Towards a Proof-Theoretic Semantics for Dynamic Logics

MSc Thesis *(Afstudeerscriptie)*

written by

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under the supervision of **Dr Alessandra Palmigiano** and **Dr Giuseppe Greco**, and submitted to the Board of Examiners in partial fulfillment of the requirements for the degree of

MSc in Logic

at the *Universiteit van Amsterdam.*

Date of the public defense:
September 27th, 2013

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Abstract

This thesis provides an analysis of the existing proof systems for dynamic epistemic logic from the viewpoint of proof-theoretic semantics. After an illustration of the basic principles of proof-theoretic semantics, we review some of the most significant proposals of proof systems for dynamic epistemic logics, and we critically reflect on them in the light of proof-theoretic semantic principles. The main original contributions of the present thesis are: (a) a revised version of the display-style calculus D.EAK [14], which we argue to be more adequate from the proof-theoretic semantic viewpoint; the main feature of this revision is that a smoother proof (so-called Belnap-style) of cut-elimination holds for it, which is problematic for the original version of D.EAK. (b) The introduction of a novel, multi-type display calculus for dynamic epistemic logic, which we refer to as Dynamic Calculus. The presence of types endows the language of the Dynamic Calculus with additional expressivity, and makes it possible to design rules with an even smoother behavior. We argue that this calculus paves the way towards a general methodology for the design of proof systems for the generality of dynamic logics, and certainly for proof systems beyond dynamic epistemic logic. We prove that the Dynamic Calculus adequately captures Baltag-Moss-Solecki's dynamic epistemic logic, and enjoys Belnap-style cut elimination.

Acknowledgements

The exceptional supervision of Alessandra Palmigiano and Giuseppe Greco exceeded all of my expectations. They provided me with all the support imaginable and I had the opportunity to be part of their research team, which turned working on this thesis into the most valuable experience I had during my master studies. I have to thank them for sharing all of their insights with me, for the time and effort they invested in working with me, and finally, for being my friends. I would like to thank a very special teacher, Alexandru Baltag, for providing me with a solid background in different areas of logic and bringing dynamic logics to my attention. He provided me with the tools which enabled me to work on this thesis. Moreover, I would like to thank him for spreading his knowledge and enthusiasm - his dedication and passion for logic are inspirational for all his students. I am grateful to Michael Moorgat for disclosing to me the possible applications of proof-theory during a course I took, and for accepting to be part of the committee from a distance. I also wish to thank Maria Aloni for taking the time to read this thesis. I am indebted to everyone from the Master of Logic who gave me a hand and made it possible for me to finish my master studies.

Being part of the Master in Logic community meant meeting logic lovers from all over the world, working as a team, and more importantly sharing visions and socio-cultural experiences. I would like to address all of my fellow logic students and thank them. I have to thank Johannes Marti in particular-the very best friend I have had in Amsterdam-for his unconditional support and presence during the difficult moments of my life.

At the end of this life chapter, what makes me most proud is my family's courage to face the worst and their capacity to keep on going without showing any weakness until the very end. Thank you for being who you are.

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Chapter 1

Introduction

In recent years, driven by applications in areas spanning from program semantics to game theory, the logical formalisms pertaining to the family of *dynamic logics* [15, 27] have been very intensely investigated, giving rise to a proliferation of variants.

Typically, the language of a given dynamic logic is an expansion of classical propositional logic with an array of modal-type *dynamic operators*, each of which takes an *action* as a parameter. The set of actions plays in some cases the role of a set of indexes or parameters; in other cases, actions form a quantale-type algebra. When interpreted in relational models, the formulas of a dynamic logic express properties of the model encoding the present state of affairs, as well as the pre- and post-conditions of a given action. Actions formalize transformations of one model into another one, the updated model, which encodes the state of affairs after the action has taken place.

Dynamic logics have been investigated mostly w.r.t. their semantics and complexity, while their proof-theoretic aspects have been comparatively not so prominent. However, the existing proposals of proof systems for dynamic logics witness a varied enough array of methodologies, that a methodological evaluation is now timely.

The present thesis is aimed at evaluating the current proposals of proof-systems for some dynamic logics from the viewpoint of *proof-theoretic semantics.*

Proof-theoretic semantics [26, 28] is an area of research in structural proof theory which aims at providing a sound alternative way of defining the meaning of logical connectives, which is to be given not in terms of denotational, truth-based procedures, but rather in terms of an analysis of the behavior of the logical connectives inside the derivations of a given proof system. Such an analysis is possible only in the context of proof systems which perform well w.r.t. certain criteria; hence one of the main themes in proof theoretic semantics is to identify design criteria which both guarantee that the proof system enjoys certain desirable properties such as normalization or cut-elimination, and which make it possible to speak about the proof-theoretic meaning for given logical connectives.

An analysis of dynamic logics from the proof-theoretic semantics viewpoint is beneficial both for dynamic logics and for proof-theoretic semantics. Indeed, such an analysis provides dynamic logics with sound methodological and foundational principles, and with an entirely novel perspective on the topic of dynamics and change, which is *independent* from the dominating model-theoretic methods. Moreover, such an analysis provides proof-theoretic semantics with a novel array of case studies against which to test the generality of its principles, and with the opportunity to extend its modus operandi to still uncharted settings, such as multi-type calculi.

The structure of this thesis goes as follows:
In Chapter 2, we sketch the basic principles of proof-theoretic semantics, and we explain their consequences and spirit, in view of their applications in the following chapters. In Chapter 3, we review some of the most significant proposals of proof systems for dynamic logics, focusing on very well known examples of dynamic epistemic logics, namely the logic of Public Announcements (PAL) [22] and the logic of Epistemic Knowledge and Actions (EAK) [7], and we critically reflect on them in the light of the principles of proof-theoretic semantics stated in Chapter 2. In Chapter 4, we expand on one display-type calculus for PAL/EAK: we highlight its critical issues—the main of which being that a smooth (*Belnap-style*) proof of cut elimination is not readily available for it; we propose a revised version, arguing why the revision is more adequate for proof-theoretic semantics, and finally prove the Belnap-style cut-elimination theorem for the revised version. In Chapter 5, we propose a novel, multi-type display calculus for EAK, which we refer to as *Dynamic Calculus*. The presence of types endows the language of the Dynamic Calculus with additional expressivity, and makes it possible to design rules with an even smoother behavior. We argue that this calculus paves the way towards a general methodology for the design of proof systems for the generality of dynamic logics, and certainly for proof systems beyond dynamic epistemic logic. We prove the soundness of this calculus w.r.t. the final coalgebra semantics, the completeness w.r.t. EAK, of which it is a conservative extension, and the Belnap-style cut elimination. In Chapter 6, we collect some conclusions and indicate further directions. Most of the proofs and derivations are collected in Chapter 7, the appendix.

Chapter 2

Proof-Theoretic Semantics

In the present chapter, we introduce the field of proof-theoretic semantics in the context of general proof theory. We will introduce and illustrate the main conceptual foundations and approach of proof-theoretic semantics, especially targeting the issues which will be needed in the further development of the thesis.

2.1 Introducing proof-theoretic semantics

The phrase 'proof theory' commonly stands for *general proof theory*, that is, the branch of mathematical logic which investigates proofs as mathematical objects in their own right through their combinatorial properties, and not as tools for analyzing the more primitive notion of consequence relation. The motivation of this line of research originates in Hilbert's program in formalizing the foundations of mathematics, and its official starting contribution is Gerhard Gentzen's paper [11]. Nowadays, additional considerations, stemming from different fields such as linguistics, artificial intelligence, computer science, keep the interest in general proof theory vivid. Of course, the first question that proof theory tries to give an answer to is 'what is a proof'. In particular, Dag Prawitz [23] specifies the following four basic topics in general proof theory:

(1) Defining the notion of a proof.

(2) Investigating the structure of different kinds of proofs.

(3) Representing proofs as derivations and investigating equivalence among them.

(4) Applying these insights to other questions in logic.

Within general proof theory, *proof-theoretic semantics* is based on the idea that a purely inferential theory of meaning is possible. That is, that the meaning of expressions (in a formal language or in natural language) can be captured purely in terms of the proofs and the inference rules which participate in the generation of the given expression, or in which the given expression participates. This *inferential* view is opposed to the mainstream *denotational* view on the theory of meaning, according to which truth values are the primary source of meaning for expressions. The inferential perspective on the theory of meaning is very influential in e.g. linguistics, and links up to the view, commonly attributed to Wittgenstein, that 'meaning is use'. That is, certain parts of language, e.g. connectives, can only be coherently explained in terms of the way they are used: the context in which they occur, the rules governing them, etc. In proof theory, this idea links up with Gentzen's very famous observation about the introduction and elimination rules of his natural deduction calculi:

> 'The introductions represent, as it were, the definitions of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions. This fact may be expressed as follows: In eliminating a symbol, we may use the formula with whose terminal symbol we are dealing only in the sense afforded it by the introduction of that symbol'. ([11] p. 80)

In proof-theoretic semantics, this observation is brought to its consequences: rather than viewing proofs as entities the meaning of which is dependent on denotation, proof-theoretic semantics assigns proofs (in the sense of formal deductions) an *autonomous semantic role*; that is, proofs are entities in terms of which meaning can be accounted for. In this sense, the expression 'proof-theoretic semantics' stands for 'semantics through proofs'.

2.2 Structural characterization of logical constants

Proof-theoretic semantics normally focuses on *logical constants*. Research in this area has focused on the so-called *structural characterization* of logical constants, which aims at characterizing logical constants purely in proof-theoretic terms. Even if a given logical constant is defined in terms of truth-conditions, its structural characterization involves capturing its inferential behavior in the setting of a proof-system.

For instance, the structural characterization of implication lies at the heart of many proof-theoretic semantics settings. In these settings, implication is typically taken as a primitive logical constant, its main feature being its intimate relationship with the concept of metalogical consequence. Indeed, implication can be viewed as expressing metalogical consequence at the sentential level, thanks to *modus ponens* and the *deduction theorem*, which together give the equivalence between $\Gamma, A \vdash B$ and $\Gamma \vdash A \to B$.

A very natural way to understand $A \to B$ is that this formula encodes the inference rule which allows to deduce B from A. Allowing the step from A to B on the basis of $A \to B$ is exactly what the rule of *modus ponens* says. Conversely, the deduction theorem establishes the rule which, having shown that B can be deduced from A, captures this fact at the level of the formula $A \to B$.
Although proof-theoretic semantics has been originally set in natural deduction, the line of research which is most immediately relevant to us aims at the proof-theoretic characterization of logical constants in the setting of sequent calculi. In particular, the contributions which are most relevant to our analysis are Belnap's [8], Wansing's [28], Goré [12] and Restall's [24].

2.3 Display calculus

Nuel Belnap introduced the first display calculus, which he calls *Display Logic* [8], as a sequent system augmenting and refining Gentzen's basic observations on structural rules. Belnap's refinement is based on the introduction of a special syntax for the constituents of each sequent. Indeed, his calculus treats sequents $X \vdash Y$ where X and Y are so-called *structures*, i.e. syntactic objects inductively defined from formulas using an array of special connectives. Belnap's basic idea is that, in the standard Gentzen formulation, the comma symbol , separating formulas in the precedent and in the succedent of sequents can be recognized as a metalinguistic connective, of which the structural rules define the behavior.

Hence, Belnap took this idea several steps further, by allowing not only the comma, but also several other connectives to keep formulas together in a structure, and called them *structural connectives*. These connectives maintain relations with one another, the most fundamental of which take the form of adjunctions and residuations. These relations make it possible for the calculus to enjoy the powerful property which gives it its name, namely, the *display property*. Before introducing it formally, let us agree on some auxiliary definitions and nomenclature: structures are defined much in the same way as formulas, taking formulas as atomic components, by applying structural connectives; therefore, each structure can be uniquely associated with and identified by a generation tree. Every node of such a generation tree defines a *substructure*.

Definition 1. A proof system enjoys the *display property* iff for every sequent $X \vdash Y$ and every substructure Z of either X or Y, the sequent $X \vdash Y$ can be equivalently transformed, using the rules of the system, into a sequent which is either of the form $Z \vdash W$ or of the form $W \vdash Z$, for some structure W. In the first case, Z is *displayed in precedent position*, and in the second case, Z is *displayed in succedent position*. The rules enabling this equivalent rewriting are called *display postulates*.

Thanks to the fact that the display postulates are based on adjunction and residuation, it can be proved that exactly one of the two alternatives mentioned in the definition above occurs. In other words, in a system enjoying the display property, any substructure of any sequent $X \vdash Y$ is always displayed either only in precedent position or only in succedent position. This is why we can talk about occurrences of substructures in precedent or in succedent position, even if they are nested deep within a given sequent.

Example 2.3.1.

$$\cfrac{\cfrac{\cfrac{Y \vdash X > Z}{X ; Y \vdash Z}}{Y ; X \vdash Z}}{X \vdash Y > Z}$$

In the example above, the structure X is on the right side of the turnstile, but it is displayable on the left, and therefore is in precedent position. As we will see next, the display property is a crucial technical ingredient for display calculi, but it is also at the basis of Belnap's methodology for characterizing operational connectives: according to Belnap, an operational connective should be introduced *in isolation*, i.e., when it is introduced, its context must be empty. The display property guarantees that this condition is not too restrictive.

In [8], a meta-theorem is proven, which gives sufficient conditions in order for a sequent calculus to enjoy the cut elimination. This meta-theorem captures the essentials of the cut-elimination procedure Gentzen-style, and is the main technical motivation for the design of the Display Logic. Belnap's analysis is inspired by a previous one, given by Curry [4]. Belnap's meta-theorem is particularly useful, since it gives a set of eight conditions, which are relatively easy to check, given that most of them are verified by inspection on the shape of the rules. When Belnap's meta-theorem can be applied, it provides a much smoother route to cut elimination than the Gentzen-style proofs, because the latter are regrettably non-modular, in the sense that, if a new rule is added to a cut-free system, cut-elimination for the resulting system cannot be deduced from the old one, and must be proved from scratch. In this perspective, Belnap's cut-elimination meta-theorem allows a greater degree of modularity. Belnap's criteria for proof-theoretic semantics might be considered rigid in comparison to the criteria proposed by other authors, but this is in a sense a price to pay to achieve Belnap's modular cut-elimination procedure. Let us discuss Belnap's conditions.

C_1: preservation of formulas. Each formula occurring in a premise of a given inference is a subformula of some formula in the conclusion of that inference. That is, structure may disappear, but not formulas. This condition is not included in the list of sufficient conditions of the meta-theorem for cut elimination, but, in the presence of cut elimination, guarantees the subformula property of a system. Condition C_1 can be verified by inspection on the shape of the rules.

$\mathbf{C}_2$: Shape-alikeness of parameters. This condition is based on the relation of *congruence* between *parameters* (i.e., non-active parts) in the rules; the congruence relation is an equivalence relation which is meant to identify the different occurrences of the same formula or substructure along the branches of a derivation. Condition C_2 is actually a condition on the definition of the congruence relation on parameters, but can be understood as a condition on the *design* of the rules of the system if the congruence relation is understood as part of the specification of each given rule; that is, each rule of the system comes with an explicit specification of which elements are congruent to which (and then the congruence relation is defined as the reflexive and transitive closure of the resulting relation). In this respect, C_2 is nothing but a sanity check, requiring that the congruence is defined in such a way that indeed identifies the occurrences which are intuitively "the same".

$\mathbf{C}_3$: Non-proliferation of parameters. Like the previous one, also this condition is actually about the definition of the congruence relation on parameters. Condition C_3 requires that, for a rule such as the following,

$$\frac{X \vdash Y}{X, X \vdash Y}$$

the structure X from the premise is congruent to *only one* occurrence of X in the conclusion sequent. Indeed, the introduced occurrence of X should be considered congruent only to itself. Moreover, given that congruence is an equivalence relation, condition C_3 implies that, within a given sequent, any substructure is congruent only to itself.

$\mathbf{C}_4$: Position-alikeness of parameters. This condition bans any rule in which a (sub)structure in precedent (resp. succedent) position in a premise is congruent to a (sub)structure in succedent (resp. precedent) position in the conclusion.

$\mathbf{C}_5$: Display of principal constituents. This condition requires that, in the conclusion of every operational rule, the *non-parametric* formula—i.e. the formula introduced by the application of the operational rule in question—occurs in isolation, i.e., it is either the entire precedent or the entire succedent of the conclusion.

The following conditions C_6 and C_7 are not reported below as they were stated in the original paper [8], but as they appear in [28, section 4.1].

$\mathbf{C_6}$: Closure under substitution for succedent parameters. This condition requires each rule to be closed under simultaneous substitution of arbitrary structures for congruent formulas which occur in succedent position. Condition C_6 can be understood as follows:

$$\frac{(X \vdash Y)[A]^{succedent}}{(X' \vdash Y')[A]^{succedent}}\, R \qquad \rightsquigarrow \qquad \frac{(X \vdash Y)[Z]^{succedent}}{(X' \vdash Y')[Z]^{succedent}}\, R.$$

Any rule R should be such that, for any parametric formula A which is in succedent position, if A is substituted for an arbitrary structure Z both in the premise(s) and in the corresponding place in the conclusion, the resulting inference should always be justified as an application of the rule R. This condition caters for the step in the cut elimination procedure in which the cut needs to be "pushed up" over rules in which the cut-formula in succedent position is parametric. Indeed, the following transformation is guaranteed go through uniformly and "canonically":

$$\dfrac{\begin{array}{c}\vdots\, \pi_1' \\ X' \vdash A \\ \vdots\, \pi_1 \\ X \vdash A\end{array} \qquad \begin{array}{c}\vdots\, \pi_2 \\ A \vdash Y\end{array}}{X \vdash Y} \qquad \rightsquigarrow \qquad \begin{array}{c}\dfrac{\begin{array}{c}\vdots\, \pi_1' \\ X' \vdash A\end{array} \qquad \begin{array}{c}\vdots\, \pi_2 \\ A \vdash Y\end{array}}{X' \vdash Y} \\ \vdots\, \pi_1[Y/A] \\ X \vdash Y\end{array}$$

if each rule in π_1 verifies condition C_6.

$\mathbf{C_7}$: Closure under substitution for precedent parameters. This condition requires each rule to be closed under simultaneous substitution of arbitrary structures for congruent formulas which occur in succedent position. Condition C_7 can be understood analogously to C_6, relative to formulas in precedent position.

$$\frac{(X \vdash Y)[A]^{precedent}}{(X' \vdash Y')[A]^{precedent}}\, R \qquad \rightsquigarrow \qquad \frac{(X \vdash Y)[Z]^{precedent}}{(X' \vdash Y')[Z]^{precedent}}\, R$$

Dually to what discussed for condition C_6, condition C_7 caters for the step in the cut elimination procedure in which the cut needs to be "pushed up" over rules in which the cut-formula in precedent position is parametric. We will return on conditions C_6 and C_7 below in this section.

$\mathbf{C_8}$: Eliminability of matching principal constituents. This condition requests a standard Gentzen-style checking, which is now limited to the case in which both cut formulas are *principal*, i.e. each of them has been introduced with the last rule application of each corresponding subdeduction. In this case, analogously to the proof Gentzen-style, condition C_8 requires being able to transform the given deduction into a deduction with the same conclusion in which either the cut is eliminated altogether, or is transformed in one or more applications of cut involving proper subformulas of the original cut formula.

Let us now return on conditions C_6 and C_7. In [28, section 4.4], Wansing reports that, in order to extend the Belnap-style cut elimination to e.g. linear logic, conditions C_6 and C_7 as given above need to be replaced by the following more general condition:

$\mathbf{C_6/C_7}$: Regularity of parametric formulas. This condition requires that in each rule, each parametric occurrence of a formula is *regular*. A parametric formula occurrence is *regular* if it is either cons-regular or ant-regular. A parametric formula occurrence is *cons-regular* if the following holds: (i) if A occurs in *succedent* position, then the analogous condition as in C_6 above should hold, as represented in the following diagram:

$$\frac{(X \vdash Y)[A]^{succedent}}{(X' \vdash Y')[A]^{succedent}}\, R \qquad \rightsquigarrow \qquad \frac{(X \vdash Y)[Z]^{succedent}}{(X' \vdash Y')[Z]^{succedent}}\, R.$$

(ii) if A occurs in *precedent* position, then the analogous condition as in C_6 above should hold, restricted to structures Z such that a derivation of the sequent $Z \vdash A$ exists, in which A is principal in the conclusion sequent. The definition of *ant-regular* parametric formula occurrence is given dually.

Like the previous C_6 and C_7, also condition C_6/C_7 caters for cases in which the cut needs to be pushed up over rules in which at least one of the cut-formulas is a parameter. In order to understand this more general condition, consider the operations *why not* (denoted by ?) and *of course* (denoted by !) in linear logic. As is well known, in linear logic, contraction is not allowed in general, but only in the following restricted form:

$$C_? \,\frac{X \vdash ?A \,;\, ?A}{X \vdash ?A} \qquad \frac{!A \,;\, !A \vdash X}{!A \vdash X}\, C_!$$

Clearly, $C_?$ does not satisfy C_6, and $C_!$ does not satisfy C_7; however, each occurrence of $?A$ is ant-regular, and each occurrence of $!A$ is cons-regular, hence the rules above satisfy C_6/C_7.

In a situation like the one below on the left-hand side, the following transformation is not viable because the application of $C_?$ is blocked for arbitrary structures:

$$\dfrac{\dfrac{\dfrac{\vdots\, \pi_1}{X \vdash ?A \,;\, ?A}}{X \vdash ?A} \qquad \dfrac{\vdots\, \pi_2}{?A \vdash Y}}{X \vdash Y} \qquad \not\rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{\vdots\, \pi_1}{X \vdash ?A \,;\, ?A} \qquad \dfrac{\vdots\, \pi_2}{?A \vdash Y}}{X \vdash Y \,;\, Y}}{X \vdash Y}\,\text{Blocked}$$

However, a more sophisticated reduction strategy is possible, which consists in tracking where the succedent occurrence of $?A$ has been introduced in the subderivation π_2. The crucial observation is that, thanks to the operational rules introducing $?$, whenever $?A$ is principal and in precedent position, the shape of the sequent in which it occurs is $?A \vdash ?Z$, and for a structure of the shape $?Z$, the application of the rule $C_?$ is allowed. Let us then rewrite the original derivation below on the left-hand side:

$$
\dfrac{\dfrac{\begin{array}{c}\vdots\ \pi_1\\ X \vdash ?A\ ;\ ?A\end{array}}{X \vdash ?A} \qquad \begin{array}{c}\vdots\ \pi_2''\\ ?A \vdash ?Z\\ \vdots\ \pi_2'\\ ?A \vdash Y\end{array}}{X \vdash Y}
\qquad \rightsquigarrow \qquad
\begin{array}{c}
\dfrac{\dfrac{\begin{array}{c}\vdots\ \pi_1\\ X \vdash ?A\ ;\ ?A\end{array} \qquad \begin{array}{c}\vdots\ \pi_2''\\ ?A \vdash ?Z\end{array}}{X \vdash ?Z\ ;\ ?Z}}{X \vdash ?Z}\ \text{Allowed}\\
\vdots\ \pi_2'[X/?A]\\
X \vdash Y
\end{array}
$$

A crucial fact for the transformation above to go through is that the rule $C_?$ is closed under the substitution of $?A$ for a structure $?Z$ such that the derivation π_2'' with conclusion $?A \vdash ?Z$—introducing $?A$ as principal formula in its conclusion—exists.

Together with the display postulates and the meta-cut elimination, display calculi provide a suitable environment to bring Gentzen design principles to their natural consequences, in particular w.r.t. a clear and explicit division of labour among structural and operational rules.

Let us expand a bit on operational rules. They typically occur in two flavors; namely, operational rules which translate one structural connective in the premises in the corresponding connective in the conclusion, and operational rules in which both structural and operational connectives are introduced in the conclusion. An example of this pattern is provided below for the case of the modal diamond connective:

$$
\dfrac{\circ A \vdash X}{\Diamond A \vdash X}\ \Diamond_L \qquad \dfrac{X \vdash A}{\circ X \vdash \Diamond A}\ \Diamond_R
$$

This introduction pattern obeys very strict criteria, which will be expanded on in the next section. From this example, it is clear that the introduction rules capture the rock bottom behavior of the logical connective in question; additional properties (for instance, normality, in the case in point), which might vary depending on the logical system, are to be captured at the level of additional and purely structural rules. This enforces a very clear-cut division of labour between operational rules, which only encode the basic proof-theoretic meaning of logical connectives, and structural rules, which account for all extra relations and properties, and which can be modularly added or removed, thus accounting for the space of logics.

In conclusion, the two main benefits of display calculi are a more modular and stream-lined proof of cut elimination, and an explicit and modular account of logical connectives.

2.4 Wansing's criteria

In [28, Section 1.3], referring to the well known idea that "a proof-theoretic semantics exemplifies the Wittgensteinian slogan that meaning is use", Wansing stresses that, for this slogan to serve as a conceptual basis for an general inferential theory of meaning, 'use' should be understood as '*correct* use'. The consequences of the idea of meaning as correct use then precipitate in the following principles for the introduction rules for operational connectives, which he discusses in the same section and which are reported below. These principles are hence to be understood as the general requirements a (sequent-style) proof system needs in order to encode the correct use, and hence for being suitable for proof-theoretic semantics.

Separation. This principle requires a non-holistic explanation of the behavior of operational connectives: the meaning of a given operational connective cannot be dependent on any other operational connectives. For instance, the following rule does not satisfy *separation*:

$$\frac{\Box\Gamma \vdash A, \Diamond\Delta}{\Box\Gamma \vdash \Box A \Diamond\Delta}$$

This criterion does not ban the possibility of defining composite connectives; however, it ensures that the dependence relation between connectives creates no vicious circles. In fact, as it is formulated, this criterion is much stronger, since it requires that every connective is independent of any other. We observe that, in order for the system to be both consistent and well grounded, we only need to require that at least one operational connective in the system is defined independently of any other.

Segregation. This is a stronger requirement than separation, and requires that the precedent (succedent) of the conclusion sequent in a left (right) introduction rule must not exhibit any structure operation. This criterion is explained with an observation of Belnap's [9], that an introduction rule with non-empty context on the principal side would fail to account for the meaning of the logical connective involved in a context-independent way.

Weak symmetry. This requirement stipulates that each introduction rule for a given connective f should either belong to a set of rules ($f \vdash$) which introduce f on the left side of $\vdash$ in the conclusion sequent, or to a set of rules ($\vdash f$) which introduce f on the right side of $\vdash$ in the conclusion sequent. Understanding the either-or as exclusive disjunction, this criterion excludes an operational connective to be introduced on both sides by the application of one and the same rule.

Symmetry. Rather than a requirement on individual rules, this principle is a requirement on the set of the introduction rules for a given connective; namely, this principle requires that the sets of left- and right- introduction rules for a given connective partition the set of introduction rules for that connective (which is our understanding of *weak symmetry*), and that moreover, each cell is non-empty. Notice that this condition does not exclude the possibility of having, for instance, two rules that introduce a connective on the left and one that introduces it on the right side of the turnstile.

Weak explicitness. An introduction rule for f is *weakly explicit* if f appears only in the conclusion of a rule and not in its premisses.

Explicitness. An introduction rule for f is *explicit* if it is weakly explicit and in addition to this, f appears *only once* in the conclusion of the rule.

The following principles are of a more global nature, which involves the proof system as a whole:

Unique characterization. This principle requires each logical connective to be uniquely characterized in the system, in the following sense. Let Λ be a logical system with a syntactic presentation S in which f occurs. Let S^* be the result of rewriting f everywhere in S as f^*, and let $\Lambda\Lambda^*$ be the system presented by the union SS^* of S and S^* in the combined language with both f and f^*. Let A_f denote a formula (in this language) that contains a certain occurrence of f, and let A_{f^*} denote the result of replacing this occurrence of f in A_f by f^*. The connectives f and f^* are *uniquely characterized* in $\Lambda\Lambda^*$ if for every formula A_f in the language of $\Lambda\Lambda^*$, A_f is provable in SS^* iff Af^* is provable in SS^*.

Došen's principle. Hilbert style presentations are modular in the following sense: if Λ_1 and Λ_2 are two finitely axiomatizable logics over the same language s.t. Λ_1 is stronger Λ_2 w.r.t. provability, then an axiomatization of Λ_2 can be obtained from one of Λ_1 by adding finitely many formulas. In this manner the whole space of finitely axiomatizable logics can be introduced. Although it is arguably more difficult to achieve an analogous degree of modularity in the sequent calculi presentation, a principle aimed to achieve it has been advocated by Wansing under the name of *Došen's principle*: "The rules for the logical operations are never changed; all changes are made in the structural rules". Thus, the whole space of logics should be constructed by adding more and more structural rules. Došen's principle is particularly suitable to the design of display calculi; as remarked early on, an added degree of modularity is guaranteed by the fact that, besides structural rules expressing properties of single structural connectives (which is the case e.g. of the rule exchange), display calculi typically feature rules which concern the interaction between different structural connectives.

Cut-eliminability. Finally, Wansing considers the eliminability of the *cut* rule an important requirement for the proof-theoretic semantics of logical connectives.

Chapter 3

Dynamic Logics

Dynamic logics form a large family of non-classical logics, which are designed to formalize change caused by actions of diverse nature: updates on the memory of a computer, displacements of moving robots in a given space, measurements in quantum physics models, belief updates, etc. In each interpretation, formulas express properties of the current model, and also the pre- and post-conditions of a given action. Actions are interpreted as transformations of one model into another one, the *updated model*, which represents the state of affairs after the action has taken place. Languages for dynamic logics are expansions of classical propositional logic with dynamic modal operators, each of which takes an action as its parameter; dynamic operators are interpreted in terms of the transformation of models corresponding to their action-parameters.

In the present chapter, we first present two of the best known logical systems belonging to this family, namely public announcement logic [22] and the logic of epistemic actions and knowledge [7]. Our presentation in sections 3.1 and 3.2 is different but equivalent to the original versions from [22] and [7], and rather follows the presentation given in [17] and in [14]. Finally, in sections 3.3 and 3.4 we discuss their existing proof-theoretic formalizations, particularly in relation to the viewpoint of proof-theoretic semantics.

3.1 The logic of public announcements

Let AtProp be a countable set of atomic propositions. The formulas of (the single-agent) public announcement logic PAL are defined inductively as follows:

$$\phi ::= p \in \mathsf{AtProp} \mid \neg\phi \mid \phi \vee \phi \mid \Diamond\phi \mid \langle\phi\rangle\phi.$$

The defined connectives $\top$, $\bot$, $\wedge$, $\rightarrow$ and $\leftrightarrow$ are defined as usual. The standard semantics for PAL consists of Kripke models $M = (W, R, V)$ such that R is an equivalence relation. The interpretation of the static fragment of the language is standard. For every Kripke frame $\mathcal{F} = (W, R)$ and every $a \subseteq W$, let the *subframe* of $\mathcal{F}$ *relativized to* a be the Kripke frame $\mathcal{F}^a = (W^a, R^a)$ defined as follows: $W^a := a$ and $R^a := R \cap (a \times a)$. Given this preliminary definition, formulas of the form $\langle\alpha\rangle\phi$ are interpreted in the following way:

$$M, w \Vdash \langle\alpha\rangle\phi \quad \text{iff} \quad M, w \Vdash \alpha \text{ and } M^\alpha, w \Vdash \phi,$$

A complete axiomatization for PAL is given by the axioms and rules for the modal logic S5 plus the following axioms:

(1) $\langle\alpha\rangle p \leftrightarrow (\alpha \wedge p)$;

(2) $\langle\alpha\rangle\neg\phi \leftrightarrow (\alpha \wedge \neg\langle\alpha\rangle\phi)$;

(3) $\langle\alpha\rangle(\phi \vee \psi) \leftrightarrow (\langle\alpha\rangle\phi \vee \langle\alpha\rangle\psi)$;

(4) $\langle\alpha\rangle\Diamond\phi \leftrightarrow (\alpha \wedge \Diamond(\alpha \wedge \langle\alpha\rangle\phi))$.

3.2 The logic of epistemic actions and knowledge

Let AtProp be a countable set of atomic propositions. The set $\mathcal{L}$ of the formulas A of (the single-agent[1] version of) the logic of epistemic actions and knowledge (EAK), and the set $\mathsf{Act}(\mathcal{L})$ of the *action structures* α *over* $\mathcal{L}$ are defined simultaneously as follows:

$$A := p \in \mathsf{AtProp} \mid \neg A \mid A \vee A \mid \Diamond A \mid \langle\alpha\rangle A \ \ (\alpha \in \mathsf{Act}(\mathcal{L})),$$

where an *action structure over* $\mathcal{L}$ is a tuple $\alpha = (K, k, \alpha, Pre_\alpha)$, such that K is a finite nonempty set, $k \in K$, $\alpha \subseteq K \times K$ and $Pre_\alpha : K \to \mathcal{L}$. Notice that, following [16], the symbol α denotes *both* the action structure *and* the accessibility relation of the action structure. Unless explicitly specified otherwise, occurrences of this symbol are to be interpreted contextually: for instance, in $j\alpha k$, the symbol α denotes the relation; in M^α, the symbol α denotes the action structure. Of course, in the multi-agent setting, each action structure comes equipped with *a collection* of accessibility relations indexed in the set of agents, and then the abuse of notation disappears.

The symbol $Pre(\alpha)$ stands for $Pre_\alpha(k)$. Let $\alpha_i = (K, i, \alpha, Pre_\alpha)$ for each action structure $\alpha = (K, k, \alpha, Pre_\alpha)$ and every $i \in K$. Intuitively, the actions α_i for $k\alpha i$ are intended to represent the uncertainty of the (unique) agent about the action that is actually taking place. The connectives $\top$, $\bot$, $\wedge$, $\to$ and $\leftrightarrow$ are defined as usual.

The standard models for EAK are relational structures $M = (W, R, V)$ such that W is a nonempty set, $R \subseteq W \times W$, and $V : \mathsf{AtProp} \to \mathcal{P}(W)$. The

[1] The multi-agent generalization of this simpler version is straightforward, and is given by taking the indexed version of the modal operators, axioms, and then by taking the interpreting relations (both in the models and in the action structures) over a set of agents.

interpretation of the static fragment of the language is standard. For every Kripke frame $\mathcal{F} = (W, R)$ and each $\alpha \subseteq K \times K$, let the Kripke frame $\coprod_\alpha \mathcal{F} := (\coprod_K W, R \times \alpha)$ be defined[2] as follows: $\coprod_K W$ is the $|K|$-fold coproduct of W (which is set-isomorphic to $W \times K$), and $R \times \alpha$ is the binary relation on $\coprod_K W$ defined as

$$(w, i)(R \times \alpha)(u, j) \quad \text{iff} \quad wRu \text{ and } i\alpha j.$$

For every model $M = (W, R, V)$ and each action structure $\alpha = (K, k, \alpha, Pre_\alpha)$, let

$$\coprod_\alpha M := (\coprod_K W, R \times \alpha, \coprod_K V)$$

be such that its underlying frame is defined as above, and $(\coprod_K V)(p) := \coprod_K V(p)$ for every $p \in \mathsf{AtProp}$. Finally, let the *update* of M with the action structure α be the submodel $M^\alpha := (W^\alpha, R^\alpha, V^\alpha)$ of $\coprod_\alpha M$ the domain of which is the subset

$$W^\alpha := \{(w, j) \in \coprod_K W \mid M, w \Vdash Pre_\alpha(j)\}.$$

Given this preliminary definition, formulas of the form $\langle\alpha\rangle A$ are interpreted as follows:

$$M, w \Vdash \langle\alpha\rangle A \quad \text{iff} \quad M, w \Vdash Pre_\alpha(k) \text{ and } M^\alpha, (w, k) \Vdash A.$$

A complete axiomatization of EAK is given by the axioms and rules for the minimal normal modal logic K plus the following axioms:

$$\langle\alpha\rangle p \leftrightarrow (Pre(\alpha) \wedge p); \tag{3.1}$$

$$\langle\alpha\rangle\neg A \leftrightarrow (Pre(\alpha) \wedge \neg\langle\alpha\rangle A); \tag{3.2}$$

$$\langle\alpha\rangle(A \vee B) \leftrightarrow (\langle\alpha\rangle A \vee \langle\alpha\rangle B); \tag{3.3}$$

$$\langle\alpha\rangle\Diamond A \leftrightarrow (Pre(\alpha) \wedge \bigvee\{\Diamond\langle\alpha_i\rangle A \mid k\alpha i\}). \tag{3.4}$$

Action structures are one among many possible ways to represent actions. Following [14], we prefer to keep a black-box perspective on actions, and to identify agents a with the indistinguishability relation they induce on actions; so, in the remainder of the thesis, the role of the action-structures α_i for $k\alpha i$ will be played by actions β such that $\alpha\mathsf{a}\beta$.

[2] This definition is of course intended to be applied to relations α which are part of the specification of some action structure; in these cases, the symbol α in $\coprod_\alpha \mathcal{F}$ will be understood as the action structure. This is why the abuse of notation turns out to be useful.

3.2.1 The intuitionistic version of EAK

In the present subsection, we report on a slightly edited (multi-agent) version of the intuitionistic counterpart of EAK, which was introduced in [16] in a single-agent version.

Let AtProp be a countable set of atomic propositions, and let Ag be a nonempty set (of agents). The set $\mathcal{L}$(m-IK) of the formulas A of the multi-modal version m-IK of Fischer Servi intuitionistic modal logic IK are inductively defined as follows:

$$A := p \in \mathsf{AtProp} \mid \bot A \mid A \vee A \mid A \wedge A \mid A \to A \mid \langle \mathsf{a} \rangle A \mid [\mathsf{a}]A \quad (\mathsf{a} \in \mathsf{Ag})$$

The logic m-IK is the smallest set of formulas in the language $\mathcal{L}$(m-IK) (where $\neg A$ abbreviates as usual $A \to \bot$) containing the following axioms and closed under modus ponens and necessitation rules:

Axioms

	$A \to (B \to A)$
	$(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
	$A \to (B \to A \wedge B)$
	$A \wedge B \to A$
	$A \wedge B \to B$
	$A \to A \vee B$
	$B \to A \vee B$
	$(A \to C) \to ((B \to C) \to (A \vee B \to C))$
	$\bot \to A$
	$[\mathsf{a}](A \to B) \to ([\mathsf{a}]A \to [\mathsf{a}]B)$
	$\langle \mathsf{a} \rangle (A \vee B) \to \langle \mathsf{a} \rangle A \vee \langle \mathsf{a} \rangle B$
	$\neg \langle \mathsf{a} \rangle \bot$
FS1	$\langle \mathsf{a} \rangle (A \to B) \to ([\mathsf{a}]A \to \langle \mathsf{a} \rangle B)$
FS2	$(\langle \mathsf{a} \rangle A \to [\mathsf{a}]B) \to [\mathsf{a}](A \to B)$

Inference Rules

MP	if $\vdash A \to B$ and $\vdash A$, then $\vdash B$
Nec	if $\vdash A$, then $\vdash [\mathsf{a}]A$

To define the language of the intuitionistic counterpart of EAK, let AtProp be a countable set of atomic propositions, and let Ag be a nonempty set. The set $\mathcal{L}$(IEAK) of the formulas A of the intuitionistic logic of epistemic actions and knowledge (IEAK), and the set $\mathsf{Act}(\mathcal{L})$ of the *action structures α over $\mathcal{L}$* are defined simultaneously as follows:

$$A := p \in \mathsf{AtProp} \mid \bot \mid A \to A \mid A \vee A \mid A \wedge A \mid \langle \mathsf{a} \rangle A \mid [\mathsf{a}]A \mid \langle \alpha \rangle A \mid [\alpha]A,$$

where $\mathsf{a} \in \mathsf{Ag}$, and an *action structure α over* $\mathcal{L}$(EAK) is defined in a completely analogous way as action structures in the classical case. Then, the logic IEAK is defined in a Hilbert-style presentation which includes the axioms and rules of m-IK plus the following axioms and rules:

Interaction Axioms

$\langle\alpha\rangle p \leftrightarrow Pre(\alpha) \wedge p$ $[\alpha]p \leftrightarrow Pre(\alpha) \to p$
$\langle\alpha\rangle\bot \leftrightarrow \bot$ $\langle\alpha\rangle\top \leftrightarrow Pre(\alpha)$ $[\alpha]\top \leftrightarrow \top$ $[\alpha]\bot \leftrightarrow \neg Pre(\alpha)$
$[\alpha](A \wedge B) \leftrightarrow [\alpha]A \wedge [\alpha]B$ $\langle\alpha\rangle(A \wedge B) \leftrightarrow \langle\alpha\rangle A \wedge \langle\alpha\rangle B$
$\langle\alpha\rangle(A \vee B) \leftrightarrow \langle\alpha\rangle A \vee \langle\alpha\rangle B$ $[\alpha](A \vee B) \leftrightarrow Pre(\alpha) \to (\langle\alpha\rangle A \vee \langle\alpha\rangle B)$
$\langle\alpha\rangle(A \to B) \leftrightarrow Pre(\alpha) \wedge (\langle\alpha\rangle A \to \langle\alpha\rangle B)$ $[\alpha](A \to B) \leftrightarrow \langle\alpha\rangle A \to \langle\alpha\rangle B$
$\langle\alpha\rangle\langle\mathsf{a}\rangle A \leftrightarrow Pre(\alpha) \wedge \bigvee\{\langle\mathsf{a}\rangle\langle\beta\rangle A \mid \alpha\mathsf{a}\beta\}$ $[\alpha]\langle\mathsf{a}\rangle A \leftrightarrow Pre(\alpha) \to \bigvee\{\langle\mathsf{a}\rangle\langle\beta\rangle A \mid \alpha\mathsf{a}\beta\}$
$[\alpha][\mathsf{a}]A \leftrightarrow Pre(\alpha) \to \bigwedge\{[\mathsf{a}][\beta]A \mid \alpha\mathsf{a}\beta\}$ $\langle\alpha\rangle[\mathsf{a}]A \leftrightarrow Pre(\alpha) \wedge \bigwedge\{[\mathsf{a}][\beta]A \mid \alpha\mathsf{a}\beta\}$

Inference Rules

vNec	if $\vdash A$, then $\vdash [\alpha]A$

3.3 Proof theoretic formalisms for PAL and DEL

In this section, we discuss the most relevant existing proof-theoretic accounts for the logic of public announcements [22] and for the logic of epistemic actions and knowledge [7].

In [5], a labelled tableaux system is proposed for public announcement logic. This system is sound and complete with respect to the semantics of PAL. Moreover, the computational complexity of this tableaux system is shown to be optimal for satisfiability checking in the language of PAL. The system manipulates triples, called labelled formulas, of the form $\langle \mu, n, \phi\rangle$ such that μ is a (possibly empty) list of PAL-formulas, n is a natural number, and ϕ is a PAL-formula. Intuitively, the tuple $\langle \mu, n\rangle$ stands for an epistemic state of the model updated with a sequence of announcements encoded by μ. To give a closer impression of this tableaux system, consider the following rule:

$$R\widehat{K}\ \frac{\langle(\alpha_1, ..., \alpha_k), n, \neg K_a A\rangle}{\langle \epsilon, n', \neg[\alpha_1]...[\alpha_k]A\rangle : \langle a, n, n'\rangle}\ n' \text{ fresh}$$

This rule can be read as follows: if a state n does not satisfy $K_a\phi$ after the sequence of announcements $\alpha_1, ..., \alpha_k$, then at least one of its R_a-successor states n' in the original model, represented by the tuple $\langle \epsilon, n'\rangle$ in the rule, must survive the updates and not satisfy ϕ. Hence, $\langle \epsilon, n'\rangle$ must satisfy the formula $\langle\alpha_1\rangle...\langle\alpha_k\rangle\neg\phi$, which is classically equivalent to $\neg[\alpha_1]...[\alpha_k]A$.

Clearly, rules such as this one incorporate the relational semantics of PAL. This fact is a shortcoming from the point of view of proof-theoretic semantics, since it prevents these rules from providing an independent contribution to the meaning of the logical connectives. A second shortcoming, of a more technical nature, is that the statement of this rule is grounded on the classical interdefinability between the box-type and diamond-type modalities. This implies that if we dispense with the classical propositional base, we would need to reformulate this rule. Hence the calculus is non-modular in the sense discussed in (2.4).

In [19] and [20], Paolo Maffezioli and Sara Negri provide cut-free labelled sequent calculi for PAL with truthful and non-truthful announcements, respectively. Also in this case, the statement of the rules of these calculi incorporates the relational semantics, for instance as illustrated here below for the case of truthful announcements.

$$\frac{w :^{\mu,\alpha} A, w :^{\mu} [\alpha]A, w :^{\mu} \alpha, \Gamma \vdash \Delta}{w :^{\mu} [\alpha]A, w :^{\mu} \alpha, \Gamma \vdash \Delta}\, L[\ ]{:}^{\mu} \qquad \frac{w :^{\mu} \alpha, \Gamma \vdash \Delta, w :^{\mu,\alpha} A}{\Gamma \vdash \Delta, w :^{\mu} [\alpha]A}\, R[\ ]{:}^{\mu}$$

In the rules above, symbols such as $w :^{\mu} A$ can be rearranged and then understood as the labelled formulas $\langle \mu, w, A\rangle$ in the tableaux system presented before. The only difference is that w is an individual variable which stands for a given state of a relational structure, and not for a natural number; however, this difference is completely nonessential. Under this interpretation, it is clear that e.g. the rule $L[\ ]{:}^{\mu}$ encodes the relational satisfaction clause of $[\alpha]A$, when α is a truthful announcement. The following rules are also part of the calculi.

$$\frac{v : A, w : K_a A, wR_a v, \Gamma \vdash \Delta}{w : K_a A, wR_a v, \Gamma \vdash \Delta}\, LK_a \qquad \frac{wR_a v, \Gamma \vdash \Delta, v : A}{\Gamma \vdash \Delta, w : K_a A}\, RK_a$$

Besides the individual variables w and v, the rules above feature the binary relation symbol R_a encoding the epistemic uncertainty of the agent a. Since the relational semantics is imported in the definitions of the rules, the same shortcomings pointed out in the case of the tableaux system appear also here. On the other hand, importing the relational semantics allows for some remarkable extra power. Indeed, the interaction axiom 3.1 can be derived from the four rules above, which deal with static and dynamic modalities in complete independence of one another.

In [6] and [10], sequent calculi have been defined for dynamic logics arising in an algebraic way, motivated by program semantics, with a methodology introduced by [Abramsky and Vickers 1993] [25]. Essentially, this approach is based on the idea of merging a linear-type logic of actions (more precisely, [18]) with a classical or intuitionistic logic of propositions. Following the treatment of [25], this logic arises semantically as the logic of certain quantale-modules, namely of maps $\star : M \times Q \to M$, preserving complete joins in each coordinate, where Q is a quantale and M is a complete join-semilattice. Each $q \in Q$ induces a completely join-preserving operation $(- \star q) : M \to M$, which, by general order-theoretic facts, has a unique right adjoint $[q] : M \to M$. That is, for every $m, m' \in M$,

$$m \star q \leq m' \ \text{ iff } \ m \leq [q]m'. \tag{3.5}$$

Intuitively, the elements of Q are actions (or rather, inverses of actions), and M is an algebra interpreting propositions, which in the best known cases arises as the complex algebra of some relational structure, and therefore will be e.g. a complete and atomic Boolean algebra with operators. Thus the framework of [6] and [10] is vastly more general than dynamic epistemic logic as it is usually understood. A remarkable feature of this setting is that the dynamic operations which are intended as the interpretation of the primitive dynamic connectives arise in this setting as adjoints of "more primitive" operations; thus, and much more importantly, every dynamic modality comes with its adjoint. Moreover, every epistemic modality (parametrized as usual with an agent) comes in two copies: one as an operation on Q and one as an operation on M, and these two copies are stipulated to interact in a suitable way. More formally, the semantic structures are defined as tuples $(M, Q, \{f_A\}_{A \in Ag})$, where M and Q are as above, and for every agent A, f_A is a pair of completely join preserving maps $(f_A^M : M \to M, f_A^Q : Q \to Q)$ such that the following three conditions hold:

$$f_A^Q(q \cdot q') \leq f_A^Q(q) \cdot f_A^Q(q') \tag{3.6}$$

$$f_A^M(m \star q) \leq f_A^M(m) \star f_A^Q(q) \tag{3.7}$$

$$1 \leq f_A^Q(1). \tag{3.8}$$

Intuitively, for every agent A, the operation f_A^M is the diamond-type modal operator encoding the epistemic uncertainty of A, and f_A^Q is the diamond-type modal operator encoding the epistemic uncertainty of A about the action that is actually taking place. Given this understanding, condition (3.7) hardcodes the following well-known DEL-axiom in the semantic structures above:

$$\bigwedge\{[A][q']m \mid qAq'\} \vdash [q][A]m. \tag{3.9}$$

where the notation qAq' means that the action q' is indistinguishable from q for the agent A. In (3.7), the element $f_A^Q(q)$ encodes the join of all such actions. Because $\star$ is bilinear, we get:

$$f_A^M(m) \star f_A^Q(q) = f_A^M(m) \star \bigvee_Q \{q' \mid qAq'\} = \bigvee_M \{f_A^M(m) \star q' \mid qAq'\}.$$

Hence, (3.7) can be equivalently rewritten in the form of a rule as follows:

$$\frac{\bigvee\{f_A^M(m) \star q' \mid qAq'\} \vdash m'}{f_A^M(m \star q) \vdash m'}$$

Applying adjunction to the premise and to the conclusion gets us to:

$$\frac{m \vdash \bigwedge\{[A][q']m' \mid qAq'\}}{m \vdash [q][A]m'}$$

Finally, rewriting the rule above back as an inequality gets us to (3.9). The first pioneering proposal is the sequent calculus developed in [6]. This calculus manipulates two kinds of sequents: Q-sequents, of the form $\Gamma \vdash_Q q$, where q is an action and Γ is a sequence of actions and agents, and M- sequents, of the form $\Gamma \vdash_M m$, where m is a proposition and Γ is a sequence of propositions, actions and agents. These different entailment relations need to be brought together by means of rules of hybrid type, such as the left one below.

$$\frac{m' \vdash_M m \qquad \Gamma_Q \vdash_Q q}{[q]m', \Gamma_Q \vdash_M m}\ Dy\mathrm{L} \qquad \frac{\Gamma, q \vdash_M m}{\Gamma \vdash_M [q]m}\ Dy\mathrm{R}$$

As to the soundness of the rule DyL, let us identify the logical symbols with their interpretation, assume that the inequalities $m \leq m'$ and $\Gamma_Q \leq q$ are satisfied on given M and Q respectively[3], and prove that $[q]m', \Gamma_Q \leq m$ in M. Indeed,

$$[q]m' \star \Gamma_Q \leq [q]m' \star q \leq m' \leq m.$$

The first inequality follows from $\Gamma_Q \leq q$ and $\star$ being order-preserving in its second coordinate; the second inequality is obtained by applying the right-to-left direction of (3.5) to the inequality $[q]m' \leq [q]m'$; the last inequality holds by assumption. The soundness of DyR follows likewise from the left-to-right direction of (3.5).

This calculus is shown to be both sound and complete w.r.t. this algebraic semantics. The setting illustrated above is powerful enough that sufficiently many epistemic actions can be encoded in it to support the formalisation of various variants of the Muddy Children Puzzle, in which children might be cheating. However, cut elimination for this system has not been proven.

In [10] a similar framework is presented, which exploits the same basic ideas, and results in a system with more explicit proof-theoretic performances

[3]where Γ_Q now stands for a suitable product in Q of the interpretations of its individual components.

and which is shown to be cut-free. However, like its predecessor, this system focuses on a logic semantically arising from an algebraic setting which is vastly more general than the usual relational setting. The issue about how it precisely restricts to the usual setting, and hence how the usual DEL-type logics can be captured within this more general calculus, is left largely implicit. The semantic setting of [6], where propositions are interpreted as elements of a right module M on a quantale Q, specialises in [10] to a setting in which $M = (\mathbb{A}, \{\Box_A, \blacklozenge_A : A \in Ag\})$, where $\mathbb{A}$ is a Heyting algebra and, for every agent A, the modalities $\Box_A$ and $\blacklozenge_A$ are adjoint to each other. Notice that $\Diamond_A$, which in the classical case is defined as $\neg\Box_A\neg$, cannot be expressed any more in this way, and needs to be added as a primitive connective, which has not been done in [10].

As mentioned before, the design of this calculus gives a more explicit account than its predecessor to certain technical aspects which come from the semantic setting; for instance, the semantic setting motivating both papers features two domains of interpretation (one for the actions and one for the propositions), which are intended to give rise to two logics which are to be treated on a par and then merged. In [6], the calculus manipulates sequents which are made of heterogeneous components; for instance, in action-sequents $\Gamma \vdash_Q q$, the precedent Γ is a sequence in which both actions and agents may occur. Since Γ is to be semantically interpreted as an element of Q, they need to resort to a rather clumsy technical solution which consists in interpreting, e.g. the sequence (q, A, q') as the element $f_A^Q(q) \cdot q'$. In [10], the calculus is given in a *deep-inference* format; namely, rules of this calculus make it possible to manipulate formulas inside a given context. This more explicit bookkeeping makes it possible to prove the cut elimination, following the original Gentzen strategy. However, the presence of two different consequence relations and the need to account for their interaction then calls for the development of an extensive theory-of-contexts, in which no less than five different types of contexts need to be introduced. This also causes a proliferation of rules, since the possibility of performing some inferences depends on the type of context under which they are to be performed.

In [1], a formal framework accounting for dynamic revisions or updates is introduced, in which the revisions/updates are formalized using the turnstile symbol. This framework has aspects similar to Hoare logic: indeed, it manipulates sequent-type structures of the form $\phi, \phi' \models \phi''$, such that ϕ and ϕ'' are formulas of proposition-type, and ϕ' is a formula of event-type. This formalism has also common aspects to [6] and [10]: indeed, both proposition-type and event-type (i.e. action-type) formulas allow epistemic modalities for each agent, respectively accounting for the agent's epistemic uncertainty about the world and about the actions actually taking place.

In [3] and [2], three formal calculi are introduced, manipulating the syntactic structures above. Given that the turnstile encodes the update rather than a consequence relation or entailment, the syntactic structures above are not sequents in a proper sense; hence, rather than sequent calculi, these calculi should be rather regarded as being of natural deduction-type. As such, the design of this calculi presents many shortcomings; to mention only one, multiple connectives are introduced at the same time, for instance in the following rule:

$$\frac{\phi, \phi' \vdash \phi''}{\langle Bj\rangle(\phi \wedge Pre(p')), \langle Bj\rangle(\phi' \wedge p') \vdash \langle Bj\rangle\phi''}\, R_5.$$

These calculi are shown to be sound and complete w.r.t three semantic consequence relations, respectively.

3.4 The calculus D.EAK

In this section, we present a slightly edited version of the display-type calculus D.EAK for the logic EAK, introduced in [14]. This calculus will be presented in detail, since it is the one which most closely approximates the criteria of proof-theoretic semantics. As is typical of display calculi, D.EAK manipulates sequents of type $X \vdash Y$, where X and Y are structures, i.e. syntactic objects inductively built from formulas using *structural connectives*, or *proxies*. Every tuple of logical connectives is associated with its proxy. Hence, occurrences of proxies are interpreted *contextually*, depending on whether they are in precedent or in succedent position (cf. Definition 1). The design of D.EAK follows Došen's principle (cf. Section 2.4); consequently, D.EAK is modular along many dimensions: for instance, the space of EAK-type logics on a nonclassical base, down to e.g. the Lambek calculus can be captured by suitably removing structural rules. Moreover, also w.r.t. static modal logic, the space of normal modal logic can be reconstructed by adding or removing structural rules in a suitable way. Finally, different types of interaction between the dynamic and the epistemic modalities can be captured by changing the relative structural rules.

In order to highlight this modularity, we will present the system piecewise. First we give rules for the propositional base, divided into structural rules and operational rules; then we do the same for the static modal operators; and finally we present the rules for the dynamic modalities.

In the table below, we give an overview of the logical connectives of the propositional base and their proxies.

Structural symbols	$<$		$>$		;		I	
Operational symbols	$\prec$	$\leftarrow$	$\succ$	$\rightarrow$	$\wedge$	$\vee$	$\top$	$\bot$

The table below contains the structural rules for the propositional base:

Structural Rules

$$Id \, \frac{}{p \vdash p} \qquad \frac{X \vdash A \quad A \vdash Y}{X \vdash Y} \, Cut$$

$$\mathrm{I}_L^1 \, \frac{X \vdash Y}{\mathrm{I} \vdash Y < X} \qquad \frac{X \vdash Y}{X < Y \vdash \mathrm{I}} \, \mathrm{I}_R^1$$

$$\mathrm{I}_L^2 \, \frac{X \vdash Y}{\mathrm{I} \vdash X > Y} \qquad \frac{X \vdash Y}{Y > X \vdash \mathrm{I}} \, \mathrm{I}_R^2$$

$$W_L^1 \, \frac{X \vdash Z}{Y \vdash Z < X} \qquad \frac{X \vdash Z}{X < Z \vdash Y} \, W_R^1$$

$$W_L^2 \, \frac{X \vdash Z}{Y \vdash X > Z} \qquad \frac{X \vdash Z}{Z > X \vdash Y} \, W_R^2$$

$$C_L \, \frac{X \,; X \vdash Y}{X \vdash Y} \qquad \frac{Y \vdash X \,; X}{Y \vdash X} \, C_R$$

$$E_L \, \frac{Y \,; X \vdash Z}{X \,; Y \vdash Z} \qquad \frac{Z \vdash X \,; Y}{Z \vdash Y \,; X} \, E_R$$

$$A_L \, \frac{X \,; (Y \,; Z) \vdash W}{(X \,; Y) \,; Z \vdash W} \qquad \frac{W \vdash (Z \,; Y) \,; X}{W \vdash Z \,; (Y \,; X)} \, A_R$$

The top-to-bottom direction of each I rule is a special case of the corresponding weakening rule. However, we state them all the same for the sake of modularity, because in a substructural logic without weakening they would still hold. The weakening is not given in the usual manner, but using the structural connective $>$ with this rule, the new structure is introduced in isolation; nevertheless, the standard version is derivable from the display postulate and the rules *exchange* in the following manner:

$$\cfrac{\cfrac{\cfrac{X \vdash Z}{Y \vdash Z < X}}{Z ; Y \vdash Z}}{Y ; Z \vdash Z}$$

Having both versions of weakening as primitive rules is useful for reducing the size of derivations. In the following table, the display postulates linking the structural connective ; with $>$ and $<$ are given:

Display Postulates

$$(;,<)\ \frac{X\,;Y \vdash Z}{X \vdash Z < Y} \qquad \frac{Z \vdash X;Y}{Z < Y \vdash X}\ (<,;)$$

$$(;,>)\ \frac{X\,;Y \vdash Z}{Y \vdash X > Z} \qquad \frac{Z \vdash X\,;Y}{X > Z \vdash Y}\ (>,;)$$

In the current presentation, more connectives with their relative rules are accounted for than in [14]. The additional rules can be proved to be derivable from the remaining ones in the presence of the rules *exchange* E_L and E_R. Likewise, as is well known, by dispensing with *contraction*, *weakening* and *associativity*, an even wider array of connectives would ensue (e.g. the additive and multiplicative versions of each connective, if we dispense with weakening and contraction, etc). We are not going to develop these well known ideas any further, but only point out that, in the context of the whole system that we are going to present below, this would chart the space of the EAK-type logics on a substructural base. The system presented in this section is clearly well suited for this line of investigation. A natural question in this respect would be to relate these ensuing proof-formalisms with the semantic settings of [6]. The classical base is obtained by adding the so-called *Grishin* rules (following e.g. [13]), which encode classical, but not intuitionistic validities:

$$Gri_L\ \frac{X\ > (Y;Z) \vdash W}{(X > Y); Z \vdash W} \qquad \frac{W \vdash X > (Y;Z)}{W \vdash (X > Y); Z}\ Gri_R$$

In this system, the operational rules remain the same, no-matter which changes are made on the structural rules. The following table presents the operational rules for the propositional base:

Operational Rules

$$\bot_L \, \frac{}{\bot \vdash \mathrm{I}} \qquad \frac{X \vdash \mathrm{I}}{X \vdash \bot} \, \bot_R$$

$$\top_L \, \frac{\mathrm{I} \vdash X}{\top \vdash X} \qquad \frac{}{\mathrm{I} \vdash \top} \, \top_R$$

$$\wedge_L \, \frac{A \,;B \vdash Z}{A \wedge B \vdash Z} \qquad \frac{X \vdash A \qquad Y \vdash B}{X \,;Y \vdash A \wedge B} \, \wedge_R$$

$$\vee_L \, \frac{A \vdash X \qquad B \vdash Y}{A \vee B \vdash X \,;Y} \qquad \frac{Z \vdash A \,;B}{Z \vdash A \vee B} \, \vee_R$$

$$\leftarrow_L \, \frac{B \vdash Y \qquad X \vdash A}{B \leftarrow A \vdash Y < X} \qquad \frac{Z \vdash B < A}{Z \vdash B \leftarrow A} \, \leftarrow_R$$

$$\prec_L \, \frac{B < A \vdash Z}{B \prec A \vdash Z} \qquad \frac{Y \vdash B \qquad A \vdash X}{Y < X \vdash B \prec A} \, \prec_R$$

$$\rightarrow_L \, \frac{X \vdash A \qquad B \vdash Y}{A \rightarrow B \vdash X > Y} \qquad \frac{Z \vdash A > B}{Z \vdash A \rightarrow B} \, \rightarrow_R$$

$$\succ_L \, \frac{A > B \vdash Z}{A \succ B \vdash Z} \qquad \frac{A \vdash X \qquad Y \vdash B}{X > Y \vdash A \succ B} \, \succ_R$$

All the operational rules above satisfy the *segregation* requirement of Belnap, i.e. all the active formulas are isolated. As remarked by Belnap, this very restrictive requirement does not affect the proof power of the system because of the presence of the display postulates. As is well known, in the presence of exchange, the connectives $\leftarrow$ and $\prec$ collapse on $\rightarrow$ and $\succ$, respectively.

The rules for the normal epistemic modalities can be added to the system above or to any of its variants discussed early on. The language is now expanded with two contextual proxies and four operational connectives for every agent $\texttt{a}$, as follows:

Structural symbols	$\{\texttt{a}\}$		$\overbrace{\underbrace{\texttt{a}}}$	
Operational symbols	$\langle\texttt{a}\rangle$	$[\texttt{a}]$	$\overset{\wedge}{\underset{\vee}{\texttt{a}}}$	$\overline{\underline{\texttt{a}}}$

the proxies are translated into a diamond-type modality when occurring in precedent position and into a box-type modality when occurring in succedent position. In the following three tables, we present the structural rules, the display postulates, and the operational rules for the static modalities:

Structural Rules

$$nec_{\{\mathtt{a}\}}\ \frac{\mathrm{I} \vdash X}{\{\mathtt{a}\}\mathrm{I} \vdash X} \qquad \frac{X \vdash \mathrm{I}}{X \vdash \{\mathtt{a}\}\mathrm{I}}\ nec_{\{\mathtt{a}\}}$$

$$nec_{\overbrace{\underbrace{\mathtt{a}}}}\ \frac{\mathrm{I} \vdash X}{\overbrace{\underbrace{\mathtt{a}}}\mathrm{I} \vdash X} \qquad \frac{X \vdash \mathrm{I}}{X \vdash \overbrace{\underbrace{\mathtt{a}}}\mathrm{I}}\ nec_{\overbrace{\underbrace{\mathtt{a}}}}$$

$$FS_L^{ep}\ \frac{\{\mathtt{a}\}Y > \{\mathtt{a}\}Z \vdash X}{\{\mathtt{a}\}(Y > Z) \vdash X} \qquad \frac{Y \vdash \{\mathtt{a}\}X > \{\mathtt{a}\}Z}{Y \vdash \{\mathtt{a}\}(X > Z)}\ FS_R^{ep}$$

$$mon_L^{ep}\ \frac{\{\mathtt{a}\}X \,;\{\mathtt{a}\}Y \vdash Z}{\{\mathtt{a}\}(X\,;Y) \vdash Z} \qquad \frac{Z \vdash \{\mathtt{a}\}Y\,;\{\mathtt{a}\}X}{Z \vdash \{\mathtt{a}\}(Y\,;X)}\ mon_R^{ep}$$

$$^{ep}FS_L\ \frac{\overbrace{\underbrace{\mathtt{a}}}Y > \overbrace{\underbrace{\mathtt{a}}}X \vdash Z}{\overbrace{\underbrace{\mathtt{a}}}(Y > X) \vdash Z} \qquad \frac{Y \vdash \overbrace{\underbrace{\mathtt{a}}}X > \overbrace{\underbrace{\mathtt{a}}}Z}{Y \vdash \overbrace{\underbrace{\mathtt{a}}}(X > Z)}\ {}^{ep}FS_R$$

$$^{ep}mon_L\ \frac{\overbrace{\underbrace{\mathtt{a}}}X\,;\overbrace{\underbrace{\mathtt{a}}}Y \vdash Z}{\overbrace{\underbrace{\mathtt{a}}}(X\,;Y) \vdash Z} \qquad \frac{Z \vdash \overbrace{\underbrace{\mathtt{a}}}Y\,;\overbrace{\underbrace{\mathtt{a}}}X}{Z \vdash \overbrace{\underbrace{\mathtt{a}}}(Y\,;X)}\ {}^{ep}mon_R$$

Display Postulates

$$(\{\mathtt{a}\}, \overbrace{\underbrace{\mathtt{a}}})\ \frac{\{\mathtt{a}\}X \vdash Y}{X \vdash \overbrace{\underbrace{\mathtt{a}}}Y} \qquad \frac{X \vdash \{\mathtt{a}\}Y}{\overbrace{\underbrace{\mathtt{a}}}X \vdash Y}\ (\overbrace{\underbrace{\mathtt{a}}}, \{\mathtt{a}\})$$

Operational Rules

$$\langle\mathtt{a}\rangle_L\ \frac{\{\mathtt{a}\}A \vdash X}{\langle\mathtt{a}\rangle A \vdash X} \qquad \frac{X \vdash A}{\{\mathtt{a}\}X \vdash \langle\mathtt{a}\rangle A}\ \langle\mathtt{a}\rangle_R$$

$$[\mathtt{a}]_L\ \frac{A \vdash X}{[\mathtt{a}]A \vdash \{\mathtt{a}\}X} \qquad \frac{X \vdash \{\mathtt{a}\}A}{X \vdash [\mathtt{a}]A}\ [\mathtt{a}]_R$$

$$\widehat{\mathtt{a}}_L\ \frac{\overbrace{\underbrace{\mathtt{a}}}A \vdash X}{\widehat{\mathtt{a}}A \vdash X} \qquad \frac{X \vdash A}{\overbrace{\underbrace{\mathtt{a}}}X \vdash \widehat{\mathtt{a}}A}\ \widehat{\mathtt{a}}_R$$

$$\boxed{\mathtt{a}}_L\ \frac{A \vdash X}{\boxed{\mathtt{a}}A \vdash \overbrace{\underbrace{\mathtt{a}}}X} \qquad \frac{X \vdash \overbrace{\underbrace{\mathtt{a}}}A}{X \vdash \boxed{\mathtt{a}}A}\ \boxed{\mathtt{a}}_R$$

The rules presented so far are essentially adaptations of display calculi of Goré [13]. Now we turn to the interesting part of the calculus D.EAK: the language is now expanded with two contextual proxies and four operational connectives for every action α, as follows:

Structural symbols	$\{\alpha\}$		$\overbrace{\underbrace{\alpha}}$	
Operational symbols	$\langle\alpha\rangle$	$[\alpha]$	$\widehat{\underset{\smallsmile}{\alpha}}$	$\overbracket{\underbracket{\alpha}}$

again, the proxies are translated into a diamond-type modality when occurring in precedent position and into a box-type modality when occurring in succedent position.
The two tables below introduce some structural rules. In what follows, $X^{-\mathsf{a}}$ and $Y^{-\mathsf{a}}$ denote structures restricted to the language in which neither the epistemic modalities nor their proxies occur.

Structural Rules

$$
atom_L \, \frac{X^{-\mathsf{a}} \vdash Y^{-\mathsf{a}}}{\{\alpha\}X^{-\mathsf{a}} \vdash Y^{-\mathsf{a}}} \qquad \frac{X^{-\mathsf{a}} \vdash Y^{-\mathsf{a}}}{X^{-\mathsf{a}} \vdash \{\alpha\}Y^{-\mathsf{a}}} \, atom_R
$$

$$
\frac{X \vdash Y}{\{\alpha\}X \vdash \{\alpha\}Y} \, balance
$$

$$
(\{\alpha\}, >) \, \frac{\{\alpha\}Y > \{\alpha\}Z \vdash X}{\{\alpha\}(Y > Z) \vdash X} \qquad \frac{Y \vdash \{\alpha\}X > \{\alpha\}Z}{Y \vdash \{\alpha\}(X > Z)} \, (>, \{\alpha\})
$$

$$
(\{\alpha\}, ;) \, \frac{\{\alpha\}X \,;\, \{\alpha\}Y \vdash Z}{\{\alpha\}(X \,;\, Y) \vdash Z} \qquad \frac{Z \vdash \{\alpha\}Y \,;\, \{\alpha\}X}{Z \vdash \{\alpha\}(Y \,;\, X)} \, (;, \{\alpha\})
$$

$$
(\overbrace{\underbrace{\alpha}}, >) \, \frac{\overbrace{\underbrace{\alpha}}Y > \overbrace{\underbrace{\alpha}}X \vdash Z}{\overbrace{\underbrace{\alpha}}(Y > X) \vdash Z} \qquad \frac{Y \vdash \overbrace{\underbrace{\alpha}}X > \overbrace{\underbrace{\alpha}}Z}{Y \vdash \overbrace{\underbrace{\alpha}}(X > Z)} \, (>, \overbrace{\underbrace{\alpha}})
$$

$$
(\overbrace{\underbrace{\alpha}}, ;) \, \frac{\overbrace{\underbrace{\alpha}}X \,;\, \overbrace{\underbrace{\alpha}}Y \vdash Z}{\overbrace{\underbrace{\alpha}}(X \,;\, Y) \vdash Z} \qquad \frac{Z \vdash \overbrace{\underbrace{\alpha}}Y \,;\, \overbrace{\underbrace{\alpha}}X}{Z \vdash \overbrace{\underbrace{\alpha}}(Y \,;\, X)} \, (;, \overbrace{\underbrace{\alpha}})
$$

Display Postulates

$$
(\{\alpha\}, \overbrace{\underbrace{\alpha}}) \, \frac{\{\alpha\}X \vdash Y}{\overline{X \vdash \overbrace{\underbrace{\alpha}}Y}} \qquad \frac{Y \vdash \{\alpha\}X}{\overline{\overbrace{\underbrace{\alpha}}Y \vdash X}} \, (\overbrace{\underbrace{\alpha}}, \{\alpha\})
$$

The structural rules such as $(\widehat{\{\alpha\}}, >)$ or FS^{ep} encode Fischer Servi-type axioms such as the following ones:

$$
\begin{array}{ll}
\langle \mathsf{a} \rangle A \rightarrow [\mathsf{a}] B \vdash [\mathsf{a}](A \rightarrow B) & \qquad \widehat{\langle \mathsf{a} \rangle} A \rightarrow \widehat{[\mathsf{a}]} B \vdash \widehat{[\mathsf{a}]} (A \rightarrow B) \\
\langle \alpha \rangle\ A \rightarrow [\alpha] B \vdash [\alpha](A \rightarrow B) & \qquad \widehat{\langle \alpha \rangle} A \rightarrow \widehat{[\alpha]} B \vdash \widehat{[\alpha]} (A \rightarrow B).
\end{array}
$$

As is well known, these axioms correspond to the fact that e.g. $\langle \mathsf{a} \rangle$ and $[\mathsf{a}]$ are interpreted over the same relation. An alternative way to express the same fact is given by the following *conjugation axioms*:

$$
\begin{array}{ll}
\langle \mathsf{a} \rangle A \wedge B \vdash \langle \mathsf{a} \rangle (A \wedge \widehat{\langle \mathsf{a} \rangle} B) & \qquad \widehat{\langle \mathsf{a} \rangle} A \wedge B \vdash \widehat{\langle \mathsf{a} \rangle} (A \wedge \langle \mathsf{a} \rangle B) \\
\langle \alpha \rangle A \wedge B \vdash \langle \alpha \rangle (A \wedge \widehat{\langle \alpha \rangle} B) & \qquad \widehat{\langle \alpha \rangle} A \wedge B \vdash \widehat{\langle \alpha \rangle} (A \wedge \langle \alpha \rangle B),
\end{array}
$$

which in their turn can be encoded in *conjugation* rules such as the following ones:

$$
{\scriptstyle conj}\ \frac{\{\alpha\}(X \,;\, \widehat{\{\alpha\}} Y) \vdash Z}{\{\alpha\} X \,; Y \vdash Z}
\qquad
\frac{\widehat{\{\alpha\}} (X \,; \{\alpha\} Y) \vdash Z}{\widehat{\{\alpha\}} X \,; Y \vdash Z}\ {\scriptstyle conj}
$$

In the presence of the display postulates, the conjugation rules are inter-derivable with the Fischer Servi rules. Indeed, let us show that the following rules:

$$
\frac{\{\alpha\}(X \,;\, \widehat{\{\alpha\}} Y) \vdash Z}{\{\alpha\} X \,; Y \vdash Z}\ {\scriptstyle conj}
\qquad
\frac{Y \vdash \widehat{\{\alpha\}} X > \widehat{\{\alpha\}} Z}{Y \vdash \widehat{\{\alpha\}} (X > Z)}\ {\scriptstyle FS}
$$

are inter-derivable:

$$
\begin{array}{c}
\{\alpha\}(X \,;\, \widehat{\{\alpha\}} Y) \vdash Z \\ \hline
X \,;\, \widehat{\{\alpha\}} Y \vdash \widehat{\{\alpha\}} Z \\ \hline
\widehat{\{\alpha\}} Y \,; X \vdash \widehat{\{\alpha\}} Z \\ \hline
X \vdash \widehat{\{\alpha\}} Y > \widehat{\{\alpha\}} Z \\ \hline
X \vdash \widehat{\{\alpha\}} (Y > Z) \quad {\scriptstyle FS} \\ \hline
\{\alpha\} X \vdash Y > Z \\ \hline
Y \,; \{\alpha\} X \vdash Z \\ \hline
\{\alpha\} X \,; Y \vdash Z
\end{array}
\qquad
\begin{array}{c}
Y \vdash \widehat{\{\alpha\}} X > \widehat{\{\alpha\}} Z \\ \hline
\widehat{\{\alpha\}} X \,; Y \vdash \widehat{\{\alpha\}} Z \\ \hline
Y \,;\, \widehat{\{\alpha\}} X \vdash \widehat{\{\alpha\}} Z \\ \hline
\{\alpha\}(Y \,;\, \widehat{\{\alpha\}} X) \vdash Z \\ \hline
{\scriptstyle conj} \quad \{\alpha\} Y \,; X \vdash Z \\ \hline
X \,; \{\alpha\} Y \vdash Z \\ \hline
\{\alpha\} Y \vdash X > Z \\ \hline
Y \vdash \widehat{\{\alpha\}} (X > Z)
\end{array}
$$

Analogously, the following rules:

$$\frac{\underbrace{\alpha}(X\,;\{\alpha\}Y) \vdash Z}{\underbrace{\alpha}X\,;Y \vdash Z}\, conj \qquad \frac{Y \vdash \{\alpha\}X > \{\alpha\}Z}{Y \vdash \{\alpha\}(X > Z)}\, FS$$

are interderivable, but the proof is omitted. Next, the standard operational rules are given in the table below:

Operational Rules

$$\langle\alpha\rangle_L\,\frac{\{\alpha\}A \vdash X}{\langle\alpha\rangle A \vdash X} \qquad \frac{X \vdash A}{\{\alpha\}X \vdash \langle\alpha\rangle A}\,\langle\alpha\rangle_R$$

$$[\alpha]_L\,\frac{A \vdash X}{[\alpha]A \vdash \{\alpha\}X} \qquad \frac{X \vdash \{\alpha\}A}{X \vdash [\alpha]A}\,[\alpha]_R$$

The rules in the following two tables capture the specific features of EAK; all of them contain the formula $Pre(\alpha)$ as a side condition:

Structural Rules with Side Conditions

$$reduce_L\,\frac{Pre(\alpha)\,;\{\alpha\}A \vdash X}{\{\alpha\}A \vdash X} \qquad \frac{X \vdash Pre(\alpha) > \{\alpha\}A}{X \vdash \{\alpha\}A}\,reduce_R$$

$$swap\text{-}in_L\,\frac{Pre(\alpha)\,;\{\alpha\}\{\mathtt{a}\}X \vdash Y}{Pre(\alpha)\,;\{\mathtt{a}\}\{\beta\}_{\alpha\mathtt{a}\beta}\,X \vdash Y} \qquad \frac{Y \vdash Pre(\alpha) > \{\alpha\}\{\mathtt{a}\}X}{Y \vdash Pre(\alpha) > \{\mathtt{a}\}\{\beta\}_{\alpha\mathtt{a}\beta}\,X}\,swap\text{-}in_R$$

$$swap\text{-}out_L\,\frac{\Big(Pre(\alpha)\,;\{\mathtt{a}\}\{\beta_j\}_{\alpha\mathtt{a}\beta}\,X \vdash Y\Big)_j}{Pre(\alpha)\,;\{\alpha\}\{\mathtt{a}\}X \vdash \boldsymbol{;}\Big(Y\Big)_j} \qquad \frac{\Big(Y \vdash Pre(\alpha) > \{\mathtt{a}\}\{\beta_j\}_{\alpha\mathtt{a}\beta}\,X\Big)_j}{\boldsymbol{;}\Big(Y\Big)_j \vdash Pre(\alpha) > \{\alpha\}\{\mathtt{a}\}X}\,swap\text{-}out_R$$

Operational Rules with Side Conditions

$$reverse_L\,\frac{Pre(\alpha)\,;\{\alpha\}A \vdash X}{Pre(\alpha)\,;[\alpha]A \vdash X} \qquad \frac{X \vdash Pre(\alpha) > \{\alpha\}A}{X \vdash Pre(\alpha) > \langle\alpha\rangle A}\,reverse_R$$

In [14], D.EAK is proven to be sound with respect to the final coalgebra semantics, complete w.r.t. EAK, of which it is a conservative extension. The syntactic behavior of all the connectives, including the dynamic modalities, is explicit, and the meaning of the rules is intuitive. D.EAK is defined independently of the relational semantics of EAK, and therefore is suitable for a fine-grained proof-theoretic analysis. All the operational rules satisfy Wansing's criteria of *symmetry* and *explicitness*. Moreover, every operational rule satisfies the criterion of *separation*, with the exception of the *reverse* rules, in which two operational connectives are introduced at the same time. Furthermore, because of the display properties, all the operational rules, with the exception of *reverse*, satisfy even a stronger version of separation, namely *segregation*, that is, all active formulas are isolated. Finally, all structural rules with no side conditions satisfy Wansing's and Belnap's requirement that the parametric variables standing for substructures should occur unrestricted in the statement of the rules.

Another important proof-theoretic feature of D.EAK is modularity. As shown at the beginning of the section, the base of D.EAK can be modified in a modular way to capture the whole space of EAK-type logics on a nonclassical base and from the rules for static modal logic the whole space of normal modal logics can be reconstructed. Finally, D.EAK is a cut-free system for which cut elimination Gentzen-style has been proven.

Chapter 4

Towards a Proof-Theoretic Semantics for Dynamic Epistemic Logic

4.1 On the adequacy of D.EAK for proof-theoretic semantics

As discussed above, the system D.EAK is modular in the sense that it satisfies Došen's principle; it adequately captures EAK, in the sense that it is complete w.r.t. EAK, of which it is also a conservative extension; moreover, the rules of D.EAK have an intuitive interpretation in terms of well known axioms and valid EAK-principles; finally, D.EAK enjoys cut-elimination.

From the point of view of proof-theoretic semantics, all the operational rules except *reverse* satisfy Wansing's criteria of *separation, symmetry and explicitness*, the former one being satisfied even in its stronger version of *segregation*. All structural rules with no side conditions, except the *atom* rules, satisfy Wansing's and Belnap's requirement that the parametric variables standing for substructures should occur unrestricted in the statement of the rules. However, the *atom* rules and both the operational and the structural contextual rules are problematic for different reasons, which we are going to detail below.

Firstly, the *atom* rules are restricted to the introduction of structures which are free of static proxies and modalities. In Chapter 4, we will discuss how this problem could be remedied.

Secondly, also the *swap-in* and *swap-out* rules violate the principle that all parametric variables should occur unrestricted. Indeed, the problem is the presence of the formula $Pre(\alpha)$, which is easily seen to be parametric, since it occurs both in the premises and in the conclusion. Since $Pre(\alpha)$ is (the metalinguistic abbreviation of) a formula, it is a structure of a very restricted shape. As to the *swap-out* rules, it is not difficult to see, e.g. semantically (cf. [16, Definition 4.2.]), that the occurrences of $Pre(\alpha)$ can be removed both in the premises and in the conclusion without affecting either the soundness of the rule or the proof power of the system; this entirely remedies the problem. Likewise, as to *swap-in*, it is not difficult to see that the occurrences of $Pre(\alpha)$ can be removed in the premises, but not in the conclusion. However, also in this new form, the *swap-in* rules are not satisfactory: indeed, the new form of *swap-in* would introduce $Pre(\alpha)$ in the conclusion; in this respect, *swap-in* would behave similarly to a *weakening* rule. According to Belnap's understanding, *weakening* is a rule in which all structures are parametric, including the one introduced in the conclusion. By analogy then, the occurrence of $Pre(\alpha)$ in the conclusion of *swap-in* should again be counted as parametric, and hence, as before, it is problematic. This problem would be solved if $Pre(\alpha)$ could be expressed, as a structure, purely in terms of the parameter α and structural constants (but no structural variables). If this was the case, *swap-in* would encode the relations between all these logical constants, and all the occurring structural variables would be unrestricted.

Thirdly, the rules *reduce* violate Belnap's condition C_1: indeed, in each of them, a formula in the premisses, namely $Pre(\alpha)$, is not a subformula of any formula occurring in the conclusion. Together with the cut elimination, condition C_1 guarantees the subformula property (cf. [8, Theorem 4.3]), but is not itself essential for the cut elimination, and indeed, cut elimination has been proven for D.EAK (albeit not á la Belnap). The specific way in which *reduce* violates C_1 is also not a very serious one. Indeed, if the formula $Pre(\alpha)$ could be expressed in a structural way, this violation would disappear.

One reason why this solution could not be implemented in D.EAK is that the language of D.EAK does not have enough expressivity to talk about $Pre(\alpha)$ in any other way than as an arbitrary formula, which needs to be introduced via weakening or via identity (if atomic). Being able to account for $Pre(\alpha)$ in a satisfactory way from a proof-theoretic perspective would require being able to state rules which would introduce $Pre(\alpha)$ specifically, thus capturing its proof-theoretic meaning. Thus, by having structural and operational rules for $Pre(\alpha)$, we would solve many problems in one stroke: on the one hand, we would gain the *practical* advantage of achieving the satisfaction of C_1, thus guaranteeing the subformula property; on the other hand, and more importantly, from a *methodological* perspective, we would be able to have a setting in which the occurrences of $Pre(\alpha)$ are not to be regarded as side formulas, but rather, they would occur as *structures*, on a par with all the other structures they would be interacting with.

Finally, the only operational rules violating Wansing's *separation* principle are the *reverse* rules:

$$rev_L \frac{Pre(\alpha); \{\alpha\}A \vdash X}{Pre(\alpha); [\alpha]A \vdash X} \qquad \frac{X \vdash Pre(\alpha) > \{\alpha\}A}{X \vdash Pre(\alpha) > \langle\alpha\rangle A} rev_R$$

Here again, the problem comes from the fact that the language is not expressive enough to capture the principles encoded in the rules above at a purely structural level. In this operational formulation, these rules are to participate, in our view improperly, in the proof-theoretic meaning of the connectives $[\alpha]$ and $\langle\alpha\rangle$. Thus, it would be desirable that the rules above could be either derived, so that they disappear altogether, or alternatively, be reformulated as structural rules.

4.2 On the adequacy of D.EAK for cut-elimination Belnap-style

In the previous section, we discussed some possible fixes for the aspects in which the system D.EAK does not seem to satisfy Wansing's and Belnap's design principles, which would both account for the proof-theoretic semantics of the logical connectives and would guarantee desirable properties such as the subformula property or the cut elimination Belnap-style. All these fixes revolve around the possibility of expanding the expressivity of the language so as to encode crucial features of the logic EAK, such as $Pre(\alpha)$, at the structural level. Before moving on and implementing the suggested fixes, in the present section we wish to outline the reasons why in our opinion there exists an understanding of the spirit of Belnap's cut elimination metatheorem which is applicable to the system D.EAK as it is. The material of this section is not going to be used in the rest of the thesis; also for this reason, we are not going to give fully fledged formal arguments, but rather we keep the discussion informal.

Indeed, one way of understanding Belnap's theory of display logic is as a meta analysis of the proof of cut-elimination. Belnap's conditions C_2-C_8 capture in a sense the Platonic ideal of a cut elimination proof, and this explains why Belnap-style cut elimination is comparatively more demanding than Gentzen-style cut elimination. Technically, the former one requires being able to perform global transformations on any given deduction tree. These transformations require being able to move the application of cut rules arbitrarily high up the branches of a deduction tree, and hence, to suitably rearrange the relevant contexts using only display postulates. In the case of D.EAK, the main issues revolve around those rules which requires a restricted shape of parametric variables. In this respect, the offending rules are of two types: the rules with the side condition $Pre(\alpha)$, above all the operational ones, and the *atom* rules. The main reason why rules with restricted shape of parametric variables are typically not amenable to the

Belnap-style cut elimination is that, whenever global transformations are called for, we might be in need of applying such a rule in places where the restriction is violated; for instance, we might be in need of applying a rule requiring side conditions in places where the required side conditions cannot be generated. An analogous problem arises with *atom*, although the restriction in the shape of the parametric variables in these rules is not to be interpreted as a side condition.

However, we observe that, whenever such a "non-behaving" deduction system has enough structural rules that the side conditions, or the restricted parameters can always be reconstructed wherever it is needed, the global transformations of the Belnap-style cut elimination can go through.

In the case of D.EAK, the only side condition is the formula $Pre(\alpha)$ occurring in precedent position. Thanks to e.g. *weakening*$_L$, this side condition $Pre(\alpha)$ can always be generated whenever needed, and thanks to *reduce*, it can be always removed whenever needed.

Moreover, as to the *atom* rule, from the the brute-force cut elimination theorem, the stronger fact emerges that the cut can be permuted over *atom* and can be performed higher up enough that the required restrictions are in place.

Hence, we conjecture that a suitable extension of Belnap metatheorem can be proven which applies to D.EAK.

4.3 Revising D.EAK

In the present section, we discuss how to revise D.EAK so as to address the issues pointed out in section 4.1. The reasons why the system D.EAK does not fully address the criteria of proof-theoretic semantics (and hence, cut elimination Belnap-style could not be smoothly argued for it) are essentially due to the presence of rules having the meta-linguistic abbreviation $Pre(\alpha)$ as a side condition. Indeed, as in EAK, also in D.EAK the symbol $Pre(\alpha)$ is a black box term standing for any arbitrary formula in the language, which we cannot break open or account for in any other way than by introducing it by means of *weakening* and axioms, and by eliminating it by the dedicated rule *reduce*. Hence, either we try and make it disappear altogether, or we promote it to the ranks of the object language, and we endow it with its own independent proof-theoretic meaning. We can try and make $Pre(\alpha)$ disappear by uniformly substituting $\langle\alpha\rangle\top$ for it. However, this is not an acceptable course of actions, since e.g. the meaningful axiom $Pre(\alpha) \leftrightarrow \langle\alpha\rangle\top$ would be reduced to the meaningless tautology $\langle\alpha\rangle\top \leftrightarrow \langle\alpha\rangle\top$.

We can alternatively stipulate that $Pre(\alpha)$ is added to the object language, e.g. as a constant 1_α for each action α. Following the common praxis in display calculi, this constant would then need to be assigned a structural counterpart. We could consider to use the structure $\{\alpha\}\mathrm{I}$ as the structural counterpart for 1_α. However, adopting the following two rules:

$$1_{\alpha L}\,\frac{\{\alpha\}\mathrm{I} \vdash X}{1_\alpha \vdash X} \qquad\qquad \frac{}{\{\alpha\}\mathrm{I} \vdash 1_\alpha}\,1_{\alpha R}$$

would make 1_α have exactly the same proof-theoretic semantics as $\langle\alpha\rangle\top$.

Therefore, our proposed solution consists in adding, for each action α, both a new operational constant 1_α and its structural counterpart Φ_α to the language of D.EAK. The corresponding rules are:

$$1_{\alpha L}\,\frac{\Phi_\alpha \vdash X}{1_\alpha \vdash X} \qquad\qquad \frac{}{\Phi_\alpha \vdash 1_\alpha}\,1_{\alpha R}$$

The structural rules involving Φ_α are:

$$comp^\alpha_L\,\frac{\{\alpha\}\overbrace{\underbrace{\alpha}}X \vdash Y}{\Phi_\alpha ; X \vdash Y} \qquad \frac{X \vdash \{\alpha\}\overbrace{\underbrace{\alpha}}Y}{X \vdash \Phi_\alpha > Y}\,comp^\alpha_R$$

Intuitively, the *comp* rules above can be regarded as capturing a restricted form of composition of actions, which results in the idle action. They are sound w.r.t. the final coalgebra semantics (cf. appendix). Notice that all the parameters in the rules above occur unrestricted.

Notice also that Φ_α occurs in precedent position in both *comp* rules. Hence, Φ_α can never be interpreted as anything else than 1_α with the rules introduced so far. However, a natural extension of this system would be to introduce a operational constant 0_α, intuitively standing for the postconditions of α for each α, and dualize the rules given so far, so as to capture the behavior of postconditions. We are working towards the definition of the new system D'.EAK.

Next, we introduce the non-problematic structural rule which is to capture the specific behaviour of epistemic actions

Atom

$$\boxed{\quad \frac{}{\Gamma p \vdash \Delta p}\,atom \quad}$$

where Γ and Δ are arbitrary finite sequences of the form $(\alpha_1)\ldots(\alpha_n)$, such that each (α_j) is of the form $\{\alpha_j\}$ or of the form $\overbrace{\underbrace{\alpha_j}}$, for $1 \leq j \leq n$. Intuitively, the *atom rule* captures the requirement that epistemic actions do not change the factual states of affairs of the world.

The rules introduced so far meet all the Wansing's and Belnap's criteria (this will be expanded on in the following section). Moreover, we have reached a nice division of labour between the operational rules defining the proof-theoretic meaning of the new connectives, and the structural rules capturing the relations entertained between the proxies of different connectives. We are now in a position to address all the remaining issues which were raised in section 4.1. Firstly, the *reduce* and *swap-in* rules can be reformulated by replacing the formula $Pre(\alpha)$, with the proxy Φ_α, as follows:

$$\textit{reduce'}_L\ \frac{\Phi_\alpha; \{\alpha\}X \vdash Y}{\{\alpha\}X \vdash Y} \qquad \frac{Y \vdash \Phi_\alpha > \{\alpha\}X}{Y \vdash \{\alpha\}X}\ \textit{reduce'}_R$$

$$\textit{swap-in'}_L\ \frac{\{\alpha\}\{\mathtt{a}\}X \vdash Y}{\Phi_\alpha; \{\mathtt{a}\}\{\beta\}_{\alpha\mathtt{a}\beta}\, X \vdash Y} \qquad \frac{Y \vdash \{\alpha\}\{\mathtt{a}\}X}{Y \vdash \Phi_\alpha > \{\mathtt{a}\}\{\beta\}_{\alpha\mathtt{a}\beta}\, X}\ \textit{swap-in'}_R$$

Notice that all the parameters in the rules above occur unrestricted, since Φ_α is a zero-ary connective, and hence does not contain any variable. This solves the issue raised about the formulation of these rules in D.EAK.
As remarked in section 4.1, the *swap-out* rules are sound also in the following formulation:

$$\textit{swap-out'}_L\ \frac{\Big(\{\mathtt{a}\}\{\beta\}\, X \vdash Y \mid \alpha\mathtt{a}\beta\Big)}{\{\alpha\}\{\mathtt{a}\}X \vdash \boldsymbol{;}\Big(Y \mid \alpha\mathtt{a}\beta\Big)} \qquad \frac{\Big(Y \vdash \{\mathtt{a}\}\{\beta\}\, X \mid \alpha\mathtt{a}\beta\Big)}{\boldsymbol{;}\Big(Y \mid \alpha\mathtt{a}\beta\Big) \vdash \{\alpha\}\{\mathtt{a}\}X}\ \textit{swap-out'}_R$$

Let us show that the contextual rules in D.EAK can be derived from these new rules and the remaining part of the calculus. The old rules *reduce*, or more precisely, their rewriting in the new notation, are derivable as follows.

$$\cfrac{\cfrac{\cfrac{\cfrac{\Phi_\alpha \vdash 1_\alpha \qquad \cfrac{\cfrac{1_\alpha; \{\alpha\}A \vdash X}{\{\alpha\}A; 1_\alpha \vdash X}}{1_\alpha \vdash \{\alpha\}A > X}}{\Phi_\alpha \vdash \{\alpha\}A > X}}{\{\alpha\}A; \Phi_\alpha \vdash X}}{\Phi_\alpha; \{\alpha\}A \vdash X}}{\{\alpha\}A \vdash X}\ reduce'_L \qquad \cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\Phi_\alpha \vdash 1_\alpha \qquad \cfrac{\cfrac{\cfrac{X \vdash 1_\alpha > \{\alpha\}A}{1_\alpha; X \vdash \{\alpha\}A}}{X; 1_\alpha \vdash \{\alpha\}A}}{1_\alpha \vdash X > \{\alpha\}A}}{\Phi_\alpha \vdash X > \{\alpha\}A}}{X; \Phi_\alpha \vdash \{\alpha\}A}}{\Phi_\alpha; X \vdash \{\alpha\}A}}{X \vdash \Phi_\alpha > \{\alpha\}A}}{X \vdash \{\alpha\}A}\ reduce'_R$$

The old *swap-in* rules are derivable in the revised calculus from the new *swap-in* rules as follows.

$$
\textit{swap-in'}_L\;\dfrac{\textit{reduce'}_L\;\dfrac{\dfrac{\dfrac{\dfrac{\Phi_\alpha \vdash 1_\alpha \qquad \dfrac{\dfrac{1_\alpha;\{\alpha\}\{\mathsf{a}\}X \vdash Y}{\{\alpha\};1_\alpha\{\mathsf{a}\}X \vdash Y}}{1_\alpha \vdash \{\alpha\}\{\mathsf{a}\}X > Y}}{\Phi_\alpha \vdash \{\alpha\}\{\mathsf{a}\}X > Y}}{\{\alpha\}\{\mathsf{a}\}X\;;\Phi_\alpha \vdash Y}}{\Phi_\alpha\;;\{\alpha\}\{\mathsf{a}\}X \vdash Y}}{\{\alpha\}\{\mathsf{a}\}X \vdash Y}}{\dfrac{\dfrac{\dfrac{\dfrac{\Phi_\alpha;\{\mathsf{a}\}\{\beta\}_{\alpha\mathsf{a}\beta}\,X \vdash Y}{\{\mathsf{a}\}\{\beta\}_{\alpha\mathsf{a}\beta}\,X;\Phi_\alpha \vdash Y}}{\Phi_\alpha \vdash \{\mathsf{a}\}\{\beta\}_{\alpha\mathsf{a}\beta}\,X > Y}}{1_\alpha \vdash \{\mathsf{a}\}\{\beta\}_{\alpha\mathsf{a}\beta}\,X > Y}}{\dfrac{\{\mathsf{a}\}\{\beta\}_{\alpha\mathsf{a}\beta}\,X;1_\alpha \vdash Y}{1_\alpha;\{\mathsf{a}\}\{\beta\}_{\alpha\mathsf{a}\beta}\,X \vdash Y}}}
$$

$$
\textit{swap-in'}_L\;\dfrac{\textit{reduce'}_L\;\dfrac{\dfrac{\dfrac{\dfrac{\Phi_\alpha \vdash 1_\alpha \qquad \dfrac{\dfrac{\dfrac{Y \vdash 1_\alpha > \{\alpha\}\{\mathsf{a}\}X}{1_\alpha;Y \vdash \{\alpha\}\{\mathsf{a}\}X}}{Y;1_\alpha \vdash \{\alpha\}\{\mathsf{a}\}X}}{1_\alpha \vdash Y > \{\alpha\}\{\mathsf{a}\}X}}{\Phi_\alpha \vdash Y > \{\alpha\}\{\mathsf{a}\}X}}{Y;\Phi_\alpha \vdash \{\alpha\}\{\mathsf{a}\}X}}{\Phi_\alpha\;;\{\alpha\}\{\mathsf{a}\}X \vdash Y}}{\{\alpha\}\{\mathsf{a}\}X \vdash Y}}{\Phi_\alpha;\{\mathsf{a}\}\{\beta\}_{\alpha\mathsf{a}\beta}\,X \vdash Y}
$$

The old *swap-out* rules (translated into D'.EAK) are derivable using the new *swap-out* rules:

$$
\dfrac{\dfrac{\dfrac{\textit{reduce'}_L\;\dfrac{\dfrac{\dfrac{\Phi_\alpha \vdash 1_\alpha \qquad \dfrac{\dfrac{1_\alpha\,;\{\mathsf{a}\}\{\beta_1\}\,X \vdash Y \mid \alpha\mathsf{a}\beta_1}{\{\mathsf{a}\}\{\beta_1\}\,X;1_\alpha \vdash Y \mid \alpha\mathsf{a}\beta_1}}{1_\alpha \vdash \{\mathsf{a}\}\{\beta_1\}\,X > Y \mid \alpha\mathsf{a}\beta_1}}{\Phi_\alpha \vdash \{\mathsf{a}\}\{\beta_1\}\,X > Y \mid \alpha\mathsf{a}\beta_1}}{\dfrac{\{\mathsf{a}\}\{\beta_1\}\,X;\Phi_\alpha \vdash Y \mid \alpha\mathsf{a}\beta_1}{\Phi_\alpha;\{\mathsf{a}\}\{\beta_1\}\,X \vdash Y \mid \alpha\mathsf{a}\beta_1}}}{\{\mathsf{a}\}\{\beta_1\}\,X \vdash Y \mid \alpha\mathsf{a}\beta_1} \quad \cdots \quad \textit{reduce'}_L\;\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{1_\alpha\,;\{\mathsf{a}\}\{\beta_n\}\,X\vdash Y \mid \alpha\mathsf{a}\beta_n}{\{\mathsf{a}\}\{\beta_n\}\,X;1_\alpha\vdash Y \mid \alpha\mathsf{a}\beta_n}}{1_\alpha\vdash\{\mathsf{a}\}\{\beta_n\}\,X > Y \mid}}{\{\mathsf{a}\}\{\beta_n\}\,X;\Phi_\alpha\vdash Y \mid \alpha\mathsf{a}\beta_n}}{\Phi_\alpha;\{\mathsf{a}\}\{\beta_n\}\,X\vdash Y \mid \alpha\mathsf{a}\beta_n}}{\{\mathsf{a}\}\{\beta_n\}\,X\vdash Y \mid \alpha\mathsf{a}\beta_n}}{\{\alpha\}\{\mathsf{a}\}X\vdash \mathbf{;}\big(Y \mid \alpha\mathsf{a}\beta\big)}\;\textit{swap-o}}{\dfrac{1_\alpha\vdash\{\alpha\}\{\mathsf{a}\}X > \mathbf{;}\big(Y \mid \alpha\mathsf{a}\beta\big)}{\{\alpha\}\{\mathsf{a}\}X;1_\alpha\vdash \mathbf{;}\big(Y \mid \alpha\mathsf{a}\beta\big)}}}{1_\alpha;\{\alpha\}\{\mathsf{a}\}X\vdash \mathbf{;}\big(Y \mid \alpha\mathsf{a}\beta\big)}
$$

$$
\dfrac{\dfrac{\dfrac{\textit{reduce'}_R\;\dfrac{\dfrac{\dfrac{\dfrac{\Phi_\alpha \vdash 1_\alpha \qquad \dfrac{\dfrac{\dfrac{Y \vdash 1_\alpha\; > \{\mathsf{a}\}\{\beta_1\}\,X \mid \alpha\mathsf{a}\beta_1}{1_\alpha\,;Y \vdash \{\mathsf{a}\}\{\beta_1\}\,X \mid \alpha\mathsf{a}\beta_1}}{Y;1_\alpha\; \vdash \{\mathsf{a}\}\{\beta_1\}\,X \mid \alpha\mathsf{a}\beta_1}}{1_\alpha\; \vdash Y > \{\mathsf{a}\}\{\beta_1\}\,X \mid \alpha\mathsf{a}\beta_1}}{\Phi_\alpha \vdash Y > \{\mathsf{a}\}\{\beta_1\}\,X \mid \alpha\mathsf{a}\beta_1}}{Y;\Phi_\alpha\; \vdash \{\mathsf{a}\}\{\beta_1\}\,X \mid \alpha\mathsf{a}\beta_1}}{\dfrac{\Phi_\alpha\,;Y \vdash \{\mathsf{a}\}\{\beta_1\}\,X \mid \alpha\mathsf{a}\beta_1}{Y \vdash \Phi_\alpha\; > \{\mathsf{a}\}\{\beta_1\}\,X \mid \alpha\mathsf{a}\beta_1}}}{Y \vdash \{\mathsf{a}\}\{\beta_1\}\,X \mid \alpha\mathsf{a}\beta_1} \quad \cdots \quad \textit{reduce'}_R\;\dfrac{\dfrac{\dfrac{\dfrac{\Phi_\alpha\vdash 1_\alpha \qquad \dfrac{\dfrac{\dfrac{Y\vdash 1_\alpha\; > \{\mathsf{a}\}\{\beta_n\}\,X \mid \alpha\mathsf{a}\beta_n}{1_\alpha\,;Y\vdash\{\mathsf{a}\}\{\beta_n\}\,X \mid \alpha\mathsf{a}\beta_n}}{Y;1_\alpha\vdash\{\mathsf{a}\}\{\beta_n\}\,X \mid \alpha\mathsf{a}\beta_n}}{1_\alpha\vdash Y > \{\mathsf{a}\}\{\beta_n\}\,X \mid \alpha\mathsf{a}\beta_n}}{\Phi_\alpha\vdash Y > \{\mathsf{a}\}\{\beta_n\}\,X \mid \alpha\mathsf{a}\beta_n}}{Y;\Phi_\alpha\vdash\{\mathsf{a}\}\{\beta_n\}\,X \mid \alpha\mathsf{a}\beta_n}}{\dfrac{\Phi_\alpha\,;Y\vdash\{\mathsf{a}\}\{\beta_n\}\,X \mid \alpha\mathsf{a}\beta_n}{Y\vdash\Phi_\alpha\; > \{\mathsf{a}\}\{\beta_n\}\,X \mid \alpha\mathsf{a}\beta_n}}}{Y\vdash\{\mathsf{a}\}\{\beta_n\}\,X \mid \alpha\mathsf{a}\beta_n}}{\mathbf{;}\big(Y \mid \alpha\mathsf{a}\beta\big)\vdash\{\alpha\}\{\mathsf{a}\}X}\;\textit{swap-out'}}{\dfrac{1_\alpha\vdash \mathbf{;}\big(Y \mid \alpha\mathsf{a}\beta\big) > \{\alpha\}\{\mathsf{a}\}X}{\mathbf{;}\big(Y \mid \alpha\mathsf{a}\beta\big);1_\alpha\vdash\{\alpha\}\{\mathsf{a}\}X}}}{\dfrac{\mathbf{;}\big(Y \mid \alpha\mathsf{a}\beta\big);1_\alpha\vdash\{\alpha\}\{\mathsf{a}\}X}{\mathbf{;}\big(Y \mid \alpha\mathsf{a}\beta\big)\vdash 1_\alpha > \{\alpha\}\{\mathsf{a}\}X}}
$$

An important benefit of the revised system is that the operational rules *reverse*, which were primitive in the old system, are now derivable using the new rules for Φ_α and 1_α and the new *reduce*. This supports our intuition that the rules *reverse* do not participate in the proof-theoretic meaning of the connectives $\langle\alpha\rangle$ and $[\alpha]$.

$$
\begin{prooftree}
\AxiomC{$\Phi_\alpha \vdash 1_\alpha$}
\AxiomC{$1_\alpha ; \{\alpha\}A \vdash X$}
\UnaryInfC{$\{\alpha\}A ; 1_\alpha \vdash X$}
\UnaryInfC{$1_\alpha \vdash \{\alpha\}A > X$}
\BinaryInfC{$\Phi_\alpha \vdash \{\alpha\}A > X$}
\UnaryInfC{$\{\alpha\}A ; \Phi_\alpha \vdash X$}
\UnaryInfC{$\Phi_\alpha ; \{\alpha\}A \vdash X$}
\UnaryInfC{$\{\alpha\}A \vdash X$}
\UnaryInfC{$A \vdash \overset{\frown}{\underset{\smile}{\alpha}} X$}
\UnaryInfC{$[\alpha]A \vdash \{\alpha\} \overset{\frown}{\underset{\smile}{\alpha}} X$}
\LeftLabel{$comp^\alpha_R$}
\UnaryInfC{$[\alpha]A \vdash \Phi_\alpha > X$}
\UnaryInfC{$\Phi_\alpha \, ; [\alpha]A \vdash X$}
\UnaryInfC{$[\alpha]A \, ; \Phi_\alpha \vdash X$}
\UnaryInfC{$\Phi_\alpha \vdash [\alpha]A > X$}
\UnaryInfC{$1_\alpha \vdash [\alpha]A > X$}
\UnaryInfC{$[\alpha]A \, ; 1_\alpha \vdash X$}
\UnaryInfC{$1_\alpha \, ; [\alpha]A \vdash X$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{$\Phi_\alpha \vdash 1_\alpha$}
\AxiomC{$X \vdash 1_\alpha > \{\alpha\}A$}
\UnaryInfC{$1_\alpha ; X \vdash \{\alpha\}A$}
\UnaryInfC{$X ; 1_\alpha \vdash \{\alpha\}A$}
\UnaryInfC{$1_\alpha \vdash X > \{\alpha\}A$}
\BinaryInfC{$\Phi_\alpha \vdash X > \{\alpha\}A$}
\UnaryInfC{$X ; \Phi_\alpha \vdash \{\alpha\}A$}
\UnaryInfC{$\Phi_\alpha ; X \vdash \{\alpha\}A$}
\UnaryInfC{$X \vdash \Phi_\alpha > \{\alpha\}A$}
\RightLabel{$reduce'_R$}
\UnaryInfC{$X \vdash \{\alpha\}A$}
\UnaryInfC{$\overset{\frown}{\underset{\smile}{\alpha}} X \vdash A$}
\UnaryInfC{$\{\alpha\} \overset{\frown}{\underset{\smile}{\alpha}} X \vdash \langle\alpha\rangle A$}
\RightLabel{$comp^\alpha_L$}
\UnaryInfC{$\Phi_\alpha \, ; X \vdash \langle\alpha\rangle A$}
\UnaryInfC{$X ; \Phi_\alpha \vdash \langle\alpha\rangle A$}
\UnaryInfC{$\Phi_\alpha \vdash X > \langle\alpha\rangle A$}
\UnaryInfC{$1_\alpha \vdash X > \langle\alpha\rangle A$}
\UnaryInfC{$X ; 1_\alpha \vdash \langle\alpha\rangle A$}
\UnaryInfC{$1_\alpha \, ; X \vdash \langle\alpha\rangle A$}
\UnaryInfC{$X \vdash 1_\alpha > \langle\alpha\rangle A$}
\end{prooftree}
$$

Finally, let us show that 1_α is equivalent to $\langle\alpha\rangle\top$.

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$\mathrm{I} \vdash \top$}
\UnaryInfC{$\overset{\frown}{\underset{\smile}{\alpha}} \, \mathrm{I} \vdash \top$}
\UnaryInfC{$\{\alpha\} \overset{\frown}{\underset{\smile}{\alpha}} \, \mathrm{I} \vdash \langle\alpha\rangle\top$}
\LeftLabel{com^α_L}
\UnaryInfC{$\Phi_\alpha \, ; \mathrm{I} \vdash \langle\alpha\rangle\top$}
\UnaryInfC{$\Phi_\alpha \vdash \langle\alpha\rangle\top$}
\UnaryInfC{$1_\alpha \vdash \langle\alpha\rangle\top$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{$\Phi_\alpha \vdash 1_\alpha$}
\UnaryInfC{$\{\alpha\}\top \vdash 1_\alpha < \Phi_\alpha$}
\UnaryInfC{$\Phi_\alpha \, ; \{\alpha\}\top \vdash 1_\alpha$}
\LeftLabel{red_L}
\UnaryInfC{$\{\alpha\}\top \vdash 1_\alpha$}
\UnaryInfC{$\langle\alpha\rangle\top \vdash 1_\alpha$}
\end{prooftree}
$$

In conclusion, the revised system D'.EAK provides adequate proof-theoretic semantics for all the connectives occurring in it. Indeed, each operational connective has both left- and right-introduction rules, and the structural proxies are in charge of the relations between different connectives, via rules in which all parameters occur safely. Indeed, the only rules in which some parameters are of a restricted shape will be shown to be regular (cf. condition

$\mathrm{C}_6/\mathrm{C}_7$ in section 2.3) in the next section. The system D'.EAK satisfies the most rigid proof-theoretic semantic criteria, such as *segregation*, since the new rules for 1_α clearly do not violate segregation, and the old offending rules *reverse* are not primitive in D'.EAK, but derived. The side condition $Pre(\alpha)$ disappears altogether in *swap-out*, and has been reformulated in the rules *reduce* and *swap-in* in such a way that each parameter now occurs unrestricted. This has been achieved by extending the language so that the meta-linguistic abbreviation $Pre(\alpha)$ has been introduced in the language as an operational constant, with its corresponding structural connective. Furthermore, this calculus also has an additional feature, namely Belnap-style cut elimination, which we will demonstrate in the next section.

4.4 Belnap-style cut-elimination for D'.EAK

The rules in the revised system D'.EAK are designed so as to satisfy the conditions C_1-C_8 which, as we know, are sufficient to guarantee the Belnap-style cut-elimination. The rules *reverse* are now derivable, and all the rules with the side condition $Pre(\alpha)$ have been reformulated in such a way that all the parameters occur unrestricted. For this reason, it is immediate to see that the new *swap-in* rules satisfy both conditions C_6 and C_7. It is easy to see that the operational rules for 1_α and the *comp* rules satisfy the criteria C_1-C_7. Also, the revised *atom* rule can be readily seen to verify conditions C_1-C_7.

Now, as to condition C_8, let us show the cases involving the new connective 1_α. All the other cases have been already treated in the Gentzen-style cut-elimination proof for D.EAK (although they do not appear in [14]) and are reported in the Appendix.

$$
\dfrac{\Phi_\alpha \vdash 1_\alpha \qquad \dfrac{\begin{array}{c}\vdots\,\pi\\ \Phi_\alpha \vdash X\end{array}}{1_\alpha \vdash X}}{\Phi \vdash X}
\quad \rightsquigarrow \quad
\begin{array}{c}\vdots\,\pi\\ \Phi_\alpha \vdash X\end{array}
$$

Chapter 5

Dynamic Calculus

5.1 A multi-type calculus

After making all the rules of D.EAK context-independent (cf. section 4.3) and having expanded the language in such a way that $Pre(\alpha)$ can be accounted for both as a formula and as a structure, hence as a first class citizen, all the criteria of proof-theoretic semantics are satisfied. However, the *swap-in* and *swap-out* rules are stated in terms of a label $\alpha \mathsf{a} \beta$, which is essentially a reformulation of a completely analogous label which was already present in the original Hilbert-style presentation of EAK in [7]. We would like to eliminate this label, not because its presence harms in any way the performances of the calculus D.EAK, but because an analysis leading to its elimination calls for general tools facilitating the build-up of a general proof-theory of dynamic logics, covering different dynamic settings (for more on the methodological aspects of such a general proof-theory, see section 5.4).

It is perhaps worth to highlight the difference between the labels in the *swap-in* and *swap-out* rules and the labels in the rules of [19] and [20]. In the rules of these papers, two kinds of labels appear; namely, relation symbols arising from the semantics of the epistemic modalities, and superscripts on the individual variables, which stand for the semantic relation of update linking each state surviving the update to the original state it comes from. Hence, both kinds of labels are inherent to the design of those calculi, and thus uneliminable. On the other hand, the labels in the *swap-in* and *swap-out* rules describe formal conditions on the parameters used in the built-up of the language of D.EAK, and appear in the form of metalinguistic conditions because the language of D.EAK is not expressive enough to capture them.

In order to provide the desired additional expressivity, we introduce a language in which not only formulas are generated from formulas and actions (as it happens in the symbol $\langle\alpha\rangle\phi$) and formulas are generated from formulas and agents (as it happens in the symbol $\langle \mathsf{a}\rangle\phi$), but also *actions* are generated from the interactions between *agents* and *actions*; indeed, this is precisely what the label $\alpha\mathsf{a}\beta$ is about.

The key idea is to define a *multi-type language*, in which each generation step mentioned above is explicitly accounted for via special connectives between different types. More than one alternative is possible in this respect; our choice for the present setting consists of the following types: Ag for agents, Fnc for functional actions, Act for actions, and Fm for formulas. The new connectives, and their types, are:

$$\vartriangle_1, \blacktriangle_1 \ : \ \mathsf{Act} \times \mathsf{Fm} \to \mathsf{Fm} \tag{5.1}$$

$$\vartriangle_2, \blacktriangle_2 \ : \ \mathsf{Ag} \times \mathsf{Fm} \to \mathsf{Fm} \tag{5.2}$$

$$\vartriangle_3, \blacktriangle_3 \ : \ \mathsf{Ag} \times \mathsf{Fnc} \to \mathsf{Act} \tag{5.3}$$

We stipulate that the interpretations of these connectives are bilinear maps with appropriate algebras as domains and codomains, suitable to interpret (functional) actions, formulas, and agents respectively. For instance, the domain of interpretation for formulas can be a complete atomic Boolean; following [6], the domain of interpretation for actions can be a quantale, or a complete join-semilattice, which is completely join-generated by a given subset (interpreting the functional actions), and the domain of interpretation of agents can be a set.[1]

In all these previously mentioned contexts, the fact that the interpretations of the connectives above are bilinear maps (i.e., they are completely join-preserving in each coordinate) means that each of them has a right adjoint in each coordinate; therefore, in particular, the following additional connectives have a natural interpretation:

$$\mathrel{-\!\!\blacktriangleright}_1, \mathrel{-\!\!\vartriangleright}_1 \ : \ \mathsf{Act} \times \mathsf{Fm} \to \mathsf{Fm} \tag{5.4}$$

$$\mathrel{-\!\!\blacktriangleright}_2, \mathrel{-\!\!\vartriangleright}_2 \ : \ \mathsf{Ag} \times \mathsf{Fm} \to \mathsf{Fm} \tag{5.5}$$

$$\mathrel{-\!\!\blacktriangleright}_3, \mathrel{-\!\!\vartriangleright}_3 \ : \ \mathsf{Ag} \times \mathsf{Act} \to \mathsf{Fnc} \tag{5.6}$$

Also, the following connectives are naturally interpreted in the setting above:

$$\mathrel{\blacktriangleleft\!\!\sim}_1, \mathrel{\vartriangleleft\!\!\sim}_1 \ : \ \mathsf{Fm} \times \mathsf{Fm} \to \mathsf{Act} \tag{5.7}$$

$$\mathrel{\blacktriangleleft\!\!\sim}_2, \mathrel{\vartriangleleft\!\!\sim}_2 \ : \ \mathsf{Fm} \times \mathsf{Fm} \to \mathsf{Ag} \tag{5.8}$$

$$\mathrel{\blacktriangleleft\!\!\sim}_3, \mathrel{\vartriangleleft\!\!\sim}_3 \ : \ \mathsf{Act} \times \mathsf{Fnc} \to \mathsf{Ag}. \tag{5.9}$$

We cannot provide as yet an intuitive understanding of the latter array of connectives, and therefore one might wonder whether they should be added

[1] However, for other dynamic logics this does not need to be the case; for instance, in the case of game logic [?], the set of agents consists of two elements, on which a negation-type operation can be assumed.

to the syntax of operational terms. However, they are important to guarantee the display property, and hence we will introduce them *only* at the structural level, that is, they will not correspond to any operational connective. We stipulate that, for every agent $\mathtt{a}$, every action α and every formula A,

$$\alpha \vartriangle_1 A \leq B \text{ iff } A \leq \alpha \mathbin{-\!\!\blacktriangleright}_1 B \qquad \alpha \blacktriangle_1 A \leq B \text{ iff } A \leq \alpha \mathbin{-\!\!\vartriangleright}_1 B \tag{5.10}$$

$$\mathtt{a} \vartriangle_2 A \leq B \text{ iff } A \leq \mathtt{a} \mathbin{-\!\!\blacktriangleright}_2 B \qquad \mathtt{a} \blacktriangle_2 A \leq B \text{ iff } A \leq \mathtt{a} \mathbin{-\!\!\vartriangleright}_2 B \tag{5.11}$$

$$\mathtt{a} \vartriangle_3 \alpha \leq \beta \text{ iff } \alpha \leq \mathtt{a} \mathbin{-\!\!\blacktriangleright}_3 \beta \qquad \mathtt{a} \blacktriangle_3 \alpha \leq \beta \text{ iff } \alpha \leq \mathtt{a} \mathbin{-\!\!\vartriangleright}_3 \beta. \tag{5.12}$$

Also, we stipulate that the following conditions hold for every agent $\mathtt{a}$, every action α and every formula A:

$$\alpha \vartriangle_1 A \leq B \text{ iff } \alpha \leq B \mathbin{\blacktriangleleft\!\!\sim}_1 A \qquad \alpha \blacktriangle_1 A \leq B \text{ iff } \alpha \leq B \mathbin{\vartriangleleft\!\!\sim}_1 A \tag{5.13}$$

$$\mathtt{a} \vartriangle_2 A \leq B \text{ iff } \mathtt{a} \leq B \mathbin{\blacktriangleleft\!\!\sim}_2 A \qquad \mathtt{a} \blacktriangle_2 A \leq B \text{ iff } \mathtt{a} \leq B \mathbin{\vartriangleleft\!\!\sim}_2 A \tag{5.14}$$

$$\mathtt{a} \vartriangle_3 \alpha \leq \beta \text{ iff } \mathtt{a} \leq \beta \mathbin{\blacktriangleleft\!\!\sim}_3 \alpha \qquad \mathtt{a} \blacktriangle_3 \alpha \leq \beta \text{ iff } \mathtt{a} \leq \beta \mathbin{\vartriangleleft\!\!\sim}_3 \alpha. \tag{5.15}$$

The intended link between the language of D.EAK and this new language is illustrated in the following table:

$\alpha \vartriangle_1 A$	stands for	$\langle\alpha\rangle A$	$\alpha \blacktriangle_1 A$	stands for	$\widehat{\underline{\alpha}}\, A$
$\mathtt{a} \vartriangle_2 A$	stands for	$\langle\mathtt{a}\rangle A$	$\mathtt{a} \blacktriangle_2 A$	stands for	$\widehat{\underline{\mathtt{a}}}\, A.$
$\alpha \mathbin{-\!\!\vartriangleright}_1 A$	stands for	$[\alpha] A$	$\alpha \mathbin{-\!\!\blacktriangleright}_1 A$	stands for	$\overline{\underline{\alpha}}\, A$
$\mathtt{a} \mathbin{-\!\!\vartriangleright}_2 A$	stands for	$[\mathtt{a}] A$	$\mathtt{a} \mathbin{-\!\!\blacktriangleright}_2 A$	stands for	$\overline{\underline{\mathtt{a}}}\, A.$

Hence, the adjunction conditions in clauses (5.11)-(5.12) account for the following required adjunction conditions in D.EAK:

$$\langle\alpha\rangle \dashv \overline{\underline{\alpha}} \qquad \widehat{\underline{\alpha}} \dashv [\alpha].$$

The type-3 connectives $\vartriangle_3$, $\blacktriangle_3$, $\mathbin{-\!\!\vartriangleright}_3$, $\mathbin{-\!\!\blacktriangleright}_3$ have no counterpart in the language of D.EAK, but particularly, the introduction of $\blacktriangle_3$ is exactly what brings the additional expressiveness we need in order to eliminate the label. Indeed, we stipulate that for every $\mathtt{a}$ and α as above,

$$\mathtt{a} \blacktriangle_3 \alpha = \bigvee \{\beta \mid \alpha \mathtt{a} \beta\}. \tag{5.16}$$

A way to understand this stipulation is to go back to the discussion in section 3.3 after clause (3.7). There, in the context of a discussion about the proof system introduced in [6], the link between clause (3.7) and the axiom (3.4)—which was left implicit in that paper—is made more explicit, by understanding the action $f_A^Q(q)$ as the join, taken in Q, of all the actions q' which are indistinguishable from q for the agent A. In the present setting, the stipulation (5.16) says that $\mathtt{a} \blacktriangle_3 \alpha$ encodes exactly the same information encoded in $f_A^Q(q)$, namely, the nondeterministic choice between all the actions that are indistinguishable from α for the agent a. Sometimes, for the sake of uniformity, we will use the symbol $\widehat{\mathsf{a}}\,\alpha$ for $\mathsf{a}\blacktriangle_3\alpha$.

Since adjoint pairs are completely determined by one of the members, it is expected that e.g. $-\!\!\vartriangleright_3$ can be defined via its being the right adjoint of $\blacktriangle_3$ in the second coordinate, and clause (5.16). Indeed, for each $\gamma \in \mathsf{Act}$ and $\mathsf{a} \in \mathsf{Ag}$,

$$\begin{array}{lll} \mathsf{a} -\!\!\vartriangleright_3\gamma & = & \bigvee\{\alpha \in \mathsf{Fnc} \mid \mathsf{a}\blacktriangle_3\alpha \leq \gamma\} \\ & = & \bigvee\{\alpha \in \mathsf{Fnc} \mid \bigvee\{\beta \in \mathsf{Act} \mid \alpha\mathsf{a}\beta\} \leq \gamma\} \\ & = & \bigvee\{\alpha \in \mathsf{Fnc} \mid (\forall \beta \in \mathsf{Act})[\alpha\mathsf{a}\beta \Rightarrow \beta \leq \gamma]\}. \end{array}$$

As already mentioned in the setting of the revised D.EAK, in order to express in this new language that e.g. $\langle\alpha\rangle$ and $[\alpha]$ are "interpreted over the same relation", we have two alternatives: one of them is that we impose the following Fischer Servi-type inequalities:

$$\begin{array}{rl} (\alpha\vartriangle_1 A) \to (\alpha -\!\!\vartriangleright_1 B) \leq \alpha -\!\!\vartriangleright_1(A \to B) & (\alpha\blacktriangle_1 A) \to (\alpha -\!\!\blacktriangleright_1 B) \leq \alpha -\!\!\blacktriangleright_1(A \to B) \\ (\mathsf{a}\vartriangle_2 A) \to (\mathsf{a} -\!\!\vartriangleright_2 B) \leq \mathsf{a} -\!\!\vartriangleright_2(A \to B) & (\mathsf{a}\blacktriangle_2 A) \to (\mathsf{a} -\!\!\blacktriangleright_2 B) \leq \mathsf{a} -\!\!\blacktriangleright_2(A \to B) \\ (\mathsf{a}\vartriangle_3 \alpha) \to (\mathsf{a} -\!\!\vartriangleright_3 \beta) \leq \mathsf{a} -\!\!\vartriangleright_3(\alpha \to \beta) & (\mathsf{a}\blacktriangle_3 \alpha) \to (\mathsf{a} -\!\!\blacktriangleright_3 \beta) \leq \mathsf{a} -\!\!\blacktriangleright_3(\alpha \to \beta). \end{array}$$

The second alternative is to impose that, for every $1 \leq i \leq 3$, the connectives $\vartriangle_i$ and $\blacktriangle_i$ yield to *conjugated* diamonds; that is, for every a and α as above, the following inequalities hold:

$$\begin{array}{rl} (\alpha\vartriangle_1 A) \wedge B \leq \alpha\vartriangle_1(A \wedge \alpha\blacktriangle_1 B) & (\alpha\blacktriangle_1 A) \wedge B \leq \alpha\blacktriangle_1(A \wedge \alpha\vartriangle_1 B) \\ (\mathsf{a}\vartriangle_2 A) \wedge B \leq \mathsf{a}\vartriangle_2(A \wedge \mathsf{a}\blacktriangle_2 B) & (\mathsf{a}\blacktriangle_2 A) \wedge B \leq \mathsf{a}\blacktriangle_2(A \wedge \mathsf{a}\vartriangle_2 B) \\ (\mathsf{a}\vartriangle_3 \alpha) \wedge \beta \leq \mathsf{a}\vartriangle_3(\alpha \wedge \mathsf{a}\blacktriangle_3 \beta) & (\mathsf{a}\blacktriangle_3 \alpha) \wedge \beta \leq \mathsf{a}\blacktriangle_3(\alpha \wedge \mathsf{a}\vartriangle_3 \beta). \end{array}$$

This finishes the informal presentation of the expressive enhancement that we are going to pursue; more formally, we introduce the formulas and actions of our base operational language by the following simultaneous induction starting with a set AtProp of atomic propositions, a set Fnc of functional actions, and a set Ag of agents:

$$\mathsf{Fm} \ni A ::= p \mid \bot \mid \top \mid A{\wedge}A \mid A{\vee}A \mid A \to A \mid A \succ\!\!\!- A \mid \mathsf{a}\vartriangle_2 A \mid \mathsf{a} -\!\!\vartriangleright_2 A \mid \gamma\vartriangle_1 A \mid \gamma -\!\!\vartriangleright_1 A \mid$$

$$\mathsf{a}\blacktriangle_2 A \mid \mathsf{a} -\!\!\blacktriangleright_2 A \mid \gamma\blacktriangle_1 A \mid \gamma -\!\!\blacktriangleright_1 A$$

$$\mathsf{Fnc} \ni \alpha ::= \alpha$$

$$\mathsf{Act} \ni \gamma ::= \alpha \mid \mathsf{a} \blacktriangle_3 \alpha \mid \mathsf{a} \mathbin{-\!\triangleright}_3 \alpha \mid \mathsf{a} \vartriangle_3 \alpha \mid \mathsf{a} \mathbin{-\!\blacktriangleright}_3 \alpha$$

$$\mathsf{Ag} \ni \mathsf{a} ::= \mathsf{a}.$$

The fundamental difference between this language and the language of D.EAK is that, in D.EAK, agents and actions are *parameters* in the construction of formulas, which are the only first-class citizens; in the present setting, however, each type lives on a par with any other. Because of the relative simplicity of our setting, two of the four types are attributed no algebraic structure. However, it is not difficult to enrich the algebraic structure of those types with sensible and intuitive operations: for instance, the skip and crash actions are functional, and parallel and sequential composition and iteration on functional actions preserve functionality, hence can be added to the array of constructors for Fnc. As a consequence of the fact that each type is a first-class citizen, as we will see shortly, four types of structures will be defined, and the turnstile symbol in the sequents of this calculus will be interpreted in the appropriate domain (we will come back to this point later on).

Now we are in a position to translate (the intuitionistic counterparts of) the following axiom (cf. (3.4)) in the new language, without labels:

$$\bigwedge\{[\mathsf{a}][\beta]A \mid \alpha\mathsf{a}\beta\} \to [\alpha][\mathsf{a}]A.$$

Indeed, by applying the stipulations above we get:

$$\bigwedge\{\mathsf{a} \mathbin{-\!\triangleright}_2 (\beta \mathbin{-\!\triangleright}_1 A) \mid \alpha\mathsf{a}\beta\} \to \alpha \mathbin{-\!\triangleright}_1 (\mathsf{a} \mathbin{-\!\triangleright}_2 A).$$

Since $\mathbin{-\!\triangleright}_2$ is completely meet preserving in the second coordinate, we can equivalently rewrite the clause above as follows:

$$\big[\mathsf{a} \mathbin{-\!\triangleright}_2 \bigwedge\{\beta \mathbin{-\!\triangleright}_1 A \mid \alpha\mathsf{a}\beta\}\big] \to \alpha \mathbin{-\!\triangleright}_1 (\mathsf{a} \mathbin{-\!\triangleright}_2 A).$$

Since $\mathbin{-\!\triangleright}_1$ is completely join reversing in its first coordinate, we can equivalently rewrite the clause above as follows:

$$\big[\mathsf{a} \mathbin{-\!\triangleright}_2 \big(\bigvee\{\beta \mid \alpha\mathsf{a}\beta\} \mathbin{-\!\triangleright}_1 A\big)\big] \to \alpha \mathbin{-\!\triangleright}_1 (\mathsf{a} \mathbin{-\!\triangleright}_2 A).$$

Now we apply the stipulation (5.16) and get the following equivalence:

$$\big[\mathsf{a} \mathbin{-\!\triangleright}_2 \big(\mathsf{a} \blacktriangle_3 \alpha \mathbin{-\!\triangleright}_1 A\big)\big] \to \alpha \mathbin{-\!\triangleright}_1 (\mathsf{a} \mathbin{-\!\triangleright}_2 A). \tag{5.17}$$

Likewise, the following axiom:

$$\langle\alpha\rangle\langle\mathsf{a}\rangle A \to \bigvee\{\langle\mathsf{a}\rangle\langle\beta\rangle A \mid \alpha\mathsf{a}\beta\}$$

translates into:

$$\alpha \vartriangle_1 (\mathsf{a} \vartriangle_2 A) \to \mathsf{a} \vartriangle_2 [(\mathsf{a} \blacktriangle_3 \alpha) \vartriangle_1 A]. \tag{5.18}$$

Since we are working towards a display-type calculus, we need to introduce the structural language, which as usual matches the operational language, although in the present case not in the same way as in D.EAK. We have *formula-type* structures, *functional action-type structures*, *action-type structures*, *agent-type structures*, defined by simultaneous recursion as follows:

$$\mathsf{FM} \ni X ::= A \mid I \mid X \,;X \mid X > X \mid \mathsf{a} \triangle_2 X \mid \mathsf{a} \mathbin{-\!\!\rhd}_2 X \mid \Gamma \triangle_1 X \mid \Gamma \mathbin{-\!\!\rhd}_1 X \mid$$

$$\mathsf{a} \blacktriangle_2 X \mid \mathsf{a} \mathbin{-\!\!\blacktriangleright}_2 X \mid \Gamma \blacktriangle_1 X \mid \Gamma \mathbin{-\!\!\blacktriangleright}_1 X$$

$$\mathsf{FNC} \ni \alpha ::= \alpha$$

$$\mathsf{ACT} \ni \Gamma ::= \gamma \mid \mathsf{a} \blacktriangle_3 \Gamma \mid \mathsf{a} \mathbin{-\!\!\rhd}_3 \Gamma \mid \mathsf{a} \triangle_3 \Gamma \mid \mathsf{a} \mathbin{-\!\!\blacktriangleright}_3 \Gamma \mid X \mathbin{\lhd\!\!\sim}_1 X \mid X \mathbin{\blacktriangleleft\!\!\sim}_1 X$$

$$\mathsf{AG} \ni \mathsf{a} ::= \mathsf{a}.$$

As in every display calculus, each operational connective corresponds to one structural connective, and in particular, the propositional base connectives behave exactly as in D.EAK, so we are not going to expand further on this. However, unlike in the setting of D.EAK, in the present setting, the heterogeneous structural connectives correspond one-to-one with the operational ones, as illustrated in the following table: for $1 \leq i \leq 3$,

Structural symbols	$\triangle_i$		$\blacktriangle_i$		$\mathbin{-\!\!\rhd}_i$		$\mathbin{-\!\!\blacktriangleright}_i$	
Operational symbols	$\vartriangle_i$		$\blacktriangle_i$			$\mathbin{-\!\!\rhd}_i$		$\mathbin{-\!\!\blacktriangleright}_i$

That is, the structural connectives are to be interpreted contextually, but the present language lacks the operational connectives which would correspond to them on one of the two sides. This is of course because in the present setting we do not need them. However, in a setting in which they would turn out to be needed, it would not be difficult to introduce the missing operational connectives. We can now introduce the operational rules for the heterogeneous connectives: Let x, y stand for structures of an undefined type, and let a, b denote operational terms of the appropriate type; then for $1 \leq i \leq 3$,

$$
\vartriangle_{iL}\ \frac{a \bigtriangleup_i b \vdash z}{a \vartriangle_i b \vdash z}
\qquad\qquad
\frac{x \vdash a \qquad y \vdash b}{x \bigtriangleup_i y \vdash a \vartriangle_i b}\ \vartriangle_{iR}
$$

$$
\blacktriangle_{iL}\ \frac{a \blacktriangle_i b \vdash z}{a \blacktriangle_i b \vdash z}
\qquad\qquad
\frac{x \vdash a \qquad y \vdash b}{x \blacktriangle_i y \vdash a \blacktriangle_i b}\ \blacktriangle_{iR}
$$

$$
-\!\!\vartriangleright_{iL}\ \frac{x \vdash a \qquad b \vdash y}{a -\!\!\vartriangleright_i b \vdash x -\!\!\rhd_i y}
\qquad\qquad
\frac{z \vdash a -\!\!\rhd_i b}{z \vdash a -\!\!\vartriangleright_i b}\ -\!\!\vartriangleright_{iR}
$$

$$
-\!\!\blacktriangleright_{iL}\ \frac{x \vdash a \qquad b \vdash y}{a -\!\!\blacktriangleright_i b \vdash x -\!\!\blacktriangleright_i y}
\qquad\qquad
\frac{z \vdash a -\!\!\blacktriangleright_i b}{z \vdash a -\!\!\blacktriangleright_i b}\ -\!\!\blacktriangleright_{iR}
$$

Clearly, the rules above for $i = 1, 2$ yield the operational rules for the dynamic and epistemic modal operators under the translation given early on. Notice that each sequent is always interpreted in one domain; however, since the connectives take arguments of different types (and hence we are justified in referring to them as *heterogeneous connectives*), premises of binary rules are of course interpreted in different domains.

Since our setting has three main types, our axioms will be given in three types, as follows:

$$
\mathsf{a} \vdash \mathsf{a} \quad \alpha \vdash \alpha \quad p \vdash p \quad \bot \vdash \mathrm{I} \quad \mathrm{I} \vdash \top \quad \Gamma p \vdash \Delta p
$$

where the first and second axioms on the top row are of type Ag and Act respectively, and the remaining ones are of type Fm.
Further, we allow the following three cut rules on the operational terms:

$$
\frac{\mathsf{a} \vdash \mathsf{a} \qquad \mathsf{a} \vdash \mathsf{a}}{\mathsf{a} \vdash \mathsf{a}}
\qquad
\frac{\Gamma \vdash \gamma \qquad \gamma \vdash \Delta}{\Gamma \vdash \Delta}
\qquad
\frac{X \vdash A \qquad A \vdash Y}{X \vdash Y}
$$

Next, we give the display postulates for the heterogeneous connectives. In what follows, let x, y, z stand for structures of an undefined type; then, for $1 \leq i \leq 3$,

$$
(\vartriangle_i, -\!\!\blacktriangleright_i)\ \frac{x \bigtriangleup_i y \vdash z}{\overline{y \vdash x -\!\!\blacktriangleright_i z}}
\qquad
\frac{x \blacktriangle_i y \vdash z}{\overline{y \vdash x -\!\!\rhd_i z}}\ (\blacktriangle_i, -\!\!\vartriangleright_i)
$$

Notice that both sequents occurring in each display postulate above are of the same type. Further, for each $1 \leq i \leq 3$

$$
(\vartriangle_i, \blacktriangleleft\!\!\sim_i)\ \frac{x \bigtriangleup_i y \vdash z}{\overline{x \vdash z \blacktriangleleft\!\!\sim_i y}}
\qquad
\frac{x \blacktriangle_i y \vdash z}{\overline{x \vdash z \lhd\!\!\sim_i y}}\ (\blacktriangle_i, \vartriangleleft\!\!\sim_i)
$$

Notice that sequents occurring in each display postulate above are *not* of the same type. Next, the conjugation rules and the Fischer Servi rules: for $i = 1, 2$,

$$
(conj_{i1})\ \frac{x \vartriangle_i (\, x \blacktriangle_i z\ ; y\ ) \vdash w}{z\ ; (x \vartriangle_i y) \vdash w}
\qquad
\frac{x \blacktriangle_i (\, x \vartriangle_i z\ ; y\ ) \vdash w}{z\ ; (x \blacktriangle_i y)\ \vdash w}\ (conj_{i2})
$$

$$
(FS_{i1})\ \frac{W \vdash (x \vartriangle_i Y) > (x \mathbin{-\!\!\vartriangleright}_i Z)}{W \vdash x \mathbin{-\!\!\vartriangleright}_i (Y > Z)}
\qquad
\frac{W \vdash (x \blacktriangle_i Y) > (x \mathbin{-\!\!\blacktriangleright}_i Z)}{W \vdash x \mathbin{-\!\!\blacktriangleright}_i (Y > Z)}\ (FS_{i2})
$$

Next, we provide the interaction axioms between the different types; in what follows we omit the subscripts, since the reading is unambiguous.

$$
\text{swap-out}_L\ \frac{(\mathsf{a} \blacktriangle \alpha) \blacktriangle (\mathsf{a} \blacktriangle X) \vdash Y}{\mathsf{a} \blacktriangle (\alpha \blacktriangle X) \vdash Y}
\qquad
\frac{X \vdash (\mathsf{a} \blacktriangle \alpha) \mathbin{-\!\!\blacktriangleright} (\mathsf{a} \mathbin{-\!\!\blacktriangleright} Y)}{X \vdash \mathsf{a} \mathbin{-\!\!\blacktriangleright} (\alpha \mathbin{-\!\!\blacktriangleright} Y)}\ \text{swap-out}_R
$$

$$
\text{swap-in}_L\ \frac{\mathsf{a} \blacktriangle (\alpha \blacktriangle X) \vdash Y}{\Phi_\alpha\ ; (\mathsf{a} \blacktriangle \alpha) \blacktriangle (\mathsf{a} \blacktriangle X) \vdash Y}
\qquad
\frac{X \vdash \mathsf{a} \mathbin{-\!\!\blacktriangleright} (\alpha \mathbin{-\!\!\blacktriangleright} Y)}{X \vdash \Phi_\alpha > (\mathsf{a} \blacktriangle \alpha) \mathbin{-\!\!\blacktriangleright} (\mathsf{a} \mathbin{-\!\!\blacktriangleright} Y)}\ \text{swap-in}_R
$$

The remaining rules can be obtained straightforwardly by translating the rules for D'.EAK into the new language. The only proviso should be made for the *balance* rule:

$$
\frac{X \vdash Y}{\alpha \vartriangle X \vdash \alpha \mathbin{-\!\!\vartriangleright} Y}
$$

which is sound only for $\alpha \in \mathsf{Fnc}$, and cannot be extended to arbitrary $\gamma \in \mathsf{Act}$.

5.2 Properties of the new rules, and completeness

Since the present setting is multi-typed, we need to check that the rules satisfy Belnap's and Wansing's conditions relative to structures of *each* type. Indeed, it is very easy to see that the rules are purely structural (recall that the variables a and α denote operational terms of their respective type, but they are also the generic *structure* terms of their respective type), and they are closed under uniform substitution of structures of the same type.

Not only have we eliminated the labels, but the new *swap-out* rules above are unary, where the old ones are of a non-fixed arity. It is easy to see that all the rules above are sound w.r.t. the semantics that we have sketched in the beginning. Since these domains have a very clear and established relationship with the usual semantic setting of EAK and D.EAK, we know that in particular these rules are sound w.r.t. the usual semantic setting.

Let us derive the axiom (5.17):

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathtt{a}\vdash\mathtt{a}\qquad\dfrac{\dfrac{\mathtt{a}\vdash\mathtt{a}\qquad\alpha\vdash\alpha}{\mathtt{a}\mathbin{\large\blacktriangle}\alpha\vdash\mathtt{a}\blacktriangle\alpha}\qquad A\vdash A}{(\mathtt{a}\blacktriangle\alpha)\mathbin{-\!\!\triangleright}A\vdash(\mathtt{a}\mathbin{\large\blacktriangle}\alpha)\mathbin{\large-\!\!\triangleright}A}}{\mathtt{a}\mathbin{-\!\!\triangleright}((\mathtt{a}\blacktriangle\alpha)\mathbin{-\!\!\triangleright}A)\vdash\mathtt{a}\mathbin{\large-\!\!\triangleright}((\mathtt{a}\mathbin{\large\blacktriangle}\alpha)\mathbin{\large-\!\!\triangleright}A)}}{\mathtt{a}\mathbin{\large\blacktriangle}(\mathtt{a}\mathbin{-\!\!\triangleright}((\mathtt{a}\blacktriangle\alpha)\mathbin{-\!\!\triangleright}A))\vdash((\mathtt{a}\mathbin{\large\blacktriangle}\alpha)\mathbin{\large-\!\!\triangleright}A)}}{((\mathtt{a}\mathbin{\large\blacktriangle}\alpha)\mathbin{\large\blacktriangle}(\mathtt{a}\mathbin{\large\blacktriangle}(\mathtt{a}\mathbin{-\!\!\triangleright}((\mathtt{a}\blacktriangle\alpha)\mathbin{-\!\!\triangleright}A)))\vdash A}}{\mathtt{a}\mathbin{\large\blacktriangle}(\alpha\mathbin{\large\blacktriangle}(\mathtt{a}\mathbin{-\!\!\triangleright}((\mathtt{a}\blacktriangle\alpha)\mathbin{-\!\!\triangleright}A)))\vdash A}\,swap-out_L}{\alpha\mathbin{\large\blacktriangle}(\mathtt{a}\mathbin{-\!\!\triangleright}((\mathtt{a}\blacktriangle\alpha)\mathbin{-\!\!\triangleright}A))\vdash\mathtt{a}\mathbin{\large-\!\!\triangleright}A}}{\alpha\mathbin{\large\blacktriangle}(\mathtt{a}\mathbin{-\!\!\triangleright}((\mathtt{a}\blacktriangle\alpha)\mathbin{-\!\!\triangleright}A))\vdash\mathtt{a}\mathbin{-\!\!\triangleright}A}}{\mathtt{a}\mathbin{-\!\!\triangleright}((\mathtt{a}\blacktriangle\alpha)\mathbin{-\!\!\triangleright}A)\vdash\alpha\mathbin{\large-\!\!\triangleright}(\mathtt{a}\mathbin{-\!\!\triangleright}A)}}{\mathtt{a}\mathbin{-\!\!\triangleright}((\mathtt{a}\blacktriangle\alpha)\mathbin{-\!\!\triangleright}A)\vdash\alpha\mathbin{-\!\!\triangleright}(\mathtt{a}\mathbin{-\!\!\triangleright}A)}
$$

Let us derive the axiom (5.18):

$$
\dfrac{\dfrac{\dfrac{s-out_R\,\dfrac{\dfrac{\dfrac{\dfrac{\mathtt{a}\vdash\mathtt{a}\qquad\dfrac{A\vdash A\qquad\dfrac{\mathtt{a}\vdash\mathtt{a}\qquad\alpha\vdash\alpha}{\mathtt{a}\mathbin{\large\blacktriangle}\alpha\vdash\mathtt{a}\blacktriangle\alpha}}{(\mathtt{a}\mathbin{\large\blacktriangle}\alpha)\mathbin{\large\vartriangle}A\vdash(\mathtt{a}\blacktriangle\alpha)\vartriangle A}}{\mathtt{a}\mathbin{\large\vartriangle}((\mathtt{a}\mathbin{\large\blacktriangle}\alpha)\mathbin{\large\vartriangle}A)\vdash\mathtt{a}\vartriangle((\mathtt{a}\blacktriangle\alpha)\vartriangle A)}}{(\mathtt{a}\mathbin{\large\blacktriangle}\alpha)\mathbin{\large\vartriangle}A\vdash(\mathtt{a}\mathbin{\large-\!\!\blacktriangleright}(\mathtt{a}\vartriangle((\mathtt{a}\blacktriangle\alpha)\vartriangle A)))}}{A\vdash(\mathtt{a}\mathbin{\large\blacktriangle}\alpha)\mathbin{\large-\!\!\blacktriangleright}(\mathtt{a}\mathbin{\large-\!\!\blacktriangleright}(\mathtt{a}\vartriangle((\mathtt{a}\blacktriangle\alpha)\vartriangle A)))}}{A\vdash\mathtt{a}\mathbin{\large-\!\!\blacktriangleright}(\alpha\mathbin{\large-\!\!\blacktriangleright}(\mathtt{a}\vartriangle((\mathtt{a}\blacktriangle\alpha)\vartriangle A))}}{\mathtt{a}\mathbin{\large\vartriangle}A\vdash\alpha\mathbin{\large-\!\!\blacktriangleright}(\mathtt{a}\vartriangle((\mathtt{a}\blacktriangle\alpha)\vartriangle A)}}{\mathtt{a}\vartriangle A\vdash\alpha\mathbin{\large-\!\!\blacktriangleright}(\mathtt{a}\vartriangle((\mathtt{a}\blacktriangle\alpha)\vartriangle A)}}{\alpha\mathbin{\large\vartriangle}(\mathtt{a}\vartriangle A)\vdash\mathtt{a}\vartriangle((\mathtt{a}\blacktriangle\alpha)\vartriangle A)}}{\alpha\vartriangle(\mathtt{a}\vartriangle A)\vdash\mathtt{a}\vartriangle((\mathtt{a}\blacktriangle\alpha)\vartriangle A)}
$$

A slight difference between the setting of [10] and the present setting is that in that paper only the dynamic boxes are allowed in the object language, even if their propositional base is taken as non classical; in the present setting however, both the dynamic boxes and diamonds are taken as primitive connectives. The present setting needs that also the interaction axioms such as the following one:

$$[\alpha]\langle\mathtt{a}\rangle A\leftrightarrow 1_\alpha\rightarrow\bigvee\{\langle\mathtt{a}\rangle\langle\beta\rangle A\mid\alpha\mathtt{a}\beta\}$$

be accounted for, and for this, the additional display postulates ($\blacktriangle_1$, $\lhd\!\!\sim_i$) and ($\vartriangle_1$, $\blacktriangleleft\!\!\sim_1$) will be needed.

The axiom above translates as:

$$\alpha\mathbin{-\!\!\triangleright}(\mathtt{a}\vartriangle A)\leftrightarrow\alpha\vartriangle\top\rightarrow\mathtt{a}\vartriangle((\mathtt{a}\blacktriangle\alpha)\vartriangle A).$$

$$
\begin{array}{c}
\texttt{a} \vdash \texttt{a} \qquad \alpha \vdash \alpha \\
\hline
\texttt{a} \blacktriangle \alpha \vdash \texttt{a} {\scriptstyle\blacktriangle} \alpha \qquad A \vdash A \\
\hline
\texttt{a} \vdash \texttt{a} \qquad (\texttt{a} \blacktriangle \alpha) \bigtriangleup A \vdash (\texttt{a} {\scriptstyle\blacktriangle} \alpha) \vartriangle A \\
\hline
\texttt{a} \bigtriangleup ((\texttt{a} \blacktriangle \alpha) \bigtriangleup A) \vdash \texttt{a} \vartriangle ((\texttt{a} {\scriptstyle\blacktriangle} \alpha) \vartriangle A) \\
\hline
(\texttt{a} \blacktriangle \alpha) \bigtriangleup A \vdash \texttt{a} \,{-}\!\blacktriangleright (\texttt{a} \vartriangle ((\texttt{a} {\scriptstyle\blacktriangle} \alpha) \vartriangle A)) \\
\hline
A \vdash (\texttt{a} \blacktriangle \alpha) \,{-}\!\blacktriangleright (\texttt{a} \,{-}\!\blacktriangleright (\texttt{a} \vartriangle ((\texttt{a} {\scriptstyle\blacktriangle} \alpha) \vartriangle A))) \\
\hline
A \vdash \texttt{a} \,{-}\!\blacktriangleright (\alpha \,{-}\!\blacktriangleright (\texttt{a} \vartriangle ((\texttt{a} {\scriptstyle\blacktriangle} \alpha) \vartriangle A))) \quad \text{swap-out}_R \\
\hline
\texttt{a} \bigtriangleup A \vdash \alpha \,{-}\!\blacktriangleright (\texttt{a} \vartriangle ((\texttt{a} {\scriptstyle\blacktriangle} \alpha) \vartriangle A)) \\
\hline
\alpha \vdash \alpha \qquad \texttt{a} \vartriangle A \vdash \alpha \,{-}\!\blacktriangleright (\texttt{a} \vartriangle ((\texttt{a} {\scriptstyle\blacktriangle} \alpha) \vartriangle A)) \\
\hline
\alpha \,{-}\!\!\vartriangleright (\texttt{a} \vartriangle A) \vdash \alpha \,{-}\!\!\bigtriangledown\!\!\!\!\rhd (\alpha \,{-}\!\blacktriangleright (\texttt{a} \vartriangle ((\texttt{a} {\scriptstyle\blacktriangle} \alpha) \vartriangle A))) \\
\hline
\alpha \blacktriangle (\alpha \,{-}\!\!\vartriangleright (\texttt{a} \vartriangle A)) \vdash \alpha \,{-}\!\blacktriangleright (\texttt{a} \vartriangle ((\texttt{a} {\scriptstyle\blacktriangle} \alpha) \vartriangle A)) \\
\hline
\alpha \bigtriangleup (\alpha \blacktriangle (\alpha \,{-}\!\!\vartriangleright (\texttt{a} \vartriangle A))) \vdash \texttt{a} \vartriangle ((\texttt{a} {\scriptstyle\blacktriangle} \alpha) \vartriangle A) \\
\hline
\Phi_\alpha \,; (\alpha \,{-}\!\!\vartriangleright (\texttt{a} \vartriangle A)) \vdash \texttt{a} \vartriangle ((\texttt{a} {\scriptstyle\blacktriangle} \alpha) \vartriangle A) \quad \text{comp} \\
\hline
\Phi_\alpha \vdash \texttt{a} \vartriangle ((\texttt{a} {\scriptstyle\blacktriangle} \alpha) \vartriangle A) < (\alpha \,{-}\!\!\vartriangleright (\texttt{a} \vartriangle A)) \\
\hline
1_\alpha \vdash \texttt{a} \vartriangle ((\texttt{a} {\scriptstyle\blacktriangle} \alpha) \vartriangle A) < (\alpha \,{-}\!\!\vartriangleright (\texttt{a} \vartriangle A)) \\
\hline
1_\alpha \,; (\alpha \,{-}\!\!\vartriangleright (\texttt{a} \vartriangle A)) \vdash \texttt{a} \vartriangle ((\texttt{a} {\scriptstyle\blacktriangle} \alpha) \vartriangle A) \\
\hline
\alpha \,{-}\!\!\vartriangleright (\texttt{a} \vartriangle A) \vdash 1_\alpha > \texttt{a} \vartriangle ((\texttt{a} {\scriptstyle\blacktriangle} \alpha) \vartriangle A) \\
\hline
\alpha \,{-}\!\!\vartriangleright (\texttt{a} \vartriangle A) \vdash 1_\alpha \rightarrow \texttt{a} \vartriangle ((\texttt{a} {\scriptstyle\blacktriangle} \alpha) \vartriangle A)
\end{array}
$$

For the other direction:

$$
\begin{array}{c}
\texttt{a} \vdash \texttt{a} \qquad A \vdash A \\
\hline
\texttt{a} \bigtriangleup A \vdash \texttt{a} \vartriangle A \\
\hline
\text{balance} \quad \alpha \bigtriangleup (\texttt{a} \bigtriangleup A) \vdash \alpha \,{-}\!\!\bigtriangledown\!\!\!\!\rhd (\texttt{a} \vartriangle A) \\
\hline
\alpha \bigtriangleup (\texttt{a} \bigtriangleup A) \vdash \alpha \,{-}\!\!\vartriangleright (\texttt{a} \vartriangle A) \\
\hline
\texttt{a} \bigtriangleup A \vdash \alpha \,{-}\!\blacktriangleright (\alpha \,{-}\!\!\vartriangleright (\texttt{a} \vartriangle A)) \\
\hline
A \vdash \texttt{a} \,{-}\!\blacktriangleright (\alpha \,{-}\!\blacktriangleright (\alpha \,{-}\!\!\vartriangleright (\texttt{a} \vartriangle A))) \\
\hline
A \vdash \Phi_\alpha > (\texttt{a} \blacktriangle \alpha) \,{-}\!\blacktriangleright (\texttt{a} \,{-}\!\blacktriangleright (\alpha \,{-}\!\!\vartriangleright (\texttt{a} \vartriangle A))) \quad \text{s-in}_R \\
\hline
A \vdash (\texttt{a} \blacktriangle \alpha) \,{-}\!\blacktriangleright (\texttt{a} \,{-}\!\blacktriangleright (\alpha \,{-}\!\!\vartriangleright (\texttt{a} \vartriangle A))) \quad \text{reduce} \\
\hline
(\texttt{a} \blacktriangle \alpha) \bigtriangleup A \vdash \texttt{a} \,{-}\!\blacktriangleright (\alpha \,{-}\!\!\vartriangleright (\texttt{a} \vartriangle A)) \\
\hline
\texttt{a} \blacktriangle \alpha \vdash (\texttt{a} \,{-}\!\blacktriangleright (\alpha \,{-}\!\!\vartriangleright (\texttt{a} \vartriangle A))) \blacktriangleleft\!\!\sim A \\
\hline
\texttt{a} {\scriptstyle\blacktriangle} \alpha \vdash (\texttt{a} \,{-}\!\blacktriangleright (\alpha \,{-}\!\!\vartriangleright (\texttt{a} \vartriangle A))) \blacktriangleleft\!\!\sim A \\
\hline
(\texttt{a} {\scriptstyle\blacktriangle} \alpha) \bigtriangleup A \vdash \texttt{a} \,{-}\!\blacktriangleright (\alpha \,{-}\!\!\vartriangleright (\texttt{a} \vartriangle A)) \\
\hline
(\texttt{a} {\scriptstyle\blacktriangle} \alpha) \vartriangle A \vdash \texttt{a} \,{-}\!\blacktriangleright (\alpha \,{-}\!\!\vartriangleright (\texttt{a} \vartriangle A)) \\
\hline
\texttt{a} \bigtriangleup ((\texttt{a} {\scriptstyle\blacktriangle} \alpha) \vartriangle A) \vdash \alpha \,{-}\!\!\vartriangleright (\texttt{a} \vartriangle A) \\
\hline
\Phi_\alpha \vdash 1_\alpha \qquad \texttt{a} \vartriangle ((\texttt{a} {\scriptstyle\blacktriangle} \alpha) \vartriangle A) \vdash \alpha \,{-}\!\!\vartriangleright (\texttt{a} \vartriangle A) \\
\hline
1_\alpha \rightarrow \texttt{a} \vartriangle ((\texttt{a} {\scriptstyle\blacktriangle} \alpha) \vartriangle A) \vdash \Phi_\alpha > (\alpha \,{-}\!\!\vartriangleright (\texttt{a} \vartriangle A)) \\
\hline
1_\alpha \rightarrow \texttt{a} \vartriangle ((\texttt{a} {\scriptstyle\blacktriangle} \alpha) \vartriangle A) \vdash \alpha \,{-}\!\!\vartriangleright (\texttt{a} \vartriangle A)
\end{array}
$$

The derivations (of the translations) of the remaining axioms closely match the derivations given in D.EAK under the translation given above, and hence we omit them. The main difference between proofs in this systems and proofs in D.EAK is that, thanks to the enhanced expressivity of the present setting, any occurrence of an action β such that $\alpha\texttt{a}\beta$ in D.EAK is now broken down to its components, and needs to be justified in terms of introduction rules of connectives of type 3.

5.3 Belnap's style cut elimination and subformula property

In the present section, we prove that the Dynamic Calculus for EAK enjoys the cut elimination and the subformula property via Belnap's metatheorems [8, Theorems 4.3 and 4.4].

Definition 2. A sequent $x \vdash y$ is *type-regular* when the structures x and y are of the same type.

Proposition 1. *Each derivable sequent in the Dynamic Calculus for EAK is type-regular.*

Proof. We prove the proposition by induction on the hight of the derivation. The base case is verified because the following axioms are type-regular by definition of their constituents:

$$\mathsf{a} \vdash \mathsf{a} \quad \alpha \vdash \alpha \quad A \vdash A \quad \bot \vdash \mathrm{I} \quad \mathrm{I} \vdash \top$$

and because the atom rule is type-regular by definition of the connectives that appear in its constituents.

Inductive hypothesis: if the claim holds for sequents in the derivation of the hight n then it also holds for the sequent in the derivation of the hight $n+1$. As for the inductive step, one can verify by inspection that for all the rules of General Dynamic Display Calculus, for all type-regular sequents in the premises also the conclusion sequent is type-regular. Now we can conclude that each derivable sequent in the calculus is type-regular. $\square$

Since all the sequents are type-regular, we can consider the eliminability of each cut rule separately. The cut rule performed on agents is immediately eliminable in the following way:

$$\frac{\mathsf{a} \vdash \mathsf{a} \qquad \mathsf{a} \vdash \mathsf{a}}{\mathsf{a} \vdash \mathsf{a}} \qquad \rightsquigarrow \qquad \mathsf{a} \vdash \mathsf{a}.$$

Let us discuss the eliminability of the cut rule defined on actions. The verification of conditions C_1-C_5 is straightforward. Conditions C_6 and C_7 are both satisfied; indeed, each rule is closed under uniform substitution of terms of each type, both in the precedent and in succedent position, and moreover, if a structure z is to be substituted for a term a of a given type in, say, antecedent position under the assumption that the sequent $z \vdash a$ is derivable, then, by Proposition 1, z would be of the same type as a.

We are only left to check whether the condition C_8 holds. Since the operational rules for the connectives $\vartriangle_3$, $\rightharpoonup\!\!\vartriangleright_3$, $\blacktriangle_3$, $\rightharpoonup\!\!\blacktriangleright_3$ are just analogous to the usual ones for conjunction and implication in intuitionistic logic, the proof goes straightforwardly like in the setting of intuitionistic logic.

The cut-elimination on propositions is analogous to the cut elimination of D'.EAK. Again, the verification of conditions C_1-C_5 is straightforward. Conditions C_6 and C_7 are satisfied for the same reasons discussed above. Condition C_8, which involves essentially operational rules, holds for reasons analogous to the one shown for D'.EAK. The proof goes by cases, which are the straightforward translation of the ones in section 7.2. This completes the proof that the Dynamic Calculus enjoys the cut-elimination property á la Belnap.

5.4 Towards a uniform proof theory for dynamic logics: *divide et impera*

Many dynamic languages exist in the literature, addressing diverse settings, and calling for a wide array of parameters such as time, agents, strategies, coalitions, events, etc. The multi-type dynamic calculus for EAK can be understood as exemplifying a promising methodology for achieving a uniform proof-theoretic account, spanning across dynamic logics but at the same time adequately capturing each of them. The basic idea of this methodology is to introduce enough syntactic devices, both at the operational and at the structural level, so that these parameters can be accounted for in the system as first class citizens. This approach appears seminally in both [6] and [10]; However, in both these papers, the agents are not treated as first class citizens. Moreover, and more importantly, in [6] there is no theory of contexts governing the interaction of different types. In [10], this interaction is clarified, but only at the metalinguistic level.

The multi-type setting is conceptually advantageous, since it can help to achieve a better grasp, and hopefully a more natural statement, of Wansing's and Belnap's requirements C_6/C_7 (cf. 2.4), via the notion of type-regularity (Definition 2). In [8], Belnap motivates his condition C_7[2] saying that "rules need not be wholly closed under substitution of structures for congruent formulas which are antecedent parts, but they must be closed enough." Then he explains that *closed enough* refers to the closure under substitution of formulas A for structures X such that a certain shape of derivation is available in the system for the sequent $X \vdash A$. The crucial observation is that, even if a system is not defined *a priori* as multi-type, it can be regarded as a multi-type setting: indeed, the type of A can be defined as consisting of all

[2]Recall that Belnap's condition C_7 corresponds to Wansing's cons-regularity for formulas occurring in precedent position. An analogous explanation holds of course for the ant-regularity condition of formulas in succedent position.

the structures X such that the shape of derivation alluded to above exists. Then, condition C_6/C_7 can be equivalently reformulated as the requirement that rules should be closed under uniform substitution within each type. Notice that, under the stipulations above, different types must me separated by at least one structural rule. For instance, in the system D'.EAK, the $atom^p$ rules separate the type "atomic propositions" from the type "formulas", and in the dynamic calculus for EAK, the rule *balance* separates Fnc from Act. In conclusion, Wansing's and Belnap's conditions C_6/C_7 require type regularity in a context in which types are not given explicitly. The observations above indicate that type regularity is a desirable design requirement for general dynamic calculi, and in particular for the development of an adequate proof theory for dynamic logics, particularly in view of a uniform path to Belnap-style cut-elimination.

Chapter 6

Conclusions and further research

6.1 Conclusions

In the present thesis, we provided an analysis, conducted adopting the viewpoint of proof-theoretic semantics, of the state-of-the-art deductive systems for dynamic epistemic logic, focusing mainly on Baltag-Moss-Solecki's logic of epistemic actions and knowledge (EAK). We started with an overview of proof-theoretic semantics, focusing on the requirements that a proof-system should satisfy to provide proof-theoretic semantics for logical constants. We then evaluated the main existing proof systems for PAL/EAK according to these criteria; then, as an original contribution, we proposed a revised version of one such system, namely of the system D.EAK (cf. section 3.4), and we argued that our revision meets the strictest proof-theoretic semantic requirements for all the logical constants involved. In particular, we showed that our revised version enjoys Belnap-style cut elimination, which was not argued for in the case of the original system. The main ingredient of this revision is an expansion of the language of D.EAK, aimed at achieving an *independent* proof-theoretic account of the preconditions $Pre(\alpha)$. This account is independent both in the sense that it is given purely in terms of the resources of the revised system, and in the sense that $Pre(\alpha)$ is treated as a first-class citizen of the revised system; indeed, $Pre(\alpha)$ is endowed with both an operational and a structural representation, both of which well-behaving.

The main original contribution of the present thesis is the definition of a multi-type calculus for the management of the different sorts involved in dynamic epistemic logic. Besides enjoying all the proof-theoretic semantic requirements (including Belnap-style cut elimination), this calculus provides an interesting and in our opinion very promising *methodological* platform, both from the point of view of the uniform development of a general proof-theoretic account of all dynamic logics, and also for clarifying and sharpening the statement of proof-theoretic semantic criteria (see section 5.4). The conclusions of the present thesis are summarized in the following table.

	Cut-elimination	**Belnap-style cut-elimination**	**Proof-theoretic Semantics**
D.EAK	✓		
D'.EAK	✓	✓	✓
Dyn. Cal.	✓	✓	✓

6.2 Further directions

Uniform proof-theoretic account for dynamic logics. At the moment, EAK is the only logic in the family which has been given a proof-theoretic account of the kind devised in the present thesis; other interesting case studies are Parikh's Game Logic [21], where the dynamic modalities are non normal and the set of agents is endowed with algebraic structure.

Revision of Belnap-style cut elimination. In section 4.2, we gave an informal argument that the cut-elimination Belnap-style could be extended so as to be applicable to the system D.EAK. A natural direction is then to formulate and prove this extension. In fact, this direction ties in with the discussion about type-regularity in section 5.4: indeed, we conjecture that Belnap metatheorem caters for the very restricted class of global transformations which rely on display postulates because—among other reasons—display postulates are type-regular; hence, the desired extension of Belnap metatheorem would cater for global transformations which rely on other structural rules which preserve type-regularity.

Types and Moore sentences. The multi-type methodology might be helpful to model or clarify issues that are not strictly proof-theoretic; for instance, the paradox of Moore sentences might be hopefully accounted for by assigning different types to "facts" and to "epistemic formulas", which makes it possible to tell factual preconditions and epistemic preconditions apart.

Chapter 7

Appendix

7.1 Soundness of *comp* rules in the final coalgebra

We address the reader to [14] for details on the final coalgebra semantics for dynamic epistemic logic.

To prove the soundness of the rules above in the final coalgebra it suffices to check that for every formula A,

$$[\alpha][\alpha^{-1}][\![A]\!]_{\mathbb{Z}} \subseteq [\![Pre(\alpha) \to A]\!]_{\mathbb{Z}} \ \text{ and } \ [\![Pre(\alpha) \to A]\!]_{\mathbb{Z}} \subseteq \langle\alpha\rangle\langle\alpha^{-1}\rangle[\![A]\!]_{\mathbb{Z}}.$$

We will make use of the following general fact:

Fact 2. *Let R be a binary relation on a set X and let R^{-1} be its converse. Then,*

$$[Dom(R) \times Dom(R)] \cap \Delta_X \subseteq R; R^{-1},$$

where $Dom(R) = \{x \in X \mid xRy \text{ for some } y \in X\}$, and $\Delta_X = \{(x,x) \mid x \in X\}$.

Proof. Straightforward. $\square$

Fact 3. *The following comp rules:*

$$\frac{Y \vdash \{\alpha\}\widehat{\alpha}\, X}{Y \vdash \underline{Pre}(\alpha) > X} \qquad \frac{\{\alpha\}\widehat{\alpha}\, X \vdash Y}{\underline{Pre}(\alpha); X \vdash Y}$$

are sound in the final coalgebra.

Proof.

$$\begin{array}{rcll}
\langle\alpha\rangle\langle\alpha^{-1}\rangle[\![A]\!]_{\mathbb{Z}} & = & \alpha^{-1}[\alpha[[\![A]\!]_{\mathbb{Z}}] & \\
& = & (\alpha;\alpha^{-1})[[\![A]\!]_{\mathbb{Z}}] & \\
& \supseteq & S[[\![A]\!]_{\mathbb{Z}}] & \text{Fact 2} \\
& = & Dom(\alpha) \cap [\![A]\!]_{\mathbb{Z}} & \\
& = & [\![Pre(\alpha) \cap A]\!]_{\mathbb{Z}}, &
\end{array}$$

$$
\begin{array}{rcll}
[\alpha][\alpha^{-1}][\![A]\!]_{\mathbb{Z}} & = & (\alpha^{-1}[(([\alpha^{-1}][\![A]\!]_{\mathbb{Z}})^c])^c & \\
& = & (\alpha^{-1}[\alpha[[\![A]\!]_{\mathbb{Z}}^c]])^c & \\
& = & ((\alpha;\alpha^{-1})[[\![A]\!]_{\mathbb{Z}}^c])^c & \\
& \subseteq & (S[[\![A]\!]_{\mathbb{Z}}^c])^c & \text{Fact 2} \\
& = & (Dom(\alpha)\cap[\![A]\!]_{\mathbb{Z}}^c])^c & \\
& = & Dom(\alpha)^c\cup[\![A]\!]_{\mathbb{Z}} & \\
& = & [\![Pre(\alpha)\rightarrow A]\!]_{\mathbb{Z}}, &
\end{array}
$$

where $S=[Dom(R)\times Dom(R)]\cap\Delta_X$. □

7.2 Cut elimination for D'.EAK

In this section, we report on the remaining cases for the verification of condition C_8 for D'.EAK; these cases are needed already for the cut elimination á la Gentzen for D.EAK, but do not appear in [14].

First we consider the atom rule:

$$
\frac{\Gamma p\vdash\Theta p \qquad \Theta p\vdash\Delta p}{\Gamma p\ \vdash\ \Delta p} \quad\rightsquigarrow\quad \Gamma p\ \vdash\ \Delta p
$$

We also treat here the cases relative to the two additional arrows $\leftarrow$ and $\succ$ added to our presentation of D.EAK. First we treat the introductions of the connectives of the propositional base:

$$
\frac{\mathrm{I}\ \vdash\ \top \qquad \dfrac{\overset{\vdots\,\pi}{\mathrm{I}\ \vdash\ X}}{\top\ \vdash\ X}}{\mathrm{I}\ \vdash\ X} \quad\rightsquigarrow\quad \overset{\vdots\,\pi}{\mathrm{I}\ \vdash\ X}
$$

$$
\frac{\dfrac{\overset{\vdots\,\pi}{X\ \vdash\ \mathrm{I}}}{X\vdash\ \bot} \qquad \bot\vdash\ \mathrm{I}}{X\ \vdash\ \mathrm{I}} \quad\rightsquigarrow\quad \overset{\vdots\,\pi}{X\vdash\ \mathrm{I}}
$$

$$
\frac{\dfrac{\overset{\vdots\,\pi_1}{X\ \vdash\ A} \qquad \overset{\vdots\,\pi_2}{Y\ \vdash\ B}}{X\,;Y\ \vdash\ A\wedge B} \qquad \dfrac{\overset{\vdots\,\pi_3}{A\,;B\ \vdash\ Z}}{A\wedge B\ \vdash\ Z}}{X\,;Y\ \vdash\ Z}
\quad\rightsquigarrow\quad
\frac{\dfrac{\overset{\vdots\,\pi_2}{Y\ \vdash\ B} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overset{\vdots\,\pi_1}{X\ \vdash\ A} \qquad \dfrac{\dfrac{\overset{\vdots\,\pi_3}{A\,;B\ \vdash\ Z}}{B\,;A\ \vdash\ Z}}{A\ \vdash\ B>Z}}{X\ \vdash\ B>Z}}{B\,;X\ \vdash\ Z}}{X\,;B\ \vdash\ Z}}{B\ \vdash\ X>Z}}{Y\ \vdash\ X>Z}}{X\,;Y\ \vdash\ Z}
$$

$$
\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_3\\ Z \vdash B\,;A\end{array}}{Z \vdash B \vee A}\qquad\dfrac{\begin{array}{c}\vdots\,\pi_1\\ B \vdash Y\end{array}\quad\begin{array}{c}\vdots\,\pi_2\\ A \vdash X\end{array}}{B \vee A \vdash Y\,;X}}{Z \vdash Y\,;X}
\quad\rightsquigarrow\quad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_3\\ Z \vdash B\,;A\end{array}}{Z \vdash A\,;B}}{A > Z \vdash B}\quad\begin{array}{c}\vdots\,\pi_1\\ B \vdash Y\end{array}}{A > Z \vdash Y}}{Z \vdash A\,;Y}}{Z \vdash Y\,;A}}{Y > Z \vdash A}\quad\begin{array}{c}\vdots\,\pi_2\\ A \vdash X\end{array}}{Y > Z \vdash X}}{Z \vdash Y\,;X}
$$

$$
\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_1\\ Y \vdash A > B\end{array}}{Y \vdash A \to B}\qquad\dfrac{\begin{array}{c}\vdots\,\pi_2\\ X \vdash A\end{array}\quad\begin{array}{c}\vdots\,\pi_3\\ B \vdash Z\end{array}}{A \to B \vdash X > Z}}{Y \vdash X > Z}
\quad\rightsquigarrow\quad
\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_2\\ X \vdash A\end{array}\quad\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_1\\ Y \vdash A > B\end{array}}{A\,;Y \vdash B}\quad\begin{array}{c}\vdots\,\pi_3\\ B \vdash Z\end{array}}{A;Y \vdash Z}}{Y\,;A \vdash Z}}{A \vdash Y > Z}}{X \vdash Y > Z}}{\dfrac{Y\,;X \vdash Z}{X\,;Y \vdash Z}}}{Y \vdash X > Z}
$$

$$
\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_1\\ Y \vdash B < A\end{array}}{Y \vdash B \leftarrow A}\qquad\dfrac{\begin{array}{c}\vdots\,\pi_2\\ B \vdash Z\end{array}\quad\begin{array}{c}\vdots\,\pi_3\\ X \vdash A\end{array}}{B \leftarrow A \vdash Z < X}}{Y \vdash Z < X}
\quad\rightsquigarrow\quad
\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_2\\ X \vdash A\end{array}\quad\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_1\\ Y \vdash B < A\end{array}}{Y;A \vdash B}\quad\begin{array}{c}\vdots\,\pi_3\\ B \vdash Z\end{array}}{Y;A \vdash Z}}{A;Y \vdash Z}}{A \vdash Z < Y}}{X \vdash Z < Y}}{\dfrac{X;Y \vdash Z}{Y;X \vdash Z}}}{Y \vdash Z < X}
$$

$$
\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_2\\ A \vdash Y\end{array}\quad\begin{array}{c}\vdots\,\pi_3\\ Z \vdash B\end{array}}{Y > Z \vdash A \mathbin{>\!\!\!-} B}\qquad\dfrac{\begin{array}{c}\vdots\,\pi_1\\ A > B \vdash X\end{array}}{A \mathbin{>\!\!\!-} B \vdash X}}{Y > Z \vdash X}
\quad\rightsquigarrow\quad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_3\\ Z \vdash B\end{array}\quad\dfrac{\begin{array}{c}\vdots\,\pi_1\\ A > B \vdash X\end{array}}{B \vdash A\,;X}}{Z \vdash A\,;X}}{Z \vdash X\,;A}}{X > Z \vdash A}\quad\begin{array}{c}\vdots\,\pi_2\\ A \vdash Y\end{array}}{X > Z \vdash Y}}{\dfrac{Z \vdash X\,;Y}{Z \vdash Y\,;X}}}{Y > Z \vdash X}
$$

$$
\dfrac{\dfrac{\overset{\vdots\,\pi_2}{Y \vdash B} \qquad \overset{\vdots\,\pi_3}{A \vdash Z}}{Y < Z \vdash B \prec A} \qquad \dfrac{\overset{\vdots\,\pi_1}{B < A \vdash X}}{B \prec A \vdash X}}{Y < Z \vdash X}
\quad \leadsto \quad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\vdots\,\pi_3}{Y \vdash B} \qquad \dfrac{\overset{\vdots\,\pi_1}{B < A \vdash X}}{B \vdash X; A}}{Y \vdash X; A}}{Y \vdash A; X}}{Y < X \vdash A} \qquad \overset{\vdots\,\pi_2}{A \vdash Z}}{Y < X \vdash Z}}{Y \vdash Z ; X}}{\dfrac{Y \vdash X ; Z}{Y < Z \vdash Y}}
$$

Now we turn to the part of D'.EAK with static modalities. We omit the proofs for $\overset{\frown}{\underset{\smile}{\mathtt{a}}}$ and $\overline{\underline{\mathtt{a}}}$, because they analogous to the transformations of $\langle \mathtt{a} \rangle$ and $[\mathtt{a}]$.

$$
\dfrac{\dfrac{\overset{\vdots\,\pi_1}{X \vdash A}}{\{\mathtt{a}\}X \vdash \langle \mathtt{a} \rangle A} \qquad \dfrac{\overset{\vdots\,\pi_2}{\{\mathtt{a}\}A \vdash Y}}{\langle \mathtt{a} \rangle A \vdash Y}}{\{\mathtt{a}\}X \vdash Y}
\quad \leadsto \quad
\dfrac{\dfrac{\overset{\vdots\,\pi_1}{X \vdash A} \qquad \dfrac{\overset{\vdots\,\pi_2}{\{\mathtt{a}\}A \vdash Y}}{A \vdash \overset{\frown}{\underset{\smile}{\mathtt{a}}} Y}}{X \vdash \overset{\frown}{\underset{\smile}{\mathtt{a}}} Y}}{\{\mathtt{a}\}X \vdash Y}
$$

$$
\dfrac{\dfrac{\overset{\vdots\,\pi_1}{X \vdash \{\mathtt{a}\}A}}{X \vdash [\mathtt{a}]A} \qquad \dfrac{\overset{\vdots\,\pi_2}{A \vdash Y}}{[\mathtt{a}]A \vdash \{\mathtt{a}\}Y}}{X \vdash \{\mathtt{a}\}Y}
\quad \leadsto \quad
\dfrac{\dfrac{\dfrac{\overset{\vdots\,\pi_1}{X \vdash \{\mathtt{a}\}A}}{\overset{\frown}{\underset{\smile}{\mathtt{a}}} X \vdash A} \qquad \overset{\vdots\,\pi_2}{A \vdash Y}}{\overset{\frown}{\underset{\smile}{\mathtt{a}}} X \vdash Y}}{X \vdash \{\mathtt{a}\}Y}
$$

The transformations of the dynamic modalities are analogous to the ones of static modalities and, again, we only show them for $\langle \alpha \rangle$ and $[\alpha]$.

$$
\dfrac{\dfrac{\overset{\vdots\,\pi_1}{X \vdash A}}{\{\alpha\}X \vdash \langle \alpha \rangle A} \qquad \dfrac{\overset{\vdots\,\pi_2}{\{\alpha\}A \vdash Y}}{\langle \alpha \rangle A \vdash Y}}{\{\alpha\}X \vdash Y}
\quad \leadsto \quad
\dfrac{\dfrac{\overset{\vdots\,\pi_1}{X \vdash A} \qquad \dfrac{\overset{\vdots\,\pi_2}{\{\alpha\}A \vdash Y}}{A \vdash \overset{\frown}{\underset{\smile}{\alpha}} Y}}{X \vdash \overset{\frown}{\underset{\smile}{\alpha}} Y}}{\{\alpha\}X \vdash Y}
$$

$$
\dfrac{\dfrac{\overset{\vdots\,\pi_1}{X \vdash \{\alpha\}A}}{X \vdash [\alpha]A} \qquad \dfrac{\overset{\vdots\,\pi_2}{A \vdash Y}}{[\alpha]A \vdash \{\alpha\}Y}}{X \vdash \{\alpha\}Y}
\quad \leadsto \quad
\dfrac{\dfrac{\dfrac{\overset{\vdots\,\pi_1}{X \vdash \{\alpha\}A}}{\overset{\frown}{\underset{\smile}{\alpha}} X \vdash A} \qquad \overset{\vdots\,\pi_2}{A \vdash Y}}{\overset{\frown}{\underset{\smile}{\alpha}} X \vdash Y}}{X \vdash \{\alpha\}Y}
$$

7.3 Completeness of D’.EAK

To prove, indirectly, the completeness of D’.IEAK it is enough to show that all the axioms and rules of H.IEAK are theorems and, respectively, derived or admissible rules of D’.IEAK. Below we show the derivations of the dynamic axioms.

- $\langle\alpha\rangle\, p \dashv\vdash 1_\alpha \wedge p$

$$
\dfrac{{\scriptstyle atom_L^p}\,\dfrac{\dfrac{\Phi_\alpha \vdash 1_\alpha \qquad p \vdash p}{\Phi_\alpha\,;p \vdash 1_\alpha \wedge p}}{\{\alpha\}\, p \vdash 1_\alpha \wedge p}}{\langle\alpha\rangle p \vdash 1_\alpha \wedge p}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{{\scriptstyle atom_L^p}\,\dfrac{\dfrac{p \vdash p}{\{\alpha\}\, p \vdash \langle\alpha\rangle p}}{\Phi_\alpha\,; p \vdash \langle\alpha\rangle p}}{\Phi_\alpha \vdash \langle\alpha\rangle p < p}}{1_\alpha \vdash \langle\alpha\rangle p < p}}{1_\alpha\,; p \vdash \langle\alpha\rangle p}}{1_\alpha \wedge p \vdash \langle\alpha\rangle p}
$$

- $[\alpha]p \dashv\vdash 1_\alpha \to p$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \vdash p}{[\alpha]p \vdash \{\alpha\}\, p}\,{\scriptstyle atom_R^p}}{[\alpha]p \vdash \Phi_\alpha > p}}{\Phi_\alpha\,; [\alpha]p \vdash p}}{\Phi_\alpha \vdash p < [\alpha]p}}{1_\alpha \vdash p < [\alpha]p}}{1_\alpha\,; [\alpha]p \vdash p}}{[\alpha]p \vdash 1_\alpha > p}}{[\alpha]p \vdash 1_\alpha \to p}
\qquad
\dfrac{\dfrac{\dfrac{\Phi_\alpha \vdash 1_\alpha \qquad p \vdash p}{1_\alpha \to p \vdash \Phi_\alpha > p}\,{\scriptstyle atom_R^p}}{1_\alpha \to p \vdash \{\alpha\}\, p}}{1_\alpha \to p \vdash [\alpha]\, p}
$$

- $[\alpha]\bot \dashv\vdash \neg 1_\alpha$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\bot \vdash \mathrm{I}}{\bot \vdash \overset{\frown}{\underset{\smile}{\alpha}}\,\mathrm{I}}}{[\alpha]\bot \vdash \{\alpha\}\overset{\frown}{\underset{\smile}{\alpha}}\,\mathrm{I}}}{[\alpha]\bot \vdash \Phi_\alpha > \mathrm{I}}\,com^{\alpha}_{R}}{\Phi_\alpha\,;[\alpha]\bot \vdash \mathrm{I}}}{\Phi_\alpha\,;[\alpha]\bot \vdash \bot}}{\Phi_\alpha \vdash \bot < [\alpha]\bot}}{1_\alpha \vdash \bot < [\alpha]\bot}}{1_\alpha\,;[\alpha]\bot \vdash \bot}}{[\alpha]\bot \vdash 1_\alpha > \bot}}{[\alpha]\bot \vdash 1_\alpha \to \bot}}{[\alpha]\bot \vdash \neg 1_\alpha}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\Phi_\alpha \vdash 1_\alpha \quad \dfrac{\dfrac{\dfrac{\dfrac{\bot \vdash \mathrm{I}}{\bot \vdash \{\alpha\}\mathrm{I}}}{\overset{\frown}{\underset{\smile}{\alpha}}\bot \vdash \mathrm{I}}}{\overset{\frown}{\underset{\smile}{\alpha}}\bot \vdash \bot}}{\bot \vdash \{\alpha\}\bot}}{1_\alpha \to \bot \vdash \Phi_\alpha > \{\alpha\}\bot}}{\neg 1_\alpha \vdash \Phi_\alpha > \{\alpha\}\bot}}{\neg 1_\alpha \vdash \{\alpha\}\bot}\,red_R}{\neg 1_\alpha \vdash [\alpha]\bot}$$

- $\langle\alpha\rangle\bot \dashv\vdash \bot$

$$\dfrac{\dfrac{\dfrac{\dfrac{\bot \vdash \mathrm{I}}{\bot \vdash \overset{\frown}{\underset{\smile}{\alpha}}\,\mathrm{I}}}{\{\alpha\}\bot \vdash \mathrm{I}}}{\{\alpha\}\bot \vdash \bot}}{\langle\alpha\rangle\bot \vdash \bot}
\qquad
\dfrac{\bot \vdash \mathrm{I}}{\bot \vdash \langle\alpha\rangle\bot}$$

- $[\alpha]\top \dashv\vdash \top$

$$\dfrac{\mathrm{I} \vdash \top}{[\alpha]\top \vdash \top}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\mathrm{I} \vdash \top}{\overset{\frown}{\underset{\smile}{\alpha}}\mathrm{I} \vdash \top}}{\mathrm{I} \vdash \{\alpha\}\top}}{\top \vdash \{\alpha\}\top}}{\top \vdash [\alpha]\top}$$

- $[\alpha](A \wedge B) \dashv\vdash [\alpha]A \wedge [\alpha]B$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A\,;B \vdash A}}{A \wedge B \vdash A}}{[\alpha](A \wedge B) \vdash \{\alpha\}A}}{[\alpha](A \wedge B) \vdash [\alpha]A}
\quad
\dfrac{\dfrac{\dfrac{\dfrac{B \vdash B}{A\,;B \vdash B}}{A \wedge B \vdash B}}{[\alpha](A \wedge B) \vdash \{\alpha\}B}}{[\alpha](A \wedge B) \vdash [\alpha]B}}{[\alpha](A \wedge B)\,;[\alpha](A \wedge B) \vdash [\alpha]A \wedge [\alpha]B}}{[\alpha](A \wedge B) \vdash [\alpha]A \wedge [\alpha]B}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{[\alpha]A \vdash \{\alpha\}A}}{\overset{\frown}{\underset{\smile}{\alpha}}[\alpha]A \vdash A}
\quad
\dfrac{\dfrac{B \vdash B}{[\alpha]B \vdash \{\alpha\}B}}{\overset{\frown}{\underset{\smile}{\alpha}}[\alpha]B \vdash B}}{\overset{\frown}{\underset{\smile}{\alpha}}[\alpha]A\,;\overset{\frown}{\underset{\smile}{\alpha}}[\alpha]B \vdash A \wedge B}}{\overset{\frown}{\underset{\smile}{\alpha}}([\alpha]A\,;[\alpha]B) \vdash A \wedge B}}{[\alpha]A\,;[\alpha]B \vdash \{\alpha\}(A \wedge B)}}{[\alpha]A \wedge [\alpha]B \vdash [\alpha](A \wedge B)}$$

- $\langle\alpha\rangle(A \wedge B) \dashv\vdash \langle\alpha\rangle A \wedge \langle\alpha\rangle B$

$$\dfrac{\begin{array}{c} A \vdash A \\ \hline A\,;B \vdash A \\ \hline A \wedge B \vdash A \\ \hline \{\alpha\}A \wedge B \vdash \langle\alpha\rangle A \\ \hline \langle\alpha\rangle(A \wedge B) \vdash \langle\alpha\rangle A \end{array} \quad \begin{array}{c} B \vdash B \\ \hline A\,;B \vdash B \\ \hline A \wedge B \vdash B \\ \hline \{\alpha\}A \wedge B \vdash \langle\alpha\rangle B \\ \hline \langle\alpha\rangle(A \wedge B) \vdash \langle\alpha\rangle B \end{array}}{\begin{array}{c} \langle\alpha\rangle(A \wedge B)\,;\langle\alpha\rangle(A \wedge B) \vdash \langle\alpha\rangle A \wedge \langle\alpha\rangle B \\ \hline \langle\alpha\rangle(A \wedge B) \vdash \langle\alpha\rangle A \wedge \langle\alpha\rangle B \end{array}}$$

$$\dfrac{\begin{array}{c} \dfrac{A \vdash A}{\{\alpha\}A \vdash \{\alpha\}A}\,bal \\ \hline \overset{\frown}{\underset{\smile}{\alpha}}\,\{\alpha\}A \vdash A \end{array} \quad \begin{array}{c} \dfrac{B \vdash B}{\{\alpha\}B \vdash \{\alpha\}B}\,bal \\ \hline \overset{\frown}{\underset{\smile}{\alpha}}\,\{\alpha\}B \vdash B \end{array}}{\begin{array}{c} \overset{\frown}{\underset{\smile}{\alpha}}\,\{\alpha\}A\,;\overset{\frown}{\underset{\smile}{\alpha}}\,\{\alpha\}B \vdash A \wedge B \\ \hline \overset{\frown}{\underset{\smile}{\alpha}}\,(\{\alpha\}A\,;\{\alpha\}B) \vdash A \wedge B \\ \hline \{\alpha\}\,\overset{\frown}{\underset{\smile}{\alpha}}\,(\{\alpha\}A\,;\{\alpha\}B) \vdash \langle\alpha\rangle(A \wedge B) \\ \hline \Phi_\alpha\,;(\{\alpha\}A\,;\{\alpha\}B) \vdash \langle\alpha\rangle(A \wedge B) \\ \hline (\Phi_\alpha\,;\{\alpha\}A)\,;\{\alpha\}B \vdash \langle\alpha\rangle(A \wedge B) \\ \hline red_L \quad \Phi_\alpha\,;\{\alpha\}A \vdash \langle\alpha\rangle(A \wedge B) < \{\alpha\}B \\ \hline \{\alpha\}A \vdash \langle\alpha\rangle(A \wedge B) < \{\alpha\}B \\ \hline \langle\alpha\rangle A \vdash \langle\alpha\rangle(A \wedge B) < \{\alpha\}B \\ \hline \langle\alpha\rangle A\,;\{\alpha\}B \vdash \langle\alpha\rangle(A \wedge B) \\ \hline \{\alpha\}B \vdash \langle\alpha\rangle A > \langle\alpha\rangle(A \wedge B) \\ \hline \langle\alpha\rangle B \vdash \langle\alpha\rangle A > \langle\alpha\rangle(A \wedge B) \\ \hline \langle\alpha\rangle A\,;\langle\alpha\rangle B \vdash \langle\alpha\rangle(A \wedge B) \\ \hline \langle\alpha\rangle A \wedge \langle\alpha\rangle B \vdash \langle\alpha\rangle(A \wedge B) \end{array}}$$

- $\langle\alpha\rangle(A \vee B) \dashv\vdash \langle\alpha\rangle A \vee \langle\alpha\rangle B$

$$\dfrac{\begin{array}{c} A \vdash A \\ \hline \{\alpha\}A \vdash \langle\alpha\rangle A \\ \hline A \vdash \overset{\frown}{\underset{\smile}{\alpha}}\,\langle\alpha\rangle A \end{array} \quad \begin{array}{c} B \vdash B \\ \hline \{\alpha\}B \vdash \langle\alpha\rangle B \\ \hline B \vdash \overset{\frown}{\underset{\smile}{\alpha}}\,\langle\alpha\rangle B \end{array}}{\begin{array}{c} A \vee B \vdash \overset{\frown}{\underset{\smile}{\alpha}}\,\langle\alpha\rangle A\,;\overset{\frown}{\underset{\smile}{\alpha}}\,\langle\alpha\rangle B \\ \hline A \vee B \vdash \overset{\frown}{\underset{\smile}{\alpha}}\,(\langle\alpha\rangle A\,;\langle\alpha\rangle B) \\ \hline \{\alpha\}A \vee B \vdash \langle\alpha\rangle A\,;\langle\alpha\rangle B \\ \hline \langle\alpha\rangle(A \vee B) \vdash \langle\alpha\rangle A\,;\langle\alpha\rangle B \\ \hline \langle\alpha\rangle(A \vee B) \vdash \langle\alpha\rangle A \vee \langle\alpha\rangle B \end{array}}$$

$$\dfrac{\begin{array}{c} A \vdash A \\ \hline A > A \vdash B \\ \hline A \vdash A\,;B \\ \hline A \vdash A \vee B \\ \hline \{\alpha\}A \vdash \langle\alpha\rangle(A \vee B) \\ \hline \langle\alpha\rangle A \vdash \langle\alpha\rangle(A \vee B) \end{array} \quad \begin{array}{c} B \vdash B \\ \hline B < B \vdash A \\ \hline B \vdash A\,;B \\ \hline B \vdash A \vee B \\ \hline \{\alpha\}B \vdash \langle\alpha\rangle(A \vee B) \\ \hline \langle\alpha\rangle B \vdash \langle\alpha\rangle(A \vee B) \end{array}}{\begin{array}{c} \langle\alpha\rangle A \vee \langle\alpha\rangle B \vdash \langle\alpha\rangle(A \vee B)\,;\langle\alpha\rangle(A \vee B) \\ \hline \langle\alpha\rangle A \vee \langle\alpha\rangle B \vdash \langle\alpha\rangle(A \vee B) \end{array}}$$

- $[\alpha](A \vee B) \dashv\vdash 1_\alpha \to (\langle\alpha\rangle A \vee \langle\alpha\rangle B)$

$$\dfrac{\begin{array}{c} A \vdash A \\ \hline \{\alpha\}A \vdash \langle\alpha\rangle A \\ \hline A \vdash \overset{\frown}{\underset{\smile}{\alpha}}\,\langle\alpha\rangle A \end{array} \quad \begin{array}{c} B \vdash B \\ \hline \{\alpha\}B \vdash \langle\alpha\rangle B \\ \hline B \vdash \overset{\frown}{\underset{\smile}{\alpha}}\,\langle\alpha\rangle B \end{array}}{\begin{array}{c} A \vee B \vdash \overset{\frown}{\underset{\smile}{\alpha}}\,\langle\alpha\rangle A\,;\overset{\frown}{\underset{\smile}{\alpha}}\,\langle\alpha\rangle B \\ \hline A \vee B \vdash \overset{\frown}{\underset{\smile}{\alpha}}\,(\langle\alpha\rangle A\,;\langle\alpha\rangle B) \\ \hline [\alpha](A \vee B) \vdash \{\alpha\}\,\overset{\frown}{\underset{\smile}{\alpha}}\,(\langle\alpha\rangle A \vee \langle\alpha\rangle B) \\ \hline [\alpha](A \vee B) \vdash \Phi_\alpha > (\langle\alpha\rangle A \vee \langle\alpha\rangle B) \quad com_R^\alpha \\ \hline \Phi_\alpha\,;[\alpha](A \vee B) \vdash \langle\alpha\rangle A \vee \langle\alpha\rangle B \\ \hline \Phi_\alpha \vdash \langle\alpha\rangle A \vee \langle\alpha\rangle B < [\alpha](A \vee B) \\ \hline 1_\alpha \vdash \langle\alpha\rangle A \vee \langle\alpha\rangle B < [\alpha](A \vee B) \\ \hline 1_\alpha\,;[\alpha](A \vee B) \vdash \langle\alpha\rangle A \vee \langle\alpha\rangle B \\ \hline [\alpha](A \vee B) \vdash 1_\alpha > (\langle\alpha\rangle A \vee \langle\alpha\rangle B) \\ \hline [\alpha](A \vee B) \vdash 1_\alpha \to (\langle\alpha\rangle A \vee \langle\alpha\rangle B) \end{array}}$$

$$\dfrac{\Phi_\alpha \vdash 1_\alpha \quad \dfrac{\begin{array}{c} A \vdash A \\ \hline \{\alpha\}A \vdash \{\alpha\}A \\ \hline \langle\alpha\rangle A \vdash \{\alpha\}A \end{array} \quad \begin{array}{c} B \vdash B \\ \hline \{\alpha\}B \vdash \{\alpha\}B \\ \hline \langle\alpha\rangle B \vdash \{\alpha\}B \end{array}}{\begin{array}{c} \langle\alpha\rangle A \vee \langle\alpha\rangle B \vdash \{\alpha\}A\,;\{\alpha\}B \\ \hline \langle\alpha\rangle A \vee \langle\alpha\rangle B \vdash \{\alpha\}(A\,;B) \\ \hline \overset{\frown}{\underset{\smile}{\alpha}}\,(\langle\alpha\rangle A \vee \langle\alpha\rangle B) \vdash A\,;B \\ \hline \overset{\frown}{\underset{\smile}{\alpha}}\,(\langle\alpha\rangle A \vee \langle\alpha\rangle B) \vdash A \vee B \\ \hline \langle\alpha\rangle A \vee \langle\alpha\rangle B \vdash \{\alpha\}(A \vee B) \end{array}}}{\begin{array}{c} 1_\alpha \to (\langle\alpha\rangle A \vee \langle\alpha\rangle B) \vdash \Phi_\alpha > \{\alpha\}(A \vee B) \quad red_R \\ \hline 1_\alpha \to (\langle\alpha\rangle A \vee \langle\alpha\rangle B) \vdash \{\alpha\}(A \vee B) \\ \hline 1_\alpha \to (\langle\alpha\rangle A \vee \langle\alpha\rangle B) \vdash [\alpha](A \vee B) \end{array}}$$

- $\langle\alpha\rangle(A \to B) \dashv\vdash 1_\alpha \wedge (\langle\alpha\rangle A \to \langle\alpha\rangle B)$

$$A \vdash A$$
$$\{\alpha\}A \vdash \{\alpha\}A$$
$$\langle\alpha\rangle A \vdash \{\alpha\}A$$
$$\widetilde{\alpha}\,\langle\alpha\rangle A \vdash A$$

$$B \vdash B$$
$$\{\alpha\}B \vdash \langle\alpha\rangle B$$
$$B \vdash \widetilde{\alpha}\,\langle\alpha\rangle B$$

$$A \to B \vdash \widetilde{\alpha}\,\langle\alpha\rangle A > \widetilde{\alpha}\,\langle\alpha\rangle B$$
$$A \to B \vdash \widetilde{\alpha}\,(\langle\alpha\rangle A > \langle\alpha\rangle B)$$
$$\{\alpha\}A \to B \vdash \langle\alpha\rangle A > \langle\alpha\rangle B$$
$$\Phi_\alpha \vdash 1_\alpha \qquad \{\alpha\}A \to B \vdash \langle\alpha\rangle A \to \langle\alpha\rangle B$$
$$\Phi_\alpha\,;\{\alpha\}A \to B \vdash 1_\alpha \wedge (\langle\alpha\rangle A \to \langle\alpha\rangle B)$$
$$red\ \{\alpha\}A \to B \vdash 1_\alpha \wedge (\langle\alpha\rangle A \to \langle\alpha\rangle B)$$
$$\langle\alpha\rangle(A \to B) \vdash 1_\alpha \wedge (\langle\alpha\rangle A \to \langle\alpha\rangle B)$$

$$A \vdash A$$
$$\{\alpha\}A \vdash \langle\alpha\rangle A$$

$$B \vdash B$$
$$\{\alpha\}B \vdash \{\alpha\}B$$
$$\langle\alpha\rangle B \vdash \{\alpha\}B$$

$$\langle\alpha\rangle A \to \langle\alpha\rangle B \vdash \{\alpha\}A > \{\alpha\}B$$
$$\langle\alpha\rangle A \to \langle\alpha\rangle B \vdash \{\alpha\}(A > B)$$
$$\widetilde{\alpha}\,\langle\alpha\rangle A \to \langle\alpha\rangle B \vdash A > B$$
$$\widetilde{\alpha}\,\langle\alpha\rangle A \to \langle\alpha\rangle B \vdash A \to B$$
$$\{\alpha\}\,\widetilde{\alpha}\,\langle\alpha\rangle A \to \langle\alpha\rangle B \vdash \langle\alpha\rangle(A \to B)$$
$$com_L^\alpha\ \Phi_\alpha\,;\langle\alpha\rangle A \to \langle\alpha\rangle B \vdash \langle\alpha\rangle(A \to B)$$
$$\langle\alpha\rangle A \to \langle\alpha\rangle B ; \Phi_\alpha \vdash \langle\alpha\rangle(A \to B)$$
$$\Phi_\alpha \vdash \langle\alpha\rangle A \to \langle\alpha\rangle B > \langle\alpha\rangle(A \to B)$$
$$1_\alpha \vdash \langle\alpha\rangle A \to \langle\alpha\rangle B > \langle\alpha\rangle(A \to B)$$
$$\langle\alpha\rangle A \to \langle\alpha\rangle B ; 1_\alpha \vdash \langle\alpha\rangle(A \to B)$$
$$1_\alpha\,;\langle\alpha\rangle A \to \langle\alpha\rangle B \vdash \langle\alpha\rangle(A \to B)$$
$$1_\alpha \wedge (\langle\alpha\rangle A \to \langle\alpha\rangle B) \vdash \langle\alpha\rangle(A \to B)$$

- $[\alpha](A \to B) \dashv\vdash \langle\alpha\rangle A \to \langle\alpha\rangle B$

$$A \vdash A$$
$$\{\alpha\}A \vdash \{\alpha\}A$$
$$\widetilde{\alpha}\,\{\alpha\}A \vdash A$$

$$B \vdash B$$
$$\{\alpha\}B \vdash \langle\alpha\rangle B$$
$$B \vdash \widetilde{\alpha}\,\langle\alpha\rangle B$$

$$A \to B \vdash \widetilde{\alpha}\,\{\alpha\}A > \widetilde{\alpha}\,\langle\alpha\rangle B$$
$$A \to B \vdash \widetilde{\alpha}\,(\{\alpha\}A > \langle\alpha\rangle B)$$
$$\{\alpha\}A \to B \vdash \{\alpha\}A > \langle\alpha\rangle B$$
$$\{\alpha\}(A \to B) \vdash \{\alpha\}A > \langle\alpha\rangle B$$
$$A \to B \vdash \widetilde{\alpha}\,(\{\alpha\}A > \langle\alpha\rangle B)$$
$$[\alpha](A \to B) \vdash \{\alpha\}\,\widetilde{\alpha}\,(\{\alpha\}A > \langle\alpha\rangle B)$$
$$com_R^\alpha\ [\alpha](A \to B) \vdash \Phi_\alpha > (\{\alpha\}A > \langle\alpha\rangle B)$$
$$\Phi_\alpha\,;[\alpha](A \to B) \vdash \{\alpha\}A > \langle\alpha\rangle B$$
$$\{\alpha\}A\,;(\Phi_\alpha\,;[\alpha](A \to B)) \vdash \langle\alpha\rangle B$$
$$(\{\alpha\}A\,;\Phi_\alpha)\,;[\alpha](A \to B) \vdash \langle\alpha\rangle B$$
$$[\alpha](A \to B)\,;(\{\alpha\}A\,;\Phi_\alpha) \vdash \langle\alpha\rangle B$$
$$\{\alpha\}A\,;\Phi_\alpha \vdash [\alpha](A \to B) > \langle\alpha\rangle B$$
$$\Phi_\alpha\,;\{\alpha\}A \vdash [\alpha](A \to B) > \langle\alpha\rangle B$$
$$red\ \{\alpha\}A \vdash [\alpha](A \to B) > \langle\alpha\rangle B$$
$$\langle\alpha\rangle A \vdash [\alpha](A \to B) > \langle\alpha\rangle B$$
$$[\alpha](A \to B)\,;\langle\alpha\rangle A \vdash \langle\alpha\rangle B$$
$$\langle\alpha\rangle A\,;[\alpha](A \to B) \vdash \langle\alpha\rangle B$$
$$[\alpha](A \to B) \vdash \langle\alpha\rangle A > \langle\alpha\rangle B$$
$$[\alpha](A \to B) \vdash \langle\alpha\rangle A \to \langle\alpha\rangle B$$

$$A \vdash A$$
$$\{\alpha\}A \vdash \langle\alpha\rangle A$$

$$B \vdash B$$
$$\{\alpha\}B \vdash \{\alpha\}B$$
$$\langle\alpha\rangle B \vdash \{\alpha\}B$$

$$\langle\alpha\rangle A \to \langle\alpha\rangle B \vdash \{\alpha\}A > \{\alpha\}B$$
$$\langle\alpha\rangle A \to \langle\alpha\rangle B \vdash \{\alpha\}(A > B)$$
$$\widetilde{\alpha}\,(\langle\alpha\rangle A \to \langle\alpha\rangle B) \vdash A > B$$
$$\widetilde{\alpha}\,(\langle\alpha\rangle A \to \langle\alpha\rangle B) \vdash A \to B$$
$$\langle\alpha\rangle A \to \langle\alpha\rangle B \vdash \{\alpha\}(A \to B)$$
$$\langle\alpha\rangle A \to \langle\alpha\rangle B \vdash [\alpha](A \to B)$$

For ease of notation, in the following derivations we assume the actions β, such that $\alpha\mathsf{a}\beta$ form the set $\{\beta_i \,|\, 1 \leq i \leq n\}$.

- $\langle\alpha\rangle\langle\mathsf{a}\rangle A \vdash 1_\alpha \wedge \bigvee\{\langle\mathsf{a}\rangle\langle\beta\rangle A \,|\, \alpha\mathsf{a}\beta\}$

$$
\begin{prooftree}
\AxiomC{$\Phi_\alpha \vdash 1_\alpha$}
\AxiomC{$A \vdash A$}
\UnaryInfC{$\{\beta_1\}A \vdash \langle\beta_1\rangle A$}
\UnaryInfC{$\{\mathsf{a}\}\{\beta_1\}A \vdash \langle\mathsf{a}\rangle\langle\beta_1\rangle A$}
\AxiomC{$\cdots$}
\AxiomC{$A \vdash A$}
\UnaryInfC{$\{\beta_n\}A \vdash \langle\beta_n\rangle A$}
\UnaryInfC{$\{\mathsf{a}\}\{\beta_n\}A \vdash \langle\mathsf{a}\rangle\langle\beta_n\rangle A$}
\LeftLabel{$s\text{-}out$}
\TrinaryInfC{$\{\alpha\}\{\mathsf{a}\}A \vdash \mathbf{;}\Big(\langle\mathsf{a}\rangle\langle\beta_i\rangle A\Big)$}
\dashedLine
\UnaryInfC{$\{\alpha\}\{\mathsf{a}\}A \vdash \bigvee\Big(\langle\mathsf{a}\rangle\langle\beta_i\rangle A\Big)$}
\BinaryInfC{$\Phi_\alpha \,;\, \{\alpha\}\{\mathsf{a}\}A \vdash 1_\alpha \wedge \bigvee\Big(\langle\mathsf{a}\rangle\langle\beta_i\rangle A\Big)$}
\LeftLabel{red}
\UnaryInfC{$\{\alpha\}\{\mathsf{a}\}A \vdash 1_\alpha \wedge \bigvee\Big(\langle\mathsf{a}\rangle\langle\beta_i\rangle A\Big)$}
\UnaryInfC{$\{\mathsf{a}\}A \vdash \overset{\frown}{\underset{\smile}{\alpha}}\, 1_\alpha \wedge \bigvee\Big(\langle\mathsf{a}\rangle\langle\beta_i\rangle A\Big)$}
\UnaryInfC{$\langle\mathsf{a}\rangle A \vdash \overset{\frown}{\underset{\smile}{\alpha}}\, 1_\alpha \wedge \bigvee\Big(\langle\mathsf{a}\rangle\langle\beta_i\rangle A\Big)$}
\UnaryInfC{$\{\alpha\}\langle\mathsf{a}\rangle A \vdash 1_\alpha \wedge \bigvee\Big(\langle\mathsf{a}\rangle\langle\beta_i\rangle A\Big)$}
\UnaryInfC{$\langle\alpha\rangle\langle\mathsf{a}\rangle A \vdash 1_\alpha \wedge \bigvee\Big(\langle\mathsf{a}\rangle\langle\beta_i\rangle A\Big)$}
\end{prooftree}
$$

- $1_\alpha \wedge \bigvee\{\langle\mathsf{a}\rangle\langle\beta\rangle A \,|\, \alpha\mathsf{a}\beta\} \vdash \langle\alpha\rangle\langle\mathsf{a}\rangle A$

$$
\begin{prooftree}
\AxiomC{$A \vdash A$}
\UnaryInfC{$\{\mathsf{a}\}A \vdash \langle\mathsf{a}\rangle A$}
\UnaryInfC{$\{\alpha\}\{\mathsf{a}\}A \vdash \langle\alpha\rangle\langle\mathsf{a}\rangle A$}
\LeftLabel{$s\text{-}in$}
\UnaryInfC{$\Phi_\alpha \,;\, \{\mathsf{a}\}\{\beta_1\}A \vdash \langle\alpha\rangle\langle\mathsf{a}\rangle A$}
\UnaryInfC{$\{\mathsf{a}\}\{\beta_1\}A \,;\, \Phi_\alpha \vdash \langle\alpha\rangle\langle\mathsf{a}\rangle A$}
\UnaryInfC{$\Phi_\alpha \vdash \{\mathsf{a}\}\{\beta_1\}A > \langle\alpha\rangle\langle\mathsf{a}\rangle A$}
\UnaryInfC{$1_\alpha \vdash \{\mathsf{a}\}\{\beta_1\}A > \langle\alpha\rangle\langle\mathsf{a}\rangle A$}
\UnaryInfC{$\{\mathsf{a}\}\{\beta_1\}A \,;\, 1_\alpha \vdash \langle\alpha\rangle\langle\mathsf{a}\rangle A$}
\UnaryInfC{$1_\alpha \,;\, \{\mathsf{a}\}\{\beta_1\}A \vdash \langle\alpha\rangle\langle\mathsf{a}\rangle A$}
\UnaryInfC{$\{\mathsf{a}\}\{\beta_1\}A \vdash 1_\alpha > \langle\alpha\rangle\langle\mathsf{a}\rangle A$}
\UnaryInfC{$\{\beta_1\}A \vdash \overset{\frown}{\underset{\smile}{\mathsf{a}}}(1_\alpha > \langle\alpha\rangle\langle\mathsf{a}\rangle A)$}
\UnaryInfC{$\langle\beta_1\rangle A \vdash \overset{\frown}{\underset{\smile}{\mathsf{a}}}(1_\alpha > \langle\alpha\rangle\langle\mathsf{a}\rangle A)$}
\UnaryInfC{$\{\mathsf{a}\}\langle\beta_1\rangle A \vdash 1_\alpha > \langle\alpha\rangle\langle\mathsf{a}\rangle A$}
\UnaryInfC{$\langle\mathsf{a}\rangle\langle\beta_1\rangle A \vdash 1_\alpha > \langle\alpha\rangle\langle\mathsf{a}\rangle A$}
\AxiomC{$\cdots$}
\AxiomC{$A \vdash A$}
\UnaryInfC{$\{\mathsf{a}\}A \vdash \langle\mathsf{a}\rangle A$}
\UnaryInfC{$\{\alpha\}\{\mathsf{a}\}A \vdash \langle\alpha\rangle\langle\mathsf{a}\rangle A$}
\LeftLabel{$s\text{-}in$}
\UnaryInfC{$\Phi_\alpha \,;\, \{\mathsf{a}\}\{\beta_n\}A \vdash \langle\alpha\rangle\langle\mathsf{a}\rangle A$}
\UnaryInfC{$\{\mathsf{a}\}\{\beta_n\}A \,;\, \Phi_\alpha \vdash \langle\alpha\rangle\langle\mathsf{a}\rangle A$}
\UnaryInfC{$\Phi_\alpha \vdash \{\mathsf{a}\}\{\beta_n\}A > \langle\alpha\rangle\langle\mathsf{a}\rangle A$}
\UnaryInfC{$1_\alpha \vdash \{\mathsf{a}\}\{\beta_n\}A > \langle\alpha\rangle\langle\mathsf{a}\rangle A$}
\UnaryInfC{$\{\mathsf{a}\}\{\beta_n\}A \,;\, 1_\alpha \vdash \langle\alpha\rangle\langle\mathsf{a}\rangle A$}
\UnaryInfC{$1_\alpha \,;\, \{\mathsf{a}\}\{\beta_n\}A \vdash \langle\alpha\rangle\langle\mathsf{a}\rangle A$}
\UnaryInfC{$\{\mathsf{a}\}\{\beta_n\}A \vdash 1_\alpha > \langle\alpha\rangle\langle\mathsf{a}\rangle A$}
\UnaryInfC{$\{\beta_n\}A \vdash \overset{\frown}{\underset{\smile}{\mathsf{a}}}(1_\alpha > \langle\alpha\rangle\langle\mathsf{a}\rangle A)$}
\UnaryInfC{$\langle\beta_n\rangle A \vdash \overset{\frown}{\underset{\smile}{\mathsf{a}}}(1_\alpha > \langle\alpha\rangle\langle\mathsf{a}\rangle A)$}
\UnaryInfC{$\{\mathsf{a}\}\langle\beta_n\rangle A \vdash 1_\alpha > \langle\alpha\rangle\langle\mathsf{a}\rangle A$}
\UnaryInfC{$\langle\mathsf{a}\rangle\langle\beta_n\rangle A \vdash 1_\alpha > \langle\alpha\rangle\langle\mathsf{a}\rangle A$}
\dashedLine
\TrinaryInfC{$\bigvee\Big(\langle\mathsf{a}\rangle\langle\beta_i\rangle A\Big) \vdash \mathbf{;}\Big(1_\alpha > \langle\alpha\rangle\langle\mathsf{a}\rangle A\Big)$}
\dashedLine
\UnaryInfC{$\bigvee\Big(\langle\mathsf{a}\rangle\langle\beta_i\rangle A\Big) \vdash 1_\alpha > \langle\alpha\rangle\langle\mathsf{a}\rangle A)$}
\UnaryInfC{$1_\alpha \,;\, \bigvee\Big(\langle\mathsf{a}\rangle\langle\beta_i\rangle A\Big) \vdash \langle\alpha\rangle\langle\mathsf{a}\rangle A$}
\UnaryInfC{$1_\alpha \wedge \bigvee\Big(\langle\mathsf{a}\rangle\langle\beta_i\rangle A\Big) \vdash \langle\alpha\rangle\langle\mathsf{a}\rangle A$}
\end{prooftree}
$$

- $[\alpha]\langle\mathsf{a}\rangle A \vdash Pre(\alpha) \to \bigvee\{\langle\mathsf{a}\rangle\langle\beta\rangle A \mid \alpha\mathsf{a}\beta\}$

$$
\begin{array}{c}
\begin{array}{c} A \vdash A \\ \hline \{\beta_1\}A \vdash \langle\beta_1\rangle A \\ \hline \{\mathsf{a}\}\{\beta_1\}A \vdash \langle\mathsf{a}\rangle\langle\beta_1\rangle A \end{array}
\quad \begin{array}{c} \cdots \\ \cdots \\ \cdots \end{array} \quad
\begin{array}{c} A \vdash A \\ \hline \{\beta_n\}A \vdash \langle\beta_n\rangle A \\ \hline \{\mathsf{a}\}\{\beta_n\}A \vdash \langle\mathsf{a}\rangle\langle\beta_n\rangle A \end{array} \\
\hline \textit{s-out} \\
\{\alpha\}\{\mathsf{a}\}A \vdash {;}\Big(\langle\mathsf{a}\rangle\langle\beta_i\rangle A\Big) \\
\hdashline
\{\alpha\}\{\mathsf{a}\}A \vdash \bigvee\Big(\langle\mathsf{a}\rangle\langle\beta_i\rangle A\Big) \\
\hline
\{\mathsf{a}\}A \vdash \overbrace{\underbrace{\alpha}} \bigvee\Big(\langle\mathsf{a}\rangle\langle\beta_i\rangle A\Big) \\
\hline
\langle\mathsf{a}\rangle A \vdash \overbrace{\underbrace{\alpha}} \bigvee\Big(\langle\mathsf{a}\rangle\langle\beta_i\rangle A\Big) \\
\hline
[\alpha]\langle\mathsf{a}\rangle A \vdash \{\alpha\}\overbrace{\underbrace{\alpha}} \bigvee\Big(\langle\mathsf{a}\rangle\langle\beta_i\rangle A\Big) \\
\hline com^{\alpha}_{R} \\
[\alpha]\langle\mathsf{a}\rangle A \vdash \Phi_\alpha > \bigvee\Big(\langle\mathsf{a}\rangle\langle\beta_i\rangle A\Big) \\
\hline
\Phi_\alpha \,;[\alpha]\langle\mathsf{a}\rangle A \vdash \bigvee\Big(\langle\mathsf{a}\rangle\langle\beta_i\rangle A\Big) \\
\hline
\Phi_\alpha \vdash \bigvee\Big(\langle\mathsf{a}\rangle\langle\beta_i\rangle A\Big) < [\alpha]\langle\mathsf{a}\rangle A \\
\hline
1_\alpha \vdash \bigvee\Big(\langle\mathsf{a}\rangle\langle\beta_i\rangle A\Big) < [\alpha]\langle\mathsf{a}\rangle A \\
\hline
1_\alpha \,;[\alpha]\langle\mathsf{a}\rangle A \vdash \bigvee\Big(\langle\mathsf{a}\rangle\langle\beta_i\rangle A\Big) \\
\hline
[\alpha]\langle\mathsf{a}\rangle A \vdash 1_\alpha > \bigvee\Big(\langle\mathsf{a}\rangle\langle\beta_i\rangle A\Big) \\
\hline
[\alpha]\langle\mathsf{a}\rangle A \vdash 1_\alpha \to \bigvee\Big(\langle\mathsf{a}\rangle\langle\beta_i\rangle A\Big)
\end{array}
$$

- $Pre(\alpha) \to \bigvee\{\langle\mathsf{a}\rangle\langle\beta_i\rangle A \mid \alpha\mathsf{a}\beta\} \vdash [\alpha]\langle\mathsf{a}\rangle A$

$$
\begin{array}{c}
\Phi_\alpha \vdash 1_\alpha
\qquad
\begin{array}{c}
\begin{array}{c}
A \vdash A \\ \hline
\{\mathsf{a}\}A \vdash \langle\mathsf{a}\rangle A \\ \hline
\{\alpha\}\{\mathsf{a}\}A \vdash \{\alpha\}\langle\mathsf{a}\rangle A \\ \hline \textit{s-in} \\
\Phi_\alpha \,; \{\mathsf{a}\}\{\beta_1\}A \vdash \{\alpha\}\langle\mathsf{a}\rangle A \\ \hline
\{\mathsf{a}\}\{\beta_1\}A \vdash \Phi_\alpha > \{\alpha\}\langle\mathsf{a}\rangle A \\ \hline \textit{red} \\
\{\mathsf{a}\}\{\beta_1\}A \vdash \{\alpha\}\langle\mathsf{a}\rangle A \\ \hline
\{\beta_1\}A \vdash \overbrace{\underbrace{\mathsf{a}}}\{\alpha\}\langle\mathsf{a}\rangle A \\ \hline
\langle\beta_1\rangle A \vdash \overbrace{\underbrace{\mathsf{a}}}\{\alpha\}\langle\mathsf{a}\rangle A \\ \hline
\{\mathsf{a}\}\langle\beta_1\rangle A \vdash \{\alpha\}\langle\mathsf{a}\rangle A \\ \hline
\langle\mathsf{a}\rangle\langle\beta_1\rangle A \vdash \{\alpha\}\langle\mathsf{a}\rangle A
\end{array}
\quad \cdots \quad
\begin{array}{c}
A \vdash A \\ \hline
\{\mathsf{a}\}A \vdash \langle\mathsf{a}\rangle A \\ \hline
\{\alpha\}\{\mathsf{a}\}A \vdash \{\alpha\}\langle\mathsf{a}\rangle A \\ \hline \textit{s-in} \\
\Phi_\alpha \,; \{\mathsf{a}\}\{\beta_n\}A \vdash \{\alpha\}\langle\mathsf{a}\rangle A \\ \hline
\{\mathsf{a}\}\{\beta_n\}A \vdash \Phi_\alpha > \{\alpha\}\langle\mathsf{a}\rangle A \\ \hline \textit{red} \\
\{\mathsf{a}\}\{\beta_n\}A \vdash \{\alpha\}\langle\mathsf{a}\rangle A \\ \hline
\{\beta_n\}A \vdash \overbrace{\underbrace{\mathsf{a}}}\{\alpha\}\langle\mathsf{a}\rangle A \\ \hline
\langle\beta_n\rangle A \vdash \overbrace{\underbrace{\mathsf{a}}}\{\alpha\}\langle\mathsf{a}\rangle A \\ \hline
\{\mathsf{a}\}\langle\beta_n\rangle A \vdash \{\alpha\}\langle\mathsf{a}\rangle A \\ \hline
\langle\mathsf{a}\rangle\langle\beta_n\rangle A \vdash \{\alpha\}\langle\mathsf{a}\rangle A
\end{array} \\
\hdashline
\bigvee\Big(\langle\mathsf{a}\rangle\langle\beta_i\rangle A\Big) \vdash {;}\Big(\{\alpha\}\langle\mathsf{a}\rangle A\Big) \\
\hdashline
\bigvee\Big(\langle\mathsf{a}\rangle\langle\beta_i\rangle A\Big) \vdash \{\alpha\}\langle\mathsf{a}\rangle A
\end{array} \\
\hline
1_\alpha \to \bigvee\Big(\langle\mathsf{a}\rangle\langle\beta_i\rangle A\Big) \vdash \Phi_\alpha > \{\alpha\}\langle\mathsf{a}\rangle A \\
\hline \textit{red} \\
1_\alpha \to \bigvee\Big(\langle\mathsf{a}\rangle\langle\beta_i\rangle A\Big) \vdash \{\alpha\}\langle\mathsf{a}\rangle A \\
\hline
1_\alpha \to \bigvee\Big(\langle\mathsf{a}\rangle\langle\beta_i\rangle A\Big) \vdash [\alpha]\langle\mathsf{a}\rangle A
\end{array}
$$

- $[\alpha][\mathtt{a}]A \vdash Pre(\alpha) \to \bigwedge\{[\mathtt{a}][\alpha_j]A \mid \alpha\mathtt{a}\beta\}$

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{[\mathtt{a}]A \vdash \{\mathtt{a}\}A}}{[\alpha][\mathtt{a}]A \vdash \{\alpha\}\{\mathtt{a}\}A}}{[\alpha][\mathtt{a}]A \vdash \Phi_\alpha > \{\mathtt{a}\}\{\beta_1\}A}\,\textit{s-in}}{\Phi_\alpha\,;[\alpha][\mathtt{a}]A \vdash \{\mathtt{a}\}\{\beta_1\}A}}{\widehat{\mathtt{a}}\,\Phi_\alpha\,;[\alpha][\mathtt{a}]A) \vdash \{\beta_1\}A}}{\widehat{\mathtt{a}}\,\Phi_\alpha\,;[\alpha][\mathtt{a}]A) \vdash [\beta_1]A}}{\Phi_\alpha\,;[\alpha][\mathtt{a}]A \vdash \{\mathtt{a}\}[\beta_1]A}}{\Phi_\alpha\,;[\alpha][\mathtt{a}]A \vdash [\mathtt{a}][\beta_1]A}
\quad \cdots \quad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{[\mathtt{a}]A \vdash \{\mathtt{a}\}A}}{[\alpha][\mathtt{a}]A \vdash \{\alpha\}\{\mathtt{a}\}A}}{[\alpha][\mathtt{a}]A \vdash \Phi_\alpha > \{\mathtt{a}\}\{\beta_n\}A}\,\textit{s-in}}{\Phi_\alpha\,;[\alpha][\mathtt{a}]A \vdash \{\mathtt{a}\}\{\beta_n\}A}}{\widehat{\mathtt{a}}\,(\Phi_\alpha\,;[\alpha][\mathtt{a}]A) \vdash \{\beta_n\}A}}{\widehat{\mathtt{a}}\,(\Phi_\alpha\,;[\alpha][\mathtt{a}]A) \vdash [\beta_n]A}}{\Phi_\alpha\,;[\alpha][\mathtt{a}]A \vdash \{\mathtt{a}\}[\beta_n]A}}{\Phi_\alpha\,;[\alpha][\mathtt{a}]A \vdash [\mathtt{a}][\beta_n]A}
}{
\dfrac{;\Big(\Phi_\alpha\,;[\alpha][\mathtt{a}]A\Big) \vdash \bigwedge\Big([\mathtt{a}][\beta_i]A\Big)}{\Phi_\alpha\,;[\alpha][\mathtt{a}]A \vdash \bigwedge\Big([\mathtt{a}][\beta_i]A\Big)}
}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Phi_\alpha\,;[\alpha][\mathtt{a}]A \vdash \bigwedge\Big([\mathtt{a}][\beta_i]A\Big)}{[\alpha][\mathtt{a}]A\,;\Phi_\alpha \vdash \bigwedge\Big([\mathtt{a}][\beta_i]A\Big)}}{\Phi_\alpha \vdash [\alpha][\mathtt{a}]A > \bigwedge\Big([\mathtt{a}][\beta_i]A\Big)}}{1_\alpha \vdash [\alpha][\mathtt{a}]A > \bigwedge\Big([\mathtt{a}][\beta_i]A\Big)}}{[\alpha][\mathtt{a}]A\,;1_\alpha \vdash \bigwedge\Big([\mathtt{a}][\beta_i]A\Big)}}{1_\alpha\,;[\alpha][\mathtt{a}]A \vdash \bigwedge\Big([\mathtt{a}][\beta_i]A\Big)}}{[\alpha][\mathtt{a}]A \vdash 1_\alpha \ > \bigwedge\Big([\mathtt{a}][\beta_i]A\Big)}}{[\alpha][\mathtt{a}]A \vdash 1_\alpha \to \bigwedge\Big([\mathtt{a}][\beta_i]A\Big)}
$$

- $Pre(\alpha) \to \bigwedge\{[\mathtt{a}][\beta]A \mid \alpha\mathtt{a}\beta\} \vdash [\alpha][\mathtt{a}]A$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Phi_\alpha \vdash 1_\alpha \qquad \dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{[\beta_1]A \vdash \{\beta_1\}A}}{[\mathtt{a}][\beta_1]A \vdash \{\mathtt{a}\}\{\beta_1\}A} \quad \cdots \quad \dfrac{\dfrac{A \vdash A}{[\beta_n]A \vdash \{\beta_n\}A}}{[\mathtt{a}][\beta_n]A \vdash \{\mathtt{a}\}\{\beta_n\}A}}{;\Big([\mathtt{a}][\beta_i]A\Big) \vdash \{\alpha\}\{\mathtt{a}\}A}\,\textit{s-out}}{\bigwedge\Big([\mathtt{a}][\beta_i]A\Big) \vdash \{\alpha\}\{\mathtt{a}\}A}}{1_\alpha \to \bigwedge\Big([\mathtt{a}][\beta_i]A\Big) \vdash \Phi_\alpha > \{\alpha\}\{\mathtt{a}\}A}}{1_\alpha \to \bigwedge\Big([\mathtt{a}][\beta_i]A\Big) \vdash \{\alpha\}\{\mathtt{a}\}A}\,\textit{red}}{\widehat{\alpha}\,(1_\alpha \to \bigwedge\Big([\mathtt{a}][\beta_i]A\Big)) \vdash \{\mathtt{a}\}A}}{\widehat{\alpha}\,(1_\alpha \to \bigwedge\Big([\mathtt{a}][\beta_i]A\Big)) \vdash [\mathtt{a}]A}}{\dfrac{1_\alpha \to \bigwedge\Big([\mathtt{a}][\beta_i]A\Big) \vdash \{\alpha\}[\mathtt{a}]A}{1_\alpha \to \bigwedge\Big([\mathtt{a}][\beta_i]A\Big) \vdash [\alpha][\mathtt{a}]A}}
$$

- $\langle\alpha\rangle[\mathsf{a}]A \vdash Pre(\alpha) \wedge \bigwedge\{[\mathsf{a}][\beta]A \mid \alpha\mathsf{a}\beta\}$

$$
\dfrac{\Phi_\alpha \vdash 1_\alpha \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{[\mathsf{a}]A \vdash \{\mathsf{a}\}A}}{\{\alpha\}[\mathsf{a}]A \vdash \{\alpha\}\{\mathsf{a}\}A}\,bal}{\{\alpha\}[\mathsf{a}]A \vdash \Phi_\alpha > \{\mathsf{a}\}\{\beta_1\}A}\,s\text{-}in}{\Phi_\alpha \,;\, \{\alpha\}[\mathsf{a}]A \vdash \{\mathsf{a}\}\{\beta_1\}A}}{\{\alpha\}[\mathsf{a}]A \vdash \{\mathsf{a}\}\{\beta_1\}A}\,red_L}{\{\alpha\}[\mathsf{a}]A \vdash \{\mathsf{a}\}\{\beta_1\}A}}{\overset{\frown}{\underset{\smile}{\mathsf{a}}}\{\alpha\}[\mathsf{a}]A \vdash \{\beta_1\}A}}{\overset{\frown}{\underset{\smile}{\mathsf{a}}}\{\alpha\}[\mathsf{a}]A \vdash [\beta_1]A}}{\{\alpha\}[\mathsf{a}]A \vdash \{\mathsf{a}\}[\beta_1]A}}{\{\alpha\}[\mathsf{a}]A \vdash [\mathsf{a}][\beta_1]A} \quad \cdots \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{[\mathsf{a}]A \vdash \{\mathsf{a}\}A}}{\{\alpha\}[\mathsf{a}]A \vdash \{\alpha\}\{\mathsf{a}\}A}\,bal}{\{\alpha\}[\mathsf{a}]A \vdash \Phi_\alpha > \{\mathsf{a}\}\{\beta_n\}A}\,s\text{-}in}{\Phi_\alpha \,;\, \{\alpha\}[\mathsf{a}]A \vdash \{\mathsf{a}\}\{\beta_n\}A}}{\{\alpha\}[\mathsf{a}]A \vdash \{\mathsf{a}\}\{\beta_n\}A}\,red_L}{\{\alpha\}[\mathsf{a}]A \vdash \{\mathsf{a}\}\{\beta_n\}A}}{\overset{\frown}{\underset{\smile}{\mathsf{a}}}\{\alpha\}[\mathsf{a}]A \vdash \{\beta_n\}A}}{\overset{\frown}{\underset{\smile}{\mathsf{a}}}\{\alpha\}[\mathsf{a}]A \vdash [\beta_n]A}}{\{\alpha\}[\mathsf{a}]A \vdash \{\mathsf{a}\}[\beta_n]A}}{\{\alpha\}[\mathsf{a}]A \vdash [\mathsf{a}][\beta_n]A}}{\boldsymbol{;}\left(\{\alpha\}[\mathsf{a}]A\right) \vdash \bigwedge\left([\mathsf{a}][\beta_i]A\right)}}{\{\alpha\}[\mathsf{a}]A \vdash \bigwedge\left([\mathsf{a}][\beta_i]A\right)}}{\dfrac{\dfrac{\Phi_\alpha \,;\, \{\alpha\}[\mathsf{a}]A \vdash 1_\alpha \wedge \bigwedge\left([\mathsf{a}][\beta_i]A\right)}{\{\alpha\}[\mathsf{a}]A \vdash 1_\alpha \wedge \bigwedge\left([\mathsf{a}][\beta_i]A\right)}\,red}{\langle\alpha\rangle[\mathsf{a}]A \vdash 1_\alpha \wedge \bigwedge\left([\mathsf{a}][\beta_i]A\right)}}
$$

- $Pre(\alpha) \wedge \bigwedge\{[\mathsf{a}][\beta]A \mid \alpha\mathsf{a}\beta\} \vdash \langle\alpha\rangle[\mathsf{a}]A$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{[\beta_1]A \vdash \{\beta_1\}A}}{[\mathsf{a}][\beta_1]A \vdash \{\mathsf{a}\}\{\beta_1\}A} \quad \cdots \quad \dfrac{\dfrac{A \vdash A}{[\beta_n]A \vdash \{\beta_n\}A}}{[\mathsf{a}][\beta_n]A \vdash \{\mathsf{a}\}\{\beta_n\}A}}{\boldsymbol{;}\left([\mathsf{a}][\beta_i]A\right) \vdash \{\alpha\}\{\mathsf{a}\}A}\,s\text{-}out}{\bigwedge\left([\mathsf{a}][\beta_i]A\right) \vdash \{\alpha\}\{\mathsf{a}\}A}}{\overset{\frown}{\underset{\smile}{\alpha}} \bigwedge\left([\mathsf{a}][\beta_i]A\right) \vdash \{\mathsf{a}\}A}}{\overset{\frown}{\underset{\smile}{\alpha}} \bigwedge\left([\mathsf{a}][\beta_i]A\right) \vdash [\mathsf{a}]A}}{\{\alpha\}\overset{\frown}{\underset{\smile}{\alpha}} \bigwedge\left([\mathsf{a}][\beta_i]A\right) \vdash \langle\alpha\rangle[\mathsf{a}]A}}{\Phi_\alpha \,;\, \bigwedge\left([\mathsf{a}][\beta_i]A\right) \vdash \langle\alpha\rangle[\mathsf{a}]A}}{\bigwedge\left([\mathsf{a}][\beta_i]A\right) ; \Phi_\alpha \vdash \langle\alpha\rangle[\mathsf{a}]A}}{\Phi_\alpha \vdash \bigwedge\left([\mathsf{a}][\beta_i]A\right) > \langle\alpha\rangle[\mathsf{a}]A}}{1_\alpha \vdash \bigwedge\left([\mathsf{a}][\beta_i]A\right) > \langle\alpha\rangle[\mathsf{a}]A}}{\bigwedge\left([\mathsf{a}][\beta_i]A\right) ; 1_\alpha \vdash \langle\alpha\rangle[\mathsf{a}]A}}{1_\alpha ; \bigwedge\left([\mathsf{a}][\beta_i]A\right) \vdash \langle\alpha\rangle[\mathsf{a}]A}}{1_\alpha \wedge \bigwedge\left([\mathsf{a}][\beta_i]A\right) \vdash \langle\alpha\rangle[\mathsf{a}]A}
$$

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