Game characterizations of function classes and Weihrauch degrees

MSc Thesis *(Afstudeerscriptie)*

written by

Hugo de Holanda Cunha Nobrega
(born September 5, 1987 in Petrópolis, Brazil)

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Members of the Thesis Committee:
Dr. Jakub Szymanik (chair)
Dr. Benno van den Berg
Prof. Dr. Stefan Geschke
Dr. Yurii Khomskii
Prof. Dr. Benedikt Löwe
Prof. Dr. Jouko Väänänen

INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

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Abstract

Games are an important tool in mathematics and logic, providing a clear and intuitive understanding of the notions they define or characterize. In particular, since the seminal work of Wadge in the 1970s, game characterizations of classes of functions in Baire space have been a rich area of research, having had significant and far-reaching development by van Wesep, Andretta, Duparc, Motto Ros, and Semmes, among others.

In this thesis we study the connections between these games and the notion of Weihrauch reducibility, introduced in the context of computable analysis to express a particular type of continuous reducibility of functions. Especially through the work of Brattka and his collaborators, Gherardi, de Brecht and Pauly, among others, several functions — commonly referred to as choice principles — have been isolated that capture the complexity of classes of functions with respect to Weihrauch reducibility.

In particular, we use the games for the corresponding classes to provide new proofs of the Weihrauch-completeness of countable choice for the Baire class 1 functions and of discrete choice for the functions preserving $\mathbf{\Delta}^0_2$ under preimages, and to introduce new Weihrauch-complete choice principles for the class of functions preserving $\mathbf{\Delta}^0_3$ under preimages and for a particular class $\mathbf{\Lambda}_{2,3}$ characterized by a games of Semmes. In the process, we recast some of these games in a different style, which also allows for a uniform intuitive view of the way each game presented is related to the class of functions it characterizes.

Chapter 1

Introduction

Games have had considerable importance in several areas of mathematical logic since at least the middle of the 20th century. They have been used, e.g., to provide semantics for languages (see [13] for a survey) and to compare structures as in the Ehrenfeucht-Fraïssé game (see, e.g., [19, p. 52ff.]), among many other applications, and have been an important tool in set theory and the foundations of mathematics, starting with early work by Banach, Borel, Zermelo, and Kőnig, among others — see, e.g., [16, §27] for a thorough historical account of the subject—, and being heavily influenced by the seminal work of Gale and Stewart [10] on the determinacy of certain types of set-theoretical infinite games. One of the reasons for such a broad range of applications of games is that they provide an intuitive and clear understanding of several structural properties of their underlying objects, and these properties may then be used to help solve problems which could otherwise be less tractable — a prototypical example is Blackwell's short and elegant game proof of Kuratowski's coreduction principle for analytic sets [2].

In particular, game characterizations of classes of functions in Baire space have been in vogue since the 1970s, starting with Wadge's characterization of continuous functions via what is now known as the Wadge game, a result which would only be published years later in his PhD thesis [27] (see also [28] for Wadge's later account of this development). New advances were made mainly through the work of van Wesep and his backtrack game [30], later proved by Andretta to characterize the functions preserving the class $\mathbf{\Delta}^0_2$ under preimages, Duparc's (unpublished) eraser game for Baire class 1 (see, e.g., [25]), and with significant recent development by Semmes and his multitape, tree, $\mathsf{G}_{1,3}$, and $\mathsf{G}_{2,3}$ games characterizing the functions preserving the class $\mathbf{\Delta}^0_3$ under preimages, the Borel measurable functions, the Baire class 2 functions and another related class, later denoted as $\mathbf{\Lambda}_{2,3}$, respectively [26]. The work of Motto Ros and his general results about such characterizations [20, 21, 22] are also worthy of special note, among the research of other people in the area.

This thesis is about the connection between these types of games and the notion of Weihrauch reducibility, a certain type of continuous reducibility introduced in the unpublished work of Weihrauch in the early 1990s (see, e.g., [5]). Weihrauch was working in the context of computable analysis, where one is interested in questions involving the computable content of classical mathematical theorems. This reduction gives rise to a very rich degree structure which in a sense extends Turing and Medvedev degrees from computability theory [6, 12].

In this context, some functions have been isolated that capture the Weihrauch-complexity of certain classes of functions. These are commonly referred to by the name *choice principles*, the reason being their close relation to the operation of choosing an element from a set, given just a certain type of representation of the set. Through the work of Brattka and his collaborators [3, 5], Weihrauch-complete choice principles for the continuous functions, the functions preserving the class $\mathbf{\Delta}^0_2$ under preimages, the Baire class $k+1$ functions with $k \in \omega$, and the Borel measurable functions have already been uncovered.

In this thesis we give a uniform treatment of the games from the literature, and then apply them both to give new proofs of the Weihrauch-completeness of some choice principles, as well as to introduce new complete choice principles for the class of functions preserving the class $\mathbf{\Delta}^0_3$ under preimages and the class $\mathbf{\Lambda}_{2,3}$.

The text is organized as follows.

We close the present chapter with a brief review of some necessary preliminaries, and fix the notation to be used in the rest of the thesis.

In Chapter 2, we review the aforementioned games from the literature. In some cases our presentation differs from the original, so we prove some general results about the equivalence of certain types of presentations of games. We close the chapter with an overall picture of the games presented, showing in an intuitive sense how varying the value of two parameters — as dictated by the values m and n of the class $\mathbf{\Lambda}_{m,n}$ being characterized, in the first six cases — one can obtain all seven games above.

In Chapter 3, we review the basics of Weihrauch reducibility of functions between represented spaces, presenting without proof the completeness of the choice principles of k-countable choice for Baire class k with $k \geq 2$ and of closed choice on Baire space for the Borel measurable functions. Furthermore, we present with new proofs the completeness of discrete choice for $\mathbf{\Lambda}_{2,2}$ — although this is a fairly straightforward adaptation of a known result — and of countable choice for Baire class 1, and present two new complete choice principles for for the classes of functions preserving the class $\mathbf{\Delta}^0_3$ under preimages and the class $\mathbf{\Lambda}_{2,3}$, respectively, obtained in a direct way from the corresponding game characterizations.

Finally, in Chapter 4 we review the results presented in the thesis and point out some possible paths for future development.

Preliminaries and notation

We assume some knowledge of descriptive set theory, although for the purposes of this thesis only the development of that theory until the definition of the Borel hierarchy is necessary. Most of the notation we use is standard and can be found in the classic book by Kechris [17]. However, let us take this opportunity to fix some of the most important notation and nomenclature used in this thesis.

Sequences

As usual in descriptive set theory, we will mainly work on *Baire space* ω^ω, the topological space composed of countably infinite sequences of natural numbers. The topology on ω^ω is the product topology of ω, itself endowed with the discrete topology; this topology has a basis of clopen sets

$$\{[\sigma] : \sigma \in \omega^{<\omega}\},$$

where $\omega^{<\omega}$ is the set of finite sequences of natural numbers, and

$$[\sigma] := \{x \in \omega^\omega : \sigma \subset x\},$$

for each $\sigma \in \omega^{<\omega}$. This topology is also generated by the complete metric

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 2^{-n}, & \text{if } n \text{ is least such that } x(n) \neq y(n). \end{cases}$$

We will usually denote elements of ω^ω by $x, y, z, \ldots$, and elements of $\omega^{<\omega}$ by $\sigma, \tau, \ldots$ Sometimes it will also be useful to have the notation $\omega^{\leq\omega} := \omega^{<\omega} \cup \omega^\omega$ of the set of all sequences of natural numbers, finite or otherwise. We will use angle brackets when we want to specify the elements of a sequence explicitly, e.g., $\langle 2, 3, 5, 7, 11\rangle$ is the sequence of the first five prime numbers. The empty sequence $\langle\rangle$ is denoted by ϵ. Given $n \in \omega$, we denote by $\vec{n}$ the infinite sequence $\langle n, n, n, \ldots\rangle \in \omega^\omega$.

Given $\sigma \in \omega^{<\omega}$ we denote by $|\sigma|$ the unique n for which $\sigma \in \omega^n$, and call it the *length* of σ. We denote by $\operatorname{last}\sigma$ the last element of σ, i.e., the number $\sigma(|\sigma| - 1)$.

Given $\sigma \in \omega^{<\omega}$ and $s \in \omega^{\leq\omega}$, we denote by $\sigma^\frown s$ the *concatenation* of σ and s, i.e., the sequence with domain $\{n \in \omega : n < |\sigma| \text{ or } n + |\sigma| \in \operatorname{dom} s\}$ and given by

$$(\sigma^\frown s)(n) = \begin{cases} \sigma(n), & \text{if } n < |\sigma|, \\ s(n + |\sigma|), & \text{otherwise;} \end{cases}$$

thus $\sigma^\frown s = \langle \sigma(0), \sigma(1), \ldots, \operatorname{last}\sigma, s(0), s(1), \ldots\rangle$, with s determining whether this sequence is finite or infinite. We will sometimes abuse notation slightly and write $n^\frown\sigma$ and $\sigma^\frown n$ instead of $\langle n\rangle^\frown\sigma$ and $\sigma^\frown\langle n\rangle$ respectively.

Given $s, t \in \omega^{\leq\omega}$, we write $s \,||\, t$ when either $s \subseteq t$ or $t \subseteq s$, and $s \perp t$ when neither is the case.

Trees

A *tree* is a set $T \subseteq \omega^{<\omega}$ that is closed under initial segments. An *infinite path* of T, or simply a *path* of T, is an element $x \in \omega^\omega$ such that $x \restriction n \in T$ for any $n \in \omega$; we then say that each $x \restriction n$ is *on* the path x. We say T is *pruned* when every $\sigma \in T$ is on some path. We denote by $[T]$ the set of paths of T, and call it the *body* of T. The following classic result will be useful.

Theorem 1.1. *A set $X \subseteq \omega^\omega$ is closed iff there exists a pruned tree T such that $X = [T]$.*

The set of all trees is denoted by $\mathcal{T}(\omega)$, the set of finite trees by $\mathcal{T}_\omega(\omega)$, and the set of trees with a unique path by UP.

Function classes

Given $X \subseteq \omega^\omega$ and countable ordinals α and β, we will denote by $\mathbf{\Lambda}_{\alpha,\beta}(X)$ the class of functions $f : X \longrightarrow \omega^\omega$ for which

$$f^{-1}(U) \in \mathbf{\Sigma}^0_\beta(X)$$

for any $U \in \mathbf{\Sigma}^0_\alpha(\omega^\omega)$, where as usual X is endowed with the subspace topology — thus, in particular, $\mathbf{\Sigma}^0_\beta(X) = \{A \cap X : A \in \mathbf{\Sigma}^0_\beta(\omega^\omega)\}$. When X is clear from the context, we will omit it from the notation and simply write $\mathbf{\Lambda}_{\alpha,\beta}$.

In this thesis, we will denote partial functions by dashed arrows, e.g., we will write $f : A \dashrightarrow B$ to denote that f is a function with $\operatorname{dom} f \subseteq A$ and $\operatorname{ran} f \subseteq B$. Then, when $A = B = \omega^\omega$, we will simply write $f \in \mathbf{\Lambda}_{\alpha,\beta}$ instead of $f \in \mathbf{\Lambda}_{\alpha,\beta}(\operatorname{dom} f)$.

Recall that, given an ordinal α, a function $f : \omega^\omega \dashrightarrow \omega^\omega$ is said to be of *Baire class* α when f is the pointwise limit of functions $f_n : \operatorname{dom} f \longrightarrow \omega^\omega$ such that either $\alpha = 1$ and each f_n is continuous, or for each f_n there exists $1 < \alpha_n < \alpha$ such that f_n is of Baire class α_n. We will make repeated implicit use of the following result, whose proof we omit.

Theorem 1.2 (Lebesgue, Hausdorff, Banach cf. [17, Theorem 24.3])**.** *Let $f : \omega^\omega \dashrightarrow \omega^\omega$. Then*

$$f \text{ is of Baire class } \alpha \quad \text{iff} \quad f \in \mathbf{\Lambda}_{1,\alpha+1}.$$

Tupling functions on ω and ω^ω

Let us briefly fix some bijective tupling functions which will be used throughout the thesis, particularly in Chapter 3. We will use the same notation for most of them, but which function is meant in each case will always be clear from the context.

– The *pairing* $\ulcorner\cdot\urcorner : \omega^2 \longrightarrow \omega$, given by

$$\ulcorner n_0, n_1 \urcorner = \frac{1}{2}(n_0 + n_1 + 1)(n_0 + n_1) + n_1,$$

which associates each number $n \in \omega$ to a pair $\ulcorner n_0, n_1 \urcorner$ in a dovetailing fashion, as seen in Figure 1.1 below. In particular, this function and its inverse are computable.

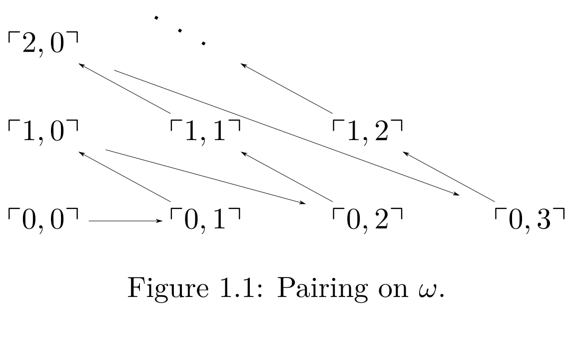


Figure 1.1: Pairing on ω.

– The *finite* tupling $\ulcorner \cdot \urcorner : \omega^k \longrightarrow \omega$, given by

$$\ulcorner n_0, \ldots, n_{k-1} \urcorner = \ulcorner n_0, \ulcorner n_1, \ulcorner \cdots, \ulcorner n_{k-2}, n_{k-1} \urcorner \cdots \urcorner \urcorner \urcorner.$$

– The *finite* tupling $\ulcorner \cdot \urcorner : (\omega^\omega)^k \longrightarrow \omega^\omega$, given by

$$\ulcorner x_0, \ldots, x_{k-1} \urcorner (n) = x_{k'}(n'),$$

where $n' = \lfloor \frac{n}{k} \rfloor$ and $k' = n \mod k$, i.e.,

$$\ulcorner x_0, \ldots, x_{k-1} \urcorner = \langle x_0(0), x_1(0), \ldots, x_{k-1}(0), x_0(1), x_1(1), \ldots, x_{k-1}(1), \ldots \rangle,$$

with inverse

$$x \mapsto \langle (x)^k_0, \ldots, (x)^k_{k-1} \rangle,$$

where $(x)^k_n = \langle x(n), x(n+k), x(n+2k), \ldots \rangle$.

– The *infinite* tupling $\ulcorner \cdot \urcorner : (\omega^\omega)^\omega \longrightarrow \omega^\omega$, given by

$$\ulcorner p \urcorner (\ulcorner n, k \urcorner) = p(n)(k),$$

with inverse

$$x \mapsto \langle (x)_0, (x)_1, \ldots \rangle,$$

where $(x)_n(k) = x(\ulcorner n, k \urcorner)$.

It will also be useful to have the following non-bijective tupling functions for $\omega^{<\omega}$.

– the injection $\ulcorner \cdot \urcorner : (\omega^{<\omega})^2 \dashrightarrow \omega^{<\omega}$, with domain $\{\langle \sigma, \tau \rangle : |\sigma| = |\tau|\}$, given by

$$\ulcorner \sigma, \tau \urcorner = \langle \sigma(0), \tau(0), \sigma(1), \tau(1), \ldots, \operatorname{last} \sigma, \operatorname{last} \tau \rangle, \text{ and}$$

– for each $n \in \omega$, the function $(\cdot)_n : \omega^{<\omega} \longrightarrow \omega^{<\omega}$, given by

$$(\sigma)_n = \langle \sigma(\ulcorner n, 0 \urcorner), \sigma(\ulcorner n, 1 \urcorner), \ldots, \sigma(\ulcorner n, k \urcorner) \rangle,$$

where k is maximum such that $\ulcorner n, k \urcorner < |\sigma|$.

Note that in particular we have

$$\begin{aligned} \ulcorner x, y \urcorner &= \bigcup_{k \in \omega} \ulcorner x \restriction k, y \restriction k \urcorner \quad \text{and} \\ (x)_n &= \bigcup_{k \in \omega} (x \restriction k)_n \end{aligned}$$

for any $x, y \in \omega^\omega$ and $n \in \omega$.

The following result will be useful in the sequel.

Theorem 1.3. *The finite tupling* $\ulcorner \cdot \urcorner : (\omega^\omega)^k \longrightarrow \omega^\omega$ *is continuous.*

Proof. Let $U \subseteq \omega^\omega$ be open, and let $(x_0, \ldots, x_{k-1}) \in (\ulcorner \cdot \urcorner)^{-1}[U]$. Then there exists $n \in \omega$ such that, for all $z \in \omega^\omega$, if $z \restriction n = \ulcorner x_0, \ldots, x_{k-1} \urcorner \restriction n$ then $z \in U$. Therefore, letting $m := \lceil \frac{n}{k} \rceil$, for any $(u_0, \ldots, u_{k-1}) \in (\omega^\omega)^k$ we have

$$\begin{aligned} d((u_0, \ldots, u_{k-1}), (x_0, \ldots, x_{k-1})) < 2^{-m} &\implies u_0 \restriction m = x_0 \restriction m, \ldots, u_{k-1} \restriction m = x_{k-1} \restriction m \\ &\implies \ulcorner u_0, \ldots, u_{k-1} \urcorner \restriction n = \ulcorner x_0, \ldots, x_{k-1} \urcorner \restriction n \\ &\implies (u_0, \ldots, u_{k-1}) \in (\ulcorner \cdot \urcorner)^{-1}[U], \end{aligned}$$

where the metric d on $(\omega^\omega)^k$ used above is the usual maximum metric. Hence $(\ulcorner \cdot \urcorner)^{-1}[U]$ is open. ■

A note on end-of-proof symbols

In this thesis, as we already did in Theorem 1.3 above, we will use the symbol ■ to signify the end of a proof. However, in order to facilitate the reading, when we structure a proof in separate claims we will number them sequentially within that proof and signify the end of each of their "mini"-proofs by the symbol $\boxed{n}$, where n is the number of the respective claim being proved. We will omit this symbol when the end of the mini-proof of the claim and of the overall proof coincide.

Chapter 2

Games

In this chapter we review the games in the literature that characterize the classes of continuous, Baire class 1, Baire class 2, $\mathbf{\Lambda}_{2,2}$, $\mathbf{\Lambda}_{2,3}$, $\mathbf{\Lambda}_{3,3}$, and Borel measurable functions, presenting some of them in a slightly novel way that will highlight the relationships between them and help us construct complete choice principles for the corresponding classes in the next chapter.

2.1 Games for functions on ω^ω

In vague terms, for our purposes a *game* is played between two players, **I** and **II**, who take turns in infinitely many rounds, each building at any given round a finitary approximation of an infinite object, and having access to one another's past moves. These infinite objects must satisfy a certain given set of rules for each player, and the winning condition of the game is then expressed in terms of whether or not the two constructed objects are related in some prescribed way. We will follow the usual convention and assume that **I** is a man and **II** is a woman.

The particular case of games for functions in Baire space has a well-established tradition in descriptive set theory, starting with Wadge's seminal work in the 1970s and 80s, and continuing with developments by van Wesep, Duparc, Andretta, and more recently through the work of Motto Ros and Semmes, among others. We will study these games extensively in this chapter, but for an intuitive idea we can say that in them, players **I** and **II** build elements of Baire space x and y, respectively, and the winning condition is then given in terms of whether or not we have $f(x) = y$, where the parameter f is a function from Baire space into itself that is known by both players from the offset. As a matter of fact, in all games we will consider the same fixed set of rules for player **I**, and so we will specify each different game by just describing the rules for **II**.

2.1.1 Definitions

Our style of presentation is largely based on Motto Ros's PhD thesis [20].

Definition 2.4. A *game* is a triple $G = (M, R, \iota)$, where M is a nonempty set of *moves*, $R \subseteq M^\omega$ is a nonempty set of *rules*, and $\iota : R \longrightarrow \omega^\omega$ is the *interpretation* function of G.

Given $f : \omega^\omega \dashrightarrow \omega^\omega$ and a game $G = (M, R, \iota)$, a *run of* G *for* f, or a *run of* $G(f)$, is played by two players, **I** and **II**, in ω *rounds*. At each round n, **I** first picks a natural number x_n and **II** then picks an element $y_n \in M$. At the (end of the) n^{th} round of the run, we denote by $\mathbf{I}(n)$ and $\mathbf{II}(n)$ the sequence of moves played so far by **I** and **II** respectively; thus, e.g., $\mathbf{I}(n) = \langle x_0, \ldots, x_n \rangle$. After ω rounds, **I** *produces* $x := \langle x_0, x_1, \cdots \rangle$ and **II** produces $y := \langle y_0, y_1, \ldots \rangle$. **II** *wins* the run when $x \notin \operatorname{dom} f$, or when $y \in R$ and $f(x) = \iota(y)$; otherwise **I** wins.

Given a game $G = (M, R, \iota)$, a *strategy for* **I** in G is a function $\phi : M^{<\omega} \longrightarrow \omega$. We say that **I** *follows* ϕ in a run of $G(f)$ when he plays $\phi(\epsilon)$ at round 0 and $\phi(\mathbf{II}(n-1))$ at round $n > 0$ of that run. Then, given $\sigma \in M^{<\omega}$, we denote by $\phi * \sigma$ the finite sequence produced by **I** after $|\sigma| + 1$ rounds of a run of $G(f)$ where he follows ϕ and **II**'s first moves produce σ; thus $\phi * \sigma = \langle \phi(\sigma \restriction 0), \phi(\sigma \restriction 1), \ldots, \phi(\sigma) \rangle$. This notation is extended to $\phi * y$, where $y \in M^\omega$, by putting $\phi * y := \bigcup_n (\phi * (y \restriction n))$. A strategy ϕ for **I** is *winning* in $G(f)$ when **I** wins all runs of $G(f)$ where he follows ϕ, i.e., when $\phi * y \in \operatorname{dom} f$ for all $y \in M^\omega$, and $f(\phi * y) \neq \iota(y)$ whenever $y \in R$.

Similarly, a *strategy for* **II** in G is a function $\phi : \omega^{<\omega} \longrightarrow M$. We say that **II** *follows* ϕ in a run of $G(f)$ when she plays $\phi(\mathbf{I}(n))$ at round n of that run. Given $\sigma \in \omega^{<\omega}$, we denote by $\sigma * \phi$ the finite sequence produced by **II** after $|\sigma|$ rounds of a run of $G(f)$ where she follows ϕ and **I**'s first moves produce σ; thus $\sigma * \phi = \langle \phi(\sigma \restriction 1), \phi(\sigma \restriction 2), \ldots, \phi(\sigma) \rangle$. This notation is extended to $x * \phi$, where $x \in \omega^\omega$, by putting $x * \phi := \bigcup_n ((x \restriction n) * \phi)$. A strategy ϕ for **II** is *legal* in $G(f)$ when $x * \phi \in R$ whenever $x \in \operatorname{dom} f$, and such a strategy is *winning* in $G(f)$ when **II** wins all runs of $G(f)$ where she follows ϕ, i.e., when $x * \phi \in R$ and $f(x) = \iota(x * \phi)$ for all $x \in \operatorname{dom} f$.

Given a class Γ of partial functions from ω^ω to ω^ω and a game G, we say that G *characterizes* Γ when for every $f : \omega^\omega \dashrightarrow \omega^\omega$ we have

$$f \in \Gamma \quad \text{iff} \quad \mathbf{II} \text{ has a winning strategy in } G(f).$$

In this chapter, our main objective is to review the games in the literature that characterize several classes of functions in the stratification $\mathbf{\Lambda}_{m,n}$, as well as the Borel measurable functions. It will turn out to be useful to present the games in a uniform fashion, each game being given in terms of player **II** being allowed to *change tapes*, *pass*, and *erase* past moves. This unified presentation of the games will help make the relationships between them clear, and will also come in handy when we are trying to extract choice principles from them in Chapter 3.

2.1.2 Default moves, rules, and interpretation

We will give player **II** countably many *tapes* on which to play, intuitively meaning that she will be able to build different potential outputs in parallel, as long as in the long run she chooses one of them as her actual output. This is implemented by a set $\mathbb{T} := \{\mathtt{t}_n : n \in \omega\}$ of moves, with $\mathtt{t}_n$ interpreted as *change to the* n^{th} *tape*.

Given a set M and $\sigma \in (M \cup \mathbb{T})^{<\omega}$, $n \in \omega$ and $m < |\sigma|$, we say that $\sigma(m)$ is *played on tape* n when $\sigma(m) \notin \mathbb{T}$, and either $\sigma(k) = \mathsf{t}_n$ for the maximum $k < m$ such that $\sigma(k) \in \mathbb{T}$, or $n = 0$ and there is no $k < m$ such that $\sigma(k) \in \mathbb{T}$. For any $n \in \omega$, the function $\mathsf{tape}_n : (M \cup \mathbb{T})^{<\omega} \longrightarrow M^{<\omega}$ then extracts the contents of the n^{th} tape of $\sigma \in (M \cup \mathbb{T})^{<\omega}$ as the finite sequence of moves in σ that are made on tape n. Formally, tape_n is defined recursively by $\mathsf{tape}_n(\epsilon) = \epsilon$ and

$$\begin{cases} \mathsf{tape}_n(\sigma^\frown s) &= \mathsf{tape}_n(\sigma)^\frown s, & \text{if } s \text{ is played on tape } n \\ \mathsf{tape}_n(\sigma^\frown s) &= \mathsf{tape}_n(\sigma), & \text{otherwise,} \end{cases}$$

and its extension $\mathsf{tape}_n : (M \cup \mathbb{T})^{\leq\omega} \longrightarrow M^{\leq\omega}$ is given by

$$\mathsf{tape}_n(x) = \bigcup_{k \in \omega} \mathsf{tape}_n(x \restriction k)$$

for $x \in (M \cup \mathbb{T})^{\omega}$. We then define $\mathsf{UT} := \{x \in (M \cup \mathbb{T})^{\omega} : \exists! n.\ \mathsf{tape}_n(x) \in M^{\omega}\}$. For $x \in \mathsf{UT}$, we denote the unique infinite sequence $\mathsf{tape}_n(x)$ by $o(x)$ and call it the *output tape* of x.

Passing and erasing moves are implemented by the sets $\mathbb{P} := \{\mathsf{p}\}$ and $\mathbb{E} := \{\mathsf{e}_n : n \in \omega\}$, with e_n interpreted as *erase all but the first n elements of the current tape.* Formally, given a set M we define a function $\mathsf{erase} : M^{<\omega} \times \omega \times \omega \longrightarrow M^{<\omega}$ recursively by $\mathsf{erase}(\epsilon, n, m) = \epsilon$, $\mathsf{erase}(\sigma, n, 0) = \sigma$, and

$$\mathsf{erase}(\sigma^\frown s, n, m+1) = \begin{cases} \mathsf{erase}(\sigma, n, m), & \text{if } s \in M \text{ and is played on tape } n, \\ \mathsf{erase}(\sigma, n, m+1)^\frown s, & \text{otherwise,} \end{cases}$$

which now allows us to define $\iota_* : (M \cup \mathbb{P} \cup \mathbb{E})^{<\omega} \longrightarrow M^{<\omega}$ recursively by $\iota_*(\epsilon) = \epsilon$ and

$$\iota_*(\sigma^\frown s) = \begin{cases} \iota_*(\sigma)^\frown s, & \text{if } s \in M, \\ \iota_*(\sigma), & \text{if } s = \mathsf{p}, \\ \mathsf{erase}(\iota_*(\sigma), n, |\mathsf{tape}_n(\iota_*(\sigma))| - m), & \text{if } s = \mathsf{e}_m \text{ is played on tape } n. \end{cases}$$

Given that they will all be defined with (subsets of) the same sets of moves $M_* := \omega \cup \mathbb{P} \cup \mathbb{E} \cup \mathbb{T}$, what will make the games we consider in this thesis significantly different will be their rules and interpretation functions. Still, the rules will always be given as subsets of

$$R_* := \{x \in (\omega \cup \mathbb{P} \cup \mathbb{E} \cup \mathbb{T})^{\omega} : x \in \mathsf{UT} \text{ and } \forall n\, \exists k > n.\ o(x)(k) \in \omega\},$$

and the interpretation functions will be given in terms of ι_*.

2.2 Classical games

In this section we review some of the more well-established games in the literature, namely the *Wadge game*, the *eraser game*, and the *backtrack game*, characterizing continuous, Baire class 1, and $\mathbf{\Lambda}_{2,2}$ functions, respectively.

2.2.1 The Wadge game

In his PhD thesis [27], William Wadge introduced a game characterizing the continuous functions. This is the simplest game we will study in this thesis, and upon which all the other games will be built.

Definition 2.5. The *Wadge game* is the game $\mathsf{G}_\mathsf{W} = (M_\mathsf{W}, R_\mathsf{W}, \iota_\mathsf{W})$, where

$$\begin{aligned} M_\mathsf{W} &= \omega \cup \mathbb{P}, \\ R_\mathsf{W} &= M_\mathsf{W}^\omega \cap R_*, \text{ and} \\ \iota_\mathsf{W}(x) &= \bigcup_{n\in\omega} \iota_*(x \restriction n). \end{aligned}$$

Thus, intuitively G_W is the game in which **II** can elect to pass — i.e., not play any natural number — for finitely many rounds before making any valid move.

Let us now prove that the Wadge game characterizes the continuous functions.

Theorem 2.6 (Wadge)**.** *Let* $f : \omega^\omega \dashrightarrow \omega^\omega$*. Then*

$$f \textit{ is continuous} \quad \textit{iff} \quad \mathbf{II} \textit{ has a winning strategy in } \mathsf{G}_\mathsf{W}(f).$$

Proof. ($\Rightarrow$) The continuity of f means that, for all $\sigma \in \omega^{<\omega}$ and for all x such that $f(x) \in [\sigma]$, there exists n such that $f[x \restriction n] \subseteq [\sigma]$. In particular, for all $i \in \omega$ and $x \in \omega^\omega$ there exists $\sigma_x^i \in \omega^{<\omega}$ such that $x \in [\sigma_x^i]$ and $f[\sigma_x^i] \subseteq [f(x) \restriction i]$.

Then the following is a winning strategy for **II**.

Strategy: At round n, let $i := |\iota_*(\mathbf{II}(n-1)|$. If $\mathbf{I}(n) \in \{\sigma_x^{i+1} : x \in \omega^\omega\}$ then play $f(\mathbf{I}(n)^\frown\vec{0})(i)$; otherwise pass.

($\Leftarrow$) By contraposition. Suppose f is not continuous, and let $\sigma \in \omega^{<\omega}$ be such that $U = f^{-1}[\sigma]$ is not open. Thus, there exists $x \in U$ such that, for all n, there exists $x^n \in [x \restriction n]$ with $x^n \notin U$, i.e, such that $f(x^n) \notin [\sigma]$. Then the following is a winning strategy for **I**.

Strategy: Play $x(0), x(1), \ldots$ until the round n in which **II** plays σ, then play $x^n(n+1), x^n(n+2), \ldots$ ■

2.2.2 The eraser game

The eraser game, usually attributed to unpublished work by Jacques Duparc (see, e.g., [25]), builds upon the Wadge game in order to characterize the Baire class 1 functions. Duparc's idea was to allow player **II** to erase her past moves, as long as for each position in the output there is a round after which it is never erased again.

Definition 2.7. The *eraser game* is the game $\mathsf{G}_\mathsf{e} = (M_\mathsf{e}, R_\mathsf{e}, \iota_\mathsf{e})$, where

$$\begin{aligned} M_\mathsf{e} &= \omega \cup \mathbb{P} \cup \mathbb{E}, \\ R_\mathsf{e} &= M_\mathsf{e}^\omega \cap \{x \in R_* : \forall n.\ \{k \in \omega : x(k) = \mathsf{e}_n\} \text{ is finite}\}, \text{ and} \\ \iota_\mathsf{e}(x) &= \lim_{n\in\omega} \iota_*(x \restriction n). \end{aligned}$$

The limit in the definition above is taken pointwise, i.e., given a sequence $\langle \sigma_n \rangle_{n\in\omega}$ in $\omega^{<\omega}$ and $x \in \omega^\omega$, we have $x = \lim_n \sigma_n$ iff $\forall k\, \exists N\, \forall n \geq N.\ |\sigma_n| > k$ and $\sigma_n(k) = x(k)$. Thus for all $x \in R_\mathsf{e}$ we have that $\lim_n \iota_*(x \upharpoonright n)$ is indeed well-defined.

Let us now move straight into the proof that the eraser game characterizes the Baire class 1 functions.

Theorem 2.8 (Duparc, folklore)**.** *Let $f : \omega^\omega \dashrightarrow \omega^\omega$. Then*

$$f \text{ is Baire Class 1} \quad \text{iff} \quad \mathbf{II} \text{ has a winning strategy in } \mathsf{G}_\mathsf{e}(f).$$

Proof. ($\Rightarrow$) Let $f_0, f_1, \ldots$ be continuous functions with domain $\operatorname{dom} f$, such that $f = \lim_n f_n$. For each n, let ϕ_n be a winning strategy for **II** in $\mathsf{G}_\mathsf{W}(f_n)$, and let ϕ be the following strategy for **II**.

Strategy: Begin by declaring that we are following f_0. At round n, when following f_N, for ease of notation let $\sigma := \iota_*(\mathbf{II}(n-1))$. If there exists a maximum $k \in \{N+1, \ldots, n\}$ such that $\sigma \perp \iota_*(\mathbf{I}(n) * \phi_k)$, then play e_m for the least m such that $\sigma(m) \neq \iota_*(\mathbf{I}(n) * \phi_k)(m)$, and declare that we now following f_k. Otherwise, if $\sigma \subset \iota_*(\mathbf{I}(n) * \phi_N)$, then play $\iota_*(\mathbf{I}(n) * \phi_N)(|\sigma|)$, else pass.

Claim 1. The strategy ϕ is winning for **II** in $\mathsf{G}_\mathsf{e}(f)$.

Indeed, let $x \in \operatorname{dom} f$ and $i \in \omega$. Since $\lim_n f_n = f$, there exists $N \in \omega$ such that for all $n \geq N$ we have $f_n(x)(i) = f(x)(i)$. If for some round k we have $|\iota_*(x \upharpoonright k * \phi)| > i$ but $\iota_*(x \upharpoonright k * \phi)(i) \neq f(x)(i)$, then at that round **II** was following some f_m with $m < N$. Therefore, at some later round **II** will erase her i^{th} output, and start following some $f_{m'}$ with $m' > m$. Thus, at some later round **II** will define her i^{th} output while following some f_n with $n \geq N$, and never again erase it. Hence

$$\begin{aligned}
\iota_\mathsf{e}(x * \phi)(i) &= \lim_{n\in\omega} \iota_*((x \upharpoonright n) * \phi)(i) \\
&= \iota_\mathsf{W}(x * \phi_N)(i) \\
&= f_N(x)(i) \\
&= f(x)(i)
\end{aligned}$$

$\boxed{1}$

($\Leftarrow$) Let ϕ be a winning strategy for **II** in $\mathsf{G}_\mathsf{e}(f)$. For each $n \in \omega$, define $f_n : \operatorname{dom} f \to \omega^\omega$ by

$$f_n(x) = \iota_*((x \upharpoonright n) * \phi)^\frown \vec{0},$$

Thus $x \upharpoonright n = y \upharpoonright n$ implies $f_n(x) = f_n(y)$, so in particular f_n is continuous for every $n \in \omega$.

Claim 2. $f = \lim_n f_n$.

Indeed, given $x \in \operatorname{dom} f$ and $i \in \omega$, let $N \in \omega$ be such that for all $n \geq N$ we have $|\iota_*((x \upharpoonright n) * \phi)| > i$; such N exists since ϕ is a legal strategy for **II** in $\mathsf{G}_\mathsf{e}(f)$. Since $\iota_\mathsf{e}(x * \phi)(i) = f(x)(i)$, it now follows that $f_n(x)(i) = f(x)(i)$ for all $n \geq N$. ∎

Allowing player II to change tapes

We have defined the eraser game so as to allow player **II** to use each of her eraser finitely often, but not change tapes. However, as we shall see in this section, also allowing her to change tapes finitely often turns out to not give her any more power — any such change of tape could have been legally mimicked by a careful use of the erasers. Although this will make the game arguably more artificial, it will turn out to be useful when we give an uniform overview of the games at the end of the chapter.

Formally, let G'_{e} be the game $(M'_{\mathsf{e}}, R'_{\mathsf{e}}, \iota'_{\mathsf{e}})$, where

$$\begin{array}{rcl} M'_{\mathsf{e}} & = & \omega \cup \mathbb{P} \cup \mathbb{E} \cup \mathbb{T}, \\ R'_{\mathsf{e}} & = & (M'_{\mathsf{e}})^{\omega} \cap \{x \in R_* : \operatorname{ran} x \cap \mathbb{T} \text{ is finite, and} \\ & & \qquad \forall n.\ \{k \in \omega : o(x)(k) = \mathsf{e}_n\} \text{ is finite}\}, \\ \iota'_{\mathsf{e}}(x) & = & \lim_{n \in \omega} \iota_*(o(x) \upharpoonright n). \end{array}$$

Theorem 2.9. *Let $f : \omega^\omega \dashrightarrow \omega^\omega$. Then*

$$\textbf{II} \textit{ has a winning strategy in } \mathsf{G}_{\mathsf{e}}(f) \quad \textit{iff} \quad \textit{she has one in } \mathsf{G}'_{\mathsf{e}}(f).$$

Proof. Clearly, for any $f : \omega^\omega \dashrightarrow \omega^\omega$, any winning strategy for **II** in $\mathsf{G}_{\mathsf{e}}(f)$ is also a winning strategy for **II** in $\mathsf{G}'_{\mathsf{e}}(f)$, so to prove the converse let ϕ' be a winning strategy for **II** in $\mathsf{G}'_{\mathsf{e}}(f)$, and let ϕ^* be the strategy for her in $\widehat{\mathsf{G}_{\mathsf{e}}}(f)$ obtained from ϕ' by copying the moves in $\omega \cup \mathbb{P} \cup \mathbb{E}$, but substituting a move of the form $\mathsf{t}_n \in \mathbb{T}$ at round k by the move $\mathsf{e}_0{}^\frown \iota_*(\mathsf{tape}_n((x \upharpoonright k) * \phi))$.

One then readily sees that $\phi = \operatorname{seq}(\phi^*)$ is a strategy for **II** in $\mathsf{G}_{\mathsf{e}}(f)$ that produces the same output as ϕ'.

Claim 1. The strategy ϕ is winning for **II** in $\mathsf{G}_{\mathsf{e}}(f)$.

Indeed, since $\operatorname{ran}(x * \phi') \cap \mathbb{T}$ is finite, each e_n is only used finitely many more times when **II** follows ϕ than when she follows ϕ', so that $\{k \in \omega : (x * \phi)(k) = \mathsf{e}_n\}$ is also finite, and thus ϕ is legal. Furthermore, after she plays e_0 (when following ϕ) corresponding to the last time ϕ' dictated that she should play in $\mathbb{T}$, all her moves are done following ϕ', and therefore ϕ is winning. ■

2.2.3 The backtrack game

The backtrack game, introduced by Robert van Wesep in his PhD thesis [30], also builds upon the Wadge game but characterizes the class $\mathbf{\Lambda}_{2,2}$. The idea is to allow player **II** to start over with the construction of her output finitely often.

Definition 2.10. The *backtrack game* is the game $\mathsf{G}_{\mathsf{bt}} = (M_{\mathsf{bt}}, R_{\mathsf{bt}}, \iota_{\mathsf{bt}})$, where

$$\begin{array}{rcl} M_{\mathsf{bt}} & = & \omega \cup \mathbb{P} \cup \mathbb{T}, \\ R_{\mathsf{bt}} & = & M_{\mathsf{bt}}^{\omega} \cap \{x \in R_* : \operatorname{ran} x \cap \mathbb{T} \text{ is finite}\}, \text{ and} \\ \iota_{\mathsf{bt}}(x) & = & \bigcup_{n \in \omega} \iota_*(o(x) \upharpoonright n). \end{array}$$

The backtrack game is more commonly defined by allowing player **II** to make moves in $\omega \cup \mathbb{P} \cup \{\mathsf{bt}\}$, where bt is a move interpreted as *ignore all moves made so far*, as long as **II** makes moves in ω infinitely often and plays bt only finitely often. Clearly, these definitions are equivalent; playing bt has exactly the same effect as playing some $t \in \mathbb{T}$ that has never been played before, and thus requiring that **II** play bt only finitely often is the same as requiring that **II** only use finitely many tapes.

The following is the most important tool when dealing with $\mathbf{\Lambda}_{2,2}$.

Theorem 2.11 (Jayne-Rogers [14]). *Let $f : \omega^\omega \dashrightarrow \omega^\omega$. Then $f \in \mathbf{\Lambda}_{2,2}$ iff $\operatorname{dom} f$ can be partitioned into relatively closed sets $\{A_n : n \in \omega\}$ in such a way that $f \restriction A_n$ is continuous for any $n \in \omega$.*

The proof of this theorem is outside the scope of this thesis, and can be found in, e.g., [23] (see also [15]). Let us now see how the Jayne-Rogers theorem can be used to prove that the backtrack game characterizes $\mathbf{\Lambda}_{2,2}$.

Theorem 2.12 (Andretta [1]). *Let $f : \omega^\omega \dashrightarrow \omega^\omega$. Then*

$$f \in \mathbf{\Lambda}_{2,2} \quad \textit{iff} \quad \mathbf{II} \textit{ has a winning strategy in } \mathsf{G}_{\mathsf{bt}}(f).$$

Proof. ($\Rightarrow$) Let $T_0, T_1, \ldots$ be trees on $\omega^{<\omega}$ such that $\{[T_n] \cap \operatorname{dom} f : n \in \omega\}$ is a partition for f as in the Jayne-Rogers theorem. For each n, let ϕ_n be a winning strategy for **II** in $\mathsf{G}_{\mathsf{W}}(f \restriction [T_n])$, and let ϕ be the following strategy for **II** in $\mathsf{G}_{\mathsf{bt}}(f)$.

Strategy: At round n, do the following.

If at the last round we did not play in $\mathbb{T}$, then let ϕ_k be the strategy we are following, m be the tape we are playing on, and $\ell = |\mathsf{tape}_m(\mathbf{II}(n-1))|$. If $\mathbf{I}(n) \in T_k$, then play $\phi_k(\mathbf{I}(\ell))$; else go to a new tape (i.e., play some $t \in \mathbb{T}$ that has never been played before).

Otherwise, if at the last round we played in $\mathbb{T}$, or if this is the first round, then play $\phi_k(\mathbf{I}(0))$, where k is least such that $\mathbf{I}(n) \in T_k$, and declare that we are now following ϕ_k.

Claim 1. The strategy ϕ is winning for **II** in $\mathsf{G}_{\mathsf{bt}}(f)$.

Indeed, suppose **I** produces $x \in \operatorname{dom} f$ and **II** follows ϕ. There is a unique n such that $x \in [T_n]$; thus, for any $k < n$ such that **II** follows ϕ_k at some round, there is a first later round m such that $x \restriction m \notin T_k$, and at that round **II** will start playing in a new tape. At round $m+1$, **II** then starts following some strategy ϕ_ℓ with $\ell > k$. Therefore, at some point **II** starts following ϕ_n, and from that point on she will never play in $\mathbb{T}$ again. Thus $o(x * \phi) = x * \phi_n$, and

$$\begin{aligned}
\iota_{\mathsf{bt}}(x * \phi) &= \bigcup_{n \in \omega} \iota_*(o(x * \phi) \restriction n) \\
&= \bigcup_{n \in \omega} \iota_*((x * \phi_n) \restriction n) \\
&= (f \restriction [T_n])(x) \\
&= f(x).
\end{aligned}$$ ⊡

($\Leftarrow$) Let ϕ be a winning strategy for **II**. Since for every $x \in \operatorname{dom} f$ we have $\operatorname{ran}(x * \phi) \cap \mathbb{T}$ finite, we can without loss of generality assume that each time ϕ tells **II** to change tape, it is into a brand new one. Now define

$$\mathsf{BT} := \phi^{-1}[\mathbb{T}],$$

i.e., BT is the set of words that cause **II** to change tape when following ϕ.

For each $\sigma \in \mathsf{BT} \cup \{\epsilon\}$, define

$$A_\sigma := \big([\sigma] \smallsetminus \bigcup\{[\tau] : \tau \in \mathsf{BT} \text{ and } \sigma \subset \tau\}\big) \cap \operatorname{dom} f,$$

i.e., A_σ is the set of $x \in [\sigma] \cap \operatorname{dom} f$ for which the last time **II** plays in $\mathbb{T}$ is when **I** plays σ. Since $[\sigma]$ is clopen for any $\sigma \in \omega^{<\omega}$, we have that A_σ is relatively closed for any $\sigma \in \mathsf{BT}$. Furthermore, for any $x \in \operatorname{dom} f$, note that $x \in A_\sigma \cap F_\tau$ implies $\sigma = \tau$, and therefore $\{A_\sigma : \sigma \in \mathsf{BT}\}$ is a countable partition of $\operatorname{dom} f$, after weeding out repetitions if necessary.

Claim 2. The function $f \restriction A_\sigma$ is continuous for each $\sigma \in \mathsf{BT}$.

Indeed, a winning strategy for **II** in $\mathsf{G_W}(f \restriction A_\sigma)$ is to pass until $\mathbf{I}(n) = \sigma$, and then start following ϕ. $\boxed{2}$

Thus, again by the Jayne-Rogers theorem, we have $f \in \mathbf{\Lambda}_{2,2}$. ■

2.3 Semmes's games

In his PhD thesis [26], Brian Semmes introduced several new games, characterizing the Borel functions, the Baire class 2 functions, and the classes $\mathbf{\Lambda}_{2,3}$ and $\mathbf{\Lambda}_{3,3}$. However, his approach was slightly different from ours; except for his *multitape game* characterizing $\mathbf{\Lambda}_{3,3}$ which fits nicely into our framework, in Semmes's games player **II** plays a finite tree in some tape at each round, and in the end her output is the unique infinite path of the infinite tree resulting from the union of the trees she played on her output tape (in particular the rules dictate that such a unique infinite path must exist). The variation between the games is then given by demanding that this resulting tree have a certain specific shape in each case.

Intuitively, the idea behind requiring that **II** play finite trees is that making such moves is equivalent to making finitely many moves in $\omega \cup \mathbb{E}$ at each round, the branching nodes of the finite tree representing corrections to **II**'s current *guess* of what the prefix of the correct output should be, and therefore corresponding to the power that **II** has by being allowed to erase past moves. And making finitely many moves at each round turns out not to be any more powerful than making a single one; as we shall see, by carefully keeping track of what she has to play, and in which order, **II** can produce the same output by making one move at each round as she could by making finitely many.

Let us make a small detour to make these observations more precise in a general setting.

Allowing II to make finitely many moves at each round

In most of the games we describe in this thesis, player **II** is only allowed to make one move at each round. However, for convenience sometimes it will be useful to allow her to make any finite number of moves at each round, which we will prove does not give player **II** any more freedom.

Given a game $G = (M, R, \iota)$, let $\widehat{G} = (\hat{M}, \hat{R}, \hat{\iota})$ be given by

$$\begin{aligned} \hat{M} &= M^{<\omega} \smallsetminus \{\epsilon\} \\ \hat{R} &= \{\hat{y} \in (\hat{M})^{\omega} : y = \hat{y}(0)^\frown\hat{y}(1)^\frown\hat{y}(2)^\frown \cdots \in R\} \\ \hat{\iota}(\hat{y}) &= \iota(y), \end{aligned}$$

i.e., $\widehat{G}$ is the version of G where **II** is allowed to make finitely many moves at each round.

Theorem 2.13. *Let $f : \omega^\omega \dashrightarrow \omega^\omega$. Then*

$$\textbf{II} \textit{ has a winning strategy in } G(f) \quad \textit{iff} \quad \textit{she has one in } \widehat{G}(f).$$

Proof. Clearly, winning strategies for **II** in G are also winning for her in $\hat{G}$. Now, given a strategy ϕ for **II** in $\hat{G}$, let $\operatorname{seq}(\phi)$ be the following strategy for her in G:

Strategy: Start with an empty queue Q.

At round n, add $\phi(\mathbf{I}(n))(0), \phi(\mathbf{I}(n))(1), \ldots, \operatorname{last}\phi(\mathbf{I}(n))$ to Q, in that order, then remove the first element s from Q and play it.

In other words, $\operatorname{seq}(\phi)$ is the strategy that makes the same moves as ϕ in the long run, and in the same order, but only makes one of them at each round. Hence, by construction, for any $x \in \omega^\omega$ we have

$$\begin{aligned} x * \operatorname{seq}(\phi) &= \phi(x\restriction 1)^\frown\phi(x\restriction 2)^\frown\phi(x\restriction 3)^\frown \cdots \\ &= x * \phi. \end{aligned}$$

Therefore, for any $f : \omega^\omega \dashrightarrow \omega^\omega$, if ϕ is winning in $\hat{G}(f)$ then so is $\operatorname{seq}(\phi)$ in $G(f)$. ■

In our framework, strategies for **II** are functions from $\omega^{<\omega}$ to $\omega \cup \mathbb{P} \cup \mathbb{E} \cup \mathbb{T}$; call these *regular strategies*, to contrast with strategies for Semmes's games, which are functions from $\omega^{<\omega}$ to $\mathcal{T}_\omega(\omega) \cup \mathbb{T}$, and which we call *tree strategies*.

Transforming tree strategies into regular ones

Given $\phi : \omega^{<\omega} \longrightarrow \mathcal{T}_\omega(\omega) \cup \mathbb{T}$, first define $\phi^* : \omega^{<\omega} \longrightarrow (\omega \cup \mathbb{P} \cup \mathbb{E} \cup \mathbb{T})^{<\omega}$ as follows (for convenience, we will describe ϕ^* using the game lingo we have used so far, i.e., in order to describe how to obtain $\phi^*(\sigma)$ we will pretend we are in the n^{th} round of a run of a game in which $\mathbf{I}(n) = \sigma$).

Strategy: Start with an empty queue Q.

At round n, let k be the tape we are currently playing on, $\tau = \iota_*(\mathsf{tape}_k(\mathbf{II}(n-1)))$, and $m = |\tau|$.

If $\phi(\mathbf{I}(n)) \in \mathbb{T}$ then add $\phi(\mathbf{I}(n))$ to the end of Q. Otherwise add all elements of $\mathsf{tape}_k(\mathbf{I}(n) * \phi)(m) \setminus \mathsf{tape}_k(\mathbf{I}(n) * \phi)(m-1)$ — or all elements of $\mathsf{tape}_k(\mathbf{I}(n) * \phi)(m)$ if $m = 0$ — to the end of Q.

Now remove the first element σ of Q. If $\sigma \in \mathbb{T}$, then play σ. Otherwise, if $\sigma \subseteq \tau$ play p, and if $\tau \subset \sigma$, then let $\ell := |\tau|$ and play

$$\langle \sigma(\ell), \sigma(\ell+1), \ldots, \operatorname{last} \sigma \rangle.$$

Finally, if $\tau \perp \sigma$, then let ℓ be least such that $\tau(\ell) \neq \sigma(\ell)$, then play

$$\langle \mathsf{e}_\ell, \sigma(\ell+1), \sigma(\ell+2), \ldots, \operatorname{last} \sigma \rangle.$$

Now define $\operatorname{sim}(\phi) := \operatorname{seq}(\phi^*) : \omega^{<\omega} \longrightarrow \omega \cup \mathbb{P} \cup \mathbb{E} \cup \mathbb{T}$, where $\operatorname{seq}(\cdot)$ is as described in the proof of Theorem 2.13.

By construction, we have the following

Theorem 2.14. *For any tree strategy ϕ, $x \in \omega^\omega$ and $n \in \omega$ we have*

$$\left\{ \iota_*\big(\mathsf{tape}_n((x \restriction k) * \operatorname{sim}(\phi))\big) : k \in \omega \right\} = \bigcup \left\{ T : T \in \operatorname{ran} \mathsf{tape}_n(x * \phi) \right\}.$$

In particular, if $\sigma_0 \subset \sigma_1 \subset \sigma_2 \subset \cdots$ is an infinite chain in

$$A := \left\{ \iota_*\big(\mathsf{tape}_n((x \restriction k) * \operatorname{sim}(\phi))\big) : k \in \omega \right\},$$

then $y = \bigcup_n \sigma_n$ is an infinite path of

$$B := \bigcup \left\{ T : T \in \operatorname{ran} \mathsf{tape}_n(x * \phi) \right\},$$

and conversely if y is an infinite path of B, then $\{ y \restriction k : k \in \omega \}$ is an infinite chain in A.

Transforming regular strategies into tree ones

Given $\phi : \omega^{<\omega} \longrightarrow \omega \cup \mathbb{P} \cup \mathbb{E} \cup \mathbb{T}$, we define $\operatorname{tree}(\phi) : \omega^{<\omega} \longrightarrow \mathcal{T}_\omega(\omega) \cup \mathbb{T}$ as follows.

Strategy: For every k, let $T^k_0 := \emptyset$.

At round n, if $\phi(\mathbf{I}(n)) \in \mathbb{T}$, then play $\phi(\mathbf{I}(n))$. Otherwise, let k be the current tape we are playing on and m be the number of moves we have made so far on this tape, then play $T^k_{m+1} := T^k_m \cup \{ \iota_*(\mathsf{tape}_k(\mathbf{I}(n) * \phi)) \}$.

Again by construction, we have the following

Theorem 2.15. *For any regular strategy ϕ, $x \in \omega^\omega$ and $n \in \omega$ we have*

$$\{\iota_*(\mathsf{tape}_n((x \restriction k) * \phi)) : k \in \omega\} = \bigcup \{T : T \in \operatorname{ran} \mathsf{tape}_n(x * \operatorname{tree}(\phi))\} .$$

In particular, if $\sigma_0 \subset \sigma_1 \subset \sigma_2 \subset \cdots$ is an infinite chain in

$$A := \{\iota_*(\mathsf{tape}_n((x \restriction k) * \phi)) : k \in \omega\} ,$$

then $y = \bigcup_n \sigma_n$ is an infinite path of

$$B := \bigcup \{T : T \in \operatorname{ran} \mathsf{tape}_n(x * \operatorname{tree}(\phi))\} ,$$

and conversely if y is an infinite path of B, then $\{y \restriction k : k \in \omega\}$ is an infinite chain in A.

The following simple observation, whose proof follows directly from the definitions and will therefore be omitted, will be useful in the sequel.

Lemma 2.16. *For any $y \in (\omega \cup \mathbb{P} \cup \mathbb{E} \cup \mathbb{T})^\omega$ and any $n \in \omega$, we have that infinite chains in*

$$\{\iota_*(\mathsf{tape}_n(y \restriction k)) : k \in \omega\}$$

are the same as infinite chains in

$$\{\sigma \in \omega^{<\omega} : \exists^\infty k.\ \sigma \subseteq \iota_*(\mathsf{tape}_n(y \restriction k))\}.$$

2.3.1 The game $\mathsf{G}_{1,3}$

The first of Semmes's games we will see is the game $\mathsf{G}_{1,3}$, which characterizes the Baire class 2 functions by building upon the eraser game, in a sense.

Definition 2.17. $\mathsf{G}_{1,3}$ is the game $(M_{1,3}, R_{1,3}, \iota_{1,3})$, where

$$\begin{aligned} M_{1,3} &= \omega \cup \mathbb{P} \cup \mathbb{E}, \\ R_{1,3} &= M_{1,3}^\omega \cap \{x \in R_* : \{\sigma \in \omega^{<\omega} : \exists^\infty k.\ \sigma \subseteq \iota_*(x \restriction k)\} \text{ is a chain } \langle \sigma_n^x \rangle_{n\in\omega}\}, \text{ and} \\ \iota_{1,3}(x) &= \bigcup_{n\in\omega} \sigma_n^x \end{aligned}$$

Thus, intuitively **II** can make the same moves as in the eraser game, but she is now allowed to use each eraser infinitely often, as long the set of words which are part of her output at infinitely many rounds forms a chain.

Originally, this game was presented by Semmes as the game $\mathsf{G}_{1,3}^S(f) = (M_{1,3}^S, R_{1,3}^S, \iota_{1,3}^S)$, where

$$\begin{aligned} M_{1,3}^S &= \mathcal{T}_\omega(\omega), \\ R_{1,3}^S &= \{x \in (M_{1,3}^S)^\omega : T_{1,3}^S(x) := \textstyle\bigcup_n x(n) \in \mathsf{UP},\ \forall n.\ x(n) \subseteq x(n+1), \text{ and} \\ &\qquad \text{all infinitely branching nodes of } T_{1,3}^S(x) \text{ lie on its infinite path}\}, \\ \iota_{1,3}^S(x) &= \text{the infinite path of } T_{1,3}^S(x) \end{aligned}$$

As promised, these games are equivalent.

Theorem 2.18. *Let $f : \omega^\omega \dashrightarrow \omega^\omega$.*

For any legal strategy ϕ for **II** *in $\mathsf{G}^S_{1,3}(f)$, we have that $\mathrm{sim}(\phi)$ is a legal strategy for her in $\mathsf{G}_{1,3}(f)$, and that $\iota_{1,3}(x*\mathrm{sim}(\phi))$ is the infinite path of $T_{1,3}(x*\phi)$ for any $x \in \mathrm{dom}\, f$.*

Conversely, for any legal strategy ϕ for **II** *in $\mathsf{G}_{1,3}(f)$, we have that $\mathrm{tree}(\phi)$ is a legal strategy for her in $\mathsf{G}^S_{1,3}(f)$, and that $\iota_{1,3}(x * \phi)$ is the infinite path of $T^S_{1,3}(x * \mathrm{tree}(\phi))$ for any $x \in \mathrm{dom}\, f$.*

Proof. Let ϕ be legal for **II** in $\mathsf{G}^S_{1,3}(f)$. To see that $\mathrm{sim}(\phi)$ is legal for her in $\mathsf{G}_{1,3}(f)$, let $x \in \mathrm{dom}\, f$. Since $T^S_{1,3}(x * \phi)$ is infinite, by Theorem 2.14 it follows that $x * \mathrm{sim}(\phi) \in R_*$. Furthermore, again by Theorem 2.14 and using Lemma 2.16, since all infinitely-branching nodes of $T^S_{1,3}(x * \phi)$ lie on its unique infinite path, we have that the set $\{\sigma \in \omega^{<\omega} : \exists^\infty k.\ \sigma \subseteq \iota_*((x * \mathrm{sim}(\phi)) \restriction k)\}$ is an infinite chain, so $x * \mathrm{sim}(\phi) \in R_{1,3}$, i.e., $\mathrm{sim}(\phi)$ is legal. Finally, once more by Theorem 2.14, we have that

$$\iota_{1,3}(x * \mathrm{sim}(\phi)) = \bigcup\{\sigma \in \omega^{<\omega} : \exists^\infty k.\ \sigma \subseteq \iota_*((x * \mathrm{sim}(\phi)) \restriction k)\}$$

is the infinite path of $T^S_{1,3}(x * \phi)$.

Conversely, let ϕ be legal for **II** in $\mathsf{G}_{1,3}(f)$. To see that $\mathrm{tree}(\phi)$ is legal for her in $\mathsf{G}^S_{1,3}(f)$, let $x \in \mathrm{dom}\, f$. Clearly for any $n \in \omega$ we have $(x * \mathrm{tree}(\phi))(n) \subseteq (x * \mathrm{tree}(\phi))(n + 1)$. Also, since $\{\sigma \in \omega^{<\omega} : \exists^\infty k.\ \sigma \subseteq \iota_*((x*\phi) \restriction k)\}$ is an infinite chain, by Theorem 2.15 and Lemma 2.16 we have that $T^S_{1,3}(x * \mathrm{tree}(\phi))$ has a unique infinite path, and all infinitely-branching nodes of $T^S_{1,3}(x*\mathrm{tree}(\phi))$ lie on this path. Thus $\mathrm{tree}(\phi)$ is legal. Finally, again by Theorem 2.15, we have that $\iota_{1,3}(x * \phi) = \bigcup\{\sigma \in \omega^{<\omega} : \exists^\infty k.\ \sigma \subseteq \iota_*((x * \phi) \restriction k)\}$ is the infinite path of $T^S_{1,3}(x * \mathrm{tree}(\phi))$. ■

We will omit the proof that $\mathsf{G}^S_{1,3}(f)$ characterizes the Baire class 2 functions, referring the reader to the original [26] for the details.

Theorem 2.19 (Semmes [26])**.** *Let $f : \omega^\omega \dashrightarrow \omega^\omega$. Then*

f is Baire Class 2 iff **II** *has a winning strategy in $\mathsf{G}^S_{1,3}(f)$.*

Corollary 2.20. *Let $f : \omega^\omega \dashrightarrow \omega^\omega$. Then*

f is Baire Class 2 iff **II** *has a winning strategy in $\mathsf{G}_{1,3}(f)$.*

Allowing player II to change tapes

As we did in the eraser game, let us now show that we can allow player **II** to change tapes in $\mathsf{G}_{1,3}$ — even infinitely often — in such a way as to not increase her overall power in the game. Again, although this will make the game arguably more artificial, it will turn out to be useful when we give an uniform overview of the games at the end of the chapter.

Formally, let $\mathsf{G}'_{1,3}$ be the game $(M'_{1,3}, R'_{1,3}, \iota'_{1,3})$, where

$$
\begin{aligned}
M'_{1,3} &= \omega \cup \mathbb{P} \cup \mathbb{E} \cup \mathbb{T}, \\
R'_{1,3} &= (M'_{1,3})^\omega \cap \{x \in R_* : \{\sigma \in \omega^{<\omega} : \exists^\infty k\, \exists m_k.\ \sigma \subseteq \iota_*(\mathsf{tape}_{m_k}(x \restriction k))\} \\
&\qquad\qquad \text{is a chain } \langle \sigma_n^x \rangle_{n \in \omega}\}, \text{ and} \\
\iota'_{1,3}(x) &= \bigcup_{n \in \omega} \sigma_n^x
\end{aligned}
$$

Theorem 2.21. *For any $f : \omega^\omega \dashrightarrow \omega^\omega$, **II** has a winning strategy in $\mathsf{G}_{1,3}(f)$ iff she has one in $\mathsf{G}'_{1,3}(f)$.*

Proof. Clearly any winning strategy for **II** in $\mathsf{G}_{1,3}(f)$ is also winning for her in $\mathsf{G}'_{1,3}(f)$, so to prove the converse let ϕ' be a winning strategy for **II** in $\mathsf{G}'_{1,3}(f)$. Let ϕ be the strategy for her in $\mathsf{G}_{1,3}$ obtained from ϕ' exactly as in the proof of Theorem 2.9.
Claim 1. The strategy ϕ is winning for **II** in $\mathsf{G}_{1,3}(f)$.

Indeed, for any $\tau \in \omega^{<\omega}$ we have

$$
\begin{aligned}
&\tau \in \{\sigma \in \omega^{<\omega} : \exists^\infty k\, \exists m_k.\ \sigma \subseteq \iota_*(\mathsf{tape}_{m_k}((x * \phi') \restriction k))\} \\
&\qquad \iff \tau \in \{\sigma \in \omega^{<\omega} : \exists^\infty k.\ \sigma \subseteq \iota_*((x * \phi) \restriction k)\},
\end{aligned}
$$

so these sets have the same unique infinite chain. ■

2.3.2 The game $\mathsf{G}_{2,3}$

The game $\mathsf{G}_{2,3}$ was introduced by Semmes in order to characterize the class $\mathbf{\Lambda}_{2,3}$. The idea by Semmes was to build upon the eraser game by allowing player **II** the further freedom of changing tapes infinitely often, as long as there is a unique tape that she makes moves on infinitely often.

Definition 2.22. $\mathsf{G}_{2,3}$ is the game $(M_{2,3}, R_{2,3}, \iota_{2,3})$, where

$$
\begin{aligned}
M_{2,3} &= \omega \cup \mathbb{P} \cup \mathbb{E} \cup \mathbb{T}, \\
R_{2,3} &= M_{2,3}^\omega \cap \{x \in R_* : \forall n.\ \{k \in \omega : o(x)(k) = \mathsf{e}_n\} \text{ is finite}\}, \text{ and} \\
\iota_{2,3}(x) &= \lim_{n \in \omega} \iota_*(o(x) \restriction n).
\end{aligned}
$$

Originally, this game was presented by Semmes as the game $\mathsf{G}^S_{1,3}(f) = (M^S_{1,3}, R^S_{1,3}, \iota^S_{1,3})$, where

$$
\begin{aligned}
M^S_{2,3} &= \mathcal{T}_\omega(\omega) \cup \mathbb{T}, \\
R^S_{2,3} &= \{x \in (M^S_{2,3})^\omega : x \in \mathsf{UT},\ \forall n \forall i, j \in \operatorname{dom} \mathsf{tape}_n(x). \mathsf{tape}_n(x)(i) \subseteq \mathsf{tape}_n(x)(j), \\
&\qquad T^S_{2,3}(x) := \textstyle\bigcup_n o(x)(n) \in \mathsf{UP}, \text{ and } T^S_{2,3}(x) \text{ is finitely branching}\}, \\
\iota^S_{2,3}(x) &= \text{the infinite path of } T^S_{2,3}(x)
\end{aligned}
$$

Again, these games are equivalent.

Theorem 2.23. *Let $f : \omega^\omega \dashrightarrow \omega^\omega$.*

For any legal strategy ϕ for **II** *in $\mathsf{G}^S_{2,3}(f)$, we have that $\mathrm{sim}(\phi)$ is a legal strategy for her in $\mathsf{G}_{2,3}(f)$, and that $\iota_{2,3}(x*\mathrm{sim}(\phi))$ is the infinite path of $T_{2,3}(x*\phi)$ for any $x \in \mathrm{dom}\, f$.*

Conversely, for any legal strategy ϕ for **II** *in $\mathsf{G}_{2,3}(f)$, we have that $\mathrm{tree}(\phi)$ is a legal strategy for her in $\mathsf{G}^S_{2,3}(f)$, and that $\iota_{2,3}(x*\phi)$ is the infinite path of $T^S_{2,3}(x*\mathrm{tree}(\phi))$ for any $x \in \mathrm{dom}\, f$.*

Proof. Let ϕ be legal for **II** in $\mathsf{G}^S_{2,3}(f)$ To see that $\mathrm{sim}(\phi)$ is legal for her in $\mathsf{G}_{2,3}(f)$, let $x \in \mathrm{dom}\, f$. Since $T^S_{2,3}(x*\phi)$ is infinite and $x*\phi \in \mathsf{UT}$, by Theorem 2.14 it follows that $x*\mathrm{sim}(\phi) \in R_*$. Furthermore, since $T^S_{2,3}(x*\phi)$ is finitely branching, and since the times **II** plays e_n when following $\mathrm{sim}(\phi)$ correspond to branching nodes on the n^{th} level of $T^S_{2,3}(x*\phi)$, it follows that for any n we have that $\{k \in \omega : o(x*\mathrm{sim}(\phi))(k) = \mathsf{e}_n\}$ is finite, and thus $\mathrm{sim}(\phi)$ is legal. Finally, again by Theorem 2.14 it is easy to see that $\iota_{2,3}(x*\mathrm{sim}(\phi))$ is the infinite path of $T_{2,3}(x*\phi)$.

Conversely, let ϕ be legal for **II** in $\mathsf{G}_{2,3}(f)$. To see that $\mathrm{tree}(\phi)$ is legal for her in $\mathsf{G}^S_{2,3}(f)$, let $x \in \mathrm{dom}\, f$. Clearly, for all $n \in \omega$ and $i, j \in \mathrm{dom}\,\mathsf{tape}_n(x*\mathrm{tree}(\phi))$ we have $\mathsf{tape}_n(x*\mathrm{tree}(\phi))(i) \subseteq \mathsf{tape}_n(x*\mathrm{tree}(\phi))(j)$. Also, since the branching nodes on the n^{th} level of $T^S_{2,3}(x*\mathrm{tree}(\phi))$ correspond to rounds k such that $o(x*\phi)(k) = \mathsf{e}_n$, and since $x*\phi \in R_{2,3}$, we have that $T^S_{2,3}(x*\mathrm{tree}(\phi))$ is finitely branching. Furthermore by Theorem 2.15 we have that $T^S_{2,3}(x*\mathrm{tree}(\phi))$ has a unique infinite path, so that $\mathrm{tree}(\phi)$ is legal and $\iota_{2,3}(x*\phi)$ is this infinite path. ■

As happened with the backtrack game, $\mathsf{G}_{2,3}$ characterizes in a natural way a class of functions defined by a partition property.

Theorem 2.24 (Semmes [26])**.** *Let $f : \omega^\omega \dashrightarrow \omega^\omega$. Then* **II** *has a winning strategy in $\mathsf{G}_{2,3}(f)$ iff $\mathrm{dom}\, f$ can be partitioned into relatively $\mathbf{\Pi}^0_2$ sets $\{A_n : n \in \omega\}$ in such a way that $f \restriction A_n \in \mathbf{\Lambda}_{1,2}$ for any $n \in \omega$.*

Proof. ($\Rightarrow$) Let ϕ be a winning strategy for **II** in $\mathsf{G}_{2,3}(f)$, and for each n define

$$A_n := \{x \in \mathrm{dom}\, f : o(x*\phi) = \mathsf{tape}_n(x*\phi)\},$$

i.e., A_n is the set of those $x \in \mathrm{dom}\, f$ such that **II**'s output tape when following ϕ against x is the n^{th} tape.

Claim 1. The function $f \restriction A_n$ is Baire class 1 for any $n \in \omega$.

Indeed, for each n let ϕ_n be the following strategy for **II** in $\mathsf{G}_{\mathsf{e}}(f \restriction A_n)$ obtained from ϕ by copying the moves made on tape n and substituting the other moves for p. Since ϕ is legal and $o(x*\phi) = \mathsf{tape}_n(x*\phi)$ for any $x \in A_n$, it follows that ϕ_n is legal, and then by construction for such x we have

$$\begin{aligned} \iota_{\mathsf{e}}(x*\phi_n) &= \iota_{\mathsf{e}}(\mathsf{tape}_n(x*\phi)) \\ &= \iota_{2,3}(x*\phi) \\ &= f(x). \end{aligned}$$ ⊡

Claim 2. The set A_n is relatively $\mathbf{\Pi}^0_2$ for any $n \in \omega$.

Indeed, for each m, n let

$$B_{m,n} := \bigcup\{[\sigma] : \sigma \in \omega^{<\omega} \text{ and } |\mathsf{tape}_n(\sigma * \phi)| \geq m\}.$$

Thus each $B_{m,n}$ is an open set, and for every $x \in \operatorname{dom} f$ and $n \in \omega$ we have

$$\begin{aligned} x \in \bigcap_m B_{m,n} &\iff \forall m\, \exists k.\ |\mathsf{tape}_n((x \restriction k) * \phi)| \geq m \\ &\iff \mathsf{tape}_n(x * \phi) \text{ is infinite} \\ &\iff x \in A_n. \end{aligned}$$

2

($\Leftarrow$) Let $\{A_n : n \in \omega\}$ be a partition of $\operatorname{dom} f$ as in the statement of the theorem, and for each n let ϕ_n be a winning strategy for **II** in $\mathsf{G_e}(f \restriction A_n)$. Finally, let $\{B_{m,n} : m, n \in \omega\}$ be open sets such that $A_n = \bigcap_m B_{m,n} \cap \operatorname{dom} f$ for each $n \in \omega$. Let ϕ be the following strategy for **II** in $\widehat{\mathsf{G}_{2,3}}(f)$, i.e., in the equivalent version of $\mathsf{G}_{2,3}(f)$ where **II** is allowed to make more than one move at each round.

Strategy: At round $n = \ulcorner m, k \urcorner$, if there exists $\ell \in \omega$ such that $[\mathbf{I}(n)] \cap B_{\ell,m} = \emptyset$ then pass. Otherwise, let

$$\begin{aligned} \sigma &:= \iota_*(\mathbf{I}(n) * \phi_m) \\ \tau &:= \iota_*(\mathsf{tape}_m(\mathbf{II}(n-1))). \end{aligned}$$

Now, if $\sigma \subseteq \tau$ then pass, else if $\tau \subset \sigma$ and we are currently playing on tape m then play

$$s := \langle \sigma(|\tau|), \sigma(|\tau|+1), \ldots, \operatorname{last} \sigma \rangle,$$

otherwise play $\mathsf{t}_m{}^\frown s$.

Finally, if $\sigma \perp \tau$, let p be least such that $\sigma(p) \neq \tau(p)$, and if we are playing on tape m play

$$s' := \mathsf{e}_p{}^\frown \langle \sigma(p), \sigma(p+1), \ldots, \operatorname{last} \sigma \rangle,$$

otherwise play $\mathsf{t}_m{}^\frown s'$.

Claim 3. The strategy ϕ is winning for **II** in $\widehat{\mathsf{G}_{2,3}}(f)$.

Let $x \in \operatorname{dom} f$. To see that $x * \phi \in \mathsf{UT}$, note that

$$\begin{aligned} \mathsf{tape}_m(x * \phi) \text{ is infinite} &\iff \forall \ell\, \forall k.\ [x \restriction \ulcorner m, k \urcorner] \cap B_{\ell,m} \neq \emptyset \\ &\iff \forall \ell.\ (\textstyle\bigcap_k [x \restriction k]) \cap B_{\ell,m} \neq \emptyset \\ &\iff x \in A_m, \end{aligned}$$

so since $\{A_n : n \in \omega\}$ partitions $\operatorname{dom} f$ we have that $\mathsf{tape}_m(x * \phi)$ is infinite for a unique m.

Furthermore, since moves in any tape m are made following the legal strategies ϕ_m, it follows that ϕ is also legal and, for any $x \in \operatorname{dom} f$,

$$\begin{aligned} \iota_{2,3}(x * \phi) &= \iota_{\mathsf{e}}(\mathsf{tape}_m(x * \phi)) && \text{for the unique } m \text{ with infinite } \mathsf{tape}_m(x * \phi) \\ &= \iota_{\mathsf{e}}(x * \phi_m) \\ &= f(x) && \text{since } x \in A_m. \end{aligned}$$

■

As happened with the Jayne-Rogers theorem for $\mathbf{\Lambda}_{2,2}$, the proof that the class of functions defined by this partition property is exactly $\mathbf{\Lambda}_{2,3}$ is outside the scope of this thesis, and will be omitted. It should be noted that this

Theorem 2.25 (Semmes [26]). *Let $f : \omega^\omega \longrightarrow \omega^\omega$. Then*

$$f \in \mathbf{\Lambda}_{2,3} \quad \textit{iff} \quad \begin{array}{l} \operatorname{dom} f \textit{ can be partitioned into relatively } \mathbf{\Pi}^0_2 \textit{ sets } \{A_n : n \in \omega\} \\ \textit{in such a way that } f \upharpoonright A_n \in \mathbf{\Lambda}_{1,2} \textit{ for any } n \in \omega. \end{array}$$

As a consequence, we have the following.

Corollary 2.26. *Let $f : \omega^\omega \longrightarrow \omega^\omega$. Then*

$$f \in \mathbf{\Lambda}_{2,3} \quad \textit{iff} \quad \mathbf{II} \textit{ has a winning strategy in } \mathsf{G}_{2,3}(f).$$

2.3.3 The multitape game

The multitape game builds upon the Wadge game to characterize the class $\mathbf{\Lambda}_{3,3}$, in an analogous way to that in which $\mathsf{G}_{2,3}$ extended the eraser game.

Definition 2.27. The *multitape game* is the game $\mathsf{G}_{\mathsf{mt}} = (M_{\mathsf{mt}}, R_{\mathsf{mt}}, \iota_{\mathsf{mt}})$, where

$$\begin{array}{rcl} M_{\mathsf{mt}} &=& \omega \cup \mathbb{P} \cup \mathbb{T}, \\ R_{\mathsf{mt}} &=& M_{\mathsf{mt}}^\omega \cap R_*, \text{ and} \\ \iota_{\mathsf{mt}}(x) &=& \displaystyle\bigcup_{n\in\omega} \iota_*(o(x) \upharpoonright n). \end{array}$$

As with the backtrack and $\mathsf{G}_{2,3}$, this game fits nicely into a partition-flavored theorem.

Theorem 2.28 (Andretta-Semmes, cf. [26]). *Let $f : \omega^\omega \dashrightarrow \omega^\omega$. Then* **II** *has a winning strategy in $\mathsf{G}_{\mathsf{mt}}(f)$ iff* $\operatorname{dom} f$ *can be partitioned into relatively $\mathbf{\Pi}^0_2$ sets $\{A_n : n \in \omega\}$ in such a way that $f \upharpoonright A_n$ is continuous for any $n \in \omega$.*

The proof is analogous but simpler than the proof of Theorem 2.24, and is therefore omitted. And, as we did for the corresponding theorems for $\mathbf{\Lambda}_{2,2}$ and $\mathbf{\Lambda}_{2,3}$, we will omit the proof that this partition property defines the class of total $\mathbf{\Lambda}_{3,3}$ functions.

Theorem 2.29 (Semmes [26]). *Let $f : \omega^\omega \longrightarrow \omega^\omega$. Then*

$$f \in \mathbf{\Lambda}_{3,3} \quad \textit{iff} \quad \begin{array}{l} \operatorname{dom} f \textit{ can be partitioned into relatively } \mathbf{\Pi}^0_2 \textit{ sets } \{A_n : n \in \omega\} \\ \textit{in such a way that } f \upharpoonright A_n \textit{ is continuous for any } n \in \omega. \end{array}$$

As a consequence, we have the following.

Corollary 2.30. *Let $f : \omega^\omega \longrightarrow \omega^\omega$. Then*

$$f \in \mathbf{\Lambda}_{3,3} \quad \textit{iff} \quad \mathbf{II} \textit{ has a winning strategy in } \mathsf{G}_{\mathsf{mt}}(f).$$

2.3.4 The tree game

The tree game was introduced by Semmes to characterize the class of Borel measurable functions, and thus it generalizes all the games we have seen so far. Naturally, it is the game in which player **II** has the most freedom among the games we will see in this thesis.

Definition 2.31. The *tree game* is the game $\mathsf{G}_\mathsf{T} = (M_\mathsf{T}, R_\mathsf{T}, \iota_\mathsf{T})$, where

$$\begin{aligned} M_\mathsf{T} &= \omega \cup \mathbb{P} \cup \mathbb{E}, \\ R_\mathsf{T} &= \{x \in R_* : \{\iota_*(x \restriction k) : k \in \omega\} \text{ has a unique maximal infinite chain } \langle \sigma_n^x \rangle_{n\in\omega}\} \\ \iota_\mathsf{T}(x) &= \bigcup_{n\in\omega} \sigma_n^x \end{aligned}$$

Originally, this game was presented by Semmes as the game $\mathsf{G}_\mathsf{T}^S = (M_\mathsf{T}^S, R_\mathsf{T}^S, \iota_\mathsf{T}^S)$, where

$$\begin{aligned} M_\mathsf{T}^S &= \mathcal{T}_\omega(\omega), \\ R_\mathsf{T}^S &= \{x \in (M_\mathsf{T}^S)^\omega : \forall n.\ x(n) \subseteq x(n+1), \text{ and } T_\mathsf{T}^S(x) := \bigcup_n x(n) \in \mathsf{UP}\}, \text{ and} \\ \iota_\mathsf{T}^S(x) &= \text{the unique infinite path of } T_\mathsf{T}^S(x) \end{aligned}$$

Once again, these games are equivalent.

Theorem 2.32. *Let $f : \omega^\omega \dashrightarrow \omega^\omega$.*

*For any legal strategy ϕ for **II** in $\mathsf{G}_\mathsf{T}^S(f)$, we have that $\mathrm{sim}(\phi)$ is a legal strategy for her in $\mathsf{G}_\mathsf{T}(f)$, and that $\iota_\mathsf{T}(x * \mathrm{sim}(\phi))$ is the infinite path of $T_\mathsf{T}^S(x*\phi)$ for any $x \in \mathrm{dom}\, f$.*

*Conversely, for any legal strategy ϕ for **II** in $\mathsf{G}_\mathsf{T}(f)$, we have that $\mathrm{tree}(\phi)$ is a legal strategy for her in $\mathsf{G}_\mathsf{T}^S(f)$, and that $\iota_\mathsf{T}(x * \phi)$ is the infinite path of $T_\mathsf{T}^S(x * \mathrm{tree}(\phi))$ for any $x \in \mathrm{dom}\, f$.*

Proof. Let ϕ be legal for **II** in $\mathsf{G}_\mathsf{T}^S(f)$. To see that $\mathrm{sim}(\phi)$ is legal for her in $\mathsf{G}_\mathsf{T}(f)$, let $x \in \mathrm{dom}\, f$. As in the proof of Theorem 2.18, since $T_\mathsf{T}^S(x)$ has an infinite path, we have $x * \mathrm{sim}(\phi) \in R_*$. Also, by Theorem 2.14, the unique infinite path in $T_\mathsf{T}^S(x)$ is the union of the unique maximal infinite chain in $\{\iota_*((x * \mathrm{sim}(\phi)) \restriction k) : k \in \omega\}$, so that $\mathrm{sim}(\phi)$ is legal, and by definition the union of this infinite chain is $\iota_\mathsf{T}(x * \mathrm{sim}(\phi))$.

Conversely, let ϕ be legal for **II** in $\mathsf{G}_\mathsf{T}(f)$. To see that $\mathrm{tree}(\phi)$ is legal for her in $\mathsf{G}_\mathsf{T}^S(f)$, let $x \in \mathrm{dom}\, f$. Clearly $(x * \mathrm{tree}(\phi))(n) \subseteq (x * \mathrm{tree}(\phi))(n+1)$ for any $n \in \omega$, and by an analogous argument as above, Theorem 2.15 implies that $\mathrm{tree}(\phi)$ is legal and $\iota_\mathsf{T}(x * \phi)$ is the infinite path of $T_\mathsf{T}^S(x * \mathrm{tree}(\phi))$. ■

We omit the proof that $\mathsf{G}_\mathsf{T}^S(f)$ characterizes the Borel measurable functions, again referring the reader to the original [26] for the details.

Theorem 2.33 (Semmes)**.** *Let $f : \omega^\omega \dashrightarrow \omega^\omega$. Then*

*f is Borel measurable iff **II** has a winning strategy in $\mathsf{G}_\mathsf{T}^S(f)$.*

Corollary 2.34. *Let $f : \omega^\omega \dashrightarrow \omega^\omega$. Then*

*f is Borel measurable iff **II** has a winning strategy in $\mathsf{G}_\mathsf{T}(f)$.*

2.4 Summary

We are now in a good position to give an intuitive summary of the games seen in this chapter. Here we use implicitly refer to the respective modified versions G'_e and $\mathsf{G}'_{1,3}$ when talking about the eraser game and $\mathsf{G}_{1,3}$.

- The games characterizing the classes $\mathbf{\Lambda}_{1,1}$ (Wadge game), $\mathbf{\Lambda}_{1,2}$ (eraser game), and $\mathbf{\Lambda}_{1,3}$ ($\mathsf{G}_{1,3}$) are obtained by varying the freedom with which player **II** can use each of her erasers — never, finitely often, and infinitely often (as long as the set of words which appear at infinitely many rounds on some tape is a chain, respectively)—, and by granting her the same level of freedom in changing tapes;
- The games characterizing the classes $\mathbf{\Lambda}_{2,2}$ (backtrack game) and $\mathbf{\Lambda}_{2,3}$ ($\mathsf{G}_{1,3}$) are obtained by varying the freedom with which player **II** can use each of her erasers — never and finitely often, respectively —, and by granting her *one degree higher* of freedom in changing tapes — finitely often and infinitely often (as long as only one is played on infinitely often), respectively;
- The game characterizing the class $\mathbf{\Lambda}_{3,3}$ ($\mathsf{G}_{1,3}$) is obtained by not allowing player **II** to erase any position, and granting her *two degrees higher* of freedom in changing tapes, i.e., infinitely often (as above).
- The game characterizing the Borel measurable functions is obtained by allowing player **II** to erase each position infinitely often, as long as the set of words which appear at infinitely many rounds contains a unique maximal infinite chain.

Summary

	Borel : tree game G_{T} Change tape: never Erase each position: inf. often[3]	
$\mathbf{\Lambda}_{1,3}$: $\mathsf{G}_{1,3}$ Change tape: inf. often[1] Erase each position: inf. often[2]	$\mathbf{\Lambda}_{2,3}$: $\mathsf{G}_{2,3}$ Change tape: inf. often[1] Erase each position: fin. often	$\mathbf{\Lambda}_{3,3}$: multitape game G_{mt} Change tape: inf. often[1] Erase each position: never
$\mathbf{\Lambda}_{1,2}$: eraser game G_{e} Change tape: fin. often Erase each position: fin. often	$\mathbf{\Lambda}_{2,2}$: backtrack game G_{bt} Change tape: fin. often Erase each position: never	—
$\mathbf{\Lambda}_{1,1}$: Wadge game G_{w} Change tape: never Erase each position: never	—	—

[1] Player **II** must play infinitely often on exactly one tape.
[2] Player **II** must erase in such a way that the set of words which appear at infinitely many rounds (on any tape) is a chain.
[3] Player **II** must erase in such a way that the set of words which appear at infinitely many rounds contains a unique maximal infinite chain.

Chapter 3

Choice principles

In this chapter we review the Weihrauch reducibility relation, a concept from computable analysis that expresses a specific notion of continuous reducibility between functions, and use the games of the last chapter to give new proofs of the completeness of discrete choice C_ω for $\mathbf{\Lambda}_{2,2}$ and countable choice C for $\mathbf{\Lambda}_{1,2}$ with respect to this relation, introduce two complete functions for $\mathbf{\Lambda}_{2,3}$ and $\mathbf{\Lambda}_{3,3}$ respectively, and briefly review other complete functions for $\mathbf{\Lambda}_{1,1}$, $\mathbf{\Lambda}_{1,k}$ with $k > 2$, and the Borel measurable functions.

3.1 Weihrauch reducibility

Computable analysis is the branch of mathematics that studies the concepts of classical mathematical analysis under the light of computability theory. Of course, since most objects studied in analysis are infinitary in nature, this has to be done with a certain amount of care.

A *represented space* is a pair (X, δ_X) where X is a set and $\delta_X : \omega^\omega \dashrightarrow\!\!\!\!\rightarrow X$. We say that δ_X *represents* X, and when $x \in \operatorname{dom} \delta_X$ we say that x is a δ_X*-name* of $\delta_X(x)$. Note that in general we don't require that δ_X be injective, so each element of X may have several δ_X-names.

The notion of a represented space is the key starting point for developing an analog to computability theory in spaces other than ω or ω^ω; with it, we are able to transfer well-understood concepts from these spaces to more general ones, as we shall see in part in this chapter. For a thorough study of this area, including aspects that are outside the scope of this thesis, we refer the reader to [8, 29].

Unless stated otherwise, in what follows ω^ω will always be represented by id, and ω will be represented by the total function $\delta_\omega : x \mapsto x(0)$.

Although in classical computability theory one usually studies partial *functions* from, say, ω to ω, in computable analysis it is useful to work in a more general setting, considering *binary relations* between represented spaces. Following the standard nomenclature, we will call such relations *partial multi-valued functions*, or partial multifunctions, or simply functions, through an arguably large abuse of nomenclature. We will, however, always make explicit exactly what kind of function is meant in each case by a careful

Notation	Meaning
$f : A \longrightarrow B$	function in the usual sense (total functional binary relation)
$f : A \twoheadrightarrow B$	surjective function (total functional surjective binary relation)
$f : A \dashrightarrow B$	partial function (functional binary relation)
$f : A \dashrightarrow\!\!\!\!\twoheadrightarrow B$	surjective partial function (functional surjective binary relation)
$f : A \Longrightarrow B$	multifunction (total binary relation)
$f : A \Longrightarrow\!\!\!\!\rightarrow B$	surjective multifunction (total surjective binary relation)
$f : A ==\Rightarrow B$	partial multifunction (binary relation)
$f : A ==\Rightarrow\!\!\!\!\rightarrow B$	surjective partial multifunction (surjective binary relation)

Table 3.1: Notation adopted for functions throughout this chapter.

use of the notation presented in Table 3.1. In summary, single or double arrows indicate whether the function is single- or (possibly) multi-valued, solid or dashed line styles whether it is total or (possibly) partial, and double arrow heads that the function is surjective. Note that this is compatible with the notation for total and partial (single-valued) functions we have used in the rest of this thesis.

One of the reasons for working with partial multifunctions is that this allows one to treat mathematical theorems of the form $\forall x \in X \, \exists y \in Y. \, P(x, y)$ as objects of the theory, and thus one can talk about one theorem being more computable than another, or reducible to it, etc., as has been extensively done in the literature, e.g., in [5, 7, 11, 24].

Given $f : X ==\Rightarrow Y$ and $x \in \operatorname{dom} f$, we will denote by $f(x)$ the set $\{y \in Y : (x, y) \in f\}$, and given $g : U ==\Rightarrow V$ with $Y \cap U \neq \emptyset$ we denote by $g \circ f$, or gf, the function with domain $\{x \in \operatorname{dom} f : f(x) \cap \operatorname{dom} g \neq \emptyset\}$, and given by

$$g \circ f(x) = \bigcup_{y \in f(x)} g(y).$$

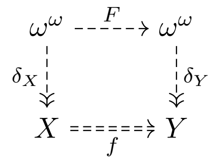
When $f(x)$ is a singleton $\{y\}$ most of the times we will write $f(x) = y$, as usual, but sometimes it will be convenient not to do this, as in Definition 3.35 below for the case when f is single-valued.

When talking about notions involving functions between represented spaces that are sensitive to what the specific representations are, most of the times we will write $f : (X, \delta_X) ==\Rightarrow (Y, \delta_Y)$ instead of $f : X ==\Rightarrow Y$, to make the representations explicit, although we may not use this convention when δ_X and δ_Y are clear from the context. Naturally, the same observation goes for the other types of functions listed in Table 3.1.

Definition 3.35. Given $f : (X, \delta_X) ==\Rightarrow (Y, \delta_Y)$, we say that $F : \omega^\omega \dashrightarrow \omega^\omega$ is a (δ_X, δ_Y)-*realizer* of f, denoted by $F \vdash f$, when

$$\delta_Y \circ F(x) \in f \circ \delta_X(x)$$

for all $x \in \operatorname{dom}(f\delta_X)$, which we will express diagrammatically by saying that



commutes.

In other words, $F \vdash f$ when $F(x)$ is a δ_Y-name of an element of $f(\delta_X(x))$, whenever x is a δ_X-name of an element of $\operatorname{dom} f$. In particular $\operatorname{dom} F$ must contain all names of elements in $\operatorname{dom} f$.

Note that, although the notion of a realizer depends on the representations of the spaces involved, this is not reflected in the notation $F \vdash f$. This will not be a problem, however, as the representations will always be clear from the context in which this notation is to be used. For the same reason, sometimes we will just say that F is a *realizer* of f. Note also that a realizer is always a single-valued function.

If f itself is of the form $f : \omega^\omega \dashrightarrow \omega^\omega$ with both domain and codomain represented by id, then we have by definition that $F \vdash f$ iff F extends f, so we can safely always consider such functions to be realized by themselves.

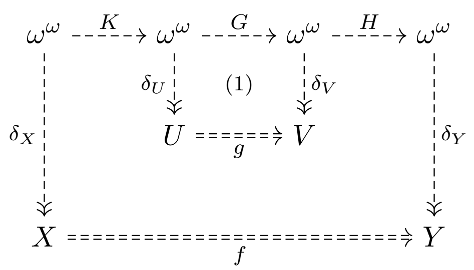
The notion of a realizer is the main tool that allows us to transfer concepts from ω^ω to general represented spaces, as in the following definition.

Definition 3.36. Let Γ be a class of partial functions from ω^ω to ω^ω, and let $f : (X, \delta_X) \rightrightarrows (Y, \delta_Y)$. We say that f is (δ_X, δ_Y)-Γ when f has a realizer $F \in \Gamma$.

Again, we will usually omit (δ_X, δ_Y) from the notation above when δ_X and δ_Y are clear from the context, and thus write things such as $f \in \mathbf{\Lambda}_{m,n}$ for $f : (X, \delta_X) \rightrightarrows (Y, \delta_Y)$, and so on. When X, Y are represented topological spaces satisfying a certain computability condition, the usual topological notion of $\mathbf{\Sigma}^0_k$-measurability for total functions coincides with that of (δ_X, δ_Y)-$\mathbf{\Sigma}^0_k$-measurability given above for any $k \in \omega$; see [3] for the details.

The first reducibility between partial multifunctions we will consider is the following.

Definition 3.37. Let $f : (X, \delta_X) \rightrightarrows (Y, \delta_Y)$ and $g : (U, \delta_U) \rightrightarrows (V, \delta_V)$. We say that f is *strongly Weihrauch-reducible* to g, denoted by $f \leq_{\mathsf{st}} g$, when there exist continuous $H, K : \omega^\omega \dashrightarrow \omega^\omega$ such that $HGK \vdash f$ for any $G \vdash g$, i.e., when



commutes for any G that makes (1) commute.

The subscript "t" stands for "topological", to contrast with a computable counterpart of this relation which we will see at the end of this section. As usual, the strict version of $\leq_{\mathsf{st}}$ is denoted by $<_{\mathsf{st}}$, and the equivalence relation induced by $\leq_{\mathsf{st}}$ is denoted by $\equiv_{\mathsf{st}}$.

Example 3.38. *The identity function* $\mathrm{id}_X : (X, \delta_X) \longrightarrow (X, \delta_X)$ *cannot be strongly Weihrauch-reduced to any constant function between represented spaces.*

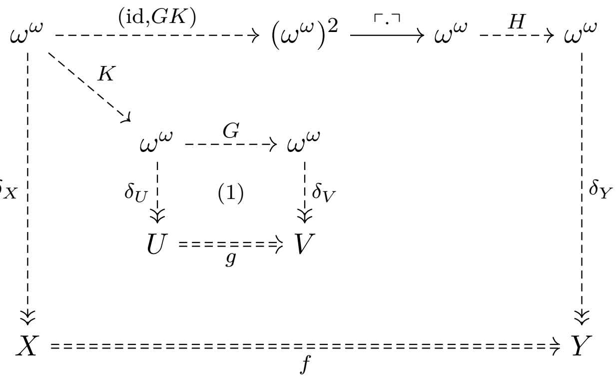
This example indicates that the reducibility $\leq_{\mathsf{st}}$ is actually *too* fine for our objectives, distinguishing between more functions than we would like. Intuitively speaking, this is because demanding that $HGK \vdash f$ whenever $G \vdash g$ means that H needs to be able to "output" a name of an element of $f(x)$ without having direct access to a name of the "input" x. However, this reducibility still has many interesting theoretical properties; see, e.g., [9].

Instead, the reducibility that will be more interesting to us is the following.

Definition 3.39. Let $f : (X, \delta_X) \rightrightarrows (Y, \delta_Y)$ and $g : (U, \delta_U) \rightrightarrows (V, \delta_V)$. We say that f is *Weihrauch-reducible* to g, denoted by $f \leq_{\mathsf{t}} g$, when there exist continuous $H, K : \omega^\omega \dashrightarrow \omega^\omega$ such that

$$H\ulcorner \mathrm{id}, GK \urcorner \vdash f$$

for all $G \vdash g$, i.e., when



commutes for any G that makes (1) commute.

Note that this is clearly weaker than $\leq_{\mathsf{st}}$; $f \leq_{\mathsf{st}} g$ implies $f \leq_{\mathsf{t}} g$, but the converse does not hold. For example, any two continuous partial functions from ω^ω to ω^ω with nonempty domains are easily seen to be Weihrauch-reducible to one another, but we saw above that id is not strongly Weihrauch-reducible to any constant function.

The following result is not as immediate as the analogous one for $\leq_{\mathsf{st}}$.

Proposition 3.40. *The relation* $\leq_{\mathsf{t}}$ *is a preorder.*

Proof. Reflexivity is clear — just take $H = (\cdot)^2_1$ and $K = \mathrm{id}$. For transitivity, suppose $f \leq_{\mathsf{t}} g \leq_{\mathsf{t}} h$. Let $A_1, A_2, B_1, B_2 : \omega^\omega \dashrightarrow \omega^\omega$ be continuous functions such that

$$\begin{aligned} &A_1\ulcorner \mathrm{id}, GB_1 \urcorner \vdash f, &&\forall G \vdash g, \text{ and} \\ &A_2\ulcorner \mathrm{id}, HB_2 \urcorner \vdash g, &&\forall H \vdash h. \end{aligned}$$

Now let $A_3(x) = A_1\ulcorner(x)_0^2, A_2\ulcorner B_1((x)_0^2), (x)_1^2\urcorner\urcorner$ and $B_3 = B_2B_1$, both of which are continuous functions. Then we have, for any $H \vdash h$,

$$\begin{aligned} A_3\ulcorner \mathrm{id}, HB_3\urcorner(x) &= A_1\ulcorner x, A_2\ulcorner B_1(x), HB_2B_1(x)\urcorner\urcorner \\ &= A_1\ulcorner x, GB_1(x)\urcorner \\ &= F(x), \end{aligned}$$

where G is the realizer $A_2\ulcorner \mathrm{id}, HB_2\urcorner$ of g, and F is the realizer $A_1\ulcorner \mathrm{id}, GB_1\urcorner$ of f. ■

The relations $<_{\mathsf{t}}$ and $\equiv_{\mathsf{t}}$ are defined from $\leq_{\mathsf{t}}$ in the usual way.

Note that where we used continuous functions to define $\leq_{\mathsf{st}}$ and $\leq_{\mathsf{t}}$, we could have used any other reasonable class of functions that is closed under compositions and contains the identity and the projections, obtaining meaningful reducibilities as results.

In particular, the following class will be used at some points in this chapter.

Definition 3.41. A function $f : \omega^\omega \dashrightarrow \omega^\omega$ is *computable* when there exists a computable (in the usual sense) winning strategy for **II** in $\mathsf{G}_{\mathsf{W}}(f)$.

Thus every computable function is continuous, but the converse does not hold – . We then define $\leq_{\mathsf{sc}}$, $\equiv_{\mathsf{sc}}$, etc., analogously to $\leq_{\mathsf{st}}$, $\equiv_{\mathsf{st}}$, etc., but with H, K computable.

The following follows directly from the definitions, and will be used throughout this chapter.

Theorem 3.42. *Let $f : (X, \delta_X) \rightrightarrows (Y, \delta_Y)$ and $g : (U, \delta_U) \rightrightarrows (V, \delta_V)$. Then, if $f \leq_{\mathsf{t}} g$ and $g \in \mathbf{\Lambda}_{m,n}$, then also $f \in \mathbf{\Lambda}_{m,n}$.*

Proof. Let $H, K : \omega^\omega \dashrightarrow \omega^\omega$ be continuous functions such that $H\ulcorner \mathrm{id}, GK\urcorner \vdash f$ for any $G \vdash g$. In particular, letting G' be a $\mathbf{\Lambda}_{m,n}$ realizer of g, since $\ulcorner\cdot\urcorner$ is continuous by Theorem 1.3, it follows that $H\ulcorner \mathrm{id}, G'K\urcorner$ is a $\mathbf{\Lambda}_{m,n}$ realizer of f. ■

3.2 Choice principles

In the theory of Weihrauch-reducibility, certain (multi-valued partial) functions have been isolated that capture the complexity of some function classes of interest. We will refer to these by the common name *choice principles.* The intuition behind this name is that most of the ones we are going to consider are restrictions of a function C_X, called *closed choice*, whose realizers *choose* elements of a set given a specific type of representation of it.

Our main goal in this chapter is to prove that certain choice principles are Weihrauch-complete for some classes in the stratification $\mathbf{\Lambda}_{m,n}$ of the Borel measurable functions. We will focus on (partial) functions from ω^ω to ω^ω, which will be convenient since this will allow us to apply the games from the preceding chapter, but this turns out to be without any loss of generality.

Theorem 3.43. *Suppose $g : (U, \delta_U) \rightrightarrows (V, \delta_V)$ is $\mathbf{\Lambda}_{m,n}$, and that for all $F : \omega^\omega \dashrightarrow \omega^\omega$ we have $F \in \mathbf{\Lambda}_{m,n}$ iff $F \leq_{\mathsf{t}} g$. Then, for any $f : (X, \delta_X) \rightrightarrows (Y, \delta_Y)$ we have*

$$f \in \mathbf{\Lambda}_{m,n} \quad \textit{iff} \quad f \leq_{\mathsf{t}} g.$$

Proof. (⇒) Let $F \vdash f$ be such that $F \in \mathbf{\Lambda}_{m,n}$. Then by hypothesis there exist continuous $H, K : \omega^\omega \dashrightarrow \omega^\omega$ such that $F \subseteq H\ulcorner \mathrm{id}, GK\urcorner$ for any $G \vdash g$, which implies $H\ulcorner \mathrm{id}, GK\urcorner \vdash f$ for any $G \vdash g$, i.e., $f \leq_{\mathsf{t}} g$.

(⇐) Let $H, K : \omega^\omega \dashrightarrow \omega^\omega$ be continuous functions such that $H\ulcorner \mathrm{id}, GK\urcorner \vdash f$ for any $G \vdash g$. Therefore, letting G' be a $\mathbf{\Lambda}_{m,n}$ realizer of g, since $\ulcorner\cdot\urcorner$ is continuous we have that $H\ulcorner \mathrm{id}, G'K\urcorner$ is a $\mathbf{\Lambda}_{m,n}$ realizer of f. ■

3.2.1 Continuous choice

As we have seen, any two continuous functions with non-empty domains are Weihrauch-reducible to one another, and thus any such function is Weihrauch-complete for the class of continuous functions. However, in order to state a general result in a later section in a more uniform way, it will be convenient to isolate one such function at this point.

Definition 3.44. The principle of *continuous choice* is the function $\mathsf{C}_0 : \omega^\omega \longrightarrow \omega^\omega$ given by

$$\mathsf{C}_0(x)(n) = \begin{cases} 0, & \text{if } x(n) \neq 0 \\ 1, & \text{otherwise.} \end{cases}$$

Thus C_0 "flips" nonzero values in its input to zero and vice versa. Hence C_0 is continuous, and we have

Theorem 3.45. *Let $f : (X, \delta_X) \rightrightarrows (Y, \delta_Y)$. Then*

$$f \text{ is continuous} \quad \text{iff} \quad f \leq_{\mathsf{t}} \mathsf{C}_0.$$

3.2.2 Discrete choice

Definition 3.46. The principle of *discrete choice* is the function $\mathsf{C}_\omega : \omega^\omega \rightrightarrows \omega$ given by

$$\mathsf{C}_\omega(x) = \omega \setminus \operatorname{ran} x,$$

with $\operatorname{dom} \mathsf{C}_\omega = \{x : \operatorname{ran} x \neq \omega\}$.

Intuitively, the infinitary nature of C_ω should indicate that it is not $(\mathrm{id}, \delta_\omega)$-continuous — given $x \in \omega^\omega$ which is a priori known to not contain every $n \in \omega$ in its range, a realizer of C_ω must then output a sequence whose first element is not in the range of x. However, C_ω does not lie very high in the stratification of the Borel functions.

Proposition 3.47. $\mathsf{C}_\omega \in \mathbf{\Lambda}_{2,2}$.

Proof. Let $F(x) = \iota_{\mathsf{bt}}(x * \phi)$, where ϕ is the following strategy for **II** in G_{bt}.

Strategy: At round n, let $k = \min(\omega \setminus \operatorname{ran} \mathbf{I}(n))$. If at the last round we played k or changed tape, then play k. Otherwise, go to a new tape.

Thus, if $x \in \operatorname{dom} \mathsf{C}_\omega$, at some point **II** will play $\min(\omega \setminus \operatorname{ran} x)$ as the first move on a tape, and from that point on she will always make that same move and never change tape again. Therefore we have $F(x)(0) \in \mathsf{C}_\omega(x)$ for any $x \in \operatorname{dom} \mathsf{C}_\omega$, i.e., $F \vdash \mathsf{C}_\omega$ ■

This section is dedicated to the proof that this choice principle captures the complexity of $\mathbf{\Lambda}_{2,2}$ exactly.

Theorem 3.48 (Brattka-de Brecht-Pauly [4])**.** *Let* $f : (X, \delta_X) \rightrightarrows (Y, \delta_Y)$*. Then*

$$f \in \mathbf{\Lambda}_{2,2} \quad \textit{iff} \quad f \leq_{\mathsf{t}} \mathsf{C}_\omega.$$

First we need some auxiliary notions and results, starting with an alternative representation $\lim_\Delta$ of ω^ω, given by

$$\lim_\Delta(x) = \lim_n \, (x)_n,$$

with $\operatorname{dom} \lim_\Delta = \{x : \exists i \, \forall j \geq i. \ (x)_j = (x)_i\}$, i.e., $\lim_\Delta(x)$ is the discrete limit of the sequence in ω^ω encoded by x, whenever this limit exists.

Proposition 3.49. $\lim_\Delta \leq_{\mathsf{t}} \mathsf{C}_\omega$.

Proof. Let $H, K : \omega^\omega \longrightarrow \omega^\omega$ be the continuous functions given by

$$\begin{aligned} H(\ulcorner x, y \urcorner) &= (x)_{y(0)} \\ K(x) &= \iota_{\mathsf{W}}(x * \phi), \end{aligned}$$

where ϕ is the following strategy for **II** in G_{W}.

Strategy: At round $n = \ulcorner m, k \urcorner$, let $\sigma := \mathbf{I}(n)$.

Let $A = \{i < m : i$ has not been played by **II** yet and there exists $j \in \{i, \ldots, m-1\}$ such that $(\sigma)_j \perp (\sigma)_{j+1}\}$.

If $A \neq \emptyset$, then play $\min A$. Otherwise, play 0.

Then $\operatorname{ran} K(x) = \{0\} \cup \{n : \exists k \geq n. \ (x)_k \neq (x)_{k+1}\}$, so

$$\begin{aligned} n \in \mathsf{C}_\omega(K(x)) &\iff n \in \omega \setminus \operatorname{ran} K(x) \\ &\iff \forall k \geq n. \ (x)_k = (x)_{k+1} \\ &\iff \lim_\Delta(x) = (x)_n. \end{aligned}$$

Recall that we have that $G : \omega^\omega \dashrightarrow \omega^\omega$ realizes C_ω iff $G(x)(0) \notin \operatorname{ran} x$ for any x such that $\operatorname{ran} x \neq \omega$. Therefore, for any such G and x we have

$$\begin{aligned} \lim_\Delta(x) &= (x)_n \quad \text{for any } n \text{ such that } \forall k \geq n. \ (x)_k = (x)_{k+1} \\ &= (x)_n \quad \text{for any } n \in \mathsf{C}_\omega(K(x)) \\ &= (x)_{GK(x)(0)} \\ &= H(\ulcorner x, GK(x) \urcorner) \end{aligned}$$

■

Theorem 3.50. *Let* $f : \omega^\omega \dashrightarrow \omega^\omega$*. Then*

$$f \textit{ is } (\mathrm{id}, \mathrm{id})\text{-}\mathbf{\Lambda}_{2,2} \quad \textit{iff} \quad f \textit{ is } (\mathrm{id}, \lim_\Delta)\textit{-continuous.}$$

Proof. (⇒) Let ϕ_{bt} be a winning strategy for **II** in $\mathsf{G}_{\mathsf{bt}}(f)$, and let $F : \omega^\omega \dashrightarrow \omega^\omega$ be given by $F(x) = \iota_{\mathsf{W}}(x * \phi)$, where ϕ is the following strategy for **II** in G_{W}.

Strategy: At round n, let $\sigma := \iota_*(\mathbf{II}(n-1))$, $|\sigma| = \ulcorner k, m \urcorner$, and finally $\tau := \iota_*(\mathsf{tape}_\ell(\mathbf{I}(n) * \phi_{\mathsf{bt}}))$, where ℓ is the tape on which $\phi_{\mathsf{bt}}(\mathbf{I}(n))$ is played.

If $(\sigma)_k$ and τ are incomparable, then play 0. Otherwise if $(\sigma)_k \supseteq \tau$ then play p, and if $(\sigma)_k \subset \tau$ then play $\tau(|(\sigma)_k|)$.

It now remains to be shown that F is an $(\mathrm{id}, \lim_\Delta)$-realizer of f. For any $x \in \operatorname{dom} f$, there exists a least N such that $\phi_{\mathsf{bt}}(x \restriction n) \notin \mathbb{T}$ for any $n \geq N$. Therefore, for any k such that $\ulcorner k, 0 \urcorner \geq N$, we have $(F(x))_k = \iota_{\mathsf{bt}}(x * \phi_{\mathsf{bt}}) = f(x)$, and therefore $\lim_\Delta \circ F(x) = f(x)$.

(⇐) Let F be a continuous $(\mathrm{id}, \lim_\Delta)$-realizer of f, and let ϕ_{W} be a winning strategy for **II** in $\mathsf{G}_{\mathsf{W}}(F)$. Let ϕ be the following strategy for **II** in $\mathsf{G}_{\mathsf{bt}}(f)$.

Strategy: Begin by declaring that we are following 0.

At round n, having made i moves in ω on the current tape and following k, let $\sigma := \iota_*(\mathbf{I}(n) * \phi_{\mathsf{W}})$.

If $|(\sigma)_k| \leq i$, then pass. Otherwise, if there exist $k' > k$ and $i' \leq i$ such that $|(\sigma)_{k'}| > i'$ and $(\sigma)_k(i') \neq (\sigma)_{k'}(i')$, then go to a new tape and start following k' instead of k. Finally, if no such k' and i' exist, then play $(\sigma)_k(i)$.

To see that ϕ is a winning strategy, just note that for any $x \in \operatorname{dom} f$ there exists a least K such that $(F(x))_k = f(x)$ for any $k > K$. Therefore, if at some point in the game **II** is following k such that $(F(x))_k \neq f(x)$, then there exist $k' > k$ and i such that $(F(x))_k(i) \neq (F(x))_{k'}(i)$, so **II** will change into a new tape at some later round and start following some $k'' > k$. Therefore, eventually **II** will be following $k \geq K$, and will then never change tape again. Since her moves will then be made copying $(F(x))_k$, it follows that $\iota_{\mathsf{bt}}(x * \phi) = f(x)$. ■

We are now in a position to prove the main result of this section.

Proof of Theorem 3.48. By Theorem 3.43, it suffices to prove the result for $f : \omega^\omega \dashrightarrow \omega^\omega$.

(⇒) Let $f \in \mathbf{\Lambda}_{2,2}$. By Theorem 3.50, f has a continuous $(\mathrm{id}, \lim_\Delta)$-realizer, i.e., $f = \lim_\Delta \circ F$ for some continuous $F : \omega^\omega \dashrightarrow \omega^\omega$. Thus, in particular, we have $f \leq_{\mathsf{st}} \lim_\Delta$, which by Proposition 3.49 implies $f \leq_{\mathsf{t}} \mathsf{C}_\omega$.

(⇐) Note that, if $f \leq_{\mathsf{t}} \mathsf{C}_\omega$, then $f \leq_{\mathsf{t}} F$ for any $F \vdash \mathsf{C}_\omega$. Therefore, the result follows from Proposition 3.47 and Theorem 3.42. ■

3.2.3 Countable choice

Definition 3.51. The principle of *countable choice* is the function $\mathsf{C} : \omega^\omega \longrightarrow \omega^\omega$ given by

$$\mathsf{C}(x)(n) = \begin{cases} 0, & \text{if } \exists k.\ x(k) = n + 1 \\ 1, & \text{otherwise.} \end{cases}$$

In other words, C transforms enumerations of subsets of ω into their characteristic functions.

Proposition 3.52. $\mathsf{C} \in \mathbf{\Lambda}_{1,2}$.

Proof. Let ϕ be the following strategy for **II** in $\mathsf{G}_{\mathsf{e}}(\mathsf{C})$.

Strategy: At round n, if $x_n = m+1$ for $m < |\iota_*(\mathbf{II}(n-1))|$ such that $\iota_*(\mathbf{II}(n-1))(m) = 1$, then play e_m. Otherwise, if there exists $k \leq n$ such that $x_k = |\iota_*(\mathbf{II}(n-1))| + 1$ then play 0, else play 1.

Claim 1. The strategy ϕ is winning for **II** in $\mathsf{G}_{\mathsf{e}}(\mathsf{C})$.

Indeed, for any $x \in \omega^\omega$, first note that each e_n is played at most once, so ϕ is legal. Furthermore, we have

$$\begin{aligned} \iota_{\mathsf{e}}(x * \phi)(n) = 0 &\iff \exists k.\ x(k) = n+1 \\ &\iff \mathsf{C}(x)(n) = 0. \end{aligned}$$

■

This section is dedicated to the proof that C is Weihrauch-complete for Baire Class 1 functions. However, it will be convenient to work instead with the strongly computably equivalent choice principle $\mathsf{C}_1 : \omega^\omega \longrightarrow \omega^\omega$, given by

$$\mathsf{C}_1(x)(n) = \begin{cases} 0, & \text{if } \exists k.\ x(\ulcorner n, k \urcorner) \neq 0 \\ 1, & \text{otherwise.} \end{cases}$$

Proposition 3.53. $\mathsf{C} \equiv_{\mathsf{sc}} \mathsf{C}_1$.

Proof. ($\mathsf{C} \leq_{\mathsf{sc}} \mathsf{C}_1$): Let

$$K(x)(\ulcorner n, k \urcorner) = \begin{cases} 1, & \text{if } x(k) = n+1 \\ 0, & \text{otherwise} \end{cases}$$

Then we have

$$\begin{aligned} \mathsf{C}(x)(n) = 0 &\iff \exists k.\ x(k) = n+1 \\ &\iff \exists k.\ K(x)(\ulcorner n, k \urcorner) = 1 \\ &\iff \mathsf{C}_1 \circ K(x)(n) = 0 \end{aligned}$$

($\mathsf{C}_1 \leq_{\mathsf{sc}} \mathsf{C}$): Let

$$K(x)(\ulcorner n, k \urcorner) = \begin{cases} n+1, & \text{if } x(\ulcorner n, k \urcorner) \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

Then we have

$$\begin{aligned} \mathsf{C}_1(x)(n) = 0 &\iff \exists k.\ x(\ulcorner n, k \urcorner) \neq 0 \\ &\iff \exists k.\ K(x)(\ulcorner n, k \urcorner) = n+1 \\ &\iff \mathsf{C} \circ K(x)(n) = 0 \end{aligned}$$

■

Therefore, the theorem we will prove in this section is the following.

Theorem 3.54 (Brattka [3]). *Let* $f : (X, \delta_X) \rightrightarrows (Y, \delta_Y)$. *Then*

$$f \in \mathbf{\Lambda}_{1,2} \quad \textit{iff} \quad f \leq_{\mathsf{t}} \mathsf{C}_1.$$

As in the proof of Theorem 3.48, we will proceed by first defining an appropriate alternative representation of ω^ω, from which it will then be easier to transform a winning strategy for **II** in $\mathsf{G_e}(f)$ into a Weihrauch-reduction of f to C_1, and vice versa, by passing through this representation as an intermediate step.

This alternative representation is $\lim : \omega^\omega \dashrightarrow \omega^\omega$, given by $\lim x = \lim_n (x)_n$, where $\operatorname{dom} \lim = \{x : \lim_n (x)_n \text{ exists}\}$, i.e., $\operatorname{dom} \lim$ is the set of all $x \in \omega^\omega$ such that $\exists x^* \, \forall m \, \exists N \, \forall n \geq N. \; x^* \restriction m \subset (x)_n$.

Proposition 3.55. $\lim \leq_{\mathsf{c}} \widehat{\mathsf{C}_\omega} \leq_{\mathsf{sc}} \mathsf{C}_1$.

Proof. ($\lim \leq_{\mathsf{c}} \widehat{\mathsf{C}_\omega}$:) Define continuous $H, K : \omega^\omega \longrightarrow \omega^\omega$ by

$$\begin{aligned} H(\ulcorner x, y \urcorner)(n) &= (x)_{y(n)}(n) \\ K(x) &= \iota_{\mathsf{W}}(x * \phi), \end{aligned}$$

where ϕ is the following strategy for **II** in $\mathsf{G_W}$.

Strategy: At round $\ulcorner n, k \urcorner$, let $A := \{m : m$ has not been played at any round $\ulcorner n, k' \urcorner$ with $k' < k$, and there exists $m' \geq m$ such that $\ulcorner m' + 1, n \urcorner < \ulcorner n, k \urcorner$ and $\mathbf{I}(\ulcorner m', n \urcorner) \neq \mathbf{I}(\ulcorner m' + 1, n \urcorner)\}$.

If $A \neq \emptyset$ then play $\min A$, otherwise play 0.

Then we have

$$\begin{aligned} M \notin \operatorname{ran} (K(x))_n &\implies \forall m \geq M. \; (x)_m(n) = (x)_M(n) \\ &\implies \lim_m (x)_m \; (n) = (x)_M(n) \\ &\implies \lim x \; (n) = (x)_M(n). \end{aligned}$$

Hence, if $G \vdash \widehat{\mathsf{C}_\omega}$, then for any $x \in \operatorname{dom} \widehat{\mathsf{C}_\omega}$ and $n \in \omega$ we have $G(x)(n) \notin \operatorname{ran} (x)_n$, hence

$$\begin{aligned} \lim x \; (n) &= (x)_{GK(x)(n)}(n) \\ &= H \ulcorner x, GK(x) \urcorner (n). \end{aligned}$$

($\widehat{\mathsf{C}_\omega} \leq_{\mathsf{sc}} \mathsf{C}_1$) Define $H, K : \omega^\omega \dashrightarrow \omega^\omega$ by

$$\begin{aligned} H(x)(n) &= \mu k. \; x(\ulcorner n, k \urcorner) = 1 \\ K(x)(\ulcorner \ulcorner n, k \urcorner, m \urcorner) &= \begin{cases} 1, & \text{if } (x)_n(m) = k \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

with $\operatorname{dom} H = \{x : \forall n \, \exists k. \; x(\ulcorner n, k \urcorner) = 1\}$ and $\operatorname{dom} K = \omega^\omega$. Then we have

$$\begin{aligned} k \notin \operatorname{ran} (x)_n &\iff \nexists m. \; K(x)(\ulcorner \ulcorner n, k \urcorner, m \urcorner) \neq 0 \\ &\iff \mathsf{C}_1 K(x)(\ulcorner n, k \urcorner) = 1, \end{aligned}$$

and therefore

$$\begin{aligned} H\mathsf{C}_1K(x)(n) &= \mu k.\ \mathsf{C}_1K(x)(\ulcorner n,k\urcorner) = 1 \\ &= \mu k.\ k \notin \operatorname{ran}(x)_n, \end{aligned}$$

thus $H\mathsf{C}_1K \vdash \widehat{\mathsf{C}_\omega}$. ■

Proposition 3.56. *The function* C_1 *is* $(\mathrm{id},\lim)$*-computable.*

Proof. Define $K : \omega^\omega \longrightarrow \omega^\omega$ by

$$K(x)(\ulcorner n,k\urcorner) = \begin{cases} 0, & \text{if } \exists k' \leq k.\ x(\ulcorner k',n\urcorner) \neq 0 \\ 1, & \text{otherwise.} \end{cases}$$

Then we have

$$\begin{aligned} \mathsf{C}_1(x)(n) = 0 &\iff \exists k_0.\ x(\ulcorner n,k_0\urcorner) \neq 0 \\ &\iff \forall k \geq k_0.\ K(x)(\ulcorner k,n\urcorner) = 0 \\ &\iff \forall k \geq k_0.\ (K(x))_k(n) = 0 \\ &\iff \lim_k (K(x))_k\ (n) = 0 \\ &\iff \lim \circ K(x)\ (n) = 0. \end{aligned}$$ ■

Corollary 3.57. $\lim \equiv_{\mathsf{c}} \widehat{\mathsf{C}_\omega} \equiv_{\mathsf{c}} \mathsf{C}_1$.

Theorem 3.58. *Let* $f : \omega^\omega \dashrightarrow \omega^\omega$ *Then*

$$f \text{ is } (\mathrm{id},\mathrm{id})\text{-}\mathbf{\Lambda}_{1,2} \quad \text{iff} \quad f \text{ is } (\mathrm{id},\lim)\text{-continuous.}$$

Proof. $(\Rightarrow)$ Let ϕ_{e} be a winning strategy for **II** in $\mathsf{G}_{\mathsf{e}}(f)$, and let $F : \omega^\omega \dashrightarrow \omega^\omega$ be given by $F(x) = \iota_{\mathsf{W}}(x * \phi)$, where ϕ is the following strategy for **II** in G_{W}.

Strategy: At round $n = \ulcorner k,m\urcorner$, let $\sigma := \mathbf{II}(n-1)$ and

$$(\sigma)_k := \langle \sigma(\ulcorner k,0\urcorner), \ldots, \sigma(\ulcorner k,m-1\urcorner)\rangle,$$

let $\sigma' := \iota_*((\sigma)_k)$, and finally $\tau := \iota_*(\mathbf{I}(n) * \phi_{\mathsf{e}})$. If σ' and τ are incomparable, then play 0. Otherwise if $|\tau| \leq m$ then play p, and if $|\tau| > m$ then play $\tau(m)$.

It now remains to be shown that F is an $(\mathrm{id},\lim)$-realizer of f. For any x and m, there exists N such that

$$\forall n \geq N.\ f(x)\restriction m \subset \iota_*((x\restriction n) * \phi_{\mathsf{e}}).$$

Therefore, for any k such that $\ulcorner k,0\urcorner \geq N$, we have that

$$\begin{aligned} (F(x))_k\restriction m &= \iota_{\mathsf{e}}(x * \phi_{\mathsf{e}})\restriction m \\ &= f(x)\restriction m, \end{aligned}$$

so $\lim F(x) = f(x)$.

$(\Leftarrow)$ Let F be a continuous $(\mathrm{id},\lim)$-realizer of f, and let ϕ_{W} be a winning strategy for **II** in $\mathsf{G}_{\mathsf{W}}(F)$. Let ϕ be the following strategy for **II** in $\mathsf{G}_{\mathsf{e}}(f)$.

Strategy: Begin by declaring that we are following 0.

At round n when following k, let $\sigma = \iota_*(\mathbf{II}(n-1))$ and $\tau := \iota_*(\mathbf{I}(n) * \phi_{\mathsf{W}})$.

If there exists $k' > k$ such that σ and $(\tau)_{k'}$ are incomparable, then let k' be the maximum such number and m be least such that $\sigma(m) \neq (\tau)_{k'}(m)$, then play e_m and declare we are now following k'.

Otherwise, if there does not exist $k' \geq k$ such that $\ulcorner k', |\sigma| \urcorner < |\tau|$ then pass, else play $(\tau)_{k'}(|\sigma|)$ for maximum such k' and declare that we are now following k'.

To see that ϕ is a winning strategy, let $x \in \operatorname{dom} f$. By induction, we will prove that for every m there exists N_m such that $f(x) \restriction m \subseteq \iota_*((x \restriction n) * \phi)$ for any $n \geq N_m$, the base case being trivial. For the induction step, note that since $\lim F(x) = f(x)$, there exists least $k_{m+1} \geq N_m$ such that $f(x) \restriction m \subseteq (F(x))_n$ for any $n \geq k_{m+1}$. Therefore, if at some round $\mathbf{II}$ follows $n < k_{m+1}$ to make her $(m+1)^{\text{th}}$ valid move, then there exists $n' > n$ such that $(F(x))_n$ and $(F(x))_{n'}$ are incomparable, and by the induction hypothesis this discrepancy does not happen in the first m elements of the sequences. Hence, after playing e_m once, $\mathbf{II}$'s next move will be made following some $n'' > n$, so that eventually at some round N_{m+1} she will follow $n''' \geq k_{m+1}$ to make her $(m+1)^{\text{th}}$ valid move, and then never erase that number in the future. Thus, for any round $n \geq N_{m+1}$, we will have $f(x) \restriction (m+1) \subseteq \iota_*((x \restriction n) * \phi)$. ■

We are now in a position to prove Theorem 3.54.

Proof. By Theorem 3.43, it is enough to prove the result for $f : \omega^\omega \dashrightarrow \omega^\omega$.

($\Rightarrow$) Let $f \in \mathbf{\Lambda}_{1,2}$. By Theorem 3.58, f has a continuous $(\mathrm{id}, \lim)$-realizer, i.e., $f = \lim \circ F$ for some continuous $F : \omega^\omega \dashrightarrow \omega^\omega$. Thus, in particular, we have $f \leq_{\mathsf{sc}} \lim$, which by Proposition 3.55 implies $f \leq_{\mathsf{t}} \mathsf{C}_1$.

($\Leftarrow$) By Theorem 3.42, it is enough to show that $\mathsf{C}_1 \in \mathbf{\Lambda}_{1,2}$, which follows from Proposition 3.56 and Theorem 3.58. ■

3.2.4 k-Countable choice

Definition 3.59. Given $k \in \omega$, the principle of *k-countable choice* is the function $\mathsf{C}_k : \omega^\omega \longrightarrow \omega^\omega$ given by

$$\mathsf{C}_k(x)(n) = \begin{cases} 0, & \text{if } \exists n_{k-1}\, \forall n_{k-2} \dots \mathrm{Q} n_0.\; x(\ulcorner n, n_{k-1}, \dots, n_0 \urcorner) \neq 0 \\ 1, & \text{otherwise} \end{cases}$$

where the quantifiers $\exists$ and $\forall$ in the condition above are alternating — thus Q is $\exists$ when k is odd and $\forall$ otherwise.

Proposition 3.60. *For each $k \in \omega$, we have that C_k is of Baire class k.*

Proof. First note that

$$\begin{aligned} \mathsf{C}_1 \circ \mathsf{C}_k \circ \mathsf{C}_0(x)(n) = 0 &\iff \exists n_k.\; \mathsf{C}_k \circ \mathsf{C}_0(x)(\ulcorner n, n_k \urcorner) \neq 0 \\ &\iff \exists n_k\, \forall n_{k-1} \dots \mathrm{Q} n_0.\; \mathsf{C}_0(x)(\ulcorner \ulcorner n, n_k \urcorner, n_{k-1}, \dots, n_0 \urcorner) = 0 \\ &\iff \exists n_k\, \forall n_{k-1} \dots \mathrm{Q} n_0.\; x(\ulcorner \ulcorner n, n_k \urcorner, n_{k-1}, \dots, n_0 \urcorner) \neq 0 \\ &\iff \mathsf{C}_{k+1}(x)(n) = 0. \end{aligned}$$

Therefore, since the composition of a Baire class 1 function with a Baire class k function is of Baire class $k+1$, and since C_1 is of Baire class 1 by Proposition 3.52, the result now follows by an easy induction. ■

The proof of the following is outside the scope of this thesis, and is therefore omitted.

Theorem 3.61 (Brattka [3]). *Let* $f : (X, \delta_X) \rightrightarrows (Y, \delta_Y)$. *Then*

$$f \in \mathbf{\Lambda}_{1,k+1} \quad \textit{iff} \quad f \leq_{\mathsf{t}} C_k.$$

3.2.5 Closed choice on ω^ω

In this section, we fix a computable enumeration $\mathsf{s} = \langle \mathsf{s}(0), \mathsf{s}(1), \ldots \rangle$ of $\omega^{<\omega} \smallsetminus \{\epsilon\}$. For example, this can be done by $\mathsf{s}(n) := \langle m_0, \ldots, m_\ell \rangle$, where $n = \ulcorner \ell, k \urcorner$ and $m_0, \ldots, m_\ell \in \omega$ are the unique numbers such that $\ulcorner m_0, \ldots, m_\ell \urcorner = k$.

We define $\mathcal{A}_\neg$ as the represented space $(\mathbf{\Pi}^0_1(\omega^\omega), \delta_\neg)$, where the total function

$$\delta_\neg(x) = \omega^\omega \smallsetminus \bigcup \{[\mathsf{s}(n)] : x(n) \neq 0\}$$

is called the *negative representation* of $\mathbf{\Pi}^0_1(\omega^\omega)$. The intuition behind the name is that $x \in \omega^\omega$ is a $\delta_\neg$-name of the closed set A when it enumerates the basic open sets that generate the complement of A.

Definition 3.62. The principle of *closed choice on* ω^ω is the function $\mathsf{C}_{\omega^\omega} : \mathcal{A}_\neg \rightrightarrows \omega^\omega$ given by

$$\mathsf{C}_{\omega^\omega}(A) = A,$$

with $\operatorname{dom} \mathsf{C}_{\omega^\omega} = \mathbf{\Pi}^0_1(\omega^\omega) \smallsetminus \{\emptyset\}$.

Although at first glance $\mathsf{C}_{\omega^\omega}$ may look like a somewhat simple operation — that of outputting A when given A as input — this impression is quickly ruled out when one considers the problem of *how* the input and output are actually given: a closed set A is given to $\mathsf{C}_{\omega^\omega}$ by an indirect enumeration of its complement, and $\mathsf{C}_{\omega^\omega}$ must then transform this enumeration into the elements of A themselves, i.e., as represented by id. Thus, in fact the intuition is that $\mathsf{C}_{\omega^\omega}$ has high complexity, and indeed this is confirmed in a precise way: $\mathsf{C}_{\omega^\omega}$ is Weihrauch-complete for the class of Borel measurable functions.

Proposition 3.63. *The function* $\mathsf{C}_{\omega^\omega}$ *is Borel measurable.*

Proof. Let $F = \iota_{\mathsf{T}}(x * \operatorname{seq}(\phi))$ with domain $\operatorname{dom}(\mathsf{C}_{\omega^\omega} \circ \delta_\neg) = \{x \in \omega^\omega : \delta_\neg(x) \neq \emptyset\}$, where ϕ is the following strategy for **II** in $\widehat{\mathsf{G}_{\mathsf{T}}}$, i.e., in the modified version of G_{T} where **II** is allowed to make any finite number of moves at each round, which is equivalent to G_{T} by Theorem 2.3.

Strategy: At round n, if $x_n \neq 0$ and $\mathsf{s}(n) \subseteq \iota_*(y_0{}^\frown \cdots {}^\frown y_{n-1})$, then play $\mathsf{e}_0{}^\frown\sigma$, where σ is the lexicographically-least word of length $n+1$ that does not extend any $\mathsf{s}(i)$ with $i \leq n$ and $x(i) \neq 0$.

Otherwise, play the least $n \in \omega$ such that $\iota_*(y_0{}^\frown \cdots {}^\frown y_{n-1}){}^\frown n$ does not extend any $\mathsf{s}(i)$ with $i \leq n$ and $x(i) \neq 0$.

Claim 1. For any $x \in \operatorname{dom} F$, we have that $A := \{\sigma \in \omega^{<\omega} : \exists n.\ \sigma \subset \iota_*((x \restriction n) * \phi)\}$ contains exactly one maximal infinite chain, whose union is the lexicographically-least element y of $\delta_\neg(x)$.

Indeed, let $n \in \omega$ and $\tau \in \omega^n$, and note that if $\tau <_{\text{lex}} y \restriction n$, then $\exists k.\ x(k) \neq 0$ and $\mathsf{s}(k) \subseteq \tau$. Therefore, after round k, by the way ϕ is defined no word played by **II** can contain τ. Hence at some point $y \restriction n$ will be played by **II**, and from that point on it will be the prefix of every word played by **II**. ⊡

Therefore ϕ is legal, which implies that $\operatorname{seq}(\phi)$ is also legal, and thus $F(x)$ is the lexicographically-least element of

$$\delta_\neg(x) \quad = \quad \mathsf{C}_{\omega^\omega} \circ \delta_\neg(x) \quad = \quad \omega^\omega \smallsetminus \bigcup \{[\mathsf{s}(n)] : x(n) \neq 0\}$$

for any $x \in \operatorname{dom}(\mathsf{C}_{\omega^\omega} \circ \delta_\neg)$, i.e., $F \vdash \mathsf{C}_{\omega^\omega}$. ■

The proof of the following result is outside the scope of this thesis, and is therefore omitted. We refer the reader to the original paper [4] for the details.

Theorem 3.64 (Brattka-de Brecht-Pauly [4]). *Let* $f : (X, \delta_X) \rightrightarrows (Y, \delta_Y)$. *Then*

$$f \text{ is Borel measurable} \quad \text{iff} \quad f \leq_{\mathsf{t}} \mathsf{C}_{\omega^\omega}.$$

3.2.6 New choice principles for $\mathbf{\Lambda}_{2,3}$ and $\mathbf{\Lambda}_{3,3}$

In this section, we will introduce Weihrauch-complete choice principles for $\mathbf{\Lambda}_{2,3}$ and $\mathbf{\Lambda}_{3,3}$. This will be done by directly *coding* the corresponding games into partial functions from ω^ω to ω^ω, which will result in functions which are arguably more artificial than the choice principles we have seen so far. However, this method of coding games into choice principles helps uncover the close connection between these two independently-developed areas, namely the games for Baire space functions on the one hand and the theory of Weihrauch-reducibility on the other, even more clearly than the other theorems we have seen so far.

Let us start by defining some auxiliary coding functions.

Define $\mathsf{skip}_0 : \omega^{<\omega} \longrightarrow \omega^{<\omega}$ by recursion, putting $\mathsf{skip}_0(\epsilon) = \epsilon$, and

$$\mathsf{skip}_0(\sigma^\frown s) = \begin{cases} \mathsf{skip}_0(\sigma)^\frown s, & \text{if } s \neq 0 \\ \mathsf{skip}_0(\sigma), & \text{otherwise,} \end{cases}$$

and extend it to $\mathsf{skip}_0 : \omega^{\leq\omega} \longrightarrow \omega^{\leq\omega}$ by putting

$$\mathsf{skip}_0(x) = \bigcup_{n \in \omega} \mathsf{skip}_0(x \restriction n)$$

for $x \in \omega^\omega$. Note that $\mathsf{skip}_0(x) \in \omega^{<\omega}$ exactly when x is eventually constant 0.

This function will be used for two goals: it will take the place of the passing move p, and it will also be used to help code the contents of the tapes of a sequence of moves.

Now define $\mathsf{info} : \omega^\omega \dashrightarrow\!\!\!\twoheadrightarrow \omega^\omega$ by

$$\mathsf{info}(x) = (x)_n$$

for the unique n such that $\mathsf{skip}_0((x)_n) \in \omega^\omega$, with $\operatorname{dom} \mathsf{info}$ exactly the set of $x \in \omega^\omega$ for which such a unique n exists.

As we saw above, 0 will be used to code *lack of information*, in a sense. The intuition behind the function info is then to extract from x the unique infinite sequence which contains an infinite amount of information.

Finally, given $x \in \omega^{\leq\omega}$ with $0 \notin \operatorname{ran} x$, denote by $(x{-}1)$ the sequence given by $(x{-}1)(n) = x(n) - 1$ for all $n \in \operatorname{dom} x$ — since 0 is reserved for "lack of information", we will need to *renormalize* sequences by subtracting one from each of their entries.

A complete choice principle for $\mathbf{\Lambda}_{2,3}$

The function $\mathsf{C}_{2,3} : \omega^\omega \dashrightarrow\!\!\!\twoheadrightarrow \omega^\omega$ is given by

$$\mathsf{C}_{2,3}(x) = \lim_{n\in\omega}(\mathsf{skip}_0[(\mathsf{info}(x))_n]{-}1),$$

with $\operatorname{dom} \mathsf{C}_{2,3}$ the set of $x \in \operatorname{dom} \mathsf{info}$ such that $\mathsf{skip}_0[(\mathsf{info}(x))_n] \in \omega^{<\omega}$ for any $n \in \omega$, and for which the limit above exists.

In other words, $\mathsf{C}_{2,3}(x)$ is obtained from x by first going to the unique $(x)_n$ that contains an infinite amount of information, extracting that information as an infinite sequence of finite words, and then computing the limit of this sequence.

Therefore, the following should intuitively hold.

Proposition 3.65. $\mathsf{C}_{2,3} \in \mathbf{\Lambda}_{2,3}$.

Proof. Let ϕ be the following strategy for **II** in $\widehat{\mathsf{G}_{2,3}}(\mathsf{C}_{2,3})$, i.e., in the modified version of $\mathsf{G}_{2,3}$ where **II** is allowed to make any finite number of moves at each round, which is equivalent to $\mathsf{G}_{2,3}$ by Theorem 2.3.

Strategy: At round $n = \ulcorner m, k, \ell \urcorner$, if $x_n = 0$ then pass. Otherwise let

$$\begin{aligned} \sigma &:= (\mathsf{skip}_0((\mathbf{I}(n))_m){-}1) \text{ and} \\ \tau &:= \iota_*(\mathsf{tape}_m(\mathbf{II}(n-1))). \end{aligned}$$

Now, if $\sigma \subseteq \tau$, then pass. Otherwise, if $\tau \subset \sigma$, then if we are on tape m play

$$s := \langle \sigma(|\tau|), \sigma(|\tau|+1), \ldots, \operatorname{last} \sigma \rangle$$

else play $\mathsf{t}_m{}^\frown s$. Finally, if $\sigma \perp \tau$, then let p be least such that $\sigma(p) \neq \tau(p)$, and if we are on tape m play

$$s' := \mathsf{e}_p{}^\frown \langle \sigma(p), \sigma(p+1), \ldots, \operatorname{last} \sigma \rangle,$$

else play $\mathsf{t}_m{}^\frown s'$.

Claim 1. The strategy ϕ is winning for **II** in $\widehat{\mathsf{G}_{2,3}}(\mathsf{C}_{2,3})$.

Indeed, let $x \in \operatorname{dom} \mathsf{C}_{2,3}$ and note that at the end of round n of a run of $\widehat{\mathsf{G}_{2,3}}(\mathsf{C}_{2,3})$ where **I** plays x and **II** follows ϕ, for each m we have

$$\iota_*(\mathsf{tape}_m((x \restriction n) * \phi)) = (\mathsf{skip}_0((x \restriction n)_m) - 1).$$

Therefore, $x \in \operatorname{dom} \mathsf{C}_{2,3}$ directly implies that ϕ is legal, and furthermore

$$\begin{aligned} \textit{theorem}\hat{\iota}_{2,3}(x * \phi) &= \lim_{n\in\omega} \iota_*(o(x * \phi) \restriction n) \\ &= \lim_{n\in\omega} (\mathsf{skip}_0[(\mathsf{info}(x))_n] - 1) \\ &= \mathsf{C}_{2,3}(x). \end{aligned}$$

■

And, as $\mathsf{C}_{2,3}$ was defined from $\mathsf{G}_{2,3}$ in such a direct way, the following should also be no surprise.

Theorem 3.66. *Let* $f : (X, \delta_X) \dashrightarrow (Y, \delta_Y)$. *Then*

$$f \in \mathbf{\Lambda}_{2,3} \quad \textit{iff} \quad f \leq_{\mathsf{t}} \mathsf{C}_{2,3}.$$

Proof. By Theorem 3.43, it is enough to prove the result for $f : \omega^\omega \dashrightarrow \omega^\omega$.

($\Rightarrow$) We will actually prove the stronger result that f is $(\mathrm{id}, \mathsf{C}_{2,3})$-continuous.

Let ϕ be a winning strategy for **II** in $\mathsf{G}_{2,3}(f)$, and let $F := \iota_{\mathsf{W}}(x * \phi_{\mathsf{W}})$, where ϕ_{W} is the following strategy for **II** in G_{W}.

Strategy: At round $n = \ulcorner m, k, \ell \urcorner$, let $\sigma := \iota_*(\mathsf{tape}_m(\mathbf{I}(k) * \phi))$ and if $\ell < |\sigma|$ then play $\sigma(\ell) + 1$, otherwise play 0.

ϕ_{W} is clearly a legal strategy and, by construction, for every $x \in \omega^\omega$ and $m, k \in \omega$ we have

$$(\mathsf{skip}_0[((x * \phi_{\mathsf{W}})_m)_k] - 1) = \iota_*(\mathsf{tape}_m((x \restriction k) * \phi)),$$

and therefore

$$\begin{aligned} \mathsf{C}_{2,3} \circ F(x) &= \mathsf{C}_{2,3} \circ \iota_{\mathsf{W}}(x * \phi_{\mathsf{W}}) \\ &= \lim_{n\in\omega} (\mathsf{skip}_0[(\mathsf{info}(x * \phi_{\mathsf{W}}))_n] - 1) \\ &= \lim_{n\in\omega} \iota_*(o(x * \phi) \restriction n) \\ &= \iota_{2,3}(x * \phi) \\ &= f(x). \end{aligned}$$

($\Leftarrow$) Follows from Proposition 3.65 and Theorem 3.42. ■

A complete choice principle for $\mathbf{\Lambda}_{3,3}$

The function $\mathsf{C}_{3,3} : \omega^\omega \dashrightarrow\!\!\!\!\rightarrow \omega^\omega$ is given by

$$\mathsf{C}_{3,3}(x) := (\mathsf{skip}_0(\mathsf{info}(x)) - 1),$$

with $\operatorname{dom} \mathsf{C}_{3,3} := \{x \in \omega^\omega : \exists! n \, \forall m \, \exists k > m. \, (x)_n(m) > 0\}$.

In other words, $\mathsf{C}_{3,3}(x)$ is obtained from x by decoding the unique sequence encoded by x that contains infinitely much information.

Proposition 3.67. $\mathsf{C}_{3,3} \in \mathbf{\Lambda}_{3,3}$.

Proof. Let ϕ be the following strategy for **II** in $\widehat{\mathsf{G_{mt}}}(\mathsf{C}_{3,3})$, i.e., in the modified version of $\mathsf{G_{mt}}$ where **II** is allowed to make any finite number of moves at each round, which is equivalent to $\mathsf{G_{mt}}$ by Theorem 2.3.

Strategy: At round $n = \ulcorner m, k \urcorner$, if $x(n) = 0$ then pass. Otherwise, if we are currently on tape m then play $x(n) - 1$, else play $\mathsf{t}_m {}^\frown x(n) - 1$.

Claim 1. The strategy ϕ is winning for **II** in $\widehat{\mathsf{G_{mt}}}(\mathsf{C}_{3,3})$.

Indeed, for any $x \in \omega^\omega$ and $n \in \omega$, by construction we have that the moves in ω that are made in $\mathsf{tape}_n(x * \phi)$ follow $(\mathsf{skip}_0((x)_n) - 1)$ exactly. Therefore, since for $x \in \operatorname{dom} \mathsf{C}_{3,3}$ we have that exactly one $(\mathsf{skip}_0((x)_n) - 1)$ is infinite, it follows that ϕ is legal and

$$\mathsf{C}_{3,3}(x) = \hat{\iota}_{\mathsf{mt}}(x * \phi). \qquad \blacksquare$$

As before, since $\mathsf{C}_{3,3}$ was defined in such an analogous way to $\mathsf{G_{mt}}$, the following holds intuitively.

Theorem 3.68. *Let $f : (X, \delta_X) \dashrightarrow (Y, \delta_Y)$. Then*

$$f \in \mathbf{\Lambda}_{3,3} \quad \textit{iff} \quad f \leq_{\mathsf{t}} \mathsf{C}_{3,3}.$$

Proof. By Theorem 3.43, it is enough to prove the result for $f : \omega^\omega \dashrightarrow \omega^\omega$.

($\Rightarrow$) We will actually prove the stronger result that f is $(\mathrm{id}, \mathsf{C}_{3,3})$-continuous.

Let ϕ be a winning strategy for **II** in $\mathsf{G_{mt}}(f)$, and let $F := \iota_{\mathsf{W}}(x * \phi_{\mathsf{W}})$, where ϕ_{W} is the following strategy for **II** in $\mathsf{G_W}$.

Strategy: At round $n = \ulcorner m, k \urcorner$, let

$$\begin{aligned} \sigma &:= \iota_*(\mathsf{tape}_m(\mathbf{I}(n) * \phi)) \\ \ell &:= \big|\{i < k : (\mathbf{II}(n-1))_m(i) \neq 0\}\big|. \end{aligned}$$

If $|\sigma| > \ell$ then play $\sigma(\ell) + 1$, otherwise play 0.

Note that if $x \in \operatorname{dom} f$, then since ϕ is legal, we have that $\mathsf{tape}_n(x * \phi)$ contains elements of ω infinitely often for exactly one n. Therefore, since for every m the non-zero moves in $(x * \phi_{\mathsf{W}})_m$ are made following the moves in ω made in $\mathsf{tape}_m(x * \phi)$, it follows that ϕ_{W} is legal and

$$\begin{aligned} \mathsf{C}_{3,3} \circ F(x) &= \mathsf{C}_{3,3} \circ \iota_{\mathsf{W}}(x * \phi_{\mathsf{W}}) \\ &= \iota_{\mathsf{mt}}(x * \phi) \\ &= f(x). \end{aligned}$$

($\Leftarrow$) Follows from Proposition 3.67 and Theorem 3.42. $\blacksquare$

Chapter 4

Conclusion and future work

Let us wrap up this thesis with a brief review of the results contained herein and an outline of some of the possible avenues for future development.

In Chapter 2, we presented the games characterizing classes of functions in Baire space currently known in the literature. In some cases, our presentation differed from that found in the original works, and for that reason we proved Theorems 2.13, 2.14, and 2.15 about the general equivalence of certain types of presentations. We closed the chapter pointing out, in an informal way, how all the currently known games can be obtained by varying two parameters — the freedom player **II** has in changing tapes and erasing past moves — as dictated by the values of m and n of the class $\mathbf{\Lambda}_{m,n}$ being characterized, or in a particular way for the game characterizing the Borel measurable functions.

In Chapter 3, we briefly introduced some basic concepts from computable analysis in order to define the relation of Weihrauch-reducibility. We then presented the known results about the completeness of certain functions, generically called choice principles, in some classes of functions. Specifically, we stated without proof the completeness of k-countable choice for Baire class k with $k \geq 2$ and of closed choice on ω^ω for the Borel measurable functions, we presented new proofs of the completeness of countable choice for Baire class 1 and of discrete choice for the class of functions preserving $\mathbf{\Delta}^0_2$ under preimages, and introduced new complete choice principles for the class of functions preserving $\mathbf{\Delta}^0_3$ under preimages and for the class $\mathbf{\Lambda}_{2,3}$ via a direct coding of the respective games.

Among the directions for future research, we can highlight the following.

Uniform description of games. Develop a formal counterpart to the intuitive relation between the rules of the currently known games and the classes of functions they characterize, as summarized at the end of Chapter 2.

New games. Develop games characterizing classes $\mathbf{\Lambda}_{m,n}$ with m or n greater than 3, possibly by also introducing new basic moves other than passing, changing tapes, or erasing, where a formalization of the relation between the currently known games and the classes they characterize could be instrumental. We also note that Louveau claimed in [18] that he, in joint work with Semmes, has provided a general method for generalizing Semmes's results to some of these classes, although this is yet to be published.

Another line of investigation is to attempt the reverse path than the one we adopted in this thesis, i.e., try to extract games from complete choice principles. This could be useful, e.g., in order to obtain new games for the Baire class n functions with $n > 3$.

Partition properties. Investigate new partition characterizations of the classes $\mathbf{\Lambda}_{\alpha,\beta}$ in the style of the Jayne-Rogers theorem, which may in turn suggest new games for them.

Composition properties. Develop a theory of composition of games that may be brought to bear on questions concerning the composition of functions. More specifically, for the moment let us denote $\mathbf{\Lambda}_{\alpha,\beta} \circ \mathbf{\Lambda}_{\alpha',\beta'} := \{f \circ g : f \in \mathbf{\Lambda}_{\alpha,\beta} \text{ and } g \in \mathbf{\Lambda}_{\alpha',\beta'}\}$. Straightforward game arguments show, for example, that $\mathbf{\Lambda}_{1,2} \circ \mathbf{\Lambda}_{2,2} = \mathbf{\Lambda}_{1,2}$ and $\mathbf{\Lambda}_{1,3} \circ \mathbf{\Lambda}_{2,2} = \mathbf{\Lambda}_{1,3}$ — since finitely many backtracking moves can be legally simulated by erasers in each case — but as far as we can tell not much else is known at this point. Relevant open questions in which games may be an important tool include, e.g., whether either one of $\mathbf{\Lambda}_{2,2} \circ \mathbf{\Lambda}_{1,3}$ and $\mathbf{\Lambda}_{3,3} \circ \mathbf{\Lambda}_{1,2}$ equals $\mathbf{\Lambda}_{2,3}$.

Connections between games and choice principles. Investigate the plausibility of a general method for obtaining a complete choice principle for a class of functions from a game characterizing that class, generalizing our results from Section 3.2.6.

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