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> First, to say concerning what and of what this investigation is: that it concerns demonstrative-proof [*apodeixis*], and is of demonstrative understanding [*apodeiktikē epistēmē*].[1]
>
> *Prior Analytics* I.1 24a10-11

1 Introduction

Aristotle holds that some—but not all—truths admit of demonstrative-proof [*apodeixis*]. That (e.g.) every triangle has internal angles summing to 180 degrees, admits of proof; that (e.g.) Socrates is currently out for a walk does not. Of course, Aristotle will not deny that one might give a sound deductive argument to the effect that Socrates is currently out for a walk. Nor will he deny that one might *know* such a truth on this or some other basis (say, by perception), without being able to prove it. Yet here Aristotle will insist that the 'proofs' or 'demonstrations' which (e.g.) geometers and metaphysicians aspire to give, are more than mere deductively sound arguments, and that not every truth is proveable in this sense. I hope it clear, that we too possess in our conceptual repertoire some such epistemically-loaded notion of 'proof'—a notion on which the proveable has a very special epistemic status, a notion of 'proof' considerably more demanding than that studied by the branch of modern logic called 'proof theory'.

Aristotle's *Analytics* investigates a notion of demonstration [*apodeixis*] on which a demonstrative-proof is an explanatory deduction of a necessary truth.[2] Being explanatory, a demonstration of φ must correctly explain *why* φ is the case; it must perspicuously *show* which features of reality are responsible for things being such that φ. Aristotle contends that an agent in possession of a demonstration that φ may be said to have scientific-understanding [*epistēmē*] of φ. For, Aristotle presumes, what science [*epistēmē*] aspires to do is just

[1] Here and in what follows, translations are always my own.

[2] Throughout, I use 'demonstration' and 'demonstrative-proof' as interchangeable renderings of *apodeixis*. The verbal noun *apodeixis* derives from *apodeiknumi*: to show forth, reveal, make known, bring to light. And by Aristotle's day *apodeixis* had already acquired a special epistemic sense.

this—to understand with respect to the permanent and non-accidental features of reality, *why* they are the way the are. Accordingly, Aristotle will see his logical researches into demonstrative-proof, as an important component of his account of *epistēmē*: his theory of science.

Posterior Analytics I.19-22 presents sustained argument that the process of (correctly) demonstrating a demonstrable truth always terminates in finitely many steps. The argument is ingenious. It records some of the most sophisticated meta-logic found anywhere in Aristotle. And passages from I.19-22 figure prominently in contemporary discussions of Aristotle's views on predication, epistemology, the essence/accident distinction, infinity, intelligibility, and the modal syllogistic. It is, then, surprising that no new studies of *Posterior Analytics* I.19-22 have appeared in the last 30 years. I suspect, that the cause for scholarly neglect of these chapters during a period of great interest in Aristotle's logic/theory of predication in general—and the *Posterior Analytics* in particular—is due more to the perceived difficulty of the material, than a sense that existing work on I.19-22 is satisfactory.

For, consider where our most recent scholarship on *Posterior Analytics* I.19-22 leaves off. On the view of Barnes (1975)[3], the argument of chapters I.19-22 is a disaster. To make any sense of Aristotle's attempted defense of the crucial lemma of I.21, Barnes maintains that we must delete seven lines from the transmitted text.[4] But even with the deletion, says Barnes, Aristotle's argument for the lemma remains obviously invalid. Aristotle's arguments in the important *Posterior Analytics* I.22, Barnes adds, are both invalid and unsound by Aristotle's own lights. So Barnes, summing up his pessimistic verdict on I.22: 'None of the arguments [of I.22] is successful; *and they cannot be reformulated in such a way as to furnish proof of Aristotle's contention*'.[5] In contrast to Barnes, Lear (1980) sees *Posterior Analytics* I.19-22 as less hopeless. A persistent goal in Lear's work is to emphasize Aristotle's positive achievements as a meta-logician. Nonetheless, Lear agrees with Barnes that Aristotle's proof of the I.21 lemma is invalid. As to whether the arguments of *Posterior Analytics* I.22 can possibly (contra Barnes) 'be reformulated in such a way as to furnish proof of Aristotle's contention', Lear omits comment. Finally, there is the short note of Scanlan (1982). Scanlan's note is mostly just an attack on Lear's reconstruction: the aim is to show that Lear's reading is thoroughly marred by a flawed understanding of the mathematics of infinite sequences. Most memorably, Scanlan objects that (what he takes to be) Lear's central interpretive claim about I.19-22—that Aristotle is trying to establish a 'proof-theoretic analogue of compactness'—is both obscure and unacceptably anachronistic. Scanlan does not develop any alternative in detail. As far as I can determine, no new studies of *Posterior Analytics* I.19-22 have appeared since.[6]

[3]Revised 1994, though Barnes' reading of I.19-22 are not significantly altered in the revision.

[4]The paragraph that Barnes proposes to delete occurs in all manuscripts of the *Analytics* in our possession, as well as those known to John Philoponus (6th century CE).

[5]Barnes 181, my emphasis.

[6]NB my search here has been limited to work in English, German, and French. Note that what follows does not take direct account of Mignucci's 1975 *L'Argomentazione dimostrativa*

I contend that the argument of *Posterior Analytics* I.19-22 has yet to be correctly understood. Existing work on the argument—both ancient and modern—is riddled with error and confusion.[7] This matters not only because this argument is worth understanding for its own sake. As noted above, fragments from I.19-22 are often adduced to support various differing positions in contemporary debates concerning Aristotle's views on predication, esssence/accident, infinity, substance, and modal syllogistic. The correct interpretation of these fragments of I.19-22 can only be settled by a reading which situates them in their global argumentative context. Moreover, as I hope shall emerge, many of the relevant issues are of considerable independent philosophical interest. So the problem of understanding the argument of *Posterior Analytics* I.19-22 has bearing on matters beyond the *Posterior Analytics* in particular.

This said, in what follows disagreement with contemporary scholars and the commentary tradition are not the primary focus. Nor will I dwell at length on the implications of my reading for issues outside *Posterior Analytics* I.19-22. The immediate aim is to develop a precise understanding of the argument witnessed by I.19-22. And to do so, I will draw on tools provided by modern mathematical logic. The interest of the project shall, I hope, emerge from the richness of material itself.

2 *Posterior Analytics* I.19-22 and compactness

We noted above that Lear (1980) characterizes the thesis of I.19-22 as a 'proof-theoretic analogue of compactness'. Now, compactness is usually thought of as a model-theoretic property. In a very (very) general way, the model-theorist will define a logic as a pair $L = (\mathscr{L}, \models_{\mathscr{L}})$, such that

1. $\mathscr{L}$ is a map from vocabularies τ (i.e. non-empty collections of symbols) to classes of sentences composed from from vocabularies: $\mathscr{L}[\tau]$ ($\mathscr{L}$-sentences)

2. $\models_{\mathscr{L}}$ ($\mathscr{L}$-satisfaction) is a binary-relation between (some specified class of extra-linguistic) τ-structures and$\mathscr{L}$-sentences[8]

As usual, given some sentences $\Gamma \subseteq \mathscr{L}[\tau]$ and a τ-structure $\mathfrak{A}$, we then say $\mathfrak{A} \models \Gamma$ ($\mathfrak{A}$ is a model of Γ) iff for all $\varphi \in \Gamma$: $\mathfrak{A} \models \varphi$. Moreover, for any $\Gamma \cup \{\varphi\} \subseteq \mathscr{L}[\tau]$, we say $\Gamma \models \varphi$ (φ is a logical consequence of Γ) iff for every

in Aristotele. Parts of Mignuicci's approach to the text of I.19-22 are known to me through reports in Barnes' 1975/1994 *Posterior Analytics* commentary. But I confess that I have not read Mignucci's book.

[7]In addition to the commentators mentioned above, I have studied the contributions of Philoponus (c. 490-530), Themistius (c. 317-88) , and Ross (1957). NB I have not made a detailed study of medieval/pre-modern commentaries on the *Posterior Analytics*.

[8]Subscripts dropped when no confusion results. On this approach to logics, familiar from Lindström, see e.g. Barwise and Feferman (esp. chapter 2). The above definition captures everything the model-theorist will normally call a 'logic'. To ensure that it does not capture more, additional conditions are normally imposed (an isomorphism condition, a renaming condition, etc.: see Barwise and Feferman 28). For our purposes, such conditions may be ignored.

$\mathfrak{A}$ such that $\mathfrak{A} \models \Gamma$, it holds that $\mathfrak{A} \models \varphi$. Finally, the model theorist defines (strong) compactness in one of the following two equivalent ways:

Compactness A Let $L = (\mathscr{L}, \models_{\mathscr{L}})$, and $\Gamma \subseteq \mathscr{L}[\tau]$, for some τ. L is compact iff Γ has a model whenever every finite subset $\Gamma_0 \subseteq \Gamma$ has a model.

Compactness B Let $L = (\mathscr{L}, \models_{\mathscr{L}})$, and $\Gamma \subseteq \mathscr{L}[\tau]$, for some τ. L is compact iff whenever $\Gamma \models_{\mathscr{L}} \varphi$, there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \models_{\mathscr{L}} \varphi$

It is **Compactness B** that Lear has in mind in calling the thesis of *Posterior Analytics* I.19-22 a 'proof-theoretic analogue of compactness'. The I.19-22 thesis is supposed to be like compactness in the sense that (according to Lear) Aristotle wants to prove in I.19-22 that every demonstrable conclusion can be demonstrated from finitely many premises.[9]

Scanlan (1982) complains of anachronism: if one does not (as Aristotle surely didn't) countenance the possibility of an actually infinite collection of sentences, nothing like the question of compactness can really arise. This is, of course, basically correct. But in focusing on this point, Scanlan is attacking a straw-man. For, despite the misleading title of his article, Lear was never claiming that Aristotle actually confronted the problem of compactness. The idea was simply that Aristotle confronted a problem *analogous* to compactness. Read more charitably, Lear's train of thought looks to be the following. For Aristotle, syllogistic deductions are finitary objects. Every individual syllogistic inference involves exactly two premises, and no chain of syllogistic inferences can be infinitely long. In *Posterior Analytics* I.19-22, Aristotle is addressing the question whether the finitary nature of syllogistic deduction imposes any limitation on the amount of *epistēmē* that can be gained by syllogistic deduction. He argues that it does not. For, everything demonstrable is demonstrable from finitely many premises. And this result is suggestively analogous to compactness.

Certainly, one might question how *helpful* Lear's comparison with compactness ultimately is. But not much of what Lear actually says about *Posterior Analytics* I.19-22 turns on the issue. The real problem with Lear (and Scanlan) is that neither interpret *Posterior Analytics* I.19-22 as addressing the problem that Aristotle himself tells us he is addressing. Contra Lear, I.19-22 is not responding to an objector who thinks that syllogistic deductions—because they are finitary—are inadequate for purposes of science [*epistēmē*], an objector who suggests that some stronger logic (with, say, infinitary rules of inference) is needed. In fact, Aristotle is engaging against (a form of) skepticism. He is responding to an objector who denies the very possibility of *epistēmē*—the possibility of a systematic scientific understanding of reality. Moreover, in arguing against to this objector, Aristotle's aim is not (pace Lear) to establish that some special property holds of syllogistic consequence in general; he is not (pace Lear) trying to show his syllogistic's proof-theoretic strength. Aristotle's aim is rather to establish something that holds of demonstrative-proof [*apodeixis*] in

[9] Corcoran also endorses something close to this reading in his work on the *Prior Analytics*, but does not argue for it.

particular—that a genuine process of demonstration can never continue ad infinitum. This is important because Aristotle insists on distinguishing the class of demonstrations (a very special type of *sound* argument) from the class of syllogisms (a certain type of *valid* argument). And in I.19 81b18-23 Aristotle allows that a processes of mere syllogistic-deduction can (in a way) continue ad infinitum. Aristotle nowhere tries to show that in such cases continual deducing ad infinitum is 'logically unnecessary'. So, if 'Aristotle's logic' *is* his syllogistic, the thesis of I.19-22 is not meta-logical in our (or Lear's) sense.[10]

In the end, getting clear on the thesis of *Posterior Analytics* I.19-22 requires getting clear on the challenge Aristotle writes I.19-22 in order to addresses. And this will demand some remarks on Aristotle's accounts of *epistēmē* in general, and demonstration in particular.

3 Demonstration and *epistēmē*: the basic picture

These days, 'epistemology' is usually practiced as an inquiry into certain rather ordinary cognitive attitudes and processes: justified belief, rational credence revision, knowledge/knowledge acquisition, etc.[11] It is, then, crucial see that in making claim about *epistēmē*, Aristotle takes himself to be characterizing a very *extra*ordinary cognitive state. In distinction to everyday forms of knowledge, *epistēmē* is an intellectual virtue or excellence [*aretē*], a rather rare form of perfected theoretical expertise. Qua excellence, it is analogous to courage or justice (non-intellectual excellences).

Roughly, '*epistēmē*' names for Aristotle the best possible cognitive state a rational agent (be it human or not) can be in with respect to an appropriately maximal systematic constellation of general, necessary truths. Accordingly, in Aristotelian contexts, *epistēmē* is normally translated not as 'knowledge' but rather as 'science' or 'scientific understanding' (the medievals used *scientia*).[12] Such translations are acceptable so long as one is mindful of the following difference between *epistēmē* and 'science' in our sense. We tend to think of a science, qua something you can practice, as a collaborative discipline that engages in various research programmes, develops progressively better theories, etc. But for Aristotle, full *epistēmē* is only achieved when no further research is necessary—when all outstanding scientific questions concerning the domain in question have received correct solution, and nothing is left to explain. So, from Aristotle's perspective, *epistēmē* is not what our biologists and physicists

[10]NB This is not to say that the argument of I.19-22 involves no meta-logical reasoning. In fact, the argument involves a great deal of meta-logical reasoning about syllogistic. So it is with some justification that Lear, in analyzing I.19-22, wants to emphasize Aristotle's activity as a meta-logician.

[11]To characterize such attitudes and processes as ordinary is not at all to deny their philosophical interest. Indeed, it is partially due to their apparent ubiquity in ordinary life that such notions are thought to merit serious philosophical attention.

[12]NB in Aristotle's Greek, *epistēmē* can refer both (i) to a systematic body of scientific knowledge—Aristotle will say, e.g., that geometry is an *epistēmē*—, and (ii) to the cognitive state of someone who has mastered a science.

are doing; it is the ultimate goal they are trying to achieve.

Posterior Analytics I.2 71b9-12 tells us that an agent A has *epistēmē* of a proposition φ iff (i) A knows [*ginōskein*] with respect to the (real) explanation [*aitia*] forφ's being the case, that it is the explanation of φ's being the case, and (ii) A knows that φ cannot be otherwise. Not only is *epistēmē* not knowledge in the ordinary sense; Aristotle is here presupposing an antecedent grasp of knowledge in the ordinary sense for explaining what *epistēmē* is. Now, as is clear from elsewhere, Aristotle's point in insisting that an object of *epistēmē* must be a necessary truth is that science is centrally about general and permanent features of reality. The biologist's goal, in the first instance, is not to understand why this or that human being is mortal. Her goal is to understand why mortality belongs to human beings in general—why being moral holds of the the *kind*: human being. And as Aristotle goes on to tell us, a scientific researcher succeeds in understanding why a necessary truth φ holds, only when she has traced φ's holding back through its intermediate causes to its ultimate explanatory grounds (ultimate here being relative to the domain in question). In this context, Aristotle finds it natural to think of an *epistēmē* structured as an axiomatic system, with explanatory facts as axioms and explainable general truths as theorems. So, immediately after providing the above analysis of *epistēmē*, Aristotle turns in I.2 to demonstrative *epistēmē*. Setting aside for later discussion the question of whether one can have *epistēmē* of some truths without demonstrative-proof, Aristotle sketches his account of demonstrative *epistēmē* and demonstrative-proof as follows (71b16-25):

> [W]e assert that there is knowing through demonstrative-proof [*di' apodeixeōs eidenai*]. By a 'demonstrative-proof' I mean an epistemic deduction [*syllogismon epistēmonikon*]; and by 'epistemic' I mean something in virtue of whose possession we have *epistēmē*. If then, to have *epistēmē* is as we have posited [i.e. immediately above in 71b9-12], it is indeed necessary that demonstrative *epistēmē* be from [premises] which are [i] true, [ii] primary, [iii] immediate [*ameson*], and [iv] better known, [v] prior to, and [vi] explanatory of [*aitia*] their conclusion; for in this manner the first-principles [*archai*] will be appropriate to what is shown. Now, there can be syllogistic-deduction [*syllogismos*] without such [premises], but not demonstrative-proof [*apodexis*]; for premises [which do not meet these conditions] will not yield *epistēmē*.

Aristotle defines a demonstration as a syllogistic-deduction [*syllogismos*] whose possession yields *epistēmē*.[13] And in so doing he seems to be assuming a notion of (explanatory) priority that is transitive, but not necessarily total.[14] His immediacy condition [iii] must be understood in the context of his syllogistic system. A syllogistic-deduction establishes a relation between two terms A and

[13] NB in Aristotle, a *syllogismos* must have at least two premises; but need not have exactly two premises. Hence, my reticence to translate '*syllogismos*' as 'syllogism'.

[14] Otherwise condition [iv] would be superfluous given condition [ii]. Recall that a relation R is total iff it satisfies trichotomy: for all x, y either xRy or yRx or $x = y$.

B through a series of one-step sub-deductions from two-sentence pairs. The premises of a syllogistic deduction are required to contain A and B. And the process of syllogistic-deduction works to link A with B through a series of 'middle terms' [*mesa*]. In the case of demonstration, the middle terms are required to be *explanatory.* Accordingly, in the text above, the condition [iii] that the premises of a demonstration be *amesa* (literally: 'lacking a middle', hence 'immediate') is supposed to amount to the requirement that the premises cannot themselves be explained by appealing to any further explanatory middle term.

4 *Posterior Analytics* I.19-22 in context: the 'skeptical' dilemma of I.3

As a sequel to Aristotle's I.2 analysis *epistēmē*, *Posterior Analytics* I.3 introduces two groups of thinkers, whose conclusions about *epistēmē* Aristotle proposes to refute. Chapter I.3 begins (72b5-18):

> Some [theorists] do not think there is *epistēmē* since [it would] require having *epistēmē* of primitives; others [think] that there is [*epistēmē*], but there demonstration is of everything. Neither of them [the views] are true, nor necessary.[15] The [first group], supposing that it is not possible to have *epistēmē* in another way [than by demonstration], contend that [in demonstrating, we] are lead up ad infinitum, seeing that we would not have *epistēmē* of the posteriors on account of the priors if there are no primitives. They argue rightly. For, it is impossible to traverse [*dielthein*] infinitely many things. But, if [the process of demonstrating] stops and there are first-principles [= primitives], these [they contend] are unknowable [*agnōstous*] since in fact there is no demonstration of them, which they say is the only [way of] having *epistēmē*. But if it is not possible to know the primitives, neither is it possible to have *epistēmē* of the things from them, without qualification and strictly speaking, rather [it is only possible to have qualified, i.e., pseudo-'*epistēmē*'] from a supposition [*ex hypotheseōs*]: 'If these things are so, [then] ...'. The [second group] agrees about having *epistēmē*, that it is through demonstration alone. But [they argue that] nothing prevents there from being demonstration of all things; for it is possible for demonstration to come about in a circle and reciprocally [*ex allēlōn*].

We may call the first group (the skeptics about unqualified, non-hypothetical *epistēmē*) **Group A**, and the second group (the anti-skeptical partisans of circular demonstration) **Group B**. From Aristotle's report (and lack of additional ancient testimony), it is not entirely clear whether **Group A** developed their argument independently of Aristotle, or whether these thinkers take themselves

[15]'Necessary' here means either: (i) that these thinkers give invalid arguments, or (ii) that the views of these thinkers are not entailed Aristotle's own view.

to be adducing through dilemma a skeptical consequence for Aristotle's particular I.2 analysis of demonstrative *epistēmē*.[16] In any case, both groups apparently accept something very close to the chapter I.2 account of demonstrative *epistēmē*. The point of the dilemma brought by **Group A** seems to be that demonstrative *epistēmē* is in principle impossible—a consequence welcome for these thinkers—not that Aristotle's analysis of demonstrative *epistēmē* should be rejected. And the anti-skeptical partisans of **Group B** (perhaps responding to **Group A**) are evidently defending the possibility of demonstrative *epistēmē* also as adherents to something like Aristotle's analysis of it.

Aristotle's rejection of **Group B's** proposal and his case against circular demonstration is the main focus in the remainder of *Posterior Analytics* I.3. But the centerpiece of Aristotle's response to **Group A** does not appear in I.3; it appears in *Posterior Analytics* I.19-22. Aristotle characterizes I.19-22 negatively as showing that it is not possible for demonstration [*apodeixis*] to continue ad infinitum.[17] He characterizes I.19-22 positively as showing that every demonstration 'terminates' or 'comes to a stop' [*histanai*], thus reaching first-principles [*archai*] or primitives [*prōta*].[18] Aristotle adds that this conclusion goes against 'what we said in the beginning that some people [*tinas*] claim' (84a32).[19] And this is clearly a back-reference to his discussion of **Group A**.

By Aristotle's lights, **Group A** reasons as follows. Either (i) there are no explanatorily basic 'primitive' truths, or (ii) there are some explanatorily basic 'primitive' truths. If (i), then for every truth φ, there are some distinct truth(s) Γ explanatorily prior to it. But explanatory priority is transitive. And an argument that φ amounts a demonstrative-proof only if it *fully* explains φ by taking account of *every* truth explanatorily prior to φ. Accordingly, a demonstrator who attempts to demonstrate any necessary truth φ will be lead back ad infinitum. For, to demonstrate φ, the demonstrator will need to assume some distinct truths Γ_0, explanatorily prior to φ. But since no $\psi \in \Gamma_0$ is explanatory primitive, the demonstrator will next have to appeal to some fresh collection of truths Γ_1 to explain the truths in Γ_0. Then the demonstrator will need to gather to a further collection of truths Γ_2 to explain the truths in Γ_1. And so on, ad infinitum. Moreover, even if (ii) is the case, and at some point the demonstrator reaches a Γ_n containing some explanatorily basic primitives $\Delta \subseteq \Gamma_n$, unqualified *epistēmē* is still impossible. For, if the 'proof' of φ is to yield *epistēmē*, these primitive truths Δ upon which φ is grounded must themselves be objects of *epistēmē* for the demonstrator. But the truths in Δ can be objects of *epistēmē* only if they themselves are demonstrated. And by assumption, since none of these truths have any truths prior to them, this is impossible.

Aristotle's response to **Group A** in the *Posterior Analytics* has two components. In I.19-22 Aristotle will rejects the first-horn, (i), of the dilemma brought by **Group A**. He argues that every demonstrable truth is fully grounded in a

[16] I suspect the former.
[17] 82a5-6, 84a32-33, 84a37-38.
[18] 84a30-32, 82a21-2, a36-37, b34-35.
[19] Cf. 82a6-8

finite collection of explanatorily primitive truths (its principles), and can indeed be demonstrated from its principles in finitely many steps. Taking the second-horn of the dilemma, (ii), Aristotle goes on to argue in II.19 that although explanatorily primitive first principles cannot be objects of *epistēmē* strictly speaking, they are not (as **Group A** charges) unknowable. For, there is a knowing state even more perfect than *epistēmē* which grasps explanatorily basic first principles. Aristotle calls it: *nous.*

5 A Zenonian aspect to the puzzle

The above characterization of **Group A**'s reasoning is abstract: it does not assume that demonstrative explanations appear any particular form. Such too is Aristotle's own presentation of **Group A** in *Posterior Analytics* I.3. But Aristotle himself, of course, does have a view as to the logical form of demonstrative-proofs. His *Posterior Analytics* is centrally concerned with demonstrative-proofs that take the form of syllogistic arguments.

From the logician's perspective, what makes *Posterior Analytics* I.19-22 seriously interesting is that it argues against the possibility that demonstrative-explanation can continue ad infinitum from within the logical framework of Aristotelian syllogistic. And it is in this context that the problem acquires, for Aristotle, its peculiar character. Note his restatement of the problem brought by **Group A** at I.19 (82a2-8):

> Further, when the extremes are fixed is it possible for the intermediates to be infinite? I mean [*legw de*], for instance, if A belongs to Γ, and B is a middle [between] them, but of A and B there are other middles, and different middles of these, is it indeed possible for these to continue ad infinitum, or is it impossible? And this is the same as to investigate whether demonstrations can continue ad infinitum and if there is demonstration of all things, or some are limited by one another. (Similarly I mean [*de legw*] also in the case of negative syllogisms and propositions...).

In Aristotelian syllogistic, every proposition is composed of two terms and a copula (copulae ≈ 'holds of all', 'holds of no', etc.). Valid syllogistic inferences establish a copulative connection between the two 'extreme' terms A, B of the conclusion, by connecting them through a 'middle term' C. For instance:

$$\frac{\text{A holds of all C} \qquad \text{C holds of all B}}{\text{A holds of all B}}$$

Now, if an argument is to constitute not merely a syllogistic-deduction, but indeed a demonstrative-proof it is required that its premises be fully explanatory of its conclusion. Every 'middle' term used to connect the two 'extreme' terms of the conclusion must be an *explanatory* middle. And in some cases more than one middle term will be required. For instance, it might occur that the premise

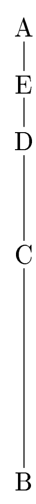
‘A holds of all C’ in the above is not explanatorily primitive. If this is so, the above cannot constitute a demonstration, and a new explanatory middle (one between A and C, explaining their connection) will be required in order to derive and more fully explain connection between A and B:

$$\dfrac{\dfrac{\text{A holds of all D} \qquad \text{D holds of all C}}{\text{A holds of all C}} \qquad \text{C holds of all B}}{\text{A holds of all B}}$$

Again the connection between A and D may or may not be primitive and explanatorily basic. If not, a new middle or ‘intermediate’ between A and D will be required:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c} ? \\ \vdots \\ \text{A holds of all E} \end{array} \qquad \text{E holds of all D}}{\text{A holds of all D}} \qquad \text{D holds of all C}}{\text{A holds of all C}} \qquad \text{C holds of all B}}{\text{A holds of all B}}$$

We can picture iterating the above process on a line. The issue is to connect A and B. So we appeal to a connecting middle C. But our task is not complete. For to connect A with C, a new middle D is required. But the connection between A and D is also non-immediate. So we appeal to a new middle E. And so on ad infinitum:



From the particular perspective of Aristotle’s system, the question of demonstration ad infinitum is the question of whether establishing a connection between two ‘extremes’ might require first going through infinitely many ‘middles’. Partisans of **Group A** charge that establishing such a connection always requires going through infinitely many middles. And as 'it is impossible to go through [*dielthein*] infinitely many things' (72b10-11) they contend that demonstration and *epistēmē* are in principle impossible.

In this light, the difficulty brought by **Group A** bears notable structural parallels to that brought by Zeno in his well-known dichotomy argument.[20] Zeno's dichotomy maintains that change [*kinēsis*] is impossible on the grounds that a changing object changing from A to B (e.g., a runner running from point A to point B) would have to perform infinitely many sub-changes (for instance, traverse infinitely many half distances). However, claims Zeno, it is impossible to go through [*dielthein*, *diexelthein*] infinitely many things. Hence, change too must be impossible. Partisans of **Group A** argue not that change [*kinēsis*] is impossible but that demonstration [*apodeixis*] is impossible. Yet these thinkers argue from analogous considerations: the impossibility of traversing the infinite due to the requirement of performing infinitely many sub-tasks (here: the impossibility of performing infinitely many sub-derivations). The difference is that whereas Zeno's challenge is derived from the alleged infinite divisibility of physical space, **Group A**'s challenge is derived from the alleged infinite divisibility of explanatory space.

Given the structural similarities between the **Group A** puzzle and the Zeno puzzle, it is interesting how structurally dissimilar Aristotle's responses to the two puzzles are. In response to Zeno, Aristotle will reject the atomistic alternative and uphold the possibility of change while preserving the infinite divisibility of the continuum. In contrast, *Posterior Analytics* I.19-22 aims to argue that explanatory space has an atomic architecture. If the connection between two universals A and B is demonstrable, that connection is grounded in a finite number of explanatorily indivisible, atomic intervals. These intervals are unmediated subject-predicate propositions, and among the first principles of the relevant science.[21]

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[20] For Aristotle's discussion of Zeno's dichotomy see esp. *Physics* VI.2, VIII.8.

[21] Compare these remarks with how Aristotle himself characterizes the conclusion of our chapters. For instance (I.22 84a32-37):

> If there are principles, it is not the case that every [scientific proposition] is demonstrable, nor is it possible to continue [demonstrating] ad infinitum: for either of these things to be the case is simply for there to be no immediate and indivisible [*ameson kai adiaireton*] intervals but for all of them to be divisible.

See also I.23 84b34-85a2:

> The middle terms will always be thickened until they become indivisible and unitary [*hen*]. They are unitary when they have become immediate, and a unitary proposition simpliciter is an immediate proposition. And just as in other things the principle [*archē*] is unitary, but not the same everywhere: a *mna* in weight, a semi-tone in a musical tune, and other things in other cases, similarly in deduction the unit [to hen] is the immediate proposition, and in demonstration and *epistēmē* it is *nous*.

6 Aristotelian syllogistic

In order to develop my interpretation of the argument of *Posterior Analytics* I.19-22, I must first introduce some formal apparatus. I begin by specifying a toy-language $\mathcal{L}$ in which we will represent Aristotelian demonstration [*apodeixis*].

Definition 1 Language $\mathcal{L}$

1. Basic expressions
   (a) a countable set of *terms* (usually represented by capitalized letters)
   (b) four syllogistic *copulae*: a, e, i, o
2. Sentences
   (a) If X and Y are terms, then the following are sentences of $\mathcal{L}$:
      i. XaY
      ii. XeY
      iii. XiY
      iv. XoY
   (b) Nothing else is a sentence of $\mathcal{L}$

Following scholarly convention, our intended interpretation is such that the right-most term of a sentence in $\mathcal{L}$ is the grammatical subject while the left-most term is the grammatical predicate. The copula a- is for universal affirmative sentences; e- is for universal negative sentences; i- is for particular affirmative sentences; and o- is for particular negative sentences. So on our intended interpretation of $\mathcal{L}$ we have:

- XaY $\Longleftrightarrow$ 'X is said of all Y' $\Longleftrightarrow$ 'Every Y is an X'
- XeY $\Longleftrightarrow$ 'X is said of no Y' $\Longleftrightarrow$ 'No Y is an X'
- XiY $\Longleftrightarrow$ 'X is said of some Y' $\Longleftrightarrow$ 'Some Y is an X'
- XoY $\Longleftrightarrow$ 'X is not said of some Y' $\Longleftrightarrow$ 'Some Y is not an X'

Definition 2 Proto-syllogism

For all terms X, Y, Z of $\mathcal{L}$ the following 'trees' constitute proto-syllogisms

First figure	Second figure	Third figure
Barbara $\dfrac{\text{XaY} \quad \text{YaZ}}{\text{XaZ}}$	Cesare $\dfrac{\text{YeX} \quad \text{YaZ}}{\text{XeZ}}$	Darapti $\dfrac{\text{XaY} \quad \text{ZaY}}{\text{XiZ}}$
Celerant $\dfrac{\text{XeY} \quad \text{YaZ}}{\text{XeZ}}$	Camestres $\dfrac{\text{YaX} \quad \text{YeZ}}{\text{XeZ}}$	Datisi $\dfrac{\text{XaY} \quad \text{ZiY}}{\text{XiZ}}$
Darii $\dfrac{\text{XaY} \quad \text{YiZ}}{\text{XiZ}}$	Festino $\dfrac{\text{YeX} \quad \text{YiZ}}{\text{XoZ}}$	Disamis $\dfrac{\text{XiY} \quad \text{ZaY}}{\text{XiZ}}$
Ferio $\dfrac{\text{XeY} \quad \text{YiZ}}{\text{XoZ}}$	Baroco $\dfrac{\text{YaX} \quad \text{YoZ}}{\text{XoZ}}$	Felapton $\dfrac{\text{XeY} \quad \text{ZaY}}{\text{XoZ}}$
		Ferison $\dfrac{\text{XeY} \quad \text{ZiY}}{\text{XoZ}}$
		Bocardo $\dfrac{\text{XoY} \quad \text{ZaY}}{\text{XoZ}}$

Each proto-syllogism is a representative of one of the 14 valid syllogistic ‘moods’ Aristotle recognizes. (Recall that Aristotle himself does not countenance a fourth figure). In first figure moods the middle term occurs as subject in the left-premise and predicate in the right-. In second figure moods, the middle term occurs as predicate in both premises. And in third figure moods, the middle term occurs as subject in both premises. Note that the traditional names for the syllogistic moods carry useful information. The first vowel corresponds to the copula of the first (left-) premise; the second vowel corresponds to the copula of the second (right-) premise; and the third vowel corresponds to the copula of the conclusion.[22] Naturally, in what follows we will sometimes use labels like ‘Barbara’ and ‘Celarent’ to describe sub-trees of syllogistic trees.

Definition 3 Basic syllogism

If

$$\mathrm{S} := \quad \frac{\varphi \qquad \psi}{\chi}$$

is a proto-syllogism such that $\varphi \neq \chi$ and $\psi \neq \chi$, then S is a basic syllogism with premises $\{\varphi, \psi\}$ and conclusion χ.[23]

[22] In fact, the traditional names also carry information on reduction. More on this later.

[23] NB we assume for syllogistic propositions φ, ψ, that $\varphi = \psi$ iff their subject terms, predicate terms, and copulas are tokens of the same type.

We take the requirement that neither premise be identical with the conclusion from Aristotle's definition of syllogism given in *Prior Analytics* I.1. Syllogistic validity is not supposed to be semantic validity in our (usual non-relevantistic) sense.

Definition 4 Syllogism

1. Any basic syllogism with premises $\{\varphi, \psi\}$ and conclusion χ, is a syllogism with premises $\{\varphi, \psi\}$ and conclusion χ.

2. If

$$\mathrm{S} := \begin{array}{c} \dfrac{\varphi \qquad \begin{array}{c}\vdots \\ \psi\end{array}}{\chi} \\ \vdots \end{array}$$

   is a syllogism with premise set $\Pi = \{\varphi, \ldots\}$ and conclusion ζ, and

$$\frac{\alpha \qquad \beta}{\varphi}$$

   is a basic syllogism such that $\alpha \neq \zeta$ and $\beta \neq \zeta$ then

$$\mathrm{S}^* := \begin{array}{c} \dfrac{\dfrac{\alpha \qquad \beta}{\varphi} \qquad \begin{array}{c}\vdots \\ \psi\end{array}}{\chi} \\ \vdots \end{array}$$

   is a syllogism with premises $(\Pi \setminus \{\varphi\}) \cup \{\alpha, \beta\}$ and conclusion ζ.

3. If

$$\mathrm{S} := \begin{array}{c} \dfrac{\begin{array}{c}\vdots \\ \varphi\end{array} \qquad \psi}{\chi} \\ \vdots \end{array}$$

   is a syllogism with premise set $\Pi = \{\varphi, \ldots\}$ and conclusion ζ, and

$$\frac{\alpha \qquad \beta}{\psi}$$

   is a proto-syllogism such that $\alpha \neq \zeta$ and $\beta \neq \zeta$ then

$$\mathrm{S}^* := \begin{array}{c} \dfrac{\begin{array}{c}\vdots\\ \phi\end{array} \qquad \dfrac{\alpha \qquad \beta}{\psi}}{\chi} \\ \vdots \end{array}$$

is a syllogism with premises $(\Pi \setminus \{\varphi\}) \cup \{\alpha, \beta\}$ and conclusion ζ.

This above constitutes the deductive system of syllogistic we will be using in what follows. The system is rather unlike the deductive systems adduced in (e.g.) the well-known papers of Corcoran and Smiley on Aristotelian syllogistic. For instance, rules of conversion and reductio are absent but all 14 syllogistic moods (not just Barbara and Celarent) are present. Smiley and Corcoran constructed their systems in the context of their work on assertoric syllogistic in *Prior Analytics* I.1-7, 23. In these chapters, Aristotle does appeal to conversion and reductio while pursuing his project of 'reduction'. Now, *Posterior Analytics* I.19-22 is dependent on *Prior Analytics* I, presupposing (e.g.) the doctrine of the figures. However, the meta-logical researches of I.19-22 make no mention of conversion or reductio. It is the above system of all 14 moods Aristotle will work with in I.19-22. It seems to me that this is significant evidence against the claim of some recent scholars that Aristotle sees his 'reduction' proof in *Prior Analytics* I in the way contemporary proof theorists see (e.g.) cut elimination. But I cannot pursue the issue here.

Definition 5 Formula height

For a sentence-occurrence φ appearing in a syllogism S define the *height of* φ *in S*, $height_{\mathrm{S}}(\varphi)$, as the number of sentence-occurrences in the syllogistic tree S dependent on φ.

Example

Where S is the syllogism: :

$$\dfrac{\dfrac{\dfrac{\mathrm{AaB} \qquad \mathrm{BaC}}{\mathrm{AaC}} \qquad \dfrac{\mathrm{CaE} \qquad \dfrac{\mathrm{EaG} \qquad \mathrm{GaD}}{\mathrm{EaD}}}{\mathrm{CaD}}}{\mathrm{AaD}} \qquad \mathrm{DiF}}{\mathrm{AiF}}$$

we have that:

- $height_{\mathrm{S}}(\mathrm{AiF}) = 0$
- $height_{\mathrm{S}}(\mathrm{DiF}) = 1$
- $height_{\mathrm{S}}(\mathrm{CaD}) = 2$
- $height_{\mathrm{S}}(\mathrm{EaG}) = 4$

Definition 6 Improvement and improvement-paths

1. If a syllogism S* may be generated from a syllogism S using a single application of **Definition 4.2** or **4.3**, we say that S* *improves* S.

2. A sequence of syllogisms $\overline{\mathrm{S}}$ = (S_0, S_1,... , $\mathrm{S}_{\text{n-1}}$, S_n, ...) with length $\leq \omega$ is an *improvement-path* originating in S* iff (a) S_0 = S*, and (b) for all $\mathrm{S}_{\mathrm{k}+1}$, $\mathrm{S}_\mathrm{k} \in \overline{\mathrm{S}}$: $\mathrm{S}_{\mathrm{k}+1}$ improves S_k.[24]

3. A syllogism S* is an *improvement* of S iff there is an improvement-path $\overline{\mathrm{S}}$ = (S_0, S_1,... , $\mathrm{S}_{\text{n-1}}$, S_n) such that S_0 = S and S_n = S*.

Ultimately, improvement-paths will be crucially important in modeling what it is for a demonstrative process to continue ad infinitum. As noted above, a demonstration in Aristotle's sense is a special sort of sound argument: not a merely valid argument. So before, introducing a formal representation of demonstration, we will be giving a formal semantics for syllogistic, and an account of truth-conditions for syllogistic propositions. But to motivate the picture I want to present, familiarity with some preliminary notions is required.

7 Predication

In Aristotle's *Analytics*, predicating [*katēgoriein*] is conceived of as a linguistic action. It is something one does with words, and it can be done truly or falsely. However, things that get predicated, one of another, are not mere representations of beings. It is beings themselves that get predicated.[25] Described from an operational perspective, predicating is well characterized as a function that takes exactly two (usually extra-linguistic) entities as input, and yields an affirmative statement as output. So, while the result of predicating, say, human being of Socrates is a simple sentence ('Socrates is a human being'), both what gets predicated (here, the species: *human being*), and what gets something predicated of it (here, the individual: *Socrates*) are beings.

Following Aristotle, let us call the result of an act of predicating: a predication [*katēgoria*]. What all genuine predications have in common is affirming that one being X underlies [*hupokeitai*] one being Y as its metaphysical subject—affirming that a being Y holds of [*huparchei*] or belongs to a being X. Predication (qua linguistic action) is a form of affirmation, not denial.

Now, unlike Plato, Aristotle systematically distinguishes between universal and individual beings. A universal (being) holds of a plurality of other beings; an individual (being) does not.[26] However, Aristotle does not systematically distinguish singular from general *terms*. And he does not see *holding-of* as a

[24] Here and in what follows we abuse notations '∈' and '⊆' harmlessly.

[25] A nice text bringing out this dual (linguistic/extra-linguistic) aspect in predication is *Prior Analytics* I.27 (43a25-6): 'Among all **beings** [*hapantōn ontōn*], some are such that they are predicated **truly** and universally [*katēgorieisthai alēthōs katholou*] of nothing else...'. We discuss 43a25-6 in detail below.

[26] See, e.g. *De Interpretatione* 7.

relation that in every case an individual bears to a universal. So the *belonging* or *holding-of* relation *animal* bears to *whale*, is not categorically different from that which *animal* bears to *Moby Dick*. For Aristotle, an individual can be truly predicated of an individual, and a universal can be truly predicated of both individuals and universals. Accordingly, in contrast to Frege-Russell, Aristotle does not think that the predications:

- 'Moby Dick is a whale.'
- 'Whales are mammals.'
- 'Moby Dick is Moby Dick.'

possess distinct 'logical forms'.

So, Aristotle evidently ignores a distinction on whose importance Frege-Russell insist. Of course, Frege-Russell are themselves ignoring a distinction on whose importance Aristotle insists.[27] For, Aristotle famously contends that there are irreducibly different kinds of predication [*genē tōn katēgoriōn*] corresponding to irreducibly different ways for one being to belong to another. Hence Aristotle's division of 'categories' [*katēgoriai*]: some predications [*katēgoriai*] indicate what a subject is ('Bucephalus is a horse', 'Triangle is a figure'), others indicate how a subject is quantified ('Bucephalus is 450kg'), or qualified ('Bucephalus is trustworthy), or related to something else ('Bucephalus is a father'), etc. These predications differ because the *way* that (say) being-a-horse belongs to Bucephalus, is irreducibly different from the way that (say) being-450kg belongs to Bucephalus. And mutandis mutatis, for the other 'categories'. So, as Aristotle ignores what Frege-Russell see as a 'categorical difference' between (e.g.): (i) 'Moby Dick is a whale' (expressing $\in$), and (ii) 'Whales are animals' (expressing $\subseteq$), Frege-Russell ignore Aristotle's 'categorical difference' between (e.g.): (iii) 'Moby Dick is a whale' (expressing what Moby Dick is), and (iv) 'Moby Dick is 60,000kg' (expressing how much Moby Dick is).

Now, Aristotle takes it that Greek speakers typically make predications by uttering simple sentences of the form: 'X is Y'. And Aristotle further assumes that a Greek speaker who says 'X is Y' standardly makes a predication indicating that the being referred to by 'X' underlies the being referred to by 'Y' as its metaphysical subject. But Aristotle also recognizes that there are exceptions. A sentence of the form 'X is Y' does not always predicate (the referent of) Y of (the referent of) X. Of special importance for us is his treatment of the issue in *Posterior Analytics* I.22 (83a1-17):

> It is possible to speak truly: [**case (a)**] 'the white is walking' and 'this large thing is a log' ; again [it is possible to speak truly]: [**case (b)**] 'the log is large' and 'the man is walking'. But to speak in the latter way is different than [to speak] in the former. For, when I

[27] The basic historical narrative of this paragraph (in one instance, where Frege-Russell see many Aristotle sees one; in another, where Frege-Russell see one Aristotle sees many) is also developed in Code (1983).

> say [*phō*] 'the white is log', then I mean [*legō*]: that to which being-white accidentally belongs is a log; I do not mean that the white is the subject for log... In contrast, when I say 'the log is white' I do not [mean]: some different thing is white and being-a-log belongs accidentally to that [i.e. the different thing],. . . Rather, [I mean that] the log is the subject, the very thing that became white, not in virtue of being something different than [a thing] which just is a log or a certain log. If, then, we must legislate, let it be that saying [*legein*] in the latter way [**case (b)**: e.g. 'the log is white'] is predicating [*katēgorein*]; but saying in the former way [**case (a)**: e.g. 'the white is a log'] is either predicating not at all [*mēdamōs*], or predicating not genuinely [*mē haplōs*] but predicating incidentally [*kata sumbebēkos*]...

This difficult text is frequently called upon to witness interpretations of Aristotle on substance as ultimate metaphysical subject-hood. But, the central contrast Aristotle wants to draw here tends to be obscured in such discussions. For, the passage's point is not to defend any view about substance or ultimate metaphysical subjects. Its point is to make a distinction between two ways of 'speaking' or 'saying' [*legein*]—in particular, two ways of speaking *truly*.[28] And to make this distinction Aristotle is just assuming a simplified version of his usual conception of metaphysical subject-hood on which, paradigmatically, perceptible substances are metaphysical subjects for qualities and quantities.[29] Let us consider, then, the two ways in which 'it is possible to speak [*eipein*] truly'. One kind of speaking Aristotle calls 'predicating' [*katēgorein*]. The other he alternatively describes as 'predicating not at all', or 'predicating not genuinely' but 'predicating incidentally' (a14ff.). Aristotle explains the difference between predicating ('The log is white') and this sort of pseudo-predicating ('The white is a log') as follows. Typically when a speaker says [*phēsi*] 'The white is log', she will not mean [*legei*] that this white is the metaphysical subject in which being-a-log inheres. Insofar as she speaks truly, the speaker presumably means that this log is a metaphysical subject in which white inheres. Indeed, the statement 'The white is a log' is true precisely because this log is a metaphysical subject of white, and therefore (per accidens) what 'the white' picks out is numerically the same as this log. (That these are the correct truth-conditions is something Aristotle simply takes for granted). So, while a speaker may utter 'The white is a log' and do so truly, there is a certain disconnect between what she means (the truth she communicates) and the grammatical structure of what she says. What is metaphysically a subject is referred to by an expression in grammatical predicate position; and what is metaphysically a predicate is referred to by an expression in grammatical subject position. Such is pseudo-predication. In

[28] That this passage's point is to contrast two ways of speaking is quite clear from its structure. 83a1-4 states the thesis; the *gar* in a5 indicates that the aim of the *hotan men* (a5)/*otan de* (a9) contrast is to explain the thesis. Finally the distinction between the two ways of speaking is labeled at a14ff.

[29] Simplified: on Aristotle's considered view, the per se subject of a color like white is not a substantial body (like log); it is the surface of a body. See, e.g., *De Sensu* 3.

contrast, for a speaker who actually predicates (saying, e.g. 'The log is white') there is no such disconnect between what is said and what is meant. In a genuine predication, the entity that the grammatical subject picks out really is the metaphysical subject for what the grammatical predicate picks out. Grammatical and metaphysical structure are well-aligned so that the truth-conditions for what the speaker who predicates says, are perfectly mirrored in the grammatical structure of her utterance. Such is predicating.

It is the custom of modern commentators to follow the usage of Philoponus, in referring to predications of the form 'The log is white' as *natural predications* and predications of the form 'The white is log' or 'The white is musical' as *unnatural predications*. But this is not entirely felicitous. In the first place, it is clear from the relevant passages of Philoponus' commentary that the distinction between natural and unnatural predication was a technical development of later Peripatetic logicians involving sub-distinctions beyond any Aristotle makes in *Posterior Analytics* I.22. Accordingly, the Peripatetics' natural/unnatural distinction ought to be studied by historians of logic in its own right, and not terminologically conflated with the distinction of I.22 83a1-17 (whatever it amounts to). Second, the natural/unnatural terminology wrongly suggests that Aristotle means to countenance two species of a genus *predication*. But Aristotle's point in 83a1-17 is that predication in the style of 'The white is log' is not, strictly speaking, predication at all. At best, he maintains, insofar as someone who says 'The white is log' successfully communicates that the log is white, we might—by courtesy and making qualifications—describe them as predicating. Such a speaker makes a predication incidentally, in the sense (perhaps) that a doctor who, with good faith, misapplies her art brings health incidentally. The desired effect is achieved, but not through the activity of a per se cause, and not without the collaboration of external factors.

So, I prefer 'genuine predication'—more often, I just say 'predication'—for what modern commentators on *Posterior Analytics* I.22 often call 'natural predication'. Of course, to insist on this would be pedantry. What actually matters here, is to see that Aristotle does not think of (genuine) predication as some special type of predication. For Aristotle, predications should not be divided into the 'natural' and 'unnatural' any more than trees should be divided into genuine and plastic. Predication in the style of 'The log is white' or 'The human being is an animal', is predication full-stop: predication in the basic and standard sense. Usually, Aristotle's only reason for even mentioning non-genuine predication is to set it aside as an irrelevant for whatever claim about predication he does want to make. And in I.22, the immediate sequel to the 83a1-17 text quoted above, is the claim that the predicational chains that reflect the real structures that demonstrative sciences make their demonstrations about—that is, the predicational chains at issue in I.22—are chains of (genuine) predications:

> So, let it be assumed that [in the chains we are considering] the predicate is always predicated of what it is predicated of genuinely, not incidentally. For this is how demonstrations demonstrate. Therefore, whenever one thing is predicated of one thing [in a demonstra-

> tion], either it is in the essence [*ti esti*] or [it indicates] that [the subject is] qualified or quantified or related to something or doing something or undergoing something or somewhere or some-when.

Note that Aristotle sees it as a consequence of the fact that propositions in demonstrative science are predicational (= express genuine predications), that his division of categories will be applicable to them. Indeed, I think Aristotle's considered view is that the the division of categories applies to genuine predications and genuine predications alone.[30]

8 Predication and metaphysical subject-hood

Posterior Analytics I.22 83a1-17 is difficult to understand partly because Aristotle's presentation is excessively digressive. To bring out the passage's central point, I translated it above with several elipses. Now I remove the elipses (83a1-17):

> It is possible to speak truly: [**case (a)**] 'the white is walking' and 'this large thing is a log' ; again [it is possible to speak truly]: [**case (b)**] 'the log is large' and 'the man is walking'. But to speak in the latter way is different than [to speak] in the former. For, when I say [*phō*] 'the white is log', then I mean [*legō*]: that to which being-white accidentally belongs is a log; I do not mean that the white is the subject for log; for neither by being white nor [a thing] which just is white did it become a log. In contrast, when I say 'the log is white' I do not [mean]: some different thing is white and being-a-log belongs accidentally to that [i.e. the different thing]. (For instance, when I utter 'the musical is white', then I mean that the man is white to whom being-musical happens to belong). Rather, [I mean that] the log is the subject, the very thing that became white, not in virtue of being something different than [a thing] which just is a log or a certain log.

As noted above, interpreters have often seen this text as articulating Aristotle's view that substances are ultimate metaphysical subjects: items that underlie everything else while nothing distinct underlies them. Unfortunately, the mistaken impression that Aristotle is making a claim about substance in 83a1-17, is often developed into an analysis of genuine predication in terms of substance. So, for instance, Hamlyn reads I.22 as arguing: 'X is Y' is genuine iff X is a primary substance (in the sense of the *Categories*). And Barnes reads I.22 as arguing: 'X is Y' is genuine iff there is no Z such that Z is Y and Z is accidentally X—making clear that he intends this to amount to: 'X is Y' is genuine iff X is

[30] What category could the predication: 'The white is a log', be in? In substance/essence because log is what the thing which underlies the white is? In quality, because the sentence just means the log is white? NB I am thinking here especially of the *Topics* I.9 division of categories which looks to divide predications (and not just predicates) into genera. (This is not to say that the same text isn't *also* dividing predicates into genera).

a primary or secondary substance (in the sense of the *Categories*). But both readings are hopeless. *Posterior Analytics* I.22 requires that all propositions which enter into demonstrative sciences are/express genuine predication. But Aristotle most definitely holds that mathematics is a demonstrative science, and most definitely denies that the subject-terms of mathematical propositions always substances. Aristotle's favorite example of a demonstrable truth is, after all, 'Every triangle has internal angles summing to 180 degrees'. And triangle, for Aristotle, is certainly no substance.

Here is the corrective. *Posterior Analytics* I.22 83a1-17 does use a notion of metaphysical subject-hood to articulate an account of genuine predication. But *ultimate* metaphysical subject-hood is not at issue. For Aristotle, 'X is Y' is a genuine predication iff X *in fact* underlies Y as its metaphysical subject. Construed rightly, this does not entail that X is a substance. Let it be that X and Y both hold of Z. On Aristotle's view, X is a metaphysical subject for Y iff it is in virtue of being X (or in virtue of being a certain X) that Z is available to receive Y. So (83a12-14): 'The log is the subject, the very thing that became white, not in virtue of being something different than [a thing] which just is a log or a certain log'. Likewise, triangle has internal angles summing to 180 degrees not in virtue of being something other than a certain triangle (e.g. bronze, wooden).[31]

In the *Analytics*, for X to be a metaphysical subject it is a sufficient that X is a what and not a such. Here, whats will include beings like Bucephalus, log, redness, triangle, walking (the type of action), night; suches will include beings like red, triangular, walks, nocturnal.[32] For Aristotle, this is a metaphysical and not a linguistic distinction. We shall meet it again when we come to the chapter I.22 account of essence (whatness) and accident (suchness).

9 Predicable semantics for syllogistic

In *Posterior Analytics* I.22 Aristotle requires that the predications involved in demonstrations be genuine predications, not pseudo-predications. However, it is not perfectly obvious what this requirement amounts to. Aristotle's examples of genuine predications are of the form 'X is Y'. But syllogistic propositions are of the form AxB (with $x \in$ {a,e,i,o}). Here I present a formal and philosophical interpretation of this requirement. The proposal is closely related to

[31] Elsewhere I develop my view on these issues further, and also in the context of *Posterior Analytics* I.4.

[32] In this context the reader is invited to recall *Topics* I.9 (103b27-35):

> It is clear... that someone indicating the what-is-it sometimes indicates a substance , sometimes a quantity, sometimes a quality, and sometimes one of the other categories. For, whenever one says of a given human that the given [thing] is a human or an animal, one says what it is and indicates a substance. But whenever one says of a given white color that it is white or is a color he says what it is and indicates quality. Likewise too if one says of a given foot-long length that it is a foot-long length he says what it is and indicates quantity. Likewise too for the other [categories].

my preferred semantics for syllogistic propositions, which I should first like to rehearse. I begin by specifying the models on which I will be interpreting syllogistic propositions. I then introduce an apparatus for formal representation of Aristotelian demonstration which builds in Aristotle's requirement that predications involved in demonstration be genuine predication. As we shall see, this requirement will be important for the argument of *Posterior Analytics* I.19-22.

Definition 7 Predication structures

A *predication structure* is a triple $K = (D, P, [\![\cdot]\!])$ such that:

1. D is a non-empty set of beings
2. P is binary relation on D that is (a) *serial* and (b) *transitive*, i.e.:
   (a) $(\forall x \in D)(\exists y \in D)\ xPy$
   (b) $(\forall x, y, z \in D)\ xPy \wedge yPz \rightarrow xPz$
3. for every term T in our language $\mathcal{L}$, $[\![\mathrm{T}]\!] \in D$

On the intended interpretation, xPy represents the relation 'x is (truly) predicated of y'. And predication here is genuine predication, not pseudo-predication in the sense of I.22. The transitivity of Aristotelain predication, is witnessed by numerous texts.[33] And I submit that the seriality of the predication relation P falls immediately out of Aristotle's basic conception of what a being is. On Aristotle's view, a being is an item which some subject *is*: Socrates is a being just in case some subject (this human being) is Socrates; likewise, the walking is a being just in case some subject is walking. So x is a being iff x is predicated of some y. But since Aristotle assumes that everything is a being, it follows straightaway that everything is predicated of something: that predication is serial.[34] Here now are semantic clauses for evaluating syllogistic propositions on predication structures.

Definition 8 Semantics for syllogistic propositions

Let $K = (D, P, [\![\cdot]\!])$. Then:

- $K \vDash \mathrm{AaB}$ iff $(\forall x \in D)([\![\mathrm{B}]\!]Px \rightarrow [\![\mathrm{A}]\!]Px)$
- $K \vDash \mathrm{AeB}$ iff $(\forall x \in D)([\![\mathrm{B}]\!]Px \rightarrow \neg[\![\mathrm{A}]\!]Px)$
- $K \vDash \mathrm{AiB}$ iff $(\exists x \in D)([\![\mathrm{B}]\!]Px \wedge [\![\mathrm{A}]\!]Px)$
- $K \vDash \mathrm{AoB}$ iff $(\exists x \in D)([\![\mathrm{B}]\!]Px \wedge \neg[\![\mathrm{A}]\!]Px)$

[33] See e.g. *Categories* 3*; Prior Analytics* 25b32-35, 44b4ff.; *Posterior Analytics* 96a12-15, *Topics* 122a35-37, etc.

[34] Here see, e.g. Metaphysics Z.1, Γ.2, Δ.6.

As usual, let us say that a syllogistic proposition φ is *true in* K iff $K \vDash \varphi$. Further, if $\Gamma \cup \{\varphi\}$ is a set of syllogistic propositions, we say $\Gamma \models \varphi$ iff for every K such that $K \models \psi$ for all $\psi \in \Gamma$, it holds that $K \models \varphi$. With 'semantic validity' so defined, Aristotle's assertoric syllogistic—represented here by the proof system given above—is sound on the class of predication structures. An inductive proof taking account of each of Aristotle's 14 basic syllogistic forms would be tedious. However, given a result due to Aristotle himself, soundness follows straightaway given: (1) the (semantic) validity of Barbara and Celarent, together with (2) the (semantic) validity of: a-, e-, i-conversion, and reductio proof. The reader can easily check that (1) and (2) hold.[35] Given soundness, predication structures can be used for representing Aristotle's method in the *Analytics* of showing invalidity and inconclusiveness via counterexamples. It will be noted, of course, that assertoric syllogistic is far from being complete on the class of predication structures. For instance, given that every syllogistic-deduction requires at least two premises, the semantic validity YaZ$\models$XaX corresponds to no syllogistic-deduction. But insofar as our purpose is the formal reconstruction of something in Aristotle, this is how things should be. For, Aristotle's notion of a syllogistic-deduction isn't supposed to capture anything like a notion of semantic validity. On Aristotle's view, it is necessary—but not sufficient—for being a syllogism that whenever the premises are true, the conclusion must be true.[36]

We note, finally, a feature of the semantics important for modeling of Aristotle's account of demonstrative-proof in particular.

Observation 1

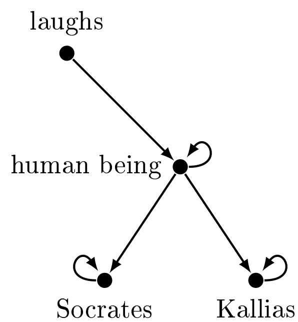
1. In any predication structure K, if $[\![\mathrm{A}]\!] P [\![\mathrm{B}]\!]$, then $K \vDash$ AaB

2. However, there are predication structures $K = (D, P, [\![\cdot]\!])$ according to which $K \vDash$ AaB but $[\![\mathrm{A}]\!] P [\![\mathrm{B}]\!] \notin P$.

It may be useful to consider a witness to **Observation 1.2** (xPy indicated by an arrow from x pointing to y; transitive arrows not pictured):

[35] Note in particular, that transitivity and seriality together with the assumption that for every term T in our language $\mathcal{L}$: $[\![\mathrm{T}]\!] \in D$, ensure a-conversion: XaY$\models$YiX, giving the 'existential import' of a-propositions.

[36] Cf. Aristotle's definition of syllogism in *Prior Analytics* I.1.



For the structure pictured above, call it K, let $[\![\mathrm{H}]\!]=$ human being and $[\![\mathrm{L}]\!]=$ laughs. Adding all transitive arrows, we have both that $K \models$LaH and $K \models$HaL: all human beings laugh and everything that laughs is a human being. But while $K \models$HaL is true, human being is not predicated of laughs. Taking a cue from Aristotle, we may describe this state of affairs as one in which (evaluating at K) a speaker can say HaL truly, but in so saying not make a (genuine) predication.

10 Interlude: comparison with other semantic approaches to Aristotelian syllogistic

The semantics of **Definition 8**, constitute a version of so-called 'intensional' or 'predicable semantics' for Aristotelian syllogistic. Support for such an approach emerges from recent work of Mario Mignucci (2000), Marko Malink (2006, 2009), and Benjamin Morison (2008).[37] The chief alternative to predicable semantics is, of course, the well-known 'set-theoretic' or 'extensional' semantics for Aristotelian syllogistic one gets (e.g.) in Smiley (1972) and Corcoran (1973).

At the center of the dispute between proponents of the predicable and set-theoretic approaches is a disagreement over the correct interpretation of Aristotle's *dictum de omni et nullo*. The *dictum* is, essentially, Aristotle's only explicit statement in the *Analytics* as to how we should understand the meaning of a-propositions and e-propositions. And is supposed to underwrite (e.g.) the 'perfection' of Barbara and Celarent's validity.[38] It is notable that no *dictum* for i- and o-propositions are given (a *dictum de aliquo et aliquo non*, as it were). And concerning e-propositions, the text in fact says next to nothing. So *Prior Analytics* I.1 24b28-30 reads:

> We say that '[A] is predicated of all [B]' when nothing of the subject [i.e., B] can be taken of which the other [i.e., A] will not be said.

[37]In the context of Barnes (2007), the latter two trace their proposals to the late Michael Frede.

[38]Warning: what contemporary scholars of Aristotle's logic call the *dictum de omni et nullo* is just the piece of text: 24b28-30, which seems to give something like semantic clauses for a- and e-propositions. In other contexts, labels like '*dictum de omni*' and '*dictum de nullo*' are used differently: for instance, as names of logical 'principles'.

And [for A is predicated] of no [B], similarly.[39]

Speaking generally (and ignoring a few small details), proponents of both the 'extensional' and 'intensional' approaches understand the *dictum de omni* as stating:

$$\text{AaB iff } \neg(\exists X)\ (\llbracket \text{B} \rrbracket\, R\, X \wedge \neg \llbracket \text{A} \rrbracket\, R\, X).$$

The disagreement is over (1) whether the varriable X (in our metalinguistic representation) should range over items of the same semantic type as $\llbracket \text{A} \rrbracket$ and $\llbracket \text{B} \rrbracket$, and (2) what precisely R is. Proponents of the set-theoretic semantics assume that X ranges over individuals while the semantic values of syllogistic terms are sets of individuals—items of of a different (semantic) type than individuals. The relation R is then given a set-theoretic interpretation. So AaB will be true just in case $\llbracket \text{A} \rrbracket \subseteq \llbracket \text{B} \rrbracket$. And more generally, any syllogistic proposition AyB (with y $\in$ {a,e,i,o}), is true just in case the sets $\llbracket \text{A} \rrbracket$ and $\llbracket \text{B} \rrbracket$ stand in the appropriate set-theoretic relation indicated by y. In contrast, proponents of the predicable semantics insist that the kind of distinction between different semantic types which the set-theoretic semantics presupposes is post-Aristotelian. We argue that one should instead take X to range over items of the same semantic type as $\llbracket \text{A} \rrbracket$ .and $\llbracket \text{B} \rrbracket$. Accordingly, we interpret R as a binary predication relation.

Many are the grounds for preferring a predicable semantics for Aristotelian syllogistic over any set-theoretic alternative, in the context of interpreting Aristotle's work. There are aspects of Aristotle's logical theory (some indeed present in I.19-22)[40] for which a set-theoretic semantics can give no plausible reconstruction. And on the final analysis, predicable semantics just provides the more realistic model for how Aristotle himself conceived of what syllogistic propositions mean. Defense of this assessment can be found elsewhere;[41] I will not rehearse it here. Instead, I propose to argue for the superiority of my particular version of predicable semantics over the alternative and closely related formal rendering of predicable semantics found in the pioneering work of Marko Malink.

As I see it, my predicable semantics differs from Malink's centrally on two issues. Firstly, the predication structures introduced in the present paper include include a semantic value function $\llbracket \cdot \rrbracket$; in contrast, Malink uses no semantic value function but (effectively) treats each term as its own semantic value. Secondly, while Malink's semantics is formulated in terms of a primitive predication relation that he requires to be a preorder (reflexive and transitive), my own approach makes the weaker assumption that P is serial and transitive.[42]

[39] Tranlsating *tou hupokeimenou* with the manuscripts at b29. But nothing much depends on this. As Mignucci (2000) argues, even if the phrase is a latter gloss, it still must be interpretively supplied to make sense of the text.

[40] Cf. our discussion of infinite predicational down-chains below.

[41] See esp. Malink (2006), Malink (2009).

[42] Malink and I also importantly differ on how we read the *dictum de omni et nullo*, and how it relates to predicable semantics. But an adequate discussion of this issue would take us too far afield.

First the question of term reference. Malink, of course, does not think that Aristotle failed to distinguish terms and (non-linguitic) beings. Malink's choice to effectively treat the semantic value of a term as the term itself is supposed to be a harmless formal simplification. Now, with respect to the parts of Aristotle's logic that Malink's system was developed to explore, I am inclined to agree that the distinction between a term and its referent plays no crucial role. However, it also seems to me that the identification of a term with its semantic value can give a misleading picture of how predicable semantics actually works. In both his 2006 and 2009, Malink claims that unlike the set-theotetic alternative, his predicable semantics demands no special assumption of existential import. For, given that Malink requires the predication relation to be reflexive, if AaB is true then B is predicated of B. So there is always at least one thing to which A belongs, namely B itself. In fact, however, Malink has not dispensed with the usual assumption of existential import. He has simply burried it. For, Malink's assumption that the semantic value of a term is the term itself is no less an added assumption of existential import, than the set-theoretic semanticist's assumption that no term picks out the empty set, or my own requirement that in a predication structure for all terms T: $[\![\mathrm{T}]\!]\in D$.

It must be admitted that my use of a semantic value function introduces a (sometimes inconvenient) degree of complexity into the formal reconstruction. Several of this paper's lemmas would be unnecessary if we followed Malink in eliding the distinction between linguistic and extra-linguistic. Nonetheless, I contend that we need such a distinction to capture important aspects of Aristotle's reasoning in the *Posterior Analytics* in general and I.19-22 in particular. Regarding the former, a specific disadvantage of Malink's approach is its inability to represent definitional a-propositions—a class of a-propositions crucial in the *Posterior Analytics*. For, on Aristotle's view, the subject and the predicate of a definitional a-proposition XaY pick out the same entity. However, the subject term (say, 'human being') and the predicate term (say, 'rational animal') of a definitional a-proposition are distinct. The present framework has no difficulty in representing definitional a-propositions as Aristotle conceives of them. For there may be instances where for distinct terms A,B: $[\![\mathrm{A}]\!]= [\![\mathrm{B}]\!]$. In all such instances it will hold that AaB (and BaA). Nothing like this is possible in Malink's framework. Finally, as to *Posterior Analytics* I.19-22 in particular I note the following. In I.19-21 Aristotle works to found his claim about the finitude of demonstrative processes, on his I.22 thesis that there are no infinite predicational chains. But Aristotle importantly conceives of this latter thesis as a thesis about the extra-linguistic architecture of reality. So Aristotle's reasoning in I.19-22 involves both considerations about the internal structure of syllogistic proofs and metaphysical considerations about extra-linguistic reality. Failing to formally represent a distinction between language and being would obscure this double aspect of I.19-22's argument.

Now to the second main difference between Malink and I: reflexivity versus (mere) seriality of the predication relation. I argued above that the seriality of (genuine) predication falls immediately out of Aristotle's basic understanding of what a being is. The claim that predication is a serial relation, I submit, has much independent motivation. The same cannot be said for reflexivity. It is important for both Malink and I that the predication relation in our models be construed as genuine predication in the sense of *Posterior Analytics* I.19 and I.22 (texts that Malink himself cites). Indeed, Malink's (2006) solution to the two Barbara problem(s) in his reconstruction of Aristotle's modal syllogistic depends on taking (his correlate to my) P as representing genuine predication. But not only is there little to no positive textual evidence for taking genuine predication as a reflexive relation. There is also a good deal of negative evidence against taking genuine predication as reflexive. For, Aristotle explicitly tells us that the 'The walking is walking' and 'The white is white' are not genuine.[43] But even without such passages, I think there are decisive reasons for preferring seriality to reflexivity. Recall the predication structure K adduced as a witness to **Observation 1.2**. Since according to K, laughs is not predicated of laughs Malink will reject K as an admissible model for assertoric syllogistic. And indeed, in Malink's system AaB does entail APB (where I have an 'if' Malink has an 'iff'). Now, both Malink and I agree that for Aristotle 'the laughing is a human' is not a genuine predication. But on Malink's semantics this entails that 'Everything that laughs is a human being' is false. Not only is this intuitively very weird. It also goes against what Aristotle actually about genuine predication in *Posterior Analytics* I.22. For recall that Aristotle's point there was to contrast genuine predicating ('the log is white') with pseudo-predicating ('the white is log') as two ways of *speaking truly.* But on Malink's semantics non-genuine predications can never be true. I note finally that this same feature of Malink's system commits him to an implausible interpretation of Aristotle's account of assymetric a-conversion—a feature of Aristotle's logic that Malink (2009) and I both take to provide strong support for adopting a predicable semantics for syllogistic. Following *Prior Analytics* II.22, let's say that that two terms A and B assymetrically a-convert iff A is predicated of everything that B is predicated of including B itself while B is predicated of everything which A is predicated of but not A itself. Since Aristotle characterizes the relevant relation as a form conversion, he must be considering cases in which both AaB and BaA are true. But on Malink's semantics, whenever A stands to B in the relation Aristotle specifies as assymetric a-conversion it can never occur that both AaB and BaA are true.

11 A formal model of Aristotelian demonstration

I am now in a position to present a formal model for Aristotelian demonstration. The model is meant to have a certain explanatory value. I propose to use it as a tool for explaining what exactly Aristotle is claiming in *Posterior Analytics* I.19-22, and showing why certain non-obvious claims he makes about his system do in fact follow. As we saw above, on Aristotle's usage a demonstrative-proof is a certain type of *sound* argument (71b16-25):

[43] See, e.g. *Posterior Analytics* I.4, *Metaphysics* N.1.

> By a 'demonstrative-proof' I mean an epistemic deduction [*syllogismon epistēmonikon*]; and by 'epistemic' I mean something in virtue of whose possession we have *epistēmē*. If then, to have *epistēmē* is as we have posited [i.e. in 71b9-12], it is indeed necessary that demonstrative *epistēmē* be from [premises] which are [i] true, [ii] primary, [iii] immediate [*ameson*], and [iv] better known, [v] prior to, and [vi] explanatory of [*aitia*] their conclusion; for in this manner the first-principles [*archai*] will be appropriate to what is shown. Now, there can be syllogistic-deduction [*syllogismos*] without such [premises], but not demonstrative-proof [*apodexis*]; for premises [which do not meet these conditions] will not yield *epistēmē*.

Our strategy will be characterize something as a demonstrative-proof only relative to a model in which its constituent propositions have the necessary properties specified by conditions [i]-[vi] in the quoted text.

Definition 9 Demonstration models

A *demonstration model* is a quadruple: $\mathfrak{M} = (D, P, [\![\cdot]\!], A)$ such that:

1. $K = (D, P, [\![\cdot]\!])$ is a predication structure
2. A is a set of sentences, such that:
   (a) if $\varphi \in A$ then $K \models \varphi$
   (b) if XaY$\in A$, then $[\![\mathrm{X}]\!] P [\![\mathrm{Y}]\!]$according to K
   (c) for all $x \in D$, there are only finitely many terms T occuring in sentences of A such that $[\![\mathrm{T}]\!]= x$

As will be clearer in the sequel, on the intended interpretation the set A contains all and only the true propositions which (relative to K) are apt to enter into demonstrative-proofs. In the language of *Posterior Analytics* I.4, they have the relevant property of per se-ity. It is required that all propositions in A are true (2a). And all a-propositions in A are required to be (or express) genuine predications (2b). Properties of (explanatory) primacy, priority, and posteriority may hold among members of A. But these are global properties. How they are fixed will be apparent in what follows. Finally we should remark on (2c). Now, (2c) does not correspond to anything Aristotle explicitly says. It captures, I believe, an implicit assumption Aristotle surely would accept. In the first book of his *Topics*, Aristotle distinguishes four types of predicable terms: genus, species, proprium, and defining-phrase [*horos*]. Consider a predication S is P. Aristotle contends that if P is a genus-term, species-term, or proprium-term with respect to S, then $[\![\mathrm{S}]\!]\neq [\![\mathrm{P}]\!]$. If, however, P is the defining-phrase of S (Aristotle assumes that for each being there is exactly one definition), then by necessity $[\![\mathrm{S}]\!]= [\![\mathrm{P}]\!]$. Aristotle's view is likely that a science contains at most two terms to pick out every being in its ontology: a name and a definition.

However, for the arguments that follow, the weaker (2c) suffices. Before offering our ultimate definition of demonstrative-proof, one final preliminary notion is required.

Definition 10 Circular syllogism

Let S be a syllogism.

1. S is *circular* iff S contains a (possibly improper) subtree S* such that:
   (a) S* is a syllogism with premise set Π and conclusion ψ
   (b) $\psi \in \Pi$

‘

We noted above that in his *Posterior Analytics* I.3 response to **Group B**, Aristotle argues against the idea that any circular argument could constitute a demonstrative-proof. In I.3 Aristotle offers several arguments to his effect. But given that explanatory priority (in Aristotle's sense) is irreflexive the impossibility of circular demonstration seems to fall immediately out of his own favored definition of demonstration. In any case, in I.19-22 Aristotle seems to be assuming that no demonstrations are circular. And we capture this in our definition of demonstrative-proof below.

Now, Aristotle requires that a demonstration be from primitives, from indemonstrable first-principles. Accordingly we represent a distinction between a demonstration and (what I'll call) a partial demonstration. On the intended interpretation, a partial demonstration is what a demonstrator produces at the intermediate stages on her way from posterior truths to explanatorily basic first-principles.

Definition 11 Partial demonstrations and demonstrable propositions

Let $\mathfrak{M} = (D, P, [\![\cdot]\!], A)$ be a demonstration model, and let S be a syllogism concluding that φ. Define *sent*(S) as the smallest set containing all sentences appearing at nodes of S.

1. S is a partial demonstration of φ in $\mathfrak{M}$ iff (i) *sent*(S) $\subseteq A$, and (ii) S is not circular

2. A sentence ψ = YxZ, with $x \in$ {a,e,i,o}, is a *demonstrable x-proposition* in $\mathfrak{M}$ iff there is a partial demonstration of ψ in $\mathfrak{M}$

Definition 12 Demonstrative improvement-path

Let $\mathfrak{M} = (D, P, [\![\cdot]\!], A)$ be a demonstration model.

1. $\overline{D}$ is a *demonstrative improvement-path* in $\mathfrak{M}$ iff
   (a) $\overline{D}$ is an improvement-path, and
   (b) for every D*$\in \overline{\mathrm{D}}$, D* is a partial demonstration in $\mathfrak{M}$.

Definition 13 Demonstration

Let $\mathfrak{M} = (D, P, [\![\cdot]\!], A)$ be a demonstration model, and let D be a partial demonstration in $\mathfrak{M}$.

1. D is a demonstration in $\mathfrak{M}$ iff there is no partial demonstration D* in $\mathfrak{M}$ such that D* improves D.

In the context of the above definitions, we can think of relations of explanatory priority between propositions in A as determined by A as a whole given Aristotle's system of syllogistic argument. We can also now formulate the following convenient representation of what it would be for a process of demonstration to continue ad infinitum.

Definition 14 Demonstration ad infinitum

1. Let $\mathfrak{M} = (D, P, [\![\cdot]\!], A)$ be a demonstration, such that $\varphi \in A$.
2. We say that *the process of demonstrating* φ *continues ad infinitum in* $\mathfrak{M}$ iff relative to $\mathfrak{M}$
   (a) there is a partial demonstration D* of φ, and
   (b) there is an infinite demonstrative improvement-path $\overline{\mathrm{D}}$ originating in D*

A feature of **Definition 14** demands special comment. Like the English 'demonstration', the Greek '*apodeixis*' may refer both to a process and a product (of such a process). One way that my interpretation of *Posterior Analytics* I.19-22 differs from others is in its focus on the process-aspect of *apodeixis*. This emerges, I hope, in my comparison of the puzzle Aristotle is confronting in these chapters with Zeno's dichotomy paradox (contrast, e.g. Lear's comparison with compactness). The issue (pace Scanlan) is not whether there can be any infinitely large proof-object that constitutes a demonstration. The issue is whether any demonstrative process can continue ad infinitum in the way that (say) in Aristotelian physics time continues ad infinitum. In other words, Aristotle's point in I.19-22 is that demonstration (qua process) is not 'potentially infinite'—is not infinite 'in capacity' [*dunamei*]). It is this conception of demonstration continuing ad infinitum that I mean to capture in **Definition 14**. It is the situation of unceasing possibilities for demonstrative improvement I mean to represent with infinite sequence: (D_0, D_1, D_2, ...). However, in so representing demonstration ad infinitum, I take myself to be relying on a meta-theory that conveniently assumes something that Aristotle in principle rejects. Following contemporary set-theorists, this paper assumes a definition of *infinite sequence* as a function from an infinite ordinal to some domain (the members of the sequence).[44] But an infinite ordinal just is an actually infinite collection. And Aristotle has principled reasons for believing there can be no actually infinite collections.

In this context, it is important to remember that the formal reconstruction adduced in the present paper is just a model. One might, I suppose, insist on formally modeling Aristotle using only meta-theoretic notions 'in principle acceptible' to Aristotle. I'm not sure what the criterion for this would be, but so long as we don't confuse our meta-theoretic representational framework with Aristotle's own framework I fail to see the interest of formal reconstruction in this style. But I also have positive reasons for assuming the usual set-theoretic definition of an infinite sequence in our meta-theory. While Aristotle in fact believes that there are no actually infinite objects, in *Posterior Analytics* I.20 and I.22 he takes himself to be considering and reasoning about the (counterfactual) possibility that such objects do exist. In effect, I.20 and I.22 will argue:

1. If demonstration continues ad infinitum there must be an actually infinite object with property F. (I.20)

2. But if there is an actually infinite object with property F, there must be another actually infinite object property G. (I.20)

3. But it is impossible for there to be an infinite object with property G. (I.22)

4. Thus demonstration cannot continue ad infinitum.

By drawing on the mathematical theory of actual infinities in our meta-theory (and the contemporary definition of infinite sequence in particular), we will be able to rigorously prove the truth of premises 1 and 2 given (our independently motivated representation of) Aristotle's system. While Aristotle is certainly aware that 1 and 2 hold, because he himself lacks a mathematical theory of the actual infinite he is in no position to provide rigorous proofs of 1 and 2. But it is important to recognize that these claims Aristotle is making about his system are true. And it is important to understand why they are true. The employment of our post-Aristotelian theory of actual infinities allows us to do this.

Before turning to this task, however, we note the following lemma. It will be required for our work on *Posterior Analytics* I.20, given our insistence on distinguishing terms from their beings. It is for proving this lemma that the assumption registered in **Definition 9.2.c** is crucial. First, however, we define a preliminary notion: the ontology of a (partial) demonstration. Intuitively, the ontology of a (partial) demonstration will be the collection of beings the demonstration refers to.

[44]For example, with $\mathbb{N}$ as the set of all natural numbers and Δ a set of demonstrations, an infinite sequence of demonstrations would be a function $\sigma : \mathbb{N} \to \Delta$ with $\sigma(0) = \mathrm{D}_0, \sigma(1) = \mathrm{D}_1$, and so on giving us a series (D_0, D_1, D_2, ...) which, like the natural numbers, has no terminal element.

Definition 15 Ontology of partial demonstrations and ontologically conservative improvement

1. Let D be a partial demonstration in $\mathfrak{M} = (D, P, [\![\cdot]\!], A)$, and let S be the smallest set containing all sentences that occur in D.

2. ont(D)= $\{z\ :\ (\exists\varphi \in \mathrm{S})(\exists v \in \{\mathrm{a,e,i,o}\})(\varphi = \mathrm{X}v\mathrm{Y} \wedge ([\![\mathrm{X}]\!]{=}z \vee [\![\mathrm{Y}]\!]{=}z)\}$, the 'ontology of D'.

3. Obviously if D* is an improvement of D, then ont(D)$\subseteq$ ont(D*). But if D* is an improvement of D, and ont(D)= ont(D*), we say that D* is an *ontologically conservative* improvement of D.

Lemma 1

1. Let $\mathfrak{M} = (D, P, [\![\cdot]\!], A)$, and let $\overline{\mathrm{D}} = (\mathrm{D}_0,\ \mathrm{D}_1,\ \mathrm{D}_2,\ ...)$ be a sequence of partial demonstrations in $\mathfrak{M}$ such that for all $\mathrm{D_n}$ and $\mathrm{D_{n+1}} \in \overline{\mathrm{D}}$: $\mathrm{D_{n+1}}$ is an ontologically conservative improvement on $\mathrm{D_n}$.

2. It follows that $\overline{\mathrm{D}}$ is of finite length.

Proof. For all $x \in D$, define the term count of x, call it: $t\#(x)$, as

$$t\#(x) = |\{\mathrm{T} \in \mathcal{L}\ : [\![\mathrm{T}]\!]{=}\ x \wedge (\exists\varphi \in A)(\exists z \in \{\mathrm{a,e,i,o}\})(\varphi = \mathrm{Y}z\mathrm{T} \vee \varphi = \mathrm{T}z\mathrm{Y})\}|.$$

By **Definition 9.2.c**, $\mathfrak{M}$ must be s.t. for some $m \in \mathbb{N}$: $max[t\#] = m$. Now, let D_0 be a demonstration relative to $\mathfrak{M}$. For some $n-1$ (where $3 \leq n < \omega$), D_0 has $n-1$ many premises. So, the number of terms that occur in D_0 is $n \in \mathbb{N}$. For any partial demonstration D, let

$$i(\mathrm{D}) = |\{\mathrm{X}z\mathrm{Y} :\ (\exists z \in \{\mathrm{a,e,i,o}\})\, \mathrm{X}z\mathrm{Y} \in A \wedge [\![\mathrm{X}]\!] \in ont(\mathrm{D}) \wedge [\![\mathrm{Y}]\!] \in ont(\mathrm{D})\}|.$$

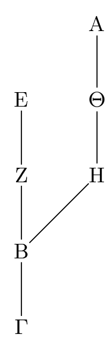
Clearly, $i(\mathrm{D}_0) < \aleph_0$ (more specifically: $i(\mathrm{D}_0) \leq 4 \times (m \times n)^2$ and in all likelihood it is considerably less). The Lemma follows given the non-circularity condition on demonstrations, and the fact that for all D, i(D) is finite. For, let $\overline{\mathrm{D}} = (\mathrm{D}_0,\ \mathrm{D}_1,\ \mathrm{D}_2,\ ...)$, as specified above. WLOG, we may assume that for all $\mathrm{D_{k+1}}$and $\mathrm{D_k} \in \overline{\mathrm{D}}$: $\mathrm{D_{k+1}}$ improves $\mathrm{D_k}$ by appending to the latter a two-premise partial demonstration of the left-most premise of $\mathrm{D_k}$ for which ontologically conservative demonstrative improvement is possible. Accordingly, we show that only finitely many ontologically conservative improvements can be done above every premise of D_0. So, let $\mathrm{D_n} \in \overline{\mathrm{D}}$ and let φ be the left-most premise in $\mathrm{D_n}$ of the required type. Due to the non-circularity requirement, if we are to conserve ontology, the number of sentences in A among which we can chose new premises for $\mathrm{D_{n+1}}$ is $k < i(\mathrm{D_n}) < \omega$ (we can't, for instance use φ). Suppose $\mathrm{D_{n+1}}$ appends to $\mathrm{D_n}$ a partial demonstration of φ that assumes ψ_L and ψ_R as fresh premises. Given the non-circularity condition, the number of propositions in A we can use to render an ontologically conservative improvement targeting ψ_j (with $j \in L, R$), is $\leq k-1$ (we can't reuse φ or ψ_j). Now, if i $\geq$ n+2 and $\mathrm{D_i} \in \overline{\mathrm{D}}$ such that it appends to to $\mathrm{D}_{\mathrm{i-1}}$ a proof of ψ_j, then D_{i}assumes new premises $\chi_L$and χ_R. Again, given the non-circularity condition, the number of propositions in A we can use in $\mathrm{D}_{\mathrm{m}}\in \overline{\mathrm{D}}$ (with m $\geq$ i+1) if D_{m} is to be ontologically conservative improvement targeting χ_j, with $j \in L, R$, is $\leq k-2$ (we can't reuse any of $\{\varphi, \psi_j, \chi_j\}$). And so on. The argument shows that for improving premises higher and higher above φ, the upper-bound of the number of propositions in A we can appeal to for ontologically conservative improvement is lower and lower. So, since k is finite, it follows that after finitely many steps no conservative ontological improvement of targeting the specific branch that arises from φ will be possible. As D_{n} was arbitrary, we can apply the argument to every premise in D_0, going from left to right. And the lemma follows.

12 The questions of *Posterior Analytics* I.19

We are finally ready to turn to text of *Posterior Analytics* I.19-22 itself. We begin at the beginning. In chapter I.19 Aristotle distinguishes three questions that focus his discussion in I.20-22. Here are the first two (81b30-37):

> Let Γ be such that it itself does not belong to anything else further, but let B belong to this in a primitive way (that is, there is no distinct intermediate between them). Again, let E belong to Z in the same way [i.e. primitively] and this [i.e. Z] to B. Must this come to a stop, or is it possible for [this] to continue ad infinitum? Again, if nothing is predicated of A per se, but A belongs to Θ in a primitive way, and between them is no prior intermediate, but Θ belongs to H and this to B, is it also necessary for this to stop, or is it in fact possible for this to continue ad infinitum?[45]

[45] Aristotle's choice of letters in developing the two questions—in particular, the repetition of beta—suggests he is referring to a single diagram:



Notably, while Aristotle explicitly specifies that the connection between A and Θ is a primative (unmediated) connection, he does not say whether or not there are further intermediates between H and B. So, Aristotle may very well mean to be asking about the possibility of an infinite down-chain beginning with A, all of whose terms are above B (where B, in this context, is something like an *infima species* as Γ is an individual not predicated of anything further). If so, Aristotle's illustration is well-chosen. For, one might naively assume that a

The pair of questions concern chains of *belonging* or, equivalently, chains of *true genuine predication*.[46] The questions can (and will later) be stated in terms of predication. But they are fundamentally about the metaphysical structure.

Question 1 Are infinite chains of 'upwards' (= increasingly more general) predication/belonging are possible?

Question 2 Are infinite chains of 'downwards' (= increasingly more particular) predication/belonging possible?

The metaphor of 'upwards' and 'downwards' is Aristotle's own. It appears in his recapitulation of these two questions at I.19 81b38-82a2. And it recurs throughout I.20-22 always with 'upwards' indicating towards the more general and 'downwards' indicating towards the more particular. Two features of the relations: upwards-of and downwards-of, stand out. Firstly, both relations are (strict) partial orders. For, nothing is upwards of itself, and if x is upwards of y while y is upwards of z then x is upwards of z (mutatis mutandis for downwards of).[47] Secondly, the pair are converses: if x is upwards of y, then y is downwards of x (and vice versa). Now, upwards-of/downwards-of share both features with more-general-than/more-particular-than. And presumably this is why Aristotle keeps returning to the metaphor of up/down in I.19-22.

One must, however, be careful here to distinguish predication (a transitive and serial relation) from more-general-than/more-particular-than (strict partial orders both). On Aristotle's considered view, there are ultimate subjects of predication which are themselves predicated of nothing else. Hence, if predication is serial, it cannot be irreflexive. While more-particular-than/more-general-than are irreflexive, I.19 is clearly not assuming that predication itself is irreflexive. In working up **Question 1**, Aristotle posits that Γ 'does not further belong to anything *else'* [*allō*]. And this leaves open (maybe calls attention to) the possibility that Γ is predicated of itself. But even if Γ is predicated of itself, an infinite sequence $(\mathrm{B}P\Gamma, \Gamma P\Gamma, \Gamma P\Gamma, \Gamma P\Gamma, ...)$ will not constitute a chain that 'goes downwards ad infinitum' [*epi to katō...eis apeiron hienai*] in the relevant sense. **Questions 1** and **2** focus exclusively on chains in which every node is both distinct from its immediate predecessor, and distinct from its immediate successor.[48] This suggests the following formal representation.

Definition 16 Predication ad infinitum: the **DCC** and **ACC** for demonstration models

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predicational structure with minimum elements can possess no infinite predicational down-chains. Aristotle's diagram would illustrate that this is not so. Indeed, the possibility of predicational down-chains in structures with minimal elements figures importantly in what follows.

[46] That genuine predication in particular is at issue, is clear from 81b22-29.

[47] Recall that a strict partial order is transitive and irreflexive (irreflexive: i.e. $(\forall x)\neg xRx$).

[48] When given the choice, Aristotle just about always prefers strict orders to reflexive non-strict orders (e.g. 'part' in Aristotle always means *proper part*). So it would be very weird for him to be allowing here that$x$ is downwards of x (x is more general than x) in some non-strict sense.

Let $\mathfrak{M} = (D, P, [\![\cdot]\!], A)$

1. $\mathfrak{M}$ satisfies the *descending chain condition* (**DCC**) iff for any infinite sequence $(x_0Px_1, x_1Px_2, \ldots, x_{k-1}Px_k, \ldots)$ consisting of P-links in $\mathfrak{M}$, there exists an $n \in \mathbb{N}$ such that $x_n = x_{n+1} = x_{n+2} = \ldots$

2. $\mathfrak{M}$ satisfies the *ascending chain condition* (**ACC**) iff for any infinite sequence $(x_1Px_0, x_2Px_1, \ldots, x_kPx_{k-1}, \ldots)$ consisting of P-links in $\mathfrak{M}$, there exists an $n \in \mathbb{N}$ such that $x_n = x_{n+1} = x_{n+2} = \ldots$

3. If $\mathfrak{M}$ fails to satisfy the **DCC**, and for $D^* \subseteq D$: the chain $(D^*, D^*|P)$ witnesses the failure, then $(D^*, D^*|P)$ is an *infinite predicational down-chain in* $\mathfrak{M}$.

4. If $\mathfrak{M}$ fails to satisfy the **ACC**, and for $D^* \subseteq D$: the chain $(D^*, D^*|P)$ witnesses the failure, then $(D^*, D^*|P)$ is an *infinite predicational up-chain in* $\mathfrak{M}$.

Examples

1. The following predicational structure$\mathfrak{M} = (D, P, [\![\cdot]\!], A)$ satisfies neither the **DCC** or the **ACC**
   - $D = \{x, y, z\}$
   - $P = \{xPy, yPz, zPy, xPz, zPz, yPy\}$

2. The following predicational structure$\mathfrak{M} = (D, P, [\![\cdot]\!], A)$ satisfies the **DCC** but not the **ACC**
   - $D = \mathbb{N}$
   - xPy iff $x \geq y$

3. The following predicational structure $\mathfrak{M} = (D, P, [\![\cdot]\!], A)$ satisfies both the **DCC** and the **ACC**
   - $D = \{x, y, z\}$
   - $P = \{vPw, xPy, yPz, xPz, zPz\}$

Aristotle's statement of his third question in *Posterior Analytics* I.19 was quoted in **Section 4** above. It concerns not predication, but demonstration I.19 (82a2-8):

> Further, when the extremes are fixed is it possible for the intermediates to be infinite? I mean [*legw de*], for instance, if A belongs to Γ, and B is a middle [between] them, but of A and B there are other middles, and different middles of these, is it indeed possible for these to continue ad infinitum, or is it impossible? And this is the same

as to investigate whether demonstrations can continue ad infinitum and if there is demonstration of all things, or some are limited by one another. (Similarly I mean [*de legw*] also in the case of negative syllogisms and propositions...).

Question 3 Can a demonstrable connection between a subject and a predicate be mediately by infinitely many (explanatory) middles?

Referring back to the chapter I.3 difficulty raised by **Group A**, I.19 remarks that to investigate **Question 3** is the same as investigating whether the process of demonstration can continue ad infinitum. Nonetheless, it is notable that in its I.19 statement **Question 3** immediately concerns the possibility of an actual infinity: an actually infinite collection of middle terms between two extremes (82a2-3).

As I explain in detail below, Aristotle's strategy in I.20-22 is to effectively reduce **Questions 3** to **Questions 1** and **2**. But before presenting this argument I must first introduce a slight refinement of demonstration models.

13 Counter-predication in I.19-22

The case of infinite cyclic chains: chains of the form $(xPy, yPz, zPx, xPy, \ldots)$[49] demands further discussion. Aristotle countenances such chains at the very end of *Posterior Analytics* I.19 (82a15-20) as a case of 'counter-predicables' [*antikatēgoroumena, antistrephonta*]. Following Aristotle, let us say that x and y *counter-predicate*, or are *counter-predicables*, iff both xPy and yPx. Aristotle raises the possibility of chains of counter-predicables at 82a15-20 in order to distinguish it from the possibilities envisioned by **Questions 1** and **2**. With respect to the (acyclic) linearly ordered chains of **Questions 1** and **2**, you can have predication ad infinitum in exactly one direction. For instance, there might be infinitely many nodes Υ linearly ordered above an absolute minimum $x \in \Upsilon$, such that every $y \in \Upsilon$ is but finitely many nodes above x. The chain consisting of Υ would then be infinite in the upwards directions without containing any infinite down-chains. In contrast, Aristotle tells us that in a cyclic chain of counter-predicables 'the things we are puzzling over are infinite in both [ways]'. If (e.g.) we add to Υ a node $z \neq x$ such that xPz and zPx, the result will be infinite both directions (cf. 82a17-19[50]).

Aristotle's discussion of cyclic chains and counter-predication in I.19 (82a15-20) is evidently an appendix. Indeed, from Aristotle's own perspective, the coherence of his pervasive up/down analogy for predicational chains falls apart in the case of cyclic chains. For, in a cyclic chain one will have a distinct x, y such that x is both 'upwards of' y, and 'downwards of' y. One might, I suppose, think of up/down as directions in the sense of east/west; (in a way, the US is both west of Japan and east of Japan). But surely Aristotle does

[49] Assuming distinctness of x, y, z

[50] Following the textual reading of manuscript n and Philoponus.

not so intend the up : down : : more-general : more-particular analogy. The Aristotelian physical universe, one should recall, possesses an absolute up and absolute down. And in this context, the idea of x being both upwards-of and downwards-of a distinct y really doesn't make sense (compare: point x is both north of point y and south of point y).

Both textual and philosophical considerations strongly suggest that the arguments of *Posterior Analytics* I.20-21 assume a notion of genuine predication according to which P is (as we would say) anti-symmetric.[51,52] Indeed, such an assumption is already signaled at the end of Aristotle's discussion of counter-predication in I.19 (82a19-20). And notably, when Aristotle returns to counter-predication in I.22 (83a36ff.), he is giving an argument which does establish the anti-symmetry of predication. Unfortunately, the details of Aristotle's case at 83a36ff. are exceptionally obscure.[53] It is probable that some of Aristotle's argument is just missing from the transmitted text. But the intended upshot is clear enough. In discussing counter-predication at 83a36ff., Aristotle is invoking his account of the four 'predicables' in the *Topics*: genus (with differentia), defining-phrase [*horos*], proprium, accident. Now, in the *Topics*, Aristotle has it that x and y counter-predicate if they are related as *definiens* and *definiendum*. Moreover, he contends that *definiens* and *definiendum* are always one in number. In *Posterior Analytics* I.22, however, Aristotle argues that x and y genuinely counter-predicate if **and only if** they are related as *definiens* and *definiendum* (in counter-predicating, say, a species and one of its propria, one can speak truly but one will not be genuinely predicating). And this entails that genuine predication is, as we would say, anti-symmetric.

With the above in mind, our subsequent formal treatment of *Posterior Analytics* I.20-22 largely restricts its focus to demonstration models $\mathfrak{M} = (D, P, \llbracket\cdot\rrbracket, A)$ on which P is anti-symmetric.

Definition 17 Anti-symmetric demonstration models

We call $\mathfrak{M} = (D, P, \llbracket\cdot\rrbracket, A)$ a *anti-symmetric demonstration model* iff $\mathfrak{M}$ is a demonstration model and P is anti-symmetric.

Unless otherwise indicated, in what follows: by 'demonstration model' we will always mean *anti-symmetric demonstration model.*[54]

[51] Recall: a relation R is anti-symmetric iff $(xRy \wedge x \neq y) \rightarrow \neg yRx$.

[52] Space is limited; I will not going into all such considerations here. I will simply assert, however, that without such an assumption Aristotle's argument in I.20 will be either formulated very strangely or invalid.

[53] So Ross (578) expressing the scholarly *opinio communis*: 'Any interpretation [of Aristotle's train of thought at 83a36ff.] must be conjectural'.

[54] NB the only part of my formal reconstruction that importantly uses this assumption is my account of the argument of chapter I.20. Again, if Aristotle is not making such an assumption in that chapter, the argument he gives is either invalid or stated both very strangely and without appropriate consideration of intermediate steps.

14 The target of *Posterior Analytics* I.20-21

Recent commentators on I.20-21 have taken these chapters to target a thesis rather limited in scope. Deleting the I.21 discussion of Bocardo, Barnes and Lear contend Aristotle wants to prove that if every predicational chain is finite, then all demonstrations of a- and e-propositions must be finite. According to these commentators, i- and o-propositions have no place in demonstration. Accordingly, Aristotle should show no interest in them in I.20-21. His argument really targets only the fragment of his syllogistic using exactly the four moods: Barbara, Celarent, Camestres, and Cesare.

Though I cannot fully defend the view here, I think this whole picture is mistaken. Philosophically, there are very good reasons for Aristotle to prefer a logic for demonstration capable of proving i- and o-propositions. And chapters I.20-21 in fact show Aristotle concerned with the possibility that demonstration of a non-universal proposition might continue ad infinitum.[55] In any case, this paper develops an interpretation of I.20-21 on which Aristotle means to establish something considerably more general than what is often assumed. For, I intend to argue that we find in I.20-21 the roots of a valid argument capable of proving that if predicational chains are finite, then no demonstrative process using any combination of the 14 syllogistic moods can continue ad infinitum. Using our apparatus of demonstration models, and with an eye to the argument-structure of the text, I formulate this interpretation in claiming that the argument witnessed by I.20-21 targets following three theorems.

Theorem 1

If $\mathfrak{M}$ is a demonstration model satisfying both the **DCC** and the **ACC**, and φ is demonstrable a-proposition in $\mathfrak{M}$, then then demonstration of φ does not continue ad infinitum.

Theorem 2

If $\mathfrak{M}$ is a demonstration model satisfying both the **DCC** and the **ACC**, and φ is demonstrable i-proposition in $\mathfrak{M}$, then then demonstration of φ does not continue ad infinitum.

Theorem 3

If $\mathfrak{M}$ is a demonstration model satisfying both the **DCC** and the **ACC**, and φ is demonstrable e or o-proposition in $\mathfrak{M}$, then then demonstration of φ does not continue ad infinitum.

The theorems are not trivial. For, there are (in theory: abstracting, as Aristotle does, from facts about what A happens to be in reality) infinitely many paths for demonstrative improvement to follow. Consider, e.g., a process of demonstration that reaches the following partial demonstration:

[55] Even independently of Bocardo in I.21, I.23 84b21-22 settles the issue against Barnes.

$$\dfrac{\dfrac{\dfrac{\begin{matrix}?\\\vdots\\\mathrm{DaA}\end{matrix}\qquad\begin{matrix}?\\\vdots\\\mathrm{DeB}\end{matrix}}{\mathrm{AeB}}\qquad\dfrac{\begin{matrix}?\\\vdots\\\mathrm{BiE}\end{matrix}\qquad\begin{matrix}?\\\vdots\\\mathrm{FaE}\end{matrix}}{\mathrm{BiF}}}{\mathrm{AoF}}\qquad\begin{matrix}?\\\vdots\\\mathrm{CaF}\end{matrix}}{\mathrm{AoC}}$$

It is far from about that a demonstration model will only support improving such a tree ad infinitum if it violates either the **DCC** or the **ACC**.

Aristotle's proof strategy in *Posterior Analytics* I.20-21 is guided by the following observation.

Observation 2

For $x \in$ {a,e,i,o} define an *x-syllogism* as a syllogism concluding in an x-proposition. And, for $x \in$ {a,e,i,o} define an *x-improvement-path* as an improvement-path $\overline{\mathrm{S}}$ originating with an x-syllogism. Accordingly, it is clear that:

1. Let $\overline{\mathrm{S}}$ be an a-improvment-path and S*$\in \overline{\mathrm{S}}$. Then (i) every premise in S* is an a-premise, and (ii) for no $y \in$ {e,i,o} does S* contain a sub-syllogism which is an y-syllogism.

2. Let$\overline{\mathrm{S}}$ be an i-improvment-path and S*$\in \overline{\mathrm{S}}$. Then (i) every premise of S* is positive while at most one premise of S* is an i-proposition, and (ii) for no $y \in$ {e,o} does S* contain a sub-syllogism which is an y-syllogism.

3. Let $\overline{\mathrm{S}}$ be an e-improvment-path and S*$\in \overline{\mathrm{S}}$. Then (i) every premise of S* is universal while exactly one premise of S* is a negative, and (ii) for no $y \in$ {i,o} does S* containa sub-syllogism which is an y-syllogism.

4. For all $y \in${a,e,i,o}: there are o-improvment-paths $\overline{\mathrm{S}}$ with S*$\in \overline{\mathrm{S}}$, such that S* contains a sub-syllogism which is an y-syllogism. Moreover, every such S* contains exactly one negative premise (and no more than one non-universal premise).

In our system, **Observation 2** admits of proof by easy induction. While Aristotle does not make **Observation 2** explicit in *Posterior Analytics* I.19-22, it follows almost immediately given the handful of more basic observations about syllogistic recorded in *Prior Analytics* I.24-5. In any case, the argumentative structure of I.19-22 certainly seems to presuppose **Observation 2**. For, here is the important upshot of **Observation 2**. Suppose a demonstrator is to demonstrate a syllogistic proposition φ. If φ is an a-proposition, the demonstrator can only proceed by appealing to Barbara at every improving step. If φ is an i-proposition, the demonstrator can only proceed by appealing to one of {Barbara, Darii, Datisi, Disamis, Darapti} at every improving step. But if φ is a negative (e- or o-) proposition, any of the 14 syllogistic moods might in principle be required.

Given **Observation 2**, Aristotle sees that the argumentative burden for proving **Theorem 2** is far lessened if **Theorem 1** is true. For, in any partial demonstration of an i-proposition, at most one premise will itself be an i-proposition. Hence, if **Theorem 1** can be taken for granted, then when in proving **Theorem 2** we consider an arbitrary partial demonstration of an i-proposition we will have no more than one premise to worry about. The argumentative burden for proving **Theorem 3** is likewise far lessened if both **Theorems 1** and **2** are true. Again since every negative partial demonstration has exactly one negative premise, if we are given **Theorems 1** and **2** and posit an arbitrary negative demonstration we will have only one premise to worry about.

Posterior Analytics I.20-21 proceeds accordingly. Chapter I.20 is devoted to an argument for **Theorem 1**. Chapter I.21 assumes that both **Theorems 1** and **2** are true. It then uses this assumption in arguing for **Theorem 3**. No explicit argument for **Theorem 2** is given in the extant[56] *Posterior Analytics* I.20-21. However, when the I.21 proof is correctly understood, it is quite easy to see how to prove that **Theorem 2** is true if **Theorem 1** is. The required argument is exactly parallel to the sub-argument for the case of e-propositions provided in I.21. And appreciating this, Aristotle might simply have decided to omit writing down his argument that if **Theorem 1** then **Theorem 2**. To sum up, then, if we supply a needed argument concerning i-improvement-paths following the argumentative pattern used in I.21 to show an analogous result e-improvement paths, chapters I.20-21 yield the following results. **Theorem 3** is true if **Theorems 1** and **2** are true (I.21). **Theorem 2** is true if **Theorem 1** is true (argument supplied from a pattern provided in I.21). But **Theorem 1** is true (I.20). So **Theorems 1-3** are all true. And the finitude of demonstrative processes follows if we can show the single antecedent shared by these three condition theorems: that reality in fact contains no infinite predicational chains. And this is exactly what Aristotle attempts in *Posterior Analytics* I.22.

15 *Posterior Analytics* I.21: interpretive issues

For purposes of defending the above view of I.20-21's global argument-structure, I propose to invert Aristotle's own order and treat the argument of *Posterior Analytics* I.21 before discussing I.20. *Posterior Analytics* I.21 argues that every process of demonstrating a (demonstrable) negative conclusion is finitely terminating, if all predicational chains are finite. As 82a36-b4 makes clear, Aristotle's strategy is to assume that every positive determination terminates if all predicational chains are finite (i.e. the upshot of **Theorems 1** and **2** above), in addition to the antecedent of the conditional I.21 aims to prove. Accordingly, I.21 will argue from the assumptional base that all predicational chains are finite and all positive demonstrative processes are finitely terminating.

Unfortunately, the text of I.21 does not contain a fully worked out proof. What the text does transmit is a mere *proof sketch*, which falls into seven sections:

[56] It is entirely possible that some part of Aristotle's original text is lost or not all of Aristotle's full oral presentation is reflected in what he wrote down.

1. Negative demonstration terminates if positive determination does (82a36-37)
2. Let predicational chains come to a stop in both directions (82a38-82b4)
3. 'Not belonging is proved in three ways'—i.e. three cases are to be considered (82b4-5)
4. Discussion of improvement by Celarent (82b5-13)
5. Discussion of improvement by Camestres (82b13-21)
6. Discussion of improvement by Bocardo (82b21-28)
7. Generalization, and statement of inferential principle meant to derive generalization (82b29-36)

The text which presents the 'generalizing' step (7), bears quotation in full (82b29-33, my emphasis):

> It is apparent that even if [a negative conclusion] is proved not using a single path [*hodos*], but rather using all [paths]—(at points from a first figure [syllogism], at points from a second or third [figure syllogism])—[it is apparent] that so too [the process of improving] will stop. *For the paths are finite, and all finite things taken a finite number of times must be finite.*

The above division of our chapter is uncontroversial; but little else concerning *Posterior Analytics* I.21 is. My own reading of I.21's argument differs significantly from other approaches in the literature. And while I have not the space here to fully defend my approach, it will be useful to rehearse what the significant interpretive issues are, and how they are usually addressed.

Most importantly, an interpreter of I.21 must answer four inter-connected questions: (A) How general is the result Aristotle is trying to prove?—specifically: is his conclusion meant to cover both demonstrations of universal negative conclusions and demonstrations of particular negative conclusions? Or does Aristotle's argument target only demonstrations of universal negative conclusions? (B) Why does Aristotle consider the three syllogistic moods that he does (Celarent, Camestres, Bocardo)? (C) Should these three moods be identified with the three ways referred to in (3)? Or are the three moods merely representative of the three ways? (D) How is the principle of inference adduced in (7) supposed to advance us from discussion of the three ways (whatever they are), to the more general conclusion Aristotle is trying to prove (whatever that is)?

In answering questions (A)-(D), the commentators cluster into two camps. Let's begin with (A)-(C). On the one side, we have Philoponus and Ross who think that Aristotle is trying to show that *every* demonstration of a negative conclusion (be it universal *or* particular) is of a finite length. Why else would Aristotle bother discussing Bocardo? And why else would he mention three figures in 82b30-1?[57] Indeed, Philoponus-Ross interpret Aristotle's statement in (3)—'not belonging is shown in three ways'—as a reminder that a negative conclusion can be proved in each of the three figures. The moods Aristotle proceeds to consider (Celarent, Camestres, and Bocardo), are then assumed to be chosen as token representatives of the three figures, so that completing Aristotle's proof sketch in fact demands parallel consideration of the six remaining syllogistic moods by which a negative conclusion may be demonstrated. On the other side, we have Barnes and Lear who hold that Aristotle's argument concerns only demonstrations of negative universal conclusions. The discussion of Bocardo (present in all our manuscripts), is spurious—a wrong-headed 'glossator's addition'. So Barnes: 'Aristotle meant to refer only to Celarent, Camestres, and Cesare, the three *tropoi* [= ways] which yield e-conclusions; he sketched the argument for the first two moods [i.e. Celarent and Cesare] and perhaps simply added "the third is proved likewise." A later improver filled this note out and filled it our wrongly.'[58] Accordingly, both the discussion of Bocardo (82b21-28), and the later reference to negative proof through all three of the figures (82b30-31) should be expunged; we should interpret the text as if they were not there. In contrast to Philoponus-Ross, Barnes-Lear take Aristotle's statement in (3)—'not belonging is shown in three ways'—as referring to the trio Celarent, Camestres, Cesare. As such, Barnes-Lear hold that in discussing Celarent and Camestres, Aristotle is not discussing two representatives of the three ways mentioned in (3), but is indeed discussing two of the ways themselves.

In answering question (D) and interpreting how Aristotle's argument in *Posterior Analytics* I.21 is supposed to actually work, we find Ross uncritically repeating the analysis of Philoponus, and Barnes referring his readers to the analysis of Lear. The Philoponus reconstruction has a vanishingly small basis in the text. One gets the impression that Philoponus didn't see how Aristotle's own argument was supposed to work, so is instead presenting an alternative and (by Philoponus' lights) simpler argument for the same conclusion.[59] Ross seems mostly uninterested in the chapter; the little he says mostly just regurgitates Philoponus. And neither Ross nor Philoponus do much of anything to ground their interpretation in I.21 itself. Most damningly, the assumption Aristotle makes at the start of I.21 from which his argument is clearly supposed to develop—(2) above—plays no role in the Philoponus-Ross reconstruction. Lear-Barnes, in contrast, are more concerned to discern Aristotle's own train of thought. However, the argument they credit Aristotle with is clearly invalid. For in reading 82b29-33 (the 'generalizing' step (7)), Lear-Barnes assume that the *hodoi* are strings of syllogisms using only one of the argument-types: Celarent, Camestres, Cesare. Aristotle's thought is supposed to be the following. For all C∈{Celarent, Camestres, Cesare} every token syllogistic string containing

[57] Recall that no universal conclusions (positive or negative) are proved in the third figure.

[58] Barnes (1994): 173

[59] Not only is the alternative most definitely not the argument Aristotle gives; it is also an argument Aristotle should not accept (endangering, as it does, the status of arithmetic as a science).

only C-proofs must be finite. Moreover, the relevant collection of argument-types: {Celarent, Camestres, Cesare}, is finite. Since no other moods show a universal negative conclusion, every process of demonstrating a universal negative must be finite. The argument is obviously invalid because it does not rule out infinite alternating improvement patterns like (e.g.) the following (with C_a = Camestres and C_e = Celarent):

$$[\text{e-conclusion}] \longleftarrow [C_e\text{-string}] \longleftarrow [C_a\text{-string}] \longleftarrow [C_e\text{-string}] \longleftarrow [C_a\text{-string}] \longleftarrow \ldots$$

Noting the argument's invalidity, Barnes and Lear locate the fault in Aristotle. Really, though, given what Aristotle actually says in I.21 (see e.g. 82b16-17) he ought to be aware of such alternating improvement patterns.

For the moment, let's leave aside the issue of whether or not *Posterior Analytics* I.21 presents a valid argument. I hope it clear from the preceding that both Philoponus-Ross and Barnes-Lear do a poor job of explaining the transmitted text. The Barnes-Lear reading is a non-starter unless we accept Barnes' conjectural excisions. And in the Philoponus-Ross interpretation, Aristotle's assumption that there are no infinite positive chains of predication plays no role.

It is not difficult to see why commentators have had such a hard time with *Posterior Analytics* I.21. As I emphasized above, the version of the argument of I.21 we have presents a bare-bones proof sketch. In order to (re)construct a proof from a proof sketch, one must have in view the overall proof strategy that the sketch's author intends his or her proof to follow. Aristotle might very well have carefully explained his overall I.21 proof strategy in oral presentation of the chapter's material. But no such detailed explanation appears in the transmitted text.

I present here a novel account of a proof strategy Aristotle might have intended the proof sketched in *Posterior Analytics* I.21 to follow. The hypothesis that Aristotle *did* intend the proof sketched in I.21 to follow (something very close to) the proof strategy I describe, is eminently plausible. The hypothesis does a good job of explaining why the sketch given in I.21 takes the exact form it does. In any case, the hypothesis does considerably better in explaining the transmitted text of I.21 than any of the alternatives available in the commentary tradition. Notably, my reading yields a valid argument for a conclusion targeting *all* negative syllogistic moods—an argument which (I contend) is both grounded in the text and sound by Aristotelian lights.

16 *Posterior Analytics* I.21: reconstruction

Let's see what sense we can make of *Posterior Analytics* I.21 if we reject Barnes' conjectural excisions and read the text as transmitted. There are, after all, no good philological reasons to accept them. Given the analysis of Bocardo (82b21-28) and the mention of three figures (82b30-31), Aristotle evidently wants to show that if there are no infinite chains of predication it follows that demonstration of e- and o-conclusions always terminates in finitely many steps. Now, Aristotle recognizes 14 valid (assertoric) syllogistic moods: nine negative (concluding an e- or o-proposition) and five positive (concluding an a- or i- proposition). In principle, all 14 moods can be used in improving a partial demonstration of a negative conclusion.

As noted above, Aristotle sees correctly that his argumentative burden is lessened if we assume **Theorems 1** and **2**: i.e. that all positive demonstrations are finitely terminating if there are no infinite predicational chains. On its face, I.21 82a36-37 seems to indicate that Aristotle is assuming exactly this. No restriction to universal positive moods is given. As Aristotle explicitly recognizes (82b6-8), the upshot of this assumption is that in considering improvement of negative partial demonstrations we can ignore possible improvement targeting positive (a- and i-) premises. Consider, e.g. a negative demonstrative process that reaches the following stage:

$$\dfrac{\dfrac{\mathrm{DaA} \qquad \mathrm{DeB}}{\mathrm{AeB}} \qquad \mathrm{BaC}}{\mathrm{AeC}}$$

Without loss of generality, we are allowed to suppose that DaA and BaC are indemonstrable principles. For, the argument is assuming that even if they are not principles, demonstration of them is finitely terminating. Similarly in other cases.

Ignoring positive improvement, we are left, then, with the nine negative syllogistic moods Aristotle recognizes: Celarent, Ferio, Camestres, Cesare, Festino, Baroco, Felapton, Bocardo, and Ferison. And the task is to show something concerning all possible processes of improvement using only combinations of these nine. For convenience, let us call any process of improvement using only a combination of the nine negative moods a process of *pure negative improvement*. Aristotle wants to show, then, that if reality is such that there are no infinite chains of predication, there can be no infinite processes of pure negative improvement.

The task is still apt to seem daunting. For, in principle there are infinitely many possible construction processes for pure negative improvement to follow. So an argument covering infinitely many distinct possibilities is needed. Here one might recall that Aristotle confronts a similarly large task in *Prior Analytics* I.23. There the issue was to show that every deduction (of any arbitrary length) can be 'perfected' with reference to one-step Barbara's and Celarent's. There are, in principle, infinitely many such deductions. And to argue for this conclusion, Aristotle uses an elegant strategy. First, Aristotle gathers a small finite collection of patterns and proves that anything conforming to one of the patterns has the relevant property. Then he argues that each of the (infinitely many) deductions to be considered are wholly constituted out of argument tokens instantiating patterns in the collection. But since all deductions are so constituted, each must itself have the relevant property. So by explicitly treating only finitely many patterns, Aristotle thus aims to establish a conclusion covering a potentially infinite collection of cases. Aristotle, I hypothesize, is using an analogous strategy in *Posterior Analytics* I.21. He wants to prove a conclusion covering a potentially infinite class of cases, by establishing something about a well-chosen finite collection of patterns. The key observation that gets the *Posterior Analytics* I.21 proof going will be that every process of pure negative improvement is wholly constituted by sub-processes that conform to one of the patterns.

Now, given his work in the *Prior Analytics* on the 'reduction' of the second and third figures to the first, the following grouping of syllogistic moods would already be salient to Aristotle[60]:

C-group: {Celarent, Cesare, Camestres} $\Longrightarrow$ e-premise, e-conclusion

F-group: {Ferio, Festino, Felapton, Ferison} $\Longrightarrow$ e-premise, o-conclusion

B-group: {Baroco, Bocardo} $\Longrightarrow$ o-premise, o-conclusion

In accordance with the above grouping of negative syllogistic moods, the following definitions will be convenient:

Definition 18

1. A *C-string* is a syllogism using only inferences in Celarent, Cesare, Camestres
2. An *F-string* is a syllogism using only inferences in Ferio, Festino, Felapton, Ferison
3. A *B-string* is a syllogism using only inferences in Baroco and Bocardo

My basic hypothesis conerning the proof strategy of *Posterior Analytics* I.21, is that the following elementary observation forms the background on the basis of which Aristotle's I.21 proof sketch is to be understood.

Observation 3

A syllogism which uses only negative syllogistic inferences must instantiate exactly one of the following forms:

1. [Conclusion]⟵[C-stiing⟵[Single negative premise]
2. [Conclusion]⟵[B-string]⟵[Single negative premise]
3. [Conclusion]⟵[F-string]⟵[Single negative premise]

[60]The salience in Aristotle of the above grouping, is conveniently reflected in the traditional names for the negative syllogistic moods given to us by the medievals. For, it was with an eye to the workings of Aristotle's 'reduction' of the other figures to the first that the initial letters of the traditional names were assigned. Hence, the important grouping for us conveniently turns out to be a partition of the collection of nine under the equivalence relation 'having a name with the same first letter'. The **C-Group** consists of Celarent (first figure) and the moods in other figures whose validitiy Aristotle establishes through the evident validity of **C**elarent and conversion; the **F-group** consists of Ferio (first figure) and the moods in other figures whose validity Aristotle establishes through the validity of **F**erio and conversion; the **B-group** consists of the moods whose validity Aristotle notes that we can only be establish through the evident validity of **B**arbara via *reductio* (or *ekthesis*).

4. [Conclusion]⟵[B-string]⟵[F-string]⟵[Single negative premise]
5. [Conclusion]⟵[B-string]⟵[F-string]⟵[C-string]⟵[Single negative premise]
6. [Conclusion]⟵[F-string]⟵[C-string]⟵[Single negative premise]

Examples

1. If a process of pure negative improvement reaches the following stage, no C-string can be appended before an F-string is appended:

$$\dfrac{\text{BaA} \qquad \text{BoC}}{\text{AoC}}$$

2. If a process of pure negative improvement reaches the following stage, every further negative improvement must use one of Celarent, Cesare, Camestres:

$$\dfrac{\dfrac{\text{AeD} \qquad \text{DiB}}{\text{AoB}} \qquad \text{CaB}}{\text{AoC}}$$

There are, I think, good grounds for believing that Aristotle was aware of **Observation 3**. In *Prior Analytics* I.24, Aristotle explicitly tells us that (a) every negative syllogistic-deduction has exactly one negative premise; also that (b) no third figure syllogistic-deduction yields a universal conclusion; also that (c) the syllogistic-deduction of a universal conclusion must have only universal premises. Now, (a)-(c) entail that no B- or F-string can ever appear be used to improve on a C-string. *Prior Analytics* I.24 also notes that a particular conclusion can follow either from two universal premises, or from one universal and one particular premise. And given all of this, for an intelligent observer **Observation 3** ought to be fairly obvious. In any case, Aristotle's idea of treating negative improvement processes by assuming a prior fact about positive improvement processes, shows that in inventing the proof of *Posterior Analytics* I.19-22 Aristotle paid very much attention to such facts.

When coupled with **Observation 2** (see above), the important upshot of **Observation 3** is that the consequent of **Theorem 3**—the thesis argued for in *Posterior Analytics* I.21—follows immediately from Aristotle's assumptional base if he can show that given this assumptional base: demonstrative C-string, F-string and B-string construction are each finitely terminating. For, what **Observation 3** makes clear is that if (i) all demonstrative C-improvement-paths are finite, (ii) all demonstrative F-improvement-paths are finite, and (iii) all demonstrative B-improvement-paths are finite, then purely negative demonstrative improvement must be finitely terminating. Contra Barnes-Lear, I hypothesize that this is the idea behind the text expressing the 'generalizing step' (7) in our outline of I.21 (82b29-33, see above).

Note that the argument from the finitude of all demonstrative C-, F-, and B-improvement-paths to the finitude of all negative demonstrative processes is valid. Let us, then, test the hypothesis. If Aristotle was pursuing this line of proof, how would he proceed? Of course, the case of F-improvement-paths can be immediately set aside. For it is obvious that one cannot improve on an F-subsyllogism by appending another F-syllogism. What all the moods in the **F-group** have in common is that the quantity of their negative premise (e-) differs from the quantity of their negative conclusion (o-). It is clear, then, that every F-string is of length one. And given **Observation 3** it follows that every negative syllogism can have at most one inference in a mood from the **F-group**.

The case is otherwise with syllogisms in the **C-group** (e-premise, e-conclusion) and **B-group** (o-premise, o-conclusion). For, as Aristotle himself notes (82b14-16, 82b23-25), one can (say) improve a syllogism in Camestres using Celarent or Cesare; and one can improve a syllogism in Bocardo using Bacoco. Accordingly, while all F-strings are of length one, C-strings and B-strings can, in principle, be of any arbitrary length. So, given the interpretive hypotheses that I.21 is trying to show something concerning all negative demonstrations (universal and particular) using the proof strategy set up by **Observation 3**, we would expect Aristotle to focus on arguing that every C-improvement-path and every B-improvement-path must finite if there are no infinite chains of a-propositions.

This, I maintain, is exactly what *Posterior Analytics* I.21 is doing. And once we see this it makes perfect sense that Aristotle discusses exactly: improvement by Celarent, improvement by Camestres, and improvement by Bocardo. For although there are five syllogistic moods in in **C-group** $\cup$ **B-group**, the five instantiate three patterns of improvement. Once the three patterns are understood, an argument that C-improvement-path and B-improvement-path construction are each finitely terminating if predicational chains are finite, is immediately available.

Observation 4

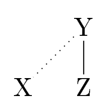
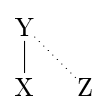
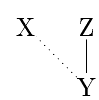
With respect to improvement, each of the five moods in **C-group** $\cup$ **B-group** instantiate one of three patterns:

1. In applying Celarent or Cesare in an improvement targeting a premise φ, we (i) assume a fresh positive premise stating that our new middle term Y holds of all of the subject of φ, and (ii) assume a fresh negative premise that relates Y negatively to the predicate of φ.

2. In applying Camestes and Baroco in an improvement targeting a premise φ, we (i) assume a fresh positive premise stating that our new middle term Y holds of all of the predicate of φ, and (ii) assume a fresh negative premise that relates Y negatively to the subject of φ.

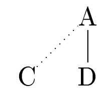
3. In applying Bocardo in an improvement targeting a premise φ, we (i) assume a fresh positive premise stating that the subject of φ holds of all of our new middle term Y, and (ii) relate Y negatively to the predicate of φ.

In **Observation 4.1-3**, both (i) and (ii) are significant. The relevant facts are perhaps clearer in diagrammatic presentation. Let us adopt a pictorial convention on which (a) we a premise GaH by placing G above H and connecting them with a single line, and (b) we represent a negative premise GxH (with $x \in$ {e,o}) by connecting G and H with a dotted line. Then **Observation 4** may be helpfully displayed:

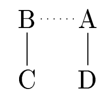
Pattern 1	Pattern 2	Pattern 3
Celarent $\dfrac{\text{XeY} \qquad \text{YaZ}}{\text{XeZ}}$ Cesare: $\dfrac{\text{YeX} \qquad \text{YaZ}}{\text{XeZ}}$	Camestres: $\dfrac{\text{YaX} \qquad \text{YeZ}}{\text{XeZ}}$ Baroco: $\dfrac{\text{YaX} \qquad \text{YoZ}}{\text{XoZ}}$	Bocardo: $\dfrac{\text{XoY} \qquad \text{ZaY}}{\text{XoZ}}$
		

Here is how to go from **Observation 4** to valid proof that if there are no infinite chains of a-propositions, then every C-improvement-path must be finite. As shall be clear soon, the case for B-improvement-paths is similar. First we introduce the basic idea behind the proof. Suppose we have a C-improvement process with initial segment (S_0, S_1, S_2) displayed below:

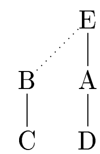
$$\frac{\mathrm{AeC} \qquad \mathrm{AaD}}{\mathrm{CeD}}$$



$$\frac{\dfrac{\mathrm{AeB} \qquad \mathrm{BaC}}{\mathrm{AeC}} \qquad \mathrm{AaD}}{\mathrm{CeD}}$$



$$\frac{\dfrac{\dfrac{\mathrm{EaA} \qquad \mathrm{EeB}}{\mathrm{AeB}} \qquad \mathrm{BaC}}{\mathrm{AeC}} \qquad \mathrm{AaD}}{\mathrm{CeD}}$$

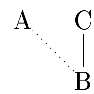


Different C-moods are used at different improving steps. But the upshot of **Observation 4** is that in each improving step our new middle term appears at the 'top' of a chain of a-linked terms—a chain which corresponds to a chain of a-propositions. More precisely, the new middle term always appears at the 'top' of a term-chain whole 'bottom' is either the subject or predicate of the original conclusion CeD. Such is the insight **Observation 4** expresses about C-improvement: it always conforms to Patters 1 and 2 (see diagram for **Observation 4** above). Accordingly, iterative C-improvement works by extending *pre-existing* chains of a-connected terms: chains which originate in the major and minor terms of the conclusion. Parts (i) and (ii) of **Observation 4.1-3** jointly explain why this is so. The negative premise that an instance of C-improvement targets always contains exactly the two terms X,Y that 'top' the two pre-existing chains of a-connected terms. Then what C-improvement does is assume a fresh middle M that it puts above one of {X,Y} and relates negatively to the other: thus yielding a new negative premise containing the exact two terms that 'top' the new term up-chains. Accordingly, it follows immediately that the C-improvement process (S_0, S_1, S_2) displayed above can only continue ad infinitum if there were infinite upwards extending chains originating in C and D.

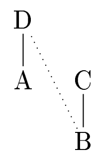
Observation 4 points a parallel insight about B-improvement. But here it is not only up-chains but also down-chains that are involved. It was, perhaps, by noting such proof-theoretic patterns that Aristotle's originally discovered the importance of both predicational up- and down-chains for the argument of *Pos-*

terior Analytics I.21. For, Iterative B-improvement always works by extending either a *pre-existing* a-proposition up-chain originating from the predicate of the conclusion (Baroco) or a *pre-existing* a-proposition down-chain originating from the subject of the conclusion (Bocardo). To see this, the reader may want to consider the following:

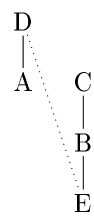
$$\dfrac{\mathrm{AoB} \qquad \mathrm{CaB}}{\mathrm{AoC}}$$



$$\dfrac{\dfrac{\mathrm{DaA} \qquad \mathrm{DoB}}{\mathrm{AoB}} \qquad \mathrm{CaB}}{\mathrm{AoC}}$$



$$\dfrac{\dfrac{\mathrm{DaA} \qquad \dfrac{\mathrm{DoE} \qquad \mathrm{BaE}}{\mathrm{DoB}}}{\mathrm{AoB}} \qquad \mathrm{CaB}}{\mathrm{AoC}}$$



Again, Aristotle's claim will be that the above improvement-path could only continue ad infinitum if there were an infinite predicational down-chain originating in C or an infinite predicational up-chain originating in A.

Admittedly, *Posterior Analytics* I.21 does not provide a complete analysis of C-improvement-path or B-improvement-path construction. Concerning the latter, Aristotle may be assuming his audience's familiarity with the account of C-string construction provided a few pages earlier in *Posterior Analytics* I.15 (79b14-18):

> If there is a middle [between A and B, for a proof of AeB], one of them [i.e. either A or B] must be in some whole. For, the syllogism [showing AeB] will be in either the first or the middle figure. So, [i] if [he syllogism] is in the first [figure], then B will be in some whole (for the premise concerning this [i.e., the premise concerning B, the

minor premise] must be made affirmative) and [ii] if [the syllogism] is in the middle [figure], then one or the other of them, [either the major term A or the minor term B], will be in some whole.

Nonetheless, we have very good reasons to believe that the train of thought outlined above lies behind the proof sketch of I.21. For not only does the interpretive hypothesis give Aristotle a valid argument for the desired conclusion. The interpretation provides an excellent explanation for why I.21 considers exactly Celarent, Camestres, and Bocardo. No other interpretation I am aware of does this. On my view, Aristotle discusses this exact trio to illustrate the three patterns at issue in **Observation 4**. To consider more would be redundant. And to consider (say) Cesare or Baroco rather than Bocardo would not serve his purpose. Contra Barnes-Lear, it is in no way surprising that I.21 discusses Bocardo. And contra Philoponus-Ross, Aristotle has no need for parallel discussions of all the other six negative moods (the **F-group**, for instance can be ignored entirely). The problem is that both Philoponus-Ross and Barnes-Lear fail to discern the overall proof strategy in the context of which Aristotle's choice of considering exactly Celarent, Camestres and Bocardo—and no others—would make perfect sense.

It is arguable that because Aristotle lacked technique of mathematical induction, he was in no position to make any of this rigorous and fully general. In oral presentation he presumably would have given a fuller explanation than what we get in the sketch preserved in *Posterior Analytics* I.21. Maybe he would have used diagrams like that appealed to in chapter I.19. I will not speculate. We, however, using the aparatus of the present paper can develop Aristotle's observations in a mathematically precise matter to establish his contention. First some definitions.

Definition 19 Term-sets

If S is a syllogism, define the term-set of S,*termset*(S), as is the smallest set containing every term used in S.

Definition 20 Major and minor a-premise-up-chains and a-premise-down-chains for C-strings and B-strings

Let S be a C-string or a B-string

1. Then the *minor a-premise-up-chain* of S, $\Pi^{\uparrow}_{min}(\mathrm{S})$, is the maximal sequence composed of premise-occurrences in S of the form: $(\mathrm{X}_1\mathrm{aX}_0, \mathrm{X}_2\mathrm{aX}_1, \mathrm{X}_3\mathrm{aX}_2, ...)$ where
   (a) X_0 is the minor term of S
   (b) If $\mathrm{n} < \mathrm{m}$, with φ and ψ the n^{th} and m^{th} members of $\Pi^{\uparrow}_{min}(\mathrm{S})$ respectively, then $height_{\mathrm{S}}(\varphi) < height_{\mathrm{S}}(\psi)$.[61]

[61] For the $height_{\mathrm{S}}$ function see **Definition 5** above.

2. The *major a-premise-up-chain* of S, $\Pi^{\uparrow}_{maj}(\mathrm{S})$, is the maximal sequence composed of premise-occurrences in S having the form $(\mathrm{X_1aX_0},\ \mathrm{X_2aX_1},\ \mathrm{X_3aX_2},\ ...)$ where
   (a) $\mathrm{X_0}$ is the major term of S
   (b) If $\mathrm{n} < \mathrm{m}$, with φ and ψ the $\mathrm{n^{th}}$ and $\mathrm{m^{th}}$ members of $\Pi^{\uparrow}_{maj}(\mathrm{S})$ respectively, then $height_{\mathrm{S}}(\varphi) < height_{\mathrm{S}}(\psi)$.

3. The *minor a-premise-down-chain* of S, $\Pi^{\downarrow}_{min}(\mathrm{S})$, is the maximal sequence composed of premise-occurrences in S having the form $(\mathrm{X_0aX_1},\ \mathrm{X_1aX_2},\ \mathrm{X_2aX_3},\ ...)$ where
   (a) $\mathrm{X_0}$ is the minor term of S
   (b) If $\mathrm{n} < \mathrm{m}$, with φ and ψ the $\mathrm{n^{th}}$ and $\mathrm{m^{th}}$ members of $\Pi^{\downarrow}_{min}(\mathrm{S})$ respectively, then $height_{\mathrm{S}}(\varphi) < height_{\mathrm{S}}(\psi)$.

4. The *major a-premise-down-chain* of S, $\Pi^{\downarrow}_{maj}(\mathrm{S})$, is the maximal sequence composed of premise-occurrences in S having the form $(\mathrm{X_0aX_1},\ \mathrm{X_1aX_2},\ \mathrm{X_2aX_3},\ ...)$ where
   (a) $\mathrm{X_0}$ is the major term of S
   (b) If $\mathrm{n} < \mathrm{m}$, with φ and ψ the $\mathrm{n^{th}}$ and $\mathrm{m^{th}}$ members of $\Pi^{\downarrow}_{maj}(\mathrm{S})$ respectively, then $height_{\mathrm{S}}(\varphi) < height_{\mathrm{S}}(\psi)$.

NB that **Definition 20.1-4** covers only B-strings and C-strings: syllogisms in which no positive syllogistic moods appear; also that using the $height_{\mathrm{S}}$ function is crucial for achieving intended results and ensuring that the Π-functions provided by **Definition 20** are well-defined.[62]

Definition 21 Major and minor (term) up-chains and down-chains for C-strings and B-strings

For all a-propositions XaY, define $f(\mathrm{XaY}) = \mathrm{X}$ and $g(\mathrm{XaY}) = \mathrm{Y}$. Let S be a C-string or a B-string with major term A and minor term C.

1. If $\Pi^{\uparrow}_{min}(\mathrm{S}) = (\varphi_0, \varphi_1, \varphi_2, ...)$, define the *minor term-up-chain* of S as $MinUp(\mathrm{S}) = (\mathrm{C}) + (f(\varphi_0), f(\varphi_1), f(\varphi_2), ...)$

[62]For instance, consider the following C-string which (given our assumptions) could very well constitute a partial demonstration. What would the minor a-premise-up-chain be, if clause (b) of **Definition 20.1** were omitted?

$$\dfrac{\dfrac{\dfrac{\mathrm{DaA} \quad \dfrac{\dfrac{\mathrm{EeD} \quad \mathrm{EaC}}{\mathrm{DeC}} \quad \mathrm{CaB}}{\mathrm{DeB}}}{\mathrm{AeB}} \quad \mathrm{BaC}}{\mathrm{AeC}} \quad \mathrm{CaE}}{\mathrm{AeE}}$$

2. If $\Pi^{\uparrow}_{maj}$(S) $= (\varphi_0, \varphi_1, \varphi_2, ...)$, define the *major-term-up-chain* of S as $MajUp$(S) $=$ (A) $+ (f(\varphi_0), f(\varphi_1), f(\varphi_2), ...)$

3. If $\Pi^{\downarrow}_{min}$(S) $= (\varphi_0, \varphi_1, \varphi_2, ...)$, define the *minor term-down-chain* of S as $MinDown$(S) $=$ (C) $+ (g(\varphi_0), g(\varphi_1), g(\varphi_2), ...)$

4. If $\Pi^{\downarrow}_{maj}$(S) $= (\varphi_0, \varphi_1, \varphi_2, ...)$, define the *major term-down-chain* of S as $MajDown$(S) $=$ (A) $+ (g(\varphi_0), g(\varphi_1), g(\varphi_2), ...)$

Example

Consider the final C-string appearing in our illustration of **Observation 4** above. Call it S. Then:

1. $\Pi^{\uparrow}_{min}$(S) ={AaD, EaA}; $\Pi^{\uparrow}_{maj}$(S) = {BaC}; $\Pi^{\downarrow}_{min}$(S) = Ø =$\Pi^{\downarrow}_{maj}$(S)
2. $MinUp$(S) = (D,A,E); $MajUp$(S) = (C,B); $MinDown$(S) = (D)

With the above definitions in hand, it is easy to prove the following two lemmas, effectively expressing the insights about C- and B-improvement construction contained in **Observation 4**.

Lemma 2

1. Let $\mathfrak{M} = (D, P, [\![\cdot]\!], A)$, and let D be a partial demonstration in $\mathfrak{M}$, which is a C-string whose negative premise is XeY.
2. Then:
   (a) $MinUp$(D) has as its final element exactly one of {X,Y}, and $MajUp$(D) has as its final element the other
   (b) Every term in *termset*(D) occurs in exactly one of: {$MinUp$(D), $MajUp$(D)}

Proof. Induction on length of syllogistic tree. Straightforward.

Lemma 3

1. Let$\mathfrak{M} = (D, P, [\![\cdot]\!], A)$, and let D be a partial demonstration in $\mathfrak{M}$ which is a B-string whose negative premise is XoY.
2. Then:
   (a) $MajUp$(D) has as its final element exactly one of {X,Y}, and $MinDown$(D) has as its final element the other
   (b) Every term in *termset*(D) occurs in exactly one of: {$MinDown$(D), $MajUp$(D)}

Proof. Induction on length of proof tree. Straightforward.

Next, using **Lemma 2** and **Lemma 3**, we prove the following intermediate result:

Lemma 4

1. Let$\mathfrak{M} = (D, P, [\![\cdot]\!], A)$ such that $\mathfrak{M}$ satisfies both the **DCC** and the **ACC**.
2. Then, for all X$\in${C,B,F}: if $\overline{\mathrm{D}}$ is an X-improvement-path, $\overline{\mathrm{D}}$ is of a finite length.

Proof. The case for X = F is trivial. So consider X$\in${C,B}. Since $\mathfrak{M}$ satisfies both the the **DCC** and the **ACC**, it follows that there is no $A_0 \subseteq A$ whose members can be organized into an infinite sequence of form $(Y_0\mathrm{a}Y_1,\ Y_1\mathrm{a}Y_2, \ldots, Y_n\mathrm{a}Y_{n+1}, \ldots)$ or $(Y_1\mathrm{a}Y_0,\ Y_2\mathrm{a}Y_1,\ldots,,Y_{n+1}\mathrm{a}Y_n,\ldots)$. The case for X = C follows immediately from **Lemma 2**. The case of X = B follows immediately from **Lemma 3**.

This gives us the thesis announced in the first sentence of *Posterior Analytics* I.21. For the following is equivalent to claim that **Theorems 1** and **2** entail **Theorem 3**.

Lemma 5

1. Let$\mathfrak{M} = (D, P, [\![\cdot]\!], A)$ such that $\mathfrak{M}$ satisfies both the **DCC** and the **ACC**. And suppose that **Theorem 1** and **Theorem 2** are true.
2. Then, if φ is a demonstrable e- or o-propoition in $\mathfrak{M}$, demonstration of φ does not continue ad infinitum in $\mathfrak{M}$.

Proof. By assumption, WLOG we can ignore positive moods. Given **Observation 3**, and **Lemma 4** the conclusion follows immediately.

17 The case of Darii, Disamis, Datisi and Darapti

I've been developing a reading of *Posterior Analytics* I.20-21 on which Aristotle's proof relies on the following lemma, analogous to **Lemma 5** above.

Lemma 6

1. Let$\mathfrak{M} = (D, P, [\![\cdot]\!], A)$ such that $\mathfrak{M}$ satisfies both the **DCC** and the **ACC**. And suppose that **Theorem 1** is true.
2. Then, if φ is a demonstrable i-propoition in $\mathfrak{M}$, demonstration of φ does not continue ad infinitum in $\mathfrak{M}$.

Effectively, **Lemma 5** says that **Theorem 1** entails **Theorem 2**. The text at I.21 82a36-39, in the context of I.20 which deals only with Barbara, certainly seems to show Aristotle relying on **Lemma 6**. But no proof of **Lemma 6** is given in the extant text. I contend that Aristotle omits a proof of **Lemma 6** while including a proof-sketch for **Lemma 5** (chapter I.21), simply because **Lemma 5** is proved by an argument exactly parallel to the sub-argument for the finitude of C-improvement-paths witnessed by the text of I.21. As in the argument of chapter I.21, and for exactly the same reasons, to prove **Lemma 6** we can without loss of generality ignore the possibility of improving a partial demonstration of an i-conclusion using Barbara. We can focus exclusively on improvement-paths which at every step use moods in {Darii, Disamis, Datisi, Darapti}: the four positive particular moods Aristotle countenances. Again the proof will be guided by an elementary observation, an observation analogous to **Observation 3** above.

Definition 22

1. A *D-string* is a syllogism using only inferences in {Disamis, Datisi, Darii}
2. A *D*-string* is a syllogism using only inferences in Darapti

Observation 5

A syllogism which uses only inferences in {Darii, Disamis, Datisi, Darapti} must instantiate exactly one of the following forms:

1. [Conclusion]⟵[D*-string]⟵[Single i-premise]
2. [i-conclusion]⟵[D-string]⟵[Single i-premise]
3. [i-conclusion]⟵[D-string]⟵[D*-string]⟵[Single i-premise]

Note that every D*-string is of length 1. One cannot improve on an inference in Darapti by reapplying Darapti, because the quantity of the conclusion (i-) does not match the quantity of either premise.

Definition 23

We say $\overline{\mathrm{S}}$ is a *D-improvement-path* iff (a) $\overline{\mathrm{S}}$ is an improvement-path, and (b) for every $\mathrm{S} \in \overline{\mathrm{S}}$, S is a D-string.

From the above, it is clear that to prove **Lemma 6**, one only needs show that given the relevant assumptions every D-improvement-path must be finite.

Now, proof-theoretically the D-string trio of Darii, Disamis, Datisi is in a close correspondence to the C-string trio Celarent, Cesare, Camestres. For our purposes, one need only note the following, analagous to **Observation 4**.

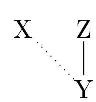
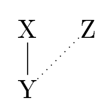
Observation 6

With respect to improvement, each of the three moods in {Darii, Disamis, Datisi} instantiate one of two patterns:

1. In applying Darii or Datisi in an improvement targeting a premise φ, we (i) assume a fresh positive premise stating that the predicate of φ holds of all of our new middle term Y, and (ii) relate Y in an i-proposition to the subject of φ.

2. In applying Disamis in an improvement targeting a premise φ, we (i) assume a fresh positive premise stating that the subject of φ holds of all of our new middle term Y, and (ii) relate Y in an i-proposition to the predicate of φ.

Again, pictoral representation may be helpful. As before we represent a premise GaH by placing G above H and connecting them with a solid line. But now we represent a particular premise GiH with a dotted line between G and H. **Observation 6** is then displayed:

Pattern 4	Pattern 5
Disamis: $\dfrac{\text{XiY} \qquad \text{ZaY}}{\text{XiZ}}$	Darii: $\dfrac{\text{XaY} \qquad \text{YiZ}}{\text{XiZ}}$ Datisi: $\dfrac{\text{XaY} \qquad \text{ZiY}}{\text{XiZ}}$
	

In a certain proof-theoretic sense, D-string construction behaves as the mirror image C-string construction. To see an aspect of this, the reader is invited to compare Patterns 1 and 2 in our pictorial rendering of **Observation 4** with Patterns 4 and 5 above. For the perspective of contemporary logic, this duality should not be surprising. After all, while e-propositions are (as we would say) left-monotone decreasing and right-monotone decreasing, i-propositions are left-monotone increasing and right-monotone increasing.

Now, **Definitions 20** and **21** above (premise and term up- and down-chains) were restricted to B- and C-strings. But nothing prevents us from extending them to D-strings. If we do so, we can state and prove an an analogue to

Lemma 4 that shows for D-string construction what **Lemma 4** showed for C-string construction.

Lemma 7

1. Let $\mathfrak{M} = (D, P, [\![\cdot]\!], A)$, and let D be a partial demonstration in $\mathfrak{M}$, which is a D-string with positive premise AiB.
2. Then:
   (a) $MinDown$(D) has as its maximum element exactly one of {A,B} and $MajDown$(D) has as its maximum element the other
   (b) Every term in T(D) occurs in exactly one of: {$MinDown$(D), $MajDown$(D)}

Proof. Induction on length of proof tree. Straightforward.

Given **Lemma 7** and **Observation 5**, **Lemma 6** follows immediately by an argument perfectly parallel to that for **Lemma 5**. And by **Lemma 6** we have that **Theorem 1** entails **Theorem 2**.

18 *Posterior Analytics* I.20: the case of Barbara

In combination **Lemma 5** and **Lemma 6**, show that **Theorem 1** entails both **Theorem 2** and **Theorem 3**. *Posterior Analytics* I.20 provides argument for **Theorem 1**: that if there are no infinite predicational up-chains and down-chains, then the process of demonstrating a (demonstrable) a-proposition is finitely terminating.

Prima facie, the task of proving **Theorem 1** might appear simpler than that of proving **Lemma 5** and **Lemma 6**. After all, since exactly one syllogistic mood (Barbara) can prove an a-proposition, we can ignore all 13 other moods. But appearances can be deceiving. In every syllogism proving an e-conclusion, there is exactly one e-premise; in every syllogism proving an o-conclusion there is at most one o-premise; and and in every syllogism proving an i-conclusion there is at most one i-premise. But for all n (with $2 \leq n < \omega$), there is a syllogism establishing an a-proposition with n many a-premises. So the structure of the problem confronting Aristotle when he attempts an argument for **Theorem 1**, is of a rather different character when compared to the problem he takes up in I.21.

The brief and highly condensed *Posterior Analytics* I.20 opens as follows (82a21-30):

> So then, that the intermediates [mediating AaZ] cannot be infinite, if the predications come to a stop upwards and downwards is clear. By upwards I mean towards the more universal, by downwards I mean towards the more particular. For if when A is predicated of Z the intermediates (the B's) are infinite, it is clear that it would be possible [for predications to continue ad infinitum]; such that [for

instance] both from A downwards one thing is predicated of another ad infinitum (for before one has reached Z there are infinitely many intermediates), and from Z upwards there are infinite [intermediates] before one has reached A; with the result that if these things are impossible, also it is impossible for there to be infinitely many intermediates.

In accord with his formulation of **Question 3** in I.19, Aristotle here represents the possibility that the process of demonstrating AaZ continues ad infinitum as a possibility on which there are infinitely many intermediates between two extremes. The first sentence of the quoted text is, then, a statement of **Theorem 1**. In what follows, Aristotle reasons by contra-position. He supposes there are infinitely many intermediates between A and C. And states that in this case then reality will violate either the **Descending Chain Condition** (**DCC**) or the **Ascending Chain Condition** (**ACC**). Aristotle finds the truth of this contra-posed conditional 'clear' [*dēlon*], and provides no substantial defense of it in the extant text of the *Posterior Analytics*. Viewing this as sufficient, he moves on to another topic.

I am not sure how close Aristotle himself ever came to a proof of the the contra-positive of **Theorem 1**: that if demonstration of a demonstrable a-proposition continues ad infinitum, then reality contains either infinite predicational up-chains or infinite predicational down-chains. However, to appreciate why he might (rightly or wrongly) thought it this conditional obvious, and on what basis he might have accepted we should try to get clearer on what in Aristotle's system the truth of the antecedent exactly amounts to. For this purpose, formal reconstruction again proves helpful.

Definition 24 Barbara (partial) demonstrations

1. Let $\mathfrak{M} = (D, P, [\![\cdot]\!], A)$
2. D is a *Barbara (partial-)demonstration* in $\mathfrak{M}$ iff D is a demonstration (partial-demonstration) in $\mathfrak{M}$ and every inference is D constitutes an instance of Barbara

So, what must be the case for a Barbara demonstrative process to continue ad infinitum? Well, to someone who has spent significant time working with Aristotelian syllogistic, the following is apt to seem obvious:

Lemma 8

1. Let S be a syllogism that uses only Barbara. And let: π(S) represent the smallest set containing all premises of S.
2. Then there is a finite *a-premise sequence* $\overrightarrow{\pi} = (\mathrm{AaB_0}, \mathrm{B_0aB_1}, \ldots, \mathrm{B_{n\text{-}1}aB_n}, \mathrm{B_naC})$, containing all and only members of Π(S) such that:
   (a) A is the major term of S (predicate of the conclusion of S)

(b) C is the minor term of S (subject of the conclusion of S)

In the formal system developed here, **Lemma 8** admits of easy demonstration by induction on syllogistic proof length. But even without the means to rigorously prove it, the lemma is obvious enough. I would be very surprised if Aristotle were not aware of the relevant fact. For what follows, a refinement of this result is convenient.

Lemma 9

1. Let S be a syllogism that uses only Barbara. Define $\overrightarrow{\pi}$(S) the smallest sequence that orders the collection of premise-occurrences in S *horizontally*: that is, with respect to their left to right position in S's tree.[63]

2. Then $\overrightarrow{\pi}$(S) is an a-premise sequence for S.

Again, proof is by easy induction. For all Barbara only syllogisms $\overrightarrow{\pi}$(S) is clearly unique. So, it will be convenient to refer to $\overrightarrow{\pi}$(S) as *the* a-premise sequence of S.

Now, as noted above, the argument of *Posterior Analytics* I.19-22 crucially assumes that every a-proposition apt for entering into demonstration must correspond to a genuine predication. We represented this assumption by requiring that in any $\mathfrak{M} = (D, P, [\![\cdot]\!], A)$, if XaY$\in A$, then $[\![\mathrm{X}]\!]P\ [\![\mathrm{Y}]\!]$. Given this assumption, and given **Lemma 9**, the following is an immediate consequence:

Lemma 10

1. Let $\mathfrak{M} = (D, P, [\![\cdot]\!], A)$ be a predicational structure, and let D be a Barbara partial-demonstration of AaC relative to $\mathfrak{M}$, such that $\overrightarrow{\pi}$(D) = (Aa$\mathrm{B_0}$, $\mathrm{B_0}$a$\mathrm{B_1}$, …, $\mathrm{B_{n\text{-}1}}$a$\mathrm{B_n}$, $\mathrm{B_n}$aC) is the premise sequence of D.

2. Then $\mathfrak{M}$ gives rise to a finite *predicational sequence* $\Sigma = ([\![\mathrm{A}]\!]P[\![\mathrm{B_0}]\!], [\![\mathrm{B_0}]\!]P[\![\mathrm{B_1}]\!], \ldots, [\![\mathrm{B_{n\text{-}1}}]\!]P[\![\mathrm{B_n}]\!], [\![\mathrm{B_n}]\!]P[\![\mathrm{C}]\!])$, which is order isomorphic to $\overrightarrow{\pi}$(D).

3. Obviously, for each $\overrightarrow{\pi}$(D) there is one such Σ. So, we can define a corresponding function, and speak of $\Sigma(\overrightarrow{\pi}(\mathrm{D}))$ as *the* predication sequence for D.

[63] For instance, if S is the tree:

$$\dfrac{\dfrac{\mathrm{FaA} \qquad \dfrac{\mathrm{AaB} \quad \mathrm{BaC}}{\mathrm{AaC}}}{\mathrm{FaC}} \qquad \dfrac{\mathrm{CaD} \qquad \dfrac{\dfrac{\mathrm{DaB} \quad \mathrm{BaA}}{\mathrm{DaA}} \qquad \mathrm{AaE}}{\mathrm{DaE}}}{\mathrm{CaE}}}{\mathrm{FaE}}$$

$\overrightarrow{\pi}$(S) = {FaA, AaB, BaC, CaD, DaB, BaA, AaE}

Effectively, what **Lemma 10** says is that every partial demonstration in Barbara requires a chain of genuine predications having the same 'length' as the premise sequence of the partial demonstration in question. The feature of Aristotle's system represented by **Lemma 10** is clearly one he was aware of. Otherwise, Aristotle's strategy of arguing from the finiteness of predicational chains to the finiteness of demonstrative processes has no sense.

Lemma 11

1. Suppose that the process of demonstrating AaB continues ad infinitum in $\mathfrak{M} = (D, P, [\![\cdot]\!], A)$.
2. Let $[\![\mathrm{A}]\!] = a$, and $[\![\mathrm{B}]\!] = b$. And let $\overline{\mathrm{D}} = (\mathrm{D}_0, \mathrm{D}_1, \mathrm{D}_2, ...)$ be an infinite Barbara-demonstrative-improvement-path, a sequence witnessing the fact that the process of demonstrating AaB continues ad infinitum in $\mathfrak{M}$.
3. For all $\mathrm{D_n} \in \overline{\mathrm{D}}$: there is a unique premise sequence $\overrightarrow{\pi}(\mathrm{D_n})$ and a correspondingly unique predication sequence $\Sigma_{\mathrm{n}} := \Sigma(\overrightarrow{\pi}(\mathrm{D_n}))$. So, order isomorphic to $\overline{\mathrm{D}} = (\mathrm{D}_0, \mathrm{D}_1, \mathrm{D}_2, ...)$, we may form an infinite sequence of finite predicational sequences $\overline{\Sigma} := (\Sigma_0, \Sigma_1, \Sigma_2, \ldots)$ such that each $\Sigma_{\mathrm{n}} \in \overline{\Sigma}$ is the predicational sequence of D_n:
   - $\Sigma_0 = (aPx_0,\ x_0Pb)$
   - $\Sigma_1 = (aPx_0,\ x_0Px_1,\ x_1Pb)$
   - $\Sigma_{n+1} = (aPx_0,\ x_0Px_1,\ \ldots,\ x_{n-1}Px_n,\ x_nPb)$,[64] for $n \geq 1$
4. Call this $\overline{\Sigma}$, the Σ-series of $\overline{\mathrm{D}}$

Note that **Lemma 11** is an immediate consequence of **Lemmas 9** and **10**.

Now for something a bit harder. Given **Lemma 11**, it is possible to prove what Aristotle in both *Posterior Analytics* I.19 and I.20 is evidently assuming: namely, that if demonstration of an a-proposition continues ad infinitum, reality must contain a (totally ordered) predicational chain with two end-points and infinitely many intermediates. Aristotle clearly believed this. And rightly so. For someone with principled reasons to reject the existence of actual infinities, his counterfactual intuitions about them are surprisingly good. Nonetheless, without a developed mathematics of actual infinities, I suggest, Aristotle had not much of an idea of how to prove the following lemma.

Lemma 12

1. Let $\mathfrak{M} = (D, P, [\![\cdot]\!], A)$ be real demonstration model such that AaB$\in A$.
2. If the process of demonstrating AaB continues ad infinitum in $\mathfrak{M}$, then for some $D^* \subseteq D$ and $P^* \subseteq P$: (D^*, P^*) is an infinite strict linear order having $[\![\mathrm{A}]\!]$ as its unique maximum and $[\![\mathrm{B}]\!]$ as its unique minimum.

[64] NB In this formalism, the value of x_0 in the above representation of $\Sigma_0$need not be the same as the value of x_0 in the above representation of Σ_1.

Proof. Let $\overline{\mathrm{D}} = (\mathrm{D}_0,\ \mathrm{D}_1,\ \mathrm{D}_2,\ ...)$ be an infinite sequence of demonstrative improvements, witnessing our assumption that demonstration of AaB continues ad infinitum in $\mathfrak{M}$. Finally, let $\overline{\Sigma} = (\Sigma_0, \Sigma_1, \Sigma_2, \ldots)$ be the Σ-series of $\overline{\mathrm{D}}$, and let $[\![\mathrm{A}]\!] = a$ with $[\![\mathrm{B}]\!]= b$. For each such $\Sigma_n \in \overline{\Sigma}$, define $relata(\Sigma_n) = \{a, b\} \cup \{x : (\exists y, z \in D)\ xPy \in \Sigma_n \wedge\ \ zPx \in \Sigma_n\}$. We set $D^* = \{x\ \ :\ \ (\exists n \in \mathbb{N})\ x \in relata(\Sigma_n)\}$, and $P^* = (P|D^*) \setminus \{xPx\ \ :\ \ x \in D^*\}$, then claim that (D^*, P^*) is an infinite strict linear order with $[\![\mathrm{A}]\!] = a$ and $[\![\mathrm{B}]\!]= b$, as its unique maximum and minimum. Irreflexivity of P^* is immediate. For transitivity, let xP^*y and yP^*z. Since P^* is irreflexive $x \neq y$ and $y \neq z$; since P is anti-symmetric it follows that $x \neq z$. So by its definition it is clear that xP^*z. To convince yourself of trichotomy, let $x, y \in D^*$ and suppose that $x \neq y$. Given the definition of D^*, there must be some $i \in \mathbb{N}$ such that $\{x, y\} \subseteq relata(\Sigma_i)$ but $\{x, y\} \nsubseteq relata(\Sigma_{i-1})$. By the transitivity of P and the structure of this Σ_i: either xPy or yPx. So either xP^*y or yP^*x. It is likewise apparent that for all $x \in D^*$, aPx and xPb. For any such x simply consider the structure of the first Σ_i in which x appears. To prove that a is the unique maximum and b the unique minimum, suppose that for some $x \in D^*$ either xP^*a or bP^*x. In both cases, transitivity (and the fact that a and b are maxima and minima respectively) ensures a violation of irreflexivity. So a and b must be unique. Finally, the demonstration that D^* is of infinite size. Now by an easy induction one can prove that for all n: $relata(\Sigma_n) = ont(\mathrm{D}_n)$. So, by **Lemma 1**, it follows that for any Σ_k in our infinite Σ-series, there is an $j > k$ such that $relata(\Sigma_k) \subsetneq relata(\Sigma_j)$. Hence, it is clear that $|D^*| \nless \aleph_0$.

When combined with **Lemma 12**, the following secures the contra-positive of **Theorem 1.**

Lemma 13

1. Let $\mathfrak{M} = (D, P, [\![\cdot]\!], A)$ such that for some $D^* \subseteq D$, and $P^* \subseteq P$: (D^*, P^*) is an infinite strict linear order with maximum element $a \in D^*$ and minimum element $b \in D^*$.

2. Then $\mathfrak{M}$ violates either the **DCC** or the **ACC**

Lemma 13, of course, immediately follows as special case of a well-known theorem of Order Theory which effectively says that a po-set contains no infinite to-sets iff it satisfies both the **DCC** and the **ACC**. We shall not rehearse a proof of that theorem here. Now, when Aristotle comes in sight of **Lemma 13** he just says that it is 'clear'. And lacking an adequate mathematical theory of the actually infinite, Aristotle was in no position to approach a rigorous proof of **Lemma 13**. However, he might very well have come to believe **Lemma 13** on the basis of something like the following non-rigorous informal argument:

- Assume the antecedent and that $\mathfrak{M}$ satisfies the **ACC**. We prove that $\mathfrak{M}$ violates the **DCC.** Since $\mathfrak{M}$ satisfies the **ACC**, our maximum element a has an immediate P^*-successor $x_0 \in D^*$. By the same argument, x_0 has an immediate P^*-successor $x_1 \in D^*$. And by the same argument x_1 has an immediate P^*-successor $x_2 \in D^*$, and x_2 has an immediate P^*-successor $x_3 \in D^*$. Generalizing from these cases one can see that since (D^*, P^*) is an **infinite** strict linear order , repetition of the argument reveals an infinite predicational down-chain $(x_0 P^* x_1, x_1 P^* x_2, \ldots, x_{k-1} P^* x_k, \ldots)$. Since $P^* \subseteq P$ it follows that $\mathfrak{M}$ violates the **DCC.**

19 *Posterior Analytics* I.22: Aristotle's problem and interpretive problems

The goal of *Posterior Analytics* I.19-22 is to establish a claim about demonstrative processes: that they never continue ad infinitum. Demonstrative processes can make use of all 14 syllogistic figures (or so Aristotle is prepared to concede in I.19-22). But the line of argument witnessed by I.20-21 shows that the only way for demonstration of AxC (with $x \in \{$a,e,i,o$\}$) to continue ad infinitum is if there is an infinite sequence of a-propositions of one of the forms:

- ($\mathrm{YaX_0}$, $\mathrm{X_0aX_1}$, $\mathrm{X_1aX_2}$, ..., $\mathrm{X_naX_{n+1}}$, ...)
- ($\mathrm{X_0aY}$, $\mathrm{X_1aX_0}$, $\mathrm{X_2aX_1}$, ..., $\mathrm{X_{n+1}aX_n}$, ...)

such that either (i) Y = A or (ii) Y = C. In other words, for any demonstrable proposition φ: demonstration of φ continues ad infinitum only if there is an infinite sequence of true a-propositions, either up or down, from either the major or minor term of φ. However, as is hinted already in *Posterior Analytics* I.19, and explicitly claimed in I.22, not just any sequence of true a-propositions will do. Demonstration works by tracing the structure of the real. Accordingly, the predications that appear in demonstrations must be genuine predications whose grammatical structure mirrors metaphysical structure. So demonstration ad infinitum is possible only if reality contains infinite predicational chains of one of the forms:

- $(x_0 P x_1, x_1 P x_2, x_2 P x_3, \ldots x_n P x_{n+1}, \ldots)$
- $(x_1 P x_0, x_2 P x_1, x_3 P x_2, \ldots x_{n-1} P x_n, \ldots)$

which gives rise to a correspondingly infinite sequence of true a-propositions. The task of *Posterior Analytics* I.22 is to show that reality contains no such infinite predicational chains. Expressed in terms of demonstration models: while I.20-21 shows that in every $\mathfrak{M}$ which satisfies the the **DCC** and **ACC**, for no φ does demonstration continue ad infinitum, I.22 looks to establish that reality (the 'real' $\mathfrak{M}$) does satisfy the **DCC** and **ACC**. The thesis of I.22, unlike those argued for in I.20-21, is thus more a claim of metaphysics than logic.

Now, *Posterior Analytics* I.22 in fact gives three (somewhat overlapping) arguments for the claim that reality contains no infinite predicational up- or down-chains. As space is limited, we focus here solely on the first argument. This first argument is the chapter's longest. Moreover, it introduces ideas on which the other arguments depend. So accordingly, we designate it the *Main Argument* of I.22.

Commentators from Philoponus, to Barnes and Lear, have found it difficult to see the Main Argument as sound by Aristotelian lights. One difficulty concerns the interpretation of genuine predication. The Main Argument evidently requires that all propositions involved in a demonstration express genuine predications. But on the reading of some interpreters (e.g. Barnes, Hamlyn) it is required for a genuine predication to have a subject-term which picks out a (primary or secondary) substance. Since Aristotle most definitely does think that some demonstrative sciences (e.g. mathematics) use propositions whose subject-terms refer to non-substances, Barnes concludes that Aristotle cannot in the end endorse this premise. Another difficulty concerns Aristotle's case for the finitude of essentialist predicational chains. The Main Argument argues from the possibility of real definition to the thesis that every essence has finitely many parts. Barnes and Lear contend that Aristotle's argument for the finitude of essence relies on an implausibly strong premise. For, Aristotle's thought is supposed to be that since we are finite creatures and we can define and know every essence, surely every essence must be finite. But what grounds could Aristotle have for such extreme epistemological optimism? A final difficulty developed at length by Barnes but found already in Philoponus, questions not the acceptability of the premises but the validity of the Main Argument. Aristotle claims to be arguing against infinite chains of predication in both the upwards and the downwards directions. But, it is charged, Main Argument only bears on predicational up-chains. Could Aristotle be inferring that there are no infinite predicational down-chains from the conclusion that there are no predicational up-chains? This is a bad inference. And there is good evidence both in the *Posterior Analytics* itself (chapter I.19-20) and elsewhere in the corpus (e.g. *De Caelo* I.5-7), that Aristotle himself knows this is a bad inference.

I maintain that the Main Argument can plausibly be read as valid. Moreover, its premises, when properly understood, are indeed acceptable by Aristotelian lights. The first difficulty adduced above has already dealt with. Aristotle's I.22 account of predication need not and should not be read as committing him to the view that all (genuine) predications have substances as subjects. I tackle the remaining two difficulties in what follows, offering a reconstruction of how I take it the Main Argument works. Unfortunately, as space is limited I cannot offer a complete defense of my reconstruction. Many delicate interpretive issues will have to be passed over.

Now *Posterior Analytics* I.19-21 discuss predication largely in the abstract;[65] effectively, these chapters appeal to the notion an unanalyzed primitive. I.19-21 do not hint at any distinction between different kinds of (genuine) predication. Chapter I.22, however, does introduce an anatomy of genuine predication, dividing it into two (exclusive and exhaustive) kinds: essential and accidental. The strategy of the Main Argument is to use this division to establish that all

[65] The only actual *example* predications we get in I.19-21, occur in I.19's brief remarks on genuine predication. Those remarks anticipate Aristotle's much fuller discussion of genuine predication in I.22.

predicational chains are finite. Much of our work in interpreting the Main Argument will involve getting clearer on Aristotle's I.22 anatomy of predication and its implications for the predicational structure of reality. Only then will we be in a position to explicitly formulate the Main Argument's premises. As these are matters more metaphysical than (what we would call) logical, I will not be giving a *formal* reconstruction of the Main Argument. Nonetheless, it should be emphasized that the anatomy of predication set forth in the Main Argument tells us much about how Aristotle conceives the extra-linguistic structures concerning which demonstrative-proofs can be given. So, having completed my informal reconstruction of the Main Argument, I conclude by explaining how the analysis of predication developed in the Main Argument can be related formally to our apparatus of demonstration models. In particular, I show that I.22's account of the predicational architecture of reality can be read as determining a class of algebraic structures with respect to which an Aristotelian predication relation P is naturally defined. Every such structure gives rise to a demonstration model.

20 Finitude, essence, intelligibility

Roughly, the text presenting I.22's Main Argument is structured as follows. First Aristotle treats essential predication, arguing that no essence is infinitely complex. Then Aristotle turns to predication 'in general' (83a1), arguing that if xPy is accidental then x can be a subject for no further predications (essential or accidental). Following Aristotle's own order we begin with essential predication. Aristotle's case for the finitude of essence is highly compressed. But it repays close scrutiny. *Posterior Analytics* I.22 opens as follows (82b27-83a1):

> So then, [**(a)**] as for [items] predicated in the what-is-it [in the *ti esti*, i.e. essentially], it is clear [that they are finite]. For, [**(b)**] if it is possible to define [*hopisasthai*], or ([**c**]) if the what-it-is-to-be [*to ti ēn einai*: i.e. essence] is knowable but it is not possible to traverse [*dielthein*] infinitely many [items], then [**(d)**] it is necessary that the [items] predicated [of a subject] in its what-is-it be finite.

For ease of discussion, I've broken the text into four. Aristotle is arguing that for every being x which has an essence, there are only finitely many items predicated of x in its what-it-is [*ti esti*]: i.e., essentially. For Aristotle, this is basically[66] equivalent to saying that every essence has finitely many parts, that no essence is infinitely complex. (I will often formulate the thesis of **(d)**/**(a)** accordingly). And **(b)** and **(c)** are the argumentative support for this thesis. Note that the text is presuming familiarity with the usual Platonist/Aristotelian conception of definition. In particular, the definability at issue in **(b)** is what a

[66] I'm bracketing here some issues about unity of definition, and essence qua form functioning as principle of unity, which arise outside the *Organon*. This is reasonable. When Aristotle himself turns to such issues in the *Metaphysics* (see e.g. Z.12), he speaks of them as unresolved in and problematized by the framework of the *Analytics*.

contemporary metaphysician would call *real definability*, where the *definiendum* of a real definition is not a word or a concept, but a being X, and what the real definition of X states is what it is for X to be. With his colleagues in the Academy, Aristotle is more specifically assuming that the real definition [*horismos*] of a being x captures the essence of X , and that *epistēmē* of x is paradigmatically a matter of grasping the (real) definition of x . Moreover, as is clear from his subsequent recapitulation of the argument (83a39-b8), Aristotle is thinking of real definitions as taking the form: genus + differentia. Note finally that Aristotle is assuming that that everything predicated in the what-is-it of x (i.e. essentially) is part of the essence of x. Accordingly, since essential predication is transitive, essential part-hood turns out to be transitive as well : i.e. if x is in the what-it-is of y (e.g. animal is in the what-is-it of human being) and y is in the what-is-it of z (e.g. human being is in the what-is-it of Socrates), then x is in the what-is-it of z (animal is in the what-is-it of Socrates). All of this is standard in the *Organon*.

Barnes and Lear (perhaps Ross as well), read **(b)**-**(c)** as arguing that every essence is finite because every essence can be comprehended by a finite human intellect. According to these interpreters, the definability assumed in **(b)** is definability *by us*; and the knowability assumed in **(c)** is knowability *by us*. Aristotle's guiding assumption is, then, supposed to be that although we human beings are finite we can define and know every essence. But defining an essence and knowing an essence requires surveying all the essence's parts. Since finite beings like us cannot go through infinitely many items, but can know every essence, every essence must be finite.

Strikingly, this reading credits Aristotle with dubious and extraordinarily strong assumption about the epistemic capacities of human beings. For a modern, I suppose, it is natural to expect the definability/knowability at issue in **(b)** and **(c)** to be definability/knowability by us. After all, our own epistemological inquiries are usually cast from a first-personal perspective (What should *I* believe?, What do *we* know?), and often reflect anthropocentric preoccupation. Moreover, Aristotle is a theist. And we are quite used to theists who affirm that God is omniscient and omnipotent, developing this philosophically by attributing to God an 'infinite intellect' outstripping our 'finite intellects'.[67] But all of this is very much post-Aristotelian. In *Metaphysics* Λ, Aristotle identifies God (or the Gods) with perfect intellects which exist for all eternity. But Aristotle nowhere characterizes God (or these Gods) as being infinite intellects, nor as having infinite intellectual powers. Indeed, the now philosophically familiar distinction between finite and infinite intellects (or minds, or whatever) is entirely absent from pre-Christian Greek philosophy. Equally post-Aristotelian is the first-personal orientation of epistemology. For Aristotle, the investigation of *epistēmē* is not conceived as an investigation into a distinctively human cognitive state.

[67] Cf. mathematicians' tendency to appeal 'metaphorically' to God in informal presentation of the construction of the set-theoretic universe.

It should give us pause, then, that neither in I.22 82b27-83a1 itself nor in its recapitulations elsewhere (e.g. 83b5-7), does Aristotle actually mention human beings. Nor do any such texts use first- or second-person pronouns/verbs (Barnes' translation is misleading on this point), or give contextual indications that they specifically concern the capacities of creatures like us. Dispensing with the philosophically unattractive assumption the Barnes-Lear reading of 82b27-83a1 attributes to Aristotle, the passage is better read in the context of Aristotle's particular conception of the infinite. Characterizing what the term 'infinte' means, *Physics* III.4 tells us: x is infinite iff (i) x is the sort of item that can be gone through one thing after another, but (ii) it is impossible to 'traverse' [*dielthein*] all of x.[68] It is important to recall that the impossibility here is impossibility full-stop. For Aristotle, the nature of the infinite is to be endless, and to admit no complete traversal. The infinite as such cannot be fully gone through: neither by us nor by anything else. Of course, since Aristotle does not himself think of God(s) as omnipotent, and indeed argues that it would be irrational for God to pay attention to anything beyond itself,[69] he never really raises the explicit question whether the unmoved movers of *Metaphysics* Λ and *Physics* VII in principle *could* perform an infinite task. But in *Physics* I.4 (188a5-11) he argues that Anaxagoras' God (which, like Aristotle's, is an eternally existing perfect intellect) cannot separate every bit of an infinitely divisible quantity since there will be no minimum magnitude, and that Anaxagoras thus absurdly has his God attempting an impossible (because infinite) task. In any case, *Posterior Analytics* I.22 82b27-83a1 really should be read alongside Aristotle's claims in both in *Physics* I.4 (187b7-13) and III.6 (207a25-32) that 'the infinite **qua infinite** is unknowable [*agnōston*]'. Linguistically, the two *Physics* texts are remarkably close to *Posterior Analytics* I.22. And in both of those contexts, the idea is clearly that the infinite is unknowable per se, by its own nature, and not just unknowable by us. I see no reason to attribute to Aristotle something different in *Posterior Analytics* I.22 82b27-83a1 .

In this light, **(b)** and **(c)** look to present two distinct but closely related arguments for the finitude of essence. Both are from the in principle intelligibility of reality: one from the possibility of definition, another from the possibility of knowledge. With respect to knowledge, Aristotle is registering his commitment to the view that every essence is in principle comprehensible—that no essence is intrinsically impenetrable by thought, and that all essences are (in principle) objects of knowledge. This is a deep Parmenidean strand in Aristotle's philosophy.[70] And Aristotle believes that his interlocutors, Platonists and Academics foremost among them, share in this commitment. There can be no infinitely complex essence because such an essence would be in principle unknowable. In

[68]The verbs *dielthein* and *diexelthein* appear frequently in Aristotle's discussions of the infinite. As mentioned above, they are used in articulating Zeno's paradoxes of the infinite in particular. So it is notable that *dielthein* appears in connection with the infinite in 82b27-83a1.

[69]Cf. *Metaphysics* Λ.9. Of the quartet of omni-'s later Christian philosophers use to characterize God (omniscience, omnipotence, omnipresence, omnibenevolence), it is only the last which might reasonably be be attributed to Aristotle's God(s).

[70]Cf. Parmenides Fragment B3.

this context, Aristotle is evidently supposing (with, e.g., Plato in the *Theaetetus*) that knowing an essence requires comprehending all of its parts. But since the parts of an essence are here taken to be actual beings prior to and principles of the whole that they compose, it follows that traversing an infinite essence would require going through each part *individually* one by one. In particular, one cannot traverse an infinite essence in the manner a body traverses an infinitely divisible (but finitely long) magnitude—a point Aristotle makes in a passage quoted below. (Nor, in this framework, could one 'comprehend' an infinite essence by means of a technique like recursive definition). From Aristotle's own perspective, the main controversial claim **(c)** relies on, is an element of his positive conception of the infinite: his claim that the infinite, by its very nature, is untraverseable and that due to the inexhaustability of its parts, must thus be incomprehensible. No strong assumption about our epistemic capacities is required.

So much for **(c)**. Now, interpreters usually to read **(b)** and **(c)** together as making a single point. This is not unreasonable, because definition and knowledge of essence are closely related for Aristotle. But the grammar of the sentence (no epexegetic *kai*) may indicate that **(b)** is referring to an independent or semi-independent argument for the finitude of essence from considerations of definability in particular. In any case, Aristotle elsewhere thus breaks his case for the finitude of essence into two. Consider *Metaphysics* α.2 994b16-25:

> And look, neither can an essence always be brought back to another definition [*horismos*] lengthening [the last definition] in formula . For it is always the case that that the prior [i.e. the further expanded definition] is more [a definition], and the later not [more a definition]; and [a sequence] which does not have a first does not have a second. Further [*eti*], those who argue in this way destroy understanding [*epistasthai*]; for it is not possible to know until one reaches atoms; and [if definitions are infinite in the way sketched above] there is no knowing, for how is it possible to think things infinite in this way? For, it is not like the case of a line which does not stop in division [i.e. is infinitely divisible], but cannot be conceived without stopping [the division], (thus one going through the infinite[ly divisible line] will not count the cuts).

In *Metaphysics* α.2, Aristotle is arguing against the possibility of infinite causal chains. Here he focuses on chains of formal causes. And since Aristotle is assuming both that the formal cause of X is its essence, and the transitivity of essential part-hood, Aristotle's target is the claim that no essence can have infinitely many parts. Note that the argument appealing to definability is kept distinct (see the *eti*) from that concerning knowability. Regarding the former, Aristotle looks to be considering this sort of case. Say we are asked what X is and reply correctly:

$$\mathrm{X} =_{def} \mathrm{G}_0\mathrm{D}_0$$

It may occur that G_0 is itself definable:

$$G_0 =_{def} G_1D_1$$

so that (substituting equivalents) a fuller definition of X would be:

$$X =_{def} G_1D_1D_0$$

Suppose it occurred that for all G_n ($n \geq 1$), there's a fresh G_{n+1} and D_{n+1} such that:

$$G_n =_{def} G_{n+1}D_{n+1}$$

This would be a case in which 'an essence [can] always be brought back to another definition lengthening [the last definition] in formula'. We would then have an infinite sequence of successively fuller definitions of X:

def 1: $X =_{def} G_0D_0$

def 2: $X =_{def} G_1D_1D_0$

def 3: $X =_{def} G_2D_2D_1D_0$

def 4: $X =_{def} G_3D_3D_2D_1D_0$

$$\vdots$$

Metaphysics α.2 argues that this cannot occur and (hence) no essence is infinitely complex. More precisely, Aristotle contends that the envisaged situation is incoherently described. It was stipulated that X was definable. But it in fact represents a possibility on which is X is indefinable. For, *the* definition of X (strictly speaking, every being has exactly one full definition) would have to capture every aspect of the essential being of X. It would have to be complete, and given in a formula which admits of no expansion. Since the sequence above has no final element, it contains no element that may rightly be called *the* definition of X. However, for every n, n+1: **def n** is less of a definition of X than **def n+1**. So, since nothing in the sequence is *the* definition of X, it turns out that no element in the sequence is even a partial definition of X.

Given the grammar of *Posterior Analytics* I.22 82b27-83a1 and the testimony of *Metaphysics* α.2 994b16-25, it is perhaps attractive to see the latter as telegraphically pointing to two semi-distinct arguments for the finitude of essence: one from the in principle definability of essence, another from in principle knowability of essence.

21 Essence and accident in *Posterior Analytics* I.22

Having thus argued that every essence has finitely many parts, the Main Argument now turns to predication in general (83a1). Aristotle's first task (83a1-23) is to clarify what predication is. This yields the discussion of genuine predication analyzed above (**Sections 6-7**). Here Aristotle insists that all the predications which appear in demonstration are genuine predications (83a17-21). And an upshot of this is supposed to be that all predications in demonstration have a predicate which falls into one of the categories (83a21-23): 'Therefore, whenever one thing is predicated of one thing [in a demonstration], either it is in the essence [*ti esti*] or [it indicates] that [the subject is] qualified or quantified or related to something or doing something or undergoing something or somewhere or some-when.'

Aristotle's next task is to sketch a distinction between essence and accident. As noted above, the Main Argument's case for the finitude of predicational chains crucially relies on this distinction. 83a24-32 reads as follows:

> Further, [a predicate Y] which indicates essence [of that X] of which it is predicated, indicates that [X] *just is* that [Y], or that [X] *just is* a certain [Y].[71] In contrast, [a predicate Y] which does not indicate essence but is said of a different subject [X] which neither *just is* that [Y] nor *just is* a certain [Y], is an accident [*sumbebēkos*]: for instance, pale [predicated] of human being. For human being neither *just is* pale nor *just is* a certain [shade of] pale; rather [human being] *just is* (perhaps) [a certain] animal: for, human being *just is* animal. But it is necessary that [items] which do not indicate essence are predicated of some [distinct] subject; and [it is necessary] that there not be some pale which not in virtue of being something else is pale.

For Aristotle, there is a fundamental difference between predicates which indicate a mere characteristic of their subject and predicates which indicate something of their subject's *identity*. What is picked out by 'human being' in 'Socrates is human being' is an element of Socrates' identity. For, part of what it is to be Socrates is to be a human being: being human is part of the nature of Socrates, part of the essence of Socrates. In contrast, when one says of the deathly-white hemlock-drunk man 'Socrates is pale' one indicates a mere characteristic of Socrates, not part of what-Socrates-is.

In general, one should note that Aristotle's essence/accident distinction is not a distinction between an object's necessary and contingent properties. For Aristotle, as for us, having internal angles summing to two right angles (call it having-2R) belongs necessary to triangle. However, Aristotle quite clearly insists that having-2R is not essential to triangle. For even while no triangle can fail to have 2R, having-2R is not part of what a triangle is. Accordingly, Aristotle will say that having-2R property is an accident of triangle. Notably, since the fact that every triangle has 2R can be proved directly from the essence of triangle (at least so Aristotle assumes), having-2R turns out to be a special kind of accident. In the terminology of the *Analytics* and the *Metaphysics*: having-2R is a 'per se accident' of triangle. Nevertheless, qua accident, having-2R attaches to triangle in a similar way that being-pale attaches to Socrates:

[71] ON TRANSLATION OF OUSIA and X, Y!!

as a characteristic external to the identity of its subject. Making the contrast in English we might affirm 'Hillary Clinton is something which *just is* a certain human being' (i.e. that is her nature), and in the same sense deny 'Hillary Clinton is something which *just is* blond'.[72] In Aristotle's Greek, the *hoper* idiom plays this very role. At any rate, in accord with his use elsewhere in the *Organon* and the *Metaphysics*, the truth condition for Aristotle's *hoper* idiom at 83a24-32 amounts to:

x is *hoper* y or *hoper* a certain y iff (i) $x = y$, or (ii) y is part of the real definition of x

So we can represent the essence/accident contrast that 83a24-32 draws as:

1. X is an essential predicate of Y iff (i) $[\![X]\!] \in \{z : z$ is in the real definition of $[\![Y]\!]\}$ or (ii) $[\![Y]\!] = [\![X]\!]$

2. X is an accidental predicate of Y iff (i) $[\![X]\!]$belongs to $[\![Y]\!]$, and (ii) X is not an essential predicate of Y.

The text above (esp 83a24-26) makes quite clear that if x is an accident of y then x is not part of the essence of y. But it also indicates a view about the nature of accidents—about beings x such that for some y: x is an accident of y. The view is further developed later in the chapter, and yields an important premise for the Main Argument. So I will draw both on 83a24-32 itself and as well as subsequent material in explaining Aristotle's view, and formulating the relevant premise of the Main Argument.

In the *Organon*, Aristotle conceives of the essence of an object as its identity. Accordingly, in some sense an essential predicate indicates nothing outside its subject's identity. The predicate of 'human being is rational animal' indicates something wholly identical to its subject; the predicate of 'human being is animal' indicates something partially identical to its subject: that is, something identical to an (essential) part of its subject. A predicate indicating accident, in contrast, indicates something wholly external to the subject's identity. And Aristotle takes this to mean that an accident is wholly distinct from ($\approx$ identical with no essential part of) its subject. *Posterior Analytics* I.22 develops this by claiming that an accident qua accident is always and only predicated of something different from it (82a26, cf. 83b23-24, *Met.* Γ.4 1007a33-b1).

But what of this different thing, an accident's x's distinct subject y? Fundamental to his metaphysics is Aristotle's multifaceted insistence that whatness (essence/essential identity) is both ontologically primary and ontologically prior to suchness (accident). And on Aristotle's conception of metaphysical subjecthood, for every such (= accident), there is a what (= essence) that underlies it. After all, for an accident to have being is simply for it to characterize some

[72] A more interesting case. One can plausibly deny 'Hillary Clinton is something which *just is* the mother of Chelsea' while admitting that 'Hillary is the mother of Chelsea' is a necessary truth.

what: for it to be a such of that what. The very existence of an accident, then, depends on its belonging to a what. Speaking from of the reverse side of the same coin, Aristotle contends that a what can receive a such as an accident only in virtue of its being a what. In a certain metaphysical sense, a what X characterized by an accident Y, *is* X prior to *being* Y. And Aristotle is putting both sides of the coin together when he says in I.22 83a32 that there is nothing pale which is not pale in virtue of its (essentially) being something else.

The above line of thought gives Aristotle the following claim about the role accidents play in predicational structures. While a what, can be a subject for both essential and accidental predications, a such can be subject for no predication: neither essential nor accidental. As Aristotle puts it towards the end of the Main Argument (83b18-24):

> As many [things] as do not [indicate] essence are not predicated, these things of themselves. For they are accidents, all of them... And we assert that all these things [accidents] are predicated of a subject. But an accident is no subject. For, we posit that none of these are such that not in virtue of being something else they are called what they are called, rather [each is] itself [predicated] of another and that of something different.

The above states a necessary condition for metaphysical subject-hood. If this X thing (e.g. pale Socrates) is X (e.g. pale) in virtue of being essentially identical to something else (i.e. a certain human) modified by X, then that X (e.g. pale, the accident) is no metaphysical subject. The philosophical grounds for this view are, of course, in need of further clarification. But for the argument of I.22 the upshot is clear enough. No accident can receive anything else (accident or essence) as subject. Hence no term which picks out an accident can be a subject term of a genuine predication. Making a related point about accidental predication in *Metaphysics* Γ.4 1007a33-b3, Aristotle writes:

> If everything were predicated accidentally, there would be no first universal [i.e. no universal predicated of only non-universals][73], (supposing that accident always indicates predication of some [other] subject). So it would necessary for [accidental predication] to continue [downwards] ad infinitum. But this is impossible; for [accidental predication in fact] does not even connect more than two things [i.e. a what and an accident]: for, there is no accident of an accident...

22 Interlude: accidents and ontology

Accidents in the above sense are not themselves subjects for any predications: essential or accidental. Hence, denizens of the Aristotelian ontology like triangles, redness, and surfaces cannot be counted as accidents in the sense of

[73]Reading the manuscript and rejecting the conjecture of Alexander (accepted by Ross, Jaeger).

Posterior Analytics I.22 (and *Metaphysics* Γ.4). For, according to Aristotle, triangle is subject to the per se accident having-2R; color is an essential predicate of redness; and surfaces are the immediate metaphysical subjects to color accidents (see *De Sensu* 3).

Posterior Analytics I.22 commits Aristotle to a distinction between, for instance, *redness* (a kind of color: a what) and *red* (a way of being colored: a such and an accident). One finds this distinction elsewhere in the corpus. See for instance *Metaphysics* Z.1 1028a20-27:

> ...[O]ne might puzzle about walking [*to badizein*], being healthy [*to hugiainein*], and sitting [*to kathēsthai*], whether each of them is a being or a non-being...and rather, if indeed [it is] the walking [*to badizon*], the sitting [*to kathēmenon*], and the healthy [*to hugiainon*] that are among beings. For these things are more clearly beings, since there is for each a certain definite subject [*hupokeimenon*]...

Note that the distinction in the passage above is not a distinction between (e.g.) walking in general and a particular trope-like individual walking property (say, Socrates' walking). Unfortunately, this is clearer in the Greek than in the English (cf. the infinitive *badizein* vs. the participle *badizon*). Expressing the relevant contrast more clearly in English, the distinction at issue is between (i) both walking (a what, a type of action) and Socrates' walking (a what, an instance of the aforementioned type of action) on one side, and (ii) walks (a such, something which a subject does to exemplify the type of action: walking). NB walks does not itself exemplify walking: it is the thing that walks which does.

Recent commentators on *Posterior Analytics* I.22 (Barnes, Hamlyn) have not endorsed this interpretation of the chapter's view of accidents. And this seems to be because they read into the chapter the ontological framework of *Categories* 2. Now in *Categories* 2 Aristotle does offer a picture of what there is. The picture, however, is not put forward as presenting a complete ontology. And indeed, the ontology of Aristotle's *Metaphysics* is considerably richer.[74] In particular, the fourfold division of beings articulated in *Categories* 2 comprehends only the sort of beings that are have essences. Both the individual substances (e.g. Socrates) and individual non-substances (e.g. an individual whiteness) of *Categories* 2, are clearly conceived of as subjects of essential predications: e.g. 'Socrates is a human being', '(this individual) whiteness is color'. And mutatis mutandis, for their species/genera ('color is a quality', 'human being is an animal'). In contrast, the items which *Posterior Analytics* I.22 and several texts in the *Metaphysics* call 'accidents' [*sumbebēkota*] are subjects of no predications: neither essential nor accidental. Hence, they simply cannot fall under the fourfold division of *Categories* 2. One might think the issue here that the *Categories* calls one sort of entity (color, triangle) 'accident', while

[74]This should not be surprising, if indeed (as several recent scholars have concluded) the *Categories* is not supposed to be a metaphysical work but was rather written as an introduction to the study of dialectic and the practice of definition.

the *Analytics* and *Metaphysics* call another sort of entity (colored, triangular) 'accident'. But this is wrong. Commentators, of course, often speak of Aristotle's *Categories* as giving a 'theory of accidents'. And from antiquity to the present, one finds interpreters who refer to the non-substances of the *Categories* which are 'in a subject' (e.g. color, triangle) as accidents.[75] But in the *Categories*, Aristotle himself never calls these entities *sumbebēkota* (= 'accidents'). The fact that Aristotle's draws a contrast between *ousia* and *sumbebēkos* in *Posterior Analytics* I.22 does not put us in the framework of the *Categories*.

In fact, the ontological framework on which *Posterior Analytics* I.22 depends, is articulated not in *Categories* 2 but in *Prior Analytics* I.27. That interpreters from Philoponus, to Ross and Barnes, have failed to recognize the importance of *Prior Analytics* I.27 for understanding *Posterior Analytics* I.22 is remarkable. For, *Prior Analytics* I.27 clearly refers forward to *Posterior Analytics* I.22. The key portion of *Prior Analytics* I.27 reads as follows (43a25-42):

> Now, among all beings: [**Group I**] some (e.g., Kleon and Kallias, the particular and perceptible) are such that truly and universally they are predicated [*katēgoreisthai*] of nothing else, but different [things] are predicated of them (for each of these is human being an animal); [**Group II**] others are themselves predicated of different [things], while different [things] are not predicated of them in a prior way; [**Group III**] others are both themselves [predicated] of different [things] and have further different [things] predicated of them (e.g., human being [is predicated] of Kallias, and animal of human being). So then, that [*hoti men oun*] some beings are by nature not predicated of anything is clear (for among perceptibles just about each is such that it is not predicated of anything [else], except in an incidental way: for we sometimes say: 'This pale is Socrates' and 'The approaching is Kallias'). But that [*hoti de*] also in the case of [chains] going upwards, at some point they stop we will argue later [i.e. in *Posterior Analytics* I.22]; for now let it be assumed. So, of these [items in **Group II**] it is not possible to demonstrate [*apodeixai*] another predicate [*katēgoroumenon*] unless it is [a demonstration] by opinion [*kata doxan*: i.e. a merely dialectical and not properly scientific demonstration in which predications need not be genuine], rather these things [can be demonstrated] of other things. Nor [is it possible to demonstrate] the particulars of other things, rather other things [can be demonstrated] of them. But it is [with respect to] the intermediates that it is possible in both ways (for they themselves are said of other things and other things of them)...

In language and structure the opening of 43a25-42 ('among all beings...some....others...', *hapantōn tōn ontōn...ta men...ta de...*) is strongly reminiscent of *Categories* 2 (1a16ff.: 'among beings...some...others' *tōn ontōn...ta men...ta de....*). The absence of 'all' in *Categories* 2, given its presence in *Prior Analytics* I.27, is perhaps

[75] For instance, Philoponus does this in commenting on *Posterior Analytics* I.22.

significant: suggesting a generality for the latter division not reached by the former. In any case, it is clear that the threefold division of *Prior Analytics* I.27 is supposed to be applicable to everything there is.

To explore the division of beings outlined by *Prior Analytics* I.27 43a25-42, let us consider some examples. The following items are mentioned in the passage:

- Kallias, Socrates
- human being, animal
- (the) pale, (the) approaching

Kallias and Socrates are adduced as items which fall into **Group I**. For, other beings are 'predicated truly and universally' of them (e.g. 'Kallias is human being'); but they themselves are predicated 'of nothing else'. Except, Aristotle later adds, incidentally. Relying on a version of the *Posterior Analytics* I.22 contrast between genuine and pseudo-predication, Aristotle remarks that we sometimes say 'The pale is Socrates'. And the implication is clearly supposed to be that in such cases we are not really predicating. Human being, in contrast, is given as an example of something in **Group III**. For, it is both the case that human being is predicated of other things (e.g. Socrates, Kallias), and the case that other things (e.g. animal) are predicated of human being. Though the above text does not exactly say so, we are presumably supposed to put animal in **Group III** as well. Aristotle's usual view would be that body is predicated (truly and universally) of animal; and certainly animal is predicated of human being. What about **Group II**? Beings in **Group II** are supposed to be those which are themselves predicated of other things, but do not have others predicated of them. Commentators traditionally conjecture that the denizens of **Group II** are the categories themselves, thinking here of the categories along Porphyrian lines as maxima genera. Now, I would agree that Aristotelian maxima genera (whatever they are) must go in **Group II**. However, I think the commentators are wrong to think of maxima genera as the only denizens of **Group II**. The text announces a distinction dividing **all** beings. And Aristotle's examples '(the) pale' and '(the) approaching' fit in no other group. These are beings which are subject for no (genuine) predication: they are accidents in the terminology of *Posterior Analytics* I.22. Since these items are predicated of other beings while nothing else is predicated of them they must belong in **Group II**. Indeed, Aristotle seems have accidents in particular on his mind in his remarks on **Group II** at 43a37ff. Issues of demonstration *kata doxan* vs. demonstration *pros alētheian* (cf. *Posterior Analytics* I.19 81b18-29) concern especially whether once can demonstrate a conclusion with an accident as its subject.

23 The Main Argument reconstructed

Prior Analytics I.27 43a25-42 refers forward to *Posterior Analytics* I.22 as a text which will establish that reality has a metaphysical top: that every maximal predicational chain has an upper bound. In contrast, *Prior Analytics* I.27 43a25-42 positively endorses the view that reality has a metaphysical bottom: that every maximal predicational chain has a lower bound.

The assumption that every maximal predicational chain has a lower bound does not entail that reality satisfies the **DCC** (consider e.g. the relation 'is a super-set of' on $\omega + 1$). So it is notable that despite the background in *Prior Analytics* I.27, Aristotle thinks in *Posterior Analytics* I.19-22 that he still needs to establish that reality has no infinite predicational down-chains. I suggested above that the diagram Aristotle refers to in raising **Questions 1** and **2** may even highlight the fact that a structure can have a lowest element with an infinite descending chain above it.

The Main Argument is not usually read in the context of the Aristotelian assumption that reality has a bottom: that every maximal predicational chain has a lower bound. However, if it is so read, the text yields a valid argument that there are no infinite predicational up-chains and no infinite predicational down-chains.

First the case for infinite up-chains. For reductio, suppose reality contains an infinite predicational up-chain $\overline{P}$ with root r. There are three options: (i) $\overline{P}$ is a purely essentialist predicational chain (every link is a link of essential predication), (ii) $\overline{P}$ is a purely accidental predicational chain (every link is a link of accidental predication), or (iii) $\overline{P}$ is a mixed chain (some links are essential, some accidental). Clearly, (i) is impossible. For, *Posterior Analytics* I.22 82b37-a1 argues that for every x the number of beings predicated essentially of x is finite. But essential predication is transitive. So if there were a purely essentialist infinite predicational up-chain originating in r, the number of beings predicated essentially of r would have to be infinite. And this, given I.22 82b37-a1 cannot be. (ii) is also impossible. After all, it is the nature of accidents that they underlie nothing else (83b17-24, a30-32). But the predicational chains at issue here are chains of genuine predication (83a17-23) which mirror the structure of the real. (Indeed, no purely accidental predication chain can have length greater than 1 if we require that the links are links of genuine predication). Finally, consider (iii). If (iii) then $\overline{P}$ has at least one node which is an accident. But since accidents can be subjects for no genuine predications, the first accident that appears in chain $\overline{P}$ must be $\overline{P}$'s terminal node. So then, if $\overline{P}$ is infinite it would have to have infinitely many nodes before its first accidental node, i.e. its terminal node. But then $\overline{P}$ would have a sub-chain which is an infinite purely essentialist predicational chain originating in r. But we've already determined that this cannot be. Hence, (i)-(iii) are all impossible and there are no infinite predicational up-chains.

Next the case for infinite predicational down-chains. For reductio, suppose reality contains an infinite predicational down-chain $\overline{P}$ with root r. Again, there are three options: (i) $\overline{P}$ is a purely essentialist predicational chain, (ii)

$\overline{P}$ is a purely accidental predicational chain, or (iii) $\overline{P}$ is a mixed chain. (i) is impossible. For grant the assumption that every maximally large predicational chain has a lower bound predicated of nothing else. Take a maximally large $\overline{P}_+ \supseteq \overline{P}$ with lower bound x. Since $\overline{P}$ is purely essentialist, by transitivity of essence it holds that for all $y \in \overline{P}$: yEx. But $\overline{P}$ is infinite. So given I.22 82b37-a1 this cannot be. Again, (ii) is clearly impossible for exactly the same reason as above. So suppose (iii). Since no accident can be subject of any predication, $\overline{P}$ must contain exactly one accident: its first node r. So, again we are back to case (i) which we've already shown cannot hold. So, (i)-(iii) are all impossible and there are no infinite predicational down-chains.

The Main Arugment leaves us, finally, with the following view of reality predicational architecture (*Posterior Analytics* I.22 83b24-28):

> So, it has been argued that neither upwards nor downwards does one thing belong to another [ad infinitum]. For the accidents are predicated of these, as many as are in the essence of each thing, but those [items in the essence] are not infinite. And upwards there are both these things [items in the essence] and accidents, neither infinite[ly].

24 Metaphysical structure in I.22: a formal model

Definition 25 Essence-accident combinatorial structures

An *essence-accident combinatorial structure* is a quadruple (W, S, d, c) such that

1. $W \cap S = \emptyset$
2. d (definition) is an injective total function $d : W \rightarrow \wp(W)$ such that for all $x, y \in W$:
   (a) $x \in d(x)$, and
   (b) if $y \in d(x)$, then $d(y) \subseteq d(x)$
3. c (composition) is a bijective partial function $c : (W \times W) \rightarrow S$ such that
   (a) $(\exists w)(w = c(x,y)) \wedge (\exists w)(w = c(y,z)) \ \rightarrow \ (\exists w)(w = c(x,z))$
   (b) $x \in d(y) \wedge (\exists w)(w = c(y,z)) \ \rightarrow \ (\exists w)(w = c(x,z))$
   (c) $(\exists w)(w = c(x,y)) \ \rightarrow \ \neg(\exists w)(w = c(y,x))$

Several features of **Definition 25** bear comment. On the intended interpretation, W is (or represents) a collection of **w**hats—that is, items which are both what something is, and themselves possess a 'real definition': *ousiai* in the sense of *Posterior Analytics* I.22. Again this is the class of beings that are metaphysical subjects, not the class of ultimate metaphysical subjects. In contrast, S is (or represents) a collection of **s**uches. These are items which have no 'real definitions' in their appropriate category and cannot be subjects for either accidental or essential predications. That is, denizens of S are accidents in the sense of *Posterior Analytics* I.22.

The function d represents real definition. It takes each $w \in W$ to what it is: i.e. the collection of items predicated essentially of it. In making $\wp(W)$ the range of d, I mean to be highlighting rather than covering over the problem of unity of essence/unity of definition. In the context of I.22 83a24-32, the intended interpretation of d may be put:

$$d(x) = \{y : x \text{ is } \textit{hoper } y \vee x \text{ is } \textit{hoper} \text{ a certain } y\}$$

which is derived simply by abstraction on a24-25. Note that **Definition 25.2.b** is supposed to capture the reflexivity of *hoper*. So we can also state the intended interpretation:

$$d(x) = \{y : x = y \vee x \text{ is in the real definition of } y\}$$

As we saw in our reading of 82b27-83a1 on the finitude of essence, I.22 is evidently assuming that essential part-hood is transitive:i.e. if x is in the what-it-is of y (e.g. animal is in the what-is-it of human being) and y is in the what-is-it of z (e.g. human being is in the what-is-it of Socrates), then x is in the what-is-it of z (animal is in the what-is-it of Socrates). Hence, **Definition 25.2.c**. Injectivity of d is meant to capture Aristotle's assumption that a *horos* is an *idion*.

In our remarks above, we emphasized that the Aristotle of *Posterior Analytics* I.22 is apparently committed to a distinction between non-substance whats (redness, triangle) and accidents (red, triangular). How are such entities related? A reading of the texts in the *Metaphysics* (esp. Z.4-6) where the same distinction is present use a notion of metaphysical composition. Metaphysically, what makes it the case that the accident *walks* belongs to *Socrates* is the composition of one what (e.g. walking) with another (e.g. Socrates). This is the basic inspiration behind **Definition 25.3**. To make c behave as a composition function we require c to be injective (if $x \neq y$ we don't want $c(x, z) = c(y, z)$). And the ontological dependence of suches (the range of c) on whats (the domain of c), is captured by requiring c to be surjective. That c should be a (strictly) partial function is obvious. **Definition 25.3.c** ensures that c plausibly represents composition in the sense of *Metaphysics* Z.4-6; **25.3.b** gives that what is composed with an essential part of an object is composed with that object; and **25.3.a** gives transitivity of composition.

We now define predication relations on Essence-accident combinatorial structures.

Definition 26

1. xEy (x belongs essentially to y) iff $x \in d(y)$
2. xAy (x belongs accidentally to y) iff $(\exists w \in d(y))(\exists z)(x = c(z, w))$
3. xPy (x is genuinely predicated of y) iff $xEy \vee xAy$

And finally we prove the following results.

Theorem 4

E is a partial order of W

- *Proof.* Reflexivity by **Def. 25.2.a**; transitivity and anti-symmetry by **25.2.b**.

Theorem 5

$xEy \to \neg xAy$

- *Proof.* Since xEy, $x \in W$. But xAy only if $x \in S$. By **Def. 25.1**, $\perp$.

Theorem 6

$xAy \to \neg(\exists z)(zPx)$

- *Proof.* Let xAy. So $x \in S$. Suppose zPx. Then $x \in W$. By **Def. 25.1**, $\perp$.

Theorem 7

$xPy \to (\exists z)(zEy)$

- *Proof.* Obvious if xEy. Suppose xAy. Then $y \in W$. But d is total on W, and $y \in d(y)$. So yPy.

Theorem 8

P is serial on$W \cup S$

- *Proof.* If $x \in W$ then $x \in d(x)$, so xPx. If $x \in S$, by surjectivity of c: $(\exists w)(\exists v)(x = c(v,w))$. But $w \in d(w)$. So xAw and xPw.

Theorem 9

P is transitive

- *Proof.* Let xPy and yPz. By **Thm. 6** either: (i) xEy and yEz, or (ii) xAy and yEz. Given **Thm. 4** we have (i). So suppose (ii). Then $(\exists w \in d(y))(\exists v)(x = c(v,w))$ and $y \in d(z)$. But **Def, 25.2** $w \in d(w) \subseteq d(y) \subseteq d(z)$. So $(\exists w \in d(z))(\exists v)(x = c(v,w))$ and xAz.

Theorem 10

P is anti-symmetric

- *Proof.* Let xPy and yPx. By **Thm. 6**: $xAy \to \neg yPx$ and $yAx \to \neg xPy$. So xEy and yEx. By **Thm. 4** : $y = x$.

Given the above, it is clear that every essence-accident combinatorial structure can be viewed as a predication structure (in the sense of **Definition 7**). It is immediate that every essence-accident combinatorial structure can be expanded into a demonstration model (in the sense of **Definition 9**).

25 Conclusion

Posterior Analytics I.19-22 work to connect two issues in Aristotelian syllogistic that are not obviously connected: the (in)finitude of predicational chains and the (in)finitude of demonstrative processes. Chapters I.20-21 establish that in Aristotle's system, the demonstration of a syllogistic proposition φ can continue ad infinitum only if there is an infinite predicational chain. And I.22 argues that reality in fact contains no such chains.

It is my view that modern studies of *Posterior Analytics* I.19-22 have severely mistaken both the character of the problem Aristotle is confronting in these chapters, as well as his means of handling of the problem. Among other things, I hope that the above presentation helps to restore the ancient view that the Aristotle of *Posterior Analytics* I.19-22 was a logician of remarkable skill and insight.

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