immediately from the definition of the canonical valuation given above (on page 32). In particular we get $V^c_{\langle\Sigma,\Delta\rangle}(P(x_1,\ldots,x_n)) = 1$ iff $\langle x_1,\ldots,x_n\rangle \in V^c_{\langle\Sigma,\Delta\rangle}(P)$ iff $P(x_1,\ldots,x_n) \in \Sigma$.

INDUCTIVE STEP

Conjunction

$V^c_{\langle\Sigma,\Delta\rangle}(\psi\wedge\varphi) = 1$ iff $\begin{cases} V^c_{\langle\Sigma,\Delta\rangle}(\psi)=1 \text{ and} \\ V^c_{\langle\Sigma,\Delta\rangle}(\varphi)=1 \end{cases}$ iff (by IH) $\begin{cases} \psi\in\Sigma \text{ and} \\ \varphi\in\Sigma \end{cases}$ iff $\psi\wedge\varphi \in \Sigma$ because $\langle\Sigma,\Delta\rangle$ is a full pair. The last equivalence holds because:

$\Rightarrow$ if $\varphi\in\Sigma$ and $\psi\in\Sigma$ then clearly $\Sigma \vdash \varphi\wedge\psi$ by conjunction introduction, so $\varphi\wedge\psi \notin \Delta$ (because $\Sigma \nvdash \Delta$) and then $\varphi\wedge\psi\in\Sigma$ by fullness.

$\Leftarrow$ if $\varphi\wedge\psi\in\Sigma$ then $\Sigma\vdash\varphi\wedge\psi$, so $\Sigma\vdash\varphi$ and $\Sigma\vdash\psi$ by conjunction elimination. Clearly $\varphi\notin\Delta$ and $\psi\notin\Delta$ (because $\Sigma\nvdash\Delta$) but then $\varphi\in\Sigma$ and $\psi\in\Sigma$ by fullness.

Therefore $V^c_{\langle\Sigma,\Delta\rangle}(\psi\wedge\varphi) = 1$ iff $\psi\wedge\varphi\in\Sigma$

Disjunction

$V^c_{\langle\Sigma,\Delta\rangle}(\psi\vee\varphi) = 1$ iff $\begin{cases} V^c_{\langle\Sigma,\Delta\rangle}(\psi)=1 \text{ or} \\ V^c_{\langle\Sigma,\Delta\rangle}(\varphi)=1 \end{cases}$ iff (by IH) $\begin{cases} \psi\in\Sigma \text{ or} \\ \varphi\in\Sigma \end{cases}$ iff $\psi\vee\varphi\in\Sigma$. The last "if and only if" holds because $\langle\Sigma,\Delta\rangle$ is a full pair and by claim 40, Σ is a prime theory (if T is a prime theory then: $\phi\vee\psi\in T$ iff $\phi\in T$ or $\psi\in T$).

$\Box$-*case:*

$V^c_{\langle\Sigma,\Delta\rangle}(\Box\varphi) = 1$ iff $\langle\Sigma,\Delta\rangle R^c_\Box \langle\Gamma,\Pi\rangle \Rightarrow V^c_{\langle\Gamma,\Pi\rangle}(\varphi) = 1$ by definition of $\Box$. By induction hypothesis $V^c_{\langle\Gamma,\Pi\rangle}(\varphi) = 1$ iff $\varphi\in\Gamma$. Hence: $V^c_{\langle\Sigma,\Delta\rangle}(\Box\varphi) = 1$ iff $\langle\Sigma,\Delta\rangle R^c_\Box \langle\Gamma,\Pi\rangle \Rightarrow \varphi\in\Gamma$.

Now we show that $\langle\Sigma,\Delta\rangle R^c_\Box \langle\Gamma,\Pi\rangle \Rightarrow \varphi\in\Gamma$ iff $\Box\varphi\in\Sigma$.

Left to right direction will be proved by contraposition. Claim: if $\Box\psi\notin\Sigma$ then there is a full quantifier suited pair $\langle a,b\rangle$ such that $\langle\Sigma,\Delta\rangle R^c_\Box \langle a,b\rangle$ and $\psi\notin a$. Assume that $\Box\psi\notin\Sigma$. Then by lemma 55.1 $\left\langle\Box^{-1}\Sigma,\{\psi\}\right\rangle$ is a quantifier suited pair, with $\Box^{-1}\Sigma := \{\varphi \mid \Box\varphi\in\Sigma\}$. Notice that $\{\psi\}$ is finite and thus we can apply the pair extension in the same language (lemma 54) which states then that there is a full quantifier-suited pair $\langle a',b'\rangle$ extending $\left\langle\Box^{-1}\Sigma,\{\psi\}\right\rangle$ in the same language. Clearly $\langle a',b'\rangle$ is such that $\langle\Sigma,\Delta\rangle R^c_\Box \langle a',b'\rangle$ because $\Box A\in\Sigma \Rightarrow A\in\Box^{-1}\Sigma \Rightarrow A\in a'$ so it fulfills the definition of $R^c_\Box$. Since $\{\psi\}\subseteq b'$ then $\psi\notin a'$ otherwise we would have $a'\vdash b'$ against the assumption that $\langle a',b'\rangle$ is a pair. Hence if $\Box\psi\notin\Sigma$ then there is a full quantifier suited pair $\langle a,b\rangle$ such that $\langle\Sigma,\Delta\rangle R^c_\Box \langle a,b\rangle$ and $\psi\notin a$, namely take $\langle a',b'\rangle = \langle a,b\rangle$ as witness.■

For the right to left direction, suppose $\Box\varphi \in \Sigma$ and assume further that $\langle \Sigma, \Delta \rangle R^c_\Box \langle \Gamma, \Pi \rangle$. Then by definition of $R^c_\Box$ we have $\varphi \in \Gamma$, so the implication $\langle \Sigma, \Delta \rangle R^c_\Box \langle \Gamma, \Pi \rangle \Rightarrow \varphi \in \Gamma$ holds, as desired.■

Hence $V^c_{\langle \Sigma, \Delta \rangle}(\Box\varphi) = 1$ iff $\Box\varphi \in \Sigma$.

$\Diamond$-*case.*

$V^c_{\langle \Sigma, \Delta \rangle}(\Diamond\varphi) = 1$ iff there exists an $\langle \Gamma, \Pi \rangle$ such that $\langle \Sigma, \Delta \rangle R^c_\Diamond \langle \Gamma, \Pi \rangle$ and $V^c_{\langle \Gamma, \Pi \rangle}(\varphi) = 1$. By IH $V^c_{\langle \Gamma, \Pi \rangle}(\varphi) = 1$ iff $\varphi \in \Gamma$.

Now we show that there exists an$\langle \Gamma, \Pi \rangle$ such that $\langle \Sigma, \Delta \rangle R^c_\Diamond \langle \Gamma, \Pi \rangle$ and $\varphi \in \Gamma$ iff $\Diamond\varphi \in \Sigma$.

Direction $\Rightarrow$ Assume there exists an $\langle \Gamma, \Pi \rangle$ such that $\langle \Sigma, \Delta \rangle R^c_\Diamond \langle \Gamma, \Pi \rangle$ and $\varphi \in \Gamma$, then by definition of $R^c_\Diamond$ we have $\Diamond\varphi \in \Sigma$, as desired.■

Direction $\Leftarrow$. Suppose $\Diamond\varphi \in \Sigma$. Then, by lemma 55.3 and since $\langle \Sigma, \Delta \rangle$ is a full quantifier suited pair, we know that $\left\langle \{\varphi\}, \Diamond^{-1}\Delta \right\rangle$ is a quantifier suited pair. Notice that $\{\varphi\}$ is finite and thus we can apply the pair extension in the same language (lemma 54) which states then that there is a full quantifier-suited pair $\langle a', b' \rangle$ extending $\left\langle \{\varphi\}, \Diamond^{-1}\Delta \right\rangle$ in the same language. Observe that $\langle \Sigma, \Delta \rangle R^c_\Diamond \langle a', b' \rangle$ by definition of $R^c_\Diamond$. Clearly, if $\psi \in a'$ then $\psi \notin b'$ by the pair condition. Hence $\psi \notin \Diamond^{-1}\Delta \subseteq b'$ and thus $\Diamond\psi \notin \Delta$. By fullness of $\langle \Sigma, \Delta \rangle$ we then get $\Diamond\psi \in \Sigma$ and the condition for $\langle \Sigma, \Delta \rangle R^c_\Diamond \langle a', b' \rangle$ is fulfilled, as desired. Hence there is an $\langle \Gamma, \Pi \rangle$ such that $\langle \Sigma, \Delta \rangle R^c_\Diamond \langle \Gamma, \Pi \rangle$ and $\varphi \in \Gamma$, namely take $\langle a', b' \rangle = \langle \Gamma, \Pi \rangle$ as witness.■

Therefore, $V^c_{\langle \Sigma, \Delta \rangle}(\Diamond\varphi) = 1$ iff $\Diamond\varphi \in \Sigma$.

$\triangleright$-*case.*

$V^c_{\langle \Sigma, \Delta \rangle}(\triangleright\varphi) = 1$ iff $\langle \Sigma, \Delta \rangle R^c_\triangleright \langle \Gamma, \Pi \rangle \Rightarrow V^c_{\langle \Gamma, \Pi \rangle}(\varphi) = 0$ by def of $\triangleright$. By induction hypothesis $V^c_{\langle \Gamma, \Pi \rangle}(\varphi) = 1$ iff $\varphi \in \Gamma$. Hence, $V^c_{\langle \Sigma, \Delta \rangle}(\triangleright\varphi) = 1$ iff $\langle \Sigma, \Delta \rangle R^c_\triangleright \langle \Gamma, \Pi \rangle \Rightarrow \varphi \notin \Gamma$. Now we show that $\langle \Sigma, \Delta \rangle R^c_\triangleright \langle \Gamma, \Pi \rangle \Rightarrow \varphi \notin \Gamma$ iff $\triangleright\varphi \in \Sigma$.

Left to right direction will be proved by contraposition. Claim: if $\triangleright\psi \notin \Sigma$ then there is a full quantifier suited pair $\langle a, b \rangle$ such that $\langle \Sigma, \Delta \rangle R^c_\triangleright \langle a, b \rangle$ and $\psi \in a$. Assume that $\triangleright\psi \notin \Sigma$. Then $\left\langle \{\psi\}, \triangleright^{-1}\Sigma \right\rangle$ is a quantifier-suited pair, by lemma 55.2. Notice that $\{\psi\}$ is finite and thus we can apply the pair extension in the same language (lemma 54) which states then that there is a full quantifier-suited pair $\langle a', b' \rangle$ extending $\left\langle \{\psi\}, \triangleright^{-1}\Sigma \right\rangle$ in the same language. Clearly $\langle a', b' \rangle$ is such that $\langle \Sigma, \Delta \rangle R^c_\triangleright \langle a', b' \rangle$ because $\triangleright A \in \Sigma \Rightarrow A \in \triangleright^{-1}\Sigma \Rightarrow A \in b' \Rightarrow A \notin a'$, where the last implication follows from the pair condition. So $\langle a', b' \rangle$ fulfills the definition of $R^c_\triangleright$. Since $\triangleright\psi \notin \Sigma$ then $\psi \notin \triangleright^{-1}\Sigma \subseteq b'$ and then $\psi \notin b'$. By fullness of $\langle a', b' \rangle$ we get $\psi \in a'$. Hence if $\triangleright\psi \notin \Sigma$ then there is a full quantifier suited pair $\langle a, b \rangle$ such that$\langle \Sigma, \Delta \rangle R^c_\triangleright \langle a, b \rangle$ and $\psi \in a$, namely take $\langle a', b' \rangle = \langle a, b \rangle$ as witness.■

Direction $\Leftarrow$ Suppose $\triangleright\varphi \in \Sigma$. Assume further that $\langle \Sigma, \Delta \rangle R^c_\triangleright \langle \Gamma, \Pi \rangle$. Then, by applying *modus ponens* to the definition of $R^c_\triangleright$, we have $\varphi \notin \Gamma$, as desired.■

Therefore: $V^c_{\langle \Sigma, \Delta \rangle}(\triangleright\varphi) = 1$ iff $\triangleright\varphi \in \Sigma$.

$\triangleleft$-*case.*

$V^c_{\langle \Sigma, \Delta \rangle}(\triangleleft\varphi) = 1$ iff there exists an $\langle \Gamma, \Pi \rangle$ such that $\langle \Sigma, \Delta \rangle R^c_\triangleleft \langle \Gamma, \Pi \rangle$ and $V^c_{\langle \Gamma, \Pi \rangle}(\varphi) = 0$. By induction hypothesis $V^c_{\langle \Gamma, \Pi \rangle}(\varphi) = 1$ iff $\varphi \in \Gamma$, thus we have:

$V^c_{\langle \Sigma, \Delta \rangle}(\triangleleft\varphi) = 1$ iff there exists an $\langle \Gamma, \Pi \rangle$ such that $\langle \Sigma, \Delta \rangle R^c_\triangleleft \langle \Gamma, \Pi \rangle$ and $\varphi \notin \Gamma$. Now we show that there exists an $\langle \Gamma, \Pi \rangle$ such that $\langle \Sigma, \Delta \rangle R^c_\triangleleft \langle \Gamma, \Pi \rangle$ and $\varphi \notin \Gamma$ iff $\triangleleft\varphi \in \Sigma$.

Direction $\Rightarrow$ Suppose that there exists an $\langle \Gamma, \Pi \rangle$ such that $\langle \Sigma, \Delta \rangle R^c_\triangleleft \langle \Gamma, \Pi \rangle$ and $\varphi \notin \Gamma$. Then by applying *modus ponens* to the definition of $R^c_\triangleleft$ we immediately have: $\triangleleft\varphi \in \Sigma$.■

Direction $\Leftarrow$ Suppose $\lhd\varphi \in \Sigma$. Then by lemma 55.4. and since $\langle \Sigma, \Delta \rangle$ is full quantifier suited, we know that $\left\langle \lhd^{-1}\Delta, \{\varphi\} \right\rangle$ is a quantifier suited pair. Notice that $\{\varphi\}$ is finite and thus we can apply the the pair extension in the same language (lemma 54) which states then that there is a full quantifier-suited pair $\langle a', b' \rangle$ extending $\left\langle \lhd^{-1}\Delta, \{\varphi\} \right\rangle$ in the same language. Since $\varphi \in \{\varphi\} \subseteq b'$ then $\varphi \notin a'$ by the pair condition and thus $\varphi \notin \lhd^{-1}\Delta \subseteq a'$, which finally leads us to $\lhd\varphi \notin \Delta$. But then by fullness, $\lhd\varphi \in \Sigma$. So, by definition of $R^c_{\lhd}$ we have: $\langle \Sigma, \Delta \rangle \, R^c_{\lhd} \, \langle a', b' \rangle$. Hence there exists an $\langle a, b \rangle$ such that $\langle \Sigma, \Delta \rangle \, R^c_{\lhd} \, \langle a, b \rangle$ and $\varphi \notin a$, namely take the witness $\langle a', b' \rangle = \langle a, b \rangle$.■

Therefore $V^c_{\langle \Sigma, \Delta \rangle} (\lhd\varphi) = 1$ iff $\lhd\varphi \in \Sigma$.

$-$-case

$V^c_{\langle \Sigma, \Delta \rangle} (\varphi - \psi) = 1$ iff there exists an $\langle \Gamma, \Pi \rangle$ such that $\langle \Gamma, \Pi \rangle \subseteq^* \langle \Sigma, \Delta \rangle$ and $\left\{ \begin{smallmatrix} V^c_{\langle \Gamma, \Pi \rangle}(\varphi)=1 \ and \\ V^c_{\langle \Gamma, \Pi \rangle}(\psi)=0 \end{smallmatrix} \right.$ which by induction hypothesis translates into $\left\{ \begin{smallmatrix} \varphi\in\Gamma \ and \\ \psi\notin\Gamma \end{smallmatrix} \right.$. Now we show that there exists an $\langle \Gamma, \Pi \rangle$ such that $\langle \Gamma, \Pi \rangle \subseteq^* \langle \Sigma, \Delta \rangle$ and $\left\{ \begin{smallmatrix} \varphi\in\Gamma \ and \\ \psi\notin\Gamma \end{smallmatrix} \right.$ if and only if $\varphi - \psi \in \Sigma$.

For the right to left direction, assume that $\varphi - \psi \in \Sigma$. By the lemma 55.6 and since $\langle \Sigma, \Delta \rangle$ is full quantifier suited, we know that $\langle \{\varphi\}, \{\psi\} \cup \Delta \rangle$ is a quantifier-suited pair. Notice that $\{\varphi\}$ is finite and thus we can apply the pair extension in the same language (lemma 54) which states then that there is a full quantifier-suited pair $\langle a, b \rangle$ extending $\langle \{\varphi\}, \{\psi\} \cup \Delta \rangle$ in the same language. Since $\Delta \subseteq (\{\psi\} \cup \Delta) \subseteq b$ then $\Delta^c \supseteq (\{\psi\} \cup \Delta)^c \supseteq b^c$, but then, since $\langle a, b \rangle$ and $\langle \Sigma, \Delta \rangle$ are full, we know that $b^c = a$ and $\Delta^c = \Sigma$ which means that $\Sigma \supseteq (\{\psi\} \cup \Delta)^c \supseteq a$. Therefore $\langle a, b \rangle \subseteq^* \langle \Sigma, \Delta \rangle$. Moreover $\{\varphi\} \subseteq a$ and $\{\psi\} \subseteq b$ so $\left\{ \begin{smallmatrix} \varphi\in a \ and \\ \psi\notin a \end{smallmatrix} \right.$, as desired.■

For the left to right direction we use contraposition. Thus assume that $\varphi - \psi \notin \Sigma$. Then by the semantic definition of substraction, we know that for all points $\langle \Gamma, \Pi \rangle$ such that $\langle \Gamma, \Pi \rangle \subseteq^* \langle \Sigma, \Delta \rangle$ it is the case that $V^c_{\langle \Gamma, \Pi \rangle} (\varphi) = 1$ implies $V^c_{\langle \Gamma, \Pi \rangle} (\psi) = 1$, but by induction hypothesis this is converted into $\varphi \in \Gamma \Rightarrow \psi \in \Gamma$. In short: there does not exist any point $\langle \Gamma, \Pi \rangle \subseteq^* \langle \Sigma, \Delta \rangle$ such that $\left\{ \begin{smallmatrix} \varphi\in\Gamma \ and \\ \psi\notin\Gamma \end{smallmatrix} \right.$, which exactly what we needed. ■

Therefore $V^c_{\langle \Sigma, \Delta \rangle} (\varphi - \psi) = 1$ iff $\varphi - \psi \in \Sigma$.

$\supset$-case

$V^c_{\langle \Sigma, \Delta \rangle} (\varphi \supset \psi) = 1$ iff for all $\langle \Gamma, \Pi \rangle$ such that $\langle \Sigma, \Delta \rangle \subseteq^* \langle \Gamma, \Pi \rangle$ we have $V^c_{\langle \Gamma, \Pi \rangle} (\varphi) = 1 \ \Rightarrow \ V^c_{\langle \Gamma, \Pi \rangle} (\psi) = 1$ which by induction hypothesis translates into $\varphi \in \Gamma \Rightarrow \psi \in \Gamma$. Now we show that all $\langle \Gamma, \Pi \rangle$ such that $\langle \Sigma, \Delta \rangle \subseteq^* \langle \Gamma, \Pi \rangle$ are such that $\varphi \in \Gamma \Rightarrow \psi \in \Gamma$ iff $\varphi \supset \psi \in \Sigma$.

The left to right direction will be verified by contraposition. We assume then that $\varphi \supset \psi \notin \Sigma$, which by 55.5 tells us that $\langle \Sigma \cup \{\varphi\}, \{\psi\} \rangle$ is a quantifier-suited pair. Notice that $\{\psi\}$ is finite and thus we can apply the pair extension in the same language (lemma 54) which states then that there is a full quantifier-suited pair $\langle a, b \rangle$ extending $\langle \Sigma \cup \{\varphi\}, \{\psi\} \rangle$ in the same language. Since $(\Sigma \cup \{\varphi\}) \subseteq a$ then $\langle \Sigma, \Delta \rangle \subseteq^* \langle a, b \rangle$ and $\varphi \in a$. But $\psi \notin a$ because $\{\psi\} \subseteq b$ and $\langle a, b \rangle$ is a pair. Thus $\varphi \in a \nRightarrow \psi \in a$ as desired.■

Assume that $\varphi \supset \psi \in \Sigma$, for the right to left direction. Then by the semantic definition of $\supset$ we have $V^c_{\langle \Gamma, \Pi \rangle} (\varphi) = 1 \ \Rightarrow \ V^c_{\langle \Gamma, \Pi \rangle} (\psi) = 1$, for all $\langle \Gamma, \Pi \rangle$ such that $\langle \Sigma, \Delta \rangle \subseteq^* \langle \Gamma, \Pi \rangle$. But this turns into $\varphi \in \Gamma \Rightarrow \psi \in \Gamma$ by induction hypothesis, which provides just what we needed.■

Therefore $V^c_{\langle \Sigma, \Delta \rangle} (\varphi \supset \psi) = 1$ iff $\varphi \supset \psi \in \Sigma$.

$\forall$-case

$V^c_{\langle \Sigma, \Delta \rangle} (\forall v\varphi) = 1$ iff $V^c_{\langle \Sigma, \Delta \rangle} (\varphi [x/v]) = 1$ for each $x \in Var^+$. By induction hypothesis we have $V^c_{\langle \Sigma, \Delta \rangle} (\varphi [x/v]) = 1$ for each $x \in Var^+$ iff $\varphi [x/v] \in \Sigma$ for each $x \in Var^+$. Now we need to show that $\varphi [x/v] \in \Sigma$ for each

$x \in Var^+$ iff $\forall v\varphi \in \Sigma$, but this already follows from lemma 43 and the fullness and quantifier suitedness of $\langle \Sigma, \Delta \rangle$.

Therefore $V^c_{\langle \Sigma,\Delta \rangle}(\forall v\varphi) = 1$ iff $\forall v\varphi \in \Sigma$.

$\exists$-*case*

$V^c_{\langle \Sigma,\Delta \rangle}(\exists v\varphi) = 1$ iff $V^c_{\langle \Sigma,\Delta \rangle}(\varphi[x/v]) = 1$ for some $x \in Var^+$. By induction hypothesis we have $V^c_{\langle \Sigma,\Delta \rangle}(\varphi[x/v]) = 1$ for some $x \in Var^+$ iff $\varphi[x/v] \in \Sigma$ for some $x \in Var^+$. Now we need to show that $\varphi[x/v] \in \Sigma$ for some $x \in Var^+$ iff $\exists v\varphi \in \Sigma$, but this already follows from lemma 43 and the fullness and quantifier suitedness of $\langle \Sigma, \Delta \rangle$.

Therefore $V^c_{\langle \Sigma,\Delta \rangle}(\exists v\varphi) = 1$ iff $\exists v\varphi \in \Sigma$.

The Truth lemma is proven. $\square$

3.3.1. Completeness theorem and proof. The moment arrived for us to present the completeness theorem.

THEOREM 63. *(completeness) Given a pair $\Sigma \nvdash_\Lambda \Delta$ and a CDML logic Λ there is a model $\mathbb{M}$ based on some DML-frame $\mathbb{F}$ such that $\Sigma \nVdash_{\mathbb{M}} \Delta$ (i.e. there exists a point $w \in W$ such that $\mathbb{M}, w \Vdash \Sigma$ and $\mathbb{M}, w \nVdash \Delta$).*

PROOF. Assume $\Sigma \nvdash \Delta$. Then $\langle \Sigma, \Delta \rangle$ is a pair which, by the Pair extension lemma on a new language (lemma 46), can be extended to a full quantifier-suited pair $\langle \Sigma', \Delta' \rangle$. Such pair is a point in the DML-canonical frame as described in definition 44, and the canonical valuation guarantees that $\langle \Sigma', \Delta' \rangle \Vdash_{\mathbb{M}^c} \Sigma$ and $\langle \Sigma', \Delta' \rangle \nVdash_{\mathbb{M}^c} \Delta$ and therefore $\Sigma \nVdash_{\mathbb{M}^c} \Delta$. Hence there is some model M in which $\Sigma \nVdash_M \Delta$, namely the canonical model. $\square$

3.3.2. Remarks on a completeness proof for substructural operators. As expected, the canonical model $\mathbb{M}^c$ for a *SDML*-logic on the language $\mathcal{L}^{+\,\prime}_{SQ}$ is entirely analogous to definition 44 with $\left\langle (W, \subseteq^*), R^c_\Box, R^c_\Diamond, R^c_\triangleright, R^c_\triangleleft, R^c_\circ, R^c_\circ, R^c_\rightarrow, R^c_\leftarrow, Var, V^c \right\rangle$, and the canonical relations $R^c_\circ$,$R^c_\rightarrow$ and $R^c_\leftarrow$ defined as follows:

- $\langle \Sigma, \Delta \rangle R^c_\circ \langle \Gamma, \Pi \rangle \langle \Theta, \Omega \rangle$ iff for each $\varphi, \psi \in \mathcal{L}^{+\,\prime}_{SQ}$ we have $\varphi \in \Gamma \,\&\, \psi \in \Theta \Rightarrow \varphi \circ \psi \in \Sigma$
- $\langle \Sigma, \Delta \rangle R^c_\rightarrow \langle \Gamma, \Pi \rangle \langle \Theta, \Omega \rangle$ iff for each $\varphi, \psi \in \mathcal{L}^{+\,\prime}_{SQ}$ we have $(\varphi \in \Gamma \;\Rightarrow\; \psi \in \Theta) \quad \Rightarrow \quad \varphi \rightarrow \psi \in \Sigma$
- $\langle \Sigma, \Delta \rangle R^c_\leftarrow \langle \Gamma, \Pi \rangle \langle \Theta, \Omega \rangle$ iff for each $\varphi, \psi \in \mathcal{L}^{+\,\prime}_{SQ}$ we have $(\psi \in \Gamma \;\Rightarrow\; \varphi \in \Theta) \quad \Rightarrow \quad \psi \leftarrow \varphi \in \Sigma$

The verification that the canonical frame is an $SDML$-frame (definition 20) would run similarly by supplying the verification of the additional accessibility relations.

(1) $(\geq) \circ R_\circ \subseteq R_\circ \circ (\geq, \geq)$ that is: $\forall t, u, v, z\,[(t \geq u \,\wedge\, R_\circ uvz) \rightarrow \exists w, w'\,(R_\circ tww' \,\wedge\, w \geq v \,\wedge\, w' \geq z)]$. In the canonical frame, such condition takes the following shape (we assume universal closure): $(\langle \Sigma, \Delta \rangle \subseteq^* \langle \Sigma', \Delta' \rangle \,\&\, \langle \Sigma, \Delta \rangle R^c_\circ \langle \Gamma, \Pi \rangle \langle \Theta, \Omega \rangle) \Rightarrow$
$\exists \langle \Gamma', \Pi' \rangle, \langle \Theta', \Omega' \rangle\,(\langle \Sigma', \Delta' \rangle R^c_\circ \langle \Gamma', \Pi' \rangle \langle \Theta', \Omega' \rangle \,\&\, \langle \Gamma, \Pi \rangle \subseteq^* \langle \Gamma', \Pi' \rangle \,\&\, \langle \Theta, \Omega \rangle \subseteq^* \langle \Theta', \Omega' \rangle)$.

PROOF. Let $\langle \Sigma, \Delta \rangle, \langle \Gamma, \Pi \rangle$, $\langle \Sigma', \Delta' \rangle$ and $\langle \Theta, \Omega \rangle$ be full quantified-suited pairs from $\mathcal{L}^{+\,\prime}_{SQ}$ such that $\langle \Sigma, \Delta \rangle R^c_\circ \langle \Gamma, \Pi \rangle \langle \Theta, \Omega \rangle$ and $\langle \Sigma, \Delta \rangle \subseteq^* \langle \Sigma', \Delta' \rangle$. Then for each $\varphi, \psi \in \mathcal{L}^{+\,\prime}_{SQ}$ we have $(\varphi \in \Gamma \,\&\, \psi \in \Theta) \Rightarrow \varphi \circ \psi \in \Sigma$, and since $\Sigma \subseteq \Sigma'$ then $(\varphi \in \Gamma \,\&\, \psi \in \Theta) \Rightarrow \varphi \circ \psi \in \Sigma \Rightarrow \varphi \circ \psi \in \Sigma'$. Therefore $\langle \Sigma', \Delta' \rangle R^c_\circ \langle \Gamma, \Pi \rangle \langle \Theta, \Omega \rangle$. So just take $\langle \Gamma', \Pi' \rangle = \langle \Gamma, \Pi \rangle$ and $\langle \Theta', \Omega' \rangle = \langle \Theta, \Omega \rangle$ as the witnesses. $\square$

(2) $(\geq)\circ R_{\rightarrow} \subseteq R_{\rightarrow}\circ(\leq,\geq)$ that is: $\forall t,u,v,z\,[(t\geq u \;\wedge\; R_{\rightarrow}uvz) \rightarrow \exists w,w'\,(R_{\rightarrow}tww' \;\wedge\; w\leq v \;\wedge\; w'\geq z)]$. In the canonical frame, such condition takes the following shape (we assume universal closure): $(\langle\Sigma,\Delta\rangle \subseteq^{*} \langle\Sigma',\Delta'\rangle \;\&\; \langle\Sigma,\Delta\rangle\, R^{c}_{\rightarrow}\, \langle\Gamma,\Pi\rangle\, \langle\Theta,\Omega\rangle) \Rightarrow$
$\exists\, \langle\Gamma',\Pi'\rangle, \langle\Theta',\Omega'\rangle\, (\langle\Sigma',\Delta'\rangle\, R^{c}_{\rightarrow}\, \langle\Gamma',\Pi'\rangle\, \langle\Theta',\Omega'\rangle \;\&\; \langle\Gamma',\Pi'\rangle \subseteq^{*} \langle\Gamma,\Pi\rangle \;\&\; \langle\Theta,\Omega\rangle \subseteq^{*} \langle\Theta',\Omega'\rangle)$.

PROOF. Let $\langle\Sigma,\Delta\rangle, \langle\Gamma,\Pi\rangle$, $\langle\Sigma',\Delta'\rangle$ and $\langle\Theta,\Omega\rangle$ be full quantified-suited pairs from ${\mathcal{L}^{+}_{SQ}}'$ such that $\langle\Sigma,\Delta\rangle\, R^{c}_{\rightarrow}\, \langle\Gamma,\Pi\rangle\, \langle\Theta,\Omega\rangle$ and $\langle\Sigma,\Delta\rangle \subseteq^{*} \langle\Sigma',\Delta'\rangle$. Then for each $\varphi,\psi \in {\mathcal{L}^{+}_{SQ}}'$ we have $(\varphi\in\Gamma \Rightarrow \psi\in\Theta) \Rightarrow \varphi\rightarrow\psi \in \Sigma$, and since $\Sigma\subseteq\Sigma'$ then $(\varphi\in\Gamma \Rightarrow \psi\in\Theta) \Rightarrow (\varphi\rightarrow\psi\in\Sigma) \Rightarrow (\varphi\rightarrow\psi\in\Sigma')$. Therefore $\langle\Sigma',\Delta'\rangle\, R^{c}_{\rightarrow}\, \langle\Gamma,\Pi\rangle\, \langle\Theta,\Omega\rangle$. So just take $\langle\Gamma',\Pi'\rangle = \langle\Gamma,\Pi\rangle$ and $\langle\Theta',\Omega'\rangle = \langle\Theta,\Omega\rangle$ as the witnesses. $\square$

(3) $(\geq)\circ R_{\leftarrow} \subseteq R_{\leftarrow}\circ(\geq,\leq)$ that is: $\forall t,u,v,z\,[(t\geq u \;\wedge\; R_{\leftarrow}uvz) \rightarrow \exists w,w'\,(R_{\leftarrow}tww' \;\wedge\; w\geq v \;\wedge\; w'\leq z)]$. In the canonical frame, such condition takes the following shape (we assume universal closure): $(\langle\Sigma,\Delta\rangle \subseteq^{*} \langle\Sigma',\Delta'\rangle \;\&\; \langle\Sigma,\Delta\rangle\, R^{c}_{\leftarrow}\, \langle\Gamma,\Pi\rangle\, \langle\Theta,\Omega\rangle) \Rightarrow$
$\exists\, \langle\Gamma',\Pi'\rangle, \langle\Theta',\Omega'\rangle\, (\langle\Sigma',\Delta'\rangle\, R^{c}_{\leftarrow}\, \langle\Gamma',\Pi'\rangle\, \langle\Theta',\Omega'\rangle \;\&\; \langle\Gamma,\Pi\rangle \subseteq^{*} \langle\Gamma',\Pi'\rangle \;\&\; \langle\Theta',\Omega'\rangle \subseteq^{*} \langle\Theta,\Omega\rangle)$.

PROOF. Let $\langle\Sigma,\Delta\rangle, \langle\Gamma,\Pi\rangle$, $\langle\Sigma',\Delta'\rangle$ and $\langle\Theta,\Omega\rangle$ be full quantified-suited pairs from ${\mathcal{L}^{+}_{SQ}}'$ such that $\langle\Sigma,\Delta\rangle\, R^{c}_{\leftarrow}\, \langle\Gamma,\Pi\rangle\, \langle\Theta,\Omega\rangle$ and $\langle\Sigma,\Delta\rangle \subseteq^{*} \langle\Sigma',\Delta'\rangle$. Then for each $\varphi,\psi \in {\mathcal{L}^{+}_{SQ}}'$ we have $(\psi\in\Gamma \Rightarrow \varphi\in\Theta) \Rightarrow \psi\leftarrow\varphi \in \Sigma$, and since $\Sigma\subseteq\Sigma'$ then $(\psi\in\Gamma \Rightarrow \varphi\in\Theta) \Rightarrow (\psi\leftarrow\varphi\in\Sigma) \Rightarrow (\psi\leftarrow\varphi\in\Sigma')$. Therefore $\langle\Sigma',\Delta'\rangle\, R^{c}_{\leftarrow}\, \langle\Gamma,\Pi\rangle\, \langle\Theta,\Omega\rangle$. So just take $\langle\Gamma',\Pi'\rangle = \langle\Gamma,\Pi\rangle$ and $\langle\Theta',\Omega'\rangle = \langle\Theta,\Omega\rangle$ as the witnesses. $\square$

When preparing the truth lemma, the first difficulties are reached. Given that we are still on the distributive setting, and since the unary diamond case in the Truth lemma works flawlessly, one may expect that the corresponding proof for fusion -a binary diamond- requires no more than a straightforward adaptation of the unary diamond case. In the truth-lemma, one direction is given by the definition of the canonical relation $R^{c}_{\circ}$, while the other direction requires to provide two witness points. Namely, we would have:

$V^{c}_{\langle\Sigma,\Delta\rangle}(\varphi\circ\psi)=1$ iff there exists an $\langle\Gamma,\Pi\rangle$ such that $V^{c}_{\langle\Gamma,\Pi\rangle}(\varphi)=1$, and a $\langle\Theta,\Omega\rangle$ such that $V^{c}_{\langle\Theta,\Omega\rangle}(\psi)=1$ with $\langle\Sigma,\Delta\rangle\, R^{c}_{\circ}\, \langle\Gamma,\Pi\rangle\, \langle\Theta,\Omega\rangle$. By induction hypothesis we have $V^{c}_{\langle\Gamma,\Pi\rangle}(\varphi)=1$ iff $\varphi\in\Gamma$ and $V^{c}_{\langle\Theta,\Omega\rangle}(\psi)=1$ iff $\psi\in\Theta$. At this stage we would need to show that there exists an $\langle\Gamma,\Pi\rangle$ such that $\varphi\in\Gamma$ and a $\langle\Theta,\Omega\rangle$ such that $\psi\in\Theta$ with $\langle\Sigma,\Delta\rangle\, R^{c}_{\circ}\, \langle\Gamma,\Pi\rangle\, \langle\Theta,\Omega\rangle$ if and only if $\varphi\circ\psi\in\Sigma$. Direction $\Rightarrow$ is the one given by the definition of $R^{c}_{\circ}$. For direction $\Leftarrow$ we suppose $\varphi\circ\psi\in\Sigma$ and then we would need an analogue of lemma 55 for fusion. Such analogue would state that since $\langle\Sigma,\Delta\rangle$ is a full quantifier suited pair, we know that $\left\langle\{\varphi\}, \bigcup_{\beta\in{\mathcal{L}^{+}_{Q}}'} \{c \mid c\circ\beta\in\Delta\}\right\rangle$ and $\left\langle\{\psi\}, \bigcup_{\alpha\in{\mathcal{L}^{+}_{Q}}'} \{c \mid \alpha\circ c\in\Delta\}\right\rangle$ are quantified-suited pairs, which then would be extended to full quantified suited pairs, to serve us as witnesses. In different words, in the truth lemma, for each formula $\varphi\circ\psi\in\Sigma$ we will need to have witness points with $\varphi\in\Gamma$ and $\psi\in\Theta$ such that $\langle\Sigma,\Delta\rangle\, R^{c}_{\circ}\, \langle\Gamma,\Pi\rangle\, \langle\Theta,\Omega\rangle$, and here, in this last condition resides the difficulty. For $\Sigma R^{c}_{\circ}\Gamma\Theta$ demands by definition (def.44) that for no formula such that $\alpha\circ\beta\notin\Sigma$ (equivalently $\alpha\circ\beta\in\Delta$) we have both $\alpha\in\Gamma$ and $\beta\in\Theta$. Thus we need either $\alpha\in\Pi$ or $\beta\in\Omega$ for each formula $\alpha\circ\beta\notin\Sigma$. This is the reason why we define the initial witness pairs above as generally as we do (notice, however, that they are more restrictive than needed but for the sake of discussion, they suffice).

We do not see clearly how to prove this claim, however:

Given $\langle\Sigma,\Delta\rangle$ a full quantifier suited pair, if $\varphi\circ\psi\in\Sigma$ then $\left\langle\{\varphi\}, \bigcup_{\beta\in{\mathcal{L}^{+}_{Q}}'} \{c \mid c\circ\beta\in\Delta\}\right\rangle$ and $\left\langle\{\psi\}, \bigcup_{\alpha\in{\mathcal{L}^{+}_{Q}}'} \{c \mid \alpha\circ c\in\Delta\}\right\rangle$ are quantified-suited pairs.

Regarding the pairness property, it can be easily shown that $\{\varphi\} \nvdash \{c \mid c\circ\beta\in\Delta\}$ for any formula β. But from this, it does not follow that $\{\varphi\} \nvdash \bigcup_{\beta\in{\mathcal{L}^{+}_{Q}}'} \{c \mid c\circ\beta\in\Delta\}$. The next idea would be to build $\bigcup_{\beta\in{\mathcal{L}^{+}_{Q}}'} \{c \mid c\circ\beta\in\Delta\}$ in stages using the enumeration of all β formulas, adding a single $\{c \mid c\circ\beta\in\Delta\}$ set (which can be empty) per stage and then run an induction proof. This does not seem to work well either. Because each $\{c \mid c \circ \beta \in \Delta\}$ set is added unconditionally, there is not enough information to push the inductive step through.

The cases for implication and co-implication face similar issues and could get even trickier as reported by [**Restall 2005**].

CHAPTER 4

The non-distributive setting: Generalized Kripke frames

We will now expose the discrete duality for non-distributive (propositional) substructural logic from [**Gehrke 2006**] as a natural outcome of the progressive generalization of classical modal logic tools to modal logics based on distributive lattices, non-distributive lattices and posets. The single most important conceptual piece underlying these results is Correspondence Theory, as it is at the root of the interplay between frames (and thus relational semantics) and their complex algebras (and thus algebraic semantics) inside the discrete duality. Discrete duality allows to explore the unfamiliar arena of Kripke frames for non-distributive logics with the safe guidance of a still transparent algebraic behaviour. Thus, the non-distributive generalization takes place on the algebraic side first, by looking at posets and their canonical extensions, which happen to be perfect lattices (4.3.1.). Both posets with monotone expansions and perfect lattices with extended operations are shown to be duals of polarities in [**Dunn, Gehrke & Palmigiano 2005**]. A subclass of polarities, called RS-frames, are shown to be the natural counterpart of Kripke frames in this wider context [**Gehrke 2006**] (4.3.2.). None of the material in this chapter is original.

4.1. Frame definability and correspondence theory

Correspondence theory exploits the observation that Kripke frames can be studied not just as models of modal logic, but also as models of first order logic. This immediately brings the possibility to define classes of Kripke frames by using either modal formulas or first order formulas (on the modal side, frame definability builds on the notion of a certain formula being valid in a given frame). We may then say that a modal formula and a first order sentence *correspond* to each other if they define the same class of Kripke frames. The following series of definitions makes all this more precise. The material of this subsection is taken from [**Blackburn, de Rijke & Venema 2001**].

Definition 64. (*validity*) Let τ be a modal similarity type. Then:

- a τ-formula φ is valid at a point w in a τ-frame $\mathbb{F}$ ($\mathbb{F}, w \Vdash \varphi$) if φ is true at that point for every valuation based on $\mathbb{F}$ ($\mathbb{F}, V, w \Vdash \varphi$ for every V)
- a τ-formula φ is valid *on* a τ-frame $\mathbb{F}$ ($\mathbb{F} \Vdash \varphi$) if φ is true at every point for every valuation based on $\mathbb{F}$ ($\mathbb{F}, V, w \Vdash \varphi$ for every V and every w)
- a τ-formula φ is valid *on* a class K of τ-frames ($\mathsf{K} \Vdash \varphi$) if φ is valid on every $\mathbb{F}$ in K

All the above concepts can be extended to sets of formulas being valid.

Remark 65. The notion of frame validity is inherently a *second-order* property as it quantifies over all possible valuations which are (assignments of) subsets of frames (i.e. "monadic predicates") -a formula φ is valid on a frame if for all sets assigned to it by valuations, such set has same extension than the frame universe.

Definition 66. (*definability*) Let τ be a modal similarity type. Then a τ-formula φ *defines* or *characterizes* a class K of τ-frames if for every frame $\mathbb{F}$, $\mathbb{F}$ is in K iff $\mathbb{F} \Vdash \varphi$. This definition can be extended to sets of formulas as well.

REMARK 67. It is usual to say that a formula φ (or a set Γ of formulas) defines a *property* of the accessibility relation (e.g. reflexivity) if it defines precisely the class of frames with such a relation. Also note that the notion of frame definability inherits its *second-order* nature from frame validity, as it builds upon it [**Blackburn, de Rijke & Venema 2001**].

REMARK 68. Given the availability of first order logic to study Kripke frames and the second-order nature of frame definability, a modal similarity type comes associated with two frame languages besides the modal one.

DEFINITION 69. (*Frame languages*) Given a modal similarity type τ,

- the first-order frame language $\mathcal{L}^1_\tau$ of τ is the first-order language that has identity symbol $=$ and an $(n+1)$-ary relation symbol $R_\triangle$ for each n-ary modal operator $\triangle \in \tau$.
- and given a set Φ of propositional letters, the monadic second-order frame language $\mathcal{L}^2_\tau(\Phi)$ of τ over Φ is the second-order language that results from adding a Φ-indexed collection of monadic-predicate variables to $\mathcal{L}^1_\tau$, thus additionally quantifying over subsets of frames.

$\mathcal{L}^1_\tau$ is often called the *first-order correspondence language* for τ, and $\mathcal{L}^2_\tau$ the *second-order correspondence language* for τ (with the restriction to monadic predicates assumed).

DEFINITION 70. (*Frame correspondence*) If a class of frames (which can informally be seen as a *property*) can be defined by a modal formula φ and by a formula ψ from one of these frame languages $\mathcal{L}^1_\tau$ or $\mathcal{L}^2_\tau$, it is said that ψ and φ are each others (global) *correspondents.*

REMARK 71. All modal formulas can be translated into a formula of $\mathcal{L}^2_\tau$, only some of these will turn out to have an equivalent in $\mathcal{L}^1_\tau$. When modal formulas have a first-order correspondent, it is always a single formula, no correspondence ever arises with a set of first-order formulas.

THEOREM 72. *If* K *is a first-order definable class of frames, then the normal modal logic* Λ_{K} *is canonical (and thus strongly complete w.r.t.* K*)*

Correspondence theory is precisely, within the model theory of modal logic, the systematic study of the correspondence phenomenon. This study provides useful methods and tools to approach modal logic problems. In particular, in the more general settings of non-Boolean and non-distributive logics, the relational semantics grow much faster in difficulty than the algebraic counterparts. In fact, it is now well-known that the algebraic theory and methods involved in these wider settings remain essentially the same than what we have for classical modal logic ([**Conradie & Palmigiano 2012, Dunn, Gehrke & Palmigiano 2005, Gehrke 2006, Gehrke, Nagahashi & Venema-2005**]). Moreover, the algebraic perspective is particularly well suited for a modular approach, which easily accommodates the expansion or reduction of a signature in several directions and the combination of these.

In recent years, there has been an increasing amount of studies on logics for which the associated algebras are not Boolean algebras, but generalizations of these which play precisely with the modular addition or removal of certain operators.

$$\begin{array}{ccccccc}
 & & & \textit{Boolean Algebra} & & & \\
 & & \nearrow & \textit{with Operators} & \searrow & & \\
 & & \textit{add operators} & & & \textit{drop negation} & \\
 & \nearrow & & \Updownarrow_{\textit{Goldblatt duality}} & & \searrow & \\
\textit{Boolean Algebra} & & & & & & \textit{Distributive Lattices} \\
 & & & \textit{Descriptive General Frames} & & & \textit{with Operators} \\
\Updownarrow_{\textit{Stone duality}} & \searrow & & & & \nearrow & \Updownarrow \\
 & & \textit{drop negation} & & & & \textit{Relational Priestley} \\
\textit{Boolean Space} & & \searrow & & & \textit{add operators} & \textit{Spaces} \\
 & & & & \nearrow & & \\
 & & & \textit{Distributive Lattices} & & & \\
 & & & \Updownarrow_{\textit{Priestley duality}} & & & \\
 & & & \textit{Priestley Spaces} & & &
\end{array}$$

We have already exploited some of these studies in the previous chapter, particularly the duality for distributive lattices expanded with modal operations as presented in [**Conradie & Palmigiano 2012**]. But a duality theory has also started to be developed for bounded lattices which are no longer distributive (for instance,[**Hartung 1992, Urquhart 1978**] and for the modal view on it [**Haim 2000**]), and finally for partially ordered sets along with a development of canonical extension theory for order-preserving and order reversing expansions [**Dunn, Gehrke & Palmigiano 2005, Gehrke 2006**]. We first provide (4.2) a brief overview of the discrete duality that arise on increasingly generalized modal settings, to finally focus on the canonical extension and discrete duality for the non-distributive setting (4.3.), as preparation for the completeness result exposition in Chapter 5.

4.2. Discrete duality from the classical case to the non-distributive case.

A brief overview on how the discrete duality emerges in the classical and distributive settings will show that the non-distributive discrete duality comes as a natural extension of a well-known strategy.

4.2.1. Stone duality. The Stone representation theorem states that every Boolean algebra $\mathbb{A}$ can be embedded in the complete and atomic Boolean algebra $\mathbb{A}^\sigma$ defined as the power set algebra of the collection of its ultrafilters[1], where the latter generalize the notion of atoms. Thus even when a Boolean algebra is not atomic nor complete, it can be embedded into a complete and atomic Boolean algebra, as every power-set algebra belongs to this category. In fact, this is often reformulated as an isomorphism (every Boolean algebra is a power-set algebra, up to isomorphism). Thus, a very obvious advantage of Representation theorems -of which this one is an example- is that they allow to reduce a class of structures to a proper subclass with a simpler behaviour. The usual shape of this type of theorems is that every element of the class of structures S is isomorphic to some element of a proper subclass $S' \subset S$ of structures. It is often the case that this subclass S' has some nice extra properties that makes it more suitable to work with than the original class S, while the isomorphism ensures that any isomorphim-invariant result proved in S' will carry over to S. But this is just one aspect of the insight gained, a fully detailed formulation of a representation theorem provides a precise definition of the embedding used to prove it.

[1] hence a *point* in this algebra is a *family* of ultrafilters

Theorem 73. *Stone representation theorem. Let $\mathbb{A} = \langle A, \wedge, \vee, \neg, 0, 1\rangle$ be a Boolean algebra and let $Uf(\mathbb{A})$ be the set of ultrafilters of $\mathbb{A}$. Then the function $v : A \longrightarrow \wp(Uf(\mathbb{A}))$ defined by $v(a) = \hat{a} = \{U \in Uf(\mathbb{A}) \mid a \in U\}$ is an injective morphism of lattices. Consequently, every Boolean algebra $\mathbb{A}$ can be embedded in the complete and atomic Boolean algebra $\mathbb{A}^\sigma$ defined as the powerset algebra of the set of ultrafilters of $\mathbb{A}$*

Such embedding provides a concrete and intuitive interpretation of otherwise abstractly defined operations. More precisely, this result links the rather abstract meaning of Boolean operations in $\mathbb{A}$ to a concrete (i.e. set theoretic) view of such meaning. Thus $a \wedge b$ can be seen as $\widehat{a \wedge b} = \hat{a} \cap \hat{b}$, for instance, with abstract meet now being interpreted as set intersection.

The main interest of representation theorems and canonical extensions in the context of this thesis, and more generally from logical viewpoint, is that they are an important source of completeness results. For instance, via the reformulation in terms of isomorphism, Stone representation theorem presents the completeness of classical logic as an essentially algebraic result. We can directly see $\mathbb{A}$ as a subalgebra of $\mathbb{A}^\sigma$ by recovering the fact that, given the embedding, $\mathbb{A}$ is isomorphic to its image under $v(\cdot)$. Since validity of equations is preserved under taking subalgebras, we can easily see this representation theorem as a powerful tool to obtain completeness results via discrete duality (validity of equations is preserved on taking subalgebras, thus logical counterexamples in the Lindenbaum algebra are preserved in the opposite direction: they are still invalid in the canonical extension of the Lindenbaum algebra).

The existence of the following two dualities of categories is crucial for the Stone representation theorem.

$$
\begin{tikzcd}
\mathsf{CABA} \arrow[r, "(\cdot)_+"', shift right] & \mathsf{Sets} \arrow[l, dashed, "(\cdot)^+"', shift right] & \textit{Discrete duality} \\
\mathsf{BA} \arrow[u, dotted, "(\cdot)^\sigma"] \arrow[ur, "(\cdot)_\bullet = U \circ (\cdot)_*"] \arrow[r, dashed, "(\cdot)_*"', shift right] & \mathsf{Stone} \arrow[l, "(\cdot)^*"', shift right] \arrow[u, dashed, "U"'] & \textit{Topological duality}
\end{tikzcd}
$$

Given a Boolean algebra $\mathbb{A}$, its associated stone space $(\mathbb{A})_*$ is formed by taking the subset of its powerset that consists only of ultrafilters and setting as the basis of the space the collection $\bigcup_{a \in A} \{U \in Uf(\mathbb{A}) \mid a \in U\}$ of families of ultrafilters selected by a common element. In general, the collection of clopens of any topological space forms a Boolean algebra, but the particular Boolean algebra that arises from a given Stone space has a special property with respect to the original BA -and this is what the Stone representation theorem states-: every Boolean algebra $\mathbb{A}$ is isomorphic to the Boolean algebra $((\mathbb{A})_*)^*$ of clopen subsets of its associated Stone space $S_{\mathbb{A}} = (\mathbb{A})_*$.

4.2.2. The modal family. We can now consider a diagram where we have Boolean algebras expanded with modal operators (BAOs) instead of raw Boolean algebras.

$$\begin{tikzcd}
\mathsf{CABACO} \arrow[r, shift right, "(\cdot)_+"'] & \mathsf{KrFr} \arrow[l, dashed, shift right, "(\cdot)^+"'] \\
\mathsf{BAO} \arrow[u, dotted, "(\cdot)^\sigma"] \arrow[ur, "(\cdot)_\bullet = U\circ(\cdot)_*"] \arrow[r, dashed, shift right, "(\cdot)_*"'] & \mathsf{DGF} \arrow[u, dashed, "U"'] \arrow[l, shift right, "(\cdot)^*"']
\end{tikzcd}$$

Discrete duality

Topological duality

A Boolean algebra with an operator or BAO is a pair $\langle \mathbb{A}, \Diamond \rangle$ where $\mathbb{A}$ is a Boolean algebra and $\Diamond : A \longrightarrow A$ is a function that preserves binary joins (and thus all finite ones). A unary operator is *complete* if it preserves arbitrary joins, and $CABACO$ is the category of complete and atomic Boolean algebras with a complete operator.[2] $KrFr$ is the category of Kripke frames, which can be seen as sets expanded with a relation on them. DGF is the category of descriptive general frames which can be seen as Stone spaces endowed with a point-closed relation (R is point-closed if for every point $w \in W$, the set of successors $R[w] = \{t \in W \mid Rwt\}$ is a closed set in the topology).

Just as the discrete and topological dualities lead to a representation theorem in the Boolean case, they also support a representation theorem in the expanded version, namely the Jonsson-Tarski representation theorem.

THEOREM 74. *(Jonsson-Tarski representation theorem). Any BAO $\mathbb{A}$ can be embedded in its canonical extension* $\mathbb{A}^\sigma$

This representation theorem tells us that any BAO, and in particular the Lindenbaum-Tarski algebra of classical modal logic, can be represented as (read “embedded into”) a concrete BAO, i.e. an algebra that comes from a frame, or more precisely, as an algebra that is linked to frames through a discrete duality and has some well-behaved properties. We may remark *en passant* that the complex algebra of the canonical frame is (up to isomorphism) identical to the canonical extension of the Lindenbaum-Tarski algebra.

[2] We call an operation $f : C \longrightarrow C'$ an *operator* if it is normal and additive in all coordinates, where normality holds if $f(\bot) = \bot'$ and additivity holds if $f(a \vee b) = f(a) \vee' f(b)$

$$\begin{tikzcd}
(Lindenbaum\,Tarski\,Alg.)^{\sigma} \cong (Canonical\,Frame)^{+} \arrow[rr, shift right, "(\cdot)_{+}"'] & & Canonical\,Frame \arrow[ll, dashed, shift right, "(\cdot)^{+}"'] \\
 & (\cdot)_{\bullet} & \\
Lindenbaum\,Tarski\,Algebra \arrow[u, dotted, "(\cdot)^{\sigma}"] \arrow[uurr] \arrow[rr, dashed, shift right, "(\cdot)_{*}"'] & & \mathsf{DGF} \arrow[ll, shift right, "(\cdot)^{*}"'] \arrow[uu, dashed, "U"']
\end{tikzcd}$$

When generalizing to distributive modal algebras, the scheme is as follows (PBLO: Perfect Bounded Distributive Lattices with Operators, OrKrFr: Ordered Kripke Frames):

$$\begin{tikzcd}
\mathsf{PBLO} \arrow[r, shift right, "(\cdot)_{+}"'] & \mathsf{OrKrFr} \arrow[l, dashed, shift right, "(\cdot)^{+}"'] & & Discrete\,duality \\
\mathsf{BLO} \arrow[u, dotted, "(\cdot)^{\sigma}"] \arrow[ur, "(\cdot)_{\bullet}=U\circ(\cdot)_{*}"'] \arrow[r, dashed, shift right, "(\cdot)_{*}"'] & \mathsf{PriestleyDGF} \arrow[l, shift right, "(\cdot)^{*}"'] \arrow[u, dashed, "U"'] & & Topological\,duality
\end{tikzcd}$$

We have detailed the discrete duality of this scheme on Chapter 1. The next step in the generalization involves considering posets (with order preserving or reversing maps) on the algebraic side and RS-polarities (with modal relations). The canonical extension of posets is abstractly defined and then proved -by purely algebraic means- to exist and to be unique in [**Dunn, Gehrke & Palmigiano 2005**]. Such canonical extensions are characterized there as perfect lattices (the corresponding category being PLat in the diagram). Then a discrete duality between perfect lattices and RS-polarities is developed in [**Gehrke 2006**]. The topological duality has not yet been established and therefore the corresponding diagram generalizing the previous ones is incomplete.

$$\begin{tikzcd}
\mathsf{PLat} \arrow[r, shift right, "(\cdot)_{+}"'] & \mathsf{RSPol} \arrow[l, dashed, shift right, "(\cdot)^{+}"'] & & Discrete\,duality \\
\mathsf{Poset} \arrow[u, dotted, "(\cdot)^{\sigma}"] \arrow[ur, "(\cdot)_{\bullet}=?"'] & & &
\end{tikzcd}$$

Given the modular nature of operations, it suffices to add the relevant operations and their dualizations to obtain the modal scheme, with PLatE the category of perfect lattices with extra operations, GKFr the category of Generalized Kripke frames (RS-frames plus modal operations) and MPE the category of monotone poset expansions (i.e., posets with operations that either preserve or reverse the order):

$$\begin{tikzcd}
\mathsf{PLatE} \arrow[rr, shift right, "(\cdot)_+"'] & & \mathsf{GKFr} \arrow[ll, dashed, shift right, "(\cdot)^+"'] \\
\mathsf{MPE} \arrow[u, dotted, "(\cdot)^\sigma"] \arrow[urr, "(\cdot)_\bullet=?"'] & &
\end{tikzcd}$$

Discrete duality

What really concerns us here is the discrete dualities, so we summarize the situation as follows.

-The dual of perfect BAOs are Kripke frames with the set of atoms as universe, which dualize back as the powerset algebra of this set of atoms.

- Given a perfect BAO $\mathbb{A} = (A, \wedge, \vee, -, \Diamond, 0, 1)$, its relational dual is $\mathbb{A}_+ = (At(\mathbb{A}), R_\Diamond)$ with $At(\mathbb{A})$ the set of atoms of $\mathbb{A}$ and $R_\Diamond ab \Leftrightarrow a \leq \Diamond b$.
- On the opposite direction, given a frame $\mathbb{F} = (W, R_\Diamond)$, its complex algebra is $\mathbb{F}^+ = \left(\wp(W), \cap, \cup, ^c, \varnothing, W, m_{R_\Diamond}\right)$ with $m_{R_\Diamond}(S) = \{w \in W | \exists s\, (R_\Diamond ws)\}$.

-The dual of a perfect DMA is an Ordered Kripke frame with the set of completely join prime irreducibles as universe, which dualize back as the powerset algebra of upsets.

- Given a perfect DMA $\mathbb{D} = (D, \wedge, \vee, \Diamond, \Box, \triangleright, \triangleleft, 0, 1)$, its relational dual is $\mathbb{D}_+ = ((J^\infty_P(\mathbb{D}), \geq), R_\Diamond, R_\Box, R_\triangleright, R_\triangleleft)$ with $J^\infty_P(\mathbb{D})$ the set of completely join-prime irreducibles of $\mathbb{D}$ with dual order and the modal relations defined as in definition 18.
- On the opposite direction, given a DML-frame $\mathbb{F} = ((W, \leq), R_\Diamond, R_\Box, R_\triangleright, R_\triangleleft)$, its complex algebra is $\mathbb{F}^+ = \left(\wp^\uparrow(W), \cap, \cup, \varnothing, W, \langle R_\Diamond\rangle, [R_\Box], [R_\triangleright\rangle, \langle R_\triangleleft]\right)$ with the modal operations defined as in definition 9

-The dual of perfect lattices are RS-polarities with the set of completely join irreducibles and the set of completely meet irreducibles as a two-sorted universe, which dualize back as the lattice of Galois-stable sets.

- Given a perfect lattice $\mathbb{L} = (L, \wedge, \vee, \Diamond, \Box, \triangleright, \triangleleft, \circ, \rightarrow, \leftarrow, 0, 1)$, its relational dual is $\mathbb{L}_+ = ((J^\infty(\mathbb{L}), M^\infty(\mathbb{L}), \leq), R_\Diamond, R_\Box, R_\triangleright, R_\triangleleft, R_\circ, R_\rightarrow, R_\leftarrow)$ with $J^\infty(\mathbb{L})$ the set of completely join irreducibles and $M^\infty(\mathbb{L})$ the set of completely meet irreducibles of $\mathbb{L}$, and the relations defined as in section 5.1.4.1.
- On the opposite direction, given a generalized Kripke-frame $\mathbb{F} = ((X, Y, \leq), R_\Diamond, R_\Box, R_\triangleright, R_\triangleleft, R_\circ, R_\rightarrow, R_\leftarrow)$, its complex algebra is $\mathbb{F}^+ = \langle \mathcal{G}(\mathbb{F}), \wedge, \vee, \varnothing, X, \langle R_\Diamond\rangle, [R_\Box], [R_\triangleright\rangle, \langle R_\triangleleft], \langle R_\circ\rangle, \langle R_\leftarrow\rangle, \langle R_\rightarrow\rangle\rangle$ with the modal operations defined as in definition 23

4.3. The non-distributive diagram in further detail

We will focus on the nuclear component of the Generalized Kripke frames,

4.3.1. Perfect lattices from posets: The canonical extension of posets.

4.3.1.1. *Abstract characterization.* In this section we will briefly display the content of [**Dunn, Gehrke & Palmigiano 2005**]. Algebraic methods are used to prove the existence and uniqueness of the canonical extension of partially ordered sets. Those algebras which are canonical extensions of partially ordered sets are furthermore characterized as *perfect lattices.*

Although distributive lattices still have a topological duality, at this level of generality the work has just started to search what the topological duality should look like. In fact not even the ultrafilter frame functor is described, cf. [**Gool 2009**]. Hence, while in the Boolean and distributive settings the existence can be obtained via the topological duality, the existence of canonical extensions of posets must be proved by other means, i.e. relying purely on algebraic methods for the construction. Such construction, when dealing with lattices, takes as key ingredients the sets of (proper) filters and ideals of the lattice. The same procedure can be extended to the more general case of posets but this requires a generalization of the lattice-oriented concepts of filter and ideal. There is more than one suitable choice regarding the more general definition of filters and ideals. In fact, as filters we can take any set of upsets $\mathcal{F}$ as long as it contains the principal upsets and likewise for the ideals, any set of downsets $\mathcal{I}$ which contains the principal downsets will do. Given this choice, the canonical extension must now be parametrized according to what definition of filters/ideals is taken as base for the construction. Hence, a canonical extension constructed out of elements in $\mathcal{F}$ and $\mathcal{I}$ is labeled the $(\mathcal{F},\mathcal{I})$-completion. For ease of notation, we will simply assume that a definition of filters and ideals has been fixed. In [**Dunn, Gehrke & Palmigiano 2005**] the definition based upon the concept of up/down-directed sets is used and it is the one we assume:

Definition 75. (down-directed subset) A subset $A \subseteq P$ of a poset $(P, \leq)$ is a down-directed subset if $A \neq \varnothing$ and for every $a, a' \in A$ there is a $c \in A$ such that $c \leq a$ and $c \leq a'$. An up-directed subset is defined order-dually.

Remark 76. A down-directed subset is not required to be upward-closed, but if it is, then we have a generalization of the lattice-notion of filter (the notion of being a upward-closed down-directed subset of a poset is weaker than the notion of being filter of the poset, since there is no requirement on c to be the greatest lower bound). Likewise, a downward-closed up-directed subset of a poset is a generalization of the notion of ideal.

Definition 77. Let $\mathbb{P}$ be a poset. A non-empty subset $F \subseteq P$ is called a *filter* if it is a down-directed upset:

- if $a, b \in F$ then there exists a c such that $c \leq a$, $c \leq b$ and $c \in F$.
- if $b \in P$ and $a \in F$ with $a \leq b$ then $b \in F$

The notion of an ideal is defined order-dually.

Remark 78. The direct poset analogue of the lattice-notion of filter, however, would be to just deal with the possibility of a non-existent meet for a given pair of elements. A non-empty subset $F \subseteq P$ is then a filter if it is an up-set closed under existing binary meets, that is, we demand that if $a, b \in P$ and $\{a, b\}$ has a greatest lower bound c in $\mathbb{P}$ then $c \in F$. [**Gool 2009**] shows that the collection of sets defined as filters along the above definitions fall all within his concept of *filter systems.* We now present the concepts of *extension* and of *canonical extension* for posets as given in [**Dunn, Gehrke & Palmigiano 2005**].

Definition 79. Let $\mathbb{P}$ be a poset. An *extension* of $\mathbb{P}$ is a pair $(e, \mathbb{Q})$ where $\mathbb{Q}$ is a poset and $e : \mathbb{P} \longrightarrow \mathbb{Q}$ a map such that for every $x, y \in P$, $x \leq_{\mathbb{P}} y$ iff $e(x) \leq_{\mathbb{Q}} e(y)$. In short, an extension of $\mathbb{P}$ is an order-embedding. A *completion* of a poset $\mathbb{P}$ is an extension $(e, \mathbb{Q})$ where $\mathbb{Q}$ is a complete lattice.

We call $K(\mathbb{Q})$ the set of *closed* elements of $\mathbb{Q}$, where an element $q \in Q$ is closed if $q = \bigwedge e[F]$ for some filter F of $\mathbb{P}$. Likewise, we call $O(\mathbb{Q})$ the set of open elements of $\mathbb{Q}$, where an element $q \in Q$ is open if $q = \bigvee e[I]$ for some ideal I of $\mathbb{P}$.

- An extension $(e, \mathbb{Q})$ of $\mathbb{P}$ is said to be *dense* when for every $q \in Q$ we have $q = \bigvee \{k \in K(\mathbb{Q}) \,|k \leq q\} = \bigwedge \{o \in O(\mathbb{Q}) \,|q \leq o\}$
- An extension $(e, \mathbb{Q})$ of $\mathbb{P}$ is said to be *compact* if whenever a non-empty down-directed set $D \subseteq P$ and a non-empty up-directed set $U \subseteq P$ are such that whenever $\bigwedge e[D] \leq \bigvee e[U]$ we also have witnesses $d \in D$ and $u \in U$ with $d \leq u$.

Definition 80. Let $\mathbb{P}$ be a poset. A *canonical extension* of $\mathbb{P}$ is a *dense* and *compact* completion of $\mathbb{P}$.

The above definition constitutes an abstract (purely algebraic) characterization of a canonical extension of a poset. It is proven in [**Dunn, Gehrke & Palmigiano 2005**] that when a poset $\mathbb{P}$ has a canonical extension, it is unique up to an isomorphism that fixes $\mathbb{P}$. Moreover, this definition fits well the *concrete* completion of $\mathbb{P}$ given by the polarity built upon the set $\mathcal{F}$ of filters of $\mathbb{P}$, the set $\mathcal{I}$ of ideals of $\mathbb{P}$ and the relation of non-disjointness between them, which ultimately proves existence (for details we refer the reader to [**Dunn, Gehrke & Palmigiano 2005**]).

It is shown in [**Dunn, Gehrke & Palmigiano 2005**] and [**Gehrke 2006**] that the RS-frames (a special kind of polarity) can be seen as perfect posets[3] and viceversa: they are two sides of the same coin and the alternative presentations can be interchanged at convenience for most purposes. While the first authors focus on a discrete duality between perfect lattices and perfect posets, the latter author shifts focus to RS-frames and their discrete duality with perfect lattices. The following diagram schematizes two algebraic methods to obtain a concrete canonical extension of a poset.

$$
\begin{tikzcd}
Perfect\,Lattices & & Perfect\,Posets \arrow[ll, "\overline{\mathbb{F}\oplus\mathbb{I}}"'] \arrow[dd, Leftrightarrow, "1{-}1_{bijection}"] \\
& & \\
Posets \arrow[uu, dotted, "(\cdot)^{\sigma}"] \arrow[uurr, "\mathbb{F}\oplus\mathbb{I}=(\mathcal{F}\uplus\mathcal{I}/\equiv,\leq)" near start] \arrow[rr, dashed, "(\mathcal{F},\mathcal{I},\perp){-}polarity"'] & & RS-Frames \arrow[uull, dashed, "Galois{-}stable\ sets"' near start]
\end{tikzcd}
$$

The first method is to take advantage of the presentation of RS-frames as perfect posets and to define a quotient on the quasi-order given by the disjoint union of filters and ideals of the starting poset $\mathbb{P}$ and then to take the MacNeille-completion of such quotient (upper path of the diagram). This purely algebraic path is briefly described below (4.3.1.2.-4.3.1.3.) but is detailed in both [**Dunn, Gehrke & Palmigiano 2005**] and [**Fulford 2009**]. The second method, detailed in [**Gool 2009**], takes the lower path and exploits the

[3]defined as: $\mathbb{Z}$ is a perfect poset iff $J^{\infty}(\mathbb{Z})$ / $M^{\infty}(\mathbb{Z})$ is join-dense / meet-dense in $\mathbb{Z}$ and $\mathbb{Z} = J^{\infty}(\mathbb{Z}) \cup M^{\infty}(\mathbb{Z})$

discrete duality between perfect lattices and RS-frames: we build a polarity out of the filters and ideals of the starting poset $\mathbb{P}$ and then take the Galois-stable lattice (details below in section 4.3.2.1-4.3.2.2.). Notice here that this works only because we restrict ourselves to RS-frames. The Galois-stable lattice of polarities in general do not give us a perfect lattice, but just a complete one.

REMARK 81. The polarities built upon the family of optimal filters and optimal ideals of a poset are in fact RS-frames.

4.3.1.2. *Concrete algebraic construction of the canonical extension.* Define a quasi-order (also known as preorder)[4] on $\mathcal{F} \uplus \mathcal{I}$ as follows: for all $F \in \mathcal{F}$ and all $I \in \mathcal{I}$,

- $F \sqsubseteq I$ iff $F \cap I \neq \varnothing$.
- $I \sqsubseteq F$ iff for all $x \in F$ and all $y \in I$: $y \leq x$.
- $F \sqsubseteq F'$ iff $F \supseteq F'$
- $I \sqsubseteq I'$ iff $I \subseteq I'$

For every $X, Y \in \mathcal{F} \uplus \mathcal{I}$ we set: $X \equiv Y$ iff $X \sqsubseteq Y$ and $X \sqsupseteq Y$. Since $\sqsubseteq$ is a preorder, $\equiv$ is clearly an equivalence relation (transitive, reflexive, and now symmetric). Now $\leq_{\equiv}$ defined as $\equiv /_{\sqsubseteq}$ is a partial order such that for every $X, Y \in \mathcal{F} \uplus \mathcal{I}$, $[X] \leq_{\equiv} [Y]$ iff $X \sqsubseteq Y$.

The principal upsets and downsets are key ingredients to guarantee that $\mathbb{P}$ embeds into $\mathbb{F} \oplus \mathbb{I}$.

LEMMA 82. *For all $p \in P$ and for all $X, Y \in \mathcal{F} \uplus \mathcal{I}$:*

(1) $[\uparrow p] = [\downarrow p]$
(2) No other sets are identified by $\equiv$ unless they are equal
(3) $[\uparrow (\_)] : \mathbb{P} \longrightarrow \mathbb{F} \oplus \mathbb{I}$ *is an order embedding.*

We can define the amalgamation $\mathbb{F} \oplus \mathbb{I}$ as the poset $(\mathcal{F} \uplus \mathcal{I} / \equiv, \leq)$. This amalgamation has the necessary properties for $([\uparrow (\_)], \mathbb{F} \oplus \mathbb{I})$ to be an $(\mathcal{F}, \mathcal{I})$-extension of $\mathbb{P}$ and has denseness and compactness, the reader can check the details in [**Fulford 2009**] but let us remark that the opens of $\mathbb{F} \oplus \mathbb{I}$ are precisely the equivalence classes of form $[I]$ for $I \in \mathcal{I}$ and the closeds are precisely the equivalence classes of shape $[F]$ for $F \in \mathcal{F}$.

Finally, by taking the Dedekind-MacNeille completion $\overline{\mathbb{F} \oplus \mathbb{I}}$ of $\mathbb{F} \oplus \mathbb{I}$, we obtain a complete extension of $\mathbb{P}$ which still is dense and compact (Dedekind-MacNeille completion preserves all existing meets and joins). This yields:

THEOREM 83. *For any poset $\mathbb{P}$, $\overline{\mathbb{F} \oplus \mathbb{I}}$ is a canonical extension of $\mathbb{P}$.*

4.3.1.3. *Extending the maps to the canonical extension.* There are canonical ways to extend an arbitrary monotone map f between two posets $\mathbb{A}$ and $\mathbb{B}$ to a map $\mathbb{A}^\sigma \longrightarrow \mathbb{B}^\sigma$ (with a map in $\mathbb{A}$ extended to $\mathbb{A}^\sigma \longrightarrow \mathbb{A}^\sigma$ as an important but particular case in which $\mathbb{B} = \mathbb{A}$). Such extension is given in two steps, we first take care of the closed and open elements of $\mathbb{A}^\sigma$. Then, we rely on the useful fact that -by denseness of the canonical extension- every element of $\mathbb{A}^\sigma$ or $\mathbb{B}^\sigma$ can be seen as the join of the closed elements below it, or the meet of the open elements above it. By definition, for every closed element k in $\mathbb{A}^\sigma$ we have: $k = \bigwedge \{a \in \mathbb{A} \mid a \geq k\}$ so if we let $f^\sigma(k) = \bigwedge \{f(a) \mid a \in \mathbb{A} : a \geq k\}$ with $f^\sigma : K(\mathbb{A}^\sigma) \cup O(\mathbb{A}^\sigma) \longrightarrow \mathbb{B}^\sigma$ we still have f^σ monotone. Observe that this definition relies on $\mathbb{B}^\sigma$ being complete, which guarantees the existence of $\bigwedge \{f(a) \mid a \in \mathbb{A} : a \geq k\}$. Likewise we have, for every open element o in $\mathbb{A}^\sigma$ we have: $o = \bigvee \{a \in \mathbb{A} \mid a \leq o\}$

[4]Recall that a quasi-order is a *reflexive* and *transitive* binary relation. When antisymmetry is added, it turns into a partial order.

so if we let $f^{\pi}(o) = \bigvee \{f(a) \mid a \in \mathbb{A} : a \leq o\}$ with $f^{\pi} : K(\mathbb{A}^{\sigma}) \cup O(\mathbb{A}^{\sigma}) \longrightarrow \mathbb{B}^{\sigma}$ we still have f^{π} monotone.. As before, this definition relies on $\mathbb{B}^{\sigma}$ being complete, which guarantees the existence of $\bigvee \{f(a) \mid a \in \mathbb{A} : a \leq o\}$.

So far we know what happens (in the extended map) with the opens and closeds of $\mathbb{A}^{\sigma}$. To determine what happens with an arbitrary (not necessarily open nor closed) element of $\mathbb{A}^{\sigma}$ we simply use denseness, which states that for all $u \in \mathbb{A}^{\sigma}$: $\bigvee \{k \in K(\mathbb{A}^{\sigma}) \mid u \geq k\} = u = \bigwedge \{o \in O(\mathbb{A}^{\sigma}) \mid o \geq u\}$.

Thus we define f^{σ}, $f^{\pi} : \mathbb{A}^{\sigma} \longrightarrow \mathbb{B}^{\sigma}$ as follows, after ([**Dunn, Gehrke & Palmigiano 2005**]):

$f^{\sigma}(u) = \bigvee \{\bigwedge \{f(a) \mid a \in \mathbb{A} : a \geq k\} \mid u \geq k \in K(\mathbb{A}^{\sigma})\}$

$f^{\pi}(o) = \bigwedge \{\bigvee \{f(a) \mid a \in \mathbb{A} : a \leq o\} \mid u \leq o \in O(\mathbb{A}^{\sigma})\}$

Once we have presented the canonical extension of a poset -which is a perfect lattice-, we can look at the objects on the other side of the discrete duality: the polarities.

4.3.2. Polarities. Now, the dual of a perfect lattice should be an object consisting of two sets and a relation between them, usually these objects are known as polarities.

Definition 84. A polarity $\mathbb{W}$ is a triple (X, Y, R) with X, Y non-empty sets and $R \subseteq X \times Y$ a binary relation.

Since a perfect lattice $\mathbb{P}^{\sigma}$ is by definition generated by $J^{\infty}(\mathbb{P}^{\sigma})$ and $M^{\infty}(\mathbb{P}^{\sigma})$, we have a straightforward way to represent $\mathbb{P}^{\sigma}$ as a polarity.

Definition 85. Let $\mathbb{L}$ be a perfect lattice. Then the polarity $\mathbb{L}_{+}$ associated with $\mathbb{L}$ is the triple $(J^{\infty}(\mathbb{L}), M^{\infty}(\mathbb{L}), \leq_{\mathbb{L}} \cap (J^{\infty}(\mathbb{L}) \times M^{\infty}(\mathbb{L})))$.

Now, since we are aiming at a duality, we should now try to see how to obtain a perfect lattice (isomorphic to) $\mathbb{L}$ from its corresponding polarity $\mathbb{L}_{+}$. This will be achieved through a Galois-connection.

4.3.2.1. *Complete lattices via Galois-connection.* First recall that given $(X, \leq)$ and $(Y, \leq)$ partial orders and $f : Y \rightarrow X$ and $g : X \rightarrow Y$ monotone maps. These form a Galois connection $f \rightleftarrows g$ between posets X and Y if the following holds:

$$f(x) \leq y \text{ iff } x \leq g(y).$$

Graphically:

$$\begin{tikzcd}
X & Y \\
x \arrow[r, "g(-)"'] & g(x) \\
f(y) \arrow[u, dashed, "\leq"'] & y \arrow[l, "f(-)"'] \arrow[u, dashed, "\leq"']
\end{tikzcd}$$

Then f is the lower adjoint of g and g the upper adjoint of f. A Galois connection induces one closure operator $f \circ g$ over X and one interior operator $g \circ f$ over Y. The original notion in Galois theory is slightly different, formulated in terms of antitone maps:

$$y \leq f(x) \text{ iff } x \leq g(y).$$

This erases the lower/upper distinction and the pair of functions is now entirely symmetric. Thus both compositions are closure operators.

It is a well-known property of complete lattices that they can be seen as polarities, closure systems and topped $\bigcap$-structures[**Davey & Priestley 2002**]:

- Every topped $\bigcap$-structure is a complete lattice and, up to isomorphism, every complete lattice is a topped $\bigcap$-structure.[5]
- The set of concepts (Galois-stable sets) of a polarity forms a complete lattice and, up to isomorphism, every complete lattice is the set of concepts of some polarity.
- Every (antitone) Galois connection induces two closure maps and thus also induces a pair of isomorphic complete lattices.
- There is a bijection between closure operators over a set X and the topped $\bigcap$-structures over X. In particular, a closure operator on X can be used to define a topped $\bigcap$-structure on X and viceversa. For this reason, a topped $\bigcap$-structure is also named as a "closure system".

These facts pave the way to a complex algebra for polarities, and the failure to guarantee an "atomic" complete lattice motivates the restriction to RS-frames. More precisely, the desire for a lattice $\mathbb{L}$ that contains the original universe and the empty set as members (i.e. as possbile interpretants), motivates the restriction to separating frames (S-frames), while the desire for a lattice $\mathbb{L}$ that is "atomic" in some sense (i.e. whose completely join-irreducibles are join-dense in $\mathbb{L}$ and whose completely meet-irreducibles are meet-dense in $\mathbb{L}$) motivates the restriction to reduced separating frames (RS-frames).

4.3.2.2. *The complex algebra of polarities.* We first present the Galois connection that will serve as base for the complex algebra construction.

DEFINITION 86. Let $\mathbb{W} = (X, Y, R)$ be a polarity. Then the functions u_R and l_R relating X and $Y$are defined as follows:

- $u_R \ : \begin{cases} \wp(X) \longrightarrow \wp(Y) \\ S \longmapsto u_R(S) = \{y \mid \forall x \in S : xRy\} \end{cases}$
- $l_R \ : \begin{cases} \wp(Y) \longrightarrow \wp(X) \\ S' \longmapsto l_R(S') = \{x \mid \forall y \in S' : xRy\} \end{cases}$

Observe that u_R retrieves the set of R-upper bounds of its input (all in Y given that $R \subseteq X \times Y$), while l_R retrieves the set of R-lower bounds (all in X given that $R \subseteq X \times Y$). Then u_R and l_R form a Galois connection between the posets $(\wp(X), \supseteq)$ and $(\wp(Y), \subseteq)$, or equivalently, $\langle u_R, l_R \rangle$ is a residuated pair. Hence, the composition $c_R = l_R \circ u_R \ : \ \wp(X) \longrightarrow \wp(X)$ is a closure operator on X, and $S \subseteq X$ is said to be $c_R - closed$ or R-Galois-stable if $c_R(S) = S$. The collection of $c_R - closed$ subsets of X form a complete lattice $(\mathbb{W})^+ = \{A \subseteq X \mid A = l_R(u_R(A))\}$ with the meet and join operations defined as follows: for any $S \subseteq X$, $\bigwedge S = \bigcap S$ and $\bigvee S = c_R(\bigcup S)$. It is also useful to remember that the set of upper bounds is always an R-upset and the set of lower bounds is always an R-downset (so stable sets are all downsets).

It is noted in [**Gehrke 2006**] that polarities are too general to provide the desired generalization of a Kripke-frame. In particular, just as Kripke-frames complex algebras are atomic -besides being complete-, we wish to have a suitable prolongation of this property in the distributive and non-distributive cases. Moreover,

[5] Observe that a topped $\bigcap$-structure is a complete meet semi-lattice. When a structure has all meets, it also has all joins and thus a topped $\bigcap$-structure turns out to be a complete lattice.

we also need to guarantee that the sets X and Y are represented in the dual algebra (or to use Gehrke's terms, to guarantee that they are potential interpretants) just as W is represented in BAOs by the set of singletons in $\wp(W)$.

DEFINITION 87. (Separating frame) A polarity $\mathbb{W} = (X, Y, R)$ is said to be a separating frame (an S-frame henceforth), if:

- $\forall x_1, x_2 \in X \ (x_1 \neq x_2 \Rightarrow u_R(\{x_1\}) \neq u_R(\{x_2\}))$
- $\forall y_1, y_2 \in Y \ (y_1 \neq y_2 \Rightarrow l_R(\{y_1\}) \neq l_R(\{y_2\}))$

REMARK 88. In an S-frame $\mathbb{W} = (X, Y, R)$, the sets X, Y are thus taken as contained in $(\mathbb{W})^+$ but in the sense that they are represented by corresponding Galois-stable sets, i.e. if $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$ then X is represented in $(\mathbb{W})^+$ by $\{R(1R), R(2R), R(3R)\}$ and Y by $\{Ra, Rb, Rc\}$.

DEFINITION 89. (Complex algebra for polarities) Given an polarity $\mathbb{W} = (X, Y, \leq)$, its complex algebra $\mathbb{W}^+ = \langle \mathcal{G}(\mathbb{W}), \wedge, \vee \rangle$ is a *complete* lattice, where $\mathcal{G}(\mathbb{W})$ is the family of Galois-stable sets of (X, Y, R) and $\wedge = \bigcap$ but $\vee = c_R \circ \bigcup$ (cf. Def. 86). When $\mathbb{W}$ is an RS-frame, then $\mathbb{W}^+ = \langle \mathcal{G}(\mathbb{W}), \wedge, \vee, \varnothing, X \rangle$.

Now we need a pair of maps that relate the elements on the two-sorted frame to the elements in the polarity-dual, its complex algebra.

DEFINITION 90. Let $\mathbb{W} = (X, Y, R)$ be a polarity. Then the functions Ξ and Υ are defined as follows:

- $\Xi : \begin{cases} X \longrightarrow (\mathbb{W})^+ \\ x \longmapsto l_R(u_R(\{x\})) \end{cases}$
- $\Upsilon : \begin{cases} Y \longrightarrow (\mathbb{W})^+ \\ y \longmapsto l_R(\{y\}) \end{cases}$

When $\mathbb{W}$ is an S-frame the functions Ξ and Υ are injective and consequently no information is lost regarding the sets X and Y. The proposition 86 below spells out the link between the partial order $\subseteq$ on the set $(\Xi[X] \cup \Upsilon[Y]) \subseteq (\mathbb{W})^+$ and the relation R. In fact, $\subseteq$ restricted to $\Xi[X] \times \Upsilon[Y]$ is entirely order-isomorphic to R: for all $x \in X$ and all $y \in Y$: $xRy \iff \Xi(x) \subseteq \Upsilon(y)$.

PROPOSITION 91. *Let $\mathbb{W} = (X, Y, R)$ be an S-frame and let $Z_1, Z_2 \in \Xi[X] \cup \Upsilon[Y]$, then the following holds:*

- $\Xi(x_1) \subseteq \Xi(x_2)$ *iff* $\forall y \in Y \, (x_2Ry \Rightarrow x_1Ry)$
- $\Upsilon(y_1) \subseteq \Upsilon(y_2)$ *iff* $\forall x \in X \, (xRy_1 \Rightarrow xRy_2)$
- $\Xi(x_1) \subseteq \Upsilon(y_2)$ *iff* x_1Ry_2
- $\Upsilon(y_1) \subseteq \Xi(x_2)$ *iff* $\forall x \in X, \forall y \in Y \, (xRy_1 \,\&\, x_2Ry \Rightarrow xRy)$

Just as we have that $\wp(W)$ is generated by W (every $S \in \wp(W)$ is the union of the singletons below it) here we have that $(\mathbb{W})^+$ is meet-generated by $\Upsilon[Y]$ and join generated by $\Xi[X]$.

PROPOSITION 92. *Let $\mathbb{W} = (X, Y, R)$ be any polarity, then $\Xi[X]$ join generates $(\mathbb{W})^+$ and $\Upsilon[Y]$ meet generates $(\mathbb{W})^+$.*

COROLLARY 93. *Let $\mathbb{W} = (X, Y, R)$ be an S-frame, then X join generates $(\mathbb{W})^+$ -thus also $\Xi[X] \cup \Upsilon[Y]$- and Y meet generates $(\mathbb{W})^+$ -thus also $\Xi[X] \cup \Upsilon[Y]$. We also have $\overline{\Xi[X] \cup \Upsilon[Y]} = (\mathbb{W})^+$, where $\overline{\Xi[X] \cup \Upsilon[Y]}$ is the Dedekind-MacNeille completion of $\Xi[X] \cup \Upsilon[Y]$.*

Join-irreducibility and meet-irreducibility take the following shape:

DEFINITION 94. (Reduced frame) Let $\mathbb{W} = (X, Y, R)$ be a frame, then $\mathbb{W}$ is reduced if the following properties hold

- $\forall x \in X \; \exists y \in Y$ with $x \nleq y$ and $\forall x' \in X$, if $x' < x$, then $x' \leq y$.
- $\forall y \in Y \; \exists x \in X$ with $x \nleq y$ and $\forall y' \in Y$, if $y < y'$, then $x \leq y'$.

When $\mathbb{W}$ is an S-frame, the first condition amounts to join-irreducibility in X -and thus in $\Xi[X] \cup \Upsilon[Y]$- of every x, while the second condition amounts to meet-irreducibility in Y -and thus in $\Xi[X] \cup \Upsilon[Y]$- of every y.

When lattices are restricted to distributive and Boolean settings, we obtain the usual family of admissible sets:

THEOREM 95. *Let $\mathbb{W} = (X, Y, R)$ be an RS-frame*[6].

- *If $\overline{\Xi[X] \cup \Upsilon[Y]}$ is distributive, then $\overline{\Xi[X] \cup \Upsilon[Y]} \cong \mathcal{O}(W) \cong \mathcal{K}(W)$*
- *If $\overline{\Xi[X] \cup \Upsilon[Y]}$ is Boolean, then $\overline{\Xi[X] \cup \Upsilon[Y]} \cong \mathcal{O}(W) \cong \wp(W)$*

Proposition 3.33. of [**Gehrke 2006**], which we reproduce below, states that there is a correspondence between binary relations that are RS-frame compatible and residuated pairs of maps between the Galois stable sets.

PROPOSITION 96. *Let $\mathbb{F}_1 = (X_1, Y_1, \leq)$ and $\mathbb{F}_2 = (X_2, Y_2, \leq)$ be RS-frames. Then the following holds:*

(1) If $\mathcal{G}(\mathbb{F}_1) \underset{h}{\overset{g}{\rightleftarrows}} \mathcal{G}(\mathbb{F}_2)$ is a residuated pair, then both maps are uniquely determined by the relation $R_h \subseteq Y_1 \times X_2$ defined as: $y_1 R_h x_2 \iff y_1 \geq h(x_2) \iff g(y_1) \geq x_2$. Since for any $x_2 \in X_2$ and any $y_1 \in Y_1$ we have $R_h[\_, x_2] = \uparrow h(x_2) \cap Y_1$ and $R_h[y_1, \_] = \downarrow g(y_1) \cap X_2$, these are stable sets and therefore $R_h \subseteq Y_1 \times X_2$ is $\mathbb{F}_1, \mathbb{F}_2$-compatible.

(2) Conversely, if a relation $R \subseteq Y_1 \times X_2$ is $\mathbb{F}_1, \mathbb{F}_2$-compatible (that is, for any $x_2 \in X_2$ and any $y_1 \in Y_1$ we have that $R_h[\_, x_2]$ and $R_h[y_1, \_]$ are stable sets) then the maps $h_R : \mathcal{G}(\mathbb{F}_2) \longrightarrow \mathcal{G}(\mathbb{F}_1)$ and $g_R : \mathcal{G}(\mathbb{F}_1) \longrightarrow \mathcal{G}(\mathbb{F}_2)$ defined by:

$$\begin{cases} h_R(x_2) = \bigwedge R[\_, x_2] & for\ x_2 \in X_2 \\ h_R(u_2) = \bigvee \{h_R(x_2) \mid u_2 \geq x_2 \in X_2\} = \bigwedge \{y_1 \mid \forall x_2 \in X_2 \, (x_2 \leq u_2 \Rightarrow y_1 R x_2)\} & for\ u_2 \in \mathcal{G}(\mathbb{F}_2) \end{cases}$$

and

$$\begin{cases} g_R(y_1) = \bigvee R[y_1, \_] & for\ y_1 \in Y_1 \\ g_R(u_1) = \bigwedge \{g_R(y_1) \mid u_1 \leq y_1 \in Y_1\} = \bigvee \{x_2 \mid \forall y_1 \in Y_1 \, (y_1 \geq u_1 \Rightarrow y_1 R x_2)\} & for\ u_1 \in \mathcal{G}(\mathbb{F}_1) \end{cases}$$

form a residuated pair.

(3) If $\mathcal{G}(\mathbb{F}_1) \underset{h}{\overset{g}{\rightleftarrows}} \mathcal{G}(\mathbb{F}_2)$ is a residuated pair, then $h_{R_h} = h$ and $g_{R_h} = g$, and if $R \subseteq Y_1 \times X_2$ is a $\mathbb{F}_1, \mathbb{F}_2$-compatible relation, then $R_{h_R} = R$.

This allows for the following definition of a dual of a complete lattice homomorphism (definition 3.34 in [**Gehrke 2006**])

PROPOSITION 97. *Let $\mathbb{F}_1 = (X_1, Y_1, \leq)$ and $\mathbb{F}_2 = (X_2, Y_2, \leq)$ be RS-frames and $h_R : \mathcal{G}(\mathbb{F}_2) \longrightarrow \mathcal{G}(\mathbb{F}_1)$ a complete lattice homomorphism. Then the dual of h is defined as the pair (R_h, S_h) where $R_h \subseteq Y_1 \times X_2$ is the relation described in 96 which arises from the fact that h is residuated, and $S_h \subseteq X_1 \times Y_2$ is the corresponding relation arising from h being dually residuated. In other words: $y_1 R_h x_2 \iff y_1 \geq h(x_2)$ and $x_1 S_h y_2 \iff h(y_2) \geq x_1$*

[6] $\mathcal{O}(\mathbb{W})$ is the traditional notation for the family of downsets of $\mathbb{W}$ ordered by inclusion. The $\mathcal{O}$ designation comes from the association of downsets with the opens of a topology and $\mathcal{K}$ from the association of upsets with the closed elements of the topology.

Definition 3.35 in [**Gehrke 2006**] extends this to n-ary relations.

DEFINITION 98. A generalized Kripke frame is a structure $\mathbb{F} = (X, Y, \leq, R)$ such that $(X, Y, \leq)$ is an $RS-frame$ and $R \subseteq X \times X \times Y$ is a compatible relation. Associated with R there is another compatible relation $R^{\downarrow} \subseteq X \times X \times X$ defined by: $(x_1, x_2, x_3) \in R^{\downarrow}$ iff $x_3 \in R[x_1, x_2, \_]^{l}$.

Finaly, as complex algebra of a generalized Krikpe frame we obtain the following definition by simply considering the extra operations:

DEFINITION 99. (Complex algebra for generalized Kripke-frames) Given a generalized Kripke-frame $\mathbb{F} = ((X, Y, \leq), R_{\Diamond}, R_{\Box}, R_{\triangleright}, R_{\triangleleft}, R_{\circ}, R_{\rightarrow}, R_{\leftarrow})$, its complex algebra $\mathbb{F}^{+} = \langle \mathcal{G}(\mathbb{F}), \wedge, \vee, \varnothing, X, \langle R_{\Diamond} \rangle, [R_{\Box}], [R_{\triangleright} \rangle, \langle R_{\triangleleft}], \langle R_{\circ} \rangle, \langle R_{\leftarrow} \rangle, \langle R_{\rightarrow} \rangle \rangle$ is a *perfect* (substructural-modal) lattice, where $\mathcal{G}(\mathbb{F})$ is the family of Galois-stable sets of the RS-polarity (X, Y, R) which constitutes $\mathbb{F}$'s universe, and $\wedge = \bigcap$ but $\vee = c_R \circ \bigcup$ (cf. Def. 86), with the modal operations defined as in definition 23

4.3.3. Polarities from perfect lattices and from posets: Optimal filters and ideals. The maximal filters of a Boolean algebra constitute the points of the dual space of such Boolean algebra. The same role is fulfilled in the non-Boolean case by the prime filters of a distributive lattice, where these filters are a generalization of the former ones. Optimal filters (and optimal ideals) push the generalization process one bit further, being the basis of dual representation of a non-distributive lattice. Optimal filters of a distributive lattice are exactly the prime ones, and prime filters of a Boolean algebra are exactly the maximal ones.

The following definition was first formulated for proper filters and ideals of bounded lattices by [**Haim 2000**] as Def.1.3.6. and then generalized for posets by [**Gool 2009**] (Def.3.1.1.).

DEFINITION 100. (maximality) Let $\mathbb{L}$ be a poset. Let F, I be a filter and ideal of $\mathbb{L}$, respectively.

- F is I-maximal iff it is $\subseteq$-maximal with respect to being disjoint from I:
  - $F \cap I = \varnothing$
  - If F' is a proper filter such that $F \subset F'$ then $F' \cap I \neq \varnothing$.
- The notion of I being F-maximal is dual.
- (Maximal filter-ideal pairs) A pair $\langle F, I \rangle$ is a *maximal filter-ideal pair* iff F is I-maximal and I is F-maximal. We also say that F and I are companions of each other.
- A filter F is *optimal* if there is an ideal I such that the pair $\langle F, I \rangle$ is a maximal filter-ideal pair. The notion of optimal ideal is defined symmetrically. The sets of optimal filters and optimal ideals are denoted $\mathcal{F}_{op}(\mathbb{L})$ and $\mathcal{I}_{op}(\mathbb{L})$ respectively.

Again, the following theorem can be found in [**Haim 2000**] for bounded lattices as Theorem .1.3.7. plus Corollary 1.3.9. and generalized for posets in [**Gool 2009**] as Theorem .3.1.2.

THEOREM 101. *(Disjoint Filter-Ideal pairs can be extended to maximal ones) Given a filter F and an ideal I in a poset $\mathbb{L}$ such that $F \cap I = \varnothing$, then:*

(1) there is an I-maximal filter F_* containing F and
(2) there is an F-maximal ideal I_* containing I.
(3) $\langle F_*, I_* \rangle$ is a maximal pair

This theorem guarantees that any disjoint filter-ideal pair can be extended to a maximal one. The reader will recognize in it a generalization of the well known facts that every filter in a Boolean algebra can be extended to a maximal one (maximal filters are ultrafilters in BAs) and that every filter in a distributive bounded lattice can be extended to a prime filter. From this theorem stems a crucial corollary which we will use later.

COROLLARY 102. *If F is a filter of $\mathbb{L}$ and $a \notin F$, then there is an $\langle F_*, I_* \rangle$ maximal pair such that $F \subseteq F_*$ and $a \in I_*$. If I is an ideal of $\mathbb{L}$ and $a \notin I$, then there is an $\langle F_*, I_* \rangle$ maximal pair such that $I \subseteq I_*$ and $a \in F_*$*

As before, it is found in [**Haim 2000**] for bounded lattices as Corollary 1.3.8. and generalized for posets in [**Gool 2009**] as Corollary .3.1.3.

REMARK 103. By definition, we already knew that:

(1) Every optimal filter is I-maximal for some ideal I.
(2) Every optimal ideal is F-maximal for some filter F.

Now, from corollary 96, the converses follows:

(1) If there is an ideal I of $\mathbb{L}$ such that the filter F is I-maximal, then F is an optimal filter (i.e., I can be extended to an F-maximal I').
(2) If there is a filter F of $\mathbb{L}$ such that the ideal I is F-maximal, then I is an optimal ideal (i.e., F can be extended to an I-maximal F').

Every prime filter is optimal. In fact, if F is a prime filter, then there is exactly one ideal I that is F-maximal. Thus, if F is a prime filter, then there is exactly one ideal I companion of F.

The following proposition states some key facts about optimal filters (Proposition 1.3.13 in [**Haim 2000**])

PROPOSITION 104. *Optimal filters, prime filters and distributivity*

- *The optimal filters of a distributive lattice are exactly the prime filters of it.*
- *Conversely, if every optimal filter of a lattice L is prime, then L is distributive.*
- *The fact that every optimal filter F of a lattice L has a unique companion does not imply that L is distributive.*

Optimal filters and ideals can be put together into a polarity as follows.

DEFINITION 105. Let $\mathbb{L}$ be a bounded lattice. Then we define the disjointness relation $\perp \subseteq \mathcal{F}_{op}(\mathbb{L}) \times \mathcal{I}_{op}(\mathbb{L})$ by $F \perp I \Longleftrightarrow F \cap I \neq \varnothing$, and we name $\mathbb{A}_+ := \langle \mathcal{F}_{op}(\mathbb{L}), \mathcal{I}_{op}(\mathbb{L}), \perp \rangle$ the *optimal polarity* of $\mathbb{L}$

Although the theory of optimal ideals and filters was formulated in [**Haim 2000**] initially for bounded lattices -thus only applicable to the discrete duality-, it has been now generalized to the wider context of posets by [**Gool 2009**], hence the title of this subsection.

DEFINITION 106. Let $\mathbb{A}$ be a poset. Then we define the disjointness relation $\perp \subseteq \mathcal{F}_{op}(\mathbb{A}) \times \mathcal{I}_{op}(\mathbb{A})$ by $F \perp I \Longleftrightarrow F \cap I \neq \varnothing$, and we name $\mathbb{A}_\bullet := \langle \mathcal{F}_{op}(\mathbb{A}), \mathcal{I}_{op}(\mathbb{A}), \perp \rangle$ the *optimal polarity* of $\mathbb{A}$[7]

To round up this section, we add the definition of a perfect non-distributive modal algebra.

[7]Terminology introduced in [**Gool 2009**]

DEFINITION 107. (Perfect non-distributive modal algebra) A lattice $\mathbb{L} = \langle L, \wedge, \vee, \top, \bot, \Diamond, \Box, \triangleright, \triangleleft, \circ, \rightarrow, \leftarrow \rangle$ is a *perfect* non-distributive modal algebra if $\langle L, \wedge, \vee, \top, \bot \rangle$ is a -non necessarily distributive- perfect lattice and for any $S, S' \subseteq L$:

- $\Diamond \left(\bigvee S\right) = \bigvee \left(\Diamond S\right) = \bigvee \{\Diamond u \mid u \in S\}$
- $\Box \left(\bigwedge S\right) = \bigwedge \left(\Box S\right) = \bigwedge \{\Box u \mid u \in S\}$
- $\triangleright \left(\bigvee S\right) = \bigwedge \left(\triangleright S\right) = \bigwedge \{\triangleright u \mid u \in S\}$
- $\triangleleft \left(\bigwedge S\right) = \bigvee \left(\triangleleft S\right) = \bigvee \{\triangleleft u \mid u \in S\}$
- $\bigvee S \circ \bigvee S' = \bigvee \left(S \circ S'\right) = \bigvee \{s \circ s' \mid s \in S \,\&\, s' \in S'\}$
- $\bigvee S \rightarrow \bigwedge S' = \bigvee \left(S \rightarrow S'\right) = \bigvee \{s \rightarrow s' \mid s \in S \,\&\, s' \in S'\}$
- $\bigwedge S \leftarrow \bigvee S' = \bigvee \left(S \leftarrow S'\right) = \bigvee \{s \leftarrow s' \mid s \in S \,\&\, s' \in S'\}$

CHAPTER 5

Completeness for non-distributive propositional case

We will now expose the completeness result for non-distributive (propositional) substructural logic from [**Gehrke 2006**], adding the unary modal operators. The focus will be on clarifying the completeness results under the light of the continuity along the classical/distributive/non-distributive line of progressive generalization. Most importantly, we attempt to present the methodology used in a more transparent way, a methodology which is greatly obscured in the original paper due to space limitations. We first draw the reader's attention to the fact that although [**Gehrke 2006**] provides a Kripke semantics completeness proof, the approach heavily relies on the algebraic side of the discrete duality. In fact, the relational semantics itself is based upon a dualization of the algebraic interpretation map. This technique is shown to retrieve the usual frame semantics on the distributive & classical settings (5.1.2.). Then we clarify the relation between the points in polarities and the usual points in the distributive setting (5.1.3.). Once this checked, we use this method (dualization of the algebraic assignments) to obtain relational satisfaction definitions for all operations $\Diamond, \Box, \rhd, \lhd, \circ, \rightarrow, \leftarrow$ in the non-distributive setting (5.1.4.). We finally expose the completeness for propositional substructural logic in the non-distributive case (5.2.).

5.1. Interpretation dualization

5.1.1. Interpretation of algebras. Given a language L, an interpretation on a perfect lattice expansion $\mathbb{C}$ is a homomorphism $V : Fm \longrightarrow \mathbb{C}$ where Fm is the formula algebra based on L. When Fm is just a propositional logic in a certain algebraic signature τ, then V is a τ-homomorphism[1]. This means that interpretations into perfect lattices can be systematically dualized into relations, just as any other homomorphism. In the Boolean and distributive setting and given any frame $\mathbb{F}$ with universe W and any previously defined satisfaction relation $\Vdash \subseteq W \times Fm$ on it, an interpretation $V : Fm \longrightarrow \mathbb{F}^+$ can be defined as an τ-homomorphism in a natural way. It suffices to take the unique homomorphic extension of the map $V := \begin{cases} \mathsf{AtProp} \longrightarrow \mathbb{F}^+ \\ p \longmapsto \Vdash^{-1}[p] = \{w \in W \mid w \Vdash p\} \end{cases}$ which is the equivalent functional representation of the relation $\Vdash$. For this equivalent functional representation to be well defined, we must ensure that the relation $\Vdash$ is $\mathbb{F}^+$ compatible, that is, it must be such that $\Vdash^{-1}[p] \in \mathbb{F}^+$ for every $p \in \mathsf{AtProp}$.[2]

The resulting interpretation on the complex algebra $\mathbb{F}^+$ is such that for every $\varphi \in Fm$ and every $x \in J^\infty(\mathbb{F}^+)$, the following condition holds:

$$x \Vdash \varphi \quad \text{iff} \quad x \leq V(\varphi) \tag{5.1.1}$$

[1] Observe that when Fm is a quantified predicate logic, then V must turn into a complete lattice homomorphism as well. To see this notice that, where g is an assignment and $g' \overset{x}{\simeq} g$ is the equivalence class of all assignment that are like g except possibly on the value of x, V must satisfy the equalities $V(\forall x\varphi(x))_g = V\left(\bigwedge_{g' \overset{x}{\simeq} g} [\varphi(x)]_{g'}\right) = \bigwedge_{g' \overset{x}{\simeq} g}\left(V\left([\varphi(x)]_{g'}\right)\right)$ and $V(\exists x\varphi(x))_g = V\left(\bigvee_{g' \overset{x}{\simeq} g} [\varphi(x)]_{g'}\right) = \bigvee_{g' \overset{x}{\simeq} g}\left(V\left([\varphi(x)]_{g'}\right)\right)$

[2] In the Boolean case this is automatically true since the complex algebra is the powerset algebra of W expanded with modal operations and thus any subset of W is in it. But already in the distributive case, this is no longer true in general: $\Vdash^{-1}[p]$ needs to be an upset since the carrier of the complex algebra is then $\wp^{\uparrow}(W)$.

where $x \in W$ on the frame side and $x \in \wp^{\uparrow}(W)$ is the dual object representing x in the perfect lattice $\mathbb{F}^{+}$.

The notion of *interpretation* on an algebra in the non distributive case extends in a transparent way the corresponding notion in the Boolean and distributive settings. The behaviour of the relational semantics at this level of generality, on the other hand, is not well understood. Or, to say it differently, it does not follow in any obvious way from the known relational semantics for classical and distributive modal logics.

Now suppose our perfect lattice $\mathbb{C}$ is the complex algebra $\mathbb{F}^{+}$ of some frame $\mathbb{F} = \left\langle (X, \leq, Y), \{R_{\triangle}\}_{\triangle \in \tau} \right\rangle$. Given an interpretation $V : Fm \longrightarrow \mathbb{F}^{+}$ is it possible to define a *satisfaction* relation on $\mathbb{F}$ associated to the algebraic *interpretation* in similar way as before, i.e. in such way as to obtain the biconditional (5.1.1)? As we will see in this section, this is indeed possible. The leading intuition is that *interpretations* on complex algebras and *satisfaction* relations on the corresponding frames are dual to one another as an instance of the more general duality between complete lattice homomorphisms and pairs of relations with special properties. The algebraic notion of homomorphism will be taken as primitive and used to build the satisfaction relation on frames at this new level of generality (i.e., in the non-distributive setting).

5.1.2. Retrieving the satisfaction relation from the interpretation function: distributive case. Let us show that not only retrieving the satisfaction relation from the interpretation is possible, but that it gives us the usual relational semantics in the distributive case (and hence also in the Boolean case). We will assume for simplicity that our signature is composed by a bounded distributive lattice expanded with a unary diamond.

We need to define $\Vdash$ inductively, in such a way that the equation (5.1.1) holds. Thus the basic step is provided by the previous desiderata, for every $x \in J^{\infty}(\mathbb{F}^{+})$ and every $p \in \mathsf{AtProp}$ we define: $x \Vdash p \quad \text{iff} \quad x \leq V(p)$. For the inductive step, let $\varphi = \Diamond\psi$ and, as inductive hypothesis, suppose that (5.1.1) holds for any ψ of strictly lower complexity than φ.

$$x \Vdash \Diamond\psi \quad \text{iff} \quad x \leq V(\Diamond\psi)$$

To unfold the inductive step we will make use of the following facts:

1. $\mathbb{F}^{+} = \left\langle \wp^{\uparrow}(W), \cap, \cup, \varnothing, W, \Diamond^{\mathbb{F}^{+}} \right\rangle$ is a *perfect* distributive lattice
2. by 1, each element in $\wp^{\uparrow}(W)$ is representable as the join of all join-irreducible elements below it or equivalently as the meet of meet-irreducibles above it.
3. $J^{\infty}(\mathbb{F}^{+}) = \{\uparrow w \mid w \in W\}$, hence in (5.1.1) x on the algebraic side is a principal upset ($x = \uparrow w$) and x on the frame side is its generator ($x = \{w\}$).
4. for all $X \in \mathbb{F}^{+}$; $\Diamond^{\mathbb{F}^{+}}(X) = R_{\Diamond}^{-1}[X]$ where $R_{\Diamond}$ is the accessibility relation that interprets $\Diamond$ on the frame $\mathbb{F}$.
5. $V(\psi) \in \mathbb{F}^{+}$ for all ψ.
6. by 1, $\Diamond^{\mathbb{F}^{+}}$ is completely join-preserving.
7. Since by assumption V is a homomorphism, then $V(\Diamond\varphi) = \Diamond^{\mathbb{F}^{+}} V(\varphi)$

Given these facts we have:

$$\begin{array}{llll}
x \leq V(\Diamond\psi) & \text{iff} & x \leq \bigvee\{\Diamond^{\mathbb{F}^+}x' \mid x' \in J^\infty(\mathbb{F}^+) \text{ and } x' \Vdash \psi\} & \\
& \text{iff} & \exists x'(x' \in J^\infty(\mathbb{F}^+) \ \&\ x' \Vdash \psi \ \&\ x \leq \Diamond^{\mathbb{F}^+}x') & \text{(a)} \\
& \text{iff} & \exists x'(x' \Vdash \psi \ \&\ x \leq \Diamond^{\mathbb{F}^+}x') & \text{(b)} \\
& \text{iff} & \exists x'(x' \Vdash \psi \ \&\ \uparrow x \subseteq R_\Diamond^{-1}[x']) & \text{(c)} \\
& \text{iff} & \exists x'(x' \Vdash \psi \ \&\ x \in R_\Diamond^{-1}[x']) & \text{(d)} \\
& \text{iff} & \exists x'(x' \Vdash \psi \ \&\ xR_\Diamond x') & \text{(e)} \\
& \text{iff} & x \Vdash \Diamond\psi & \text{(f)}
\end{array}$$

PROOF. We expand the successive "iffs" as follows:

(a) $V(\psi) =_{by\ 5\&2} \bigvee\{z \in J^\infty(\mathbb{F}^+) \mid z \leq V(\psi)\} =_{by\ IH} \bigvee\{z \in J^\infty(\mathbb{F}^+) \mid z \Vdash \psi\}$ and then $V(\Diamond\psi) = \Diamond^{\mathbb{F}^+}V(\psi) = \Diamond^{\mathbb{F}^+}\bigvee\{z \in J^\infty(\mathbb{F}^+) \mid z \Vdash \psi\} =_{by\ 6} \bigvee\left\{\Diamond^{\mathbb{F}^+}z \mid z \in J^\infty(\mathbb{F}^+) \text{ and } z \Vdash \psi\right\}$ where the first equality holds because V is an homomorphism by assumption. From such equality and for any $x \in J^\infty(\mathbb{F}^+)$ we have the following:

$$x \leq V(\Diamond\psi) \ \text{ iff } \ x \leq \bigvee\{\Diamond^{\mathbb{F}^+}x' \mid x' \in J^\infty(\mathbb{F}^+) \text{ and } x' \Vdash \psi\}$$

.

Now because x is not only completely join-irreducible but in fact completely join-prime[3], the second inequality implies that $x \leq \Diamond^{\mathbb{F}^+}x'$ for some $x' \in J^\infty(\mathbb{F}^+)$ such that $x' \Vdash \psi$. The converse follows by lattice theoretic laws, if x is below one element of the join, then it is certainly below the join itself, since a join is above all its elements and the order induced by lattice operations is transitive. This settles (a).

(b) Clearly $\exists x'(x' \in J^\infty(\mathbb{F}^+) \ \&\ x' \Vdash \psi \ \&\ x \leq \Diamond^{\mathbb{F}^+}x')$ implies $\exists x'(x' \Vdash \psi \ \&\ x \leq \Diamond^{\mathbb{F}^+}x')$ by just forgetting a property, but the converse is not so obvious. Suppose there is an x' such that $x' \Vdash \psi$ and $x \leq \Diamond^{\mathbb{F}^+}x'$, then using the representation of x' as a join of completely join-irreducibles we get $x \leq \Diamond^{\mathbb{F}^+}\bigvee\{z \in J^\infty(\mathbb{F}^+) \mid z \leq x'\}$ and by fact 6, $x \leq \bigvee\left\{\Diamond^{\mathbb{F}^+}z \mid z \in J^\infty(\mathbb{F}^+) \ \&\ z \leq x'\right\}$. As before, because x is completely join-prime we have $x \leq \Diamond^{\mathbb{F}^+}z$ for some $z \in J^\infty(\mathbb{F}^+)$ such that $z \leq x'$. But now we can apply the induction hypothesis to $x' \Vdash \psi$ and get $x' \leq V(\psi)$, so $z \leq x' \leq V(\psi)$ and applying IH again, $z \Vdash \psi$. Therefore $\exists y(y \in J^\infty(\mathbb{F}^+) \ \&\ y \Vdash \psi \ \&\ x \leq \Diamond^{\mathbb{F}^+}y)$, namely take $y = z$.

(c) Now observe that the condition $x \leq \Diamond^{\mathbb{F}^+}x'$ is really the same as $\uparrow x \subseteq R_\Diamond^{-1}[x']$. The order $\leq$ is no more than $\subseteq$ itself, by fact 1; $\Diamond^{\mathbb{F}^+}x' = R_\Diamond^{-1}[x']$ by 4; and x must be in fact the principal upset $\uparrow x$ by 3.

(d) Now clearly $\uparrow x \subseteq R_\Diamond^{-1}[x']$ implies $x \in R_\Diamond^{-1}[x']$, and on the other hand if $x \in R_\Diamond^{-1}[x']$ then $\uparrow x \subseteq R_\Diamond^{-1}[x']$, because $R_\Diamond^{-1}[x'] = \Diamond^{\mathbb{F}^+}x'$ and so by 1 it must be an upset since $\Diamond^{\mathbb{F}^+}x' \in \mathbb{F}^+$.

(e) and (f) follow immediately from the definitions of $R_\Diamond^{-1}[x']$ and the standard semantics for frames. □

This gives us the inductive step for the unary diamond. The above proof not only shows that it is possible to retrieve a frame *satisfaction* relation from an algebraic *interpretation* function through a *correspondence* argument, but also that the conditions obtained coincide exactly with the usual frame semantics.

5.1.3. The relation between the elements of Gehrke's RS-polarities and the elements of usual frames. Given the claimed algebraic continuity between the Boolean, the distributive and the non-distributive cases, it may look odd to see the non-distributive case complex algebra realized as a Galois connection construction while no such construction seems to have been used on previous cases. Accordingly, it can be confusing to see the elements of the distributive complex algebra materialized as principal upsets while the corresponding elements in the Galois connection construction are downsets rather than the expected upsets. This section is aimed at clarifying these matters.

[3] In distributive lattices the join-irreducible elements are exactly the join-prime ones

In fact, the Galois construction used in the non-distributive case does specialize correctly in the distributive and Boolean cases. In the distributive case, for each $x \in X$ in the frame, we have $\uparrow x$ in the complex algebra. By using the Galois-stable sets construction we would have ${}^{R}\left(\left(\{x\}\right)^{R}\right)$ in the complex algebra as representative of $x \in X$.

Corollaries 2.18 and 2.19 in [**Gehrke 2006**] link the standard presentation of Kripke frames, in the Boolean and distributive settings, with the generalized two-sorted presentations (polarities). Given a Kripke frame with poset $(X, \leq)$ as universe, there is a corresponding polarity-frame $\mathbb{F} = (X, X, \ngeq)$ whose complex algebra $\mathcal{G}(\mathbb{F})$ is -as expected- a complete distributive lattice.

If R is $\leq$ then ${}^{R}\left(\left(\{x\}\right)^{R}\right)$ is a downset for all $x \in X$. [**Gehrke 2006**] uses $\ngeq$ as the relation R of the polarity (xRy iff $x \ngeq y$), so for a subset $Z \subseteq X$ we have (in what follows x is assumed to range over the set X and y is assumed to range over the set Y):

$$
\begin{aligned}
(Z)^{R} &= \{y|\forall z\,(z \in Z \Rightarrow zRy)\} \\
&= \{y|\forall z\,(z \in Z \Rightarrow z \ngeq y)\} \\
&= \{y|\forall z\,(z \in Z \Rightarrow z \notin\uparrow y)\} \\
&= \{y|\forall z\,(z \in Z \Rightarrow z \in (\uparrow y)^{c})\} \\
&= \{y|\forall z\,(z \in Z \Rightarrow y \in (\downarrow z)^{c})\} \\
&= \bigcap_{z \in Z} (\downarrow z)^{c} \\
&= \left(\bigcup_{z \in Z} \downarrow z\right)^{c}
\end{aligned}
$$

and then we have (with $B = \left(\bigcup_{z \in Z} \downarrow z\right)^{c}$):

$$
\begin{aligned}
{}^{R}\left((Z)^{R}\right) &= \left\{x|\forall y\left(y \in \left(\bigcup_{z \in Z} \downarrow z\right)^{c} \Rightarrow xRy\right)\right\} \\
&= \{x|\forall y\,(y \in B \Rightarrow x \ngeq y)\} \\
&= \{x|\forall y\,(y \in B \Rightarrow y \notin\downarrow x)\} \\
&= \{x|\forall y\,(y \in B \Rightarrow y \in (\downarrow x)^{c})\} \\
&= \{x|\forall y\,(y \in B \Rightarrow x \in (\uparrow y)^{c})\} \\
&= \bigcap_{y \in B} (\uparrow y)^{c} \\
&= \left(\bigcup_{y \in B} \uparrow y\right)^{c}
\end{aligned}
$$

On the particular case in which $Z = \{x\}$ for some $x \in X$, this reduces to:

$$\left(\{x\}\right)^{R} = (\downarrow x)^{c}$$

and then

$$
{}^{R}\left(\left(\{x\}\right)^{R}\right) = \bigcap_{y \in (\downarrow x)^{c}} (\uparrow y)^{c} = \left(\bigcup_{y \in (\downarrow x)^{c}} \uparrow y\right)^{c} = \left(\bigcup_{y \nleq x} \uparrow y\right)^{c}
$$

Now let us compare ${}^{R}\left(\left(\{x\}\right)^{R}\right)$ to $\downarrow x$. Clearly, $\downarrow x \subseteq \left(\bigcup_{y \nleq x} \uparrow y\right)^{c}$. Now $s \in \left(\bigcup_{y \nleq x} \uparrow y\right)^{c}$ implies $s \notin \bigcup_{y \nleq x} \uparrow y$ implies $y \nleq x \Rightarrow y \nleq s$ implies $y \leq s \Rightarrow y \leq x$. Now by substitution ($y = s$) we obtain $s \leq s \Rightarrow s \leq x$ which implies $s \in\downarrow x$. Thus $\downarrow x \supseteq \left(\bigcup_{y \nleq x} \uparrow y\right)$ and ${}^{R}\left(\left(\{x\}\right)^{R}\right) = \downarrow x$.

REMARK 108. Observe that while $\left(\bigcup_{y\in(\downarrow x)^c} \uparrow y\right)^c$ is a $\leq$-downset, it is also an R-upset given that xRy iff $x \ngeq y$. Thus, the representation of worlds in the lattice of Galois-stable sets coincides with their representations in the standard complex algebra formed by taking the lattice of upsets of the universe of ordered Kripke frame.

Now let us check the Boolean case. There $\neq$ is used as the relation R of the polarity (xRy iff $x \neq y$), so for a subset $Z \subseteq X$ we have (in what follows x is assumed to range over the set X and y is assumed to range over the set Y):

$$\begin{aligned} (Z)^R &= \{y|\forall z\,(z\in Z \Rightarrow zRy)\} \\ &= \{y|\forall z\,(z\in Z \Rightarrow z\neq y)\} \\ &= (Z)^c \end{aligned}$$

and then we have:

$$\begin{aligned} {}^R\left((Z)^R\right) &= \{x|\forall y\,(y\in (Z)^c \Rightarrow xRy)\} \\ &= \{x|\forall y\,(y\in (Z)^c \Rightarrow x\neq y)\} \\ &= ((Z)^c)^c = Z \end{aligned}$$

Clearly, we obtain the same representation of each $x \in X$ in the Galois-stable lattice than with the usual powerset construction: the singleton $\{x\} = {}^R\left((\{x\})^R\right)$.

5.1.4. Retrieving the satisfaction relation from the interpretation function: non-distributive case. The interpretants of a formula φ is a set of worlds in classical modal logic, an upset of worlds in distributive modal logic and a pair composed of an upset of worlds and a downset of co-worlds in non-distributive modal logic. In this last setting, accordingly, we get a two-sorted satisfaction: the usual satisfaction relation and a co-satisfaction relation.

5.1.4.1. *Preliminaries: the duals of modal operations.* As preliminary for the dualization of interpretation, we will here treat the dualization of modal operations. The first operation to be treated will be taken as an opportunity to illustrate correspondence method in more detail, details which will be mostly omitted in the remaining cases. We start with the unary diamond. For the sake of generality we will use three complex algebras $\mathbb{C}_1, \mathbb{C}_2$ and $\mathbb{C}_3$, even though we are just interested in the particular case where $\mathbb{C}_1 = \mathbb{C}_2 = \mathbb{C}_3$.

Let $\Diamond : \begin{cases} \mathbb{C}_1 \longrightarrow \mathbb{C}_2 \\ u \longmapsto \Diamond u \end{cases}$ then the relational dual has the shape $R_\Diamond \subseteq \mathbb{C}_2 \times \mathbb{C}_1$, or more precisely $R_\Diamond \subseteq M^\infty(\mathbb{C}_2) \times J^\infty(\mathbb{C}_1)$ as it only covers sets of generators. Both $\mathbb{C}_1$ and $\mathbb{C}_2$ are perfect, which means that we have a set of generators for each. We know $\Diamond$ (and thus each of its values) because the algebras over which it operates are given. However, when we are interested in dualization to the frame side, the general definition of $\Diamond$ which is built upon all elements of $\mathbb{C}_1$ (for the inputs) and all elements of $\mathbb{C}_2$ (for the values) needs to be couched exclusively in terms of generators, because these are precisely the only elements that travel through dualization. We have two sets of generators, the completely join irreducibles and the completely meet irreducibles. We can represent any element in terms of either set as follows.

(1) for any $u \in \mathbb{C}_1$ we have $u = \bigwedge\{y_1|y_1 \in M^\infty(\mathbb{C}_1) \text{ and } u \leq y_1\} = \bigvee\{x_1|x_1 \in J^\infty(\mathbb{C}_1) \text{ and } u \geq x_1\}$.
(2) for any $\Diamond u \in \mathbb{C}_2$ we have $\Diamond u = \bigwedge\{y_2|y_2 \in M^\infty(\mathbb{C}_2) \text{ and } \Diamond u \leq y_2\} = \bigvee\{x_2|x_2 \in J^\infty(\mathbb{C}_2) \text{ and } \Diamond u \geq x_2\}$.

Therefore, by 1, $\Diamond u = \Diamond \bigvee\{x_1|x_1 \in J^\infty(\mathbb{C}_1) \text{ and } u \geq x_1\}$, and we choose the representation in terms of the join of lower generators, because $\Diamond$ is join preserving, which allows us to write $\Diamond u = \Diamond \bigvee\{x_1|x_1 \in J^\infty(\mathbb{C}_1) \text{ and } u \geq x_1\} = \bigvee\{\Diamond x_1|x_1 \in J^\infty(\mathbb{C}_1) \text{ and } u \geq x_1\}$. This reduces the problem of knowing where an arbitrary element of $\mathbb{C}_1$ is mapped to the problem of knowing where the generators below it are mapped. There is no guarantee that for a given generator x_1 the value $\Diamond x_1$ will still be a generator in $\mathbb{C}_2$, however. Therefore we need to use 2. to approximate this value $\Diamond x_1$ in terms of generators in $\mathbb{C}_2$. Such approximation provides us with the condition associated to the relational dual of $\Diamond$: since $\Diamond x_1 = \bigwedge\{y_2 | y_2 \in M^\infty(\mathbb{C}_2) \text{ and } \Diamond x_1 \leq y_2\}$ then:[4]

$$y_2 R_\Diamond x_1 \text{ iff } \Diamond x_1 \leq y_2 \tag{5.1.2}$$

Similarly we let $\lhd : \begin{cases} \mathbb{C}_1 \longrightarrow \mathbb{C}_2 \\ u \longmapsto \lhd u \end{cases}$ and the relational dual has the shape $R_\lhd \subseteq M^\infty(\mathbb{C}_2) \times M^\infty(\mathbb{C}_1)$. For an arbitrary element $u \in \mathbb{C}_1$, $\lhd u = \lhd \bigwedge\{y_1 | y_1 \in M^\infty(\mathbb{C}_1) \text{ and } u \leq y_1\}$, and we choose the representation in terms of the meet of upper generators, because $\lhd$ turns meets into joins, which gives us $\lhd u = \bigvee\{\lhd y_1 | y_1 \in M^\infty(\mathbb{C}_1) \text{ and } u \leq y_1\}$. As before, there is no guarantee that for a given generator y_1 the value $\lhd y_1$ will still be a generator in $\mathbb{C}_2$. Therefore we need to approximate this value $\lhd y_1$ in terms of generators in $\mathbb{C}_2$. Such approximation provides us with the condition associated to the relational dual of $\lhd$: since $\lhd y_1 = \bigwedge\{y_2 | y_2 \in M^\infty(\mathbb{C}_2) \text{ and } \lhd y_1 \leq y_2\}$ then:

$$y_2 R_\lhd y_1 \text{ iff } \lhd y_1 \leq y_2 \tag{5.1.3}$$

Given $\Box : \begin{cases} \mathbb{C}_1 \longrightarrow \mathbb{C}_2 \\ u \longmapsto \Box u \end{cases}$ the relational dual has the shape $R_\Box \subseteq J^\infty(\mathbb{C}_2) \times M^\infty(\mathbb{C}_1)$. For an arbitrary element $u \in \mathbb{C}_1$, $\Box u = \Box \bigwedge\{y_1 | y_1 \in M^\infty(\mathbb{C}_1) \text{ and } u \leq y_1\}$, and we choose the representation in terms of the meet of upper generators, because $\Box$ is meet preserving, which leads to $\Box u = \bigwedge\{\Box y_1 | y_1 \in M^\infty(\mathbb{C}_1) \text{ and } u \leq y_1\}$. For a given generator y_1 , we approximate $\Box y_1$ in terms of generators in $\mathbb{C}_2$ as follows: $\Box y_1 = \bigvee\{x_2 | x_2 \in J^\infty(\mathbb{C}_2) \text{ and } \Box y_1 \geq x_2\}$. This gives us the condition:

$$x_2 R_\Box y_1 \text{ iff } x_2 \leq \Box y_1 \tag{5.1.4}$$

Taking $\rhd : \begin{cases} \mathbb{C}_1 \longrightarrow \mathbb{C}_2 \\ u \longmapsto \rhd u \end{cases}$ as given, the relational dual has the shape $R_\rhd \subseteq J^\infty(\mathbb{C}_2) \times J^\infty(\mathbb{C}_1)$.

For an arbitrary element $u \in \mathbb{C}_1$, $\rhd u = \rhd \bigvee\{x_1 | x_1 \in J^\infty(\mathbb{C}_1) \text{ and } u \geq x_1\}$, and we choose the representation in terms of the join of lower generators, because $\rhd$ turns joins into meets, property used to obtain $\rhd u = \bigwedge\{\rhd x_1 | x_1 \in J^\infty(\mathbb{C}_1) \text{ and } u \geq x_1\}$. Again, for a given generator x_1 we approximate the value $\rhd x_1$ as follows: $\rhd x_1 = \bigvee\{x_2 | x_2 \in J^\infty(\mathbb{C}_2) \text{ and } \rhd x_1 \geq x_2\}$, then:

[4] Notice that we choose $R_\Diamond \subseteq M^\infty(\mathbb{C}_2) \times J^\infty(\mathbb{C}_1)$, but we could have equivalently chosen $R_\Diamond \subseteq J^\infty(\mathbb{C}_2) \times J^\infty(\mathbb{C}_1)$ and taken $\Diamond x_1 \geq x_2$ as the condition associated to $R_\Diamond$. The reason for which we made such choice ($R_\Diamond \subseteq - \times J^\infty(\mathbb{C}_1)$) is because we have chosen to represent an arbitrary element of $\mathbb{C}_1$ as the join of generator below (instead of the set of generators above), choice motivated by the property of $\Diamond$ to preserve joins. We can thus see the chain of dependencies in the choices being made: first we see whether the operation in the algebra to be dualized is meet or join-preserving, then we choose a representation of arbitrary elements in the algebra as joins, if it was join preserving, or meets if it was meet-preserving. Finally, these $\mathbb{C}_1$ -generators are mapped by $\Diamond$ to $\mathbb{C}_2$, where these values need once more to be approximated in terms of generators of $\mathbb{C}_2$ as we have no longer the guarantee that they are generators themselves. Finally we choose to represent $R_\Diamond$ as a relation from upper (or lower) generators to bottom generators, if arbitrary $\mathbb{C}_1$-elements were choosen to be represented in terms of bottom generators and otherwise if represented in terms of upper generators. Thus, the way to represent $R_\Diamond$ is dictated by our choice on how to represent or approximate our arbitrary elements of $\mathbb{C}_1$ (from above or from below), dictated in turn by the preservation properties of the operation treated.

$$x_2 R_{\triangleright} x_1 \text{ iff } x_2 \leq \triangleright x_1 \tag{5.1.5}$$

We know proceed with the binary diamond, better known as fusion. Let $\circ : \begin{cases} (\mathbb{C}_1 \times \mathbb{C}_2) \longrightarrow \mathbb{C}_3 \\ (u,v) \longmapsto u \circ v \end{cases}$ then its dual is the corresponding relation $R_\circ \subseteq \mathbb{C}_3 \times (\mathbb{C}_1 \times \mathbb{C}_2)$ where the value of the operation will be taken as first argument of the relation. The pair which makes up the input of the operation is taken as a unit and thus the sequence is not altered when representing the relational dual. As before, all $\mathbb{C}_1, \mathbb{C}_2$ and $\mathbb{C}_3$ are perfect, which means both have a set of generators for each.

(1) for any $u \in \mathbb{C}_1$ and any $v \in \mathbb{C}_2$ we have
   (a) $u = \bigwedge \{y_1 | y_1 \in M^\infty(\mathbb{C}_1) \text{ and } u \leq y_1\} = \bigvee \{x_1 | x_1 \in J^\infty(\mathbb{C}_1) \text{ and } u \geq x_1\}$ and
   (b) $v = \bigwedge \{y_2 | y_2 \in M^\infty(\mathbb{C}_2) \text{ and } v \leq y_2\} = \bigvee \{x_2 | x_2 \in J^\infty(\mathbb{C}_2) \text{ and } v \geq x_2\}$
(2) for any $u \circ v \in \mathbb{C}_3$ we have $u \circ v = \bigwedge \{y_3 | y_3 \in M^\infty(\mathbb{C}_3) \;\&\; u \circ v \leq y_3\} = \bigvee \{x_3 | x_3 \in J^\infty(\mathbb{C}_3) \;\&\; u \circ v \geq x_3\}$.

Therefore, by 1, $u \circ v = \bigvee \{x_1 | x_1 \in J^\infty(\mathbb{C}_1) \text{ and } u \geq x_1\} \circ \bigvee \{x_2 | x_2 \in J^\infty(\mathbb{C}_2) \text{ and } v \geq x_2\}$, and we choose the representation in terms of the join of lower generators, because $\circ$ is join preserving on both coordinates, which allows us to write

$$\begin{aligned} u \circ v &= \bigvee \{x_1 | x_1 \in J^\infty(\mathbb{C}_1) \text{ and } u \geq x_1\} \circ \bigvee \{x_2 | x_2 \in J^\infty(\mathbb{C}_2) \text{ and } v \geq x_2\} \\ &= \bigvee \{x_1 \circ x_2 | x_1 \in J^\infty(\mathbb{C}_1) \text{ and } x_2 \in J^\infty(\mathbb{C}_2) \text{ and } v \geq x_2 \text{ and } u \geq x_1\} \end{aligned}$$

This reduces the problem of knowing where an arbitrary element (u,v) of $\mathbb{C}_1 \times \mathbb{C}_2$ is mapped to the problem of knowing where the generators of each coordinate are mapped. There is no guarantee that for a given pair of generators (x_1, x_2) that the value $x_1 \circ x_2$ will still be a generator in $\mathbb{C}_3$. Therefore we need to use 2. to approximate this value $x_1 \circ x_2$ in terms of generators in $\mathbb{C}_3$. Such approximation provides us with the condition associated to the relational dual of $\circ$, since $x_1 \circ x_2 = \bigwedge \{y_3 | y_3 \in M^\infty(\mathbb{C}_3) \text{ and } x_1 \circ x_2 \leq y_3\}$ then:

$$R_\circ (y_3, x_1, x_2) \text{ iff } x_1 \circ x_2 \leq y_3 \tag{5.1.6}$$

We continue with implication. Given $\rightarrow : \begin{cases} \left(\mathbb{C}_1^\partial \times \mathbb{C}_2\right) \longrightarrow \mathbb{C}_3 \\ (u,v) \longmapsto u \rightarrow v \end{cases}$ the corresponding relation has the shape $R_\rightarrow \subseteq \mathbb{C}_1 \times \mathbb{C}_3 \times \mathbb{C}_2$. Because $\rightarrow$ is antitone on the first coordinate and monotone on the second, we flip $\mathbb{C}_1$ order to simulate an operation which is monotone (in fact, meet-preserving) in both coordinates. As before, all $\mathbb{C}_1, \mathbb{C}_2$ and $\mathbb{C}_3$ are perfect, which means both have a set of generators for each.

(1) for any $u \in \mathbb{C}_1$ and any $v \in \mathbb{C}_2$ we have
   (a) $u = \bigwedge \{y_1 | y_1 \in M^\infty(\mathbb{C}_1) \text{ and } u \leq y_1\} = \bigvee \{x_1 | x_1 \in J^\infty(\mathbb{C}_1) \text{ and } u \geq x_1\}$ and
   (b) $v = \bigwedge \{y_2 | y_2 \in M^\infty(\mathbb{C}_2) \text{ and } v \leq y_2\} = \bigvee \{x_2 | x_2 \in J^\infty(\mathbb{C}_2) \text{ and } v \geq x_2\}$
(2) for any $u \rightarrow v \in \mathbb{C}_3 :$ $u \rightarrow v = \bigwedge \{y_3 | y_3 \in M^\infty(\mathbb{C}_3) \;\&\; u \rightarrow v \leq y_3\} = \bigvee \{x_3 | x_3 \in J^\infty(\mathbb{C}_3) \;\&\; x_3 \leq u \rightarrow v\}$.

Therefore, by 1, $u \rightarrow v = \bigvee \{x_1 | x_1 \in J^\infty(\mathbb{C}_1) \text{ and } u \geq x_1\} \rightarrow \bigwedge \{y_2 | y_2 \in M^\infty(\mathbb{C}_2) \text{ and } v \leq y_2\}$, and we choose the representation in terms of the join of lower generators and the meet of upper generators, because $\rightarrow : \left(\mathbb{C}_1^\partial \times \mathbb{C}_2\right) \longrightarrow \mathbb{C}_3$ is meet preserving on both coordinates (with the first coordinate turning into a meet first by the order flip, as it is a join in $\mathbb{C}_1$ but a meet in $\mathbb{C}_1^\partial$), which allows us to push $\rightarrow$ inside.

$$\begin{aligned} u \rightarrow v &= \bigvee \{x_1 | x_1 \in J^\infty(\mathbb{C}_1) \text{ and } u \geq x_1\} \rightarrow \bigwedge \{y_2 | y_2 \in M^\infty(\mathbb{C}_2) \text{ and } v \leq y_2\} \\ &= \bigvee \{x_1 \rightarrow y_2 | x_1 \in J^\infty(\mathbb{C}_1) \text{ and } y_2 \in M^\infty(\mathbb{C}_2) \text{ and } u \geq x_1 \text{ and } v \leq y_2\} \end{aligned}$$

This reduces the problem of knowing where an arbitrary element (u,v) of $\mathbb{C}_1^\partial \times \mathbb{C}_2$ is mapped to the problem of knowing where the generators of each coordinate are mapped. There is no guarantee that for a given pair of generators (x_1, y_2) that the value $x_1 \rightarrow y_2$ will still be a generator in $\mathbb{C}_3$. Therefore we need to use 2. to approximate this value $x_1 \rightarrow y_2$ in terms of generators in $\mathbb{C}_3$. Such approximation provides us with the condition associated to the relational dual of $\rightarrow$, since $x_1 \rightarrow y_2 = \bigvee \{x_3 | x_3 \in J^\infty(\mathbb{C}_3) \text{ and } x_3 \leq x_1 \rightarrow y_2\}$ then:

$$R_\circ (x_3, x_1, y_2) \text{ iff } R_\rightarrow (x_1, x_3, y_2) \text{ iff } x_3 \leq x_1 \rightarrow y_2 \tag{5.1.7}$$

We continue with coimplication. Let $\leftarrow : \begin{cases} (\mathbb{C}_2 \times \mathbb{C}_1^\partial) \longrightarrow \mathbb{C}_3 \\ (u,v) \longmapsto u \leftarrow v \end{cases}$ then its dual is the corresponding relation $R_\leftarrow \subseteq \mathbb{C}_3 \times (\mathbb{C}_2 \times \mathbb{C}_1)$. Because $\leftarrow$ is antitone on the second coordinate and monotone on the first, we flip $\mathbb{C}_1$ order to simulate an operation which is monotone (in fact, meet-preserving) in both coordinates. The value of arbitrary elements are represented in terms of the join of lower generators and meet of upper generators, because $\leftarrow : (\mathbb{C}_2 \times \mathbb{C}_1^\partial) \longrightarrow \mathbb{C}_3$ is join preserving on both coordinates (with the second coordinate turning into a join first by the order flip, as it is a meet in $\mathbb{C}_1$ but a join in $\mathbb{C}_1^\partial$), which allows us to push $\leftarrow$ inside.

$$\begin{aligned} u \leftarrow v &= \bigwedge \{y_2 | y_2 \in M^\infty(\mathbb{C}_2) \text{ and } u \leq y_2\} \leftarrow \bigvee \{x_1 | x_1 \in J^\infty(\mathbb{C}_1) \text{ and } v \geq x_1\} \\ &= \bigvee \{y_2 \leftarrow x_1 | x_1 \in J^\infty(\mathbb{C}_1) \text{ and } y_2 \in M^\infty(\mathbb{C}_2) \text{ and } u \leq y_2 \text{ and } v \geq x_1\} \end{aligned}$$

There is no guarantee that for a given pair of generators (y_2, x_1) that the value $y_2 \leftarrow x_1$ will still be a generator in $\mathbb{C}_3$. Therefore we approximate $y_2 \leftarrow x_1$ in terms of generators in $\mathbb{C}_3$. Such approximation provides us with the condition associated to the relational dual of $\leftarrow$, since $y_2 \leftarrow x_1 = \bigvee \{x_3 | x_3 \in J^\infty(\mathbb{C}_3) \text{ and } x_3 \leq y_2 \leftarrow x_1\}$ then:

$$R_\circ (x_3, y_2, x_1) \text{ iff } R_\leftarrow (x_3, x_1, y_2) \text{ iff } x_3 \leq y_2 \leftarrow x_1 \tag{5.1.8}$$

REMARK 109. As the complex algebra $\mathbb{F}^+$ is given, so are all the operations on it. We can thus rely on them to define the corresponding relations on the frame $\mathbb{F}$. This is what we just did with the correspondence equivalences from 5.1.2 to 5.1.8. Now we can build upon them the dual of the algebraic interpretation, which will be done in the next section.

5.1.4.2. *Dualization of interpretation.* The standard habit grown in the classical setting is to dualize from frames to algebras, but when reaching a sufficiently general setting, as the non-distributive one, it is no longer clear how will the relational semantics behave. On the algebraic side, however, the semantics is still reasonably transparent since the operations and the building blocks involved match the general pattern found in the Boolean and distributive settings and thus the overall jump in complexity is not a radical one. For this reason, the natural method to approach the subject is to stand on the algebraic side where the terrain is fairly familiar and then dualize to the relational side, hoping to reach a more manageable understanding on the new relational semantics.

This dualization works through correspondence theory: in the algebra side properties are captured by equations in the Boolean case (or quasi-equations in the non-Boolean generalization). Such (quasi-)equations then correspond to a first order formula in the language of frames. This first order formula provides the conditions of satisfaction of the associated modal formula. Such method is what will be described here, but in a nutshell the method is: define a mapping from propositions into the complex algebra $\mathbb{F}^+$ and then dualize.

REMARK 110. We will have to deal with *satisfaction* and *co-satisfaction*, however it turns out that these two parts of the dual of *interpretation* are not symmetric. The operations that are left adjoints will have co-satisfaction as primitive and satisfaction as derived, while operations that are right adjoints will have satisfaction as primitive and co-satisfaction as derived.

Let us now apply the same trick used in section 5.1.2. to the non-distributive setting to obtain the frame satisfaction relation from the interpretation on the algebra side. In this setting, the interpretation V can be systematically dualized into a tuple of relations $(\Vdash_V, \succ_V)$ with $\Vdash_V \subseteq J^\infty(\mathbb{F}^+) \times Fm$ and $\succ_V \subseteq M^\infty(\mathbb{F}^+) \times Fm$.

We will have to modify slightly our approach, however, because by dropping distributivity the completely join-irreducible elements are no longer guaranteed to be completely join-prime. We used this crucial property in two steps (a & b) of our distributive variant of the proof. Fortunately, by using the approximation from above (i.e. using $M^\infty(\mathbb{F}^+)$) instead of the one from below, this gap can be circumvented.

As expected, for the basic case, for every $x \in J^\infty(\mathbb{F}^+)$ and every $y \in M^\infty(\mathbb{F}^+)$ and every $p \in \mathsf{AtProp}$ we define:[5]

$$x \Vdash p \quad \text{iff} \quad x \leq V(p) \tag{5.1.9}$$

$$y \succ p \quad \text{iff} \quad y \geq V(p) \tag{5.1.10}$$

Now we will treat each inductive step to retrieve the (co-)satisfaction conditions for each modal operator. We will make use of the following facts:

(1) $\mathbb{F}^+ = \left\langle \mathcal{G}(\mathbb{F}), \wedge, \vee, \varnothing, X, \Diamond^{\mathbb{F}^+}, \Box^{\mathbb{F}^+}, \rhd^{\mathbb{F}^+}, \lhd^{\mathbb{F}^+}, \circ^{\mathbb{F}^+}, \rightarrow^{\mathbb{F}^+}, \leftarrow^{\mathbb{F}^+} \right\rangle$ is a *perfect* (substructural-modal) lattice[6], where $\mathcal{G}(\mathbb{F})$ is the family of Galois-stable sets of the RS-polarity (X, Y, R) which constitutes $\mathbb{F}$'s universe, and $\wedge = \bigcap$ but $\vee = c_R \circ \bigcup$ (cf. Def. 86).

(2) by 1, each element in $\mathcal{G}(\mathbb{F})$ is representable as the join of all join-irreducible elements below it or equivalently as the meet of meet-irreducibles above it.

(3) $J^\infty(\mathbb{F}^+) = \{c_R(x) \mid x \in X\}$ and $M^\infty(\mathbb{F}^+) = \{l_R(y) \mid y \in Y\}$ hence in (4.2.2) x on the algebraic side is the downset $c_R(x)$.[7]

(4) for all $S \in \mathbb{F}^+$; $\Diamond^{\mathbb{F}^+}(S) = R_\Diamond^{-1}[S]$ where $R_\Diamond$ is the accessibility relation that interprets $\Diamond$ on the frame $\mathbb{F}$; $\Box^{\mathbb{F}^+}(S) = \left(R_\Box^{-1}[S^c]\right)^c$ where $R_\Box$ is the accessibility relation that interprets $\Box$ on the frame $\mathbb{F}$; $\lhd^{\mathbb{F}^+}(S) = R_\lhd^{-1}[S^c]$ where $R_\lhd$ is the accessibility relation that interprets $\lhd$ on the frame $\mathbb{F}$; $\rhd^{\mathbb{F}^+}(S) = \left(R_\rhd^{-1}[S]\right)^c$ where $R_\rhd$ is the accessibility relation that interprets $\rhd$ on the frame $\mathbb{F}$.

(5) $V(\psi) \in \mathbb{F}^+$ for all ψ.

[5]Throughout what remains of 4.2.4. we will assume that all variables x range over $J^\infty(\mathbb{F}^+)$, and all variables y range over $M^\infty(\mathbb{F}^+)$

[6]Therefore, the following holds for any $S, S' \subseteq \mathcal{G}(\mathbb{F})$:

- $\Diamond(\bigvee S) = \bigvee(\Diamond S) = \bigvee\{\Diamond u \mid u \in S\}$
- $\Box(\bigwedge S) = \bigwedge(\Box S) = \bigwedge\{\Box u \mid u \in S\}$
- $\rhd(\bigvee S) = \bigwedge(\rhd S) = \bigwedge\{\rhd u \mid u \in S\}$
- $\lhd(\bigwedge S) = \bigvee(\lhd S) = \bigvee\{\lhd u \mid u \in S\}$
- $\bigvee S \circ \bigvee S' = \bigvee(S \circ S') = \bigvee\{s \circ s' \mid s \in S \,\&\, s' \in S'\}$
- $\bigvee S \rightarrow \bigwedge S' = \bigvee(S \rightarrow S') = \bigvee\{s \rightarrow s' \mid s \in S \,\&\, s' \in S'\}$
- $\bigwedge S \leftarrow \bigvee S' = \bigvee(S \leftarrow S') = \bigvee\{s \leftarrow s' \mid s \in S \,\&\, s' \in S'\}$

[7]In fact, X join generates $\mathbb{F}^+$ and Y meet generates $\mathbb{F}^+$, as can be checked in [**Gool 2009**]'s Proposition 2.2.3. or in [**Gehrke 2006**] Proposition 2.10.

(6) by 1, $\Diamond^{\mathbb{F}^+}$ is completely join-preserving, $\Box^{\mathbb{F}^+}$ is completely meet-preserving, $\rhd^{\mathbb{F}^+}$ is completely join-reversing and $\lhd^{\mathbb{F}^+}$ is completely meet-reversing.

(7) Since by assumption V is a homomorphism, then $V(\Diamond\varphi) = \Diamond^{\mathbb{F}^+}V(\varphi)$, $V(\Box\varphi) = \Box^{\mathbb{F}^+}V(\varphi)$, $V(\rhd\varphi) = \rhd^{\mathbb{F}^+}V(\varphi)$ and $V(\lhd\varphi) = \lhd^{\mathbb{F}^+}V(\varphi)$. Also, $V(\varphi\circ\psi) = V(\varphi)\circ^{\mathbb{F}^+}V(\psi)$, $V(\varphi\to\psi) = V(\varphi)\to^{\mathbb{F}^+}V(\psi)$ and $V(\varphi\leftarrow\psi) = V(\varphi)\leftarrow^{\mathbb{F}^+}V(\psi)$.

Let us start with the inductive step for the satisfaction of $\Diamond$.

Inductive step for the satisfaction of $\Diamond$:

$$x \Vdash \Diamond\psi \quad \text{iff} \quad x \leq V(\Diamond\psi)$$

Then we have:

$$\begin{array}{llll}
x \Vdash \Diamond\psi \quad \text{iff} \quad x \leq V(\Diamond\psi) & \text{iff} & x \leq \bigwedge\{y \in M^\infty(\mathbb{F}^+) \mid V(\Diamond\psi) \leq y\} & \\
& \text{iff} & \forall y(V(\Diamond\psi) \leq y \;\Rightarrow\; x \leq y) & \text{(a)}\\
& \text{iff} & \forall y(\bigvee\{\Diamond^{\mathbb{F}^+}x' \mid x' \in J^\infty(\mathbb{F}^+) \text{ and } x' \leq V(\psi)\} \leq y \;\Rightarrow\; x \leq y) & \text{(b)}\\
& \text{iff} & \forall y\left[\forall x'\left[x' \leq V(\psi) \;\Rightarrow\; \Diamond^{\mathbb{F}^+}x' \leq y\right] \;\Rightarrow\; x \leq y\right] & \text{(c)}\\
& \text{iff} & \forall y[\forall x'[x' \Vdash \psi \;\Rightarrow\; yR_\Diamond x'] \;\Rightarrow\; x \leq y] & \text{(d)}
\end{array}$$

We expand the equivalences as follows. The first added equivalence relies on the fact 2., namely, that our complex algebra $\mathbb{F}^+$ is a perfect lattice and thus has two sets of generators, $M^\infty(\mathbb{F}^+)$ and $J^\infty(\mathbb{F}^+)$. In fact that is precisely the reason why we use $M^\infty(\mathbb{F}^+)$: because of its generation properties, while the additional fact that these elements are completely meet irreducibles is not used at all. Notice that since we have established that all variables x range over $J^\infty(\mathbb{F}^+)$, and all variables y range over $M^\infty(\mathbb{F}^+)$, we can simply omit the explicit mention of this fact for a shorter notation. So we have $V(\Diamond\psi) = \bigwedge\{y \in M^\infty(\mathbb{F}^+) \mid V(\Diamond\psi) \leq y\}$ and we may abbreviate as $\bigwedge\{y \mid V(\Diamond\psi) \leq y\}$. Then (a) stems from order-theoretic properties of meets. For (b) we observe that the equality $V(\Diamond\psi) = \Diamond^{\mathbb{F}^+}V(\psi) = \Diamond^{\mathbb{F}^+}\bigvee\{x' \in J^\infty(\mathbb{F}^+) \mid x' \leq V(\psi)\} = \bigvee\left\{\Diamond^{\mathbb{F}^+}x' \mid x' \in J^\infty(\mathbb{F}^+) \text{ and } x' \leq V(\psi)\right\}$ is still valid in the non-distributive setting, since it only uses the fact that $\Diamond^{\mathbb{F}^+}$ is completely join-preserving and that V is a homomorphism. We obtain (c) by order-theoretic properties of joins and notational abbreviation. By IH (5.1.9) we have $x' \leq V(\psi)$ iff $x' \Vdash \psi$ and (5.1.2) gives us the remaining step to get (d)

Inductive step for the co-satisfaction of $\Diamond$:

$$y \succ \Diamond\psi \quad \text{iff} \quad y \geq V(\Diamond\psi)$$

The case $y \geq V(\Diamond\psi)$ has already been developed above (a)-(d) as the antecedent of a conditional. Thus we have

$$y \geq V(\Diamond\psi) \quad \text{iff} \quad \forall x'[x' \Vdash \psi \;\Rightarrow\; yR_\Diamond x']$$

Remark 111. We have started the diamond treatment by its satisfaction relation to parallel the distributive setting, but notice that it is the co-satisfaction relation that is primitive in this case while the satisfaction is derived from it. Indeed, the condition to satisfy a diamond should be written as $x \leq V(\Diamond\psi)$ iff $\forall y[y \succ \Diamond\psi \;\Rightarrow\; x \leq y]$ where one can clearly see the dependency.

Now we continue with $\Box$, but this time we start with the main component of the dual of $\Box$-interpretation (satisfaction) and leave the derived component ($\Box$-co-satisfaction) for afterwards.

INDUCTIVE STEP FOR THE SATISFACTION OF $\Box$:

$$x \Vdash \Box\psi \quad \text{iff} \quad x \leq V(\Box\psi)$$

Then we have:

$$\begin{array}{lllll} x \Vdash \Box\psi & \text{iff} & x \leq V(\Box\psi) & \text{iff} & x \leq \bigwedge\{\Box^{\mathbb{F}^+} y | y \in M^\infty(\mathbb{F}^+) \text{ and } V(\psi) \leq y\} \\ & & & \text{iff} & \forall y (V(\psi) \leq y \ \Rightarrow \ x \leq \Box^{\mathbb{F}^+} y) \qquad \text{(a)} \\ & & & \text{iff} & \forall y (y \succ \psi \ \Rightarrow \ x R_\Box y) \qquad \text{(b)} \end{array}$$

The first equivalence added relies on the representation of $V(\psi)$ as meet of upper generators, giving $V(\psi) = \bigwedge\{y | y \in M^\infty(\mathbb{F}^+) \text{ and } V(\psi) \leq y\}$, and since V is an homomorphism and $\Box$ preserves all meets, then: $V(\Box\psi) = \Box^{\mathbb{F}^+} V(\psi) = \Box^{\mathbb{F}^+} \bigwedge\{y | y \in M^\infty(\mathbb{F}^+) \text{ and } V(\psi) \leq y\} = \bigwedge\{\Box^{\mathbb{F}^+} y | y \in M^\infty(\mathbb{F}^+) \text{ and } V(\psi) \leq y\} = \bigwedge\{\Box^{\mathbb{F}^+} y | V(\psi) \leq y\}$ with the last equality amounting to a notational abbreviation, which we make effective in the next step. (a) stems from order-theoretic properties of meets, and (b) relies on the IH (5.1.10) for the antecedent and on (5.1.4) for the consequent.

INDUCTIVE STEP FOR THE CO-SATISFACTION OF $\Box$:

$$y \succ \Box\psi \quad \text{iff} \quad y \geq V(\Box\psi)$$

Then we have:

$$\begin{array}{lllll} y \succ \Box\psi & \text{iff} & y \geq V(\Box\psi) & \text{iff} & y \geq \bigvee\{x \in J^\infty(\mathbb{F}^+) \mid V(\Box\psi) \geq x\} \\ & & & \text{iff} & \forall x (V(\Box\psi) \geq x \ \Rightarrow \ x \leq y) \qquad \text{(a)} \\ & & & \text{iff} & \forall x (x \Vdash \Box\psi \ \Rightarrow \ x \leq y) \qquad \text{(b)} \end{array}$$

The first equivalence added relies on the representation of $V(\Box\psi)$ as join of lower generators, then (a) stems from order-theoretic properties of joins and notational abbreviation (x is assumed to be lower generator), and finally (b) relies on the satisfaction relation for $\Box$ which was just defined above.

We continue with $\lhd$, starting with the main component of the dual of $\lhd$-interpretation (co-satisfaction) while the derived component ($\lhd$-satisfaction) is left for afterwards.

INDUCTIVE STEP FOR THE CO-SATISFACTION OF $\lhd$:

$$y \succ \lhd\psi \quad \text{iff} \quad y \geq V(\lhd\psi)$$

Then we have: $y \succ \lhd\psi \quad \text{iff} \quad \forall y' \, [(y' \succ \psi) \Rightarrow y R_\lhd y']$

$$\begin{array}{lllll} y \succ \lhd\psi & \text{iff} & y \geq V(\lhd\psi) & \text{iff} & y \geq \bigvee\{\lhd^{\mathbb{F}^+} y' | y' \in M^\infty(\mathbb{F}^+) \text{ and } V(\psi) \leq y'\} \\ & & & \text{iff} & \forall y' \left[V(\psi) \leq y' \ \Rightarrow \ \lhd^{\mathbb{F}^+} y' \leq y\right] \qquad \text{(a)} \\ & & & \text{iff} & \forall y' \, [y' \succ \psi \ \Rightarrow \ y R_\lhd y'] \qquad \text{(b)} \end{array}$$

The argument proceeds as before, just recall that $\lhd$ turns meets into joins, thus: $V(\lhd\psi) = \lhd^{\mathbb{F}^+} V(\psi) = \lhd^{\mathbb{F}^+} \bigwedge\{y | y \in M^\infty(\mathbb{F}^+) \text{ and } V(\psi) \leq y\} = \bigvee\{\lhd^{\mathbb{F}^+} y | y \in M^\infty(\mathbb{F}^+) \text{ and } V(\psi) \leq y\} = \bigwedge\{\lhd^{\mathbb{F}^+} y | V(\psi) \leq y\}$. Last line uses (5.1.10) as IH for the antecedent and (5.1.3) for the consequent.

INDUCTIVE STEP FOR THE SATISFACTION OF $\lhd$:

$$x \Vdash \lhd\psi \quad \text{iff} \quad x \leq V(\lhd\psi)$$

Then we have:

$$
\begin{aligned}
x \Vdash \lhd\psi \quad \text{iff} \quad x \leq V(\lhd\psi) \quad & \text{iff} \quad x \leq \bigwedge\{y' | y' \in M^\infty(\mathbb{F}^+) \text{ and } V(\lhd\psi) \leq y'\} \\
& \text{iff} \quad \forall y' \, [V(\lhd\psi) \leq y' \;\Rightarrow\; x \leq y'] \\
& \text{iff} \quad \forall y' \, [y' \succ \lhd\psi \;\Rightarrow\; x \leq y']
\end{aligned}
$$

The reader may notice how the entire previous unraveling of co-satisfaction conditions could be plugged in as equivalent of $y' \succ \lhd\psi$ in the antecedent of the last line. Now we continue with $\rhd$, the main component of the dual of $\rhd$-interpretation being the satisfaction relation while co-satisfaction is derived.

Inductive step for the satisfaction of $\rhd$:

$$x \Vdash \rhd\psi \quad \text{iff} \quad x \leq V(\rhd\psi)$$

Then we have:

$$
\begin{aligned}
x \Vdash \rhd\psi \quad \text{iff} \quad x \leq V(\rhd\psi) \quad & \text{iff} \quad x \leq \bigwedge\{\rhd^{\mathbb{F}^+} x' | x' \in J^\infty(\mathbb{F}^+) \text{ and } V(\psi) \geq x'\} \\
& \text{iff} \quad \forall x' (V(\psi) \geq x' \;\Rightarrow\; x \leq \rhd^{\mathbb{F}^+} x') \\
& \text{iff} \quad \forall x' \, [x' \Vdash \psi \Rightarrow x R_{\rhd} x']
\end{aligned}
$$

For the last line we use (5.1.9) as IH for the antecedent and (5.1.5) for the consequent.

Inductive step for the co-satisfaction of $\rhd$:

$$y \succ \rhd\psi \quad \text{iff} \quad y \geq V(\rhd\psi)$$

Then we have:

$$
\begin{aligned}
y \succ \rhd\psi \quad \text{iff} \quad y \geq V(\rhd\psi) \quad & \text{iff} \quad y \geq \bigvee\{x' \in J^\infty(\mathbb{F}^+) \mid V(\rhd\psi) \geq x'\} \\
& \text{iff} \quad \forall x' (V(\rhd\psi) \geq x' \;\Rightarrow\; x' \leq y) \\
& \text{iff} \quad \forall x' \, [x' \Vdash \rhd\psi \Rightarrow y \geq x']
\end{aligned}
$$

The conditions of satisfaction for $\rhd$ have been established previously so the antecedent of the conditional in the last line can be unraveled properly. Now we continue with $\circ$, the main component being the co-satisfaction relation, but to parallel the unary diamond we will start with the satisfaction conditions.

Inductive step for the satisfaction of $\circ$:

$$x \Vdash \varphi \circ \psi \quad \text{iff} \quad x \leq V(\varphi \circ \psi)$$

$$
\begin{aligned}
x \Vdash \varphi \circ \psi \quad \text{iff} \quad x \leq V(\varphi \circ \psi) \quad & \text{iff} \quad x \leq \bigwedge\{y \mid y \in M^\infty(\mathbb{F}^+) \text{ and } V(\varphi \circ \psi) \leq y\} \\
& \text{iff} \quad \forall y \, [V(\varphi \circ \psi) \leq y \Rightarrow x \leq y] && \text{(a)} \\
& \text{iff} \quad \forall y \left[\bigvee\{x_1 \circ^{\mathbb{F}^+} x_2 \mid x_1 \leq V(\varphi) \text{ and } x_2 \leq V(\psi)\} \leq y \Rightarrow x \leq y\right] && \text{(b)} \\
& \text{iff} \quad \forall y \left[\forall x_1, x_2 \left[(x_1 \leq V(\varphi) \text{ and } x_2 \leq V(\psi)) \Rightarrow x_1 \circ^{\mathbb{F}^+} x_2 \leq y\right] \Rightarrow x \leq y\right] && \text{(c)} \\
& \text{iff} \quad \forall y \left[\forall x_1, x_2 \left[(x_1 \Vdash \varphi \text{ and } x_2 \Vdash \psi) \Rightarrow x_1 \circ^{\mathbb{F}^+} x_2 \leq y\right] \Rightarrow x \leq y\right] && \text{(d)} \\
& \text{iff} \quad \forall y \, [\forall x_1, x_2 \, [(x_1 \Vdash \varphi \text{ and } x_2 \Vdash \psi) \Rightarrow R_\circ(y, x_1, x_2)] \Rightarrow x \leq y] && \text{(e)}
\end{aligned}
$$

This is entirely analogue to the $\Diamond$ case above. Just notice that V is an homomorphism and thus $V(\varphi \circ \psi) = V(\varphi) \circ^{\mathbb{F}^+} V(\psi)$, and both $V(\varphi)$ and $V(\psi)$ lie in the algebra, which is perfect. This allows us to represent $V(\varphi)$ and $V(\psi)$ in terms of generators, either from above or from below. In this case, since $\circ^{\mathbb{F}^+}$ is join preserving in both coordinates, the representation as a join is the most convenient. Thus, $V(\varphi \circ \psi) = V(\varphi) \circ^{\mathbb{F}^+} V(\psi) = (\bigvee\{x_1 \mid x_1 \in J^\infty(\mathbb{F}^+) \text{ and } V(\varphi) \geq x_1\}) \circ^{\mathbb{F}^+} (\bigvee\{x_2 \mid x_2 \in J^\infty(\mathbb{F}^+) \text{ and } V(\psi) \geq x_2\}) = \bigvee\{x_1 \circ^{\mathbb{F}^+} x_2 \mid V(\varphi) \geq x_1 \text{ and } V(\psi) \geq x_2\}$(a) simply translates the order theoretic properties of meets (x is below a meet iff it is below all its elements), and (c) does the same but this time order-dually (y is above a join iff it is above all its elements). (d) follows from the induction hypothesis and (e) relies on the dualization to the frame side of the function $\circ^{\mathbb{F}^+}$ as developed in (5.1.6).

INDUCTIVE STEP FOR THE CO-SATISFACTION OF $\circ$:

$$y \succ \varphi \circ \psi \quad \text{iff} \quad y \geq V(\varphi \circ \psi)$$

The case $y \geq V(\varphi \circ \psi)$ was already developed above (a)-(e) as the antecedent of a conditional. Thus we have:

$$y \succ \varphi \circ \psi \quad \text{iff} \quad \forall x_1, x_2\, [(x_1 \Vdash \varphi \text{ and } x_2 \Vdash \psi) \Rightarrow R_\circ\, (y, x_1, x_2)]$$

INDUCTIVE STEP FOR THE SATISFACTION OF $\to$:

$$x \Vdash \varphi \to \psi \quad \text{iff} \quad x \leq V(\varphi \to \psi)$$

$$\begin{aligned}
x \Vdash \varphi \to \psi \quad \text{iff} \quad x \leq V(\varphi \to \psi) \quad & \text{iff} \quad x \leq \bigwedge \left\{x' \to^{\mathbb{F}^+} y \,|\, x' \leq V(\varphi) \text{ and } y \geq V(\psi)\right\} \\
& \text{iff} \quad \forall x', y \left[[x' \leq V(\varphi) \text{ and } y \geq V(\psi)] \Rightarrow x \leq x' \to^{\mathbb{F}^+} y\right] \qquad \text{(a)} \\
& \text{iff} \quad \forall x', y\, [[x' \Vdash \varphi \text{ and } y \succ \psi] \Rightarrow R_\to\, (x', x, y)] \qquad \text{(b)}
\end{aligned}$$

First observe that:

$V(\varphi \to \psi) = V(\varphi)^\partial \to^{\mathbb{F}^+} V(\psi) = \bigwedge\{x' \mid x' \in J^\infty(\mathbb{F}^+) \text{ and } x' \leq V(\varphi)\} \to^{\mathbb{F}^+} \bigwedge\{y \mid y \in M^\infty(\mathbb{F}^+) \text{ and } y \geq V(\psi)\} = \bigwedge \left\{x' \to^{\mathbb{F}^+} y \,|\, x' \leq V(\varphi) \text{ and } y \geq V(\psi)\right\}$

Then, the last lines uses the IH on the antecedent and relies on (5.1.7) for the consequent.

INDUCTIVE STEP FOR THE CO-SATISFACTION OF $\to$:

$$y \succ \varphi \to \psi \quad \text{iff} \quad y \geq V(\varphi \to \psi)$$

Then we have:

$$\begin{aligned}
y \succ \varphi \to \psi \quad \text{iff} \quad y \geq V(\varphi \to \psi) \quad & \text{iff} \quad y \geq \bigvee\{x \mid x \in J^\infty(\mathbb{F}^+) \text{ and } x \leq V(\varphi \to \psi)\} \\
& \text{iff} \quad \forall x\, [x \leq V(\varphi \to \psi) \Rightarrow x \leq y] \\
& \text{iff} \quad \forall x\, [x \Vdash \varphi \to \psi \Rightarrow x \leq y]
\end{aligned}$$

Where $x \Vdash \varphi \to \psi$ in the last line can be replaced by the associated satisfaction conditions as established above.

INDUCTIVE STEP FOR THE SATISFACTION OF $\leftarrow$:

$$x \Vdash \psi \leftarrow \varphi \quad \text{iff} \quad x \leq V(\psi \leftarrow \varphi)$$

$$\begin{aligned}
x \Vdash \psi \leftarrow \varphi \quad \text{iff} \quad x \leq V(\psi \leftarrow \varphi) \quad & \text{iff} \quad x \leq \bigwedge \left\{y \leftarrow^{\mathbb{F}^+} x' \,|\, x' \leq V(\varphi) \text{ and } y \geq V(\psi)\right\} \\
& \text{iff} \quad \forall x', y \left[[x' \leq V(\varphi) \text{ and } y \geq V(\psi)] \Rightarrow x \leq y \leftarrow^{\mathbb{F}^+} x'\right] \\
& \text{iff} \quad \forall x', y\, [[x' \Vdash \varphi \text{ and } y \succ \psi] \Rightarrow R_\leftarrow\, (x, x', y)]
\end{aligned}$$

First observe that:

$V(\psi \leftarrow \varphi) = V(\psi) \leftarrow^{\mathbb{F}^+} V(\varphi)^\partial = \bigwedge\{y \mid y \in M^\infty(\mathbb{F}^+) \text{ and } y \geq V(\psi)\} \leftarrow^{\mathbb{F}^+} \bigwedge\{x' \mid x' \in J^\infty(\mathbb{F}^+) \text{ and } x' \leq V(\varphi)\} = \bigwedge \left\{y \leftarrow^{\mathbb{F}^+} x' \,|\, x' \leq V(\varphi) \text{ and } y \geq V(\psi)\right\}$

Then, the last lines uses the IH on the antecedent and relies on (5.1.8) for the consequent.

Inductive step for the co-satisfaction of $\leftarrow$:

$$y \succ \psi \leftarrow \varphi \quad \text{iff} \quad y \geq V(\psi \leftarrow \varphi)$$

Then we have:

$$\begin{aligned} y \succ \psi \leftarrow \varphi \quad \text{iff} \quad y \geq V(\psi \leftarrow \varphi) \quad & \text{iff} \quad y \geq \bigvee\{x \mid x \in J^{\infty}(\mathbb{F}^{+}) \text{ and } x \leq V(\psi \leftarrow \varphi)\} \\ & \text{iff} \quad \forall x\,[x \leq V(\psi \leftarrow \varphi) \Rightarrow x \leq y] \\ & \text{iff} \quad \forall x\,[x \Vdash \psi \leftarrow \varphi \Rightarrow x \leq y] \end{aligned}$$

Where $x \Vdash \psi \leftarrow \varphi$ in the last line can be replaced by the associated satisfaction conditions as established above.

Given the previous unraveling of satisfaction conditions, for any given interpretation $V : \mathsf{AtProp} \longrightarrow \mathcal{G}((X, Y, \leq))$ into the complex algebra $\mathbb{F}^{+}$, and with $\mathbb{M} = (\mathbb{F}, V)$ and $\mathbb{F} = ((X, Y, \leq), R_{\Box}, R_{\Diamond}, R_{\rhd}, R_{\lhd}, R_{\circ}, R_{\rightarrow}, R_{\leftarrow})$, we can define the associated two-sorted relational semantics by induction as follows: for $x \in X$, $y \in Y$ and $p \in \mathsf{AtProp}$ we let $\begin{cases} \mathbb{M}, x \Vdash p & \text{iff} \quad x \leq V(p) \\ \mathbb{M}, y \succ p & \text{iff} \quad y \geq V(p) \end{cases}$. For $\varphi, \psi \in Form(\mathsf{AtProp})$

$$\mathbb{M}, y \succ_V \Diamond\psi \;\text{ iff }\; \forall x'[x' \Vdash_V \psi \;\Rightarrow\; yR_{\Diamond}x'] \tag{5.1.11}$$

$$\mathbb{M}, x \Vdash_V \Diamond\psi \;\text{ iff }\; \forall y[y \succ_V \Diamond\psi \;\Rightarrow\; x \leq y]. \tag{5.1.12}$$

$$\mathbb{M}, x \Vdash_V \Box\psi \;\text{ iff }\; \forall y[y \succ_V \psi \;\Rightarrow\; xR_{\Box}y]. \tag{5.1.13}$$

$$\mathbb{M}, y \succ_V \Box\psi \;\text{ iff }\; \forall x[x \Vdash_V \Box\psi \Rightarrow\; x \leq y]. \tag{5.1.14}$$

$$\mathbb{M}, y \succ_V \lhd\psi \;\text{ iff }\; \forall y'[y' \succ \psi \;\Rightarrow\; yR_{\lhd}y']. \tag{5.1.15}$$

$$\mathbb{M}, x \Vdash_V \lhd\psi \;\text{ iff }\; \forall y[y \succ_V \lhd\psi \;\Rightarrow\; x \leq y]. \tag{5.1.16}$$

$$\mathbb{M}, x \Vdash_V \rhd\psi \;\text{ iff }\; \forall x'[x' \Vdash_V \psi \;\Rightarrow\; xR_{\rhd}x']. \tag{5.1.17}$$

$$\mathbb{M}, y \succ_V \rhd\psi \;\text{ iff }\; \forall x[x \Vdash_V \rhd\psi \Rightarrow\; x \leq y]. \tag{5.1.18}$$

$$\mathbb{M}, y \succ_V \varphi \circ \psi \quad \text{iff} \quad \forall x_1, x_2\,[(x_1 \Vdash \varphi \text{ and } x_2 \Vdash \psi) \Rightarrow R_{\circ}(y, x_1, x_2)] \tag{5.1.19}$$

$$\mathbb{M}, x \Vdash_V \varphi \circ \psi \quad \text{iff} \quad \forall y\, [y \succ_V \varphi \circ \psi \Rightarrow x \leq y] \tag{5.1.20}$$

$$\mathbb{M}, x \Vdash_V \varphi \to \psi \quad \text{iff} \quad \forall x', y\, [[x' \Vdash_V \varphi \text{ and } y \succ_V \psi] \Rightarrow R_{\to}(x', x, y)] \tag{5.1.21}$$

$$\mathbb{M}, y \succ_V \varphi \to \psi \quad \text{iff} \quad \forall x\, [x \Vdash_V \varphi \to \psi \Rightarrow x \leq y] \tag{5.1.22}$$

$$\mathbb{M}, x \Vdash_V \psi \leftarrow \varphi \quad \text{iff} \quad \forall x', y\, [[x' \Vdash_V \varphi \text{ and } y \succ_V \psi] \Rightarrow R_{\leftarrow}(x, x', y)] \tag{5.1.23}$$

$$\mathbb{M}, y \succ_V \psi \leftarrow \varphi \quad \text{iff} \quad \forall x\, [x \Vdash_V \psi \leftarrow \varphi \Rightarrow x \leq y] \tag{5.1.24}$$

We now proceed to build the canonical model for non-distributive logic with modal/substructural signature type.

5.2. Propositional substructural logic completeness on non-distributive setting

A model is a pair $\mathbb{M} = (\mathbb{F}, V)$ where $\mathbb{F}$ is a frame, $V : \mathsf{AtProp} \longrightarrow \mathcal{G}((X, Y, \leq))$ is an interpretation.

Our canonical model will be based on an RS-polarity $(X, Y, \leq)$ where the elements of X will be certain theories Σ and the elements of Y will be certain co-theories Δ (see claim 115 below). For every theory $\Sigma \in X$ and every co-theory $\Delta \in Y$ we set $\Sigma \leq \Delta$ iff $\Sigma \vdash \Delta$. The truth lemma now will have to treat both satisfaction and co-satisfaction relations, and reducing them to the set-theoretic belong-to relation, in short $(\Vdash^c, \succ^c) =_{truth\, lemma} (\ni, \ni)$.

Definition 112. Let $\mathbb{M}^c = (\mathbb{F}^c, V^c)$ be the canonical model for non-distributive substructural modal logic (*SML* henceforth), based on the canonical frame $\mathbb{F}^c = \big((X, Y, \leq), R^c_{\Box}, R^c_{\Diamond}, R^c_{\triangleright}, R^c_{\triangleleft}, R^c_{\circ}, R^c_{\to}, R^c_{\leftarrow}\big)$, where:

- X is the set of all optimal theories, i.e. $X = \{\Sigma \mid \langle \Sigma, \Delta \rangle$ is a maximal filter-ideal pair for some $\Delta\}$;
- Y is the set of all optimal co-theories, i.e. $Y = \{\Delta \mid \langle \Sigma, \Delta \rangle$ is a maximal filter-ideal for some $\Sigma\}$;
- $\leq \subseteq X \times Y$, is s.t. $\Sigma \leq \Delta$ iff $\Sigma \cap \Delta \neq \varnothing$;
- The canonical relations $R^c_{\Box}$, $R^c_{\Diamond}$, $R^c_{\triangleright}$,$R^c_{\triangleleft}$ and $R^c_{\circ}$ are defined as follows:
  - $R^c_{\Diamond} \subseteq Y \times X$, s.t. $\Delta R^c_{\Diamond} \Sigma$ iff $\Diamond[\Sigma] \cap \Delta \neq \varnothing$;
  - $R^c_{\Box} \subseteq X \times Y$, s.t. $\Sigma R^c_{\Box} \Delta$ iff $\Box[\Delta] \cap \Sigma \neq \varnothing$;
  - $R^c_{\triangleleft} \subseteq Y \times Y$, s.t. $\Delta R^c_{\triangleleft} \Delta'$ iff $\triangleleft[\Delta'] \cap \Delta \neq \varnothing$;
  - $R^c_{\triangleright} \subseteq X \times X$, s.t. $\Sigma R^c_{\triangleright} \Sigma'$ iff $\triangleright[\Sigma'] \cap \Sigma \neq \varnothing$;
  - $R^c_{\circ} \subseteq Y \times X \times X$, s.t. $\Delta R^c_{\circ} \Sigma, \Sigma'$ iff $\Sigma \circ \Sigma' \cap \Delta \neq \varnothing$, with $\Sigma \circ \Sigma' = \{\varphi \circ \psi | \varphi \in \Sigma \,\&\, \psi \in \Sigma'\}$
- The canonical valuation $V^c : \mathsf{AtProp} \to \mathbb{F}^+$ is s.t. $V^c(p) := \bigvee\{\Sigma \mid \Sigma \in X$ and $p \in \Sigma\} = \bigwedge\{\Delta \mid \Delta \in Y$ and $p \in \Delta\}$.

Claim 113. The equality $\bigvee\{\Sigma \mid \Sigma \in X$ and $p \in \Sigma\} = \bigwedge\{\Delta \mid \Delta \in Y$ and $p \in \Delta\}$ indeed holds.

PROOF. Let $A = \{\Sigma \mid \Sigma \in X \text{ and } p \in \Sigma\}$ and $B = \{\Delta \mid \Delta \in Y \text{ and } p \in \Delta\}$. Clearly, for any $\Sigma \in A$ and any $\Delta \in B$ we have $\Sigma \cap \Delta \neq \varnothing$ with p as witness. So $\Sigma \leq \Delta$ by definition. Thus, $A^u = B$ and $B^l = A$. Hence $\bigvee A$ is the least element in B and $\bigwedge B$ is the greatest element in A. Therefore $\bigvee A = \bigwedge B$, as desired. □

REMARK 114. Our definitions closely follow [**Gehrke 2006**] work. We have swapped Y coordinate on $R^c_\circ$ to look more alike the usual approach with diamond like interpretation. Likewise we omit $R^c_\rightarrow$ and $R^c_\leftarrow$ definitions as they are simply swappings of $R^c_\circ$. We refer the reader to [**Gehrke 2006**]:267-68 for the proofs that the canonical frame is of the right kind (based on a polarity which is an RS-frame) and whose relations are compatible (this is proven for $R^c_\circ$ but it is straightforward to see it carries over to unary modal relations).

CLAIM 115. Given $\langle \Sigma, \Delta \rangle$ a maximal pair, then Σ is a theory and Δ a co-theory.

PROOF. To show that Σ is a theory amounts to prove it is closed under derivability. So suppose that $\Sigma \vdash \varphi$ and assume towards a contradiction that $\varphi \notin \Sigma$. Then $\Sigma' := \Sigma \cup \{\varphi\}$ is a proper extension of Σ. Now observe that $\Sigma \vdash \varphi$ implies $C(\Sigma) = C(\Sigma')$ -they have the same set of consequences-. Then $\Sigma \nvdash \Delta$ implies $\Sigma' \nvdash \Delta$. But this contradicts the assumption that $\langle \Sigma, \Delta \rangle$ is maximal. Therefore $\varphi \in \Sigma$. The proof that Δ is a co-theory is order-dual. □

The following lemma will need later on to create witness points. An *optimal theory* is a generalization of the notion of prime theory (cf. definition 39), and simply states that the theory fulfills the definition of filter for posets (definition 77). Co-theory is the dual generalization.

LEMMA 116. *If Σ is an optimal theory and Δ is an optimal cotheory then*

- $\Box^{-1}[\Sigma]$ *is an optimal theory and*
- $\triangleright^{-1}[\Sigma]$ *an optimal cotheory.*
- $\Diamond^{-1}[\Delta]$ *is an optimal cotheory and*
- $\triangleleft^{-1}[\Delta]$ *is an optimal theory.*
- $\Delta_1 = \{\chi' | \chi' \circ \psi \in \Delta\}$ *is an ideal (an optimal cotheory).*
- $\Delta_2 = \{\psi' | \exists \chi' (\chi' \in \Sigma \,\&\, \chi' \circ \psi' \in \Delta)\}$ *is an ideal (an optimal cotheory).*

REMARK 117. To prove such lemma we will need to consider the residual operations for all the unary modal operations -fusion already has it own residuals-. So let $\blacklozenge, \blacksquare, \blacktriangleleft, \blacktriangleright$ be the residual operations of $\Box, \Diamond, \triangleright, \triangleleft$ respectively. As [**Gehrke 2006**] points out, although $\Box, \Diamond, \triangleright, \triangleleft$ are not generally stipulated to be residuated, they become so in the canonical extension.

PROOF. Suppose Σ is a theory. Then it is closed under derivability, i.e. if $\Sigma \vdash \varphi$ then $\varphi \in \Sigma$, and it is downdirected.

We show $\Box^{-1}[\Sigma]$ is closed under derivability. So suppose $\Box^{-1}[\Sigma] \vdash \psi$. Then there are $\varphi_1, \ldots, \varphi_n \in \Box^{-1}[\Sigma]$ such that $\bigwedge_{i=1}^{n} \varphi_i \vdash \psi$. But then $\Box \bigwedge_{i=1}^{n} \varphi_i \vdash \Box\psi$ and thus $\bigwedge_{i=1}^{n} \Box\varphi_i \vdash \Box\psi$ since $\Box$ preserves meets. From $\varphi_1, \ldots, \varphi_n \in \Box^{-1}[\Sigma]$ it immediately follows that $\Box\varphi_1, \ldots, \Box\varphi_n \in \Sigma$ and hence $\Sigma \vdash \Box\psi$, with Σ being a theory. Then $\Box\psi \in \Sigma$ and thus $\psi \in \Box^{-1}[\Sigma]$. Since ψ was an arbitrary formula, we proved that $\Box^{-1}[\Sigma]$ is closed under derivability, i.e. it is a theory. It is downdirected: Suppose $\varphi, \psi \in \Box^{-1}[\Sigma]$, then $\Box\varphi, \Box\psi \in \Sigma$ and since Σ is downdirected, then there is some $\beta \in \Sigma$ with $\begin{cases} \beta \vdash \Box\varphi \\ \beta \vdash \Box\psi \end{cases}$. Therefore $\begin{cases} \blacklozenge\beta \vdash \varphi \\ \blacklozenge\beta \vdash \psi \end{cases}$ with $\blacklozenge\beta \in \Box^{-1}[\Sigma]$, as $\blacklozenge = \Box^{-1}$. Since φ, ψ were arbitrary, this shows that $\Box^{-1}[\Sigma]$ is downdirected.

We show $\triangleright^{-1}[\Sigma]$ is closed under inverse of derivability. Now suppose $\psi \vdash \triangleright^{-1}[\Sigma]$. Then there are $\varphi_1, \ldots, \varphi_n \in \triangleright^{-1}[\Sigma]$ such that $\psi \vdash \bigvee_{i=1}^{n} \varphi_i$. But then $\triangleright \bigvee_{i=1}^{n} \varphi_i \vdash \triangleright\psi$ and thus $\bigwedge_{i=1}^{n} \triangleright \varphi_i \vdash \triangleright\psi$ since $\triangleright$ turns joins into meets.

From $\varphi_1, \ldots, \varphi_n \in \rhd^{-1}[\Sigma]$ it immediately follows that $\rhd\varphi_1, \ldots, \rhd\varphi_n \in \Sigma$ and hence $\Sigma \vdash \rhd\psi$, with Σ being a theory. Then $\rhd\psi \in \Sigma$ and thus $\psi \in \rhd^{-1}[\Sigma]$. Since ψ was an arbitrary formula, we proved that $\rhd^{-1}[\Sigma]$ is closed under inverse of derivability, i.e. it is a cotheory. It is updirected: Suppose $\varphi, \psi \in \rhd^{-1}[\Sigma]$, then $\rhd\varphi, \rhd\psi \in \Sigma$ and since Σ is downdirected, then there is some $\beta \in \Sigma$ with $\begin{cases} \beta \vdash \rhd\varphi \\ \beta \vdash \rhd\psi \end{cases}$. Therefore $\begin{cases} \varphi \vdash \blacktriangleleft \beta \\ \psi \vdash \blacktriangleleft \beta \end{cases}$ with $\blacktriangleleft \beta \in \rhd^{-1}[\Sigma]$, as $\blacktriangleleft = \rhd^{-1}$. Since φ, ψ were arbitrary, this shows that $\rhd^{-1}[\Sigma]$ is updirected.

Suppose Δ is a co-theory. Then it is closed under inverse of derivability, i.e. if $\varphi \vdash \Delta$ then $\varphi \in \Delta$ and it is updirected.

So suppose $\psi \vdash \Diamond^{-1}[\Delta]$. Then there are $\varphi_1, \ldots, \varphi_n \in \Diamond^{-1}[\Delta]$ such that $\psi \vdash \bigvee_{i=1}^{n} \varphi_i$. But then $\Diamond\psi \vdash \Diamond \bigvee_{i=1}^{n} \varphi_i$ and thus $\Diamond\psi \vdash \bigvee_{i=1}^{n} \Diamond\varphi_i$ since $\Diamond$ preserves joins. From $\varphi_1, \ldots, \varphi_n \in \Diamond^{-1}[\Delta]$ it immediately follows that $\Diamond\varphi_1, \ldots, \Diamond\varphi_n \in \Delta$ and hence $\Diamond\psi \vdash \Delta$, with Δ being a co-theory. Then $\Diamond\psi \in \Delta$ and thus $\psi \in \Diamond^{-1}[\Delta]$. Since ψ was an arbitrary formula, we proved that $\Diamond^{-1}[\Delta]$ is closed under inverse of derivability, i.e. it is a co-theory. It is updirected: Suppose $\varphi, \psi \in \Diamond^{-1}[\Delta]$, then $\Diamond\varphi, \Diamond\psi \in \Delta$ and since Δ is updirected, then there is some $\beta \in \Delta$ with $\begin{cases} \Diamond\varphi \vdash \beta \\ \Diamond\psi \vdash \beta \end{cases}$. Therefore $\begin{cases} \varphi \vdash \blacksquare\beta \\ \psi \vdash \blacksquare\beta \end{cases}$ with $\blacksquare\beta \in \Diamond^{-1}[\Delta]$, as $\blacksquare = \Diamond^{-1}$. Since φ, ψ were arbitrary, this shows that $\Diamond^{-1}[\Delta]$ is updirected.

Now suppose $\lhd^{-1}[\Delta] \vdash \psi$. Then there are $\varphi_1, \ldots, \varphi_n \in \lhd^{-1}[\Delta]$ such that $\bigwedge_{i=1}^{n} \varphi_i \vdash \psi$. But then $\lhd\psi \vdash \lhd \bigwedge_{i=1}^{n} \varphi_i$ and thus $\lhd\psi \vdash \bigvee_{i=1}^{n} \lhd \varphi_i$ since $\lhd$ turns meets into joins. From $\varphi_1, \ldots, \varphi_n \in \lhd^{-1}[\Delta]$ it immediately follows that $\lhd\varphi_1, \ldots, \lhd\varphi_n \in \Delta$ and hence $\lhd\psi \vdash \Delta$, with Δ being a cotheory, i.e. closed under the inverse of derivability. Then $\lhd\psi \in \Delta$ and thus $\psi \in \lhd^{-1}[\Delta]$. Since ψ was an arbitrary formula, we proved that $\lhd^{-1}[\Delta]$ is closed under derivability, i.e. it is a theory. It is downdirected: Suppose $\varphi, \psi \in \lhd^{-1}[\Delta]$, then $\lhd\varphi, \lhd\psi \in \Delta$ and since Δ is updirected, then there is some $\beta \in \Delta$ with $\begin{cases} \lhd\varphi \vdash \beta \\ \lhd\psi \vdash \beta \end{cases}$. Therefore $\begin{cases} \blacktriangleright \beta \vdash \varphi \\ \blacktriangleright \beta \vdash \psi \end{cases}$ with $\blacktriangleright \beta \in \lhd^{-1}[\Delta]$, as $\blacktriangleright = \lhd^{-1}$. Since φ, ψ were arbitrary, this shows that $\lhd^{-1}[\Delta]$ is downdirected.

Now we show that $\Delta_1 = \{\chi' | \chi' \circ \psi \in \Delta\}$ is an ideal (a cotheory). It is down-closed. Let $a \in \Delta_1$ and $b \vdash a$, then $a \circ \psi \in \Delta$ and $b \circ \psi \vdash a \circ \psi$ since fusion is order preserving on both coordinates. Therefore $b \circ \psi \in \Delta$ since Δ is a cotheory. Then $b \in \Delta_1$. It is updirected: if $a, b \in \Delta_1$ then there is some $z \in \Delta_1$ with $a \vdash z$ and $b \vdash z$. To see this, let $a, b \in \Delta_1$ then $\begin{cases} a \circ \psi \in \Delta \\ b \circ \psi \in \Delta \end{cases}$ and thus $\begin{cases} a \circ \psi \vdash c \\ b \circ \psi \vdash c \end{cases}$ for some $c \in \Delta$ because Δ is updirected. Then by residuation, $\begin{cases} a \vdash c \leftarrow \psi \\ b \vdash c \leftarrow \psi \end{cases}$ and $(c \leftarrow \psi) \circ \psi = c \in \Delta$. Thus $c \leftarrow \psi \in \Delta_1$ and $c \leftarrow \psi$ is our witness z.

Finally, we show $\Delta_2 = \{\psi' | \exists \chi' \, (\chi' \in \Sigma \,\&\, \chi' \circ \psi' \in \Delta)\}$ is an ideal (a cotheory). Suppose $\alpha \in \Delta_2$, then there is some $z \in \Sigma$ s.t. $z \circ \alpha \in \Delta$, which is a downset. Suppose further that $\beta \vdash \alpha$, then $z \circ \beta \vdash z \circ \alpha$ as fusion is order preserving on both coordinates. Therefore $z \circ \beta \in \Delta$ and $\beta \in \Delta_2$. Thus Δ_2 is a downset. Now we show it is updirected. Assume $\psi', \psi" \in \Delta_2$, then there exist $\varphi', \varphi" \in \Sigma$ with $(*) \begin{cases} \varphi' \circ \psi' \in \Delta \\ \varphi" \circ \psi" \in \Delta \end{cases}$. Since Σ is down-directed then there exists $\theta \in \Sigma$ such that $\begin{cases} \theta \vdash \varphi' \\ \theta \vdash \varphi" \end{cases}$. But then $(**) \begin{cases} \theta \circ \psi' \vdash \varphi' \circ \psi' \\ \theta \circ \psi" \vdash \varphi" \circ \psi" \end{cases}$ since fusion is order preserving on both coordinates. Now from $(*)$ and $(**)$ we have $\begin{cases}\theta \circ \psi' \in \Delta \\ \theta \circ \psi" \in \Delta\end{cases}$ because Δ is a downset. Since Δ is up-directed, then there is some $\theta' \in \Delta$ with $(***)\begin{cases}\theta \circ \psi' \vdash \theta' \\ \theta \circ \psi" \vdash \theta'\end{cases}$. Now $\theta \to \theta' \in \Delta_2$ because $\theta \circ (\theta \to \theta') = \theta' \in \Delta$ and $\theta \in \Sigma$. Now $\begin{cases}\psi' \vdash \theta \to \theta' \\ \psi" \vdash \theta \to \theta'\end{cases}$ by applying residuation to $(***)$, so Δ_2 is updirected with $\theta \to \theta'$ as witness. $\square$

The following lemma is the key to the truth-lemma and thus to the completeness result:

LEMMA 118. ***(Existence lemma)*** *Let Σ be an optimal theory and Δ an optimal co-theory of our language*

- *if $\Diamond\psi \notin \Delta$, then $\left\langle\{\psi\}, \Diamond^{-1}[\Delta]\right\rangle$ can be extended to a maximal pair $\langle\Sigma', \Delta'\rangle$.*
- *if $\Diamond\psi \notin \Sigma$, then $\langle\Sigma, \{\Diamond\psi\}\rangle$ can be extended to a maximal pair $\langle\Sigma', \Delta'\rangle$.*
- *if $\Box\psi \notin \Sigma$, then $\left\langle\Box^{-1}[\Sigma], \{\psi\}\right\rangle$ can be extended to a maximal pair $\langle\Sigma', \Delta'\rangle$.*
- *if $\Box\psi \notin \Delta$, then $\langle\{\Box\psi\}, \Delta\rangle$ can be extended to a maximal pair $\langle\Sigma', \Delta'\rangle$.*
- *if $\lhd\psi \notin \Delta$ then $\left\langle\lhd^{-1}[\Delta], \{\psi\}\right\rangle$ can be extended to a maximal pair $\langle\Sigma', \Delta'\rangle$.*
- *if $\lhd\psi \notin \Sigma$, then $\langle\Sigma, \{\lhd\psi\}\rangle$ can be extended to a maximal pair $\langle\Sigma', \Delta'\rangle$.*
- *if $\rhd\psi \notin \Sigma$, then $\left\langle\{\psi\}, \rhd^{-1}[\Sigma]\right\rangle$ can be extended to a maximal pair $\langle\Sigma', \Delta'\rangle$.*
- *if $\rhd\psi \notin \Delta$, then $\langle\{\rhd\psi\}, \Delta\rangle$ can be extended to a maximal pair $\langle\Sigma', \Delta'\rangle$.*
- *if $\chi \circ \psi \notin \Delta$, then*
  - *$\langle\{\chi\}, \Delta_1\rangle$ with $\Delta_1 = \{\chi' | \chi' \circ \psi \in \Delta\}$ can be extended to a maximal pair $\langle\Sigma', \Delta'\rangle$.*
  - *$\langle\{\psi\}, \Delta_2\rangle$ with $\Delta_2 = \{\psi' | \exists\chi' (\chi' \in \Sigma' \,\&\, \chi' \circ \psi' \in \Delta)\}$ can be extended to a maximal pair $\langle\Sigma", \Delta"\rangle$.*

PROOF. Given the case per case assumptions (regarding modal formulas not belonging to Σ or Δ), all the pairs to be extended are indeed disjoint. Now given the global assumption about Σ and Δ being respectively a theory and cotheory, these are in fact a filter and an ideal. Now a straightforward application of corollary 102 will suffice. $\square$

5.2.1. The truth lemma. Let $(\Vdash^c, \succ^c)$ be the satisfaction and co-satisfaction relations associated with V^c (or more precisely, with its unique homomorphic extension), then:

LEMMA 119. *(Truth lemma) For every $\varphi \in Fm$, every $\Sigma \in X$ and every $\Delta \in Y$:*

- *$\Sigma \Vdash^c \varphi$ iff $\varphi \in \Sigma$;*
- *$\Delta \succ^c \varphi$ iff $\varphi \in \Delta$.*

PROOF. By induction on the complexity of φ.

BASE CASE:

If $\varphi = p \in \mathsf{AtProp}$, then:

$$\begin{array}{lll} \Sigma \Vdash^c p & \text{iff} & \Sigma \leq V^c(p) = \bigwedge\{\Delta \mid \Delta \in Y \text{ and } p \in \Delta\} \\ & \text{iff} & \forall\Delta[(\Delta \in Y \text{ and } p \in \Delta) \Rightarrow \Sigma \leq \Delta] \\ & \text{iff} & \forall\Delta[(\Delta \in Y \text{ and } \Sigma \nleq \Delta) \Rightarrow p \notin \Delta] \\ & \text{iff} & \forall\Delta[(\Delta \in Y \text{ and } \Sigma \cap \Delta = \varnothing) \Rightarrow p \notin \Delta]. \end{array}$$

So suppose that $\Sigma \Vdash^c p$ and assume towards a contradiction that $p \notin \Sigma$. Since $\Sigma \in X$, then it is an optimal filter, so in particular Σ is maximal w.r.t. some $\Delta' \in Y$. Since $\Delta' \in Y$ and $\Sigma \cap \Delta' = \varnothing$ then we conclude by the above equivalences that $p \notin \Delta'$. But then $\Sigma' := \Sigma \cup \{p\}$ would be a proper extension of Σ and

$\Sigma' \cap \Delta' \neq \varnothing$. Since Δ' is a co-theory, this implies that $\Sigma' \nvdash \Delta'$, against the maximality of Σ w.r.t. Δ'. This shows that $p \in \Sigma$. The proof that $\Delta \succ^c p$ iff $p \in \Delta$ is analogous. Therefore, for every $p \in \mathsf{AtProp}$:

- $\Sigma \Vdash^c p$ iff $p \in \Sigma$;
- $\Delta \succ^c p$ iff $p \in \Delta$.

INDUCTIVE STEP:

As for the inductive step, we need to consider various cases:

The primary cases do the hard job of breaking down the complexity of the formula, reason for which they need the existence lemma, and the secondary cases rely on the primary ones.

$\Diamond$-case

Primary subcase.

Assume that $\varphi = \Diamond\psi$ and that for every $\Sigma \in X$ and every $\Delta \in Y$, $\Sigma \Vdash^c \psi$ iff $\psi \in \Sigma$ and $\Delta \succ^c \psi$ iff $\psi \in \Delta$.

Let us fix $\Delta \in Y$ and let us show that:

$$\Delta \succ^c \Diamond\psi \ \text{ iff } \ \Diamond\psi \in \Delta.$$

($\Leftarrow$) Assume that $\Diamond\psi \in \Delta$. By definition of $\Delta \succ^c \Diamond\psi$ in (5.1.11), we need to show that if $\Sigma' \in X$ and $\Sigma' \Vdash^c \psi$, then $\Diamond[\Sigma'] \cap \Delta \neq \varnothing$. By induction hypothesis, $\Sigma' \Vdash^c \psi$ means that $\psi \in \Sigma'$, so $\Diamond\psi \in \Diamond[\Sigma']$, and since by assumption $\Diamond\psi \in \Delta$, then indeed $\Diamond[\Sigma'] \cap \Delta \neq \varnothing$.

($\Rightarrow$) Conversely, assume that $\Diamond\psi \notin \Delta$. We need to show that there exists some $\Sigma' \in X$ such that $\psi \in \Sigma'$ and $\Diamond[\Sigma'] \cap \Delta = \varnothing$, i.e. $\Sigma' \cap \Diamond^{-1}[\Delta] = \varnothing$. Since $\Diamond\psi \notin \Delta$, by the Existence Lemma (118) $\langle \{\psi\}, \Diamond^{-1}[\Delta] \rangle$ can be extended to a maximal pair $\langle \Sigma', \Delta' \rangle$. Then $\Sigma' \in X$, $\Delta' \in Y$ and $\psi \in \Sigma'$. Moreover, $\Diamond^{-1}[\Delta] \subseteq \Delta'$ and $\Sigma' \cap \Delta' = \varnothing$ implies that $\Sigma' \cap \Diamond^{-1}[\Delta] = \varnothing$. So result is proven.

Secondary subcase.

Now let us fix $\Sigma \in X$ and show that:

$$\Sigma \Vdash^c \Diamond\psi \ \text{ iff } \ \Diamond\psi \in \Sigma.$$

($\Leftarrow$) Assume that $\Diamond\psi \in \Sigma$. By definition of $\Sigma \Vdash^c \Diamond\psi$ in (5.1.12), we need to show that if $\Delta \in Y$ and $\Delta \succ^c \Diamond\psi$ then $\Sigma \leq \Delta$, that is $\Sigma \cap \Delta \neq \varnothing$. So suppose $\Delta \in Y$ and $\Delta \succ^c \Diamond\psi$. By the previous case this means that $\Diamond\psi \in \Delta$. But then clearly $\Sigma \cap \Delta \neq \varnothing$ since we assumed $\Diamond\psi \in \Sigma$.

($\Rightarrow$) Assume that $\Diamond\psi \notin \Sigma$. We need to show that there is some $\Delta \in Y$such that $\Delta \succ^c \Diamond\psi$ and $\Sigma \nleq \Delta$, that is $\Sigma \cap \Delta = \varnothing$. Since $\Diamond\psi \notin \Sigma$, by the Existence Lemma (118) $\langle \Sigma, \{\Diamond\psi\} \rangle$ can be extended to a maximal pair $\langle \Sigma', \Delta' \rangle$. Then $\Sigma' \in X$, $\Delta' \in Y$ and $\Diamond\psi \in \Delta'$, that is $\Delta' \succ^c \Diamond\psi$. Moreover, $\Sigma \subseteq \Sigma'$ and $\Sigma' \cap \Delta' = \varnothing$ implies that $\Sigma \cap \Delta' = \varnothing$. So result is proven.

$\Box$-*case*

Primary subcase.

Let $\Sigma \in X$. We shall show that: $\nVdash$

$$\Sigma \Vdash^c \Box\psi \text{ iff } \Box\psi \in \Sigma.$$

($\Leftarrow$) For the easy direction, from right to left, assume that $\Box\psi \in \Sigma$. Then by definition of $\Sigma \Vdash^c \Box\psi$ in (5.1.13), we have to show $\forall\Delta[\Delta \succ_V \psi \Rightarrow \Sigma R^c_\Box \Delta]$. So let $\Delta \in Y$ and suppose that $\Delta \succ_V \psi$. Then by IH $\psi \in \Delta$ and thus $\Box[\Delta] \cap \Sigma \neq \varnothing$ so $\Sigma R^c_\Box \Delta$ by definition of $R^c_\Box$. Since $\Delta \in Y$ was arbitrary, the implication is shown and therefore $\Sigma \Vdash^c \Box\psi$.

($\Rightarrow$) For the other direction, assume that $\Box\psi \notin \Sigma$. To show that $\Sigma \nVdash^c \Box\psi$ we have to find a $\Delta \in Y$ such that $\Delta \succ_V \psi$ and such that $\Sigma R^c_\Box \Delta$ doesn't hold, that is $\Box[\Delta] \cap \Sigma = \varnothing$. Since $\Box\psi \notin \Sigma$, by the Existence Lemma (118) $\langle \Box^{-1}[\Sigma], \{\psi\} \rangle$ can be extended to a maximal pair $\langle \Sigma', \Delta' \rangle$. Then $\Sigma' \in X$, $\Delta' \in Y$ and $\psi \in \Delta'$. Thus by IH $\Delta' \succ_V \psi$. Moreover, $\Box^{-1}[\Sigma] \subseteq \Sigma'$ so $\Sigma' \cap \Delta' = \varnothing$ implies that $\Box^{-1}[\Sigma] \cap \Delta' = \varnothing$. Hence Δ' is the desired counterexample and $\Sigma \Vdash^c \Box\psi$ doesn't hold.

Secondary subcase.

Let $\Delta \in Y$. We shall show that:

$$\Delta \succ^c \Box\psi \text{ iff } \Box\psi \in \Delta.$$

($\Leftarrow$) For the easy direction, from right to left, assume that $\Box\psi \in \Delta$. Then by definition of $\Delta \succ^c \Box\psi$ in (5.1.14), we have to show $\forall\Sigma[\Sigma \Vdash^c \Box\psi \Rightarrow \Sigma \leq \Delta]$. So let $\Sigma \in X$ and suppose $\Sigma \Vdash^c \Box\psi$. Then $\Box\psi \in \Sigma$ as we just have shown above and therefore $\Delta \cap \Sigma \neq \varnothing$ so $\Sigma \leq \Delta$ by definition of $\leq$. Since $\Sigma \in X$ was arbitrary, the implication is shown and therefore $\Delta \succ^c \Box\psi$.

($\Rightarrow$) For the other direction, assume that $\Box\psi \notin \Delta$. To show that $\Delta \not\succ \Box\psi$ we have to find a $\Sigma \in X$ such that $\Sigma \Vdash^c \Box\psi$ but $\Sigma \nleq \Delta$. Since $\Box\psi \notin \Delta$, by the Existence Lemma (118) $\langle \{\Box\psi\}, \Delta \rangle$ can be extended to a maximal pair $\langle \Sigma', \Delta' \rangle$. Then $\Sigma' \in X$, $\Delta' \in Y$ and $\Box\psi \in \Sigma'$. Moreover, $\Delta \subseteq \Delta'$ so $\Sigma' \cap \Delta' = \varnothing$ implies that $\Delta \cap \Sigma' = \varnothing$ which means $\Sigma \nleq \Delta$, as desired.

$\lhd$-*case*

Primary subcase.

Let $\Delta \in Y$. We shall show that:

$$\Delta \succ^c \lhd\psi \text{ iff } \lhd\psi \in \Delta.$$

($\Leftarrow$) Let $\lhd\psi \in \Delta$. By definition of co-satisfaction (5.1.15) we have to show $\forall\Delta'[\Delta' \succ^c \psi \Rightarrow \Delta R^c_\lhd \Delta']$. So let $\Delta' \succ^c \psi$ for some $\Delta' \in Y$, then by IH $\psi \in \Delta'$ and thus $\Delta \cap \lhd[\Delta'] \neq \varnothing$ which by definition leads to the desired result : $\Delta R^c_\lhd \Delta'$. Thus $\Delta \succ^c \lhd\psi$.

($\Rightarrow$) Let $\lhd\psi \notin \Delta$. By the definitions of co-satisfaction and of $R_\lhd$ we have to show $\exists\Delta'[\Delta' \succ^c \psi \;\&\; \Delta \cap \lhd[\Delta'] = \varnothing]$, with $\Delta' \succ^c \psi$ rewritten as $\psi \in \Delta'$ by IH. Since $\lhd\psi \notin \Delta$ then $\langle \lhd^{-1}[\Delta], \{\psi\} \rangle$ is a pair, which by the Existence Lemma (118) can be extended to a maximal pair $\langle \Sigma', \Delta' \rangle$. Then $\Sigma' \in X$, $\Delta' \in Y$ and $\psi \in \Delta'$. Moreover, $\lhd^{-1}[\Delta] \subseteq \Sigma'$ so $\Sigma' \cap \Delta' = \varnothing$ implies that $\lhd^{-1}[\Delta] \cap \Delta' = \varnothing$ which means $\Delta \cap \lhd[\Delta'] = \varnothing$, as desired.

Secondary subcase.

Let $\Sigma \in X$. We shall show that:

$$\Sigma \Vdash^c \lhd\psi \text{ iff } \lhd\psi \in \Sigma.$$

($\Leftarrow$) Let $\lhd\psi \in \Sigma$. By (5.1.16) we have to show $\forall\Delta\,[\Delta \succ^c \lhd\psi \Rightarrow \Sigma \leq \Delta]$. Let us fix a $\Delta \in Y$ and assume $\Delta \succ^c \lhd\psi$ then by the previous proof $\lhd\psi \in \Delta$ and thus $\Sigma \cap \Delta \neq \varnothing$. Therefore $\Sigma \leq \Delta$ as desired. Since Δ was arbitrary, this shows the implication.

($\Rightarrow$) Let $\lhd\psi \notin \Sigma$. We have to show that there exists $\Delta \succ^c \lhd\psi$ and $\Sigma \nleq \Delta$, that is $\Sigma \cap \Delta = \varnothing$. Since Σ is a theory, it is clear that $\Sigma \nvdash^c \lhd\psi$, hence we can try and extend $\langle \Sigma, \{\lhd\psi\}\rangle$ to a maximal pair $\langle \Sigma', \Delta'\rangle$ by the Existence lemma (118). Then $\Sigma' \in X$, $\Delta' \in Y$ and $\lhd\psi \in \Delta'$, thus by the previous results $\Delta' \succ^c \lhd\psi$. Since $\Sigma \subseteq \Sigma'$ then $\Sigma' \cap \Delta' = \varnothing$ implies $\Sigma \cap \Delta' = \varnothing$ and thus $\Sigma \nleq \Delta'$ as desired.

$\rhd$-*case*

Primary subcase.

Let $\Sigma \in X$. We shall show that:

$$\Sigma \Vdash^c \rhd\psi \text{ iff } \rhd\psi \in \Sigma.$$

($\Leftarrow$) For the easy direction, from right to left, assume that $\rhd\psi \in \Sigma$. Then by definition of $\Sigma \Vdash^c \rhd\psi$ in (5.1.17), we have to show $\forall\Sigma'[\Sigma' \Vdash^c \psi \;\Rightarrow\; \Sigma R^c_{\rhd}\Sigma']$. So let $\Sigma' \in X$ and suppose that $\Sigma' \Vdash^c \psi$, which by IH means $\psi \in \Sigma'$. But then $\rhd[\Sigma'] \cap \Sigma \neq \varnothing$, as desired.

($\Rightarrow$) For the other direction, assume that $\rhd\psi \notin \Sigma$. To show that $\Sigma \nVdash^c \rhd\psi$ we have to find a $\Sigma' \in X$ such that $\Sigma' \Vdash^c \psi$ (by IH $\psi \in \Sigma'$) but $\Sigma R^c_{\rhd}\Sigma'$ doesn't hold, that is $\rhd[\Sigma'] \cap \Sigma = \varnothing$. Since $\rhd\psi \notin \Sigma$ then $\left\langle \{\psi\}, \rhd^{-1}[\Sigma]\right\rangle$ is a pair and by the Existence lemma (118), it can be extended to a maximal pair $\langle \Sigma', \Delta'\rangle$. Then $\Sigma' \in X$, $\Delta' \in Y$ and $\psi \in \Sigma'$. Since $\rhd^{-1}[\Sigma] \subseteq \Delta'$ then $\Sigma' \cap \Delta' = \varnothing$ implies $\Sigma' \cap \rhd^{-1}[\Sigma] = \varnothing$, that is $\rhd[\Sigma'] \cap \Sigma = \varnothing$ as desired.

Secondary subcase.

Let $\Delta \in Y$. We shall show that:

$$\Delta \succ^c \rhd\psi \text{ iff } \rhd\psi \in \Delta.$$

($\Leftarrow$) Assume $\rhd\psi \in \Delta$. Given the definition in (5.1.18) we have to show $\forall\Sigma\,[\Sigma \Vdash^c \rhd\psi \Rightarrow \Sigma \leq \Delta]$. So fix an $\Sigma \in X$ and let $\Sigma \Vdash^c \rhd\psi$, then $\rhd\psi \in \Sigma$ by previous proof, and then $\Sigma\cap\Delta \neq \varnothing$ which gives $\Sigma \leq \Delta$ by definition. Since Σ was arbitrary, the implication is proven.

($\Rightarrow$) Assume $\rhd\psi \notin \Delta$. We have to show $\exists\Sigma\,[\Sigma \Vdash^c \rhd\psi \,\&\, \Sigma \nleq \Delta]$. By the previous result $\Sigma \Vdash^c \rhd\psi$ amounts to $\rhd\psi \in \Sigma$. Since $\rhd\psi \notin \Delta$ then $\langle\{\rhd\psi\}, \Delta\rangle$ is disjoint and can be extended to a maximal pair $\langle \Sigma', \Delta'\rangle$ by the Existence Lemma (118). Then $\Sigma' \in X$, $\Delta' \in Y$ and $\rhd\psi \in \Sigma'$. Since $\Delta \subseteq \Delta'$, then $\Sigma' \cap \Delta' = \varnothing$ implies $\Sigma' \cap \Delta = \varnothing$, that is $\Sigma' \nleq \Delta$ as desired.

$\circ$-*case*

Primary subcase.

Assume that $\varphi = \chi \circ \psi$ and that for every $\Sigma \in X$ and every $\Delta \in Y$:

- $\Sigma \Vdash^c \chi$ iff $\chi \in \Sigma$ and $\Delta \succ^c \chi$ iff $\chi \in \Delta$.
- $\Sigma \Vdash^c \psi$ iff $\psi \in \Sigma$ and $\Delta \succ^c \psi$ iff $\psi \in \Delta$.

Let us fix $\Delta \in Y$ and let us show that:

$$\Delta \succ^c \chi \circ \psi \text{ iff } \chi \circ \psi \in \Delta.$$

($\Leftarrow$) Assume that $\chi \circ \psi \in \Delta$. By definition of $\Delta \succ^c \chi \circ \psi$ in (5.1.19), we need to show that if (a) $\Sigma \in X$ and $\Sigma \Vdash^c \chi$ and (b) $\Sigma' \in X$ and $\Sigma' \Vdash^c \psi$, then $\Sigma \circ \Sigma' \cap \Delta \neq \varnothing$, with $\Sigma \circ \Sigma' = \{\varphi \circ \psi | \varphi \in \Sigma \,\&\, \psi \in \Sigma'\}$. By induction hypothesis, (a) means that $\chi \in \Sigma$, and likewise (b) means that $\psi \in \Sigma'$, so $\chi \circ \psi \in \Sigma \circ \Sigma'$ and since by assumption $\chi \circ \psi \in \Delta$, then indeed $\Sigma \circ \Sigma' \cap \Delta \neq \varnothing$.

($\Rightarrow$) Conversely, assume that $\chi \circ \psi \notin \Delta$. We need to show that there exists some $\Sigma, \Sigma' \in X$ such that $\varphi \in \Sigma$ and $\psi \in \Sigma'$ and $\Sigma \circ \Sigma' \cap \Delta = \varnothing$. Let $\Delta_1 = \{\chi' | \chi' \circ \psi \in \Delta\}$, then $\chi \notin \Delta_1$. Since $\chi \notin \Delta_1$ then $\langle \{\chi\}, \Delta_1 \rangle$ is disjoint pair and by lemma 118 it can be extended to a maximal pair $\langle \Sigma_1, \Delta_1' \rangle$, our first witness point. Now let $\Delta_2 = \{\psi' | \exists \chi' (\chi' \in \Sigma_1 \,\&\, \chi' \circ \psi' \in \Delta)\}$ be our starting base for our 2nd witness. Notice that $\psi \notin \Delta_2$. For suppose otherwise, if $\psi \in \Delta_2$ then there exists $\beta \in \Sigma_1$ such that $\beta \circ \psi \in \Delta$. But then $\beta \in \Delta_1$ and therefore $\beta \in \Sigma_1 \cap \Delta_1$. However $\langle \Sigma_1, \Delta_1' \rangle$ is a maximal pair with $\Delta_1 \subseteq \Delta_1'$ and thus $\Sigma_1 \cap \Delta_1 = \varnothing$, contradiction! Therefore $\langle \{\psi\}, \Delta_2 \rangle$ is a disjoint pair which can be extended to a maximal pair $\langle \Sigma_2, \Delta_2' \rangle$ by lemma 118. Now we have $\begin{cases} \langle \Sigma_1, \Delta_1' \rangle \text{ with } \Delta_1 \subseteq \Delta_1' \,\&\, \chi \in \Sigma_1 \\ \langle \Sigma_2, \Delta_2' \rangle \text{ with } \Delta_2 \subseteq \Delta_2' \,\&\, \psi \in \Sigma_2 \end{cases}$. Let $\delta \in \Sigma_1$. If there is some δ' such that $\delta \circ \delta' \in \Delta$ then $\delta' \in \Delta_2 \subseteq \Delta_2'$. Therefore $\delta' \notin \Sigma_2$ because $\langle \Sigma_2, \Delta_2' \rangle$ is a disjoint pair. As δ, δ' were arbitrary then $\Sigma_1 \circ \Sigma_2 \cap \Delta = \varnothing$, as desired.

Secondary subcase.

Now let us fix $\Sigma \in X$ and show that:

$$\Sigma \Vdash^c \chi \circ \psi \;\text{ iff }\; \chi \circ \psi \in \Sigma.$$

By definition of $\Sigma \Vdash^c \chi \circ \psi$ in (5.1.20), $\Sigma \Vdash^c \chi \circ \psi$ if and only if $\forall \Delta\, [\Delta \succ_V \varphi \circ \psi \Rightarrow \Sigma \leq \Delta]$, that is $\forall \Delta\, [\varphi \circ \psi \in \Delta \Rightarrow \Sigma \leq \Delta]$ by the previous result. But clearly $(\varphi \circ \psi \in \Delta) \Rightarrow \Sigma \leq \Delta$ iff $(\varphi \circ \psi \in \Delta) \Rightarrow \Sigma \cap \Delta \neq \varnothing$ iff $\varphi \circ \psi \in \Sigma$. So $\Sigma \Vdash^c \chi \circ \psi$ iff $\chi \circ \psi \in \Sigma$, as desired.

The Truth lemma is proven. □

5.2.2. Completeness theorem and proof. The moment arrived for us to present the completeness theorem.

THEOREM 120. *(completeness) Given a pair $\Sigma \nvdash_\Lambda \Delta$ and a SML logic Λ there is a model $\mathbb{M}$ based on some SML-frame $\mathbb{F}$ such that $\Sigma \nVdash_\mathbb{M} \Delta$ (i.e. there exists a two-sorted point $\langle x, y \rangle$ which is a maximal pair with $x \in X$ and $y \in Y$ and such that $\mathbb{M}, x \Vdash \Sigma$ and $\mathbb{M}, y \succ \Delta$).*

PROOF. Assume $\Sigma \nvdash \Delta$. Then $\langle \Sigma, \Delta \rangle$ is a disjoint pair which via corollary (102) can be extended to a maximal pair $\langle \Sigma', \Delta' \rangle$. Consequently $\Sigma' \in X$ and $\Delta' \in Y$ in the DML-canonical frame as described in definition 112, and the canonical valuation guarantees that $\Sigma' \Vdash_{\mathbb{M}^c} \Sigma$ and $\Delta' \succ_{\mathbb{M}^c} \Delta$ and therefore $\Sigma \nVdash_{\mathbb{M}^c} \Delta$. Hence there is some model M in which $\Sigma \nVdash_M \Delta$, namely the canonical model. □

CHAPTER 6

Conclusion and future work

The present thesis was motivated by the idea of extending the completeness results in [**Restall 2005**] to substructural operators. We did not succeed in this task, however a small flaw in the original proof was corrected and the material was presented in more accessible way. The propositional non-distributive completeness result from [**Gehrke 2006**] was presented in similar fashion, with clarification of the methods used and with the addition of unary modal operators to the completeness proof (only binary ones are treated in the original paper). The overall picture emerging from the thesis is one in which classical, distributive and non-distributive settings share a non-negligible amount of features. This looks quite obvious when classical and distributive settings are seen as particular cases of posets. In the distributive setting, the universe of a Kripke-frame is a non-empty set with a (possibly) non-trivial order over it. Such order, in the classical case, boils down to the degenerate order given by identity. In fact, the two-sorted nature of Generalized Kripke frames is present all the way down to classical modal logic, but just in a less explicit way. We have seen in Chapter 3 that the absence of Boolean negation in the language required from us to explicitly bring into our table an element usually hidden in the background: the (order-) dual side of the theories, i.e. the co-theories. While the *theory* of a point in a frame is the set of all sentences satisfied in it -which constitute some sort of finger-print of the point-, the co-theory is the set of all unsatisfied (or refuted) sentences. Algebraically, in the poset of the formulas ordered by deducibility, the theories are filters while the co-theories are ideals. More precisely, while we did not have negation in the language we simply treated it directly on the structures being interpreted by considering both the (filter-shaped) *positive* side and the (ideal-shaped) *negative* side of a point in the canonical frame and then by talking about these in the metalanguage. Since we were in a distributive setting, though, this order-dual structure of ideals mirrors perfectly the structure of filters (the family of *positive sides* of points). To state it differently: a theory still uniquely determines its co-theory, and thus both sorts can still be seen as a two-sided monolith. Quantifiers are not disturbed by the increased generality at this level: the distributive setting looks very much as the classical setting although its canonical frame makes reference to co-theories. Its discrete duality indeed only needs to treat completely join irreducibles of the algebra. The treatment of binary modal operators seems unexpectedly harder in this distributive setting than with the unary analogues or than the non-distributive treatment. The two-sided monolith aspect of theories/co-theories without the two-sorted discrete duality and associated two-sorted satisfaction relations makes it presumably harder to track information.

We can summarize the humble contribution of our thesis as the parentheticals in the following table, along with the systematization in the presentation of the results.

	modal operators	Distributive setting completeness	Non-distributive setting completeness
Propositional logic	unary	subsumed under the quantified result	(added unary ops.)
	binary		[**Gehrke 2006**] (methods clarified)
Constant domains Quantified logic	unary	[**Restall 2005**] (flaw fixed and detail increased)	remains to do
	binary	remains to do	

We leave for future research the extension of [**Restall 2005**]'s proof to accommodate substructural operations (seen as binary modal operations), and the extension of the propositional completeness result in non-distributive setting to account for constant domain quantification. This might require a revised interpretation of quantifiers, as the standard interpretation forces distributivity.

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