A Final Coalgebra Theorem in the Context of Algebraic Set Theory

MSc Thesis *(Afstudeerscriptie)*

written by

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Chapter 1

Introduction

In his book *Non-Well-Founded Sets*, Aczel forms a model for his system of non-well-founded set theory **ZFCA** by constructing a final coalgebra for the powerclass functor acting on $Classes$, the category of classes and functions. Relying on this result, he included a proof showing that one can obtain such a final coalgebra for every endofunctor on $Classes$ that is set-based, monotone, and preserves inclusions and weak pullbacks. Such final coalgebras have been used in the semantics of coinductive types [DM05] and in modelling the behaviour of programs [JR97,Rut00]. Later on, in a paper together with Mendler [AM89], Aczel gave an improvement of this result by showing that it is still valid when the assumptions of monotonicity and of preservation are dropped. In this thesis, we shall further generalise by proving a Final Coalgebra Theorem in the context of Algebraic Set Theory.

Algebraic Set Theory, first presented by André Joyal and Ieke Moerdijk [JM95], is a novel framework where models of set theory are determined algebraically. This new approach claims to have the potential of describing various set theories in one uniform fashion. Indeed, theories such as **CZF**, **IZF**, **ZF**, **BIST**, **CST** and so on have been successfully modelled, as can be seen from, e.g., the work by Awodey et al. [ABSS07], [AF05], and [AW05].

However, it turns out that only Van den Berg and De Marchi [vdBM07] made the attempt to use the Joyal-Moerdijk axiomatisation to model a set theory containing Aczel's Anti-Foundation Axiom **AFA**. In the aforementioned paper, Van den Berg and De Marchi prove a final coalgebra theorem for a certain class of endofunctors on a Heyting pretopos with a class of small maps. As a special instance of this result, they construct a final coalgebra for the P_s functor determined by the class of small maps, and prove that it is a model of the weak non-well-founded set theory $\mathbf{CZF}_0$ + **AFA**. The proof of their theorem is, however, very indirect. Using the formalism of indexed categories they prove an Indexed Final Object Theorem from which their recover the Final Coalgebra Theorem as a special case. The aim of this thesis is to find a new, more insightful proof of this latter theorem.

In order to give the intuition for the algebraic proof, we start by reviewing Aczel and Mendler's result. The review that we provide improves their result by showing that the axiom of choice is unnecessary to their proof. We show it by using the Relation Reflection Scheme **RRS**, introduced and showed to be provable in **ZF** by Aczel [Acz08]. In fact, our proof is essentially constructive and can be carried in **CZF+RRS** modulo the fact that we have to assume the existence of coequalizers in *Classes* in order to form quotients freely.

After having reviewed Aczel and Mendler's paper, we recast their proof in the internal logic of a Heyting pretopos with a class of small maps satisfying the Joyal-Moerdijk axiomatisation from [JM95], equipped with a stable natural number object, and satisfying **RRS**. The functors to which the result applies are assumed to be indexed, set-based, and monomorphism preserving. Although Van den Berg and De Marchi do not need **RRS**, they use the preservation of weak pullbacks. Our proof is an improvement in this respect for we only require the preservation of monomorphisms, which is a weaker assumption.

In Chapter 2, we provide an introduction to categorical logic: many-sorted first-order languages are defined and their categorical interpretation is presented. Chapter 3 introduces the Joyal-Moerdijk small maps axiomatisation. We review Aczel and Mendler's result in Chapter 4 and dedicate Chapter 5 to the proof of our theorem.

Chapter 2

Categorical Logic

The aim of this chapter is to give an introduction to categorical logic. After recalling the definition of a many-sorted first-order language and defining several syntactic features, we present the categorical generalisation of the standard set-theoretic semantics for first-order languages. Next, we explain how a functor can be safely used in the internal logic of a category. We finish with several examples of characterisation of diagrammatic facts. This presentation is based on [Joh02b, Section D1.4] and [MR77].

2.1 Many-Sorted First-Order Languages

We start with the definition of a first-order signature.

Definition 2.1.1. A first-order *signature* Σ consists of the following data:

(i) A set Σ-Sort of *sorts*.

(ii) A set Σ-Fun of *function symbols*; we write

$$f : A_1, \ldots, A_n \to B$$

to indicate that f has type $A_1, \ldots, A_n, B$.

(iii) A set Σ-Rel of *relation symbols*; we write

$$R \rightarrowtail A_1, \ldots, A_n$$

to indicate that R has type $A_1, \ldots, A_n$.

For each sort A in Σ-Sort, we assume to be given an unbounded supply of variables of sort A.

Definition 2.1.2. The collection of *terms* over a signature Σ is defined recursively as follows:

(i) If x is a variable of sort A, then $x : A$ is a term of sort A.

(ii) If $f : A_1, \ldots, A_n \to B$ is a function symbol and $t_1 : A_1, \ldots, t_n : A_n$ are terms, then $f(t_1, \ldots, t_n) : B$ is a term of sort B.

We go on with the formulas over Σ.

Definition 2.1.3. The class F of *formulas* over a signature Σ is recursively defined as follows:

(i) *Relations*: If $R \rightarrowtail A_1, \ldots, A_n$ and $t_1 : A_1, \ldots, t_n : A_n$ are terms, then $R(t_1, \ldots, t_n)$ is in F.

(ii) *Equality*: If s and t are terms of the same sort, then $s = t$ is in F.

(iii) *Truth*: $\top$ is in F.

(iv) *Binary conjunction*: If ϕ and ψ are in F, then $\phi \wedge \psi$ is in F.

(v) *Falsity*: $\bot$ is in F.

(vi) *Binary disjunction*: If ϕ and ψ are in F, then $\phi \vee \psi$ is in F.

(vii) *Implication*: If ϕ and ψ are in F, then $\phi \to \psi$ is in F.

(viii) *Negation*: If ϕ is in F, then $\neg\phi$ is in F.

(ix) *Existential quantification*: If ϕ is in F and x is a variable, then $(\exists x)\phi$ is in F.

(x) *Universal quantification*: If ϕ is in F and x is a variable, then $(\forall x)\phi$ is in F.

We shall not consider terms and formulas by themselves, but rather terms and formulas-in-context.

Definition 2.1.4. A *context* is a finite tuple $\bar{x} = x_1, \ldots, x_n$ of distinct variables. Given two disjoint contexts $\bar{x}$ and $\bar{y}$, the result of their concatenation is denoted $\bar{x}\bar{y}$. The *type* of a context $\bar{x}$ is the finite list of sorts of the variable appearing in $\bar{x}$ which preserves the order of the context.

Given a formula ϕ, a context $\bar{x}$ is called *suitable* for ϕ if all free variables of ϕ appear in $\bar{x}$. A *formula-in-context* is an expression $\bar{x}.\phi$, where ϕ is a formula and $\bar{x}$ is a suitable context for ϕ. Similarly, one defines a *term-in-context* $\bar{x}.t$ as a term t and a context $\bar{x}$ where all the variables of t appear in $\bar{x}$.

We end this section with the definition of a sequent, which expresses the notion of logical entailment.

Definition 2.1.5. A *sequent* over a signature Σ is an expression of the form

$$\phi \vdash_{\bar{x}} \psi,$$

where ϕ and ψ are formulas over Σ and $\bar{x}$ is a context suitable for both of them.

2.2 Categorical Semantics

We start by generalising the usual definition of set-theoretical structure for a many-sorted first-order signature Σ to any category with finite products.

Definition 2.2.1. Let $\mathcal{C}$ be a category with finite products and Σ a signature. A Σ-*structure* M on $\mathcal{C}$ is specified by:

(i) An assignment sending each sort $A \in \Sigma$-Sort to an object MA of $\mathcal{C}$.

(ii) An assignment sending each function symbol $f : A_1, \ldots, A_n \to B \in \Sigma$-Fun to a morphism $Mf : MA_1 \times \ldots \times MA_n \to MB$ of $\mathcal{C}$.

(iii) An assigment sending each relation symbol $R : A_1, \ldots, A_n \in \Sigma$-Rel to a subobject $MR \rightarrowtail MA_1 \times \ldots \times MA_n$ in $\mathcal{C}$.

We continue with the interpretation of terms and formulas-in-context in a Σ-structure. The interpretation of the function symbols naturally extends to the interpretation of the terms-in-context over Σ, and nothing more than finite products is required on the underlying category.

Definition 2.2.2. Let M be a Σ-structure on a category $\mathcal{C}$ with finite products. A term-in-context $\bar{x}.t$ over Σ with $\bar{x} = x_1, \ldots, x_n$, $x_i : A_i$, and $t : B$, is interpreted as a morphism

$$[\![\bar{x}.t]\!]_M : MA_1 \times \ldots \times MA_n \to MB$$

in $\mathcal{C}$, recursively defined as follows (we omit the subscript M from $[\![\bar{x}.t]\!]$ when it is clear which structure is being referred to):

(i) If t is a variable, it must appear in $\bar{x}$, say $t = x_i$, then $[\![\bar{x}.t]\!] = \pi_i$ is the ith projection
$$MA_1 \times \ldots \times MA_n \xrightarrow{p_i} MB.$$
Note that in the case where the context consists only in the variable x_i, $[\![\bar{x}.t]\!]$ is simply 1_{A_i}.

(ii) If t is $f(t_1, \ldots, t_m)$ with $t_i : C_i$, then $[\![\bar{x}.t]\!]$ is the composition

$$MA_1 \times \ldots \times MA_n \xrightarrow{\langle [\![\bar{x}.t_1]\!], \ldots, [\![\bar{x}.t_m]\!] \rangle} MC_1 \times \ldots \times MC_m \xrightarrow{Mf} MB.$$

Next, we turn to the interpretation of formulas-in-context in a Σ-structure. The number of logical operators that we are able to interpret depends on the amount of structure in $\mathcal{C}$.

Definition 2.2.3. Let M be a Σ-structure on a category $\mathcal{C}$ with at least finite limits. A formula-in-context $\bar{x}.\phi$ over Σ with $\bar{x} = x_1, \ldots, x_n$ and $x_i : A_i$ is interpreted as a subobject

$$[\![\bar{x}.\phi]\!] \rightarrowtail MA_1 \times \ldots \times MA_n,$$

recursively defined as follows:

(i) If ϕ is $R(t_1, \ldots, t_m)$ where R is a relation symbol of type $B_1, \ldots, B_m$, then $[\![\bar{x}.\phi]\!]$ is the pullback

$$\begin{tikzcd}
[\![\bar{x}.\phi]\!] \arrow[r] \arrow[d, tail] & MR \arrow[d, tail] \\
MA_1 \times \ldots \times MA_n \arrow[r, "\langle [\![\bar{x}.t_1]\!], \ldots, [\![\bar{x}.t_m]\!] \rangle"] & MB_1 \times \ldots \times MB_m.
\end{tikzcd}$$

(ii) If ϕ is $s = t$, where s and t are terms of sort B, then $[\![\bar{x}.\phi]\!]$ is the equalizer of

$$\begin{tikzcd}
MA_1 \times \ldots \times MA_n \arrow[r, shift left, "[\![\bar{x}.s]\!]"] \arrow[r, shift right, "[\![\bar{x}.t]\!]"'] & MB.
\end{tikzcd}$$

We could reduce the semantics of equality to that of relations by introducing a symbol $=_A$ for each sort A and requiring $M(=_A)$ to be the diagonal of MA; a formula-in-context of the form $\bar{x}.s =_A t$ would then be interpreted as the pullback of the diagonal along $\langle [\![\bar{x}.s]\!], [\![\bar{x}.t]\!] \rangle$.

(iii) If ϕ is $\top$, then $[\![\bar{x}.\phi]\!]$ is the top element of $\mathrm{Sub}(MA_1 \times \ldots \times MA_n)$.

(iv) If ϕ is $\psi \wedge \chi$, then $[\![\bar{x}.\phi]\!]$ is the meet of $[\![\bar{x}.\psi]\!]$ and $[\![\bar{x}.\chi]\!]$ in $\mathrm{Sub}(MA_1 \times \ldots \times MA_n)$.

(v) If ϕ is $\bot$ and $\mathrm{Sub}(X)$ has a bottom element for each X in $\mathcal{C}$, then $[\![\bar{x}.\phi]\!]$ is the bottom element of $\mathrm{Sub}(MA_1 \times \ldots \times MA_n)$.

(vi) If ϕ is $\psi \vee \chi$ and $\mathrm{Sub}(X)$ has a finite joins for each X in $\mathcal{C}$, then $[\![\bar{x}.\phi]\!]$ is the join of $[\![\bar{x}.\psi]\!]$ and $[\![\bar{x}.\chi]\!]$ in $\mathrm{Sub}(MA_1 \times \ldots \times MA_n)$.

(vii) If ϕ is $\psi \to \chi$ and $\mathrm{Sub}(X)$ is a Heyting algebra for each X in $\mathcal{C}$, then $[\![\bar{x}.\phi]\!]$ is the Heyting implication $[\![\bar{x}.\psi]\!] \to [\![\bar{x}.\chi]\!]$ in $\mathrm{Sub}(MA_1 \times \ldots \times MA_n)$.

(viii) If ϕ is $\neg\psi$ and $\mathrm{Sub}(X)$ is a Heyting algebra for each X in $\mathcal{C}$, then $[\![\bar{x}.\phi]\!]$ is the Heyting negation $\neg[\![\bar{x}.\psi]\!]$ in $\mathrm{Sub}(MA_1 \times \ldots \times MA_n)$.

(ix) If ϕ is $(\exists y)\psi$ and pulling back along π has a left adjoint $\exists_\pi$ satisfying the Beck-Chevalley condition for each projection π in $\mathcal{C}$, then $[\![\bar{x}.\phi]\!]$ is $\exists_\pi([\![\bar{x}y.\psi]\!])$ where π is the projection

$$MA_1 \times \cdots \times MA_n \times MB \xrightarrow{p} MA_1 \times \cdots \times MA_n.$$

(x) If ϕ is $(\forall y)\psi$ and pulling back along π has a right adjoint $\forall_\pi$ satisfying the Beck-Chevalley condition for each projection π in $\mathcal{C}$, then $[\![\bar{x}.\phi]\!]$ is $\forall_\pi([\![\bar{x}y.\psi]\!])$ where π is the projection of (ix).

We have required that the quantifiers satisfy the Beck-Chevalley condition. This condition says that for any pullback square

$$\begin{tikzcd} X \times_Y Z \arrow[r, "f'"] \arrow[d, "g'"'] & Z \arrow[d, "g"] \\ X \arrow[r, "f"] & Y, \end{tikzcd}$$

the diagram

$$\begin{tikzcd} \mathrm{Sub}(X \times_Y Z) \arrow[d, "Q_{g'}"'] & \mathrm{Sub}(Z) \arrow[l, "f'^*"'] \arrow[d, "Q_g"] \\ \mathrm{Sub}(X) & \mathrm{Sub}(Y), \arrow[l, "f^*"] \end{tikzcd}$$

satisfies $f^*Q_gA = Q_{g'}f'^*A$, for all subobjects A in $\mathrm{Sub}(Z)$, with $Q = \exists, \forall$, i.e., it says that quantification commutes with pullback.

Satisfaction of a sequent in a model is defined as follows.

Definition 2.2.4. Let M be a Σ-structure on a category $\mathcal{C}$.

(i) A sequent $\phi \vdash_{\bar{x}} \psi$ over Σ interpretable in $\mathcal{C}$ with $x_i : A_i$ is satisfied in M, denoted $\phi \vDash_{\mathcal{M},\bar{x}} \psi$, if $[\![\bar{x}.\phi]\!]_M \leq [\![\bar{x}.\psi]\!]_M$ in $\mathrm{Sub}(MA_1 \times \ldots \times MA_n)$.

(ii) $\mathcal{M}$ is a model of a theory T over Σ interpretable in $\mathcal{C}$ if M satisfies all the axioms of T.

In order for the connectives to commutes with substitution, we shall ask $\mathcal{C}$ not only to possess the structure required by the previous definition, but also its stability under pullback, i.e., we require pullback functors to be homomorphisms for the subobject categories of $\mathcal{C}$. Similarly, the Beck-Chevalley condition is imposed on the left and right adjoints to pullback to ensure the commutation of the quantifiers with substitution as long as no bound variable is captured.

The Substitution Rule for sequent calculi is the following rule:

$$\frac{\phi \vdash_{\bar{x}} \psi}{\phi[\bar{s}/\bar{x}] \vdash_{\bar{y}} \psi[\bar{s}/\bar{x}],}$$

where $\bar{y}$ is any string of variables including all the variables occurring in the string of terms $\bar{s}$. Semantically, substitution is interpreted by pullback along projections. Indeed, we have the following two substitution properties from which soundness of the above rule follows from the fact that taking pullback is order preserving:

Proposition 2.2.5. *Let $\bar{y}$ be a suitable context for a term $t : C$ with $y_i : B_i$, $\bar{s}$ be a string of terms of the same length and type as $\bar{y}$, and $\bar{x}$ be a suitable context for each s_i with $x_i : A_i$. Then $[\![\bar{x}.t[\bar{s}/\bar{y}]]\!]$ is the composite*

$$MA_1 \times \ldots \times MA_n \xrightarrow{\langle [\![\bar{x}.s_1]\!] \ldots [\![\bar{x}.s_m]\!]\rangle} MB_1 \times \ldots \times MB_m \xrightarrow{[\![\bar{y}.t]\!]} MC.$$

Proof. We go by induction on the structure of t. If t is a variable, then it has to be among $\bar{y}$, say $t = y_i$. In this case, $\bar{y}.t[\bar{s}/\bar{y}] = \bar{x}.s_i$. By definition,

$$[\![\bar{y}.y_i]\!] = MB_1 \times \ldots \times MB_n \xrightarrow{p_i} MB_i,$$

and

$$[\![\bar{x}.s_i]\!] = MA_1 \times \ldots \times MA_n \xrightarrow{[\![\bar{x}.s_i]\!]} MB_i.$$

Therefore,

$$[\![\bar{x}.s_i]\!] = MA_1 \times \ldots \times MA_n \xrightarrow{\langle [\![\bar{x}.s_1]\!], \ldots, [\![\bar{x}.s_m]\!]\rangle} MB_1 \times \ldots \times MB_m \xrightarrow{p_i = [\![\bar{y}.t]\!]} MB_i.$$

If $t = f(t_1, \ldots, t_k)$ with $t_i : C_i$. Then

$$\bar{x}.f(t_1, \ldots, t_k)[\bar{s}/\bar{y}] = \bar{x}.f(t_1[\bar{s}/\bar{y}], \ldots, t_k[\bar{s}/\bar{y}]).$$

By definition, $[\![\bar{x}.f(t_1[\bar{s}/\bar{y}], \ldots, t_k[\bar{s}/\bar{y}])]\!]$ is the morphism

$$MA_1 \times \ldots \times MA_n \xrightarrow{\langle [\![\bar{x}.t_1[\bar{s}/\bar{y}]]\!], \ldots, [\![\bar{x}.t_k[\bar{s}/\bar{y}]]\!]\rangle} MC_1 \times \ldots \times MC_m \xrightarrow{[\![Mf]\!]} MC,$$

but, by induction hypothesis, for each t_i, $[\![\bar{x}.t_i[\bar{s}/\bar{y}]]\!]$ is the morphism

$$MA_1 \times \ldots \times MA_n \xrightarrow{\langle [\![\bar{x}.s_1]\!] \times \ldots \times [\![\bar{x}.s_m]\!]\rangle} MB_1 \times \ldots \times MB_m \xrightarrow{[\![\bar{y}.t_i]\!]} MC_i.$$

Therefore, $[\![\bar{x}.f(t_1[\bar{s}/\bar{y}], \ldots, t_k[\bar{s}/\bar{y}])]\!]$ equals

$$MA_1 \times \ldots \times MA_n \xrightarrow{\langle [\![\bar{x}.s_1]\!] \times \ldots \times [\![\bar{x}.s_m]\!]\rangle} MB_1 \times \ldots \times MB_m \xrightarrow{[\![\bar{y}.t_i]\!]} MC_1 \times \ldots \times MC_k \xrightarrow{[\![Mf]\!]} MC,$$

as required. $\square$

Requiring the structure of the subobject categories of $\mathcal{C}$ to be stable under pullback, the substitution property extends to formulas.

Proposition 2.2.6. *Let $\bar{y}.\phi$ be a formula-in-context over Σ with $y_i : B_i$ and $\mathcal{C}$ a Heyting category, $\bar{s}$ be a string of terms of the same length and type as $\bar{y}$, and $\bar{x}$ be a context suitable for all the terms in $\bar{s}$ with $x_i : A_i$. Then, for any Σ-structure M on $\mathcal{C}$, $[\![\bar{x}.\phi[\bar{s}/\bar{y}]]\!]$ is given by the pullback square*

$$\begin{tikzcd}
{[\![\bar{x}.\phi[\bar{s}/\bar{y}]]\!]} \arrow[r] \arrow[d, tail] & {[\![\bar{y}.\phi]\!]} \arrow[d, tail] \\
MA_1 \times \ldots \times MA_n \arrow[r, "{\langle [\![\bar{x}.s_1]\!], \ldots, [\![\bar{x}.s_m]\!] \rangle}"] & MB_1 \times \ldots \times MB_m.
\end{tikzcd}$$

Proof. By induction on ϕ. If $\phi := R(t_1, \ldots, t_k)$ with $R : C_1, \ldots, C_k$. Then,

$$\bar{x}.R(t_1, \ldots, t_k)[\bar{s}/\bar{y}] = \bar{x}.R(t_1[\bar{s}/\bar{y}], \ldots, t_k[\bar{s}/\bar{y}]).$$

By definition, $[\![\bar{x}.R(t_1[\bar{s}/\bar{y}], \ldots, t_k[\bar{s}/\bar{y}])]\!]$ is given by the pullback square

$$\begin{tikzcd}
{[\![\bar{x}.R(t_1[\bar{s}/\bar{y}], \ldots, t_k[\bar{s}/\bar{y}])]\!]} \arrow[r] \arrow[d, tail] & MR \arrow[d, tail] \\
MA_1 \times \ldots \times MA_n \arrow[r, "{\langle [\![\bar{x}.t_1[\bar{s}/\bar{y}]]\!], \ldots, [\![\bar{x}.t_k[\bar{s}/\bar{y}]]\!] \rangle}"] & MC_1 \times \ldots \times MC_k,
\end{tikzcd}$$

hence, by Proposition 2.2.5, $[\![\bar{x}.R(t_1[\bar{s}/\bar{y}], \ldots, t_k[\bar{s}/\bar{y}])]\!]$ is the pullback

$$\begin{tikzcd}
{[\![\bar{x}.R(t_1[\bar{s}/\bar{y}], \ldots, t_k[\bar{s}/\bar{y}])]\!]} \arrow[rr] \arrow[d, tail] & & MR \arrow[d, tail] \\
MA_1 \times \ldots \times MA_n \arrow[r, "{\langle [\![\bar{x}.s_1]\!], \ldots, [\![\bar{x}.s_m]\!] \rangle}"] & MB_1 \times \ldots \times MB_m \arrow[r, "{\langle [\![\bar{y}.t_1]\!], \ldots, [\![\bar{y}.t_k]\!] \rangle}"] & MC_1 \times \ldots \times MC_k,
\end{tikzcd}$$

and since $[\![\bar{y}.R(t_1, \ldots, t_k)]\!]$ is the pullback

$$\begin{tikzcd}
{[\![\bar{y}.R(t_1, \ldots, t_k)]\!]} \arrow[r] \arrow[d, tail] & MR \arrow[d, tail] \\
MB_1 \times \ldots \times MB_m \arrow[r, "{\langle [\![\bar{y}.t_1]\!], \ldots, [\![\bar{y}.t_k]\!] \rangle}"] & MC_1 \times \ldots \times MC_k,
\end{tikzcd}$$

by the pullback lemma, we have the pullback square

$$\begin{tikzcd}
{[\![\bar{x}.R(t_1[\bar{s}/\bar{y}], \ldots, t_k[\bar{s}/\bar{y}])]\!]} \arrow[r] \arrow[d, tail] & {[\![\bar{y}.R(t_1, \ldots, t_k)]\!]} \arrow[d, tail] \\
MA_1 \times \ldots \times MA_n \arrow[r, "{\langle [\![\bar{x}.s_1]\!], \ldots, [\![\bar{x}.s_k]\!] \rangle}"] & MB_1 \times \ldots \times MB_m.
\end{tikzcd}$$

The cases of the logical connectives follow readily from the fact that the structure of the subobject categories of $\mathcal{C}$ is stable under pullback.

If $\phi := (\exists z)\psi$ with z not occurring in $\bar{y}$, then $((\exists z)\phi)[\bar{s}/\bar{y}] = (\exists z)\phi[\bar{s}/\bar{y}]$. Hence, $[\![(\exists z)\phi[\bar{s}/\bar{y}]]\!] = \exists_z [\![\phi[\bar{s}/\bar{y}]]\!]$. Recall that in Definition 2.2.3, we required both quantifiers to satisfy the Beck-Chevalley condition so as to make them commutes with pullback. By induction hypothesis, $[\![\bar{x}.\phi[\bar{s}/\bar{y}]]\!]$ is the pullback of $[\![\bar{y}.\phi]\!]$ along $\langle [\![\bar{x}.s_1]\!], \ldots, [\![\bar{x}.s_m]\!], 1_Z \rangle$, thus $\exists_z [\![\phi[\bar{s}/\bar{y}]]\!]$ is the pullback of $[\![\bar{y}.\exists_z \phi]\!]$ along $\langle [\![\bar{x}.s_1]\!], \ldots, [\![\bar{x}.s_m]\!] \rangle$, as required.

The case of the universal quantifier is similar. $\square$

Requiring the structure of the subobject categories to be stable under pullback, we obtain the commutation of the logical connectives with substitution:

$$\begin{aligned}
[\![\bar{x}.\psi \wedge \chi[\bar{t}/\bar{y}]]\!] &= [\![\bar{x}.\psi[\bar{t}/\bar{y}]]\!] \wedge [\![\bar{x}.\chi[\bar{t}/\bar{y}]]\!], \\
[\![\bar{x}.(\psi \vee \chi)[\bar{t}/\bar{y}]]\!] &= [\![\bar{x}.\psi[\bar{t}/\bar{y}]]\!] \vee [\![\bar{x}.\chi[\bar{t}/\bar{y}]]\!], \\
[\![\bar{x}.(\psi \to \chi)[\bar{t}/\bar{y}]]\!] &= [\![\bar{x}.\psi[\bar{t}/\bar{y}]]\!] \to [\![\bar{x}.\chi[\bar{t}/\bar{y}]]\!], \text{ and} \\
[\![\bar{x}.\neg\psi[\bar{t}/\bar{y}]]\!] &= \neg [\![\bar{x}.\psi[\bar{t}/\bar{y}]]\!].
\end{aligned}$$

Regarding quantification, the Beck-Chevaley condition implies that quantification commutes with substitution, provided that no bound variable are captured. Indeed, starting with the pullback square

$$\begin{array}{ccc}
M\bar{X} \times MY & \xrightarrow{\langle [\![\bar{x}.t_1]\!], \ldots, [\![\bar{x}.t_n]\!] \rangle \times 1_{MY}} & M\bar{Z} \times MY \\
{\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_1} \\
M\bar{X} & \xrightarrow{\langle [\![\bar{x}.t_1]\!], \ldots, [\![\bar{x}.t_n]\!] \rangle} & M\bar{Z},
\end{array}$$

the Beck-Chevalley condition implies that $[\![\bar{x}.Q(\phi)[\bar{t}/\bar{x}]]\!] = [\![\bar{x}.Q(\phi[\bar{t}/\bar{x}])]\!]$.

The foregoing shows that the substitution rule is sound. For the rest of the soundness proof, the rest of the soundness proof can be found in [Joh02b, Section D1.3, Proposition 1.3.2]. One of the advantages of being able to build models out of abstract categories is that the completeness argument becomes simpler than in the standard case; the reader can find the proof in [Joh02b, Section D1.4, Theorem 1.4.11], and a simplified version in [Pit00, Section 5.5.7].

We define the following for future reference.

Definition 2.2.7. Let $\mathcal{C}$ be a category with finite limits.

(i) A category $\mathcal{C}$ is called a *regular category* if it is finitely complete, the kernel pair of any morphism – see Definition 3.1.4 below – has a coequalizer which is stable under pullback. Equivalently, each subobject category has a stable top element, stable finite meets, and pullback functors have a left adjoint satisfying the Beck-Chevalley condition; i.e., $\mathcal{C}$ has the required structure to interpret *regular logic*, the fragment of many-sorted first-order logic closed under top, conjunction, and existential quantification.

(ii) A category $\mathcal{C}$ is called a *coherent category* if it is a regular category in which each subobject category has a stable bottom element and stable finite joins; i.e., $\mathcal{C}$ has the required structure to interpret *coherent logic*, the fragment of many-sorted first-order logic closed under top, conjunction, bottom, disjunction, and existential quantification.

(iii) A category $\mathcal{C}$ is called a *Heyting category* if each subobject category is a Heyting algebra such that pullback functors are Heyting algebra homomorphisms and have both a left and a right adjoint satisfying the Beck-Chevalley condition; i.e., $\mathcal{C}$ has the required structure to interpret full intuitionistic many-sorted first-order logic.

(iv) A category $\mathcal{C}$ is called a *Boolean category* if it is a Heyting category in which each subobject category is a boolean algebra such that pullback functors are Boolean algebra homomorphisms; i.e., $\mathcal{C}$ has the required structure to interpret full classical many-sorted first-order logic.

The categorical treatment of the logical connectives brings no new insights, it only replies on the fact that propositional calculi can be interpreted in Heyting and Boolean algebras. The original contribution of categorical logic is due to Lawvere's observation – see [Law65] – that quantification can be treated as the adjoint of pullback along the substitution. Let us expand on this observation.

Syntactically, a quantifier Q can be seen as an operator which takes a formula-in-context $\bar{x}y.\phi$ to return the formula-in-context $\bar{x}.(\exists y)\phi$ with, say, $x_i : A_i$ and $y : B$. Semantically, if a similar treatment is possible, Q has to be an operator sending $[\![\bar{x}y.\phi]\!]$ to $[\![\bar{x}.(\exists y)\phi]\!]$, therefore, it must be a map from $\mathrm{Sub}(MA_1 \times \ldots \times MA_n \times MB)$ to $\mathrm{Sub}(MA_1 \times \ldots \times MA_n)$. To see that this mapping is functorial, consider the following two rules:

$$\frac{\phi \vdash_{\bar{x}y} \psi}{(\exists y)\phi \vdash_{\bar{x}} (\exists y)\psi} \qquad \frac{\phi \vdash_{\bar{x}y} \psi}{(\forall y)\phi \vdash_{\bar{x}} (\forall y)\psi.}$$

As these rules are derivable in intuitionistic logic, by soundness, we must have that, in $\mathrm{Sub}(MA_1 \times \ldots \times MA_n) \times MB$, $[\![\bar{x}y.\phi]\!] \leq [\![\bar{x}y.\psi]\!]$ implies $[\![\bar{x}.(Qy)\phi]\!] \leq [\![\bar{x}.(Qy)\psi]\!]$, i.e., Qy be monotone, so it is a functor.

Let us now show that $\exists y$ and $\forall y$ are the left and right adjoints to pullback. We are familiar with the following two-way rules on introduction and elimination of quantifiers:

$$\frac{\phi \vdash_{\bar{x}y} \psi}{\overline{(\exists y)\phi \vdash_{\bar{x}} \psi}} \qquad \frac{\phi \vdash_{\bar{x}y} \psi}{\overline{\phi \vdash_{\bar{x}} (\forall y)\psi.}}$$

Note that the usual requirement that y must not occur free in ψ and ϕ for existential and universal quantification respectively is implicit, since under the inference line both formulas appear in the context $\bar{x}$ which it suitable for both of them.

Although, y does not occur in ψ, above the inference line, ψ does appear in the context $\bar{x}y$. This occurance of ψ in this context can be seen as an occurrence of ψ in the context $\bar{x}$ when $\bar{x}$ is extended to $\bar{x}y$. This extension is made possible by the *weakening rule* which is nothing but the following trivial substitution:

$$\frac{\phi \vdash_{\bar{x}} \psi}{\phi[\bar{x}/\bar{x}] \vdash_{y\bar{x}} \psi[\bar{x}/\bar{x}].}$$

Semantically, the weakening rule is interpreted as the following pullback:

$$\begin{tikzcd}
[\![\bar{x}y.\phi[\bar{x}y/\bar{x}]]\!] \arrow[rr] \arrow[d, tail] & & [\![\bar{x}.\phi]\!] \arrow[d, tail] \\
MA_1 \times \ldots \times MA_n \times MB \arrow[rr, "\pi"'] & & MA_1 \times \ldots \times MA_n,
\end{tikzcd}$$

where π is the projection of the first n factors of $MA_1 \times \ldots \times MA_n \times MB$. Note in passing that this implies that the interpretation of a formula in the context consisting of its free variable determines its interpretation in any other context. Thus, $[\![\bar{x}y.\phi]\!]$ can be written as $\pi^*[\![\bar{x}.\phi]\!]$ where π^* is the pullback functor from $\mathrm{Sub}(MA_1 \times \ldots \times MA_n)$ to $\mathrm{Sub}(MA_1 \times \ldots \times MA_n \times MB)$ determined by π. This begin given, soundness of the rules of introduction and elimination of quantifiers means that

$$\frac{[\![\bar{x}y.\phi]\!] \leq p^*[\![\bar{x}.\psi]\!]}{\overline{[\![\bar{x}.\exists y(\phi)]\!] \leq [\![\bar{x}.\psi]\!],}}$$

and

$$\frac{p^*[\![\bar{x}.\phi]\!] \leq [\![\bar{x}y.\psi]\!]}{\overline{[\![\bar{x}.\phi]\!] \leq [\![\bar{x}.\forall y(\psi)]\!];}}$$

i.e., they precisely say that the existential and universal quantifiers are, respectively, left and right adjoints to π^*.

Next, we show how an endofunctor on $\mathcal{C}$ can be used in the internal logic of $\mathcal{C}$. In defining the categorical semantics, we have been very cautious about preservation under pullback; indeed, we have required the structure of the subobject categories to be stable under pullback and the quantifiers to satisfy the beck-Chevalley. We have seen that the reason for this requirement as to do with soundness of the substitution rule. When a variable $x : X$ is introduced, we implicitly start to work in the slice $\mathcal{C}/X$. Now if we wish to introduce a new variable $y : Y$, i.e., move to the slice $\mathcal{C}/YX$, we want to make sure that what we have proved in the context x, i.e. in $\mathcal{C}/X$, is still correct in the extended context yx, i.e. in $\mathcal{C}/YX$. This prerequisite corresponds to soundness of the instance of the substitution rule that we have called weakening of contexts.

Given an endofunctor F on $\mathcal{C}$, suppose that we wish to consider, e.g., the formula-in-context $x.FPx$, for a predicate $P : X$, and next the formula-in-context $yx.FPx$ where the previous context is extended to yx. Given the foregoing, we want to be sure that pulling back FP along the projection $\pi_2 : Y \times X \to X$ is the same as applying F to the pullback of P along π_2. This property can be forced by assuming that F is *indexed*.

Definition 2.2.8. A endofunctor F on $\mathcal{C}$ is *indexed* if it is given as a collection

$$(F^A : \mathcal{C}/A \to \mathcal{C}/A \mid A \in \mathcal{C})$$

such that for all $f : J \to I$ we have

$$f^* \circ F^I \cong F^J \circ f^*.$$

Remark 2.2.9. This definition is a special case of endofunctor on an indexed category $\mathcal{C}$ where $\mathcal{C}$ is self-indexed.

We end this section by showing how the internal logic of a category $\mathcal{C}$ can be used to express diagrammatic facts in $\mathcal{C}$.

Definition 2.2.10. Given a category $\mathcal{C}$ with finite limits, we define $\mathcal{L}(\mathcal{C})$ the *canonical language* of $\mathcal{C}$ consisting of a sort for each object of $\mathcal{C}$, a function symbol for each morphism of $\mathcal{C}$, and a relation symbol for each subobject of $\mathcal{C}$.

Then the "identity" map from $\mathcal{L}(\mathcal{C})$ to $\mathcal{C}$ gives rise to the obvious canonical model $\mathcal{M}_\mathcal{C}$ of $\mathcal{L}(\mathcal{C})$, which we think as $\mathcal{C}$ itself, where each sort, function, and relation symbol gets interpreted by "itself". The following proposition gives a list of characterisations of diagrammatic facts in $\mathcal{C}$ via the satisfaction of sequents of $\mathcal{L}(\mathcal{C})$ in $\mathcal{M}_\mathcal{C}$. The logical expression of these facts arise naturally, and it enables to work in $\mathcal{C}$ as if it were the category *Sets*. The proofs of these characterisations can be found in [MR77, Chapters 2 & 3]. As we think of $\mathcal{M}_\mathcal{C}$ as $\mathcal{C}$ itself, when a sequent $\phi \vDash_{\bar{x}} \psi$ is satisfied in $\mathcal{M}_\mathcal{C}$, we say that $\mathcal{C}$ satisfies it or that it holds in the internal logic of $\mathcal{C}$.

Proposition 2.2.11. *(i) An object A in $\mathcal{C}$ is an initial object if and only if the sequent*

$$x = x \vdash_{x:A} \top$$

holds in $\mathcal{C}$.

(ii) *An object A in $\mathcal{C}$ is a terminal object if and only if the sequents*

$$\vdash_{x:A,y:A} x = y$$

and

$$\vdash_{x:A} (\exists x)(x = x)$$

hold in $\mathcal{C}$.

(iii) *An morphism $m : A \to B$ in $\mathcal{C}$ is a monomorphism if and only if the sequent*

$$m(x) = m(y) \vdash_{x:A,y:A} x = y$$

holds in $\mathcal{C}$.

(iv) *An morphism $e : A \to B$ in $\mathcal{C}$ is a cover (see Definition 3.1.2) if and only if the sequent*

$$\vdash_{x:B} \exists y(e(y) = x)$$

holds in $\mathcal{C}$.

(v) *An morphism $m : E \to A$ in $\mathcal{C}$ equalizes $f, g : A \to B$ if and only if the sequents*

$$m(x) = m(y) \vdash_{xy:E} x = y$$

and

$$f(x) = g(x) \vdash_{x:A} \exists y(m(y) = x)$$

hold in $\mathcal{C}$.

(vi) *The following diagram is a pullback square in $\mathcal{C}$*

$$\begin{array}{ccc} A & \xrightarrow{b} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle d} \\ C & \xrightarrow{c} & D \end{array}$$

if and only if the sequents

$$c(x) = d(y) \vdash_{x:By:C} \exists z(a(z) = x \wedge b(z) = y)$$

and

$$a(x) = a(y) \wedge b(x) = b(y) \vdash_{xy:A} x = y$$

hold in $\mathcal{C}$.

vii *A morphism* $q : A \to Q$ *in* $\mathcal{E}$ *is the quotient (see Definition* 3.1.5*) of an equivalence relation* $R \rightarrowtail A \times A$ *(see Definition* 3.1.3*) if and only if the sequents*

$$Rxy \vdash_{xy:A} q(x) = q(y),$$

$$\vdash_{x:Q} \exists y(q(y) = x),$$

and

$$q(x) = q(y) \vdash_{xy:A} Rxy$$

hold in $\mathcal{C}$*.*

We finish with this useful characterisation of morphisms in $\mathcal{C}$.

Definition 2.2.12. For any morphism $f : A \to B$ in $\mathcal{C}$, call G the *graph* of f if G is the subobject $[\![f(a) = b]\!]$ of $A \times B$.

Proposition 2.2.13. *Call* R *a functional relation if the sequents*

$$Rxy \wedge Rxy' \vdash y = y'$$

and

$$\vdash (\forall x)(\exists y)Rxy$$

hold in $\mathcal{C}$*. Then* R *is a functional relation if and only if it is the graph of some morphism in* $\mathcal{C}$*.*

Proof. See [MR77, Theorem 2.4.4]. $\square$

Chapter 3

Categories of Classes

This chapter presents the material on topos theory and algebraic set theory theory necessary for Chapter 5. In the first section, we define the notion of pretopos, and derive the structure required to their internal logic. The presentation of this material is based on [Joh02a, Chapter A1] and [MR77]. In the second section, we introduce the Joyal-Moerdijk axiomatisation for small maps [JM95], and develop an internal version of inclusion maps that we shall use in Chapter 5.

3.1 Pretopoi

We start by generalising the usual notions of image, equivalence relation, and kernel to any category.

Definition 3.1.1. A monomorphism m is called an *image* of f if f factors through m, and, whenever f factors through a monomorphism n, so does m; m is then the least subobject of the codomain of f through which f factors.

With images, we can defined *covers*, which are characterised by surjectivity.

Definition 3.1.2. A morphism f is called a *cover* if it cannot factor through any proper subobject of it codomain.

Definition 3.1.3. A relation $R \rightarrowtail A \times A$ on A with projections $r_1, r_2 : R \to A$ is an *equivalence relation* on A if it satisfies the following three properties:

(i) It is *reflexive*, i.e., there exists an arrow $r : A \to R$ which is a section to both r_1 and r_2, i.e., such that $r_1 r = r_2 r = 1_A$.

(ii) It is *symmetric*, i.e., there exists an arrow $s : R \to R$ such that $r_1 s = r_2$ and $r_2 s = r_1$.

(iii) It is *transitive*, i.e., there exists an arrow $t : P \to R$, where P the pullback

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & R \arrow[d, "r_2"] \\ R \arrow[r, "r_1"'] & A, \end{tikzcd}$$

such that $r_1 t = r_1 p_2$ and $r_2 t = r_2 p_1$.

Definition 3.1.4. Let $f : A \to B$ be an arrow in any category $\mathcal{C}$, if it exists, the *kernel pair* of f is the pullback of f along itself, as displayed in the following diagram:

$$\begin{tikzcd} A \times_B A \arrow[r] \arrow[d] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B. \end{tikzcd}$$

It follows easily that kernel pairs are equivalence relations. We give three extra definitions in order to be able to state what pretopoi are.

Definition 3.1.5. A morphism $q : A \to Q$ is called a *quotient* of the equivalence relation $R \to A \times A$ if q coequalizes the pair of projections $r_1, r_2 : R \to A$ and (r_1, r_2) is at the same time the kernel pair of q.

Definition 3.1.6. A category $\mathcal{C}$ is called *exact* if every equivalence relation in $\mathcal{C}$ has a stable quotient.

Definition 3.1.7. A coproduct $A+B$ is called *disjoint* if the coprojections $q_1 : A \to A+B$ and $q_2 : B \to A+B$ are monomorphisms and their pullback is an initial object.

We are now ready for the following definition.

Definition 3.1.8. A category $\mathcal{E}$ is a *pretopos* if it is finitely complete, has disjoint and stable coproducts, and is exact.

They are many examples of pretopoi that the reader can find in [Joh14] and [MR77]. Our first task will to derive the structure required to interpret regular logic. As it is the logic of regular categories, we work in full generality with a regular category – see Definition 2.2.7 for the definitions of regular, coherent, Heyting, and Boolean categories.

In *Sets*, every function has an image through which it factors as an injection composed with a surjection; this fact generalises to any regular category.

Proposition 3.1.9. *In a regular category* E*, any arrow* $f : A \to B$ *has an image* m*, and factors as* $f = me$ *with* e *an epimorphism.*

Proof. Let $k_1, k_2 : K \to A$ be the kernel pair of f. As $\mathcal{E}$ is regular, the pair (k_1, k_2) has a quotient $e : A \to Q$, and so we obtain an arrow $m : M \to B$ such that $f = me$ by universality, as displayed in the following diagram:

$$\begin{tikzcd}
K \arrow[r, shift left, "k_1"] \arrow[r, shift right, "k_2"'] & A \arrow[rr, "f"] \arrow[dr, two heads, "e"'] & & B \\
& & Q. \arrow[ur, "m"'] &
\end{tikzcd}$$

In order to show that m is mono, take its kernel pair $k_1', k_2' : K' \to Q$. As mek_1, mek_2, we obtain a unique arrow $u : K \to K'$ by universality of K' such that $k_1'u = ek_1$ and $k_2'u = ek_2$, as depicted below:

$$\begin{tikzcd}
K \arrow[r, shift left, "k_1"] \arrow[r, shift right, "k_2"'] \arrow[d, dashed, "u"'] & A \arrow[r, "f"] \arrow[d, two heads, "e"'] & B \\
k' \arrow[r, shift left, "k_1'"] \arrow[r, shift right, "k_2'"'] & Q. \arrow[ur, "m"'] &
\end{tikzcd}$$

This u must be epi because it is given as follows:

$$\begin{tikzcd}
K \arrow[rr, "k_2"] \arrow[dr, dashed, "u"] \arrow[dd, "k_1"'] & & A \arrow[d, two heads, "e"] \\
& K' \arrow[r, "k_2'"] \arrow[d, "k_1'"] & Q \arrow[d, "m"] \\
A \arrow[r, two heads, "e"'] & Q \arrow[r, "m"'] & M,
\end{tikzcd}$$

where the two squares are pullbacks. So by multiple application of the pullback lemma we obtain

$$\begin{tikzcd}
K \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, dashed, "u"] & . \arrow[r] \arrow[d, two heads] & A \arrow[d, two heads, "e"] \\
. \arrow[r, two heads] \arrow[d] & K' \arrow[r, "k_2'"] \arrow[d, "k_1'"] & Q \arrow[d, "m"] \\
A \arrow[r, two heads, "e"] & Q \arrow[r, "m"] & M,
\end{tikzcd}$$

where all squares are pullbacks. Hence, u is epi since it is the composite of two epimorphisms. From this fact, we can easily derive that m is mono. Firstly, u being epi implies that $k_1' = k_2'$; secondly, any two arrows $x, y : C \to M$ such that $mx = my$ gives an unique arrow k with $x = k_1'k$ and $y = k_2'k$, as displayed in the following diagram:

$$
\begin{tikzcd}
C \arrow[drr, bend left, "y"] \arrow[dr, dashed, "k"] \arrow[ddr, bend right, "x"'] & & \\
& K' \arrow[r, "k_2'"] \arrow[d, "k_1'"'] & Q \arrow[d, "m"] \\
& Q \arrow[r, "m"] & B.
\end{tikzcd}
$$

As $k_1' = k_2'$, we have $x = k_1'k = k_2'k = y$. Thus, m is a monomorphism.

We have proved that M is a subobject of B, remains to show that it is also the image of f. For that matter, suppose f factors through another subobject on B as in

$$
\begin{tikzcd}
K \arrow[r, shift left, "k_1"] \arrow[r, shift right, "k_2"'] & A \arrow[rr, "f"] \arrow[dr, two heads, "e"'] \arrow[dd, equal] & & B \arrow[dd, equal] \\
& & M \arrow[ur, tail, "m"'] & \\
& A \arrow[rr, "f"] \arrow[dr, "e'"'] & & B \\
& & M' \arrow[ur, tail, "m'"'] &
\end{tikzcd},
$$

then m' being mono we have that $e'k_1 = e'k_2$, so we obtain $u : M \to M'$ with $e' = ue$. Hence, $me = m'ue$, which implies that $m = m'u$ since e is epi. Thus, we have $M \leq M'$ in $\mathrm{Sub}(B)$. $\square$

We shall also need epi-mono factorisations to be stable under pullback.

Proposition 3.1.10. *Images are stable under pullback, i.e., for any two $f : A \to B$ and $g : B' \to B$ with the same codomain, $Img(g^*(f)) \cong g^*(Img(f))$ where $g^* : \mathrm{Sub}(B) \to \mathrm{Sub}(B')$ is the pullback functor determined by g between the subobject categories* $\mathrm{Sub}(B)$ *and* $\mathrm{Sub}(B')$*. Consequently, the image factorizations of Proposition* 3.1.9 *are stable under pullback.*

Proof. Consider the pullback

$$
\begin{tikzcd}
K' \arrow[rr] \arrow[d, shift right, "g^*k_1"'] \arrow[d, shift left, "g^*k_2"] & & K \arrow[d, shift right, "k_1"'] \arrow[d, shift left, "k_2"] & \\
A' \arrow[rr] \arrow[dr, two heads, "e'"] \arrow[dd, "g^*f"'] & & A \arrow[dr, two heads, "e"] \arrow[dd, "f"] & \\
& Img(g^*f) \arrow[dl, tail, "m'"] & & Img(f) \arrow[dl, tail, "m"] \\
B' \arrow[rr, "g"] & & B &
\end{tikzcd}
$$

of f and its kernel pair k_1, k_2 along g, with me the factorization of f and $m'e'$ the factorization of g^*f. As the pullback of a pullback square is a pullback square, g^*k_1, g^*k_2 is the kernel pair of g^*f. As $\mathcal{E}$ is regular, quotients of kernel pairs are stable, hence the pullback of $m : Img(f) \rightarrowtail B$ along g is a quotient of the kernel pair g^*k_1, g^*k_2. Therefore, $Img(g^*f)$ and $g^*Img(g)$ are isomorphic. $\square$

The structure derived so far is enough to interpret regular logic. To prove it, we have to check that each subobject category has a stable top element, stable finite meets, and that pullback functors have a left adjoint satisfying the Beck-Chevalley condition.

Proposition 3.1.11. *For any X in a pretopos $\mathcal{E}$, $\mathrm{Sub}(X)$ has a top element; hence, the internal logic of $\mathcal{E}$ has top.*

Proof. Simply take 1_X. $\square$

Proposition 3.1.12. *For any X in a pretopos $\mathcal{E}$, $\mathrm{Sub}(X)$ has finite meets; hence, the internal logic of $\mathcal{E}$ has conjunction.*

Proof. Let A, B be two subobjects of X. Take the pullback $A \times_X B$ of A and B in $\mathcal{E}$. As monos are stable under pullback, $A \times_X B$ is a subobject of X; moreover, its universal property makes sure that it is the least upperbound of A and B in $\mathrm{Sub}(X)$. $\square$

The top element and the meet operation are stable under stable under pullback.

Proposition 3.1.13. *For any X in a pretopos $\mathcal{E}$, the top element and the meets of $\mathrm{Sub}(X)$ are stable under pullback.*

Proof. Let $f : Y \to X$ be any morphism with codomain X. The top element is preserved by f^* because we have the following pullback square:

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "1_Y"] & X \arrow[d, "1_X"] \\
Y \arrow[r, "f"] & X;
\end{tikzcd}$$

meets are preserved because they are given by pulling back in $\mathcal{E}$, and pullback functors preserve pullbacks. $\square$

We now show that regular categories have existential quantification.

Proposition 3.1.14. *For any arrow $f : X \to Y$ in a pretopos $\mathcal{E}$, there exists a left adjoint $\exists_f$ to f^*; moreover, this adjoint satisfies the Beck-Chevalley condition. In particular, pullback functors along projections have a left adjoint satisfying Beck-Chevalley, hence the internal logic of $\mathcal{E}$ has existential quantification.*

Proof. Define $\exists_f : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ to be the functor sending $m_A : A \rightarrowtail Y$ to the image of fm_A. This map $\exists_f$ is indeed a functor because if $A \le B$ for any two A and B in $\mathrm{Sub}(Y)$ then fm_A factors through $\exists f B$, and so $\exists_f A \le \exists_f B$ since $\exists_f A$ is the least subobject of X with this property.

To show that $\exists_f$ is left adjoint to f^*, it suffices to show that

$$\exists_f A \le B \text{ if and only if A } \le f^* B,$$

for any subobjects A and B of X and Y, respectively. The situation is depicted in the following diagram:

$$\begin{tikzcd}
A \arrow[dr, "m_A"'] \arrow[rrr, bend left, "e"] & & f^*B \arrow[dl, "m_{f^*B}"] \arrow[rrr, bend left] & \exists_f A \arrow[dr, "m_{\exists_f A}"'] & & B \arrow[dl, "m_B"] \\
& X \arrow[rrr, "f"'] & & & Y & .
\end{tikzcd}$$

Suppose $\exists_f A \le B$, then we have an arrow $h : \exists_f A \to B$ such that $m_{\exists_f A} = m_B h$. So the span (m_A, he) is a cone for

$$X \xrightarrow{f} Y \xleftarrow{m_B} B,$$

therefore we obtain an arrow $h' : A \to f^* B$ in $\mathrm{Sub}(X)$, i.e., $A \le f^* B$. For the converse, suppose $A \le f^* B$, then fm_A factors through B, and hence we must have $\exists_f A \le B$, since $\exists_f A$ is the least object of $\mathrm{Sub}(Y)$ with this property.

Let us now check the Beck-Chevalley condition. Considering any pullback square

$$\begin{tikzcd} X \times_Y Z \arrow[r, "f'"] \arrow[d, "g'"'] & Z \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y, \end{tikzcd}$$

we need to show that the diagram

$$\begin{tikzcd} \mathrm{Sub}(X \times_Y Z) \arrow[d, "\exists g'"'] & \mathrm{Sub}(Z) \arrow[l, "f'^*"'] \arrow[d, "\exists_g"] \\ \mathrm{Sub}(X) & \mathrm{Sub}(Y) \arrow[l, "f^*"] \end{tikzcd}$$

satisfies $f^*\exists_g A = \exists_{g'} f'^* A$, for all subobjects Z. For any such subobject $m_A : A \rightarrowtail Z$. As $X \times_Y Z$ is the pullback of Z along f, by the pullback lemma, f'^*A is the pullback of gm_A along f, and since we have stable images by Proposition 3.1.10, $f^*\exists_g A = \exists'_g f'^* A$ holds in $\mathrm{Sub}(X)$, as required. $\square$

Using the existence of stable coproducts, we can interprete more connectives, namely, we can show that each subobject category in a pretopos also has a stable bottom element and stable finite joins, i.e., pretopoi are able to interpret coherent logic.

Lemma 3.1.15. *(i) For each X in a pretopos $\mathcal{E}$,* $\mathrm{Sub}(X)$ *has a bottom element.*

(ii) Any $f : A \to 0_1$ is an isomorphism, where 0_1 is the bottom element of $\mathrm{Sub}(0)$ *and A any object of $\mathcal{E}$.*

(iii) 0_1 is the initial object of $\mathcal{E}$.

Proof. For *(i)*, take 0 the empty coproduct and consider any $m_A : A \rightarrowtail X$ in $\mathrm{Sub}(X)$. As 0 is initial, there is $x : 0 \to X$ and $a : 0 \to A$ such that $x = m_A a$. Moreover, x has an image 0_X, therefore, as x factors through m_A, we must have $0_X \leq A$, i.e., the image of x is the bottom element of $\mathrm{Sub}(X)$.

Next, to prove *(ii)*, as 0_1 is the initial object of $\mathrm{Sub}(1)$ and coproducts are assumed to be stable, we have the pullback diagram

$$\begin{tikzcd} 0_A \arrow[r] \arrow[d, tail, "m_A"'] & 0_1 \arrow[d, tail, "m_1"] \\ A \arrow[ur, "f"] \arrow[r, "g"'] & 1 \end{tikzcd}$$

where 0_A is the initial object of $\mathrm{Sub}(A)$. As 1 is final, g factors as $g = m_1 f$ which implies that 0_A is the equalizer of g and $m_1 f$. Indeed, for any x as displayed below

$$\begin{tikzcd}
0_A \arrow[r, tail, "m_A"] & A \arrow[r, shift left, "m_1 f"] \arrow[r, shift right, "g"'] & 1 \\
Z \arrow[u, dashed, "u"] \arrow[ur, "x"'] & &
\end{tikzcd}$$

with $m_1 f x = g x$, 0_A being the pullback of m_1 along g, we obtain a unique $u : Z \to 0_A$ such that $x = m_A u$. Now as $m_1 f$ and g are equal we have $0_A \cong A$, and consequently A has no proper subobject. By post-composing with any of the two projections, the existence of g entails the existence of an arrow from $A \times A$ to 1, and it follows by the above argument that $A \times A$ has no proper subobject. In particular, the diagonal Δ_A (i.e. the equalizer of the projections) is isomorphic to $A \times A$, and hence the projections of $A \times A$ are equal. It follows that any morphism having A as its domain is a monomorphism, so in particular g is mono, and therefore $A \leq 0_1$ in $\mathrm{Sub}(1)$, i.e., $A \cong 0_1$.

Finally, for the last item, $\pi_1 : 0_1 \times A \to 0_1$ is an isomorphism, hence there is a morphism $\pi_2 \pi_1^{-1} : 0_1 \to 0_1 \times A \to A$. Moreover, it is unique because for any two $f, g : 0 \to A$, by (ii) again, their equalizer is an isomorphism, therefore they must be equal. $\square$

Proposition 3.1.16. *For each X in a pretopos $\mathcal{E}$, the initial object 0 is the bottom element of $\mathrm{Sub}(X)$; hence, the internal logic of $\mathcal{E}$ has bottom.*

Proof. From Lemma 3.1.15 (iii), $0 \cong 0_1$, and since $0_1 \rightarrowtail 1$ is mono, so is $0 \rightarrowtail 1$ and $0 \rightarrowtail X$ must be mono as well. Thus, 0 is the bottom element of $\mathrm{Sub}(X)$. $\square$

Proposition 3.1.17. *For any X in a pretopos $\mathcal{E}$, $\mathrm{Sub}(X)$ has finite joins; hence, the internal logic of $\mathcal{E}$ has disjunction.*

Proof. Let A and B be two subobjects of X. Using the assumption that $\mathcal{E}$ has coproducts, we can form $A + B$

$$\begin{tikzcd}
A \arrow[r, "p_1"] \arrow[dr, bend right, "m_A"'] & A + B \arrow[d, "u"] & B \arrow[l, "q_2"'] \arrow[dl, bend left, "m_B"] \\
& X &
\end{tikzcd}.$$

By universality of $A + B$, we obtain an arrow $u : A + B \to X$ unique with the property that $m_A = u q_1$ and $m_B = u q_2$. However, nothing ensures that u is a monomorphism. To obtain a subobject of X, we take me the image-factorization of u, as indicated below:

$$
\begin{tikzcd}
& Z \arrow[ddd, bend left, "m_Z"] & \\
A \arrow[ur, "z_1"] \arrow[r, "p_1"] \arrow[ddr, bend right, "m_A"'] & A+B \arrow[u, dashed, "v"'] \arrow[d, two heads, "e"'] & B \arrow[ul, "z_2"'] \arrow[l, "q_2"'] \arrow[ddl, bend left, "m_B"] \\
& M \arrow[d, tail, "m"'] & \\
& X & .
\end{tikzcd}
$$

We claim that M is the join of A and B. Suppose that there is some Z in $\mathrm{Sub}(X)$ as depicted above. Then we obtain $v : A + B \to X$ by universality and have $m_A = m_Z v p_1$ and $m_B = m_Z v p_2$, but since me is the unique arrow with this property, it follows that $me = m_Z v$, i.e., me factors through Z. Therefore $M \leq Z$. □

We have showed that the subobject categories of a pretopos are bounded lattices, the next two propositions show that there are distributive and stable under pullback.

Proposition 3.1.18. *For any X in a pretopos $\mathcal{E}$, the top element, meets, the bottom element, and joins in* $\mathrm{Sub}(X)$ *are stable under pullback.*

Proof. By Proposition 3.1.13 the top element and the meets are stable. For the rest, let $f : Y \to X$ be any morphism with codomain X. By Proposition 3.1.16, the bottom element is the empty coproduct, and hence it is stable by assumption. As the join of A and B is constructed by taking the image of their coproduct, stability of joins follows from that of coproducts and images, given by assumption and Proposition 3.1.10, respectively. □

Proposition 3.1.19. *For any X in a pretopos $\mathcal{E}$,* $\mathrm{Sub}(X)$ *is distributive.*

Proof. We prove distributivity of meets over joins. Let A be any subobject of X. We can define the operation $A \wedge (-)$ as the composite

$$\mathrm{Sub}(X) \xrightarrow{m_A^*} \mathrm{Sub}(A) \xrightarrow{\exists_{m_A}} \mathrm{Sub}(X)$$

so that $A \wedge (-)$ preserves joins because there are stable under pullback by Proposition 3.1.18 and preserved by left adjoints since joins are the coproducts of $\mathrm{Sub}(X)$. □

Here is a useful property of pretopoi that we shall need later on.

Proposition 3.1.20. *In a pretopos $\mathcal{E}$, covers coincide with regular epimorphisms and every epimorphism is regular.*

Proof. See [Joh02a, Proposition 1.3.4 and Corollary 1.4.9]. □

With an arbitrary pretopos, we cannot obtain better than coherent logic. In order to interpret full many-sorted first-order logic, we need to assume that pullback has a right adjoint. The following definition introduces the kind of pretopoi with which we shall work later on.

Definition 3.1.21. A pretopos $\mathcal{E}$ is a *Heyting pretopos* if all pullback functors have a right adjoint.

The fact the left adjoints satisfy Beck-Chevalley implies that the right ones satisfy the condition as well. The next proposition shows that Heyting pretopoi interpret full many-sorted first-order intuitionistic logic.

Proposition 3.1.22. *Any subobject category in a Heyting pretopos $\mathcal{E}$ is a Heyting algebra; moreover, pullback functors between subobject categories are Heyting algebra homomorphisms.*

Proof. As $\mathrm{Sub}(X)$ is a (distributive) bounded lattice whose structure is preserved by pullback functors, we only need to show that for any X in $\mathcal{E}$, $\mathrm{Sub}(X)$ has stable Heyting implications. Let A and B be two subobjects of X. We claim that $A \Rightarrow B$ is given by $\forall_m(A \wedge B)$ with m is the inclusion of A in X. We must check that for any C in $\mathrm{Sub}(X)$, the following holds:

$$C \wedge A \leq B \Leftrightarrow C \leq \forall_m(A \wedge B).$$

From right to left, suppose $C \wedge A \leq B$. As $\forall_m$ is the right adjoint of m^*, $C \leq \forall_m(A \wedge B)$ is equivalent to $m^*C = C \wedge A \leq A \wedge B$. So we need to show that $C \wedge A \leq A \wedge B$, but, by assumption, $C \wedge A \leq B$ and, by definition, $C \wedge A \leq A$, hence $C \wedge A \leq A \wedge B$. For the converse, suppose $C \leq \forall_m(A \wedge B)$. Then, by the same equivalence, $C \wedge A \leq A \wedge B$, so $C \wedge A \leq B$.

In order to prove stability, let A and B be two subobjects of X, $m : A \rightarrowtail X$ be the inclusion of A into X, and $f : Y \to X$ be any morphism with codomain X. We have to show that

$$f^*(A \to B) \cong f^*(A) \to f^*(B),$$

that is

$$f^*\forall_m(A \wedge B) \cong \forall_{f^*m}(f^*(A) \wedge f^*(B)).$$

As $\forall_m$ satisfies the Beck-Chevalley condition, given the pullback square

$$\begin{tikzcd} f^*A \arrow[r, "g"] \arrow[d, tail, "f^*m"'] & A \arrow[d, tail, "m"] \\ Y \arrow[r, "f"'] & X, \end{tikzcd}$$

the following diagram commutes:

$$
\begin{tikzcd}
\mathrm{Sub}(f^*A) \arrow[d, "\forall_{f^*m}"'] & \mathrm{Sub}(A) \arrow[l, "g^*"'] \arrow[d, "\forall_m"] \\
\mathrm{Sub}(Y) & \mathrm{Sub}(X). \arrow[l, "f^*"]
\end{tikzcd}
$$

Since meets are constructed by pullback and pullbacks preserve monomorphisms, $A \wedge B$ is a subobject of A, so $f^*\forall_m(A \wedge B) \cong \forall_{f^*m}g^*(A \wedge B)$. Moreover, the pullback lemma implies that the outer square of the diagram below is a pullback:

$$
\begin{tikzcd}
g^*(A \wedge B) \arrow[r] \arrow[d, tail] & A \wedge B \arrow[d, tail] \\
f^*A \arrow[r, tail, "g"] \arrow[d, tail, "f^*m"'] & A \arrow[d, "m"] \\
Y \arrow[r, "f"'] & X,
\end{tikzcd}
$$

hence $f^*(A \wedge B) \cong g^*(A \wedge B)$. Therefore,

$$
\begin{aligned}
\forall_{f^*m}(g^*(A \wedge B)) &\cong \forall_{f^*m}(f^*(A \wedge B)) \\
&\cong \forall_{f^*m}(f^*(A) \wedge f^*(B)),
\end{aligned}
$$

from which follows that

$$f^*\forall_m(A \wedge B) \cong \forall_{f^*m}(f^*(A) \wedge f^*(B)).$$

□

Although we shall work with the more general Heyting pretopoi, we introduce their Boolean version for sake of completeness.

Definition 3.1.23. A *Boolean pretopos* is a Heyting pretopos in which every subobject category $\mathrm{Sub}(X)$ is *complemented*, i.e., for all A in $\mathrm{Sub}(X)$ there is a (necessarily unique) A in $\mathrm{Sub}(X)$ such that $A \wedge A' = \perp$ and $A \vee A' = \top$.

This last proposition shows that Boolean pretopoi interpret full many-sorted classical first-order logic.

Proposition 3.1.24. *For any X in a Heyting pretopos, if A a subobject of X has a complement A', then $A' = \neg A$. Consequently, the internal logic of Boolean pretopoi satisfies the law of excluded middle.*

Proof. Let A' be the complement of A, we have to show that $A' \leq \neg A$ and $\neg A \leq A'$. For the first one, from $A \wedge A' = \perp$ we obtain

$$
\begin{aligned}
A \wedge A' \leq \perp &\Rightarrow A' \leq Q \to 0 \\
&\Rightarrow A' \leq \neg A;
\end{aligned}
$$

for the second one, from $A \vee A' = \top$ and distributivity we obtain

$$\begin{aligned}\neg A = \neg A \wedge (A \vee A') &\Rightarrow \neg A = (\neg A \wedge A) \vee (\neg A \wedge A') \\ &\Rightarrow \neg A = \neg A \wedge A' \\ &\Rightarrow \neg A \leq A'.\end{aligned}$$

□

3.2 The Small Maps Axiomatisation

Here we introduce the notion of *category of classes* in the sense of Algebraic Set Theory. It will consists of a Heyting pretopos $\mathcal{E}$ satisfying the Joyal-Moerdijk axiomatisation for small maps [JM95]. This set of axioms is intended to distinguish in $\mathcal{E}$ a class a morphisms $\mathcal{S}$ referred to as the *class of small maps* of $\mathcal{E}$. The intention is that a map $f : A \to B$ is considered *small* if its fibres are small. When $\mathcal{E}$ is the (true) category of classes according to some model of set theory and small is defined as being a set, a canonical example of class of small maps $\mathcal{S}$ satisfying the axiomatisation is the collection of functions whose fibres are sets.

We shall assume that a category of classes satisfies the following axioms.

(A1) (Pullback Stability) In any pullback square

$$\begin{tikzcd} Y' \arrow[r] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ X' \arrow[r, "p"'] & X \end{tikzcd}$$

if f belongs to $\mathcal{S}$ then so does g.

(A2) (Descent) In any pullback square as above, if g belongs to $\mathcal{S}$ and p is an epimorphism then f belongs to $\mathcal{S}$.

(A3) (Finiteness) The maps $0 \to 1$ and $1 \to 1$ belong to $\mathcal{S}$.

(A4) (Sums) If $Y \to X$ and $Y' \to X'$ belong to $\mathcal{S}$ then so does their sum $Y + Y' \to X + X'$.

(A5) (Composition) $\mathcal{S}$ is closed under composition, and any isomophism belongs to $\mathcal{S}$.

(A6) (Quotients) In any commutative diagram

$$\begin{tikzcd} z \arrow[rr, two heads, "p"] \arrow[dr, "g"'] & & Y \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

if p is an epimorphism and g belongs to $\mathcal{S}$ then so does f.

(A7) (Strong Collection Axiom) For any two arrows $p : Y \rightarrowtail X$ and $f : X \to A$ where p is an epimorphism and f belongs to $\mathcal{S}$, there exists a quasi-pullback square[1] of the form

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "g"'] & Y \arrow[r, two heads, "p"] & X \arrow[d, "f"] \\ B \arrow[rr, two heads, "h"'] & & A \end{tikzcd}$$

(S1) (Exponentiability Axiom) Every map in $\mathcal{C}$ is exponentiable, i.e., $f : A \to X$ belongs to $\mathcal{S}$ then it is exponentiable as an object of $\mathcal{E}/X$.

(S2) (Representability Axiom) There exists a map $\pi : E \to U$ in $\mathcal{S}$ which is universal in the following sense: for any $f : Y \to X$ in $\mathcal{S}$ there exists a diagram

$$\begin{tikzcd} Y \arrow[d] & Y' \arrow[l] \arrow[r] \arrow[d, "f'"] & E \arrow[d, "\pi"] \\ X & X' \arrow[l, two heads, "p"] \arrow[r, "c"'] & U \end{tikzcd}$$

in which both squares are pullbacks.

(NNO) (Natural Numbers Object) There exists an object N in $\mathcal{E}$ and two arrows $0 : 1 \to N$ and $s : N \to N$ with the following universal property: for all $f : A \to B$ and $g : B \to B$ there exists a unique morphism $u : N \to Z$ such that the following diagram commutes:

$$\begin{tikzcd} 1 \times A \arrow[r, "0\times 1_A"] \arrow[d, "\cong"'] & N \times A \arrow[r, "s\times 1_A"] \arrow[d, "u"] & N \times A \arrow[d, "u"] \\ A \arrow[r, "f"'] & B \arrow[r, "g"'] & B. \end{tikzcd}$$

Axioms **(A1)**-**(A3)** say that being small is a property of the fibres of the maps in $\mathcal{S}$. Axioms **(A4)-(A7)** are set-theoretic: **(A4)** says that collections containing 0 and 1 elements are small; **(A5)** is a union axiom, it says that the union of a small disjoint family of small object is again small; **(A6)** is a replacement axiom, it says that the image of a small object is a small object; and, **(A7)** is the categorical version of the strong collection axiom,

[1]Recall that such a square is said to be a quasi-pullback if the unique arrow $Z \to B \times_A X$ is an epimorphism.

it says that for any epimorphism $p : Y \twoheadrightarrow X$ with small codomain there is an epimorphism $Z \twoheadrightarrow X$ with small domain that factors through p. One can check in particular that $\mathcal{E}$ satisfies **(A7)** if and only if the more familiar version of strong collection holds in the internal logic of $\mathcal{E}$:

$$\mathcal{E} \models (\forall a : P_s(X))((\forall x \in a)(\exists y : Y)\varphi(x,y) \rightarrow (\exists b : P_s(Y))collec(x \in a, y \in b, \varphi(x,y))),$$

where $collec(x \in a, y \in b, \varphi(x,y))$ abbreviates

$$(\forall x \in a)(\exists y \in b)(\varphi(x,y)) \wedge (\forall y \in B)(\exists x \in a)(\varphi(x,y)).$$

Axiom **(S1)** expresses that if A is small and X is a class then there is a object of arrow from A to X. Finally, **(S2)** means that every small object is isomorphic to E_u some fibre of π.

We cite two consequences of this axiomatisation from [JM95]. The first one says that the axioms **(A1)-(A7)**, **(S1)**, and **(S2)** are stable under slicing; the second one that their exists a power object for any class.

Proposition 3.2.1. *If $\mathcal{S}$ is a class of small maps in $\mathcal{E}$ then $\mathcal{S}_X$, the class of maps of $\mathcal{S}$ with codomain X, is a class of small maps for $\mathcal{E}/X$.*

A relation $R \rightarrowtail X \times Y$ is said to be small, if the composite $R \rightarrowtail X \times Y \xrightarrow{\pi_2} Y$ is small.

Proposition 3.2.2. *For any object X in $\mathcal{E}$ there exists a power object $P_s(X)$ and a small relation $\epsilon_X \rightarrowtail X \times P_s(X)$ such that for any Z and small relation $R \rightarrowtail X \times Z$ there exists a unique morphism $u : Z \rightarrow P_s(X)$ such that the following square is a pullback:*

$$\begin{tikzcd} R \arrow[r] \arrow[d, tail] & \epsilon_X \arrow[d, tail] \\ X \times Z \arrow[r, "1_X \times u"'] & X \times P_s(X). \end{tikzcd}$$

Moreover, the assignment $X \mapsto P_s(X)$ is functorial and gives rise to a monad.

We are in in position to give to define categories of classes.

Definition 3.2.3. *A* category of classes *is a Heyting pretopos $\mathcal{E}$ satisfying the axioms **(A1)-(A7)**, **(S1)**, **(S2)**, and **(NNO)**.*

We prove one property of our categories of classes that we shall need later on.

Proposition 3.1. *Any Heyting pretopos $\mathcal{E}$ with a natural numbers object and a class of small maps has stable finite colimits.*

Proof. We need to show that $\mathcal{E}$ has stable finite coequalizers. The coequalizer of any diagram

$$A \overset{f}{\underset{g}{\rightrightarrows}} B,$$

is given as the quotient of the equivalence relation generated by

$$\{(x, y) : A \times A \mid f(x) = g(y)\}.$$

Since in a pretopos all equivalence relations have a quotient, all we need to do it to show how to form the reflexive, symmetric, and transitive closure of a relation. Using the natural numbers object, [vdBM07, Proposition 3.4] shows how to do that. $\square$

In the remainder of this section, we define a notion of inclusion maps that we shall use in Chapter 5. For any X in a category of classes $\mathcal{E}$, we want to have an inclusion map i_A^X for any $A : P_s(X)$. As $\in_X$ is a small relation, the morphism q in the diagram below is small by definition:

$$\begin{tikzcd} \in_X \arrow[d, tail, "\epsilon"'] \arrow[dr, "q"] & \\ X \times P_s(X) \arrow[r, "\pi_2"'] & P_s(X). \end{tikzcd} \tag{3.1}$$

By **(S1)**, we can form the exponential

$$\begin{tikzcd} (X \times P_s(X) \xrightarrow{\pi_2} P_s(X))^{(\in_X \xrightarrow{q} P_s(X))} \arrow[d] \\ P_s(X) \end{tikzcd}$$

in $\mathcal{C}/P_s(X)$. Recall that in *Sets* the exponential β^α in the slice *Sets*$/I$ of two maps $\alpha : A \to I$ and $\beta : B \to I$ is given by the set of maps between slices of α and β:

$$\{(f_i : \alpha^{-1}(i) \to \beta^{-1}(i)) \mid i \in I\}.$$

Hence, in $\mathcal{E}/P_s(X)$, π_2^q is given internally by

$$\{(i_A^X : A \cong q^{-1}(A) \to \pi_2^{-1}(A) \cong X) \mid A : P_s(X)\}.$$

Its universal property is illustrated by the following diagram:

$$\begin{tikzcd} \in_X \times_{p_s(X)} \pi_2^q \arrow[dr, "ev"] & \\ \in_X \times_{P_s(X)} Z \arrow[u, "1_{\in_X} \times_{p_s(X)} \bar{f}"] \arrow[r, "f"'] & X \times_{P_s(X)} P_s(X), \end{tikzcd}$$

expressing the one-to-one correspondence in $\mathcal{E}/P_s(X)$

$$\frac{f :\in_X \times_{P_s(X)} Z \to X \times_{P_s(X)} P_s(X)}{\bar{f} : Z \to \pi_2^q.}$$

between maps $f :\in_X \times Z \to X \times P_s(X)$ and their transposes $\bar{f} : Z \to \pi_2^q$. In $\mathcal{E}/P_s(X)$, the terminal object is the identity $1_{P_s(X)}$, so we have in particular the following one-to-one correspondence:

$$\frac{f :\in_X \to X \times_{P_s(X)} P_s(X)}{\bar{f} : P_s(X) \to \pi_2^q.}$$

For the morphism $\epsilon :\in_X \rightarrowtail X \times P_s(X)$ specifically, we have the transpose $i^X_{(\text{-})} : P_s(X) \to \pi_2^q$, as displayed below:

$$\begin{tikzcd}
\in_X \times_{P_s(X)} \pi_2^q \arrow[drr, "ev"] & & \\
\in_X \times_{P_s(X)} P_s(X) \arrow[u, "1_{\in_X} \times i^X_{(\text{-})}"] \arrow[rr, "\epsilon"'] & & X \times_{P_s(X)} P_s(X).
\end{tikzcd}$$

For each $A : P_s(X)$, we can think of i^X_A as the inclusion of A into X in the sense that $ev((a, A), i^X_A) = (a, A)$.

Now consider $A, B : P_s(X)$, we also want to have an inclusion maps between A and B. We obtain obtain them in a similar way. Define

$$I_X = \{(A, B) : X \times X \mid (\forall x : X)(x \in A \to x \in B)\},$$

and consider the following pullback of diagram (3.1):

$$\begin{tikzcd}
\pi_1^* \in_X \arrow[rr, tail, bend left] \arrow[dr, "\pi_1^* q"'] & \pi_2^* \in_X \arrow[r, tail] \arrow[d, "\pi_2^* q"] & X \times P_s(X)^2 \arrow[dl] & \in_X \arrow[r, tail, "\epsilon"] \arrow[d, "q"'] & X \times P_s(X) \arrow[dl, "\pi_2"] \\
& P_s(X)^2 \arrow[rr, shift left, "\pi_1"] \arrow[rr, shift right, "\pi_2"'] & & P_s(X), &
\end{tikzcd}$$

where $\pi_1^* \in_X$ and $\pi_2^* \in_X$ consist of

$$\{(x, A, B) : X \times P_s(X) \times P_s(X) \mid x \in A\}$$

and

$$\{(x, A, B) : X \times P_s(X) \times P_s(X) \mid x \in B\},$$

respectively. In this situation, there need not be a morphism $\pi_1^* \in_X \rightarrowtail \pi_2^* \in_X$, but by pulling back these two triangles along I_X

$$\begin{tikzcd}
l^*\pi_1^* \in_X \arrow[r, tail, "m"] \arrow[rr, bend left] \arrow[dr, "l^*\pi_1^* q"'] & l^*\pi_2^* \in_X \arrow[r, tail] \arrow[d, "l^*\pi_2^* q"] & X \times I_X \arrow[dl] & \pi_1^* \in_X \arrow[rr, tail, bend left] \arrow[dr, "\pi_1^* q"'] & \pi_2^* \in_X \arrow[r, tail] \arrow[d, "\pi_2^* q"] & X \times P_s(X)^2 \arrow[dl] \\
& I_X \arrow[rrr, "l"'] & & & P_s(X)^2,
\end{tikzcd}$$

where $l^*\pi_1^* \in_X$ and $l^*\pi_2^* \in_X$ consist of

$$\{(x, A, B) : X \times P_s(X) \times P_s(X) \mid (\forall x : X)(x \in A \to x \in B) \wedge (x \in A)\}$$

and

$$\{(x, A, B) : X \times P_s(X) \times P_s(X) \mid (\forall x : X)(x \in A \to x \in B) \wedge (x \in B)\},$$

respectively, we obtain a monomorphism $m : l^*\pi_1^* \in_X \rightarrowtail l^*\pi_2^* \in_X$, since in this case if $(x, A, B) \in l^*\pi_1^* \in_X$ then $A \subseteq B$ and so $(x, A, B) \in l^*\pi_2^* \in_X$.

As m is a monomorphism, it is small; hence, by **(S1)**, we can form the exponential $l^*\pi_2^* q^{l^*\pi_1^* q}$ in $\mathcal{C}/l^*\pi_2^* q$, internally given by

$$\{i^X_{A,B} : A \cong l^*\pi_1^* q^{-1}(A, B) \cong B \to l^*\pi_2^* q^{-1}(A, B) \mid (A, B) : I_X\}.$$

By its universal property and the fact that in $\mathcal{C}/I_X$ the terminal object is the indentity 1_{I_X}, we have the following one-to-one correspondence:

$$\frac{l^*\pi_1^* \in_X \to l^*\pi_2^* \in_X}{\overline{I_X \to l^*\pi_2^* q^{l^*\pi_1^* q}},}$$

and we obtain $i^X_{(\text{-}),(\text{-})}$ as the transpose of m.

Note that these two kind of inclusion interact well in the sense that we have $ev(ev(x, i^X_{A,B}), i^X_B(x)) = ev(x, i^X_A(x))$, i.e., $i^X_B i^X_{A,B} = i^X_A$. Since an inclusion i^X_A or $i^X_{A,B}$ is only a term of type π_2^q or $l^*\pi_2^* q^{l^*\pi_1^* q}$, they need not be symbol in the internal language of $\mathcal{E}$ corresponding to it. However, we will pretend that we have such symbol and write, e.g., $i^X_A(a) = a$ while we really mean $ev(a, i^X_A)(a) = a$.

Besides begin able to use the inclusions i^X_A and $i^X_{A,B}$ is the internal language of $\mathcal{E}$, given an indexed endofunctor on $\mathcal{E}$, we also want to form Fi^X_A and $Fi^X_{A,B}$. This is done in the following way. Consider any exponential B^A in a category with finite products $\mathcal{C}$. We have the following diagram in $\mathcal{C}/B^A$:

$$\begin{tikzcd}
A \times_{B^A} B^A \arrow[rr, "\langle ev, \pi_1 \rangle"] \arrow[dr, "\pi_2"'] & & A \times_{B^A} B^A \arrow[dl, "\pi_2"] \\
& B^A &
\end{tikzcd}.$$

So, we have the morphism

$$F^{B^A}(A \times B^A) \xrightarrow{F^{B^A}(\langle ev, \pi_1 \rangle)} F^{B^A}(A \times B^A),$$

and, by indexation, $F^{B^A}(A \times_{B^A} B^A) \cong FA \times B^A$. Hence, in $\mathcal{C}$, we have the morphism

$$FA \times B^A \xrightarrow{F(\langle ev, \pi_1 \rangle)} FB \times B^A \xrightarrow{\quad \pi_1 \quad} FB,$$

which takes a pair $(x : FA, f : B^A)$ and evaluates Ff at x. If FB was small, which need not be the case even though B is assumed to be so, we could form the exponential FB^{FA} in $\mathcal{C}/B^A$. The morphism $\pi_1 F(\langle ev, \pi_1 \rangle)$ would then have a transpose

$$B^A \to FB^{FA}$$

taking $f : A \to B$ to $Ff : FA \to FB$. Although this transpose need not exists, $\pi_1 F(\langle ev, \pi_1 \rangle)$ is enough to evaluate Ff, and that is all we need.

The foregoing holds in particular for the exponentials π_2^Q and $l^* \pi_2^* q^{l^* \pi_1^* q}$ in $\mathcal{E}/P_s(X)$ and $\mathcal{E}/I_X$, therefore, we can form the image under F of the inclusion maps that we have defined.

Chapter 4

A ZF Proof of the Final Coalgebra Theorem

In this chapter, we provide a review of Aczel and Mendler's Final Coalgebra Theorem [AM89]. The theorem concerns coalgebras for an endofunctor F acting on $Classes$, any category of classes and functions in the sense of any model of **ZF**. We shall work under the assumption that F satisfies the following assumption:

Definition 4.1. A endofunctor F on *Classes* is called *set-based* if for every F-coalgebra (A, α) and every $x \in A$ there exists a sub*set* Y of A and some $y \in FY$ such that $Fi_{Y,A}(y) = \alpha(x)$, where $i_{Y,A}$ is the inclusion map of Y into A.

Our aim is to prove the following result:

Theorem. (The Final Coalgebra Theorem) *Every set-based endofunctor F on* Classes *admits a final F-coalgebra.*

Our reviewed version improves Aczel and Mendler's result in two ways: first, our notion of set-based functor presupposes the existence of such a $Y \subseteq A$ only for elements of the form $\alpha(x)$ with $x \in A$, while Aczel and Mendler requires it for every $x \in FA$. Second, whereas they assume their category of classes to be defined over **ZFC**, we show that the Axiom of Choice is unnecessary and can be avoided with the Relation Reflection Scheme [Acz08]. This scheme is formulated as follows:

RRS For any class A and R with $R \subseteq A \times A$ such that $(\forall x \in A)(\exists y \in A)R(x, y)$, if a is a subset of A then there exists b a subset of A such that $a \subseteq b$ and $(\forall x \in a)(\exists y \in b)R(x, y)$.

Its intuitive reading is that given a class-sized relation on A which is total, for any element $x \in A$ we can bound the collection of elements related to x

by R to a set-sized collection. Regarding the strength of this scheme, Aczel argues for its free use in constructive mathematics for each of its instance is satisfied in the interpretation of **CZF** in Martin-Löfs constructive type theory, cf. [Acz08] and [Acz78].

The main purpose of this review is to provide intuitions for the next chapter. Although we work over **ZF**, the proof is essentially constructive and, if you assume the preservation of monomorphisms, it can be carried in **CZF+RRS** modulo the fact that we assume the existence of coequalizers in *Classes*. Over **ZF**, coequalizers do exist and are constructed using Scott's trick. Unfortunately, Scott's trick is not applicable in a constructive setting due to the fact that the least number principle fails constructively. Nonetheless, the result could possibly be extended to the exact completion of *Classes* on **CZF+RRS** if, following [dBM08], one manages to show that **RRS** is preserved under exact completion.

We give one more definition before embarking in the proof.

Definition 4.2. We call (B, β) a *subcoalgebra* of (A, α) if B is a subclass of A and the inclusion map $i_{B,A}$ is a coalgebra homomorphism; a coalgebra (A, α) *small* if A is a set.

The first step of the strategy consists in proving the following key proposition which says that large coalgebras for a set-based functor are in a sense determined by their small subcoalgebras. Indeed, we shall crucially use it to prove 4.4 below where we show that such large coalgebras are nothing but the colimit of their small subcoalgebras.

Proposition 4.3. Let (A, α) be an F-coalgebra for a set-based functor F. Then for any set $X \subseteq A$ there exists (B, β) a small subcoalgebra of (A, α) such that $X \subseteq B$.

Proof. We want to construct some set $B \subseteq A$ with $X \subseteq B$ and such that for all $x \in B$ there exists some $y \in FB$ with $Fi_{B,A}(y) = \alpha(x)$. Let R be the following class-sized relation:

$$R = \{(X, Y) \in \mathcal{P}(A) \times \mathcal{P}(A) \mid (\forall x \in X)(\exists y \in FY)(Fi_{Y,A}(y) = \alpha(x))\},$$

where $\mathcal{P}(A)$ is the class of sub*sets* of A. We claim that R is total. Let $X \subseteq A$ be any subset of A, and $\varphi(x, Y)$ be the following formula:

$$\varphi(x, Y) = x \in X \wedge Y \in \mathcal{P}(A) \wedge (\exists y \in FY)(Fi_{Y,A}(y) = a(x)).$$

The assumption that F is set-based ensures that for every $x \in X$ there exists some Y for which $\varphi(x, Y)$ holds. By the Strong Collection Axiom (which is provable in **ZF**, see, e.g., [Kun80]), we obtain a set B such that for every $x \in X$ there is some $Y \in B$ such that $\varphi(x, Y)$ holds, and for every $Y \in B$ there exists some $x \in X$ such that $\varphi(x, Y)$ holds. Define

$$C = \bigcup B.$$

Then for any $x \in X$, there is some $Y \in B$ such that $Fi_{Y,A}(y) = \alpha(x)$ for some $y \in FY$, and since $Y \subseteq C \subseteq A$ we have $Fi_{Y,A} = Fi_{C,A}Fi_{Y,C}$, so $Fi_{C,A}(Fi_{Y,C}(y))) = \alpha(x)$. Hence, $\varphi(x, C)$ holds. Therefore $(X, C) \in R$ and we conclude that R is total.

Considering $\{X\}$ for any $X \subseteq A$, as R is total, **RRS** gives us a set B such that $\{X\} \subseteq B \subseteq \mathcal{P}(A)$ on which R is total. Once more, define

$$C = \bigcup B.$$

Then for any $x \in C$ there is some $y \in FC$ such that $Fi_{C,A}(y) = \alpha(x)$. Indeed, if $x \in C$ then $x \in Z$ for some $Z \in B$, and so by totality of R on B there is some $Y \in B$ such that $Fi_{Y,A}(y) = \alpha(x)$ for some $y \in FY$. Thus, we have $Y \subseteq C \subseteq A$, and obtain $Fi_{C,A}(Fi_{Y,C}(y))) = \alpha(x)$.

It remains to define a coalgebra map γ on C making $i_{C,A}$ a coalgebra homomorphism. We use Excluded Middle to make the following case distinction. If C is empty, γ can be set to be the empty map. If C is inhabited, as $i_{C,A}$ is injective, using Excluded Middle again we can define a left-inverse to $i_{C,A}$, i.e., a map $f : A \to C$ such that $fi_{C,A} = 1_C$. By functoriality, we have $FfFi_{C,A} = 1_{FC}$ which implies that $Fi_{C,A}$ is injective as well. So γ can be define to be the mapping sending x to be unique y such that $Fi_{C,A}(y) = \alpha(x)$. These two uses of Excluded Middle can be avoided by assuming that F preserves monomorphisms. Indeed, this assumption implies that $Fi_{C,A}$ is injective and γ can then be defined as above. □

We shall stick to the terminology of Aczel and Mendler [AM89] and call (A, α) *complete* if for every small coalgebras (B, β) there exists exactly one coalgebra homomorphism from (B, β) to (A, α), *weakly complete* if there exists at least one such homomorphism, and *strongly extensional* if there exists at most one such homomorphism. We now prove that our tasks boils down to finding a complete F-coalgebra.

Theorem 4.4. Every complete F-coalgebra for a set-based functor F is final.

Proof. Assuming that (Z, ζ) is complete, we have to show that there exists a unique coalgebra homomorphism from any large coalgebra (A, a) to (Z, ζ). Let $\{(B_i, \beta_i) \mid i \in I\}$ be an indexation of the small subcoalgebras of (A, α) and $\{z_i : (B_i, \beta_i) \to (Z, \zeta)\}$ be the collection of unique homomorphisms from each (B_i, β_i) to (Z, ζ). We want to define a map $u : A \to Z$ unique with the property that $F(u)\alpha = \zeta u$. Consider any $x \in A$, Proposition 4.3 ensures that there exists (B, β) some small subcoalgebra of (A, α) such that $x \in B$. Suppose that there exist (B_1, β_1) and (B_2, β_2) such that x belongs to both B_1 and B_2. We want to make sure that z_1 and z_2 map x to the same element of Z so that $u : A \to Z$ can be defined to be $\bigcup_{i \in I} z_i$. By Proposition 4.3 again, there exists a small subcoalgebra (C, γ) such that $B_1 \cup B_2 \subseteq C$. Hence, for $j = 1, 2$, we have

$$
\begin{tikzcd}
B_j \arrow[r, "i_{B_j,C}"] \arrow[rr, bend left, "z_j"] \arrow[d, "\beta_j"'] & C \arrow[r, "z_c"] \arrow[d, "\beta"] & Z \arrow[d, "\zeta"] \\
FB_j \arrow[r, "Fi_{B_j,C}"'] \arrow[rr, bend right, "Fz_j"'] & FC \arrow[r, "Fz_c"'] & FZ
\end{tikzcd}
$$

where the right and outer squares commute. To show that the left square commutes as well, we use Excluded Middle one more time to define a left-inverse to z_c and conclude that $F(z_c)$ is a monomorphism. As in the proof of Proposition 4.3, this can be avoided by assuming that F preserves monomorphisms. Indeed, we have the following diagram where the right and outer squares commute:

$$
\begin{tikzcd}
B_j \arrow[r, "i_{B_j,C}"] \arrow[rr, bend left, "i_{B_j,A}"] \arrow[d, "\beta_j"'] & C \arrow[r, "i_{C,A}"] \arrow[d, "\beta"] & A \arrow[d, "\alpha"] \\
FB_j \arrow[r, "Fi_{B_j,C}"'] \arrow[rr, bend right, "Fi_{B_j,A}"'] & FC \arrow[r, "Fi_{C,A}"'] & FA.
\end{tikzcd}
$$

By preservation of monomorphisms, $Fi_{C,A}$ is mono, therefore, the left square commutes. As (Z, ζ) is complete, we have $z_c i_{B_j,C} = z_j$, and since $z_c i_{B_1,C}(x) = z_c i_{B_2,C}(x)$, we conclude that $z_1(x) = z_2(x)$.

To see that u is indeed a coalgebra morphism, consider any $x \in A$. By Proposition 4.3, there exists (B, β) a small subcoalgebra of (A, α) such that $x \in B$, as displayed below:

$$
\begin{tikzcd}
B \arrow[r, "i_{B_j,A}"] \arrow[rr, bend left, "z_b"] \arrow[d, "\beta"'] & A \arrow[r, "i_{A,Z}"] \arrow[d, "\alpha"] & Z \arrow[d, "\zeta"] \\
FB \arrow[r, "Fi_{B,A}"'] \arrow[rr, bend right, "Fz_b"'] & FA \arrow[r, "Fi_{A,Z}"'] & FZ.
\end{tikzcd}
$$

By construction, $ui_{A,B} = z_b$, hence $FuFi_{A,B} = Fz_b$, and we have:

$$\begin{aligned}
\zeta u(x) &= \zeta z_b(x) \\
&= Fz_b(x)\beta(x) \\
&= FuFi_{A,B}\beta(x) \\
&= Fu\alpha i_{A,B}(a) \\
&= Fu\alpha(a).
\end{aligned}$$

The map u is unique for suppose that they are two homomorphisms $u, u' : (A, \alpha) \to (Z, \zeta)$. Then for any $x \in A$, Proposition 4.3 gives (B, β) some small subcalgebra of (A, α) such that $x \in B$:

$$\begin{tikzcd}
B \arrow[r, "i_B^A"] \arrow[d, "\beta"'] & A \arrow[r, shift left, "u"] \arrow[r, shift right, "u'"'] \arrow[d, "\alpha"] & Z \arrow[d, "\zeta"] \\
FB \arrow[r, "F(i_B^A)"'] & FA \arrow[r, shift left, "Fu"] \arrow[r, shift right, "Fu'"'] & FZ.
\end{tikzcd}$$

As (Z, ζ) is complete, $ui_{B,A} = u'i_{B,A}$. Therefore, $u(x) = u'(x)$. $\square$

It remains to construct such a complete coalgebra. We start by constructing a weakly final coalgebra and then show how to make it complete.

Proposition 4.5. There exists a weakly complete F-coalgebra.

Proof. Let $(B_i, \beta_i)_{i \in I}$ be an indexation of the class of all small coalgebras. We can form the disjoint union $B := \biguplus_{i \in I}(B_i)$. Then for each (B_i, β_i) we have an map $b_i : B_i \to B$ sending x to (x, i). Define $\beta : B \to FB$ to be the coalgebra map sending (x, i) to $F(b_i)\beta_i(x)$. Then each b_i a coalgebra homomorphism. $\square$

The most technical part of this proof is the following proposition where we show how to turn an F-coalgebra (A, α) into a strongly complete one. We shall need to take quotients of (A, α) with respect to relations on A respecting the coalgebra map in the following sense.

Definition 4.6. A *congruence* R on a coalgebra (A, α) is a relation on A such that if $(x, y) \in R$ then $F(q)\alpha(x) = F(q)\alpha(y)$, with $q : A \to A/R$ the coequalizer of R.

Remark 4.7. In the usual sense, a congruence is required to be an equivalence relation. We work with the above definition for we shall need to consider small congruences on large coalgebras which cannot be small if required to be equivalence relations due to the reflexivity condition.

The reader could have expected that we would work with bisimulations; however, as we only ask F to satisfy the assumption of being set-based, it need not preserve weak pullbacks. The problem in this situation is that a congruence on an F-coalgebra (A, α) may not be a bisimulation. This problem prevents us to use bisimulations because we want to work with quotients with respect to arbitrary relations preserving coalgebra maps in the sense of the above definition. One can note indeed that all results involving quotients with respect to bisimulations in [Rut00] are proved under the assumption that F preserves weak pullbacks. An example of congruence on an F-coalgebra where F is a set-based endofunctor on *Classes* which is not a bisimulation can be found in [AM89, Section 6].

To turn a coalgebra into a strongly extensional one, the strategy is to define the largest congruence on (A, α) so as to form the smallest quotient of (A, α). In such a quotient, all elements that can possibly be identified are identified (in reference to the previous remark, note that it need not be the case if we form the quotient of (A, α) with respect to the largest bisimulation on it) which enables to prove that if there is some homomorphism from a (small) coalgebra into such quotient, then there can only be one such homomorphism. Since homomorphisms into (A, α) are carried to the quotient via the quotient map, applying this construction to a weakly complete coalgebra produces a complete coalgebra.

Proposition 4.8. For every F-coalgebra (A, α) for a set-based functor F there exists a strongly extensional F-coalgebra (B, β) and a surjective coalgebra homomorphism $(A, \alpha) \twoheadrightarrow (B, \beta)$.

We divide the proof in the following three lemmas.

Lemma 4.9. For every F-coalgebra (A, α) there exists a maximal congruence M on (A, α), i.e., there exists a congruence M on (A, α) such that every congruence on (A, α) is a subclass of M.

Proof. Let M be the union of all small congruences on (A, α). We claim that M is the (necessarily unique) maximal congruence on (A, α). To check that M is a congruence, take any $(x, y) \in M$. Then there exists R some small congruence such that $(x, y) \in R$. As R is a congruence on (A, α), we have $F(q_R)\alpha(x) = F(q_R)\alpha(y)$, and, since $R \subseteq M$, there is a (unique) coalgebra homomorphism $u : A/R \to A/M$ such that $uq_R = q_M$, as displayed in the following:

$$\begin{tikzcd}
 & R \arrow[d, shift left] \arrow[d, shift right] & & \\
M \arrow[r, shift left] \arrow[r, shift right] & A \arrow[rr, "q_M"] \arrow[dr, "q_M"'] & & A/M \\
 & & A/R \arrow[ur, dashed, "u"'] &
\end{tikzcd},$$

where q_M and q_R are the equalizers of M and R, respectively. Hence, we have

$$\begin{aligned}
F(q_M)\alpha(x) &= F(u)F(q_R)\alpha(x) \\
&= F(u)F(q_R)\alpha(y) \\
&= F(q_M)\alpha(y).
\end{aligned}$$

Therefore, M is a congruence. To check that M is maximal, let R be some large congruence on (A, α). For any $(x, y) \in R$, by Proposition 4.3, there exists (B, β) some small subcoalgebra of (A, α) such that $x, y \in B$. To show that $(x, y) \in M$, we prove that $R_{\upharpoonright B}$ is a congruence on (A, α), where $R_{\upharpoonright B}$ is the restriction of R to B. For that matter, consider the situation depicted below:

$$\begin{tikzcd}
R_{\upharpoonright B} \arrow[rr, "i_{R_{\upharpoonright B},R}"] \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] & & R \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & \\
B \arrow[rr, "i_{B,A}"] \arrow[dr, "q_{R_{\upharpoonright B}}"] \arrow[dd, "\beta"'] & & A \arrow[dr, "q_R"] \arrow[dd, "\alpha"] & \\
& B/R_{\upharpoonright B} \arrow[rr, dashed, "j"'] & & A/R \\
FB \arrow[rr, "Fi_{B,A}"] \arrow[dr, "Fq_{R_{\upharpoonright B}}"'] & & FA \arrow[dr, "Fq_R"] & \\
& FB/R_{\upharpoonright B} \arrow[rr, dashed, "Fj"'] & & FA/R
\end{tikzcd}$$

where q_R and $q_{R_{\upharpoonright B}}$ are the quotients of (A, α) with respect to R and $R_{\upharpoonright B}$, and the two upper squares are pullbacks. We first show that $R_{\upharpoonright B}$ is a congruence on (B, β) and then deduce that it is also a congruence on (A, α). As the upper squares are pullbacks, we have

$$q_R i s_1 = q_R i s_2$$

and we obtain the j by the universal property of $q_{R_{\upharpoonright B}}$. We claim that j is a monomorphism. Consider any $x, y \in B/R_{\upharpoonright B}$ and suppose that $j(x) = j(y)$, we have to show that $x = y$. Since $q_{R_{\upharpoonright B}}$ is surjective, x and y are given by $q_{R_{\upharpoonright B}}(a)$ and $q_{R_{\upharpoonright B}}(b)$ for some $a, b \in B$. As

$$j q_{R_{\upharpoonright B}} = q_R i_{B,A},$$

we have

$$j q_{R_{\upharpoonright B}}(a) = j q_{R_{\upharpoonright B}}(b),$$

hence

$$q_R(a) = q_R(b),$$

from which $(a,b) \in R$ and so $(a,b) \in R_{\restriction B}$. It follows that

$$q_{R_{\restriction B}}(a) = q_{R_{\restriction B}}(b),$$

as

required. As j is mono, preservation of monomorphisms gives that Fj is mono as well, besides, by functoriality,

$$F(j)F(q_{R_{\restriction B}}) = F(q_R)F(i).$$

Hence,

$$\begin{aligned}
& F(q_R)\alpha i_{B,A}s_1 = F(q_R)\alpha i_{B,A}s_2 \\
\Leftrightarrow\ & F(q_R)F(i_{B,A})\beta s_1 = F(q_R)F(i_{B,A})\beta s_2 \\
\Leftrightarrow\ & F(j)F(q_{R_{\restriction B}})\beta s_1 = F(j)F(q_{R_{\restriction B}})\beta s_2.
\end{aligned}$$

But

$$F(q_R)\alpha i_{B,A}s_1 = F(q_R)\alpha i_{B,A}s_2$$

holds since R is a congruence, so we have

$$F(j)F(q_{R_{\restriction B}})\beta s_1 = F(j)F(q_{R_{\restriction B}})\beta s_2,$$

and, since Fj is a monomorphism, we conclude that

$$F(q_{R_{\restriction B}})\beta s_1 = F(q_{R_{\restriction B}})\beta s_2,$$

i.e., $R_{\restriction B}$ is a congruence on (B,β).

To see that $R_{\restriction B}$ is a congruence on (A,α), consider the diagram below:

$$\begin{tikzcd}
R_{\restriction B} \arrow[rr, "1_{R_{\restriction B}}"] \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] & & R_{\restriction B} \arrow[d, shift right, "t_1"'] \arrow[d, shift left, "t_2"] & \\
B \arrow[rr, "i_{B,A}"] \arrow[dr, "q_B"'] \arrow[dd, "\beta"'] & & A \arrow[dr, "q_A"] \arrow[dd, "\alpha"] & \\
& B/R_{\restriction B} \arrow[rr, dashed, "k"'] & & A/R_{\restriction B} \\
FB \arrow[rr, "Fi_{B,A}"] \arrow[dr, "Fq_B"'] & & FA \arrow[dr, "Fq_A"] & \\
& FB/R_{\restriction B} \arrow[rr, dashed, "Fk"'] & & FA/R_{\restriction B},
\end{tikzcd}$$

where $t_1 = i_{B,A}s_1$, $t_2 = i_{B,A}s_2$, and k is obtained in the same way as j in the previous diagram. $R_{\restriction B}$ is a congruence on B, so

$$F(q_B)\beta s_1 = F(q_B)\beta s_2.$$

Hence,

$$\begin{aligned} & F(k)F(q_B)\beta s_1 = F(k)F(q_B)\beta s_2 \\ \Leftrightarrow\ & F(q_A)F(i_{B,A})\beta s_1 = F(q_A)F(i_{B,A})\beta s_2 \\ \Leftrightarrow\ & F(q_A)\alpha i_{B,A}s_1 = F(q_A)\alpha i_{B,A}s_2 \\ \Leftrightarrow\ & F(q_A)\alpha t_1 = F(q_A)\alpha t_2. \end{aligned}$$

Thus, $R_{\restriction B}$ is a congruence on (A,α). $\square$

Quotienting some F-coalgebra with respect to its maximal congruence results in an F-coalgebra that cannot be further quotiented.

Lemma 4.10. Let $q : A \to Q$ be the quotient of (A,α) with respect to its maximal congruence M and ρ be the unique coalgebra map on Q making q a coalgebra homomorphism. Then (Q,ρ) has no proper quotient, i.e., for any congruence R on (Q,ρ), if $(x,y) \in R$ then $x = y$.

Proof. Let R be any congruence on (Q,ρ). Take $\bar{R}$ the reflexive, symmetric, and transitive closure of R and define the equivalence relation

$$S = \{(x,y) \in A \times A \mid (q(x),q(y)) \in \bar{R}\}.$$

Consider the quotient $q_{\bar{R}} : Q \to Q/\bar{R}$ of (Q,ρ) with respect to $\bar{R}$. Suppose $(x,y) \in S$, then $(q(x),q(y)) \in \bar{R}$, so $q_{\bar{R}}q(x) = q_{\bar{R}}q(y)$. Conversely, if $q_{\bar{R}}q(x) = q_{\bar{R}}q(y)$, then $(q(x) = q(y)) \in \bar{R}$ since it is an equivalence relation, so $(x,y) \in S$. We conclude that $Q/\bar{R}$ is a quotient of (A,α) with respect to S. Illustrating the situation with the following diagram:

$$\begin{tikzcd} S \arrow[d, shift left] \arrow[d, shift right] & \bar{R} \arrow[d, shift left] \arrow[d, shift right] & \\ A \arrow[r, "q"] \arrow[d, "\alpha"] & Q \arrow[r, "q_{\bar{R}}"] \arrow[d, "\rho"] & Q/\bar{R} \\ FA \arrow[r, "Fq"'] & FQ \arrow[r, "Fq_{\bar{R}}"'] & FQ/R, \end{tikzcd}$$

we see that if $(x,y) \in S$ then $(q(x),q(y)) \in \bar{R}$ which implies that

$$F(q_{\bar{R}})\rho q(x) = F(q_{\bar{R}})\rho q(y)$$

since $\bar{R}$ is a congruence. Hence

$$F(q_{\bar{R}}q)\alpha(x) = F(q_{\bar{R}}q)\alpha(y),$$

i.e., S is a congruence on (A, a). As M is the maximal congruence on (A, α), $S \subseteq M$. Hence for any $(x, y) \in R$, $(x, y) \in \bar{R}$ and there exists $a, b \in A$ such that $q(a) = x$ and $q(b) = y$. Hence, $(a, b) \in S$, so $(a, b) \in M$ and so $q(a) = q(b)$, i.e., $x = y$. □

We conclude the proof of the proposition with this last lemma.

Lemma 4.11. If (A, α) has no proper quotient then it is strongly extensional.

Proof. Suppose that there exists two coalgebra homomorphisms $\tau_1, \tau_2 : (B, \beta) \to (A, \alpha)$. We want to show that $\tau_1 = \tau_2$. By Lemma 4.10, it suffices to check that the relation

$$R = \{(\tau_1(x), \tau_2(x)) \mid x \in B\}$$

is a congruence on (A, α). Let $q_R : A \to A/R$ be the quotient of A with respect to R. The following diagram depicts the situation:

$$\begin{tikzcd}
 & R \arrow[d, shift left] \arrow[d, shift right] & \\
B \arrow[r, shift left, "\tau_1"] \arrow[r, shift right, "\tau_2"'] \arrow[d, "\beta"'] & A \arrow[r, "q_R"] \arrow[d, "\alpha"] & A/R \\
FB \arrow[r, shift left, "F\tau_1"] \arrow[r, shift right, "F\tau_2"'] & FA \arrow[r, "Fq_R"'] & FA/R.
\end{tikzcd}$$

Then $q_R \tau_1 = q_R \tau_2$, so we have

$$F(q_R)F(\tau_1) = F(q_R)F(\tau_2)$$

by functoriality, hence

$$F(q_R)F(\tau_1)\beta = F(q_R)F(\tau_2)\beta,$$

and since τ_1 and τ_2 are coalgebra homomorphisms

$$F(q_R)\alpha\tau_1 = F(q_R)\alpha\tau_2.$$

Thus, R is a congruence on (A, α). □

Chapter 5

The Final Coalgebra Theorem in AST

Our aim in this chapter is to prove the following result.

Theorem. (The Final Coalgebra Theorem in AST) *Every indexed, set-based, and monomorphism preserving endofunctor F on a category of classes $\mathcal{E}$ satisfying the Relation Reflection Scheme admits a final coalgebra.*

Besides the Moerdijk-Joyal axioms for small maps introduced in 3.2, we assume that our category of classes $\mathcal{E}$ satisfies the Relation Reflection Scheme.

RRS For all I in $\mathcal{E}$, and for all A and R in the slice $\mathcal{E}/I$ such that $R \subseteq A \times_I A$ the following holds in $\mathcal{E}/I$:

$$\begin{aligned}\mathcal{E}/I \vDash & (\forall x : A)(\exists y : A)R(x, y) \\ & \rightarrow (\forall a : P_s(A))(\exists b : P_s(A))((\forall x : A)(x \in a \rightarrow x \in b) \\ & \wedge (\forall x \in a)(\exists y \in b)R(x, y)).\end{aligned}$$

The above formulation forces preservation under slicing so that Proposition 3.2.1 holds as well for **RRS**, which is necessary to work with the internal logic of $\mathcal{E}$.

In the standard case, the definitions of set-based functor and small subcoalgebra are given in terms of inclusion maps. In our context, we shall define them using the inclusion maps introduced in Section 3.2.

Definition 5.1. We call an endofunctor F on a category of classes $\mathcal{E}$ *set-based* if for every F-coalgebra (A, α) it holds in $\mathcal{E}$ that for all $x : A$, there exists some $Y : P_s(A)$ for which there is some $y : FY$ such that $Fi^A_Y(y) = \alpha(x)$.

Definition 5.2. An internal small subcoalgebras of (A, α) is, internally, a pair of variables (B, β) of type, respectively, $P_s(A)$ and FB^B such that $\alpha i^A_B = F(i^A_B)\beta$ holds in $\mathcal{E}$.

We start with the following key proposition.

Proposition 5.3. *Let F be an indexed, set-based, and monomorphism preserving endofunctor on a category of classes $\mathcal{E}$ satisfying the Relation Reflection Scheme. Then for every F-coalgebra (A, α) it holds in $\mathcal{E}$ that for all $X : P_s(A)$ there exists a small subcoalgebra (B, β) of (A, α) such that $X \subseteq B$.*

Proof. Define the relation

$$R = \{(X, Y) : P_s(A) \times P_s(A) \mid (\forall x : X)(\exists y : FY)(Fi^A_Y(y) = \alpha(x))\}.$$

We claim that R is total. Let $\varphi(x, Y)$ be the following formula where x and Y are of type A and $P_s(A)$, respectively:

$$\varphi(x, Y) = (\exists y : FY)(Fi^A_Y(y) = \alpha(x)).$$

Let $collec(x \in X, Y \in B, \varphi(x, Y))$ be the abbreviation of the following formula:

$$(\forall x \in X)(\exists Y \in B)(\varphi(x, Y)) \wedge (\forall Y \in B)(\exists x \in X)(\varphi(x, Y)).$$

By the assumption that F is set-based, we have

$$\mathcal{E} \models (\forall x \in X)(\exists Y : P_s(A))\varphi(x, Y),$$

and by **(A7)**, $\mathcal{E}$ satisfies the Strong Collection Schema – cf. Section 3.2. Hence, for any $X : P_s(A)$, we have:

$$\mathcal{E} \models (\exists B : P_s(A))(collec(x \in X, Y \in B, \varphi(x, Y)).$$

Define

$$C = \bigcup B.$$

To show that $R(X, C)$ holds, consider any $x \in X$. Then there exists some $Y : P_s(A)$ and some $y : FY$ such that $Fi^A_Y(y) = \alpha(x)$. As $i^A_Y = i^A_C i^A_{Y,C}$, by functoriality, we have $Fi^A_Y = Fi^A_C Fi^A_{Y,C}$, hence $Fi^A_C Fi^A_{Y,C}(y) = Fi^A_Y(y) = \alpha(x)$. Thus, $R(X, C)$ holds and we conclude that R is total.

Now we can use **RRS** to obtain some $B : P_s(P_s(A))$ such that R is total on B. To this aim, for any $X : P_s(A)$ consider the singleton $\{X\}$ which is given by the map $single : P_s(A) \to P_s(P_s(A))$ obtained by the universal property of $\in_{P_s(A)}$, as displayed below:

$$\begin{tikzcd}
\llbracket x = y \rrbracket \arrow[rr] \arrow[d, tail] & & \in_{P_s(A)} \arrow[d, tail] \\
P_s(A) \times P_s(A) \arrow[rr, "1_A \times single"'] & & P_s(A) \times P_s(P_s(A)).
\end{tikzcd}$$

Then **RRS** says that there exists some $B : P_s(P_s(A))$ such that $\{X\} \subseteq B$ and on which R is total. Define

$$C = \bigcup B.$$

Then for any $x : A$ such that $x \in C$, there exists some $Y : P_s(A)$ such that $Y \subseteq C$ and some $y : FY$ such that $Fi^A_Y(y) = \alpha(x)$, hence $Fi^A_C(Fi^A_{Y,C}(y)) = Fi^A_Y(y) = \alpha(x)$. Moreover, as i^A_C is mono the assumption that F preserves monomorphisms implies that Fi^A_C is a monomorphism. Hence, the relation

$$\{(x, y) : C \times FC \mid Fi^A_C(y) = \alpha(x)\}$$

is functional, and it is therefore correspond to the graph of an morphism $\gamma : C \to FC$ by Proposition 2.2.13. Thus, (C, γ) is a small subcoalgebra of (A, α) containing X, as required. $\square$

We adapt the terminology of the previous chapter to the algebraic setting.

Definition 5.4. We call an F-coalgebra (A, α) *complete* if it holds in $\mathcal{E}$ that for every small F-coalgebras (B, β) there exists exactly one coalgebra homomorphism from (B, β) to (A, α), *weakly complete* if there exists at least one such homomorphism, and *strongly extensional* if there exists at most one such homomorphism.

Remark 5.5. To quantify over all small F-coalgebras we use the fact that internally every small object is isomorphic to some fibre E_u of the universal small map $\pi : E \to U$ given by **(S2)**. A small coalgebra is then internally specified by some $u : U$ and $\alpha : FE_u^{E_u}$ which allows to define an object of all small coalgebras on which we can quantify. We shall go through this construction in more detail in the proof of Proposition 5.7.

As in the standard case, our task boils down to finding a complete F-coalgebra.

Theorem 5.6. *Complete F-coalgebras are final.*

Proof. Let (Z, ζ) be complete, and consider any large coalgebra (A, α). We must show that there exist a unique coalgebra homomorphism u from (A, α) to (Z, ζ). Define the following relation:

$$U = \{(x, z) : A \times Z \mid (\exists B : P_s(A))(\exists \beta : FB^B)(\exists z_b : Z^B)(\alpha i^A_B = Fi^A_B \beta \\ \wedge \zeta z_b = F z_b \beta \wedge x \in B \wedge z_b(x) = z)\}.$$

We claim that U is functional. For any $x : A$, we have $\{x\} : P_s(A)$. Applying Proposition 5.3, we obtain (B, β) a small subcoalgebra of (A, α) such that $x \in B$. As (Z, ζ) is complete, there exists a unique homomorphism $z_b : (B, \beta) \to (Z, \zeta)$, hence $U(x, z_b(x))$ holds. To check the uniqueness part of functionality, suppose the existence of (B_1, β_1) and (B_2, β_2) two small subcoalgebras of (A, α) such that $x \in B_1$ and $x \in B_2$. We want to show that $z_1(x) = z_2(x)$, where z_1 and z_2 are the unique homomorphisms from (B_1, β_1) and (B_2, β_2) into (Z, ζ). By Proposition 5.3 again, there exists (C, γ) a small subcoalgebra of (A, α) such that $B_1 \cup B_2 \subseteq C$. With $j = 1, 2$, we have

$$\begin{tikzcd}
B_j \arrow[r, "i^A_{B_j,C}"] \arrow[rr, bend left, "z_j"] \arrow[d, "\beta_j"] & C \arrow[r, "z_c"] \arrow[d, "\beta"] & Z \arrow[d, "\zeta"] \\
FB_j \arrow[r, "Fi^A_{B_j,B}"'] \arrow[rr, bend right, "Fz_j"'] & FC \arrow[r, "Fz_c"'] & FZ,
\end{tikzcd}$$

where the right and outer square commute. To show that the left square commutes as well, consider the diagram below:

$$\begin{tikzcd}
B_j \arrow[r, "i^A_{B_j,C}"] \arrow[rr, bend left, "i^A_{B_j}"] \arrow[d, "\beta_j"'] & C \arrow[r, "i^A_C"] \arrow[d, "\gamma"] & A \arrow[d, "\alpha"] \\
FB_j \arrow[r, "Fi^A_{B_j,C}"'] \arrow[rr, bend right, "Fi^A_{B_j}"'] & FC \arrow[r, "Fi^A_C"'] & FA.
\end{tikzcd}$$

The inclusion i^A_C is a monomorphism and so is Fi^A_C by preservation of monomorphisms. It then follows that the left square commutes. As (Z, ζ) is complete, we have $z_j = z_c i^A_B$, and since $z_c i^A_{Bj,C}(x) = z_c i^A_{Bj,C}(x)$, we conclude that $z_1(x) = z_2(x)$. Thus, U is a functional relation and is therefore the graph of a morphism $u : A \to Z$ by Proposition 2.2.13.

To see that u is indeed a coalgebra morphism, consider any $x : A$. By Proposition 5.3, there exists (B, β) a small subcoalgebra of (A, α) such that $x \in B$, as displayed below:

$$
\begin{tikzcd}
B \arrow[r, "i_B^A"] \arrow[d, "\beta"'] \arrow[rr, bend left, "z_b"] & A \arrow[r, "u"] \arrow[d, "\alpha"] & Z \arrow[d, "\zeta"] \\
FB \arrow[r, "Fi_B^A"'] \arrow[rr, bend right, "Fz_b"'] & FA \arrow[r, "Fu"'] & FZ.
\end{tikzcd}
$$

By construction, $ui_B^A = z_b$, hence $FuFi_B^A = Fz_b$, and we have:

$$
\begin{aligned}
\zeta u(x) &= \zeta z_b(x) \\
&= Fz_b(x)\beta(x) \\
&= FuFi_B^A\beta(x) \\
&= Fu\alpha i_B^A(a) \\
&= Fu\alpha(a).
\end{aligned}
$$

To prove uniqueness of u, suppose that we are given two such homomorphisms u and u'. For any $x : A$, by Proposition 5.3 again, we have (B, β) with $x \in B$:

$$
\begin{tikzcd}
B \arrow[r, "i_B^A"] \arrow[d, "\beta"'] & A \arrow[r, shift left, "u"] \arrow[r, shift right, "u'"'] \arrow[d, "\alpha"] & Z \arrow[d, "\zeta"] \\
FB \arrow[r, "F(i_B^A)"'] & FA \arrow[r, shift left, "Fu"] \arrow[r, shift right, "Fu'"'] & FZ.
\end{tikzcd}
$$

As (Z, ζ) is complete, $ui_B^A = u'i_B^A$. Therefore $u(x) = u'(x)$. $\square$

It remains to construct a complete F-coalgebra. We start with the construction of weakly complete F-coalgebra and then show how to turn it into a complete one.

Proposition 5.7. *There exists a weakly complete F-coalgebra.*

Proof. By **(S2)**, there exists a universal small map $\pi : E \to U$ such that, internally, there is an isomorphism from every small object A to E_u for some $u \in U$. As π is small, we can use **(S1)** to form the following exponential in the slice $\mathcal{E}/U$:

$$
\begin{tikzcd}
F^U(\pi : E \to U)^{(\pi : E \to U)} \arrow[d] \\
U.
\end{tikzcd}
$$

Being given that exponentials in slices consists in the objects of maps between fibres, this exponential that we denote S is the object of pairs

$$(u : U, \alpha : E_u \to (F^U E)_u).$$

But by indexation of F (see Definition 2.2.8) $(F^U E)_u) \cong F(E_u)$, so S is in fact the object

$$\{(u : U, \alpha : E_u \to FE_u)\},$$

i.e., the object of all small coalgebras.

Using S, we define the internal sum

$$T = \sum_{(u,\alpha)\in S} E_u,$$

the coproduct of all small objects on which there is a coalgebra morphism. We want to define a coalgebra map on T. Consider the following diagram:

$$\begin{tikzcd}
\sum_{(u,\alpha)\in S} E_u \arrow[d, "\sum_{(u,\alpha)\in S} \alpha"'] & E_u \arrow[l, "i"'] \arrow[d, "\alpha"] \\
\sum_{(u,\alpha)\in S} FE_u \arrow[d, dashed, "u"'] & FE_u \arrow[l, "j"] \arrow[dl, "Fi"] \\
F \sum_{(u,\alpha)\in S} E_u &
\end{tikzcd}$$

where i and j are coprojections. Then the collection of Fi's for each $(u, \alpha) \in S$ forms a cocone and we obtain u by universality of $\sum_{(u,\alpha)\in S} FE_u$. The composite $\tau = u \sum_{(u,\alpha)\in S} \alpha$ is a coalgebra morphism on T, moreover, by **(S2)** every small coalgebra is given as some $(u, \alpha) \in U$, therefore (T, τ) is weakly extensional. □

As in the standard case, the next proposition ensures that a weakly complete F-coalgebra can be made complete.

Proposition 5.8. *For any F-coalgebra (A, α), there exists a strongly extensional F-coalgebra (Z, ζ) and an epimorphism $(A, \alpha) \twoheadrightarrow (Z, \zeta)$.*

Before we embark in the proof, we give the internal and an external version of the congruences that we have worked with in the previous chapter.

Definition 5.9. We call $R : P_s(A \times A)$ a *internal congruence* on (A, α) if it holds in $\mathcal{E}$ that if $R(x, y)$ then $F(q)\alpha(x) = F(q)\alpha(y)$, where $q : A \twoheadrightarrow Q_R$ is the coequalizer of R. We call an external relation $R \rightarrowtail A \times A$ an *external congruence* if it holds in $\mathcal{E}$ that if $R(x, y)$ then $F(q)\alpha(x) = F(q)\alpha(y)$, where $q : A \twoheadrightarrow Q_R$ is the coequalizer of R.

We define maximality for external congruences; moreover, we require maximality to be a stable property.

Definition 5.10. An external congruence M on (A, α) is *stably maximal* on (A, α) if for every object I and every external congruence R on I^*A in $\mathcal{E}/I$, it holds in $\mathcal{E}/I$ that $R \subseteq I^*M$, where I^*A and I^*M are respectively the pullbacks of R and M along $I \to 1$.

We start by showing that such a maximal congruence on (A, α) exists.

Lemma 5.11. *There exists a maximal internal congruence on every F-coalgebra.*

We prove this lemma in two steps: first we construct an object of all internal congruences on (A, α) and next we show that the internal union of this object is a stably maximal congruence on (A, α).

Sublemma 5.12. *There exists a object of all internal small congruences on (A, α).*

Proof. Consider the composite p

$$\begin{tikzcd} \in_{P_s(A^2)} \arrow[d, tail] \arrow[dr, "q"] & \\ A^2 \times P_s(A^2) \arrow[r, "\pi_2"'] & P_s(A^2), \end{tikzcd}$$

which is small by definition of $\in_{P_s(A \times A)}$. As $A \times A \times P_s(A \times A)$ fits in the pullback square

$$\begin{tikzcd} A \times A \times P_s(A \times A) \arrow[r, "\langle \pi_2, \pi_3 \rangle"] \arrow[d, "\langle \pi_1, \pi_3 \rangle"'] & A \times P_s(A \times A) \arrow[d, "\pi_2"] \\ A \times P_s(A \times A) \arrow[r, "\pi_2"'] & P_s(A \times A), \end{tikzcd}$$

q is a subobject of $A \times P_s(A \times A) \times_{P_s(A \times A)} A \times P_s(A \times A)$ in the slice $\mathcal{E}/P_s(A \times A)$, and we can form its coequalizer:

$$\in_{P_s(A \times A)} \rightrightarrows A \times P_s(A \times A) \xrightarrow{q} Q,$$

in $\mathcal{E}/P_s(A \times A)$. We shall define a coalgebra morphism on $A \times P_s(A \times A)$ in $\mathcal{E}/P_s(A \times A)$. To this end, consider the following pullback of α:

$$\begin{tikzcd} A \times P_s(A^2) \arrow[rr, "\alpha \times 1"] \arrow[dr, "\pi_2"'] & & FA \times P_s(A^2) \arrow[dl, "\pi_2"] & A \arrow[r, "\alpha"] \arrow[d] & FA \arrow[dl] \\ & P_s(A^2) \arrow[rr] & & 1 & \end{tikzcd} .$$

As

$$P_s(A \times A) \times FA = P_s(A \times A)^* FA,$$

by indexation, we have

$$P_s(A^2)^* FA \cong F^{P_s(A\times A)}(P_s(A^2)^* A) = F^{P_s(A\times A)}(P_s(A^2) \times A),$$

which provides $A \times P_s(A\times A)$ with a coalgebraic structure $\bar{\alpha}$ in $\mathcal{E}/P_s(A\times A)$ by composition of the above isomorphism with $\alpha \times 1_{P_s(A\times A)}$:

$$\begin{tikzcd}
\in_{P_s(A^2)} \arrow[r, shift left] \arrow[r, shift right] & A \times P_s(A^2) \arrow[r, "q"] \arrow[d, "\bar{\alpha}"'] & Q \\
& F^{P_s(A^2)}(A \times P_s(A^2)) \arrow[r, "F^{P_s(A^2)}(q)"] & F^{P_s(A^2)}Q.
\end{tikzcd}$$

We defined this coalgebra morphism on $A \times P_s(A \times A)$ to obtain the above diagram which lives in the slice $\mathcal{E}/P_s(A \times A)$ and has the property that its fibre over any $R : P_s(A \times A)$ is

$$\begin{tikzcd}
R \arrow[r, shift left] \arrow[r, shift right] & A \arrow[r, "q_R"] \arrow[d, "\alpha"'] & A/R \\
& FA \arrow[r, "Fq_R"] & F(A/R).
\end{tikzcd} \tag{5.1}$$

Using $\bar{\alpha}$, we can define C the object of congruences on (A, α):

$$C = \{R : P_s(A^2) \mid \forall((x,R),(y,R)) :\in_{P_s(A^2)} (F^{P_s(A^2)}(q)\bar{\alpha}(x,R) = F^{P_s(A^2)}(q)\bar{\alpha}(y,R))\},$$

□

The stably maximal congruence is constructed by taking the internal union of C.

Sublemma 5.13. *The internal union of the object of all internal congruences on (A, α) is stably maximal.*

Proof. Let M be the following internal union:

$$M = \{(x,y) : A \times A \mid (\exists R : C)((x,y) \in R)\},$$

i.e., M is the image of $\pi_2\langle\pi_2, \pi_3\rangle$, as displayed in the following diagram where the square on the right-hand side is a pullback:

$$\begin{tikzcd}
M \arrow[d, tail] & \bullet \arrow[l, two heads] \arrow[r] \arrow[d, tail, "m"] & \in_{A\times A} \arrow[d, tail] \\
A \times A & A \times A \times C \arrow[l, "\langle\pi_2,\pi_3\rangle"] \arrow[r] & A \times A \times P_s(A \times A).
\end{tikzcd}$$

We claim that M is stably maximal. First, we check that it is a congruence. Suppose Mxy holds, then there exists some internal congruence R such that $R(x, y)$ holds. Let q_M and q_R be the coequalizers of M and R. Note that the coequalizer of R is internally given by the property describe in diagram 5.1. Although R is not an external object, we illustrate the situation with the following diagram:

$$\begin{tikzcd}
 & R \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & & \\
M \arrow[r, shift left] \arrow[r, shift right] & A \arrow[rr, "q_M"] \arrow[dr, "q_R"] \arrow[dd, "\alpha"] & & Q_M \\
 & & Q_R \arrow[ur, dashed, "u"] \arrow[dd] & \\
 & FA \arrow[rr, "Fq_M"] \arrow[dr, "Fq_R"'] & & FQ_M \\
 & & FQ_R \arrow[ur, dashed, "Fu"] &
\end{tikzcd} .$$

By definition, $R(x, y)$ implies Mxy, so $q_M(x) = q_M(y)$, and so externally we have $q_M r_1 = q_M r_2$. Hence, we obtain a morphism u by universality of q_R making the triangle commute. By functoriality, $Fq_M = FuFq_R$, and, since R is a congruence,

$$\begin{aligned}
&Fq_R\alpha(x) = Fq_R\alpha(y) \\
\Rightarrow\ &FuFq_R\alpha(x) = FuFq_R\alpha(y) \\
\Rightarrow\ &Fq_M\alpha(x) = Fq_M\alpha(y),
\end{aligned}$$

i.e., M is a congruence on (A, α).

To check maximality, let R be some external congruence on (A, α). For any $(x, y) : A \times A$ such that $R(x, y)$, consider $\{x, y\}$ which is given by the map *pair* $: A \times A \to P_s(A\times)$ obtained by the universal property of $\in_A$, as displayed below:

$$\begin{tikzcd}
\llbracket x = y \vee x = z \rrbracket \arrow[r] \arrow[d, tail] & \in_A \arrow[d, tail] \\
A \times (A \times A) \arrow[r, "1_A \times pair"'] & A \times P_s(A).
\end{tikzcd}$$

By Proposition 5.3, there exists (B, β) some small subcoalgebra of (A, α) such that $\{x, y\} \subseteq B$. To show that Mxy holds, we prove that $R_{\restriction B}$ the restriction of R to B given by pulling back R along i_B^A is a congruence on (A, α). For that matter, consider the situation depicted below:

$$
\begin{tikzcd}
R_{\restriction B} \arrow[rr] \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] & & R \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & \\
B \arrow[rr, "i_B^A"] \arrow[dr, "q_{R_{\restriction B}}"] \arrow[dd, "\beta"'] & & A \arrow[dr, "q_R"] \arrow[dd, "\alpha"] & \\
& B/R_{\restriction B} \arrow[rr, dashed, "j"'] & & A/R \\
FB \arrow[rr, "Fi_B^A"] \arrow[dr, "Fq_{R_{\restriction B}}"'] & & FA \arrow[dr, "Fq_R"] & \\
& FB/R_{\restriction B} \arrow[rr, dashed, "Fj"'] & & FA/R,
\end{tikzcd}
$$

where q_R and $q_{R_{\restriction B}}$ are the coequalizers of R and $R_{\restriction B}$. We first show that $R_{\restriction B}$ is a congruence on (B, β), and then deduce that it is a congruence on (A, α). As the upper squares are pullbacks, we have

$$q_R i_B^A s_1 = q_R i_B^A s_2,$$

and we obtain j by the universal property of $q_{R_{\restriction B}}$. We claim that j is a monomorphism. Suppose that $j(x) = j(y)$, we have to show that $x = y$. Since $q_{R_{\restriction B}}$ is a regular epimorphism, it is surjective by Proposition 2.2.11(iv), so $x = q_{R_{\restriction B}}(a)$ and $y = q_{R_{\restriction B}}(b)$ for some $a, b : B$. As

$$j(x) = j(y),$$

$$j q_{R_{\restriction B}}(a) = j q_{R_{\restriction B}}(b),$$

and, as

$$j q_{R_{\restriction B}} = q_R i_B^A,$$

we have

$$q_R(a) = q_R(b),$$

so that $R(a, b)$, so $R_{\restriction B}(a, b)$ and we conclude that

$$q_{R_{\restriction B}}(a) = q_{R_{\restriction B}}(b),$$

as required. Since j is mono, preservation of monomorphisms gives that Fj is mono as well, besides, by functoriality,

$$F(j)F(q_{R_{\restriction B}}) = F(q_R)F(i_B^A).$$

So we have

$$
\begin{aligned}
& F(q_R)\alpha i_B^A s_1 = F(q_R)\alpha i_B^A s_2 \\
\Leftrightarrow\ & F(q_R)F(i_B^A)\beta s_1 = F(q_R)F(i_B^A)\beta s_2 \\
\Leftrightarrow\ & F(j)F(q_{R_{\restriction B}})\beta s_1 = F(j)F(q_{R_{\restriction B}})\beta s_2.
\end{aligned}
$$

But

$$F(q_R)\alpha i_{B,A}s_1 = F(q_R)\alpha i_{B,A}s_2$$

holds since R is a congruence, so

$$F(j)F(q_{R_{\restriction B}})\beta s_1 = F(j)F(q_{R_{\restriction B}})\beta s_2,$$

and, since Fj is a monomorphism, we conclude that

$$F(q_{R_{\restriction B}})\beta s_1 = F(q_{R_{\restriction B}})\beta s_2,$$

i.e., $R_{\restriction B}$ is a congruence on (B, β).

To see that $R_{\restriction B}$ is a congruence on (A, α), consider the diagram below:

$$\begin{tikzcd}
R_{\restriction B} \arrow[rr, "1_{R_{\restriction B}}"] \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] & & R_{\restriction B} \arrow[d, shift right, "t_1"'] \arrow[d, shift left, "t_2"] & \\
B \arrow[rr, "i^A_B"] \arrow[dr, "q_B"] \arrow[dd, "\beta"'] & & A \arrow[dr, "q_A"] \arrow[dd, "\alpha"] & \\
& B/R_{\restriction B} \arrow[rr, dashed, "k"'] & & A/R_{\restriction B} \\
FB \arrow[rr, "Fi^A_B"] \arrow[dr, "Fq_B"'] & & FA \arrow[dr, "Fq_A"] & \\
& FB/R_{\restriction B} \arrow[rr, dashed, "Fk"'] & & FA/R_{\restriction B},
\end{tikzcd}$$

where $t_1 = i^A_B s_1$, $t_1 = i^A_B s_1$, and k is obtained by the universal property of q_B. $R_{\restriction B}$ is a congruence on B, so

$$F(q_B)\beta s_1 = F(q_B)\beta s_2.$$

Hence,

$$\begin{aligned}
&F(k)F(q_B)\beta s_1 = F(k)F(q_B)\beta s_2 \\
\Rightarrow\ &F(q_A)F(i^A_B)\beta s_1 = F(q_A)F(i^A_B)\beta s_2 \\
\Rightarrow\ &F(q_A)\alpha i^A_B s_1 = F(q_A)\alpha i^A_B s_2.
\end{aligned}$$

Thus, $R_{\restriction B}$ is a congruence on (A, α). The fact that M is stably maximal simply follows from the fact that any argument in the internal logic is stable under slicing. $\square$

Next, we prove that the quotient of (A, α) with respect to its maximal congruence has no proper quotient.

Lemma 5.14. *Let $q_M : A \to Q$ is the quotient of (A, α) with respect to its maximal congruence and ρ be the unique coalgebra morphism on Q making q a coalgebra homomorphism. Then (Q, ρ) has no proper quotient, i.e., it holds in $\mathcal{E}$ that for every internal congruence R on (Q, ρ), $R(x, y)$ implies $x = y$.*

Proof. Let q_M be the coequalizer of the maximal congruence on (A, α), R be any congruence on Q, and $\bar{R}$ be the reflexive, symmetric, and transitive closure of R, constructed as in Proposition 3.1. Further, let S be the equivalence relation given by

$$S = \{(x, y) : A \times A \mid \bar{R}(q_M(x), q_M(y))\}.$$

The situation is depicted below where $q_{\bar{R}}$ is the coequalizer of $\bar{R}$:

$$\begin{tikzcd}
S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] & \bar{R} \arrow[d, shift right, "t_1"'] \arrow[d, shift left, "t_2"] & \\
A \arrow[r, two heads, "q_M"] \arrow[d, "\alpha"] & Q \arrow[r, two heads, "q_{\bar{R}}"] \arrow[d, "\rho"] & Q/\bar{R} \\
FA \arrow[r, "Fq_M"'] & FQ \arrow[r, "Fq_{\bar{R}}"'] & FQ/\bar{R}.
\end{tikzcd}$$

We show that $q_{\bar{R}} q_M$ is a coequalizer of S, and next that S is a congruence on (A, α). As S is an equivalence relation, by Proposition 2.2.11(vii), we have to check that the following sequents hold in $\mathcal{E}$:

$$\vdash (\forall x : Q/\bar{R})(\exists y : Q)(q_{\bar{R}} q_M(y) = x),$$

$$S(x, y) \vdash q_{\bar{R}} q_M(x) = q_{\bar{R}} q_M(y),$$

and

$$q_{\bar{R}} q_M(x) = q_{\bar{R}} q_M(y) \vdash S(x, y).$$

The first sequent holds because $q_{\bar{R}} q_M$ is an epimorphism, and the second because $q_{\bar{R}} q_M s_1 = q_{\bar{R}} q_M s_2$. The last sequent holds as well for if $q_R q_M(x) = q_R q_M(y)$, then $\bar{R}(q_M(x), q_M(y))$ holds since $q_{\bar{R}}$ is the coequalizer of $\bar{R}$ and $\bar{R}$ is an equivalence relation, so by definition $S(x, y)$ holds.

To show that S is a congruence on (A, α), suppose $S(x, y)$. Then $\bar{R}(q_M(x), q_M(y))$ holds. Since R is a congruence, using **(NNO)** we can prove by induction on the R-paths that $\bar{R}$ is a congruence as well. Hence

$$F(q_{\bar{R}})\rho q_M(x) = F(q_{\bar{R}})\rho q_M(x),$$

and as

$$\rho q_M = F q_M \alpha,$$

we have

$$F(q_{\bar{R}})F(q_M)\alpha(x) = F(q_{\bar{R}})F(q_M)\alpha(y),$$

as required.

To conclude the proof of this lemma, suppose $R(x, y)$ holds. As $q_M : A \to Q$ is a regular epimorphism, it is surjective by Proposition 2.2.11(iv). Therefore, $q_M(a) = x$ and $q_M(b) = y$ for some $a, b : A$ so that $S(a, b)$ holds. But, since S is a congruence on (A, α), we have Mab. Thus $q_M(a) = q_M(b)$, i.e., $x = y$. □

Finally, we show that F-coalgebras which have no proper quotient are strongly extensional.

Lemma 5.15. *If (A, α) has no proper quotient, then it is strongly extensional.*

Proof. Suppose (A, α) has no proper quotient, and let $\tau_1, \tau_2 : B \to A$ be two coalgebra homomorphisms with B a small object. We have to show that $\tau_1 = \tau_2$. Consider the relation R given by

$$R = \{(x, y) : A \times A \mid (\exists b : B)(x = \tau_1(b) \wedge y = \tau_2(b))\},$$

and the following diagram:

$$\begin{tikzcd}
 & R \arrow[d, shift left] \arrow[d, shift right] & \\
B \arrow[r, shift left, "\tau_1"] \arrow[r, shift right, "\tau_2"'] \arrow[d, "\beta"'] & A \arrow[r, "q_R"] \arrow[d, "\alpha"] & A/R \\
FB \arrow[r, shift left, "F\tau_1"] \arrow[r, shift right, "F\tau_2"'] & FA \arrow[r, "Fq_R"] & FA/R,
\end{tikzcd}$$

where q_R is the equalizer of R. For any $b : B$, $R(\tau_1(b), \tau_2(b))$, so $q_R\tau_1(b) = q_R\tau_2(b)$ from which we have $q_R\tau_1 = q_R\tau_2$. By functoriality,

$$F(q_R)F(\tau_1)\beta = F(q_R)F(\tau_2)\beta,$$

and since $F\tau_i\beta = \alpha\tau_i$, we have $Fq_R\alpha\tau_1 = Fq_R\alpha\tau_2$. So R is a internal congruence on (A, α). Consequently, as (A, α) is assumed to have no proper quotient, $R(\tau_1(b), \tau_2(b))$ implies $\tau_1(b) = \tau_2(b)$. Thus, as $R(\tau_1(b), \tau_2(b))$ holds for all $b : B$, $\tau_1 = \tau_2$. □

Chapter 6

Further Research

In this thesis, we started with a review of Aczel and Mendler's Final Coalgebra Theorem. We have shown that the Axiom of Choice was not necessary to their result and could be avoided with the Relation Reflection Scheme, a principle provable in **ZF**.

The main result of this work is a new, more insightful proof of the Final Coalgebra Theorem in AST, originally proved by Van den Berg and De Marchi. While they provide a general result granting the existence of final objects in indexed categories from which the recover the theorem as a special case, we managed to recast Aczel and Mendler's argument directly in the internal logic of our category of classes. Although in the original proof the authors do not need to assume the Relation Reflection Scheme, their result requires the preservation of weak pullbacks. Our result is an improvement in this respect for we only needed the preservation of monomorphisms.

Building upon our theorem, an interesting goal for further research would be to prove a dual version, namely, that indexed set-based functors preserving monomorphisms also admits an initial algebra. This dual result could possibly find an application in type theory as it would entail the existence of so-called W-types, see [MP00]. The following seems to be a promising strategy: given an endofunctor F satisfying our assumptions, our result ensures the existence of a final F-coalgebra (A, α). By Lambek's lemma, α is an isomorphism, so it has an inverse α^{-1} which provides A with an algebraic structure. As α^{-1} is an isomorphism, it is a monomorphism; the idea at this point would be to adapt to our setting a folklore result, see, e.g., [jib97], claiming that in this case the smallest subalgebra of (A, α^{-1}) is an initial algebra.

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