Operations on a Wadge-Type Hierarchy of Ordinal-Valued Functions

MSc Thesis *(Afstudeerscriptie)*

written by

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Contents

Abstract

In this thesis we investigate – under the assumption of the Axiom of Determinacy ($\mathbf{AD}$) – the structure of the *hierarchy of regular norms*, a Wadge-type hierarchy of ordinal-valued functions that originally arose from Moschovakis's proof of the First Periodicity Theorem in descriptive set theory. We introduce the notion of a *very strong better quasi-order* that will provide a framework to treat the hierarchy of regular norms and the well-known Wadge hierarchy uniformly. From this we can establish classical results for both hierarchies with uniform proofs. Among these are the Martin-Monk and the Steel-Van Wesep Theorem. After that we define operations on the hierarchy of regular norms which are used to show closure properties of this hierarchy. Using these closure properties, we can significantly improve the best formerly known lower bound for the order type of the hierarchy of regular norms.

Keywords: Descriptive Set Theory, Order Theory, First Periodicity Theorem, Axiom of Determinacy, Wadge hierarchy

Chapter 1

Introduction and Acknowledgments

1.1 Outline of the Thesis

In this section I will give a short overview about the topics in each section in Chapters 2 to 5 of this thesis as well as an overview over which of the notions and results of this thesis are my original contribution and which ones only serve expositional purposes.

In **Chapter 2** I will introduce concepts and state results which will form the basis for later chapters.

Section 2.1 will treat standard concepts from set theory, the theory of sequences, order theory and topology. Also the axiomatic framework for this thesis will be fixed in this section. The main sources for this section are the book [Kan09] and the book draft [And01].

The aim of **Section 2.2** will be to introduce the notion of infinite games as well as the Axiom of Determinacy. I will introduce the notion of a global play, will be used for the proofs of some of the main results in Chapter 3. Also I will show how the Axiom of Determinacy relates to other commonly used set theoretic axioms. Again the content of this section is mostly standard; it can be found in the book [Kan09]. The notion of a flip set, the result that the Axiom of Determinacy proves the non-existence of flip sets and the concept of a global game, however, are taken from the book draft [And01] and the article [And07].

Finally in **Section 2.3** I will introduce more advanced concepts and results from descriptive set theory, which will lay the groundwork in particular for Chapter 3. The original Wadge hierarchy and the hierarchy of strictly regular norms, which is the same as the hierarchy of regular norms considered in Chapter 4, are introduced in this section.

In **Chapter 3** I will introduce a general framework that will unify the notion of the original Wadge hierarchy and the notion of the hierarchy of regular norms, the main subject of Chapter 4, which we will also introduce here. This unified notion will allow us to uniformly prove classical theorems about both hierarchies.

Section 3.1 will, starting from the original Wadge hierarchy and the hierarchy of strictly regular norms, introduce the general notions of a $(Q, \leq)$-Wadge hierarchy and a $(Q, \leq)$-Lipschitz hierarchy for a general quasi-order $(Q, \leq)$. Then I will elaborate on how this new notion relates to the hierarchies I started from. Also in this section I will introduce the hierarchy of regular norms and show that it is isomorphic to the hierarchy of regular norms. Since it is moreover more amenable to analysis with the methods in Chapter 4, I will from then on only consider the hierarchy of regular norms. The notion of a $(Q, \leq)$-Wadge hierarchy was introduced by Benedikt Löwe in the article [Löw05a] for the special case of linear orders $(Q, \leq)$. The general case for arbitrary quasi-orders $(Q, \leq)$ is considered here for the first time.

In **Section 3.2** I will introduce games to characterize $(Q, \leq)$-Wadge and $(Q, \leq)$-Lipschitz hierarchies and use these to prove – assuming the Axiom of Determinacy – a generalization of Wadge's Lemma, from which Wadge's Lemma for the original Wadge hierarchy and the linearity of the hierarchy of norms follow as corollaries. The content of this section is a generalization of notions and results in the case of the original Wadge hierarchy as they can be found in the book [Kan09] and the book draft [And01], but has not been considered in this general setting before.

In **Section 3.3** I will introduce the notion of a very strong better quasi-order (vsBQO), a strengthening of the notion of a better quasi-order, which I will use to show – under the assumption of the Axiom of Determinacy and a fragment of the Axiom of Choice – that for every vsBQO $(Q, \leq)$ the corresponding $(Q, \leq)$-Wadge and $(Q, \leq)$-Lipschitz hierarchies are well-founded. Furthermore I will re-establish the well-foundedness of the original Wadge hierarchy and the hierarchy of regular norms by showing that suitable well-orders are vsBQOs. Our analysis in the general setting then allows us to realize that the proof for the original Wadge hierarchy uses an extra assumption not needed for the proof of the hierarchy of regular norms: the former uses the assumption "the order with two incomparable elements is a vSBQO" whereas the latter doesn't.[1] The main result of this section is a generalization of a result from the article [vEMS87], in which the authors consider the relationship between better quasi-orders and a hierarchy that is a variant of my notion of a $(Q, \leq)$-Wadge hierarchy.

In **Section 3.4** I will first introduce for any vsBQO $(Q, \leq)$ a generalized notion of the notion of self-duality in the sense of the $(Q, \leq)$-Wadge hierarchy and in the sense of the $(Q, \leq)$-Lipschitz hierarchy, known from the context of the original Wadge hierarchy. I then show how this notion of self-duality can be characterized in the context of the hierarchy of norms. Finally I prove a generalization of the Steel-Van Wesep theorem, which states that – assuming the Axiom of Determinacy – for any better quasi-order the notion of self-duality for the $(Q, \leq)$-Wadge hierarchy exactly coincides with the notion of self-duality for the $(Q, \leq)$-Lipschitz hierarchy. As corollaries I then get the classical Steel-Van Wesep Theorem and a version of this theorem for the hierarchy of norms. My proof of the generalized version of the Steel-Van Wesep Theorem is roughly based on the proof of the Steel-Van Wesep Theorem for the Wadge hierarchy as it can be found in the book draft [And01]. The particular notion of self-duality for the hierarchy of norms was singled out by Benedikt Löwe in unpublished notes [Löw10] based on an analysis of the article [Dup03] by Duparc, in which a variant of the hierarchy of regular norms is investigated.

In **Section 3.5** I will give an alternative characterization of self-duality in the context of $(Q, \leq)$-Lipschitz and $(Q, \leq)$-Wadge hierarchies in terms of well-foundedness of certain trees. This result – in its instance for the hierarchy of norms – will be of technical importance throughout Chapter 4. The proof of this result is very similar to the proof for the corresponding theorem for the original Wadge hierarchy as it can be found in the book draft [And01].

In **Chapter 4** I will analyze the structure of the hierarchy of regular norms in more detail. In light of its well-foundedness as established in Section 3.3 the guiding question will be the question for the value of the order type of the hierarchy of regular norms, which I will denote by Σ. The order type of a regular norm in the hierarchy of regular norms will be called its Wadge rank. For all results noted in the following we will assume the Axiom of Determinacy and possibly some fragment of the Axiom of Choice.

In **Section 4.1** I will recapitulate results from the article [Löw05b], which establish the formerly best known lower and upper bounds for Σ, to be explicit that $\Theta^2 \leq \Sigma < \Theta^+$.

Section 4.2 is roughly composed of two parts. The first part encompasses the content up to Proposition 4.2.9. In this part I introduce the join-operation assigning one regular norm to a countable family of regular norm, show that the join acts as a supremum on the level of Wadge ranks and using this show that a regular norm is self-dual if and only if it has a limit Wadge rank of countable cofinality. The second part of this section encompasses the content after Proposition 4.2.9. In this part I introduce an operation on regular norms that acts on the level of Wadge ranks as a successor operation. Furthermore this operation assigns to every regular norm a with respect to Wadge ranks minimal non-self-dual norm strictly above it. Then I show how to iterate the successor operation just defined in order to get an operation $\varphi \mapsto \varphi^{+\omega}$ on regular norms such that on the level of Wadge ranks this operation corresponds to addition with ω. Then we show that this operation assigns to any regular norm a minimal self-dual regular norm strictly above it, thus showing that there are unboundedly many self-dual regular norms. After that I will give another operation on regular norms acting as addition with ω on the level of Wadge rank for non-self-dual degrees. This restriction to non-self-dual degrees, however, will make it possible to give a less complex description of this operation when compared to the operation $\varphi \mapsto \varphi^{+\omega}$ of this section.

In **Section 4.3** I will endeavor to provide a lower bound for Σ improving the lower bound from Section 4.1 considerably. For the idea I first note that by a result from Section 4.1 we can consider Σ as a limit of ordinals Σ_α with $\alpha < \Theta$, which are the order types of certain well-behaved initial segments $\mathcal{N}_\alpha$ of the

[1] In the article [Löw05a] Benedikt Löwe considers an analogue of the hierarchy of regular norms defined via Blackwell games, i.e., games with imperfect information, instead of perfect information games as considered in this thesis. He establishes that this Blackwell hierarchy of norms is well-founded. However, the analogous statement for the original Wadge hierarchy is still an open problem. Löwe notes that this difficulty to establish well-foundedness of the Blackwell variant of the original Wadge hierarchy is connected to the additional assumption needed to establish well-foundedness of the original Wadge hierarchy.

hierarchy of norms. Then I note that by a result from Section 4.2 there are unboundedly many self-dual norms in each $\mathcal{N}_\alpha$. Now fixing some $\alpha < \Theta$ and proceed as follows.

In **Subsection 4.3.1** I construct an operation $(\varphi, \psi) \mapsto \varphi \dotplus \psi$ on $\mathcal{N}_\alpha$ such that for any self-dual regular norm φ in $\mathcal{N}_\alpha$ and any regular norm ψ in $\mathcal{N}_\alpha$ I get, denoting Wadge ranks by $|\cdot|_W$

$$|\varphi \dotplus \psi|_W = |\varphi|_W + 1 + |\psi|_W$$

and such that the $\dotplus$-operation is monotone in both arguments. But then, since there are unboundedly many self-dual regular norms in $\mathcal{N}_\alpha$, I get as an immediate corollary that Σ_α is closed under addition.

In **Subsection 4.3.2** I construct an operation $\varphi \mapsto \varphi^\natural$ on $\mathcal{N}_\alpha$ such that for any self-dual regular norm φ in $\mathcal{N}_\alpha$ I get that

$$|\varphi^\natural|_W = |\varphi|_W \cdot \omega_1$$

and such that this operation is monotone with respect to Wadge degrees. In particular I get an explicit description of a regular norm of Wadge rank ω_1. Together with the $\dotplus$-operation constructed in Subsection 4.3.1 I then get an explicit operation on $\mathcal{N}_\alpha$ corresponding to addition with ω_1, which is one way to see that in $\mathcal{N}_\alpha$ there are unboundedly many regular norms φ whose Wadge rank is a limit ordinal of uncountable cofinality.

In **Subsection 4.3.3** I then construct an operation $(\varphi, \psi) \mapsto \varphi \odot \psi$ on $\mathcal{N}_\alpha$ such that for any self-dual regular norm φ in $\mathcal{N}$ and any regular norm ψ in $\mathcal{N}$, whose Wadge rank is a limit ordinal of uncountable cofinality, we have that

$$|\varphi \odot \psi|_W = |\varphi|_W \cdot \omega_1 \cdot |\psi|_W$$

and such that – at least as long as the second argument is non-self-dual – this operation is monotone in both arguments. But since it was already established in the last sections that there are unboundedly many self-dual regular norms in $\mathcal{N}_\alpha$ as well as unboundedly many regular norms with limit Wadge ranks of uncountable cofinalities, it follows that each Σ_α is closed under multiplication.

In **Subsection 4.3.4** I then show that it is possible to iterate the $\odot$-operation to obtain that each Σ_α is closed under exponentiation with γ for any $\gamma < \Theta$. Based on this I establish that $\Sigma_\alpha \geq \Theta^{(\Theta^\alpha)}$ for any $\alpha < \Theta$ and so that $\Sigma \geq \Theta^{(\Theta^\Theta)}$, which is an improved lower bound for Σ.

In **Chapter 5** I finally pose some questions which are related to but left open by this thesis and could lead to further work on this topic in the future.

1.2 Original Contribution

In this section I list the main original contributions of this thesis. The first such original contribution is the notion of a vSBQO introduced in Chapter 3, which gives us an abstract framework to uniformly prove classical theorems for the original Wadge hierarchy and the hierarchy of regular norms. In particular working in this abstract framework in Section 3.3 I give an explanation for why the Martin-Monk Theorem for the original Wadge hierarchy seems to use a bigger fragment of the Axiom of Determinacy compared to the Martin-Monk theorem for the hierarchy of norm. Also the general notion of self-duality in this abstract setting as introduced in Section 3.4 is original[2] as is the statement and proof of Theorem 3.4.4, which is a general version of the classical Steel-Van Wesep Theorem.

Now I get to the original contributions in Chapter 4. Three new notions are the operations $\varphi \mapsto \varphi^{\mathrm{succ}}$, $\varphi \mapsto \varphi^{+\omega}$ and $\varphi \mapsto \varphi^\vee$ that are introduced in Section 4.2. The first of these corresponds to the successor operation on Wadge ranks, while the other two correspond to addition with ω. All three operations play a central role in the calculation of the action of the operation $(\varphi, \psi) \mapsto \varphi \odot \psi$ on the level of Wadge degrees in Subsection 4.3.3, which leads to Theorem 4.3.40. This theorem in turn is an original result. The last main original contribution of this thesis is Theorem 4.3.43, which provides the best currently known lower bound for the order type of the hierarchy of regular norms.

[2]After the defense of this thesis, but before the submission of a final version for online publication, I found that this notion of self-duality was actually already present in the article [LSR90], although in a slightly different context and with a very different purpose compared to this thesis.

1.3 Acknowledgments

It was a rather long road to my graduation from the Master's programme in Logic, but fortunately there were several people who by intent or by accident helped me along the way. First I have to thank Benedikt Löwe who not only supervised my thesis, but also has been my first teacher in mathematical logic back when I was still a Bachelor's student at the University of Hamburg. Also throughout my studies he was always eager to lend me help and support in academic matters whenever I needed it; among other things he was the one who originally pointed me to the possibility to study in Amsterdam to specialize in logic as an interdisciplinary field. During the preparation of my Master's thesis he took a lot of time to discuss my research and no matter how many ideas I had, he always pointed to yet another interesting direction of research. Shortly before I handed in my Master's thesis he furthermore sacrificed a part of his winter holiday to read the (pre-)final version of my thesis, although at that point I was already later with providing a stable version than we originally agreed on.

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Last but definitely not least I thank my parents and the "Studienstiftung des deutschen Volkes" who made my stay in Amsterdam financially possible.

Chapter 2

Prerequisites

2.1 Basic Notation and Results

2.1.1 Axiomatic Framework and Set Theory

The basic axiomatic framework throughout this thesis will be $\mathbf{ZF}$, i.e., Zermelo-Fraenkel set theory without the Axiom of Choice, short $\mathbf{AC}$. A list of the axioms of this theory can be found in the book [Jec03]. Whenever we will assume any set theoretic principles surpassing $\mathbf{ZF}$, we will explicitly note it.

Our set theoretic notation will be standard. In particular we will for given sets X and Y use $\mathcal{P}(X)$ to denote the power set of X, $X \setminus Y$ to denote the difference between X and Y, X^Y to denote the set of all functions with domain Y and range X and χ_X to denote the characteristic function of a set X. For the last notation we note that we will also use the letter χ, possibly with subscripts, in other contexts. However, the intended meaning will always be clear from the context.

For any two sets X, Y, any function $f \in X^Y$ and any subset $X' \subseteq X$ and $Y' \subseteq Y$ we define the *image of X' under f* as

$$f[X'] := \{f(x) \mid x \in X'\}$$

and the *preimage of Y' under f* as

$$f^{-1}[Y'] := \{x \in X \mid \exists y \in Y' \, (f(x) = y)\}.$$

As usual we say that a set is an ordinal iff it is transitive and linearly ordered by the $\in$-relation. We will use lower case Greek letters except for φ, ψ, χ and ω, possibly with various sub- and superscripts, to refer to arbitrary ordinals and will write $\alpha < \beta$ for two ordinals α, β to refer to $\alpha \in \beta$. Then $\alpha \leq \beta$, $\alpha > \beta$ and $\alpha \geq \beta$ are defined in the usual way.

We let ω be the smallest infinite ordinal and refer to its elements as natural numbers. We will refer to natural numbers by lower case letters from the middle of the latin alphabet, i.e., $m, n, k, \ell, \ldots$, again possibly with various sub- and superscripts.

For α an ordinal and $P(x)$ a first-order formula of set theory we write $\forall^\infty \beta < \alpha \, (P(\beta))$ to abbreviate $\exists \gamma \in \alpha \forall \beta > \gamma \, (\beta < \alpha \to P(\beta))$. Furthermore we write $\exists^\infty \beta \in \alpha \, (P(\beta))$ to abbreviate $\forall \gamma \in \alpha \, \exists \beta > \gamma \, (\beta < \alpha \wedge P(\beta))$. In the special case of the ordinal ω we spell out $\forall^\infty n \in \omega \, (P(n))$ as "Cofinitely many $n \in \omega$ have the property $P(n)$" and $\exists^\infty n \in \omega \, (P(n))$ as "There are infinitely many $n \in \omega$ with the property $P(n)$". It is clear by definition that $\forall^\infty$ and $\exists^\infty$ are dual in the sense that $\neg \exists^\infty \beta \in \alpha \, (P(\beta))$ is equivalent to $\forall^\infty \beta \in \alpha \, (\neg P(\beta))$.

We refer to the class of ordinals by $\mathbf{Ord}$. We define an operation $S : \mathbf{Ord} \to \mathbf{Ord}, \alpha \mapsto \alpha \cup \{\alpha\}$, which assigns to every ordinal its successor. We call an ordinal α a successor ordinal iff there is an ordinal β such that $\alpha = S(\beta)$. We call an ordinal α a limit ordinal iff $\alpha \neq 0$ and α is not a successor ordinal.

One of the most important features of ordinals is that we can define functions on ordinals by recursion. Using recursion we will define arithmetical operations on the class of ordinals. For any two ordinals α, β we define their ordinal sum $\alpha + \beta$ by setting

$$\alpha + \beta := \begin{cases} \alpha, & \text{if } \beta = 0, \\ S(\alpha + \gamma), & \text{if } \beta \text{ is successor ordinal and } \beta = \gamma + 1, \\ \bigcup_{\beta' < \beta} (\alpha + \beta'), & \text{if } \beta \text{ is a limit ordinal.} \end{cases}$$

For any two ordinals α, β we define their ordinal product $\alpha \cdot \beta$ by setting

$$\alpha \cdot \beta := \begin{cases} 0, & \text{if } \beta = 0, \\ (\alpha \cdot \gamma) + \alpha, & \text{if } \beta \text{ is successor ordinal and } \beta = \gamma + 1, \\ \bigcup_{\beta' < \beta} (\alpha \cdot \beta'), & \text{if } \beta \text{ is a limit ordinal.} \end{cases}$$

For any two ordinal α, β we define their ordinal exponentiation α^β by setting

$$\alpha^\beta := \begin{cases} 0, & \text{if } \beta = 0, \\ (\alpha^\gamma) \cdot \alpha, & \text{if } \beta \text{ is successor ordinal and } \beta = \gamma + 1, \\ \bigcup_{\beta' < \beta} (\alpha^{\beta'}), & \text{if } \beta \text{ is a limit ordinal.} \end{cases}$$

A β-sequence of ordinals is a function $f : \beta \to \mathbf{Ord}$. Such a β-sequence f we often denote by $\langle f(\alpha) \mid \alpha < \beta \rangle$. We call a β-sequence f of ordinals strictly increasing iff for all $\alpha, \alpha' < \beta$ we have that $\alpha < \alpha'$ implies $f(\alpha) < f(\alpha')$. Let X be a set of ordinals. Then we call a β-sequence f an enumeration of X iff it is strictly increasing and $f[\beta] = X$. Let λ be a limit ordinal; a β-sequence f of ordinals is cofinal in λ iff it is strictly increasing, for all $\gamma < \beta$ we have $f(\gamma) < \lambda$ and for all $\beta < \lambda$ there is $\gamma < \beta$ such that $\beta < f(\gamma)$. For a limit ordinal λ we define an ordinal $\mathrm{cf}(\lambda)$, called the cofinality of λ, to be the least $\beta \leq \lambda$ such that there is a β-sequence of ordinals cofinal in λ. For any limit ordinal λ we have that $\mathrm{cf}(\lambda) \leq \lambda$ and that $\mathrm{cf}(\lambda)$ is a limit ordinal (see [Jec03]). We call a limit ordinal λ *regular* iff $\mathrm{cf}(\lambda) = \lambda$. Otherwise we call it *singular*.

An ordinal β is a cardinal if and only if for all $\alpha < \beta$ there is no injection $f : \beta \to \alpha$. We call a set X *countable* if and only if it is empty or there is a surjection $s : \omega \to X$. Otherwise we call it *uncountable*. We denote by ω_1 the smallest uncountable cardinal, which exists by Hartog's Theorem (see [Jec03]). More generally by the same theorem for any cardinal κ there is a least cardinal strictly greater than κ and we denote it by κ^+.

In the following we will assemble some rules of ordinal arithmetic that we are going to use. Most of them can be found in [Kun11, Table I.1] and in [Jec03].

Lemma 2.1.1. *Let α, β, γ be arbitrary ordinals and let $\lambda \geq \omega$. Then we have:*

- $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.
- $\left(\alpha^\beta\right)^\gamma = \alpha^{\beta \cdot \gamma}$.
- $1 + \lambda = \lambda$.
- $\omega \cdot \omega_1 = \omega_1$.
- $(\alpha + \omega) \cdot \omega_1$.
- $\alpha \cdot \omega_1 \cdot \beta + \omega + \alpha + \alpha \cdot \omega_1 \cdot \gamma = \alpha \cdot \omega_1 \cdot (\beta + \gamma)$.

For any subset X of an ordinal α there indeed is a unique cardinal κ such that X and α are in bijection. We denote this cardinal the cardinality of X and denote it by $\mathrm{card}(X)$. However, we note that in absence of $\mathbf{AC}$ it is for arbitrary sets in general not true that there is a unique cardinal in bijection with it.

For any set X we denote by $[X]^\omega$ the set of countable subsets of X. Thus in particular $[\omega]^\omega$ is the set of infinite subsets of ω.

Now we will look at fragments of $\mathbf{AC}$ that we will employ in this thesis. For any set X we let $\mathbf{AC}_\omega(X)$, the Axiom of Countable Choices on X, be the statement that for every function $f : \omega \to \mathcal{P}(X) \setminus \{\emptyset\}$ there is a function $f : \omega \to X$ such that for all $n \in \omega$ we have that $f(n) \in F(n)$. We let $\mathbf{AC}_\omega$, the Axiom of Countable Choice, be the axiom expressing that $\mathbf{AC}_\omega(X)$ holds for all sets X. Next for any nonempty set X we let $\mathbf{DC}(X)$, the Axiom of Dependent Choices on X, be the statement

$$\forall R \subseteq X \times X \; [\forall x \in X \, \exists y \in X \, (\langle x, y \rangle \in R) \Rightarrow \exists f \in X^\omega \, \forall n \in \omega \, (\langle f(n), f(n+1) \rangle \in R)].$$

Then $\mathbf{DC}$ is the statement asserting that for any non-empty set X we have that $\mathbf{DC}(X)$ holds.

If for two sets X, Y there is a surjection $f : X \to Y$, then we have that $\mathbf{AC}_\omega(X)$ implies $\mathbf{AC}_\omega(Y)$ and that $\mathbf{DC}(X)$ implies $\mathbf{DC}(Y)$. Furthermore we have that $\mathbf{DC}$ implies $\mathbf{AC}_\omega$.

Now we give two basic set theoretic results – a version of the Pigeonhole Principle and of Cantor's Normal Form Theorem – which are provable in $\mathbf{ZF}$ alone.

Lemma 2.1.2 (Pigeonhole Principle for Successor Cardinals)**.** *If κ is an infinite cardinal, then for every function $f : \kappa^+ \to \kappa$ there is a set S and an ordinal $\alpha < \kappa$ such that $f[S] = \{\alpha\}$.*

The proof for this particular instance of the pigeonhole principle can be found in the article [Löw05b, Lemma 2.2].

To state Cantor's Normal Form Theorem we define for any $k \in \omega$ and any sequence $\langle \nu_m \mid m < k \rangle$ of ordinals an ordinal $\sum_{m<k} \nu_m$ by setting recursively on m:

- $\sum_{m \leq 0} \nu_m := \nu_0$.
- For any $k \in \omega$: $\sum_{m \leq k+1} \nu_m := \left(\sum_{m \leq k} \nu_m \right) + \nu_{k+1}$.

Using this we now state the Cantor Normal Form Theorem as follows:

Lemma 2.1.3 (Cantor's Normal Form Theorem for basis ω_1)**.** *For any ordinal α with $\omega_1 \leq \alpha$ there are unique $k \in \omega$ as well as unique sequences $\langle \xi_m \mid m \leq k \rangle$ and $\langle \eta_m \mid m \leq k \rangle$ of ordinals such that for all $m < k$ we have that $\xi_m > \xi_{m+1}$ and for all $m \leq k$ we have that $0 < \eta_m < \omega_1$ and such that*

$$\alpha = \sum_{m \leq k} \omega_1^{\xi_m} \cdot \eta_m.$$

The Cantor Normal Form Theorem for basis ω and its proof can be found in [Jec03, Theorem 2.26]. The proof of Cantor's Normal Form Theorem for basis ω_1 is completely analogous.

2.1.2 Sequences

For any set X we have that X^ω is the set of functions with domain ω and range X. We call X^ω the set of infinite sequences on X. Furthermore we define $X^{<\omega} := \bigcup X^n$ and call this set the set of finite sequences on X. Finally we define $X^{\leq\omega} := X^\omega \cup X^{<\omega}$ and call this the set of sequences on X.

Following common set-theoretic practice we will denote ω^ω, the set of infinite sequences of natural numbers, by $\mathbb{R}$ and refer to its elements as reals.

Let X be an arbitrary set. For any $x \in X^{\leq\omega}$ we define $\mathrm{lh}(x)$, the length of x, to be the domain of x. For $x \in X^{\leq\omega}$ and $n < \mathrm{lh}(x)$ we sometimes write x_n to refer to $x(n)$ in cases where this cannot lead to confusion. For $x \in X^\omega$ we denote by x^+ the shift of x, which is the unique element of X^ω defined by setting for all $n \in \omega$

$$x^+(n) = x(n+1).$$

Although it follows standard set theoretic notation we note that $\emptyset$ denotes the empty sequence. Also for two $x, y \in X^{\leq\omega}$ we have that $x \subseteq y$ iff $\mathrm{lh}(x) \leq \mathrm{lh}(y)$ and for all $n < \mathrm{lh}(x)$ we have that $x(n) = y(n)$. For $x \in X^{\leq\omega}$ and $m < \mathrm{lh}(x)$ we have that $x\restriction_m$ is the unique sequence on x of length m such that for all $n < m$ we have that $x\restriction_m(n) = x(n)$. Now given a set $\Gamma \subseteq X^{\leq\omega}$ such that for any two $x, y \in \Gamma$ we have that either $x \subseteq y$ or $y \subseteq x$ we use $\lim \Gamma$ to denote to the unique $z \in X^{\leq\omega}$ of minimal length such that for all $x \in \Gamma$ we have that $x \subseteq z$.

For $s \in X^{<\omega}$ and $t \in X^{\leq\omega}$ we define $s \frown t$, the concatenation of s and t, to be a sequence of length $\mathrm{lh}(s) + \mathrm{lh}(t)$ such that for all $n < \mathrm{lh}(s)$ we have $(s \frown t)(n) = s(n)$ and for any $n < \mathrm{lh}(s) + \mathrm{lh}(t)$ such that $\mathrm{lh}(s) \leq n$ we have that $(s \frown t)(n) = t(n - \mathrm{lh}(s))$. Then for any $x \in X^{\leq\omega}$ and any $s \in X^{<\omega}$ such that $s \subseteq x$ we let $x \setminus s$ be the unique sequence on X such that $x = s \frown (x \setminus s)$. We call an $x \in X^{\leq\omega}$ an initial segment of $y \in X^{\leq\omega}$ iff $x \subseteq y$. We call an $x \in X^{\leq\omega}$ an end-segment of $y \in X^{\leq\omega}$ iff there is an $s \in X^{<\omega}$ such that $y = s \frown x$.

For any two $x, y \in X^\omega$ we define their interleaving $x * y$ to be the unique infinite sequence on X such that for all $n \in \omega$ we have that $(x * y)(2n) = x(n)$ and $(x * y)(2n+1) = y(n)$.

If X is a set of the form $X = Y \times Z$, then we define the projection $p_{\mathbf{I}} : X^{\leq\omega} \to Y^{\leq\omega}$ by setting for any $x \in X^{\leq\omega}$ and any $n < \mathrm{lh}(x)$: If $x(n) = \langle a, b \rangle$, then $p_{\mathbf{I}}(x)(n) = a$. Analogously we define the projection $p_{\mathbf{II}} : X^{\leq\omega} \to Z^{\leq\omega}$ by setting for any $x \in X^{\leq\omega}$ and any $n < \mathrm{lh}(x)$: If $x(n) = \langle a, b \rangle$, then $p_{\mathbf{II}}(x)(n) = b$.

Abusing notation, if X is not of the form $X = Y \times Z$ for two other sets, then we define projections $p_{\mathbf{I}}, p_{\mathbf{II}} : X^\omega \to X^\omega$ by letting for all $x \in X^\omega$ the two infinite sequences $p_{\mathbf{I}}(x)$ and $p_{\mathbf{II}}(x)$ over X be the unique ones such that $x = p_{\mathbf{I}}(x) * p_{\mathbf{II}}(x)$.

For an arbitrary $a \in X$ we define $a^{(n)}$ for $n \in \omega \cup \{\omega\}$ to be the sequence with $\mathrm{lh}(a^{(n)}) = n$ such that for all $m < n$ we have $a^{(n)}(m) = a$.

If X is a set such that $\omega \subseteq X$, then we define for any $x \in X^{\leq\omega}$ a sequence $(x+1) \in X^{\leq\omega}$ of the same length as x by setting for all $n < \mathrm{lh}(x)$

$$(s+1)(x) = \begin{cases} s(x)+1, & \text{if } s(x) \in \omega, \\ s(x), & \text{otherwise.} \end{cases}$$

Furthermore for any $x \in X^{\leq\omega}$ such that for all $n < \mathrm{lh}(x)$ we have that $x(n) \neq 0$ we define a sequence $(x-1) \in X^{\leq\omega}$ of the same length as x by setting for all $n < \mathrm{lh}(x)$

$$(s-1)(x) = \begin{cases} s(x)-1, & \text{if } s(x) \in \omega \setminus \{0\}, \\ s(x), & \text{otherwise.} \end{cases}$$

Given any two sets X, Y and a function $\sigma : X^{<\omega} \to Y$ we define a function $\overline{\sigma} : X^{\leq\omega} \to Y^{\leq\omega}$, *the lift of* σ recursively as follows; we set $\overline{\sigma}(\emptyset) = \emptyset$; for any $s \in X^{<\omega}$ and $n \in \omega$ we set $\overline{\sigma}(s ^\frown \langle n\rangle) = \overline{\sigma}(s) ^\frown \langle \sigma(s ^\frown \langle n\rangle)\rangle$; finally for any $x \in X^{\omega}$ we set $\overline{\sigma}(x) = \lim\{\overline{\sigma}(x{\restriction}_n) \mid n \in \omega\}$. Completely analogously we define for a function $\tau : X^{<\omega} \setminus \{\emptyset\} \to Y$ the lift $\overline{\tau} : X^{\leq\omega} \to Y^{\leq\omega}$ of τ.

Finally for any set X, any subset $A \subseteq X^{\omega}$ and any $s \in X^{<\omega}$ we set $s ^\frown A := \{y \in X^{\omega} \mid \exists x \in A\,(y = s ^\frown y)\}$.

2.1.3 Order Theory

A quasi-ordering is a binary relation $(\leq) \subseteq Q \times Q$ on some set Q that is reflexive and transitive. Then we call the structure $(Q, \leq)$ a quasi-order. We will usually denote quasi-orderings by $\leq_a^b$, where a and b are arbitrary sub- and superscripts. Given such a quasi-order $(Q, \leq_a^b)$ we will defined associated binary relations $<_a^b, >_a^b, \geq_a^b, \perp_a^b, \equiv_a^b$ by setting for all $q, q' \in Q$:

$$\begin{aligned} q <_a^b q' \quad &:\Leftrightarrow \quad q \leq_a^b q' \text{ and } q' \not\leq_a^b q, \\ q \geq_a^b q' \quad &:\Leftrightarrow \quad q' \leq_a^b q, \\ q >_a^b q' \quad &:\Leftrightarrow \quad q' <_a^b q, \\ a \perp_a^b q' \quad &:\Leftrightarrow \quad q \not\leq_a^b q' \text{ and } q' \not\leq_a^b q, \\ a \equiv_a^b q' \quad &:\Leftrightarrow \quad q \leq_a^b q' \text{ and } q' \leq_a^b q. \end{aligned}$$

For any quasi-ordering $\leq$ the relation $\equiv$ is an equivalence relation. For a set Q and an equivalence relation $\equiv$ we denote the set of $\equiv$-equivalence classes for Q by $Q/_{\equiv}$. The equivalence class for a given element $q \in Q$ will then be denoted by $[q]_{\equiv}$. Then for $(Q, \leq)$ a quasi-order we can define a relation $\leq'$ on $Q/_{\equiv}$ by setting for all $q, q' \in Q$:

$$[q]_{\equiv} \leq' [q']_{\equiv} \quad \Leftrightarrow \quad q \leq q'.$$

Then the structure $(Q/_{\equiv}, \leq')$ is a partial order. Abusing notation we will denote $\leq'$ just by $\leq$ from now on and say that $(Q/_{\equiv}, \leq)$ is a partial order.

Given two quasi-orders $(Q, \leq)$ and $(Q', \leq')$. Then we call $(Q', \leq')$ *a substructure of* $(Q, \leq)$ iff there is a subset $X \subseteq Q$ such that $(Q', \leq') = \big(X, \leq{\restriction}_{X \times X}\big)$. Abusing notation we will from now on refer to quasi-orders of the form $\big(X, \leq{\restriction}_{X \times X}\big)$ simply as $(X, \leq)$, whenever no confusion is to be expected.

Let $(Q, \leq)$ be a quasi-order. Then we call an element $q \in Q$ *minimal* if and only if for all $q' \in Q$ with $q' \leq q$ we have that $q' \equiv q$. We call an element $q \in Q$ *maximal* if and only if for all $q' \in Q$ with $q \leq q'$ we have that $q \equiv q'$.

We call a quasi-ordering $\leq$ on some set Q *well-founded* if and only if for every non-empty subset $X \subseteq Q$ the quasi-order $(X, \leq)$ has a minimal element. Otherwise we call it *ill-founded*. We call a well-founded linear order $(L, \leq)$ a *well-order*. We call a quasi-order $(Q, \leq)$ a *pre-well-order* if and only if $(Q/_{\equiv}, \leq)$ is a well-order. For $(W, \leq)$ a well-order and $X \in \mathcal{P}(W) \setminus \{W\}$ we let $\sup X$, the *supremum of* X, be the unique minimal $w \in W$ such that for all $x \in X$ we have that $x \leq w$; if we even have $x \in X$, then we also write $\max X$ for this. Analogously for any $X \in \mathcal{P}(W)$ we define $\min X$ to be the unique minimal element in X.

Given a well-founded quasi-order $(Q, \leq)$ we define an ordinal $\|q\|_{\leq}$, its *rank in* $(Q, \leq)$ recursively by setting

$$\|q\|_{\leq} := \sup\{\|q'\|_{\leq} \mid q' < q\}.$$

Then we define $\mathrm{otyp}(Q, \leq)$, the order-type of Q to be

$$\mathrm{otyp}(Q, \leq) := \sup\{\|q\|_{\leq} + 1 \mid q \in Q\},$$

where here and in the definition of $\|q\|_{\leq}$ the supremum is taken in the class of ordinals.

We call a quasi-order $(Q, \leq)$ a *well-quasi-order*, short *WQO*, if and only if for any infinite sequence $\langle q_i \mid i \in \omega \rangle \in Q^\omega$ there are two $i, j \in \omega$ such that $i < j$ and $q_i \leq q_j$.

Given a quasi-order $(Q, \leq)$ we call a subset $A \subseteq Q$ an *antichain* if and only if for all $a, b \in A$ we have that $a \perp b$. Furthermore we call a sequence $\langle q_i \mid i \in \omega \rangle \in Q^\omega$ *stricly $\leq$-decreasing* if and only if for all $i, j \in \omega$ with $i < j$ we have that $q_j < q_i$.

We then see that any WQO $(Q, \leq)$ has the property that any antichain $A \subseteq Q$ is finite and that there is no strictly decreasing infinite sequence over Q. If we have the full Axiom of Choice at our disposal we can show that a quasi-order is well-founded if and only if it has no strictly decreasing infinite sequence. However, in $\mathbf{ZF}$ alone, we can only show that a well-founded quasi-order has no strictly decreasing infinite sequence. We can show the other direction, too, if we assume the axiom $\mathbf{DC}$. Thus assuming $\mathbf{DC}$ we get that every WQO is a well-founded quasi-order.

Again let a quasi-order $(Q, \leq)$ be given. In analogy to strictly $\leq$-decreasing sequences we call a sequence $\langle q_i \mid i \in \omega \rangle \in Q^\omega$ *strictly $\leq$-increasing* if and only if for all $i, j \in \omega$ with $i < j$ we have that $q_i < q_j$. We say that a subset $Q' \subseteq Q$ is $\leq$-unbounded if and only if there is no $\leq$-maximal element in Q.

Let $(Q, \leq)$ and $(Q', \leq')$ be two quasi-orders. Then we call a map $f : Q \to Q'$ monotone if and only if for all $q, r \in Q$ we have that $q \leq r$ implies that $f(q) \leq' f(r)$. We call a map $f : Q \to Q'$ order-preserving if and only if we have for all $q, r \in Q$ that $q \leq r$ holds if and only if $f(q) \leq' f(r)$ holds. We say that $(Q, \leq)$ *embeds into* $(Q', \leq')$ if and only if there is an injective order preserving map $f : Q \to Q'$, which we call an *embedding*. We say that $(Q, \leq)$ *is isomorphic to* $(Q', \leq')$ if and only if there is a bijective order preserving map $f : Q \to Q'$, which we call an *isomorphism*. Two isomorphic quasi-orders have the same order theoretic properties.

Let $(W, \leq)$ be a well-order. Then we call a set $X \subseteq W$ an initial segment of W if and only if there is $w \in W$ such that $X = \{w' \in W \mid w' < w\}$. Given two well-orders $(W, \leq)$ and $(W', \leq')$ we note that if there is an isomorphism $f : W \to W'$ between W and an initial segment of W', then $(W, \leq)$ and $(W', \leq')$ are not isomorphic.

Finally concluding this subsection we introduce a particularly important partial order, namely the the partial order $(\mathbf{TV}, \leq_{\mathbf{TV}})$ of truth values, which is defined by setting $\mathbf{TV} = \{0, 1\}$ and $0 \not\leq_{\mathbf{TV}} 1$ and $1 \not\leq_{\mathbf{TV}} 0$.

2.1.4 Trees and Topology

In this subsection we use $\mathcal{R}$ to denote the real numbers in the sense of real analysis, defined via Dedekind cuts.

For any set X we endow the set X^ω with a metric $d : X^\omega \times X^\omega \to \mathcal{R}$ by setting for any two $a, b \in X^\omega$

$$d(a, b) := \begin{cases} 0, & \text{if } a = b, \\ 2^{-\min\{n \in \omega \mid a(n) \neq b(n)\}}, & \text{otherwise.} \end{cases}$$

We call this metric the *standard metric for X^ω*. Special cases of this then are the metric spaces $(\mathbb{R}, d)$, the *Baire space*, and $(2^\omega, d)$, the *Cantor space*, where by abuse of notation in both cases we denote by d the respective standard metric as just defined.

For any set X and any $s \in X^{<\omega}$ we have that the set

$$\mathbf{N}_s := \{a \in X^\omega \mid s \subseteq a\} = \{a \in X^\omega \mid d\left(a, s^\frown 0^{(\omega)}\right) < 2^{-\mathrm{lh}(s)+1}\}$$

is open in (X^ω, d).

Next we note that there is a bijection $b : \mathcal{P}(\omega) \to 2^\omega, A \mapsto \chi_A$ and so we can define a metric d' on $\mathcal{P}(\omega)$ by setting for any two $A, B \in \mathcal{P}(\omega)$

$$d'(A, B) \quad :\Leftrightarrow \quad d(\chi_A, \chi_B).$$

Then in particular for any $i \in \omega$ the set $\{X \in \mathcal{P} \mid \forall n < i(n \notin X) \wedge i \in X\}$ is open in $(\mathcal{P}(\omega), d')$, since if $s \in 2^\omega$ is such that $\mathrm{lh}(s) = i + 1$ and for all $n < i$ we have $s(n) = 0$, but $s(i) = 1$, then we get that

$$b[\{X \in \mathcal{P} \mid \forall n < i(n \notin X) \wedge i \in X\}] = N_s.$$

Now since $[\omega]^\omega \subseteq \mathcal{P}(\omega)$ and we can endow the set $[\omega]^\omega$ with the metric $d'\restriction_{[\omega]^\omega \times [\omega]^\omega}$, which we notationally identify with d'. Now whenever we refer to $[\omega]^\omega$ as a metric space, we mean $([\omega]^\omega, d')$. Using what we have just noted about $(\mathcal{P}(\omega), d')$ we obtain the following fact:

Lemma 2.1.4. *For any $i \in \omega$ we have that the set*

$$\{X \in [\omega]^\omega \mid \forall n < i(n \notin X) \wedge i \in X\}$$

is open in $[\omega]^\omega$.

We call a function $g : \mathbb{R} \to \mathbb{R}$ *continuous*, if it is continuous in the sense of the metric space $(\mathbb{R}, d)$. We call a function $f : \mathbb{R} \to \mathbb{R}$ *Lipschitz* if and only if for any two $x, y \in \mathbb{R}$ we have that $d(f(x), f(y)) \leq d(x, y)$.

An important feature of continuous and Lipschitz function is that we can approximate them by functions on finite sequences as follows. We call a function $h : \omega^{<\omega} \to \omega^{<\omega}$ *monotone* if and only if it is monotone in the sense of the partial order $(\omega^{<\omega}, \subseteq)$. We call a function $h : \omega^{<\omega} \to \omega^{<\omega}$ *infinitary* if and only if for any $x \in \mathbb{R}$ and any $n \in \omega$ there is some $m \in \omega$ such that $\mathrm{lh}(h(x\restriction_m)) \geq n$. We call a function $h : \omega^{<\omega} \to \omega^{<\omega}$ *strictly infinitary* if and only if for any $s \in \omega^{<\omega}$ we have that $\mathrm{lh}(h(s)) = \mathrm{lh}(s)$.

For a monotone function $h : \omega^{<\omega} \to \omega^{<\omega}$ and a function $f : \mathbb{R} \to \mathbb{R}$ we say that f *is induced by* h if and only if for all $x \in \mathbb{R}$ we have that

$$f(x) = \lim\{h(x\restriction_n) \mid n \in \omega\}.$$

Then we get that every monotone and infinitary function $h : \omega^{<\omega} \to \omega^{<\omega}$ induces a unique continuous function $g : \mathbb{R} \to \mathbb{R}$. Conversely for every continuous function $g : \mathbb{R} \to \mathbb{R}$ there is a monotone and infinitary function $h : \omega^{<\omega} \to \omega^{<\omega}$ inducing it.

Furthermore every monotone and strictly infinitary function $h : \omega^{<\omega} \to \omega^{<\omega}$ induces a unique Lipschitz function $f : \mathbb{R} \to \mathbb{R}$. Conversely for every Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ there is a monotone and strictly infinitary function $h : \omega^{<\omega} \to \omega^{<\omega}$ inducing it.

On a related note we have surjections from $\mathbb{R}$ onto the set of continuous functions $g : \mathbb{R} \to \mathbb{R}$ and from $\mathbb{R}$ onto the set of continuous function $f : \mathbb{R} \to \mathbb{R}$. We will by $\mathbf{g}_x$ denote the continuous function that is the image of $x \in \mathbb{R}$ under the former surjection and by $\mathbf{f}_x$ the Lipschitz function that is the image of $x \in \mathbb{R}$ under the latter surjection.

As important examples we note that the projection functions $\varphi_{\mathbf{I}}, \varphi_{\mathbf{II}} : \mathbb{R} \to \mathbb{R}$ as well as any function $f : \mathbb{R} \to \mathbb{R}$ such that there is $y \in \mathbb{R}$ such that for all $x \in \mathbb{R}$ we have $f(x) = y * x$ are continuous functions.

We call a subset $T \subseteq \omega^{<\omega}$ a tree if and only if $\emptyset \in T$ and for all $t \in T$ and any $s \in \omega^{<\omega}$ with $s \subseteq t$ we have $s \in T$. We call a tree T *well-founded* if and only if the partial order $(T, \supseteq)$ is well-founded. Otherwise we call it *ill-founded*. We call a tree T *pruned* if for every $t \in T$ there is an $n \in \omega$ such that $t^\frown \langle n \rangle \in T$.

Given a tree T we call an $x \in \mathbb{R}$ a *path through* T if and only if for all $n \in \omega$ we have that $x\restriction_n \in T$. We denote the set of paths through a tree T by $[T]$ and call it the *body of* T. We note that a tree that has a path through it is not ill-founded. Assuming $\mathbf{DC}$ also the converse is true.

For a tree T we define the *boundary of* T as

$$\partial T := \{s \in \omega^{<\omega} \mid s \notin T \text{ and } \forall t \subsetneq s\,(t \in T)\}.$$

Furthermore we call an element $t \in T$ *a terminal node* iff for all $n \in \omega$ we have that $t^\frown \langle n \rangle \in \partial T$.

Now we return to the topology of $\mathbb{R}$. We note that for any pruned tree T we have that $[T] \neq \emptyset$ and that $[T] \subseteq \mathbb{R}$ is a closed set in $\mathbb{R}$. Furthermore for any non-empty closed set $C \subseteq \mathbb{R}$ there is a pruned tree T such that $[T] = C$. We note without proof the following fact about closed sets and Lipschitz functions:

Proposition 2.1.5. *For any non-empty closed set $C \subseteq \mathbb{R}$ there is a surjective Lipschitz function $f : \mathbb{R} \to C$ such that the function $f\restriction_C$ is just the identity function on C.*

The last definition of this section is that of a Borel set. For any set D we call a set $\mathcal{A} \subseteq \mathcal{P}(D)$ a *σ-algebra over D* if and only if it satisfies the following three properties:

1. $D \in \mathcal{A}$;
2. For all $X \in \mathcal{A}$ we have that $D \setminus X \in \mathcal{A}$;
3. For any countable set $\mathcal{B} \subseteq \mathcal{A}$ we have that $\bigcup \mathcal{B} \in \mathcal{A}$.

For any topological space D the *σ-algebra of Borel sets for D* is the smallest σ-algebra $\mathcal{A}$ such that for any open set $O \subseteq D$ we have that $O \in \mathcal{A}$. We call its elements *Borel sets*. As a special case we denote the σ-algebra of Borel sets for $\mathbb{R}$ by Borel.

2.2 Games and Determinacy

All games considered throughout this thesis will be games for two players such that both players are perfectly informed about their opponent's moves, there is no element of chance and in every run of these games exactly one player wins, i.e., there is never a draw. Such a kind of game is called a two-player zero-sum perfect information game. The games considered here will have more restrictions, namely that both players can always make ω-many moves before the game is evaluated, Player $\mathbf{I}$ always begins and furthermore Player $\mathbf{I}$ always makes moves in ω. We formalize the type of game that we will consider with the following definition:

Definition 2.2.1 A *game* is a tuple $\langle M, W\rangle$ of a countable set M with $\omega \subseteq M$, called the *set of* $\mathbf{II}$*'s moves*, and a non-empty relation $W \subseteq \omega^\omega \times M^\omega$, called the *winning condition for Player* $\mathbf{II}$.

Given a game $\mathcal{G}$ we refer by $M_\mathcal{G}$ to its set of $\mathbf{II}$'s moves and by $W_\mathcal{G}$ to its winning condition for Player $\mathbf{II}$.

A *position* in a game $\mathcal{G}$ is an element $p \in \bigcup_{n\in\omega} \left(\omega^n \times M^n \cup \omega^{n+1} \times M^n\right)$. The intuition behind this definition is that $p_\mathbf{I}$ are Player $\mathbf{I}$'s moves up to some point and $p_\mathbf{II}$ are the reaction of Player $\mathbf{II}$ to these moves. However, we leave it open in this definition whether Player $\mathbf{II}$ has already reacted to Player $\mathbf{I}$'s last move already played. We thus call $p_\mathbf{I}$ *Player* $\mathbf{I}$*'s partial play up to* p and we call $p_\mathbf{II}$ *Player* $\mathbf{II}$*'s partial play up to* p.

A *play* of a game $\mathcal{G}$ is an element $P \in \omega^\omega \times M_\mathcal{G}^\omega$, where $P_\mathbf{I}$ is $\mathbf{I}$*'s part of the play* P and $P_\mathbf{II}$ is $\mathbf{II}$*'s part of the play* P. A play P of a game $\mathcal{G}$ is *winning for Player* $\mathbf{II}$ iff $P \in W_\mathcal{G}$. Otherwise it is *winning for Player* $\mathbf{I}$.

A *strategy for Player* $\mathbf{I}$ in a game $\mathcal{G}$ is a map $\sigma : M_\mathcal{G}^{<\omega} \to \omega$; a *strategy for Player* $\mathbf{II}$ in a game $\mathcal{G}$ is a map $\tau : \omega^{<\omega} \setminus \{\emptyset\} \to M_\mathcal{G}$. A strategy σ for Player $\mathbf{I}$ in a game $\mathcal{G}$ is a *winning strategy for Player* $\mathbf{I}$ *in* $\mathcal{G}$ iff for all $y \in M^\omega$ the play $\langle\overline{\sigma}(x), y\rangle$ is winning for Player $\mathbf{I}$ in $\mathcal{G}$. Analogously a strategy τ for Player $\mathbf{II}$ in a game $\mathcal{G}$ is a *winning strategy for player* $\mathbf{I}$ *in* $\mathcal{G}$ iff for all $x \in \omega^\omega$ the play $\langle x, \overline{\tau}(x)\rangle$ is winning for Player $\mathbf{II}$. We note that there are surjections from $\mathbb{R}$ to the set of strategies for $\mathbf{I}$ and from $\mathbb{R}$ to the set of strategies for $\mathbf{II}$, when we fix a set $M_\mathcal{G}$ for these strategies. Thus whenever we assume $\mathbf{AC}_\omega(\mathbb{R})$ we can freely choose countably many strategies.

We say that Player $\mathbf{I}$ wins a game $\mathcal{G}$ iff Player $\mathbf{I}$ has a winning strategy in $\mathcal{G}$. Analogously for Player $\mathbf{II}$. We say that a game $\mathcal{G}$ is determined if either $\mathbf{I}$ wins $\mathcal{G}$ or $\mathbf{II}$ wins $\mathcal{G}$.

Given a strategy σ for Player $\mathbf{I}$ or Player $\mathbf{II}$ in a game $\mathcal{G}$, we let $\mathrm{Player}(\sigma)$ denote the unique player among $\mathbf{I}$ and $\mathbf{II}$ for whom σ is a strategy. We then let $\mathrm{Opponent}(\sigma)$ denote the other player.

Next we single out an important class of games. But before we can do that we need an auxiliary notion. Given a set M with $\omega \subseteq M$ let $\mathrm{filter} : M^\omega \to \omega^{\leq\omega}$ be the map that strips any sequence $x \in M^\omega$ of all occurrences of elements not in ω. To define filter we let for a given $x \in M^\omega$ the number $\alpha \in \omega + 1$ be the number of occurrences of elements of natural numbers in x. Then we define $\mathrm{filter}(x) \in \omega^\alpha$ by letting for any $n < \alpha$, $\mathrm{filter}(x)(n)$ be the n-th occurrence of a natural number in x.

Definition 2.2.2 We call a game $\mathcal{G}$ an *ω-game* iff there is a relation $R \subseteq \omega^\omega \times \omega^\omega$ such that

$$W_\mathcal{G} = \{\langle x, y\rangle \in \omega^\omega \times M^\omega \mid \mathrm{filter}(y) \in \omega^\omega \wedge \langle x, \mathrm{filter}(y)\rangle \in R\}.$$

We call a sequence $\mathfrak{G} = \langle\langle\mathcal{G}_n, \sigma_n\rangle \mid n \in \omega\rangle$ a *game sequence* iff for every $n \in \omega$, $\mathcal{G}_n$ is an ω-game, and σ_n is a winning strategy either for Player **I** or for Player **II** in the game $\mathcal{G}_n$. For a game sequence $\mathfrak{G} = \langle\langle\mathcal{G}_n, \sigma_n\rangle \mid n \in \omega\rangle$ we then let $\mathrm{Player}^{\mathfrak{G}}(n) := \mathrm{Player}(\sigma_n)$ for all $n \in \omega$.

Given a game sequence $\mathfrak{G} = \langle\langle\mathcal{G}_n, \sigma_n\rangle \mid n \in \omega\rangle$ and an $m \in \omega$ such that $\mathrm{Player}^{\mathfrak{G}}(m) = \mathbf{I}$, we define the *finite play* for $\mathfrak{G}$ starting at m to be the function

$$f_m^{\mathfrak{G}} : m+1 \longrightarrow \omega^{<\omega},$$

$$k \longmapsto \begin{cases} \langle\sigma_m(\emptyset)\rangle, & \text{if } k = m, \\ \mathrm{filter}(\overline{\sigma_m}(f_m^{\mathfrak{G}}(k+1))), & \text{if } k < m. \end{cases}$$

We note that in the above definition the application of filter disappears whenever σ_m is a strategy for Player **I**.

For a game sequence $\mathfrak{G} = \langle\langle\mathcal{G}_n, \sigma_n\rangle \mid n \in \omega\rangle$ let $m, m' \in \omega$ such that $m < m'$ and $\mathrm{Player}^{\mathfrak{G}}(m) = \mathrm{Player}^{\mathfrak{G}}(m') = \mathbf{I}$. Then we note that for all $k \leq m$ we have that $f_m^{\mathfrak{G}} \subseteq f_{m'}^{\mathfrak{G}}$. We can show this by induction on $m - k$ as follows. For $k = m$ we note that for all $s \in \omega^{<\omega}$ we have $\langle\sigma_n(\emptyset)\rangle \subseteq \overline{\sigma_n}(s)$ and so in particular

$$f_m^{\mathfrak{G}}(m) = \langle\sigma_m(\emptyset)\rangle \subseteq \overline{\sigma_n}(f_{m'}^{\mathfrak{G}}(m+1)) = f_{m'}^{\mathfrak{G}}(m).$$

Now assume the claim was shown for k with $0 < k \leq m$. Then for $k - 1$ we get using this induction hypothesis

$$f_m^{\mathfrak{G}}(k-1) = \mathrm{filter}(\overline{\sigma_m}(f_m^{\mathfrak{G}}(k))) \subseteq \mathrm{filter}(\overline{\sigma_m}(f_{m'}^{\mathfrak{G}}(k))) = f_{m'}^{\mathfrak{G}}(k-1),$$

concluding the induction. The claim just shown justifies the following definition.

Given a game sequence $\mathfrak{G}$ we call a function $F : \omega \to \omega^\omega$ a *global play* for $\mathfrak{G}$ iff for all $n \in \omega$ we have that

$$F(n) = \lim\{f_m^{\mathfrak{G}}(n) \mid n \leq m < \omega, \mathrm{Player}^{\mathfrak{G}}(m) = \mathbf{I}\}.$$

Of course given a game sequence $\mathfrak{G}$ there need not be a global play for $\mathfrak{G}$; for instance there never is a global play if there are cofinitely many $n \in \omega$ such that $\mathrm{Player}^{\mathfrak{G}}(n) = \mathbf{II}$. But when a global game exists, it is unique. However we will give some criteria for the existence of a global play.

Lemma 2.2.3. *Let $\mathfrak{G} = \langle\langle\mathcal{G}_n, \sigma_n\rangle \mid n \in \omega\rangle$ be a game sequence such that* $\mathrm{Player}^{\mathfrak{G}}(n) = \mathbf{I}$ *for all $n \in \omega$. Then there is a global play for $\mathfrak{G}$.*

Proof. We need to show that for all $n \in \omega$ we have that $\lim\{f_m^{\mathfrak{G}}(n) \mid n \leq m < \omega\} \in \omega^\omega$. To do this we note that since for all $n \in \omega$ we have that $\mathrm{Player}(\sigma_n) = \mathbf{I}$ and so $\mathrm{dom}(\sigma_n) = \omega^{<\omega}$, which in turn implies that for all $s \in \omega^{<\omega}$ we have that $\mathrm{lh}(\overline{\sigma_n}(s)) = \mathrm{lh}(s) + 1$. Using this we show that for all $n, k \in \omega$ with $k \leq n$ we have that $\mathrm{lh}\left(f_n^{\mathfrak{G}}(n-k)\right) = k + 1$ by induction on k as follows. For $k = 0$ we note that $\mathrm{lh}\left(f_n^{\mathfrak{G}}(n)\right) = \mathrm{lh}(\langle\sigma_n(\emptyset)\rangle) = 0$. Next we assume that the claim was already shown for some $k < n$. Then using this induction hypothesis we can calculate that

$$\mathrm{lh}\left(f_n^{\mathfrak{G}}(n-k-1)\right) = \mathrm{lh}\left(\overline{\sigma_n}(f_n^{\mathfrak{G}}(n-k))\right) = \mathrm{lh}(f_n^{\mathfrak{G}}(n-k)) + 1 = k + 2.$$

But then for any $n, k \in \omega$ we get that $\mathrm{lh}\left(f_{n+k}^{\mathfrak{G}}(n)\right) = k$ and so that $\lim\{f_m^{\mathfrak{G}}(n) \mid n \leq m < \omega\} \in \omega^\omega$ as claimed. □

Proposition 2.2.4. *Let $\mathfrak{G} = \langle\langle\mathcal{G}_n, \sigma_n\rangle \mid n \in \omega\rangle$ be a game sequence such that there are cofinitely many $n \in \omega$ such that* $\mathrm{Player}^{\mathfrak{G}}(n) = \mathbf{I}$. *Then there is a global play for $\mathfrak{G}$.*

Proof. Since there are cofinitely many $n \in \omega$ such that $\mathrm{Player}^{\mathfrak{G}}(n) = \mathbf{I}$, we can fix a maximal $i^* \in \omega$ such that $\mathrm{Player}^{\mathfrak{G}}(i^*) = \mathbf{II}$. But then we consider the game sequence

$$\mathfrak{G}' := \langle\langle\mathcal{G}_{n+i^*}, \sigma_{n+i^*}\rangle \mid n \in \omega\rangle,$$

for which by Lemma 2.2.3 there is a global play. This implies that for all $n > i^*$ we have that

$$\lim\{f_m^{\mathfrak{G}}(n) \mid n \leq m < \omega\} \in \omega^\omega.$$

Now we assume towards a contradiction that there is no global play for $\mathfrak{G}$ and fix $i \in \omega$ such that we have that

$$s := \lim\{f_m^{\mathfrak{G}}(i) \mid i \leq m < \omega, \text{Player}^{\mathfrak{G}}(m) = \mathbf{I}\} \in \omega^{<\omega}.$$

By the remark just made we have that $i \leq i^*$; so we can take i to be maximal with above property. Then we can define a real $x \in \mathbb{R}$ by setting $x := \lim\{f_m^{\mathfrak{G}}(i+1) \mid i \leq m < \omega, \text{Player}^{\mathfrak{G}}(m) = \mathbf{I}\}$. By definition of the finite plays we then have that $s = \text{filter}(\overline{\sigma_i}(x))$. To arrive at a contradiction we now distinguish two cases.

Case 1 is that $\text{Player}^{\mathfrak{G}}(i+1) = \mathbf{I}$. Then we have that $s = \text{filter}(\overline{\sigma_i}(x)) = \overline{\sigma_i}(x)$ and thus, since $x \in \mathbb{R}$ also $s \in \mathbb{R}$, contradicting the fact that $s \in \omega^{<\omega}$.

Case 2 is that $\text{Player}^{\mathfrak{G}}(i+1) = \mathbf{II}$. Then we have that $\text{lh}(\text{filter}(\overline{\sigma_i}(x))) < \omega$, meaning that the play $\langle x, \overline{\sigma_i}(x)\rangle$ is winning for Player $\mathbf{I}$ in $\mathcal{G}_i$, since $\mathcal{G}_i$ is an ω-game. But this is a contradiction to the assumption that σ_i is a winning strategy for $\mathbf{II}$ in $\mathcal{G}_i$.

Thus we have shown that for all $i \in \omega$ we have that $\lim\{f_m^{\mathfrak{G}}(i) \mid i \leq m < \omega, \text{Player}^{\mathfrak{G}}(m) = \mathbf{I}\} \in \omega^\omega$ and thus we can define a global game for $\mathfrak{G}$. □

The usefulness of the existence of global games lies in the following fact. Let a game sequence $\mathfrak{G} = \langle\langle\mathcal{G}_n, \sigma_n\rangle \mid n \in \omega\rangle$, and relations R_n for all $n \in \omega$ such that

$$M_{\mathcal{G}_n} = \{\langle x, y\rangle \mid \text{filter}(y) \in \omega^\omega \mid \langle x, \text{filter}(y)\rangle \in R_n\}$$

be given. Then for a global game $F : \omega \to \omega^\omega$ for $\mathfrak{G}$ we get for any $m \in \omega$ the following:

$$\begin{aligned} \text{Player}^{\mathfrak{G}}(m) = \mathbf{I} \quad &\Rightarrow \quad \langle F(m), F(m+1)\rangle \notin R_m, \\ \text{and} \quad \text{Player}^{\mathfrak{G}}(m) = \mathbf{II} \quad &\Rightarrow \quad \langle F(m), F(m+1)\rangle \in R_m. \end{aligned}$$

We will use this fact in the proofs of our main results in Sections 3.3 and 3.4.

Next we work towards introducing the Axiom of Determinacy and fragments of it, which are assumptions that we will use for many results in this thesis. For this we first introduce an archetypical class of games.

Definition 2.2.5 Let $A \in \mathcal{P}(\mathbb{R})$ be arbitrary. Then we define a game $G(A)$, the *canonical game on A*, by setting $M_{G(A)} := \omega$ and

$$W_{G(A)} := \{\langle x, y\rangle \mid x * y \notin A\}.$$

We can think of the canonical game on a set $A \in \mathcal{P}(\mathbb{R})$ as the game in which Player $\mathbf{I}$ and Player $\mathbf{II}$ alternatingly play natural numbers, thereby constructing a real number. Player $\mathbf{I}$ then wins if a real number in A results and Player $\mathbf{II}$ wins otherwise. On the basis of this game we define $\mathbf{AD}$.

Definition 2.2.6 Let $A \in \mathcal{P}(\mathbb{R})$ be arbitrary. Then $\mathbf{Det}(A)$ is the statement asserting that the game $G(A)$ is determined.

Let $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ be arbitrary. Then $\mathbf{Det}(\Gamma)$ is the statement asserting that $\mathbf{Det}(A)$ holds for every $A \in \Gamma$.

Finally $\mathbf{AD}$ is the statement $\mathbf{Det}(\mathcal{P}(\mathbb{R}))$.

Next we are going to show that under the assumption of $\mathbf{AD}$ all games in the sense of Definition 2.2.1 are determined. For this we first introduce the notion of game equivalence.

Definition 2.2.7 We call two games $\mathcal{G}$ and $\mathcal{G}'$ game-equivalent iff

$$\begin{aligned} \mathbf{I} \text{ wins } \mathcal{G} \quad &\Leftrightarrow \quad \mathbf{I} \text{ wins } \mathcal{G}' \\ \text{and} \quad \mathbf{II} \text{ wins } \mathcal{G} \quad &\Leftrightarrow \quad \mathbf{II} \text{ wins } \mathcal{G}' \end{aligned}$$

In particular if $\mathcal{G}$ is determined and $\mathcal{G}$ and $\mathcal{G}'$ are game-equivalent, then also $\mathcal{G}'$ is determined. We now show that up to game-equivalence all games are of the form $G(A)$ for some $A \in \mathcal{P}(\mathbb{R})$:

Proposition 2.2.8. *Let $\mathcal{G}$ be an arbitrary game. Then there is $A \in \mathcal{P}(\mathbb{R})$ such that $\mathcal{G}$ and $G(A)$ are game-equivalent.*

Proof. We fix a bijection $f : M_{\mathcal{G}} \to \omega$. Then we lift f to a bijection $\check{f} : M_{\mathcal{G}}^{\leq\omega} \to \omega^{\leq}$ by defining for all $x \in M_{\mathcal{G}}^{\leq\omega}$ the sequence $\check{f}(x) \in \omega^{\mathrm{lh}(x)}$ by setting for all $n < \mathrm{lh}(x)$

$$\check{f}(x)(n) = f(x(n)).$$

Now we let $A := \{x * y \in \mathbb{R} \mid \langle x, \check{f}^{-1}(y)\rangle \notin W_{\mathcal{G}}\}$. Now we claim that $\mathcal{G}$ and $G(A)$ are game-equivalent. We will only show that **II** wins $\mathcal{G}$ if and only if **II** wins $\mathcal{G}'$. The case for Player **I** is completely analogous.

For the left-to-right direction we let $\sigma : \omega^{<\omega} \setminus \{\emptyset\} \to M_{\mathcal{G}}$ be a winning strategy for Player **II** in the game $\mathcal{G}$. Then we have for all $x \in \mathbb{R}$ that $\langle x, \overline{\sigma}(x)\rangle \in W_{\mathcal{G}}$. But then we define a strategy $\sigma' : \omega^{<\omega} \setminus \{\emptyset\} \to \omega$ for Player **II** in $G(A)$ by setting for all $s \in \omega^{<\omega}$

$$\sigma'(s) := f(\sigma(s)).$$

But then σ' is winning for Player **II** in $G(A)$, since for all $x \in \mathbb{R}$ we have that $\overline{\sigma'}(x) = \check{f}(\overline{\sigma}(x))$ and so $\langle x, \overline{\sigma'}(x)\rangle \notin A$.

For the right-to-left direction we let $\tau : \omega^{<\omega} \setminus \{\emptyset\} \to \omega$ be a winning strategy for Player **II** in the game $G(A)$. Then for all $x \in \mathbb{R}$ we have that $(x * (\overline{\sigma}(x))) \notin A$ and so $\langle x, \check{f}^{-1}(\overline{\sigma}(x))\rangle \in W_{\mathcal{G}}$. Thus the strategy $\tau' : \omega^{<\omega} \setminus \{\emptyset\} \to M_{\mathcal{G}}$ defined by setting for all $s \in \omega^{<\omega}$

$$\tau'(s) := f^{-1}(\tau(s))$$

is winning for player **II** in $\mathcal{G}$, since for any $x \in \mathbb{R}$ we have that $\overline{\tau'}(x) = \check{f}^{-1}(\overline{\sigma}(x))$. □

An immediate corollary of this is that under $\mathbf{AD}$ all games are determined.

Now we will say a bit about the relation of the theory $\mathbf{ZF} + \mathbf{AD}$ to other set theoretic theories. First we note that the consistency strength of $\mathbf{ZF} + \mathbf{AD}$ is fairly high. It is proved in [Kan09, Theorem 32.16] that $\mathbf{ZF} + \mathbf{AD}$ is equiconsistent to the the theory $\mathbf{ZF} + \mathbf{AC}$ plus the assertion that there are infinitely many Woodin cardinals. Woodin cardinals are a certain type of large cardinals; information on them can also be found in [Kan09]. Next we collect some results regarding the interplay of $\mathbf{AD}$ with some choice principles.

Proposition 2.2.9. *1. Assume* $\mathbf{AD}$. *Then* $\mathbf{AC}_\omega(\mathbb{R})$ *holds.*
2. Assume that $\mathbf{ZF} + \mathbf{AD}$ *is consistent. Then also* $\mathbf{ZF} + \mathbf{AD} + \mathbf{DC}$ *is consistent.*
3. $\mathbf{ZF} + \mathbf{AC} + \mathbf{AD}$ *is inconsistent.*

Part 1 of this proposition is proved in [Kan09, Proposition 27.10]. Part 2 follows from the fact that for any model of $\mathbf{ZF} + \mathbf{AD}$ there is an inner model $L(\mathbb{R})$ of $\mathbf{ZF} + \mathbf{AD} + \mathbf{DC}$ as is remarked in [Kan09, p. 378]. Part 3 is proved in [Kan09, Proposition 27.2]. This last part is the reason why we avoid the full Axiom of Choice throughout this thesis.

Finally we note a consequence of $\mathbf{AD}$ that will become important in the next section and in Chapter 3. We call a set $F \subseteq 2^\omega$ a flip-set if and only if for all $x, y \in 2^\omega$, if there exists exactly one $k \in \omega$ such that $x(k) \neq y(k)$, then $x \in F \Leftrightarrow y \notin F$. Then we have the following:

Proposition 2.2.10. *1. Assume* $\mathbf{AD}$. *Then there is* no *flip-set* $F \subseteq 2^\omega$.
2. Assume $\mathbf{AC}$. *Then there is a flip-set* $F \subseteq 2^\omega$.

Part 1 of this proposition is proved in [And07, p. 14]. Part 2 is contained in [And01, Exercise 5.22.(ii)]. An important consequence of part 2 is that the non-existence of flip-sets cannot be established in $\mathbf{ZF}$ alone, since else $\mathbf{ZF}$ would prove the negation of $\mathbf{AC}$, which is a contradiction.

2.3 Descriptive Set Theory

In this section we will introduce some basic notions of descriptive set theory that we are either going to use later on or that motivate the investigation we undertake in this thesis. Subsection 2.3.1 will be concerned with the original Wadge hierarchy as introduced by William Wadge in the early 70's and the ordinal Θ that plays a central role in descriptive set theory, in particular since under $\mathbf{AD}$ it is the order type of the Wadge hierarchy. In Subsection 2.3.2 we will then introduce the hierarchy of a strictly regular norms as implicitly introduced by Yiannis Moschovakis in the article [Löw10] for his proof of the First

Periodicity Theorem. This notion will be central for this thesis and will be our object of investigation throughout Chapter 4.

In both subsections we will not only introduce the relevant notions, but also give a short overview over the historical developments leading towards them. Regarding descriptive set theory this overview is based on Chapter 6 of the book [Kan09], regarding the original Wadge hierarchy it is based on the overview articles [Wad12] and [And07] and regarding the hierarchy of strictly regular norms it is based on the article [Löw05b].

2.3.1 The Original Wadge Hierarchy and the Ordinal Θ

Descriptive set theory is concerned with the structure of $\mathcal{P}(\mathbb{R})$, the set of subsets of $\mathbb{R}$. Of particular interest are boldface pointclasses, which we will now define. We call any set $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ a pointclass. We call a pointclass Γ *boldface* if and only if for any $x \in \mathbb{R}$ and any $A \in \Gamma$ we have that $\mathbf{g}_x^{-1}[A] \in \Gamma$. Thus boldface pointclasses are exactly the pointclasses which are closed under continuous pre-images. The reason why they are interesting is that many properties of interest in descriptive set theory are preserved under continuous pre-images. A very basic example is that a continuous pre-image of any open set is open and a continuous pre-image of any closed set is closed. A slightly more advanced example is the set Borel of Borel sets, which is also a boldface pointclass.

The original Wadge hierarchy, which we will define in the following, is a tool to understand boldface pointclasses. For any two sets $A, B \in \mathcal{P}(\mathbb{R})$ we say that A Wadge-reduces to B and write $A \leq^*_{\mathrm{W}} B$ iff there is a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $A = f^{-1}[B]$. Since the identity function on $\mathbb{R}$ is continuous and the composition of two continuous functions is continuous we have that $\leq^*_{\mathrm{W}}$ is a quasi-order. Thus denoting the equivalence relation corresponding to $\leq^*_{\mathrm{W}}$ by $\equiv^*_{\mathrm{W}}$ we have that the structure $(\mathcal{P}(\mathbb{R})/_{\equiv^*_{\mathrm{W}}}, \leq_{\mathrm{W}})$ is a partial order, which we call *the original Wadge hierarchy*. This hierarchy, which in the literature usually is just refered to as the Wadge hierarchy, was originally introduced and investigated by William Wadge in the early 70's, although his PhD thesis [Wad83] containing his results was only published in 1984. A central result regarding the structure of the original Wadge hierarchy under the assumption of $\mathbf{AD}$ is Wadge's Lemma, which was already present in Wadge's original work.

Theorem 2.3.1 (Wadge's Lemma)**.** *Assume* $\mathbf{AD}$. *For any two* $A, B \in \mathcal{P}(\mathbb{R})$ *we have that either* $A \leq^*_{\mathrm{W}} B$ *or* $B \leq^*_{\mathrm{W}} \mathbb{R} \setminus A$.

Another major result is that under $\mathbf{AD}$ and $\mathbf{DC}$ the original Wadge hierarchy is well-founded. This result, known today as the Martin-Monk Theorem, was first published in 1972 by Donald Martin in the article [Mar70], which was based on previous work of Leonhard Monk.

Theorem 2.3.2 (Martin-Monk)**.** *Assume* $\mathbf{AD}$ *and* $\mathbf{DC}$. *The quasi-order* $(\mathcal{P}(\mathbb{R}), \leq^*_{\mathrm{W}})$ *is well-founded.*

For any $A \in \mathcal{P}(\mathbb{R})$ we denote its equivalence class in $\mathcal{P}(\mathbb{R})/_{\equiv^*_{\mathrm{W}}}$ by $[A]^*_{\mathrm{W}}$ Then Wadge's Lemma implies that $(\mathcal{P}(\mathbb{R})/_{\equiv^*_{\mathrm{W}}}, \leq_{\mathrm{W}})$ has antichains of length at most 2 and that all such antichains are of the form $\{[A]^*_{\mathrm{W}}, [\mathbb{R} \setminus A]^*_W\}$ for some $A \in \mathcal{P}(\mathbb{R})$, while the Martin-Monk Theorem implies that we can consider for any $A \in \mathcal{P}(\mathbb{R})$ the ordinal $|A|^*_{\mathrm{W}} := \|A\|_{\leq^*_{\mathrm{W}}}$, which we call the *Wadge rank of A*. Thus we can stratify $\mathcal{P}(\mathbb{R})$ into layers of constant Wadge rank:

Definition 2.3.3 For any ordinal $\alpha < \mathrm{otyp}(\mathcal{P}(\mathbb{R}), \leq^*_{\mathrm{W}})$ we set

$$\mathcal{P}^{(\alpha)}(\mathbb{R}) := \left\{A \in \mathcal{P}(\mathbb{R}) \mid |A|^*_{\mathrm{W}} = \alpha\right\}.$$

Then assuming $\mathbf{AD}$ and $\mathbf{DC}$ clearly we get that $\mathcal{P}(\mathbb{R}) = \bigcup_{\alpha < \mathrm{otyp}(\mathcal{P}(\mathbb{R}), \leq^*_{\mathrm{W}})} \mathcal{P}^{(\alpha)}(\mathbb{R})$. Furthermore again assuming $\mathbf{AD}$ and $\mathbf{DC}$ we have that there is a surjection $f : \mathbb{R} \to \mathcal{P}^{(\alpha)}$ for any $\alpha < \Theta$, which is proved in the article [Löw05b, Proposition 2.6].

Another consequence of the well-foundedness of $(\mathcal{P}(\mathbb{R}), \leq^*_{\mathrm{W}})$ is that for any boldface pointclass Γ there is a $\leq^*_{\mathrm{W}}$-minimal (of course in general not unique) $A \in \mathcal{P}(\mathbb{R})$ with $A \notin \Gamma$ and so for any boldface pointclass Γ there is an $A \in \mathcal{P}(\mathbb{R})$ such that $\Gamma = \{B \in \mathcal{P}(\mathbb{R}) \mid B <^*_{\mathrm{W}} A\}$, which shows that boldface pointclasses are just initial segments of the Wadge hierarchy.

After having considered a few consequences of the Martin-Monk theorem we now consider the question which fragment of $\mathbf{AD}$ is actually used in the proof of the Martin-Monk Theorem. We first note that for any two $A, B \in \mathcal{P}(\mathbb{R})$ there is a game $\mathrm{G}^*_{\mathrm{W}}(A, B)$ such that

$$\text{Player } \mathbf{II} \text{ wins } \mathrm{G}^*_{\mathrm{W}}(A, B) \quad \Leftrightarrow \quad A \leq^*_{\mathrm{W}} B.$$

We will not define it here, since we will define a more general type of game encompassing this game in Section 3.2. However, the definition of $\mathrm{G}^*_{\mathrm{W}}(A, B)$ can be found in the article [And07, Subsection 2.3]. Now we denote by $\mathbf{AD}^*_{\mathrm{W}}$ the claim that every game of the form $\mathrm{G}^*_{\mathrm{W}}(A, B)$ is determined. The axiom $\mathbf{AD}^*_{\mathrm{W}}$ is used in the proof of the Martin-Monk theorem. However, also the non-existence of flip-sets is used.

Theorem 2.3.4. *Assume* $\mathbf{DC}$ *and* $\mathbf{AD}^*_{\mathrm{W}}$ *and that there is no flip set* $F \subseteq 2^\omega$. *Then the quasi-order* $(\Gamma, \leq^*_{\mathrm{W}})$ *is well-founded.*

This result can be found in the article [And01, Theorem 11.17].

The fact that any antichain in the Wadge hierarchy is of the form $\{[A]_W, [\mathbb{R} \setminus A]_W\}$ for some $A \in \mathcal{P}(\mathbb{R})$ suggests the definition of self-duality; we call an $A \in \mathcal{P}(\mathbb{R})$ W-self-dual iff $A \equiv^*_{\mathrm{W}} \mathbb{R} \setminus A$. Otherwise we call it W-non-self-dual. Now we note that we can define another quasi-ordering $\leq^*_{\mathrm{L}}$ on $\mathcal{P}(\mathbb{R})$ by setting for any two $A, B \in \mathcal{P}(\mathbb{R})$, $A \leq^*_{\mathrm{L}} B$ iff there is a Lipschitz function f such that $A = f^{-1}[B]$. Then we let $\equiv^*_{\mathrm{L}}$ be the equivalence relation corresponding to $\leq_{\mathrm{L}}$ and call $(\mathcal{P}(\mathbb{R})/_{\equiv^*_{\mathrm{L}}}, \leq^*_{\mathrm{L}})$ *the original Lipschitz hierarchy*. Results analogous to Wadge's Lemma and the Martin-Monk Theorem hold for the original Lipschitz hierarchy, i.e., any antichain in the Lipschitz hierarchy is of the form $\{[A]_L, [\mathbb{R} \setminus A]_L\}$ and the original Lipschitz hierarchy is well-founded. Proofs for this can be found in the article [And07]. But then we also have a notion of self-duality for the original Lipschitz hierarchy; we say that $A \in \mathcal{P}(\mathbb{R})$ is L-self-dual iff $A \equiv_{\mathrm{L}} \mathbb{R} \setminus A$. Otherwise A is L-non-self-dual. Clearly, since every Lipschitz function is continuous, we have that any L-self-dual $A \in \mathcal{P}(\mathbb{R})$ is also W-self-dual. But – assuming $\mathbf{AD}$ – the other direction also holds, as was proved by John Steel and Robert Van Wesep in 1978 and published in the article [VW78]:

Theorem 2.3.5 (Steel-Van Wesep)**.** *Assume* $\mathbf{AD}$ *and* $\mathbf{DC}$. *Then for any* $A \in \mathcal{P}(\mathbb{R})$ *we have that* A *is* W*-self-dual if and only if* A *is* L*-self-dual.*

The Lipschitz hierarchy is useful to obtain results about the Wadge hierarchy and the Steel-Van Wesep Theorem relates these two hierarchies.

Since under $\mathbf{AD}$ and $\mathbf{DC}$ by the Martin-Monk Theorem the original Wadge hierarchy is well-founded, it is natural to ask for the value of its order type. This question was settled in 1978 by Solovay in the article [Sol78]. Since the order type of the Wadge hierarchy will be used in the following, we are going to introduce a few facts about it. We define an ordinal Θ by setting

$$\Theta := \sup\{\alpha \mid \text{There is a surjection } f : \mathbb{R} \to \alpha\}.$$

Basic facts about Θ are that Θ is a cardinal and that there is no surjection $f : \mathbb{R} \to \Theta$. Furthermore in $\mathbf{ZFC}$ we can prove that Θ is simply identical to the cardinal $(2^{\aleph_0})^+$, which is the cardinal successor if the cardinality of $\mathbb{R}$. Working in $\mathbf{ZF} + \mathbf{AD}$, however, Θ has a far richer structure as discussed in the book [Kan09, pp. 396–399]. There is an extensive literature on it, but we will only note the following result:

Theorem 2.3.6 (Solovay)**.** *Assume* $\mathbf{AD}$ *and* $\mathbf{DC}$. *Then* $\Theta = \mathrm{otyp}(\mathcal{P}(\mathbb{R}), \leq^*_{\mathrm{W}})$.

A proof of this result can be found in the article [And07, Lemma 16].

One other property of Θ that will be important to us is its cofinality. However, the question whether Θ is regular or singular is subtle. We have that $\mathbf{AC}_\omega$ implies $\mathrm{cf}(\Theta) > \omega$. Furthermore, if $\mathbf{ZF} + \mathbf{AD}$ is consistent, then also $\mathbf{ZF} + \mathbf{AD} + \mathbf{DC} +$ "Θ is regular" is consistent. The consistency strength of $\mathbf{ZF} + \mathbf{AD} + \mathbf{DC} +$ "$\Theta$is singular", however is strictly higher than the consistency strength of $\mathbf{ZF} + \mathbf{AD}$, but was not proved to be outright inconsistent. For more details on this we refer to the book draft [And01] and again to the book [Kan09, pp. 396–399].

2.3.2 Regular Norms

From another question in descriptive set theory arises the main notion that we will investigate in this thesis, the hierarchy of regular norms, which we will motivate and introduce in the following. For $A \in \mathcal{P}(\mathbb{R})$ a *strictly regular norm on* A is a surjection $\varphi : \mathbb{R} \to \alpha$ onto some ordinal α. As a special case a *strictly* *regular norm* is a surjection $\varphi : \mathbb{R} \to \alpha$ onto some ordinal α. We denote the set of strictly regular norms by $\mathcal{N}'$.

Clearly every strictly regular norm on a subset $A \in \mathcal{P}(\mathbb{R})$ corresponds to a pre-wellorder $\leq_\varphi$ on A defined for $x, y \in A$ as

$$x \leq_\varphi y \quad :\Leftrightarrow \quad \varphi(x) \leq \varphi(y).$$

Next we note that for any boldface pointclass Γ we can define another boldface pointclass $\breve{\Gamma}$, the *dual of* Γ, by setting $\breve{\Gamma} := \{A \in \mathcal{P}(\mathbb{R}) \mid \mathbb{R} \setminus A \in \Gamma\}$. Furthermore we define a pointclass Δ_Γ by setting $\Delta_\Gamma := \Gamma \cap \breve{\Gamma}$. For $A \in \mathcal{P}(\mathbb{R})$ and a boldface pointclass Γ we call a strictly regular norm φ on A *a strictly regular* Γ*-norm on* A if and only if there are sets $R^+, R^- \in \mathcal{P}(\mathbb{R})$ such that $R^+ \in \Gamma$ and $R^- \in \breve{\Gamma}$ and furthermore for any $y \in \mathbb{R}$ we have that

$$x \in A \wedge \varphi(x) \leq \varphi(y) \quad \Leftrightarrow \quad x * y \in R^+ \quad \Leftrightarrow \quad x * y \in R^-.$$

Now we say that a boldface pointclass Γ has the *pre-wellordering property* if and only if for every $A \in \Gamma$ there is a strictly regular Γ-norm on A. In 1968 Yiannis Moschovakis and John Addison proved their first Periodicity Theorem in the article [AJM68]. The First Periodicity Theorem gives under $\mathbf{AD}$ a characterization of when certain boldface pointclasses have the pre-wellordering property. The exact statement of this Theorem is not important here and would require the introduction of concepts not used anywhere else in this thesis. Of importance to this thesis, however, is an aspect of its proof. Given a boldface pointclass Γ a set $B \in \Gamma$ and a Γ-norm φ on B we define $\overline{p}B := \{x \mid \forall z (x * z \in B)\}$ and define a quasi-order $\preceq'_\varphi$ on $\overline{p}B$ by setting for all $x, y \in \overline{p}B$

$$\begin{aligned} x \preceq'_\varphi y \quad :\Leftrightarrow \quad & \exists z \in \mathbb{R}\, (y * z \notin B) \\ & \text{or } \forall y \in \mathbb{R}(y * z \in B) \text{ and there is a Lipschitz } f : \mathbb{R} \to \mathbb{R} \text{ s.t. for all } z \in \mathbb{R}: \\ & \qquad \varphi(x * z) \leq \varphi(y * (f(z))). \end{aligned}$$

In terms of these notions then Addison and Moschovakis essentially establish the following result:

Theorem 2.3.7. *Let* Γ *be a boldface pointclass. Assume* $\mathbf{DC}$ *and* $\mathbf{Det}(\Delta_\Gamma)$*. Then for any* $B \in \Gamma$ *and for any* Γ*-norm* φ *on* B *we have that* $(\overline{p}B, \preceq'_\varphi)$ *is a pre-well-order.*

A proof of this can be found in the book [Kan09, p. 412] using the methods of the proof of our Proposition 3.2.2 to relate the game occurring there to Lipschitz functions.

Inspired by the definition of quasi-orderings $\leq_\varphi$ we can define a quasi-ordering $\leq_N$ on $\mathcal{N}'$ by setting for any two strictly regular norms $\varphi, \psi \in \mathcal{N}'$

$$\varphi \leq_N \psi \quad :\Leftrightarrow \quad \text{There is a continuous function } f : \mathbb{R} \to \mathbb{R} \text{ s.t. for all } x \in \mathbb{R} : \ \varphi(x) \leq \varphi(f(x)).$$

The quasi-order $(\mathcal{N}', \leq_N)$ was in its full generality for the first time in 2005 considered by Benedikt Löwe in his article [Löw05b]. He notes that method of the proof of Theorem 2.3.7 can be immediately transfered to establish the following two results.

Theorem 2.3.8. *Assume* $\mathbf{AD}$*. Then the partial order* $(\mathcal{N}'/_{\equiv_N}, \leq_N)$ *is a linear order.*

Theorem 2.3.9. *Assume* $\mathbf{AD}$ *and* $\mathbf{DC}$*. Then the partial order* $(\mathcal{N}'/_{\equiv_N}, \leq_N)$ *is a well-order.*

We will note give proofs of these results here, since we will re-establish them in Chapter 3 in a more general context.

We note that Theorem 2.3.9 is a direct analogue of the Martin-Monk Theorem and that Theorem 2.3.8 is an analogue of Wadge's Lemma, which is stronger than the one for the original Wadge hierarchy. We call $(\mathcal{N}'/_{\equiv_N}, \leq_N)$ *the hierarchy of strictly regular norms* and get by Theorem 2.3.9 that assuming $\mathbf{AD}$ and $\mathbf{DC}$ this hierarchy is a well-order. We note that in [Löw05b] this hierarchy is just called the hierarchy of norms, but we adopt our nomenclature to distinguish it from other hierarchies of norms that we are going to consider in Chapter 3.

Now we can ask – as in the case of the original Wadge hierarchy – which fragment of $\mathbf{AD}$ suffices to prove Theorem 2.3.9. Again we note without proof that for any two norms $\varphi, \psi \in \mathcal{N}'$ there is a game $G_N(\varphi, \psi)$ such that

$$\text{Player } \mathbf{II} \text{ wins } G_N(\varphi, \psi) \quad \Leftrightarrow \quad \varphi \leq_N \psi.$$

Then we denote by $\mathbf{AD}_N$ the assertion that all games of the form $G_N(\varphi, \psi)$ are determined. A careful analysis of the proof of Theorem 2.3.9 now gives us the following result.

Theorem 2.3.10. *Assume* $\mathbf{DC}$ *and* $\mathbf{AD}_N$*. Then the quasi-order* $(\mathcal{N}', \leq_N)$ *is well-founded.*

We note that there is a clear difference between Theorem 2.3.10 and Theorem 2.3.4, the analogous result for the original Wadge hierarchy, in that the former only uses determinacy of the games corresponding directly to the considered quasi-order, while the latter additionally assumes the non-existence of flip sets, which does not seem directly related to the considered quasi-order.

The analogies between the original Wadge hierarchy and the hierachy of strictly regular norms suggest the question whether there is a general framework encompassing these two notions and in which we can prove all the theorems just listed. The answer to this is affirmative and will be the subject of Chapter 3 of this thesis. In this general framework we will also give a concrete explanation for the difference in the fragments of $\mathbf{AD}$ assumed in Theorem 2.3.10 and Theorem 2.3.4.

Another question that we asked for the original Wadge hierarchy and that we can – in light of Theorem 2.3.9 – also ask for the hierarchy of strictly regular norms is the question of the value of $\mathrm{otyp}(\mathcal{N}'/_{\equiv_N}, \leq_N)$. This question is still open, although a lower and an upper bounds have been determined by Benedikt Löwe in the article [Löw05b]. In Chapter 4 we will lead an investigation of the structure of the hierarchy of strictly regular norms, based on which we will be able to improve the previously known lower bound for $\mathrm{otyp}(\mathcal{N}'/_{\equiv_N}, \leq_N)$.

Chapter 3

Induced Hierarchies of Norms

3.1 Original Wadge Hierarchy and Hierarchy of Strictly Regular Norms Revisited

In this section we work towards obtaining a general notion that will encompass the notions of the original Wadge hierarchy and of the hierarchy of strictly regular norms and will allow us to uniformly prove theorems analogous to Wadge's Lemma, the Martin–Monk Theorem and the Steel-Van Wesep Theorem for these notions.

For the original Wadge hierarchy we have that $\mathcal{P}(\mathbb{R})$ is in bijection to $2^{\mathbb{R}}$ via the map

$$b : \mathcal{P}(\mathbb{R}) \to 2^{\mathbb{R}},\ X \mapsto \chi_X.$$

But denoting the quasi-ordering induced by $\leq^*_{\mathrm{W}}$ via b on $2^{\mathbb{R}}$ by $\leq^{**}_{\mathrm{W}}$ we get for any two $x, y \in 2^{\mathbb{R}}$:

$$\begin{aligned}
x \leq^{**}_{\mathrm{W}} y \quad &\Leftrightarrow \quad b^{-1}(x) \leq^*_{\mathrm{W}} b^{-1}(y). \\
&\Leftrightarrow \quad \text{There is a continuous } f : \mathbb{R} \to \mathbb{R} \text{ s.t. } b^{-1}(x) = f^{-1}[b^{-1}(y)]. \\
&\Leftrightarrow \quad \text{There is a continuous } f : \mathbb{R} \to \mathbb{R} \text{ s.t. for all } z \in \mathbb{R}\text{: } z \in b^{-1}(x) \Leftrightarrow z \in b^{-1}(y). \\
&\Leftrightarrow \quad \text{There is a continuous } f : \mathbb{R} \to \mathbb{R} \text{ s.t. for all } z \in \mathbb{R}\text{: } \chi_{b^{-1}(x)}(z) = \chi_{b^{-1}(y)}(z). \\
&\Leftrightarrow \quad \text{There is a continuous } f : \mathbb{R} \to \mathbb{R} \text{ s.t. for all } z \in \mathbb{R}\text{: } x(z) = y(z). \\
&\Leftrightarrow \quad \text{There is a continuous } f : \mathbb{R} \to \mathbb{R} \text{ s.t. for all } z \in \mathbb{R}\ x(z) \leq_{\mathbf{TV}} y(z).
\end{aligned}$$

Clearly by construction we have that $(\mathcal{P}(\mathbb{R}), \leq^*_{\mathrm{W}})$ is isomorphic to $(2^{\mathbb{R}}, \leq^{**}_{\mathrm{W}})$, where we recall that $\mathbf{TV} = 2$. A generalization of this idea leads to the following definition:

Definition 3.1.1 Let $(Q, \leq)$ be an arbitrary quasi-ordering. Then we define the *set of Q-norms* to be the set of all functions from $\mathbb{R}$ to Q. Furthermore we define binary relations $\leq_{\mathrm{W}}$ and $\equiv_{\mathrm{W}}$ on $Q^{\mathbb{R}}$ by setting for any two $\varphi, \psi \in Q^{\mathbb{R}}$:

$$\begin{aligned}
\varphi \leq_{\mathrm{W}} \psi \quad &:\Leftrightarrow \quad \text{There is a continuous } f : \mathbb{R} \to \mathbb{R} \text{ s.t. for all } x \in \mathbb{R} :\ \varphi(x) \leq \psi(f(x)). \\
\varphi \equiv_{\mathrm{W}} \psi \quad &:\Leftrightarrow \quad \varphi \leq_{\mathrm{W}} \psi \text{ and } \psi \leq_{\mathrm{W}} \varphi.
\end{aligned}$$

Then clearly by what we have noted above up to an isomorphism we have that $(\mathcal{P}(\mathbb{R}), \leq^*_{\mathrm{W}})$ is just $(\mathbf{TV}^{\mathbb{R}}, \leq_{\mathbf{TV},\mathrm{W}})$ in the sense just defined and thus the Wadge hierarchy itself is isomorphic to $(\mathbf{TV}^{\mathbb{R}}/_{\equiv_{\mathbf{TV},\mathrm{W}}}, \leq_{\mathbf{TV},\mathrm{W}})$. We now note that of course we can also define an analogous notion with continuous functions replaced by Lipschitz functions.

Definition 3.1.2 Let $(Q, \leq)$ be an arbitrary quasi-ordering. Then we define the *set of Q-norms* to be the set of all functions from $\mathbb{R}$ to Q. Furthermore we define binary relations $\leq_{\mathrm{L}}$ and $\equiv_{\mathrm{L}}$ on $Q^{\mathbb{R}}$ by setting for any two $\varphi, \psi \in Q^{\mathbb{R}}$:

$$\begin{aligned}
\varphi \leq_{\mathrm{L}} \psi \quad &:\Leftrightarrow \quad \text{There is a Lipschitz } f : \mathbb{R} \to \mathbb{R} \text{ s.t. for all } x \in \mathbb{R} :\ \varphi(x) \leq \psi(f(x)). \\
\varphi \equiv_{\mathrm{L}} \psi \quad &:\Leftrightarrow \quad \varphi \leq_{\mathrm{W}} \psi \text{ and } \psi \leq_{\mathrm{L}} \varphi.
\end{aligned}$$

Now we note that these relation $\leq_{\mathrm{W}}$ and $\leq_{\mathrm{L}}$ as just defined are always quasi-orderings.

Lemma 3.1.3. *For an arbitrary quasi-order $(Q, \leq)$ we have that $(Q^{\mathbb{R}}, \leq_{\mathrm{L}})$ and $(Q^{\mathbb{R}}, \leq_{\mathrm{W}})$ are quasi-orders and therefore that $\equiv_{\mathrm{L}}$ and $\equiv_{\mathrm{W}}$ are equivalence relations on $Q^{\mathbb{R}}$.*

Proof. Reflexivity of $\leq_{\mathrm{L}}$ and $\leq_{\mathrm{W}}$ follows from the fact that the identity function on $\mathbb{R}$ is Lipschitz and therefore continuous. Transitivity of $\leq_{\mathrm{L}}$ and $\leq_{\mathrm{W}}$ follows from the fact that both the class of Lipschitz functions from $\mathbb{R}$ to $\mathbb{R}$ and the class of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ are closed under composition. $\square$

Lemma 3.1.4. *Let $(Q, \leq)$ be an arbitrary quasi-order φ, ψ two Q-norms. Then we have:*

$$\varphi \leq_{\mathrm{L}} \psi \ \Rightarrow \ \varphi \leq_{\mathrm{W}} \psi$$

Proof. This follows immediately from the fact that every Lipschitz function is in particular a continuous function. $\square$

We refer to the elements of $Q^{\mathbb{R}}/_{\equiv_{\mathrm{L}}}$ as $(Q, \leq)$-*Wadge degrees* and to the elements of $Q^{\mathbb{R}}/_{\equiv_{\mathrm{W}}}$ as $(Q, \leq)$-*Lipschitz degrees*. More specifically for any $\varphi \in Q^{\mathbb{R}}$ we denote the equivalence class of φ in $Q^{\mathbb{R}}/_{\equiv_{\mathrm{W}}}$ by $[\varphi]_{\mathrm{W}}$ and call it the $(Q, \leq)$-*Wadge degree of* φ. We denote the equivalence class of φ in $Q^{\mathbb{R}}/_{\equiv_{\mathrm{L}}}$ by $[\varphi]_{\mathrm{L}}$ and call it the $(Q, \leq)$-*Lipschitz degree of* φ. Furthermore we will refer to the partial order $(Q^{\mathbb{R}}/_{\equiv_{\mathrm{W}}}, \leq_{\mathrm{W}})$ as the $(Q, \leq)$-*Wadge hierarchy* and to $(Q^{\mathbb{R}}/_{\equiv_{\mathrm{L}}}, \leq_{\mathrm{L}})$ as the $(Q, \leq)$-*Lipschitz hierarchy*.

So subsuming what we showed above we have the following correspondence to the Wadge hierarchy:

Proposition 3.1.5. *The Wadge hierarchy $(\mathcal{P}(\mathbb{R})/_{\equiv_{\mathrm{W}}}, \leq_{\mathrm{W}})$ is isomorphic to the $(\mathbf{TV}, \leq_{\mathbf{TV}})$-Wadge hierarchy $(\mathbf{TV}^{\mathbb{R}}/_{\equiv_{\mathbf{TV},\mathrm{W}}}, \leq_{\mathbf{TV},W})$.*

Next we are going to show that in some sense also the hierarchy of strictly regular norms can be subsumed under the notion of $(Q, \leq)$-Wadge hierarchy for some quasi-order $(Q, \leq)$. For this we first note that by definition of Θ we have that $\mathcal{N}' = \{f : \mathbb{R} \to \Theta \mid \exists \alpha < \Theta\, (f[\mathbb{R}] = \alpha)\}$ and so $\mathcal{N}' \subseteq \Theta^{\mathbb{R}}$, i.e., every strictly regular norm is also a $(\Theta, \leq)$-norm. Considering the well-order $(\Theta, \leq)$, we now note that in fact for all $\varphi, \psi \in \mathcal{N}'$ we have that the definition of $\leq_N$ in the sense of $(\mathcal{N}', \leq_N)$ coincides with the definition of $\leq_{\mathrm{W}}\restriction_{\mathcal{N}' \times \mathcal{N}'}$ in the sense of $(\Theta^{\mathbb{R}}, \leq_{\mathrm{W}})$. So the quasi-order $(\mathcal{N}', \leq_N)$ is a substructure of the quasi-order $(\Theta^{\mathbb{R}}, \leq_{\mathrm{W}})$ and thus the hierarchy of strictly regular norms is a substructure of the $(\Theta, \leq)$-Wadge hierarchy. In this light we will from now on also denote the quasi-ordering $\leq_N$ by $\leq_{\mathrm{W}}$ to avoid an unnecessary cluttering of our notation.

Clearly it is not true that $\Theta^{\mathbb{R}} \subseteq \mathcal{N}'$, since there are functions $f : \mathbb{R} \to \Theta$ that are not surjective onto any ordinal. However, we can ask the question whether the hierarchy of strictly regular norms is isomorphic to the $(\Theta, \leq)$-Wadge hierarchy. The answer to this question is subtle and the rest of this section will be devoted to it.

First we introduce the notion of a regular norm, which is a slight generalization of the notion of a strictly regular norm.

Definition 3.1.6 We call $\mathcal{N} := \{\varphi \in \Theta^{\mathbb{R}} \mid \exists \alpha < \Theta\, (\varphi[\mathbb{R}] \subseteq \alpha)\}$ the *set of regular norms* and call any $\varphi \in \mathcal{N}$ a *regular norm*.

It is immediate from this definition that $\mathcal{N}' \subsetneq \mathcal{N}$. Also, again fixing the order $(\Theta, \leq)$ we get that $(\mathcal{N}'/_{\equiv_{\mathrm{W}}}, \leq_{\mathrm{W}})$ is a substructure of $(\mathcal{N}/_{\equiv_{\mathrm{W}}}, \leq_{\mathrm{W}})$. We will also show the reverse, but before we can do that we need an additional definition.

Definition 3.1.7 For any Θ-norm $\varphi \in \Theta^{\mathbb{R}}$ we let define the *length of* φ as

$$\mathrm{lh}(\varphi) := \sup\{\alpha + 1 \mid \alpha \in \varphi[\mathbb{R}]\}.$$

In particular for any $\varphi \in \mathcal{N}'$ we have that $\mathrm{lh}(\varphi) = f[\mathbb{R}]$. Also we have that

$$\mathcal{N} = \{\varphi \in \Theta^{\mathbb{R}} \mid \mathrm{lh}(\varphi) < \Theta\}.$$

Proposition 3.1.8. *Fix the order* $(\Theta, \leq)$. *For any* $\varphi \in \mathcal{N}$ *there is a* $\psi \in \mathcal{N}'$ *such that* $\varphi \equiv_{\mathrm{W}} \psi$ *and furthermore* $\mathrm{lh}(\varphi) = \mathrm{lh}(\psi)$. *Hence* $(\mathcal{N}/_{\equiv_{\mathrm{W}}}, \leq_{\mathrm{W}})$ *and* $(\mathcal{N}'/_{\equiv_{\mathrm{W}}}, \leq_{\mathrm{W}})$ *are isomorphic.*

Proof. Let $\varphi \in \mathcal{N}$ and let $\alpha := \mathrm{lh}(\varphi)$. Then $\alpha < \Theta$ and since Θ is a limit ordinal also $\alpha + 1 < \Theta$ and so by definition of Θ we can fix a surjection $f : \mathbb{R} \to \alpha + 1$, for which we furthermore assume that $f(0^{(\omega)}) = \alpha$. Then we define $\psi : \mathbb{R} \to \Theta$ by setting for any two $x, y \in \mathbb{R}$:

$$\psi(x * y) := \begin{cases} f(x), & \text{if } f(x) \leq \varphi(y), \\ \varphi(y), & \text{otherwise.} \end{cases}$$

Then we have for any $x, y \in \mathbb{R}$ that $\psi(x * y) \leq \varphi(y)$ and so that $\mathrm{lh}(\psi) \leq \mathrm{lh}(\varphi)$. But also for any $y \in \mathbb{R}$ we have that $\psi(0^{(\omega)} * y) = \varphi(y)$ and so indeed $\mathrm{lh}(\psi) = \mathrm{lh}(\varphi)$, as claimed.

To see that $\varphi \leq_{\mathrm{W}} \psi$ we note that the function $g : \mathbb{R} \to \mathbb{R}, x \mapsto 0^{(\omega)} * x$ is continuous and that for all $x \in \mathbb{R}$ we have that

$$\varphi(x) = \varphi(0^{(\omega)} * x) = \varphi(g(x)).$$

To see that $\psi \leq_{\mathrm{W}} \varphi$ we note that the function $h : \mathbb{R} \to \mathbb{R}, z \mapsto p_{\mathbf{II}}(z)$ is continuous and that for all $x, y \in \mathbb{R}$ we have that

$$\psi(x * y) \leq \varphi(y) = \varphi(p_{\mathbf{II}}(x * y)).$$

Thus we have that $\varphi \equiv_{\mathrm{W}} \psi$, as claimed.

Now to see that $(\mathcal{N}'/_{\equiv_{\mathrm{W}}}, \leq_{\mathrm{W}})$ and $(\mathcal{N}/_{\equiv_{\mathrm{W}}}, \leq_{\mathrm{W}})$ are isomorphic, we denote for every $\varphi \in \mathcal{N}'$ its equivalence class in $\mathcal{N}'/_{\equiv_{\mathrm{W}}}$ by $[\varphi]'_{\mathrm{W}}$ and its equivalence class in $\mathcal{N}/_{\equiv_{\mathrm{W}}}$ by $[\varphi]_{\mathrm{W}}$. Then we define a map $b : \mathcal{N}'/_{\equiv_{\mathrm{W}}} \to \mathcal{N}/_{\equiv_{\mathrm{W}}}, [\varphi]'_{\mathrm{W}} \to [\varphi]_{\mathrm{W}}$. Since we have that $(\mathcal{N}', \leq_{\mathrm{W}})$ is a substructure of $(\mathcal{N}, \leq_{\mathrm{W}})$ it is clear that b is well-defined, order-preserving and injective. To see that b is also surjective, we take any $\varphi \in \mathcal{N}$ and note that by what we have just shown there is a $\psi \in \mathcal{N}'$ such that $\varphi \equiv_{\mathrm{W}} \psi$ and so $b([\psi]'_{\mathrm{W}}) = [\varphi]'_{\mathrm{W}}$. This concludes the proof. □

Now taking this result into account and already noting the fact that the partial order $(\mathcal{N}/_{\equiv_{\mathrm{W}}}, \leq_{\mathrm{W}})$, which we call the *hierarchy of regular norms*, is far more amenable to the methods employed in Chapter 4 than the hierarchy of strictly regular norms, we will from now on solely work with the former, since we can directly transfer all results to the latter.

Then the question, which we are trying to answer, translates to the question whether the hierarchy of regular norms is isomorphic to the $(\Theta, \leq)$-Wadge hierarchy. To answer this we will first give a result about how the length of a regular norm relates to its place in the hierarchy of regular norms.

Proposition 3.1.9. *For any two regular norms* φ, ψ *with* $\varphi \leq_{\mathrm{W}} \psi$ *we have that* $\mathrm{lh}(\varphi) \leq \mathrm{lh}(\psi)$.

Proof. If $\varphi \leq_{\mathrm{W}} \psi$, then there is a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$ we have that $\varphi(x) \leq \psi(f(x))$. But then in particular we have that for any $x \in \mathbb{R}$ there is $y \in \mathbb{R}$ such that $\varphi(x) \leq \psi(y)$. This immediately implies that $\mathrm{lh}(\varphi) \leq \mathrm{lh}(\psi)$. □

This result just shows that for any $(\Theta, \leq)$-Wadge degree $c \in \Theta^{\mathbb{R}}/_{\equiv_{\mathrm{W}}}$ we have that for any two $\varphi, \psi \in c$, $\mathrm{lh}(\varphi) = \mathrm{lh}(\psi)$ and so we can extend the definition of length from $(\Theta, \leq)$-norms to $(\Theta, \leq)$-Wadge degrees by setting $\mathrm{lh}([\varphi]_{\mathrm{W}}) = \mathrm{lh}(\varphi)$ for any $\varphi \in \Theta^{\mathbb{R}}$. Thus again noting that $\mathcal{N} = \{\varphi \in \Theta^{\mathbb{R}} \mid \mathrm{lh}(\varphi) < \Theta\}$ we get that $\mathcal{N}/_{\equiv_{\mathrm{W}}} = \{c \in \Theta^{\mathbb{R}}/_{\equiv_{\mathrm{W}}} \mid \mathrm{lh}(c) < \Theta\}$. But this means that a sufficient criterion for the existence of an isomorphism between the hierarchy of regular norms and the $(\Theta, \leq)$-Wadge hierarchy is that there is no $(\Theta, \leq)$-norm φ with $\mathrm{lh}(\varphi) = \varphi$. In Section 3.3, assuming **AD** and **DC**, we will see that this criterion is also necessary. However, the validity of this criterion depends on the cofinality of Θ.

Proposition 3.1.10. *There is a* $(\Theta, \leq)$*-norm* φ *with* $\mathrm{lh}(\varphi) = \Theta$ *if and only if* Θ *is a singular cardinal. Therefore if* Θ *is a regular cardinal, then* $(\mathcal{N}/_{\equiv_{\mathrm{W}}}, \leq_{\mathrm{W}}) = (\Theta^{\mathbb{R}}/_{\equiv_{\mathrm{W}}}, \leq_{\mathrm{W}})$.

Proof. For the left-to-right direction let $\varphi \in \Theta^{\mathbb{R}}$ be such that $\mathrm{lh}(\varphi) = \Theta$. Then since there is no surjection $f : \mathbb{R} \to \Theta$ we have that $\varphi[\mathbb{R}] \subsetneq \Theta$ and so $\mathrm{otyp}(\varphi[\mathbb{R}]) < \Theta$. We let $\beta := \mathrm{otyp}(\varphi[\mathbb{R}])$ and let $\langle \nu_\alpha \mid \alpha < \beta \rangle$ be an enumeration of all ordinals in $\varphi[\mathbb{R}]$. Then since $\mathrm{lh}(\varphi) = \Theta$ we have that $\langle \nu_\alpha \mid \alpha < \beta \rangle$ is a sequence of ordinals cofinal in Θ. But since $\beta < \Theta$ we then have that $\mathrm{cf}(\Theta) \leq \beta < \Theta$ and so Θ is singular.

For the right-to-left direction we assume that Θ is singular. Then let $\langle \nu_\alpha \mid \alpha < \mathrm{cf}(\Theta) \rangle$ be a sequence of ordinals cofinal in Θ. Since $\mathrm{cf}(\Theta) < \Theta$ by assumption, we can fix a surjection $f : \mathbb{R} \to \mathrm{cf}(\Theta)$ and define a $(\Theta, \leq)$-norm φ by setting for all $x \in \mathbb{R}$,

$$\varphi(x) := \nu_{f(x)}.$$

But by the fact that $\langle \nu_\alpha \mid \alpha < \mathrm{cf}(\Theta) \rangle$ is cofinal in Θ then clearly $\mathrm{lh}(\varphi) = \Theta$. □

So, if we do not exclude the possibility of Θ being singular, it could be that the notion of the hierarchy of regular norms is not exactly captured by the notion of the $(\Theta, \leq)$-Wadge hierarchy. However, the main theorems of the next section, among them generalizations of Wadge's Lemma, the Martin–Monk Theorem and the Steel-Van Wesep Theorem, all transfer from a given $(Q, \leq)$-Wadge hierarchy to any substructure. So, since in any case the hierarchy of regular norms is a substructure of the $(\Theta, \leq)$-Wadge hierarchy and even an initial segment, we will be able to capture the hierarchy of norms in our framework in the sense that the theorems to be established immediately transfer to the hierarchy of norms via the $(\Theta, \leq)$-Wadge hierarchy.

Since we will also need to establish the results of the next sections for the partial order $(\mathcal{N}/_{\equiv_{\mathrm{L}}}, \leq_{\mathrm{L}})$, which we call the *Lipschitz hierarchy of regular norms*, we note that $(\mathcal{N}/_{\equiv_{\mathrm{L}}}, \leq_{\mathrm{L}})$ is a substructure of $(\Theta^{\mathbb{R}}/_{\equiv_{\mathrm{L}}}, \leq_{\mathrm{L}})$ by an argument completely analogous to the one for the hierarchy of regular norms.

3.2 Games and Wadge's Lemma

In this section we will show – assuming $\mathbf{AD}$ – a generalization of Wadge's Lemma for $(Q^{\mathbb{R}}, \leq_{\mathrm{L}})$ and $(Q^{\mathbb{R}}, \leq_{\mathrm{W}})$, from which we can obtain Wadge's Lemma for the original Wadge hierarchy and the linearity of the hierarchy of regular norms as immediate corollaries. As a first step we will define games that will characterize the relations $\leq_{\mathrm{L}}$ and $\leq_{\mathrm{W}}$ for any given quasi-order $(Q, \leq)$.

Definition 3.2.1 Let Q be an arbitary set, $R \subseteq Q \times Q$ an arbitrary binary relation on Q and $\varphi, \psi \in Q^{\mathbb{R}}$ arbitrary Q-norms.

Then we define a game $\mathrm{G}^R_{\mathrm{L}}(\varphi, \psi)$ by setting $M_{\mathrm{G}^R_{\mathrm{L}}(\varphi,\psi)} := \omega$ and

$$W_{\mathrm{G}^R_{\mathrm{L}}(\varphi,\psi)} := \{\langle x, y \rangle \in \mathbb{R} \times \mathbb{R} \mid \langle \varphi(x), \psi(y) \rangle \in R\}.$$

Furthermore we define a game $\mathrm{G}^R_{\mathrm{W}}(\varphi, \psi)$ by setting $M_{\mathrm{G}^R_{\mathrm{W}}(\varphi,\psi)} := \omega \cup \{\mathfrak{p}\}$ and

$$W_{\mathrm{G}^R_{\mathrm{W}}(\varphi,\psi)} := \{\langle x, y \rangle \in \mathbb{R} \times (\omega \cup \{\mathfrak{p}\})^\omega \mid \mathrm{filter}(y) \in \mathbb{R} \text{ and } \langle \varphi(x), \psi(\mathrm{filter}(y)) \rangle \in R\}.$$

It is clear that the games just defined are ω-games in the sense of Definition 2.2.2.

We can understand games of the form $\mathrm{G}^R_{\mathrm{L}}(\varphi, \psi)$ as follows. Player **I** and Player **II** alternate at playing natural numbers for ω-many turns as in the following schematic:

$$\begin{array}{lcccccccccc} \mathbf{I} & a_0 & & a_1 & & \ldots & a_n & & a_{n+1} & & \\ & & & & & & & & & & \ldots\ldots \\ \mathbf{II} & & b_0 & & b_1 & \ldots & & b_n & & b_{n+1} & \end{array}$$

The winning conditions are then given as follows. Let $a = \langle a_i \mid i \in \omega \rangle$ be the string of natural numbers played by Player **I** and $b = \langle b_i \mid i \in \omega \rangle$ the string of natural numbers played by Player **II**. Then player **II** wins if and only if $\langle \varphi(a), \psi(b) \rangle \in R$.

Analogously we can understand games of the form $\mathrm{G}^R_{\mathrm{W}}(\varphi, \psi)$ as follows. As before Player **I** and Player **II** alternate at playing. Thereby Player **I** can play natural numbers and Player **II** has the choice between playing a natural number or playing a non-integer move $\mathfrak{p}$ representing passing for one turn. A play of $\mathrm{G}^R_{\mathrm{W}}(\varphi, \psi)$ then looks as follows:

$$\begin{array}{lccccccccccccccc} \mathbf{I} & a_0 & & a_1 & & \ldots & & a_n & & a_{n+1} & & a_{n+2} & & \ldots & & a_m \\ & & & & & & & & & & & & & & & \ldots\ldots \\ \mathbf{II} & & \mathfrak{p} & & \mathfrak{p}\ldots\mathfrak{p} & & & & b_0 & & \mathfrak{p} & & \mathfrak{p}\ldots\mathfrak{p} & & b_1 & \end{array}$$

Then Player $\mathbf{II}$ looses if she makes non-$\mathfrak{p}$ moves only finitely often. Otherwise let $a = \langle a_i \mid i \in \omega \rangle$ be the string of natural numbers played by Player I and $b = \langle b_i \mid i \in \omega \rangle$ the string of natural numbers played by Player $\mathbf{II}$ (ignoring all $\mathfrak{p}$-moves). Then Player $\mathbf{II}$ wins if and only if $\langle \varphi(a), \psi(b) \rangle \in R$.

Next we will show how the games just defined relate to Lipschitz and Wadge hierarchies.

Proposition 3.2.2. *Let Q be an arbitrary set, $R \subseteq Q \times Q$ an arbitrary relation on Q and $\varphi, \psi \in Q^{\mathbb{R}}$ two Q-norms.*

Then we have that Player $\mathbf{II}$ wins the game $\mathrm{G}_{\mathrm{L}}^{R}(\varphi, \psi)$ if and only if there is a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$ we have $\langle x, f(x) \rangle \in R$.

Also we have that Player $\mathbf{II}$ wins the game $\mathrm{G}_{\mathrm{W}}^{R}(\varphi, \psi)$ if and only if there is a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$ we have $\langle x, g(x) \rangle \in R$.

Proof. We only show the second part of the proposition. The proof of the first part is completely analogous.

For the left-to-right direction we take a winning strategy $\sigma : \omega^{<\omega} \setminus \{\emptyset\} \to \omega \cup \{\mathfrak{p}\}$ for Player $\mathbf{II}$ in the game $\mathrm{G}_{\mathrm{W}}^{R}(\varphi, \psi)$. Then we have for any $x \in \mathbb{R}$ that $\langle x, \mathrm{filter}(\overline{\sigma}(x)) \rangle \in R$. Now we consider the function $h : \omega^{<\omega} \to \omega^{<\omega}, s \mapsto \mathrm{filter}(\overline{\sigma}(s))$. We note that h is monotone by construction. Furthermore it is infinitary, since σ is a winning strategy for $\mathbf{II}$ and therefore for any $x \in \mathbb{R}$ we have that $\mathrm{filter}(\overline{\sigma}(x)) \in \mathbb{R}$. Next we note that the map $g : \mathbb{R} \to \mathbb{R}, x \mapsto \mathrm{filter}(\overline{\sigma}(x))$ is induced by h and so a continuous function. Furthermore by what we have noted above, $\langle x, g(x) \rangle \in R$ for any $x \in \mathbb{R}$.

For the right-to-left direction we take a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$ we have that $\langle x, g(x) \rangle \in R$. Then we let $h : \omega^{<\omega} \to \omega^{<\omega}$ be a monotone and infinitary function inducing g. Now we define a strategy $\tau : \omega^{<\omega} \setminus \{\emptyset\} \to \omega \cup \{\mathfrak{p}\}$ for Player $\mathbf{II}$ in the game $\mathrm{G}_{\mathrm{W}}^{R}(\varphi, \psi)$ by setting for any $s \in \omega^{<\omega} \setminus \{\emptyset\}$:

$$\tau(s) := \begin{cases} (h(s))(\mathrm{lh}(h(s)) - 1), & \text{if } h(s\restriction_{\mathrm{lh}(s)-1}) \subsetneq h(s), \\ \mathfrak{p}, & \text{otherwise.} \end{cases}$$

Then by construction clearly for any $x \in \mathbb{R}$ we have that $\mathrm{filter}(\overline{\tau}(x)) = g(x)$ and so $\langle x, \mathrm{filter}(\overline{\tau}(x)) \rangle \in R$, which shows that τ is indeed winning for player $\mathbf{II}$. □

Corollary 3.2.3. *Let $(Q, \leq)$ be a quasi-order.*

Then we have that Player $\mathbf{II}$ wins the game $\mathrm{G}_{\mathrm{L}}(\varphi, \psi)$ if and only $\varphi \leq_{\mathrm{L}} \psi$. Furthermore we have that Player $\mathbf{II}$ wins the game $\mathrm{G}_{\mathrm{W}}(\varphi, \psi)$ if and only if $\varphi \leq_{\mathrm{W}} \psi$.

Now using this game-characterization of $\leq_{\mathrm{W}}$ and $\leq_{\mathrm{L}}$ we can prove a general analogue of Wadge's Lemma.

Theorem 3.2.4 (Wadge's Lemma)**.** *Assume* $\mathbf{AD}$. *Let $(Q, \leq)$ be an arbitrary quasi-order. Then for any two $\varphi, \psi \in Q^{\mathbb{R}}$ we have that either $\varphi \leq_{\mathrm{W}} \psi$ or there is a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$ we have $\varphi(f(x)) \not\leq \psi(x)$.*

Proof. We assume that $\varphi \not\leq_{\mathrm{W}} \psi$. Then Player $\mathbf{II}$ does not win the game $\mathrm{G}_{\mathrm{W}}^{\leq}(\varphi, \psi)$. Hence there is a winning strategy $\sigma : (\omega \cup \{\mathfrak{p}\})^{<\omega} \to \omega$ for Player $\mathbf{I}$ in $\mathrm{G}_{\mathrm{W}}^{\leq}(\varphi, \psi)$. Then in particular for any $x \in \mathbb{R}$, $\varphi(\overline{\sigma}(x)) \not\leq \psi(x)$. Since $\sigma\restriction_{\omega^{<\omega}}$ is clearly strictly infinitary, the map $\overline{\sigma}\restriction_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ is Lipschitz, which concludes the proof. □

Now we show how this general version of Wadge's Lemma implies Wadge's original lemma as stated in Theorem 2.3.1:

Corollary 3.2.5. *Assume* $\mathbf{AD}$. *Then for any $A, B \in \mathcal{P}(\mathbb{R})$ we have that either $A \leq_{\mathrm{W}}^{*} B$ or $B \leq_{\mathrm{L}}^{*} \mathbb{R} \setminus A$.*

Proof. Let $\chi_A, \chi_B \in \mathbf{TV}^{\mathbb{R}}$ be characteristic functions for the sets $A, B \in \mathcal{P}(\mathbb{R})$. We assume that $\chi_A \not\leq_{\mathbf{TV},\mathrm{W}} \chi_B$. Then by Wadge's Lemma there is a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$ we have $\chi_A(f(x)) \not\leq_{\mathbf{TV}} \chi_B(x)$ and so for all $x \in \mathbb{R}$ we have $x \in B$ if and only if $f(x) \notin A$. Thus for all $x \in \mathbb{R}$ we have that $\chi_B \leq_{\mathbf{TV},\mathrm{L}} \chi_{\mathbb{R}\setminus A}$. But by the correspondence between $\mathbf{TV}^{\mathbb{R}}$ and $\mathcal{P}(\mathbb{R})$ this shows the claim. □

Next we show how we can obtain the linearity of the hierarchy of regular norms, which is the statement of Theorem 2.3.8, from Wadge's Lemma as follows:

Corollary 3.2.6. *Assume* $\mathbf{AD}$*. If* $(Q, \leq)$ *is a linear order, then* $(Q^{\mathbb{R}}, \leq_{\mathrm{L}})$ *and* $(Q^{\mathbb{R}}, \leq_{\mathrm{W}})$ *are linear quasi-orders. Thus in particular the hierarchy of regular norms and the Lipschitz hierarchy of regular norms are linear orders.*

Proof. We show the claim for $(Q^{\mathbb{R}}, \leq_{\mathrm{L}})$. The argument for $(Q^{\mathbb{R}}, \leq_{\mathrm{W}})$ is analogous. Let φ, ψ be two Q-norms such that $\varphi \not\leq_{\mathrm{L}} \psi$. Then in particular $\varphi \not\leq_{\mathrm{W}} \psi$. So by Wadge's Lemma there is a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$ we have $\varphi(f(x)) \not\leq \psi(x)$ and thus by linearity of Q we have $\varphi(f(x)) > \psi(x)$ and so in particular $\psi(x) \leq \varphi(f(x))$, which shows that in fact $\psi \leq_{\mathrm{L}} \varphi$. Thus we have for any two Q-norms φ, ψ that $\varphi \leq_{\mathrm{L}} \psi$ or $\psi \leq_{\mathrm{L}} \varphi$, establishing the linearity of $(Q^{\mathbb{R}}, \leq_{\mathrm{L}})$.

In particular in case of $(\Theta, \leq)$ we get from this that $(\Theta^{\mathbb{R}}/_{\equiv_{\mathrm{L}}}, \leq_{\mathrm{L}})$ is a linear order, but since every substructure of a linear order is also a linear order we get that the Lipschitz hierarchy of regular norms is linear. The same argument works for the hierarchy of regular norms. □

3.3 Better Quasi-Orders and Well-Foundedness

In this section we are going to establish a generalization of the Martin–Monk Theorem, i.e., Theorem 2.3.2, stating under the assumption of $\mathbf{DC}$ and $\mathbf{AD}$ that the Wadge hierarchy is well-founded. It is, however, clearly not true that $(Q^{\mathbb{R}}, \leq_{\mathrm{L}})$ and $(Q^{\mathbb{R}}, \leq_{\mathrm{W}})$ are well-founded for every quasi-order $(Q, \leq)$, as can be seen from the example of a quasi-order that is itself ill-founded.

Proposition 3.3.1. *If* $(Q, \leq)$ *is a quasi-order such that there is a strictly* $\leq$*-decreasing sequence* $\langle q_i \mid i \in \omega \rangle \in Q^{\omega}$*, then both* $(Q^{\mathbb{R}}, \leq_{\mathrm{L}})$ *and* $(Q^{\mathbb{R}}, \leq_{\mathrm{W}})$ *are ill-founded.*

Proof. For any $q \in Q$ we let $c_q : \mathbb{R} \to Q, x \mapsto q$. Then the identity function on $\mathbb{R}$ witnesses for any $i \in \omega$ that $c_{q_{i+1}} \leq_{\mathrm{L}} c_{q_i}$ and $c_{q_{i+}} \leq_{\mathrm{W}} c_{q_i}$. On the other hand by assumption for any $i \in \omega$ and for any two $x, y \in \mathbb{R}$ we also have that $c_{q_{i+1}}(x) < c_{q_i}(y)$ and thus surely for no $i \in \omega$ there can be any function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$ we have $c_{q_{i+1}}(x) < c_{q_i}(f(x))$. But this shows that for any $i \in \omega$ we have that $c_{q_i} \not\leq_{\mathrm{L}} c_{q_{i+1}}$ and $c_{q_i} \not\leq_{\mathrm{W}} c_{q_{i+1}}$. Thus in total for all $i \in \omega$ we have that $c_{q_{i+1}} <_{\mathrm{L}} c_{q_i}$ and $c_{q_{i+1}} <_{\mathrm{W}} c_{q_i}$, which shows that the set $\{c_{q_i} \mid i \in \omega\}$ has neither a $\leq_{\mathrm{L}}$-minimal nor a $\leq_{\mathrm{W}}$-minimal element and thus that $(Q^{\mathbb{R}}, \leq_{\mathrm{L}})$ and $(Q^{\mathbb{R}}, \leq_{\mathrm{W}})$ are ill-founded. □

Thus we have to find a suitable restriction on $(Q, \leq)$ that ensures that the $(Q, \leq)$-Wadge hierarchy and the $(Q, \leq)$-Lipschitz hierarchy are well-founded. A clue to the right notion for this can be extracted from the article [vEMS87] by van Engelen, Miller and Steel. In the following definition, lemma and proposition I will repeat their result.

Definition 3.3.2 For any $X \in [\omega]^{\omega}$ arbitrary we define $X^* \in [\omega]^{\omega}$ by setting $X^* := X \setminus \{\min X\}$.

Let $(Q, \leq)$ be a quasi-order. Then a $(Q, \leq)$*-array* is a map $f : [\omega]^{\omega} \to Q$ such that $f[[\omega]^{\omega}]$ is countable and for any $q \in Q$ the set $f^{-1}[\{q\}]$ is a Borel set in $[\omega]^{\omega}$. We call a $(Q, \leq)$-array *good* iff there is $X \in [\omega]^{\omega}$ such that $f(X) \leq f(X^*)$. Otherwise it is *bad*.

Now we call a quasi-order $(Q, \leq)$ a *better quasi-order* (short: *BQO*) iff every $(Q, \leq)$-array is good.

Given any quasi-order $(Q, \leq)$ we define Q^* to be the set of all functions $\ell \in Q^{\mathbb{R}}$ such that $\ell[\mathbb{R}]$ is a countable set and for every $q \in Q$ the set $\ell^{-1}[\{q\}]$ is a Borel set in $\mathbb{R}$. Then we define a binary relation $\leq^*$ on Q^* by setting for any two $\ell_1, \ell_2 \in Q^*$

$$\ell_1 \leq^* \ell_2 \quad :\Leftrightarrow \quad \text{There is a continuous } f : \mathbb{R} \to \mathbb{R} \text{ such that for all } x \in \mathbb{R}\text{: } \ell_1(x) \leq \ell_1(f(x)).$$

First we repeat a remark made in the article [vEMS87] that every BQO is a WQO.

Lemma 3.3.3. *If* $(Q, \leq)$ *is a BQO, then it is a WQO.*

Proof. Let $(Q, \leq)$ be a quasi-order that is no WQO. Then there is a sequence $\langle q_i \mid i \in \omega \rangle$ of elements in Q such that for all $i, j \in \omega$ such that $i < j$ we have $i \not\leq j$. Now we construct a bad $(Q, \leq)$-array $f : [\omega]^{\omega} \to Q$ by setting

$$f(X) := q_{\min X}.$$

Clearly $f[[\omega]^{\omega}] = \{q_i \mid i \in \omega\}$ and so $f[[\omega]^{\omega}]$ is countable. Furthermore for any $q \in Q$, we either have $q \notin \{q_i \mid i \in \omega\}$ and so $f^{-1}[\{q\}] = \emptyset$, which is clearly an open set, or we have that $q = q_i$ for some

$i \in \omega$ and so $f^{-1}[\{q\}] = \{X \in [\omega]^\omega \mid \forall n < i\, (n \notin X) \wedge i \in X\}$, which by Lemma 2.1.4 is an open set in $[\omega]^\omega$. Now since every open set is a Borel set it follows that f is indeed a $(Q, \leq)$-array.

Then since for any $X \in [\omega]^\omega$ we have $\min X < \min X^*$ we get that $f(X) \not\leq f(X^*)$, which shows that f is bad and thus that $(Q, \leq)$ is no BQO. □

Now the following proposition is an immediate corollary of Theorem 3.2. in the article [vEMS87].

Proposition 3.3.4. *Assume* $\mathbf{AD}(\mathrm{Borel})$. *If* $(Q, \leq)$ *is a BQO, then* $(Q^*, \leq^*)$ *is a BQO and so* $(Q^*, \leq^*)$ *is a WQO. Thus additionally assuming* $\mathbf{DC}$ *we have that for every BQO* $(Q, \leq^*)$ *the quasi-order* $(Q^*, \leq^*)$ *is well-founded.*

We will modify the notion of a BQO to get an analogous result to Proposition 3.3.4 in our context. More precisely we will modify the notion of $(Q, \leq)$-array to our context. Considering the reliance of Proposition 3.3.4 on $\mathbf{AD}(\mathrm{Borel})$ we should drop the insistence that a $(Q, \leq)$-array should have Borel preimages. Furthermore the decision whether to drop the insistence on the countability of $(Q, \leq)$-arrays or not will give rise to two natural generalizations of the notion of a BQO, both of which will be of use in the following.

Definition 3.3.5 Let $(Q, \leq)$ be a quasi-order. Then a generalized $(Q, \leq)$-*array* is a map $f : [\omega]^\omega \to Q$. We call a generalized $(Q, \leq)$-array *countable* iff its image is countable. A (countable) generalized $(Q, \leq)$-array is *good* iff there is $X \in [\omega]^\omega$ such that $f(X) \leq f(X^*)$. Otherwise it is *bad*.

We call a quasi-order $(Q, \leq)$ a *strong better quasi-order* (short *sBQO*) iff every countable generalized $(Q, \leq)$-array is good. We call a quasi-order $(Q, \leq)$ a *very strong better quasi-order* (short *vsBQO*) iff every generalized $(Q, \leq)$-array is good.

The nomenclature of very strong better quasi-orders and strong better quasi-orders actually make sense, since the former is indeed a strengthening of the latter and both are a strengthening of the concept of a better quasi-order, as is expressed in the next lemma.

Lemma 3.3.6. *Let* $(Q, \leq)$ *be a quasi-order. Then we have the following chain of implications:*

$$\begin{aligned} (Q, \leq) \text{ is a vsBQO} \quad &\Rightarrow \quad (Q, \leq) \text{ is an sBQO.} \\ &\Rightarrow \quad (Q, \leq) \text{ is a BQO.} \\ &\Rightarrow \quad (Q, \leq) \text{ is a WQO.} \end{aligned}$$

Proof. For the first implication we note that every countable generalized $(Q, \leq)$-array is already a generalized $(Q, \leq)$-array. If now $(Q, \leq)$ is no sBQO, then there is a bad countable generalized $(Q, \leq)$-array, which already is a bad generalized $(Q, \leq)$-array. So $(Q, \leq)$ also is no vsBQO.

The second implication follows analogously, noting that every $(Q, \leq)$-array is a countable generalized $(Q, \leq)$-array.

The last implication, finally, is just Lemma 3.3.3. □

Our generalization of the Martin–Monk theorem will use a slight modification of the proof of Proposition 3.3.4 and will state that under the assumption of $\mathbf{AD}$ for every vsBQO $(Q, \leq)$ we have that $(Q^{\mathbb{R}}, \leq_{\mathrm{W}})$ and $(Q^{\mathbb{R}}, \leq_{\mathrm{L}})$ are sBQOs and therefore WQOs. But before we get to the proof of this statement we will check that the quasi-orders $(\mathbf{TV}, \leq_{\mathbf{TV}})$ and $(\Theta, \leq)$ are vsBQOs to see that said result will be applicable to the original Wadge hierarchy and the hierarchy of regular norms. First we note that any well-order and thus in particular $(\Theta, \leq)$ is a vsBQO.

Lemma 3.3.7. *Every well-order is a vsBQO.*

Proof. We assume for a contradiction that $(W, \leq)$ is a well-order that is no vsBQO. Then we take a bad generalized $(W, \leq)$-array f. Since $\leq$ is linear, this is just a map $f : [\omega]^\omega \to Q$ such that for all $X \in [\omega]^\omega$ we have $f(X) > f(X^*)$. So we take an arbitrary $X \in [\omega]^\omega$ and define a sequence $\langle X_n \mid n \in \omega \rangle$ on $[\omega]^\omega$ by recursively setting $X_0 := X$ and $X_{n+1} := X_n^*$ for any $n \geq 1$. Then by choice of f the sequence $\langle f(X_n) \mid n \in \omega \rangle$ is a strictly decreasing sequence on $(W, \leq)$, which is absurd, since $(W, \leq)$ is a well-order. □

Interestingly the case of $(\mathbf{TV}, \leq_{\mathbf{TV}})$ is not so straightforward. We already need a fragment of $\mathbf{AD}$ to establish that $(\mathbf{TV}, \leq_{\mathbf{TV}})$ is a vsBQO.

Lemma 3.3.8. *Then* $(\mathbf{TV}, \leq_{\mathbf{TV}})$ *is a vsBQO if and only if there is no flip-set* $F \subseteq 2^\omega$.

Proof. For the left-to-right direction we assume that there is a flip-set $F \subseteq 2^\omega$. Then we define a generalized $(\mathbf{TV}, \leq_{\mathbf{TV}})$-array $f : [\omega]^\omega \to \mathbf{TV}$ by setting for any $X \in [\omega]^\omega$

$$f(X) := \begin{cases} 0, & \text{if } \chi_X \in F, \\ 1, & \text{if } \chi_X \notin F. \end{cases}$$

Since for any $X \in [\omega]^\omega$ we have that the sequences $\chi_X \in 2^\omega$ and $\chi_{X^*} \in 2^\omega$ differ in exactly one bit and F is a flip set, we get that $\chi_X \in F$ if and only if $\chi_{X^*} \notin F$. So for any $X \in [\omega]^\omega$ we have that $f(X) \neq f(X^*)$ and since $(\mathbf{TV}, \leq_{\mathbf{TV}})$ is just an antichain of length 2 thus $f(X) \not\leq_{\mathbf{TV}} f(X^*)$, which shows that f is a bad generalized $(\mathbf{TV}, \leq_{\mathbf{TV}})$-array and thus that $(\mathbf{TV}, \leq_{\mathbf{TV}})$ is no vsBQO.

For the right-to-left direction we assume towards a contradiction that that there is no flip-set and $(\mathbf{TV}, \leq_{\mathbf{TV}})$ is no vsBQO. Then we take a bad generalized $(\mathbf{TV}, \leq_{\mathbf{TV}})$-array f, i.e., a map $f : [\omega]^\omega \to \mathbf{TV}$ such that for all $X \in [\omega]^\omega$ we have $f(X) \not\leq_{\mathbf{TV}} f(X^*)$ and so $f(X) \neq f(X^*)$. Now we define a set $F \subseteq 2^\omega$ by setting

$$\begin{aligned} F := &\{a \in 2^\omega \mid \exists^\infty n(a(n) = 1) \wedge f(\{n \in \omega \mid a(n) = 1\}) = 1\} \\ &\cup \{a \in 2^\omega \mid \forall^\infty n(a(n) = 0) \wedge \operatorname{card}(\{n \in \omega \mid a(n) = 1\}) \text{ is odd}\}. \end{aligned}$$

We show that F is a flip set. For this we take $a, b \in 2^\omega$ such that there is exactly one $k \in \omega$ such that $a(k) \neq b(k)$ and show that then we have that $a \in F$ if and only if $b \notin F$. This is immediate if $\forall^\infty n(a(n) = 0)$, since then also $\forall^\infty n(b(n) = 0)$ and furthermore $\operatorname{card}(\{n \in \omega \mid a(n) = 1\})$ is odd if and only if $\operatorname{card}(\{n \in \omega \mid b(n) = 1\})$ is even.

Now we consider the case that $\exists^\infty n(a(n) = 0)$. Then also $\exists^\infty n(b(n) = 0)$. Now let k be the index at which a and b differ. Without loss of generality we assume that $a = 0$ and $b = 1$. Then clearly $\{n \geq k \mid a(n) = 1\} = \{n > k \mid b(n) = 1\} = \{n \geq k \mid b(n) = 1\}^*$ and so by our choice of f:

$$f(\{n \geq k \mid b(n) = 1\}) \neq f(\{n \geq k \mid a(n) = 1\}).$$

Next we inductively show that for any $m \leq k$ we have

$$f(\{n \geq k - m \mid b(n) = 1\}) \neq f(\{n \geq k - m \mid a(n) = 1\})$$

by noting that the base case for $m = 0$ was just shown and that for any $m < k$ for which the claim holds we can distinguish the following two cases.

Case 1 is that $\{n \geq k - m - 1 \mid a(n) = 1\} = \{n \geq k - m \mid a(n) = 1\}$. Then we also have $\{n \geq k - m - 1 \mid b(n) = 1\} = \{n \geq k - m \mid b(n) = 1\}$, since a and b only differ at index k. By induction hypothesis we thus get

$$\begin{aligned} f(\{n \geq k - m - 1 \mid b(n) = 1\}) &= f(\{n \geq k - m \mid b(n) = 1\}) \\ &\neq f(\{n \geq k - m \mid a(n) = 1\}) = f(\{n \geq k - m - 1 \mid a(n) = 1\}). \end{aligned}$$

Case 2 is that $\{n \geq k - m - 1 \mid a(n) = 1\} \neq \{n \geq k - m \mid a(n) = 1\}$. Then

$$\{n \geq k - m \mid a(n) = 1\} = \{n \geq k - m - 1 \mid a(n) = 1\}^*$$

and also

$$\{n \geq k - m \mid b(n) = 1\} = \{n \geq k - m - 1 \mid b(n) = 1\}^*,$$

since a and b only differ at k. Thus we get that

$$f(\{n \geq k - m - 1 \mid a(n) = 1\}) \neq f(\{n \geq k - m \mid a(n) = 1\})$$

and

$$f(\{n \geq k - m - 1 \mid b(n) = 1\}) \neq f(\{n \geq k - m \mid b(n) = 1\}).$$

But since $\mathbf{TV}$ has exactly two elements and by induction hypothesis $f(\{n \geq k - m \mid b(n) = 1\}) \neq f(\{n \geq k - m \mid a(n) = 1\})$, we again get

$$f(\{n \geq k - m - 1 \mid b(n) = 1\}) \neq f(\{n \geq k - m - 1 \mid a(n) = 1\}),$$

concluding the induction.

Thus in particular we have shown for $m = k$ that

$$f(\{n \in \omega \mid b(n) = 1\}) \neq f(\{n \in \omega \mid a(n) = 1\})$$

and thus we get that $a \in F$ if and only if $b \notin F$, which shows that F is a flip-set. □

Corollary 3.3.9. *Assume* $\mathbf{AD}$. *Then* $(\mathbf{TV}, \leq_{\mathbf{TV}})$ *is a vsBQO.*

Proof. This is immediate, since by Proposition 2.2.10 $\mathbf{AD}$ implies the non-existence of flip-sets. □

Now we are ready to state and proof the main theorem of this section. As announced its proof is based on the proof of Proposition 3.3.4, which can be found in the article [vEMS87].

Theorem 3.3.10 (Van Engelen-Miller-Steel)**.** *Let* $(Q, \leq)$ *be a vsBQO. If all games of the form* $\mathrm{G}_{\mathrm{L}}^{\leq}(\varphi, \psi)$ *for* $\varphi, \psi \in Q^{\mathbb{R}}$ *are determined, then* $(Q^{\mathbb{R}}, \leq_{\mathrm{L}})$ *and* $(Q^{\mathbb{R}}, \leq_{\mathrm{W}})$ *are sBQOs.*

Thus if we assume $\mathbf{AD}$*, then for any vsBQO* $(Q, \leq)$ *we have that* $(Q^{\mathbb{R}}, \leq_{\mathrm{L}})$ *and* $(Q^{\mathbb{R}}, \leq_{\mathrm{W}})$ *are sBQOs.*

Proof. First we show by contraposition that if $(Q, \leq)$ is a vsBQO, then $(Q^{\mathbb{R}}, \leq_{\mathrm{L}})$ is an sBQO. So we assume that $(Q^{\mathbb{R}}, \leq_{\mathrm{L}})$ is no sBQO and let $\langle \psi_X \mid X \in [\omega]^\omega \rangle$ be a bad countable generalized $(Q^{\mathbb{R}}, \leq_{\mathrm{L}})$-array. So for any $X \in [\omega]^\omega$ we have that $\psi_X \not\leq_{\mathrm{L}} \psi_{X^*}$ and so Player $\mathbf{II}$ has no winning strategy in the games $\mathrm{G}_{\mathrm{L}}^{\leq}(\psi_X, \psi_{X^*})$, which by $\mathbf{AD}$ implies that Player $\mathbf{I}$ has a winning strategy in $\mathrm{G}_{\mathrm{L}}^{\leq}(\psi_X, \psi_{X^*})$. Using this we produce a bad generalized $(Q, \leq)$-array. By the definition of countable generalized arrays the set $\{\psi_X \mid X \in [\omega]^\omega\}$ is countable. So using $\mathbf{AC}_\omega(\mathbb{R})$ we can choose for any $X \in [\omega]^\omega$ a winning strategy $\sigma_X : \omega^{<\omega} \to \omega$ for Player $\mathbf{I}$ in the game $\mathrm{G}_{\mathrm{L}}^{\leq}(\psi_X, \psi_{X^*})$. Now let $X \in [\omega]^\omega$ be arbitrary. Then we define a sequence $\langle X_n \mid n \in \omega \rangle$ on $[\omega]^\omega$ recursively by setting $X_0 := X$ and $X_{n+1} = X_n^*$ for any $n \geq 1$. Then for each $n \in \omega$ we let $\sigma_n^X := \sigma_{X_n}$.

Based on this we get for any $X \in [\omega]^\omega$ a game sequence $\mathfrak{G}_X := \langle \langle \mathrm{G}_{\mathrm{L}}^{\leq}(\psi_{X_n}, \psi_{X_{n+1}}), \sigma_n^X \rangle \mid n \in \omega \rangle$. Since for all $X \in [\omega]^\omega$ and any $n \in \omega$ we have that σ_n^X is a strategy for Player $\mathbf{I}$ we get by Lemma 2.2.3 that for any $\mathfrak{G}_X$ there is a unique global play F^X.

Now we define a generalized $(Q, \leq)$-array $f : [\omega]^\omega \to Q$ by setting

$$f(X) := \psi_X(F^X(0)).$$

Furthermore this generalized array is bad, since for any $X \in [\omega]^\omega$ by construction of the global plays F^X we have that $f(X^*) = \psi_{X^*}(F^{X^*}(0)) = \psi_{X^*}(F^X(1))$. But $\overline{\sigma_0^X}(F^X(1)) = F^X(0)$ and σ_0^X is winning for $\mathbf{I}$ in $\mathrm{G}_{\mathrm{L}}^{\leq}(\psi_X, \psi_{X^*}$. So $\psi_X(F^X(0)) \not\leq \psi_{X^*}(F^X(1)) = \psi_{X^*}(F^{X^*}(0))$, i.e., $f(X) \not\leq f(X^*)$ and so f is a bad generalized $(Q, \leq)$-array, as claimed. This shows that if $(Q, \leq)$ is a vsBQO, then $(Q^{\mathbb{R}}, \leq_{\mathrm{L}})$ is an sBQO.

To conclude the proof we show that if $(Q^{\mathbb{R}}, \leq_{\mathrm{L}})$ is an sBQO, then also $(Q^{\mathbb{R}}, \leq_{\mathrm{W}})$ is an sBQO. To do this we consider an arbitrary countable generalized $(Q^{\mathbb{R}}, \leq_{\mathrm{W}})$-array f. Then f is also a countable generalized $(Q^{\mathbb{R}}, \leq_{\mathrm{L}})$-array. Furthermore by assumption f is a good countable generalized $(Q^{\mathbb{R}}, \leq_{\mathrm{L}})$-array, i.e., there is an $X \in [\omega]^\omega$ such that $f(X) \leq_{\mathrm{L}} f(X^*)$ and so a fortiori $f(X) \leq_{\mathrm{W}} f(X^*)$. Hence f is also a good countable generalized $(Q^{\mathbb{R}}, \leq_{\mathrm{W}})$-array. But since f was an arbitrary countable generalized $(Q^{\mathbb{R}}, \leq_{\mathrm{W}})$-array, this shows that indeed $(Q^{\mathbb{R}}, \leq_{\mathrm{W}})$ is an sBQO. □

As an immediate corollary we now get the following generalization of Theorem 2.3.2.

Corollary 3.3.11 (Martin – Monk)**.** *We assume* $\mathbf{AD}$ *and* $\mathbf{DC}$*. If* $(Q, \leq)$ *is a vsBQO, then both* $(Q^{\mathbb{R}}, \leq_{\mathrm{L}})$ *and* $(Q^{\mathbb{R}}, \leq_{\mathrm{W}})$ *are well-founded.*

The reason why we list the fragment of $\mathbf{AD}$ needed for Theorem 3.3.10 in such a fine-grained way, is that by this theorem and the fact that every substructure of a WQO is itself a WQO, we immediately get the following two corollaries:

Corollary 3.3.12. *We assume* $\mathbf{DC}$ *and that all games of the form* $\mathrm{G}_{\mathrm{L}}^{\leq_{\mathbf{TV}}}(\chi_A, \chi_B)$ *for* $\chi_A, \chi_B \in \mathbf{TV}^{\mathbb{R}}$ *are determined and that there is no flip set* $F \subseteq 2^\omega$*. Then the original Lipschitz hierarchy and the original Wadge hierarchy are well-founded.*

Corollary 3.3.13. *We assume* $\mathbf{DC}$ *and that all games of the form* $\mathrm{G}_{\mathrm{L}}^{\leq}(\varphi,\psi)$ *for* $\varphi,\psi \in \Theta^{\mathbb{R}}$ *are determined. Then the Lipschitz hierarchy of regular norms and the hierarchy of regular norms are well-founded. Furthermore the Lipschitz hierarchy of regular norms and the hierarchy of regular norms are well-orders.*

These corollaries look suspiciously like Theorem 2.3.4 and Theorem 2.3.10 and indeed we can obtain Theorem 2.3.4 and Theorem 2.3.10 by virtually the same argument we used to prove these corollaries. This reveals, why there is such a difference between the fragments of $\mathbf{AD}$ used in Theorem 2.3.4 for the original Wadge hierarchy and in Theorem 2.3.10 for the hierarchy of norms. The difference is that by Lemma 3.3.7 the well-order $(\Theta,\leq)$ can be proved to be a vsBQO in $\mathbf{ZF}$ alone, while to show the analogous result for $(\mathbf{TV},\leq_{\mathbf{TV}})$ in Lemma 3.3.8 we assumed the non-existence of flip-sets, which by Proposition 2.2.10 we cannot prove in $\mathbf{ZF}$ alone. From now on we will cease looking at fragments of $\mathbf{AD}$ in detail and just assume full $\mathbf{AD}$, whenever appropriate. In this spirit we can re-establish Theorem 2.3.2 and Theorem 2.3.9 by restating Corollaries 3.3.12 and 3.3.13 as follows:

Proposition 3.3.14. *Assume* $\mathbf{DC}$ *and* $\mathbf{AD}$. *Then the original Lipschitz hierarchy and the original Wadge hierarchy are well-founded.*

Proposition 3.3.15. *Assume* $\mathbf{DC}$ *and* $\mathbf{AD}$. *Then the hierarchy of regular norms and the Lipschitz hierarchy of regular norms are well-orders.*

Now whenever we assume $\mathbf{AD}$ and $\mathbf{DC}$ we can for any vsBQO $(Q,\leq)$ and any Q-norm $\varphi \in Q^{\mathbb{R}}$ define its *Wadge rank* $|\varphi|_{\mathrm{W}}$ by setting $|\varphi|_{\mathrm{W}} := \|\varphi\|_{\leq_{\mathrm{W}}}$ and its *Lipschitz rank* $|\varphi|_{\mathrm{L}}$ by setting $|\varphi|_{\mathrm{L}} := \|\varphi\|_{\leq_{\mathrm{L}}}$.

As final remark in this section we can conclude the comparison of the hierarchy of regular norms with $(\mathbb{R}^{\Theta}/_{\equiv_{\mathrm{W}}},\leq_{\mathrm{W}})$ we began in Section 3.1 with the following result:

Proposition 3.3.16. *Assume* $\mathbf{AD}$ *and* $\mathbf{DC}$. *Then* Θ *is a singular cardinal if and only if the hierarchy of norms* $(\mathcal{N}/_{\equiv_{\mathrm{W}}},\leq_{\mathrm{W}})$ *is isomorphic to* $(\mathbb{R}^{\Theta}/_{\equiv_{\mathrm{W}}},\leq_{\mathrm{W}})$.

Proof. We have already shown the left-to-right direction in Proposition 3.1.10.

For the right-to-left direction we assume that Θ is regular. Then also by Proposition 3.1.10 we have that there is no $(\Theta,\leq)$-norm φ with $\mathrm{lh}(\varphi)=\Theta$. But then by Proposition 3.1.9 $(\mathcal{N}/_{\equiv_{\mathrm{W}}},\leq_{\mathrm{W}})$ is a proper initial seqment of $(\mathbb{R}^{\Theta}/_{\equiv_{\mathrm{W}}},\leq_{\mathrm{W}})$. But since both $(\mathcal{N}/_{\equiv_{\mathrm{W}}},\leq_{\mathrm{W}})$ and $(\mathbb{R}^{\Theta}/_{\equiv_{\mathrm{W}}},\leq_{\mathrm{W}})$ are well-orders, this implies that $(\mathcal{N}/_{\equiv_{\mathrm{W}}},\leq_{\mathrm{W}})$ and $(\mathbb{R}^{\Theta}/_{\equiv_{\mathrm{W}}},\leq_{\mathrm{W}})$ are not isomorphic. □

3.4 Self-Duality and the Steel-Van Wesep Theorem

The notion of self-duality is a central notion for the original Wadge hierarchy. We recall that we call a set $A \in \mathcal{P}(\mathbb{R})$ self-dual if and only if $A \equiv_{\mathrm{W}}^{*} \mathbb{R}\backslash A$, which in fact is equivalent to $A \leq_{\mathrm{W}}^{*} \mathbb{R}\backslash A$. The distinction between self-dual and non-self-dual sets is important for the Wadge hierarchy in particular when we try to define operations on Wadge degrees, since often their properties differ depending on whether they are applied to self-dual or non-self-dual Wadge degrees. Many examples for this can be found in the book drafts [And01] and [And00].

On first glance one could conjecture that the usefulness of this distinction between self-dual and non-self-dual Wadge degrees is intimately connected to the fact that the original Wadge hierarchy has antichains of length 2 and that these are all of the form $\{[A]_{\mathrm{W}}^{*},[\mathbb{R}\setminus A]_{\mathrm{W}}^{*}\}$ for some $A \in \mathcal{P}(\mathbb{R})$. As a consequence there would be no such distinction for the hierarchy of norms, which under $\mathbf{AD}$ is a linear order. However, this is not true. Benedikt Löwe observed in unpublished notes [Löw10] based on ideas from Duparc's article [Dup03] that there is a notion of self-duality for the hierarchy of norms sharing many properties with the notion of self-duality in the case of the original Wadge hierarchy. He makes the following definition.

Definition 3.4.1 We call a regular norm φ *self-dual* iff Player $\mathbf{II}$ has a winning strategy in the game $\mathrm{G}_{\mathrm{W}}^{<}(\varphi,\varphi)$.

We will use this notion throughout Chapter 4, in which we will define several operations on regular norms which have certain desirable properties only when restricted to self-dual regular norms. Unifying the concept of self-duality for the original Wadge hierarchy and the hierarchy of regular norms, we now define the following general notion of self-duality.

Definition 3.4.2 Let $(Q, \leq)$ be a quasi-order. We call a Q-norm φ *L-self-dual* iff Player **II** has a winning strategy in the game $\mathrm{G}_{\mathrm{L}}^{\ngeq}(\varphi, \varphi)$. We call a Q-norm φ *W-self-dual* iff Player **II** has a winning strategy in the game $\mathrm{G}_{\mathrm{W}}^{\ngeq}(\varphi, \varphi)$. We call a Q-norm φ *self-dual* iff it is both L-self-dual and W-self-dual.

This notions extends to degrees as follows.

Lemma 3.4.3. *Let $(Q, \leq)$ be a quasi-order. Let φ and ψ be Q-norms such that $\varphi \equiv_{\mathrm{L}} \psi$. Then φ is L-self-dual if and only if ψ is L-self-dual. The same is true for L replaced with W.*

Proof. We only show the claim for the Lipschitz case. The Wadge case is completely analogous. We assume that $\varphi \equiv_{\mathrm{L}} \psi$ and φ is L-self-dual. Then we have three Lipschitz functions $f, g, h : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$

$$\begin{aligned} \varphi(f(x)) &\nleq \varphi(x), \\ \varphi(x) &\leq \psi(g(x)), \\ \psi(x) &\leq \varphi(h(x)). \end{aligned}$$

Then we consider the Lipschitz function $g \circ f \circ h : \mathbb{R} \to \mathbb{R}$ and claim that for all $x \in \mathbb{R}$ we have that $\psi(x) \ngeq \psi(g \circ f \circ h(x))$. To see this we assume for a contradiction that it is not the case, i.e., we take an $x \in \mathbb{R}$ such that $\psi(g \circ f \circ h(x)) \leq \psi(x)$. But then we get by the properties of the function f, g, h that

$$\varphi(f \circ h(x)) \leq \psi(g \circ f \circ h(x)) \leq \psi(x) \leq \varphi(h(x))$$

and so we have that $\varphi(f(h(x))) \leq \varphi(h(x))$, contradicting our choice of f. □

In light of this result we call a $(Q, \leq)$-Wadge degree c self-dual iff every $\varphi \in c$ is W-self-dual and we call a $(Q, \leq)$-Lipschitz degree c self-dual iff every every $\varphi \in c$ is L-self-dual. Next we see that these general notion of self-duality coincides with the ones for the original Wadge hierarchy and for the hierarchy of norms. For the original Wadge hierarchy we note the following sequence of equivalences

$$\begin{aligned} A \text{ is self-dual} \quad &\Leftrightarrow \quad A \leq_{\mathrm{W}}^{*} \mathbb{R} \setminus A \\ &\Leftrightarrow \quad \text{there is continuous } f : \mathbb{R} \to \mathbb{R} \text{ such that for all } x \in \mathbb{R} :\ x \in A \Leftrightarrow x \notin A \\ &\Leftrightarrow \quad \text{there is continuous } f : \mathbb{R} \to \mathbb{R} \text{ such that for all } x \in \mathbb{R} :\ \chi_A(x) \ngeq_{\mathbf{TV}} \chi_A(f(x)) \\ &\Leftrightarrow \quad \text{Player } \mathbf{II} \text{ wins the game } \mathrm{G}_{\mathrm{W}}^{\ngeq_{\mathbf{TV}}}(\chi_A, \chi_A), \end{aligned}$$

which shows that the new notion of W-self-duality in this case coincides with the old notion. For the hierarchy of norms we note that $(\Theta, \leq)$ is a linear order and so for any two $\alpha, \beta \in \Theta$ we have that $\alpha < \beta$ if and only if $\alpha \ngeq \beta$ and so vacuously we have for any $\varphi \in N$ that

$$\text{Player } \mathbf{II} \text{ wins } \mathrm{G}_{\mathrm{W}}^{<}(\varphi, \varphi) \quad \Leftrightarrow \quad \text{Player } \mathbf{II} \text{ wins } \mathrm{G}_{\mathrm{W}}^{\ngeq}(\varphi, \varphi).$$

Next we will state and prove the natural generalization of the Steel-Van Wesep theorem to this context. This proof is original work.

Theorem 3.4.4 (Steel-Van Wesep)**.** *Let $(Q, \leq)$ be a vsBQO. Then a Q-norm φ is L-self-dual if and only if it is W-self-dual.*

Proof. Clearly for any Q-norm φ, if Player **II** has a winning strategy in the game $\mathrm{G}_{\mathrm{L}}^{\ngeq}(\varphi, \varphi)$, then this is also a winning strategy in the game $GW^{\ngeq}(\varphi, \varphi)$. Hence if φ is L-self-dual, then φ is also W-self-dual.

We show the other direction by contraposition. For this we assume that there is a Q-norm φ such that φ is W-self-dual, but not L-self-dual. Then we show that $(Q, \leq)$ is no vsBQO.

We fix a winning strategy $\sigma_2 : \omega^{<\omega} \setminus \{\emptyset\} \to \omega \cup \{\mathfrak{p}\}$ for Player **II** in the game $\mathcal{G}_2 := \mathrm{G}_{\mathrm{W}}^{\ngeq}(\varphi, \varphi)$ and by $\mathbf{AD}$ furthermore a winning strategy $\sigma_0 : \omega^{<\omega} \to \omega$ for Player **I** in the game $\mathcal{G}_0 := \mathrm{G}_{\mathrm{L}}^{\ngeq}(\varphi, \varphi)$. Finally we define a winning strategy $\sigma_1 : \omega^{\omega} \setminus \{\emptyset\} \to \omega$ for Player **II** in the game $\mathcal{G}_1 := \mathrm{G}_{\mathrm{L}}^{=}(\varphi, \varphi)$ by setting for all $s \in \omega^{\omega} \setminus \{\emptyset\}$

$$\sigma_1(s) = s(\mathrm{lh}(s) - 1).$$

Now for any $x \in 3^\omega$ we define a game sequence $\mathfrak{G}_x := \langle\langle \mathcal{G}_{x(n)}, \sigma_{x(n)}\rangle \mid n \in \omega\rangle$. To construct a bad generalized $(Q, \leq)$-array we first construct a sequence $\langle M_k \mid k \in \omega\rangle$ of natural numbers such that for all $x \in 3^\omega$ with the property that $x(M_k) \in \{1, 2\}$ for all $k \in \omega$ and that $x(n) = 0$ for all $n \notin \{M_k \mid k \in \omega\}$ the game sequence $\mathfrak{G}_x$ admits a global play.

We define this sequence recursively. First we set $M_0 := 0$. Then we assume that we have already constructed the sequence $\langle M_k \mid k \leq j\rangle$ for some $j \in \omega$. Now for any $s \in 3^{M_j+1}$ such that

$$\begin{aligned} & \forall i \leq j\ (s(M_i) \in \{1, 2\}), && (*) \\ \text{and}\quad & \forall m \leq M_j\ (m \notin \{M_k \mid k \leq j\} \Rightarrow s(m) = 0), && (**) \end{aligned}$$

we define a natural number m_s as follows. Let $x_s := s ^\frown 0^{(\omega)}$. Then by Proposition 2.2.4 the game sequence $\mathfrak{G}_{x_s}$ admits a global play $F_s : \omega \to \mathbb{R}$. Now let $z_s := F_s(M_j + 1)$. Then for any $i \leq M_j$ we define a function $g_i : \omega^{\leq\omega} \to \omega^{\leq\omega}$ such that $g_i{\restriction}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ is continuous by setting for any $y \in \omega^{\leq\omega}$

$$g_i(y) := \mathrm{filter}(\overline{\sigma_{s(i)}}(y)).$$

Then we set $g := g_0 \circ g_1 \circ \cdots \circ g_{M_j}$. As a composition of continuous functions $g{\restriction}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ is also a continuous function and furthermore we have that $F_s(0) = g(z_s)$. The continuous function $g{\restriction}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ is induced by $g{\restriction}_{\omega^{<\omega}} : \omega^{<\omega} \to \omega^{<\omega}$ and thus we let m_s be the least $m \in \omega$ such that $g\,(z_s{\restriction}_m) \supseteq g(z_s){\restriction}_{M_j+1}$. Using this construction we finally define M_{j+1} by setting

$$M_{j+1} := M_j + \max\{m_s \mid s \in 3^{M_j+1} \text{ and } s \text{ satisfies } (*) \text{ and } (**)\},$$

concluding the construction of $\langle M_k \mid k \in \omega\rangle$.

Now for any $X \in \mathcal{P}(\omega)$ we define a sequence $z_X \in 3^\omega$ by setting for any $i \in \omega$

$$z_X(i) := \begin{cases} 1, & \text{if there is } k \in \omega \text{ s.t. } i = M_k \text{ and } k \notin X, \\ 2, & \text{if there is } k \in \omega \text{ s.t. } i = M_k \text{ and } k \in X, \\ 0, & \text{otherwise.} \end{cases}$$

But then by construction of the sequence $\langle M_k \mid k \in \omega\rangle$ we have that for every $X \in \mathcal{P}(\omega)$ the game sequence $\mathfrak{G}_{z_X}$ admits a unique global play, which we call F^X. Using this we define a generalized $(Q, \leq)$-array $G : [\omega]^\omega \to Q$ by setting for any $X \in [\omega]^\omega$

$$G(X) := \varphi(F^X(M_{\min X} + 1)).$$

Now we show that G is bad. For this we fix $X \in [\omega]^\omega$ and note that since for all $j \geq \min X^*$ we have $j \in X$ if and only if $j \in X^*$, we get that $F^X(M_{\min X^*} + 1) = F^{X^*}(M_{\min X^*} + 1)$ and therefore that $\varphi(F^X(M_{\min X^*} + 1)) = \varphi(F^{X^*}(M_{\min X^*} + 1))$. Now we show that for all $j \in \omega$ with

$$M_{\min X} + 1 \leq j \leq M_{\min X^*}$$

we have that $\varphi(F^X(M_{\min X^*})) \leq \varphi(F^X(j))$ using induction on $M_{\min X^*} - k$. If $j = \min X^*$, then nothing is to show. Now we assume for some j with $M_{\min X} + 1 < j \leq M_{\min X^*}$ that the claim has been established, i.e., that we have that $\varphi(F^X(M_{\min X^*})) \leq \varphi(F^X(j))$. Now we have to distinguish two cases.

Case 1 is that $(j - 1) \in \{M_k \mid k \in \omega\}$, say $j - 1 = M_i$. Then since $M_{\min X} < M_i < M_{\min X^*}$ we have that $i \notin X$ and so $z_X(j-1) = 1$, which implies that $F^X(j-1) = \mathrm{filter}(\overline{\sigma_1}(F^X(j)))$. But since σ_1 is a winning strategy for Player **II** in the game $G^{=}_{\mathrm{L}}(\varphi, \varphi)$, this implies that $\varphi(F^X(j-1)) = \varphi(F^X(j))$ and so by induction hypothesis $\varphi(F^X(M_{\min X^*})) \leq \varphi(F^X(j-1))$.

Case 2 is that $(j - 1) \notin \{M_k \mid k \in \omega\}$. Then $z_X(j-1) = 0$ and so $F^X(j-1) = \mathrm{filter}(\overline{\sigma_0}(F^X(j)))$. But since σ_0 is a winning strategy for Player **I** in the game $G^{\not\geq}_{\mathrm{W}}(\varphi, \varphi)$, this implies that $\varphi(F^X(j-1)) \geq \varphi(F^X(j))$ and so by induction hypothesis $\varphi(F^X(M_{\min X^*})) \leq \varphi(F^X(j-1))$.

This concludes the induction, in particular showing that

$$\varphi(F^X(M_{\min X^*})) \leq \varphi(F^X(M_{\min X} + 1)).$$

But now we note that since clearly $\min X^* \in X^*$ we have that

$$F(M_{\min X^*}) = \mathrm{filter}(\overline{\sigma_2}(F^X(M_{\min X^*} + 1))).$$

But since σ_2 is a winning strategy for Player **II** in the game $G^{\not\geq}(\varphi,\varphi)$ we thus get that $\varphi(F^X(M_{\min X^*}+1)) \not\geq \varphi(F^X(M_{\min X^*}))$. Since $\varphi(F^X(M_{\min X^*}+1)) = \varphi(F^{X^*}(M_{\min X^*}+1))$ we thus in total have that

$$\varphi(F^{X^*}(M_{\min X^*}+1)) \not\geq \varphi(F^X(M_{\min X^*}))$$

and

$$\varphi(F^X(M_{\min X^*})) \leq \varphi(F^X(M_{\min X}+1)).$$

Now we assume towards a contradiction that $\varphi(F^X(M_{\min X}+1)) \leq \varphi(F^{X^*}(M_{\min X^*}+1))$. Then we have $\varphi(F^X(M_{\min X^*})) \leq \varphi(F^{X^*}(M_{\min X^*}+1))$, a contradiction. Hence we have that

$$\varphi(F^X(M_{\min X}+1)) \not\leq \varphi(F^{X^*}(M_{\min X^*}+1)),$$

i.e., $G(X) \not\leq G(X^*)$, which shows that G is a bad generalized $(Q,\leq)$-array and hence that Q is no vsBQO, concluding the proof. □

The two corollaries for the original Wadge hierarchy and the hierarchy of norms re-establishing Theorem 2.3.5 and an analogous result for the hierarchy of regular norms are the following.

Corollary 3.4.5. *For any $A \in \mathcal{P}(\mathbb{R})$ we have that A is W-self-dual if and only if A is L-self-dual. Thus $A \leq^*_{\mathrm{W}} \mathbb{R}\setminus A$ if and only if $A \leq^*_{\mathrm{L}} \mathbb{R}\setminus A$.*

Corollary 3.4.6. *For any regular norm φ we have that φ is W-self-dual if and only if φ is L-self-dual. Thus Player **II** wins the game $\mathrm{G}^{<}_{\mathrm{W}}(\varphi,\varphi)$ if and only if Player **II** wins the game $\mathrm{G}^{<}_{\mathrm{L}}(\varphi,\varphi)$.*

We conclude this section with noting a very useful property of non-self-dual Q-norms, which we are going to use in Chapter 4.

Proposition 3.4.7. *Assume **AD**. Let $(Q,\leq)$ be an arbitrary vsBQO. For ψ a non-self-dual Q-norm and φ an arbitrary Q-norm we have that $\varphi \not\geq_{\mathrm{W}} \psi$ if and only if Player **II** wins the game $\mathrm{G}^{\not\geq}_{\mathrm{W}}(\varphi,\psi)$. Also we have that $\varphi \not\geq_{\mathrm{L}} \psi$ if and only if Player **II** wins the game $\mathrm{G}^{\not\geq}_{\mathrm{L}}(\varphi,\psi)$.*

Proof. We only show the first part of the proposition. The second part is completely analogous. For the left-to-right direction we assume that Player **II** wins the game $\mathrm{G}^{\not\geq}_{\mathrm{W}}(\varphi,\psi)$, but $\varphi \geq_{\mathrm{W}} \psi$. Then there are continuous functions $f, g : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$ we have $\varphi(x) \not\geq \psi(f(x))$ and $\psi(x) \leq \varphi(g(x))$. We claim that thus for all $x \in \mathbb{R}$ we have $\varphi(x) \not\geq \varphi(g \circ f(x))$. To see this we take towards a contradiction an $x \in \mathbb{R}$ such that $\varphi(x) \geq \varphi(g \circ f(x))$. Then in particular

$$\varphi(x) \geq \varphi(g(f(x))) \geq \psi(f(x)),$$

a contradiction. But since $g \circ f$ is a continuous function it now follows that Player **II** wins the game $\mathrm{G}^{\not\geq}_{\mathrm{W}}(\varphi,\varphi)$, contradicting the non-self-duality of φ.

For the right-to-left direction we assume that Player **II** loses the game $\mathrm{G}^{\not\geq}_{\mathrm{W}}(\varphi,\psi)$. But then by **AD** Player **I** has a winning strategy $\sigma : (\omega \cup \{\mathfrak{p}\})^{<\omega} \to \omega$ in $\mathrm{G}^{\not\geq}_{\mathrm{W}}(\varphi,\psi)$. Then we can define a winning strategy $\tau : \omega^{<\omega}\setminus\{\emptyset\} \to \omega$ for Player **II** in the game $\mathrm{G}^{\leq}_{\mathrm{L}}(\psi,\varphi)$ by setting for any $s \in \omega^{<\omega}\setminus\{\emptyset\}$

$$\tau(s) := \sigma(s\restriction_{\mathrm{lh}(s)-1}).$$

This is indeed winning for **II** since for any $x \in \mathbb{R}$ we have $\overline{\sigma}(x) = \overline{\tau}(x)$ and so

$$\psi(x) \leq \varphi(\overline{\sigma}(x)) = \varphi(\overline{\tau}(x)).$$

Thus we have $\psi \leq_{\mathrm{L}} \varphi$ and so $\psi \leq_{\mathrm{W}} \varphi$. □

An important corollary of this proposition is that for any non-self-dual $(Q,\leq)$-norm we have that $[\varphi]_{\mathrm{W}} = [\varphi]_{\mathrm{L}}$.

Corollary 3.4.8. *Assume **AD**. Let $(Q,\leq)$ be an arbitrary vsBQO. Then for any non-self-dual Q-norm φ and any Q-norm ψ we have that $\varphi \equiv_{\mathrm{L}} \psi$ if and only if $\varphi \equiv_{\mathrm{W}} \psi$.*

Proof. The left-to-right direction is obvious, since $\varphi \leq_{\mathrm{L}} \psi$ implies $\varphi \leq_{\mathrm{W}} \psi$ and $\psi \leq_{\mathrm{L}} \varphi$ implies $\psi \leq_{\mathrm{L}} \psi$.

For the right-to-left direction we assume towards a contradiction that $\varphi \equiv_{\mathrm{W}} \psi$, but $\varphi \not\equiv_{\mathrm{L}} \psi$. Since $\varphi \equiv_{\mathrm{W}} \psi$ and φ is non-self-dual by assumption we get that also ψ is non-self-dual. Without loss of generality we now assume that $\varphi \not\geq_{\mathrm{L}} \psi$. Then we have by Proposition 3.4.7 that Player **II** wins the game $\mathrm{G}_{\mathrm{L}}^{\not\geq}(\varphi, \psi)$, i.e., that there is a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$ we have that $\varphi(x) \not\geq \psi(f(x))$. But since every Lipschitz function is also continuous we note that Player **II** also wins the game $\mathrm{G}_{\mathrm{W}}^{\not\geq}(\varphi, \psi)$, which implies that $\varphi \not\geq_{\mathrm{W}} \psi$, but this contradicts the assumption that $\varphi \equiv_{\mathrm{W}} \psi$. □

3.5 An Alternative Characterization of Self-Duality

In this section we will provide an alternative characterization of self-duality in terms of well-foundedness of a certain tree. We will make use of this characterization in several places in Chapter 4. The proof that we are going to give is a generalization of the proof of this result in the context of the original Wadge hierarchy as presented in the book draft [And01].

Throughout this section we fix a vsBQO $(Q, \leq)$ and a minimal element $0_Q \in Q$, i.e., an element such that there is no $q \in Q$ with $q \neq 0_Q$ such that $q \leq 0_Q$.

Definition 3.5.1 1. For any $s \in \omega^{<\omega}$ and any Q-norm φ we define a Q-norm $\varphi_{\lfloor s \rfloor}$ by setting for all $x \in \mathbb{R}$:

$$\varphi_{\lfloor s \rfloor}(x) := \varphi(s \frown x)$$

2. For any Q-norm φ we define a tree $\mathbf{T}(\varphi) \subseteq \omega^{<\omega}$ by setting

$$\mathbf{T}(\varphi) := \{s \in \omega^{<\omega} \mid \varphi_{\lfloor s \rfloor} \equiv_{\mathrm{W}} \varphi\}.$$

3. For any Q-norm φ and any $s \in \omega^{<\omega}$ we define a Q-norm $\varphi^{\lceil s \rceil}$ by setting for any $x \in \mathbb{R}$:

$$\varphi^{\lceil s \rceil}(x) := \begin{cases} \varphi(x \setminus s), & \text{if } s \subseteq x, \\ 0_Q, & \text{otherwise.} \end{cases}$$

When no confusion is to be expected we will for any $n \in \omega$ write $\varphi_{\lfloor n \rfloor}$ to abbreviate $\varphi_{\lfloor \langle n \rangle \rfloor}$ and we will write $\varphi^{\lceil n \rceil}$ to abbreviate $\varphi^{\lceil \langle n \rangle \rceil}$.

We call a Q-norm φ *usual* iff there is an $x \in \mathbb{R}$ such that $0_Q \leq \varphi(x)$. The notion of usualness is a property of degrees as follows from the next result.

Lemma 3.5.2. *Let φ, ψ be Q-norms. If φ is usual and $\varphi \leq_{\mathrm{L}} \psi$, then also ψ is usual.*

Proof. By assumption there is a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$ we have $\varphi(x) \leq \psi(f(x))$. So taking x to be such that $0_Q \leq \varphi(x)$ we get that also $0_Q \leq \psi(f(x))$ and so ψ is usual. □

Lemma 3.5.3. *1. For any Q-norm φ and any $s, t \in \omega^{<\omega}$ we have $\left(\varphi_{\lfloor s \rfloor}\right)_{\lfloor t \rfloor} = \varphi_{\lfloor s \frown t \rfloor}$.*

2. For any Q-norm φ and for any $s \in \omega^{<\omega}$ we have $\varphi_{\lfloor s \rfloor} \leq_{\mathrm{L}} \varphi$.

3. For any self-dual Q-norm φ and any $n \in \omega$ we have that $\varphi <_{\mathrm{L}} \varphi^{\lceil n \rceil}$.

4. For any usual Q-norm φ and for any $n \in \omega$ we have that $\left(\varphi_{\lfloor n \rfloor}\right)^{\lceil n \rceil} \leq_{\mathrm{L}} \varphi$.

5. For any self-dual usual Q-norm φ and for any $n \in \omega$ we have that $\varphi_{\lfloor n \rfloor} <_{\mathrm{L}} \varphi$.

Proof. 1. We have for any $x \in \mathbb{R}$ that

$$\left(\varphi_{\lfloor s \rfloor}\right)_{\lfloor t \rfloor}(x) = \varphi_{\lfloor s \rfloor}(t \frown x) = \varphi(s \frown t \frown x) = \varphi_{\lfloor s \frown t \rfloor}.$$

2. The function $f : \mathbb{R} \to \mathbb{R}, x \mapsto s \frown x$ is Lipschitz and we have that for all $x \in \mathbb{R}$ that

$$\varphi_{\lfloor s \rfloor}(x) = \varphi(s \frown x) = \varphi(f(x))$$

and thus $\varphi_{\lfloor s \rfloor} \leq_{\mathrm{L}} \varphi$.

3. To see that $\varphi \leq_{\mathrm{L}} \varphi^{\lceil n\rceil}$ we take the Lipschitz function $f : \mathbb{R} \to \mathbb{R}, x \mapsto \langle n\rangle \frown s$. Then we have for any $x \in \mathbb{R}$

$$\varphi(x) = \varphi^{\lceil n\rceil}(\langle n\rangle \frown x) = \varphi^{\lceil n\rceil}(f(x))$$

and so $\varphi \leq_{\mathrm{L}} \varphi^{\lceil n\rceil}$.

To see that $\varphi^{\lceil n\rceil} \not\leq_{\mathrm{L}} \varphi$ we give a winning strategy $\sigma : \omega^{<\omega} \to \omega$ for Player **I** in the game $\mathrm{G}^{\leq}_{\mathrm{L}}\left(\varphi^{\lceil n\rceil}, \varphi\right)$ as follows. We note that by assumption φ is self-dual and thus Player **II** has a winning strategy $\tau : \omega^{<\omega} \setminus \{\emptyset\} \to \omega$ in $G^{\not\geq}_{\mathrm{L}}(\varphi, \varphi)$ and define σ by setting for any $s \in \omega^{<\omega}$

$$\sigma(\langle x_0, \ldots, x_n\rangle) := \begin{cases} n, & \text{if } s = \emptyset, \\ \tau(s), & \text{otherwise.} \end{cases}$$

Then for any $x \in \mathbb{R}$ we have that $\overline{\sigma}(x) = \langle n\rangle \frown \overline{\tau}(x)$ and so

$$\varphi^{\lceil n\rceil}(\overline{\sigma}(x)) = \varphi^{\lceil n\rceil}(\langle n\rangle \frown \overline{\tau}(x)) = \varphi(\overline{\tau}(x)) \not\leq \varphi(x),$$

where the last inequality follows from the fact that τ is winning for Player **II** in $G^{\not\geq}_{\mathrm{L}}(\varphi, \varphi)$. In total this shows that σ is indeed winning for **I** in $G^{\leq}_{\mathrm{L}}\left(\varphi^{\lceil n\rceil}, \varphi\right)$ and so $\varphi^{\lceil n\rceil} \not\leq_{\mathrm{L}} \varphi$ and thus in total $\varphi <_{\mathrm{L}} \varphi^{\lceil n\rceil}$, as claimed.

4. For any $x \in \mathbb{R}$ we have that

$$(\varphi_{\lfloor n\rfloor})^{\lceil n\rceil}(x) = \begin{cases} x, & \text{if } \langle n\rangle \subseteq x, \\ 0_Q, & \text{otherwise.} \end{cases}$$

Since φ is usual by assumption, we can fix a $z \in \mathbb{R}$ such that $0_Q \leq \varphi(z)$. Now we define a function $f : \mathbb{R} \to \mathbb{R}$ by setting for any $x \in \mathbb{R}$

$$f(x) := \begin{cases} x, & \text{if } x(0) = n, \\ y, & \text{otherwise.} \end{cases}$$

Then f is Lipschitz and furthermore we have for any $x \in \mathbb{R}$ that $(\varphi_{\lfloor n\rfloor})^{\lceil n\rceil}(x) \leq \varphi(f(x))$, which shows that $(\varphi_{\lfloor n\rfloor})^{\lceil n\rceil} \leq_{\mathrm{L}} \varphi$.

5. Putting together part 3 and part 4 of this lemma we get that $\varphi_{\lfloor n\rfloor} <_{\mathrm{L}} (\varphi_{\lfloor n\rfloor})^{\lceil n\rceil} \leq_{\mathrm{L}} \varphi$. ☐

Now we are ready to state and prove the main result of this section.

Proposition 3.5.4. *Assume* **AD**. *For any usual Q-norm φ, $\mathbf{T}(\varphi)$ is ill-founded if and only if φ is non-self-dual.*

Proof. For the left-to-right direction we assume that φ is non-self-dual and let $\sigma : (\omega \cup \{\mathfrak{p}\})^{<\omega} \to \omega$ be a winning strategy for Player **I** in $\mathrm{G}^{\not\geq}_{\mathrm{W}}(\varphi, \varphi)$. Then we consider the sequence $x := \overline{\sigma}\left(\mathfrak{p}^{(\omega)}\right)$ and note that for any $n \in \omega$ we can define a winning strategy $\sigma_n : (\omega \cup \{\mathfrak{p}\})^{<\omega} \to \omega$ for Player **I** in the game $\mathrm{G}^{\geq}_{\mathrm{W}}(\varphi_{\lfloor x\restriction_n\rfloor}, \varphi)$ by setting for any $s \in (\omega \cup \{\mathfrak{p}\})^{<\omega}$

$$\sigma_n(s) := \sigma(\mathfrak{p}^{(n)} \frown s).$$

This is indeed winning for Player **I**, since by construction we have for all $x \in (\omega \cup \{p\})^{\omega}$ that

$$\varphi_{\lfloor x\restriction_n\rfloor}(\overline{\sigma_n}(x)) = \varphi_{\lfloor x\restriction_n\rfloor}(\overline{\sigma}(\mathfrak{p}^{(n)} \frown s)) \geq \varphi(\mathrm{filter}((\mathfrak{p}^{(n)} \frown s)) = \varphi(\mathrm{filter}(s)).$$

From this we can for any $n \in \omega$ construct a winning strategy $\tau_n : \omega^{<\omega} \setminus \{\emptyset\} \to \omega$ for Player **II** in the game $\mathrm{G}^{\leq}_{\mathrm{L}}(\varphi, \varphi_{\lfloor x\restriction_n\rfloor})$ by setting for all $s \in \omega^{<\omega} \setminus \{\emptyset\}$

$$\tau_n(s) := \sigma_n(s\restriction_{\mathrm{lh}(s)-1}).$$

This is indeed winning for Player **II**, since for all $x \in \mathbb{R}$ we have that $\overline{\tau_n}(x) = \overline{\sigma_n}(x)$ and so

$$\varphi(x) \leq \varphi_{\lfloor x\restriction_n\rfloor}(\overline{\sigma_n}(x)) = \varphi_{\lfloor x\restriction_n\rfloor}(\overline{\tau_n}(x)).$$

This shows that for any $n \in \omega$ we have $\varphi \leq_{\mathrm{L}} \varphi_{\lfloor x\restriction_n \rfloor}$ and thus by Lemma 3.5.3 that $\varphi \leq_{\mathrm{L}} \varphi_{\lfloor x\restriction_n \rfloor}$ and a fortiori that $\varphi \leq_{\mathrm{W}} \varphi_{\lfloor x\restriction_n \rfloor}$, which shows that x is a path in $\mathbf{T}(\varphi)$. Thus $\mathbf{T}(\varphi)$ is ill-founded.

For the right-to-left direction we assume towards a contradiction that $\mathbf{T}(\varphi)$ is ill-founded and φ is self-dual. Then we take a path $x \in [\mathbf{T}(\varphi)]$; so we have $\varphi_{\lfloor x\restriction_n \rfloor} \equiv_{\mathrm{W}} \varphi$ for all $n \in \omega$. By Lemma 3.4.3 and Lemma 3.5.2 it then follows that for any $n \in \omega$ the Q-degree $\varphi_{\lfloor x\restriction_n \rfloor}$ is self-dual and usual. But by Lemma 3.5.3 we then get for any $n \in \omega$ that

$$\varphi_{\lfloor x\restriction_{n+1} \rfloor} = \varphi_{\lfloor x\restriction_n {}^\frown \langle x(n) \rangle \rfloor} = \left(\varphi_{\lfloor x\restriction_n \rfloor}\right)_{\lfloor x(n) \rfloor} <_{\mathrm{L}} \varphi_{\lfloor x\restriction_n \rfloor}$$

and thus we have an infinite $<_{\mathrm{L}}$-descending sequence in $Q^{\mathbb{R}}$, contradicting the well-foundedness of $\leq_{\mathrm{L}}$. □

The two corollaries for the original Wadge hierarchy and the hierarchy of norms are as follows, taking $0_{\mathbf{TV}}$ to be the element $0 \in \mathbf{TV}$ for the case of the original Wadge hierarchy.

Corollary 3.5.5. *Assume* $\mathbf{AD}$. *For any $A \in \mathcal{P}(\mathbb{R})$ with $A \neq \mathbb{R}$ we have that $\mathbf{T}(\chi_A)$ is well-founded if and only if A is self-dual.*

Proof. This follows from Proposition 3.5.4, since for any $A \in \mathcal{P}(\mathbb{R}) \setminus \{\mathbb{R}\}$ there is $x \in \mathbb{R}$ such that $\chi_A(x) = 0$ and so χ_A is usual. □

Corollary 3.5.6. *Assume* $\mathbf{AD}$. *For any regular norm we have that $\mathbf{T}(\varphi)$ is well-founded if and only if φ is self-dual.*

Proof. This follows from Proposition 3.5.4, since for any $\varphi \in \Theta^{\mathbb{R}}$ and any $x \in \mathbb{R}$ we have that $\varphi(x) \geq 0$ and so in particular any $\varphi \in \mathcal{N}$ is usual. □

Chapter 4

The Hierarchy of Regular Norms

In this chapter we will analyze the structure of the hierarchy of regular norms in detail. We will especially focus on establishing a lower bound for the order type of the hierarchy of regular norms. Using the abstract approach from Chapter 3 many results could be stated abstractly, which would thus provide the analogous result for the original Wadge hierarchy, but throughout this chapter we will only consider the hierarchy of regular norms.

4.1 The Ordinal Σ

As shown in Proposition 3.3.15 we have that, assuming $\mathbf{AD}$ and $\mathbf{DC}$, the hierarchy of regular norms $(\mathcal{N}/_{\equiv_{\mathrm{W}}}, \leq_{\mathrm{W}})$ is a well-order. So we define an ordinal Σ, the *length of the hierarchy of norms*, by setting $\Sigma := \mathrm{otyp}(\mathcal{N}/_{\equiv_{\mathrm{W}}}, \leq_{\mathrm{W}})$. Our main goal in this chapter will be to investigate the value of Σ. Before we get to new results we first repeat the formerly known best lower and upper bound for Σ. All results of this section can be found in the article [Löw05b].

4.1.1 A Lower Bound

We note that since by Proposition 3.1.9 we have for any two regular norms φ, ψ that $\mathrm{lh}(\varphi) < \mathrm{lh}(\psi)$ implies that $|\varphi|_{\mathrm{W}} < |\psi|_{\mathrm{W}}$. Therefore $(\mathcal{N}/_{\equiv_{\mathrm{W}}}, \leq_{\mathrm{W}})$ is stratified into Θ-many blocks each of which contains all norms of a given length. Furthermore we have the following result regarding regular norms of non-limit length.

Proposition 4.1.1. *Let $\alpha < \Theta$ be arbitrary. Then for any two regular norms φ, ψ with $\mathrm{lh}(\varphi) = \mathrm{lh}(\psi) = \alpha + 1$ we have $\varphi \equiv_{\mathrm{L}} \psi$.*

Proof. To see that $\varphi \leq_{\mathrm{L}} \psi$ we take an $y \in \mathbb{R}$ such that $\psi(y) = \alpha$ and consider the constant Lipschitz function $f : \mathbb{R} \to \mathbb{R}, x \mapsto y$. Then for all $x \in \mathbb{R}$ we have that $\varphi(x) \leq \psi(f(x))$, since $\mathrm{lh}(\varphi) = \alpha + 1$, and so $\varphi \leq_{\mathrm{L}} \psi$. By symmetry of the argument we get that $\varphi \equiv_{\mathrm{L}} \psi$. □

Thus for any ordinal α that is not a limit ordinal, the regular norms of length α form a single Lipschitz degree and therefore a single Wadge degree. Thus we can expect that the regular norms of non-limit length do not contribute much to the overall value of Σ. So we now focus on regular norms of limit length. We note that since Θ is a cardinal there are Θ-many limit ordinals strictly below Θ. This leads to the following definition:

Definition 4.1.2 We let $\langle \lambda_\alpha \mid \alpha < \Theta \rangle$ be an enumeration of all limit ordinals strictly below Θ.
For any $\alpha < \Theta$ we let $\vec{\alpha}$ denote the regular norm $\vec{\alpha} : \mathbb{R} \to \Theta, x \mapsto \alpha$, which is of length $\alpha + 1$.
Assuming $\mathbf{AD}$ and $\mathbf{DC}$ we define for any $\alpha < \Theta$ an ordinal Σ_α by setting

$$\Sigma_\alpha := \sup\{|\varphi|_{\mathrm{W}} \mid \mathrm{lh}(\varphi) \leq \lambda_\alpha\}.$$

In light of Proposition 3.1.9 and Proposition 4.1.2 – assuming $\mathbf{AD}$ and $\mathbf{DC}$ – we have for any $\alpha < \Theta$ that $\Sigma_\alpha = \left|\vec{\lambda_\alpha}\right|_{\mathrm{W}}$. Furthermore directly from the definition of the Σ_αs we get that

$$\Sigma = \sup\{\Sigma_\alpha \mid \alpha < \Theta\}.$$

Thus if we are able to obtain lower bounds for the Σ_αs, we immediately get a lower bound for Σ. Our goal throughout the rest of this subsection will be to show that for any $\alpha < \Theta$ we have that $\Sigma_\alpha \geq \Theta \cdot \alpha$, which already implies that $\Sigma \geq \Theta^2$.

Definition 4.1.3 For any regular norm φ we define a regular norm φ^+ by setting for all $x \in \mathbb{R}$

$$\varphi^+(x) := \begin{cases} \varphi(\mathbf{g}_{x^+}(x)) + 1, & \text{if } x(0) \neq 0, \\ \varphi(x^+), & \text{otherwise.} \end{cases}$$

Proposition 4.1.4. *If φ is a regular norm such that* $\mathrm{lh}(\varphi)$ *is a limit ordinal then* $\varphi <_{\mathrm{W}} \varphi^+$.

Proof. We assume towards a contradiction that $\varphi^+ \leq_{\mathrm{W}} \varphi$. Let $x \in \mathbb{R}$ be such that the continuous function $\mathbf{g}_x$ witnesses this, i.e., for all $y \in \mathbb{R}$, $\varphi^+(y) \leq \varphi(\mathbf{g}_x(y))$. Then we get

$$\varphi(\mathbf{g}_x(\langle 1\rangle ^\frown x)) \geq \varphi^+(\langle 1\rangle ^\frown x) = \varphi(g_x(\langle 1\rangle ^\frown x)) + 1 > \varphi(\mathbf{g}_x(\langle 1\rangle ^\frown x)),$$

which is absurd. □

Definition 4.1.5 For two regular norms φ, ψ we say that φ is *embedded in* ψ iff there is some $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, $\psi(x * y) = \varphi(y)$.

Lemma 4.1.6. *If φ is embedded in ψ, then $\varphi \leq_{\mathrm{L}} \psi$.*

Proof. Let $x \in \mathbb{R}$ be such that for all $y \in \mathbb{R}$ we have $\psi(x * y) = \varphi(y)$. Then we give a winning strategy $\sigma : \omega^{<\omega} \setminus \{\emptyset\} \to \omega$ for Player $\mathbf{II}$ in the game $\mathrm{G}_{\mathrm{L}}^{\leq}(\varphi, \psi)$ by setting for all $s \in \omega^{<\omega} \setminus \{\emptyset\}$

$$\sigma(s) := \begin{cases} x\left(\frac{\mathrm{lh}(s)}{2}\right), & \text{if } \mathrm{lh}(s) \text{ is even,} \\ s\left(\frac{\mathrm{lh}(s)-1}{2}\right), & \text{if } \mathrm{lh}(s) \text{ is odd.} \end{cases}$$

This strategy is indeed winning for $\mathbf{II}$ since for any $y \in \mathbb{R}$ we have $\overline{\sigma}(y) = x * y$ and so $\varphi(y) = \psi(\overline{\sigma}(y))$. □

Now we can establish a lower bound for the number of regular norms of a given limit length.

Lemma 4.1.7. *If $\lambda < \Theta$ is a limit ordinal and $\alpha < \Theta$ arbitrary, then there is a strictly $\leq_{\mathrm{W}}$-increasing sequence $\langle \varphi_\nu \mid \nu < \alpha\rangle$ such that for all $\nu < \alpha$ we have* $\mathrm{lh}(\varphi_\nu) = \lambda$.

Proof. Since $\alpha < \Theta$ we fix a surjection $s : \mathbb{R} \to \alpha$. Furthermore we fix a regular norm φ with $\mathrm{lh}(\varphi) = \lambda$. Now we define a strictly $\leq_{\mathrm{W}}$-increasing sequence $\langle \varphi_\nu \mid \nu < \alpha\rangle$ of norms of length λ recursively on ν as follows. We set $\varphi_0 := \varphi$. Then we assume that φ_ξ is already defined for $\xi < \nu$ and define a regular norm φ_ν^* by setting for all $x, y \in \mathbb{R}$

$$\varphi_\nu^*(x * y) := \begin{cases} \varphi_{s(x)}(y), & \text{if } s(x) < \nu, \\ \varphi(y), & \text{if } \varphi(y). \end{cases}$$

Since by induction hypothesis we have $\mathrm{lh}(\varphi_\xi) = \lambda$ this definition ensures that also $\mathrm{lh}(\varphi_\nu^*) = \lambda$. Furthermore for all $\xi < \nu$ we have by Lemma 4.1.6 that $\varphi_\xi \leq_{\mathrm{W}} \varphi_\nu^*$. Now we let $\varphi_\nu := (\varphi_\nu^*)^+$. Then we still have $\mathrm{lh}(\varphi_\nu) = \lambda$ and furthermore by Lemma 4.1.4 we get that $\varphi_\xi <_{\mathrm{W}} \varphi_\nu$ for all $\xi < \nu$, concluding the proof. □

Theorem 4.1.8. *We assume* **AD** *and* **DC**. *Let $\alpha < \Theta$. Then we have $\Sigma_\alpha \geq \Theta \cdot \alpha$. For $\alpha < \omega$ we even have that $\Sigma_\alpha \geq \Theta \cdot (\alpha + 1)$.*

Proof. We assume towards a contradiction that this claim is not true. Then let α be the least ordinal such that there is a regular norm φ with $\mathrm{lh}(\varphi) = \lambda_\alpha + 1$ and such that $|\varphi|_{\mathrm{W}} < \Theta \cdot (\alpha + 1)$, if $\alpha < \omega$, and $|\varphi|_{\mathrm{W}} < \Theta \cdot \alpha$ otherwise. We fix such a regular norm φ and distinguish the following cases.

Case 1 is that $\alpha = 0$. Then $\mathrm{lh}(\varphi) = \omega + 1$ and $|\varphi|_{\mathrm{W}} < \Theta$. But then by Lemma 4.1.7 there is a strictly $\leq_{\mathrm{W}}$-increasing sequence $\langle \psi_\nu \mid \nu < |\varphi|_{\mathrm{W}} + 1 \rangle$ of regular norms with $\mathrm{lh}(\psi_\nu) = \omega$ for all $\nu < |\varphi|_{\mathrm{W}} + 1$. But then for any $\nu < |\varphi|_{\mathrm{W}} + 1$ we can inductively establish that we have $|\psi_\nu| \geq \nu$. So in particular for $\psi := \psi_{|\varphi|_{\mathrm{W}}}$ we have $|\psi|_{\mathrm{W}} \geq |\varphi|_{\mathrm{W}}$, while $\mathrm{lh}(\psi) < \mathrm{lh}(\varphi)$, which is absurd.

Case 2 is that α is a successor ordinal, say $\gamma + 1$. Without loss of generality we assume that $\alpha > \omega$. The argument for $\alpha < \omega$ is very similar. Let ψ be a norm of length $\lambda_\gamma + 1$. Then by minimality of α we have that $|\psi|_{\mathrm{W}} \geq \Theta \cdot (\gamma + 1)$. Since, however, α was a counterexample, there is an ordinal $\zeta < \Theta$ such that $|\varphi|_{\mathrm{W}} = \Theta \cdot (\gamma + 1) + \zeta$. But then again by Lemma 4.1.7 we get a strictly $\leq_{\mathrm{W}}$-increasing sequence $\langle \psi_\nu \mid \nu < \zeta + 1 \rangle$ of regular norms with $\mathrm{lh}(\psi_\nu) = \lambda_\alpha$ for all $\nu < \zeta + 1$. Then since $|\psi_0|_{\mathrm{W}} \geq \Theta \cdot \gamma$ we get that for all $\nu < \zeta + 1$, $|\psi_\nu|_{\mathrm{W}} \geq \Theta \cdot \gamma + \nu$ and so in particular $|\psi_\zeta|_{\mathrm{W}} \geq \Theta \cdot \gamma + \zeta = |\varphi|_{\mathrm{W}}$, while $\mathrm{lh}(\psi) < \mathrm{lh}(\varphi)$, which is absurd.

Case 3 is that α is a limit ordinal. Then $\Theta \cdot \alpha = \bigcup_{\gamma < \alpha} \Theta \cdot \gamma$. So if $|\varphi|_{\mathrm{W}} < \Theta \cdot \alpha$ there is $\gamma < \alpha$ such that $|\varphi|_{\mathrm{W}} \leq \Theta \cdot \gamma$. But by minimality of γ we can take a regular norm ψ with $\mathrm{lh}(\psi) = \lambda_\gamma + 1$ and get that $|\psi|_{\mathrm{W}} \geq \Theta \cdot \gamma \geq |\varphi|_{\mathrm{W}}$ and so $\varphi \leq_{\mathrm{W}} \psi$, while $\mathrm{lh}(\psi) < \mathrm{lh}(\varphi)$, which is absurd. □

This now immediately gives us the desired lower bound for Σ.

Theorem 4.1.9. *We assume* **AD** *and* **DC**. *Then we have that $\Theta^2 \leq \Sigma$.*

Proof. Using Theorem 4.1.8 we can calculate that

$$\Sigma = \sup\{\Sigma_\alpha \mid \alpha < \Theta\} \geq \sup\{\Theta \cdot \alpha \mid \alpha < \Theta\} = \Theta^2.$$

□

4.1.2 An Upper Bound

In this subsection we will give a combinatorial argument that $\Sigma < \Theta^+$, thereby providing an upper bound for Σ. As in the last subsection the content of this subsection is taken from the article [Löw05b].

We start by introducing a few new pieces of notation that we are going to use in the following.

Definition 4.1.10 For any ordinal ξ we let $\Phi_\xi := \{\varphi \in \mathcal{N}' \mid |\varphi|_{\mathrm{W}} = \xi\}$.

For any strictly regular norm φ we define $X_\varphi \subseteq \mathbb{R}$ to be $X_\varphi := \{x * y \mid \varphi(x) \leq \varphi(y)\}$.

We now note that for the assignment of strictly regular norms φ to X_φ is injective.

Lemma 4.1.11. *For two strictly regular norms φ, ψ we have that $X_\varphi = X_\psi$ implies that $\varphi = \psi$.*

Proof. Let φ, ψ two strictly regular norms such that $X_\varphi = X_\psi$. Then defining binary relations $\leq_\varphi, \leq_\psi$ by setting for all $x, y \in \mathbb{R}$

$$\begin{aligned} x \leq_\varphi y \quad &:\Leftrightarrow \quad \varphi(x) \leq \varphi(y), \\ x \leq_\psi y \quad &:\Leftrightarrow \quad \psi(x) \leq \psi(y), \end{aligned}$$

we get for all $x, y \in \mathbb{R}$ that $x \leq_\varphi y$ if and only if $x \leq_\psi y$. Using this we show now that $\varphi = \psi$ by showing inductively that for all $\alpha < \omega$ we have that $\varphi^{-1}[\{\alpha\}] = \psi^{-1}[\{\alpha\}]$ as follows. For any given $\alpha < \Theta$ we assume that for all $\alpha' < \alpha$ we already have that $\varphi^{-1}[\{\alpha'\}] = \psi^{-1}[\{\alpha'\}]$. Then by surjectivity of φ we get that

$$\varphi^{-1}[\{\alpha\}] = \left\{ x \in \mathbb{R} \setminus \left(\bigcup_{\alpha' < \alpha} \varphi^{-1}[\{\alpha'\}] \right) \;\middle|\; x \text{ is } \leq_\varphi\text{-minimal in } \mathbb{R} \setminus \left(\bigcup_{\alpha' < \alpha} \varphi^{-1}[\{\alpha'\}] \right) \right\}.$$

Analogously we get

$$\psi^{-1}[\{\alpha\}] = \left\{ x \in \mathbb{R} \setminus \left(\bigcup_{\alpha' < \alpha} \psi^{-1}[\{\alpha'\}] \right) \;\middle|\; x \text{ is } \leq_\psi\text{-minimal in } \mathbb{R} \setminus \left(\bigcup_{\alpha' < \alpha} \psi^{-1}[\{\alpha'\}] \right) \right\}.$$

But then since for all $\alpha' < \alpha$ we have that $\varphi^{-1}[\{\alpha'\}] = \psi^{-1}[\{\alpha'\}]$ and since the relations $\leq_\varphi$ and $\leq_\psi$ are identical, this already implies that $\varphi^{-1}[\{\alpha\}] = \psi^{-1}[\{\alpha\}]$. □

Now we note that as shown in Section 3.1 we have that $(\mathcal{N}/_{\equiv_{\mathrm{W}}}, \leq_{\mathrm{W}})$ and $(\mathcal{N}'/_{\equiv_{\mathrm{W}}}, \leq_{\mathrm{W}})$ are isomorphic and therefore $\mathrm{otyp}(\mathcal{N}'/_{\equiv_{\mathrm{W}}}, \leq_{\mathrm{W}}) = \Sigma$. Using this we obtain the following result.

Theorem 4.1.12. *Assume* $\mathbf{AD}$ *and* $\mathbf{DC}$. *Then we have that* $\Sigma < \Theta^+$.

Proof. We assume towards a contradiction that $\Phi_\xi \neq \emptyset$ for all $\xi < \Theta^+$. Then we define a map $w : \Theta^+ \to \Theta$ by setting for all $\xi < \Theta$:

$$w(\xi) := \min\{\alpha \mid \exists \varphi \in \mathcal{N}' \ \left(|\varphi|_{\mathrm{W}} = \xi \wedge |X_\varphi|^*_{\mathrm{W}} = \alpha\right)\}.$$

Now by Lemma 2.1.2 we have that there is an ordinal $\alpha < \Theta$, a subset $S \subseteq \Theta^+$ and a bijection $b : \Theta \to S$ such that we have that $w[S] = \{\alpha\}$.

Now for any $\xi \in S$ we set

$$H_\xi := \{X_\varphi \in \mathcal{P}^{(\alpha)}(\mathbb{R}) \mid |\varphi|_{\mathrm{W}} = \xi \wedge |X_\varphi|^*_{\mathrm{W}} = \alpha\}.$$

Then the sequence $\langle H_{b(\gamma)} \mid \gamma < \Theta \rangle$ is a Θ-sequence of elements of $\mathcal{P}^{(\alpha)}(\mathbb{R})$. But then we see that there must be two $\gamma_0, \gamma_1 < \Theta$ such that $H_{b(\gamma_0)} \cap H_{b(\gamma_1)} \neq \emptyset$, since otherwise the function $f : \mathcal{P}^{(\alpha)}(\mathbb{R}) \to \Theta, A \mapsto \min\{\gamma \mid A \in H_{b(\gamma)}\}$ would be a surjection and since there is a surjection $g : \mathbb{R} \to \mathcal{P}^{(\alpha)}$ we would get that there is a surjection $f \circ g : \mathbb{R} \to \Theta$, yielding a contradiction.

Thus we take two ordinals $\gamma_0, \gamma_1 < \Theta$ such that $H_{b(\gamma_0)} \cap H_{b(\gamma_1)} \neq \emptyset$. But if we take $X_\varphi \in H_{b(\gamma_0)} \cap H_{b(\gamma_1)}$, then we have

$$|\varphi|_{\mathrm{W}} = b(\gamma_0) \neq b(\gamma_1) = |\varphi|_{\mathrm{W}},$$

a contradiction. □

Thus in total we have – assuming $\mathbf{AD}$ and $\mathbf{DC}$ – established that $\Theta^2 \leq \Sigma < \Theta^+$. The rest of this thesis will be devoted to improving the lower bound for Σ, while we will note refer to the upper bound again before Chapter 5.

4.2 Self-Duality

The aim of this section will be twofold. First, we are going to characterize self-dual degrees in the hierarchy of regular norms by properties of their Wadge rank. Secondly, we are going to show that there are both unboundedly many self-dual and non-self-dual degrees in the hierarchy of regular norms. Towards this end we will introduce several operations on $\mathcal{N}$ which will be of use later on.

First we begin with introducing the join operation, which on the level of Wadge ranks as well as Lipschitz ranks corresponds to taking suprema of countably many ranks.

Definition 4.2.1 We call a subset $\mathcal{U} \subseteq \omega^{<\omega}$ a partition of $\omega^{<\omega}$ iff for any two $u, u' \in \mathcal{U}$ with $u \neq u'$ we have $u ^\frown \omega^{<\omega} \cap u' ^\frown \omega^{<\omega} = \emptyset$ and we have $\bigcup_{u \in \mathcal{U}} u ^\frown \omega^{<\omega} = \omega^{<\omega}$.

Given a partition $\mathcal{U}$ of $\omega^{<\omega}$ and a family $\{\varphi_u \mid u \in \mathcal{U}\}$ of regular norms we define a regular norm $\bigoplus_{u \in \mathcal{U}} \varphi_u$, the *join of* $\{\varphi_u \mid u \in \mathcal{U}\}$ by setting for any $v \in \mathcal{U}$ and any $x \in \mathbb{R}$:

$$\left(\bigoplus_{u \in \mathcal{U}} \varphi_u\right)(v ^\frown x) := \varphi_v(x).$$

An important special case for a partition of $\omega^{<\omega}$ is the set $\mathcal{U}_\omega = \{\langle n \rangle \mid n \in \omega\}$. For a family $\{\varphi_n \mid n \in \omega\}$ of regular norms we define a family $\{\varphi'_u \mid u \in \mathcal{U}_\omega\}$ by setting $\varphi'_{\langle n \rangle} = \varphi_n$ for all $n \in \omega$. We then write $\bigoplus_{n \in \omega} \varphi_n$ to denote the regular norm $\bigoplus_{u \in \mathcal{U}_\omega} \varphi'_u$.

Now we show that the join operation respects $\leq_{\mathrm{L}}$ and $\leq_{\mathrm{W}}$ and therefore is an operation on degrees.

Lemma 4.2.2. *Assume* $\mathbf{AC}_\omega(\mathbb{R})$. *If* $\mathcal{U}$ *is a partition of* $\omega^{<\omega}$ *and* $\{\varphi_u \mid u \in \mathcal{U}\}$ *and* $\{\varphi'_u \mid u \in \mathcal{U}\}$ *are two families of regular norms such that for all* $u \in \mathcal{U}$ *we have that* $\varphi_u \leq_{\mathrm{L}} \varphi'_u$, *then we have that* $\bigoplus_{u \in \mathcal{U}} \varphi_u \leq_{\mathrm{L}} \bigoplus_{u \in \mathcal{U}} \varphi'_u$. *The same is true with* $\leq_{\mathrm{L}}$ *replaced everywhere with* $\leq_{\mathrm{W}}$.

Proof. We show the claim for $\leq_{\mathrm{L}}$. The argument for $\leq_{\mathrm{W}}$ is analogous. We assume that $\varphi_u \leq_{\mathrm{L}} \varphi'_u$ for all $u \in \mathcal{U}$. Since $\mathcal{U} \subseteq \omega^{<\omega}$ is a countable set, we can then by $\mathbf{AC}_\omega(\mathbb{R})$ choose a winning strategy $\sigma_u : \omega^{<\omega} \setminus \{\emptyset\} \to \omega$ for Player **II** in the game $\mathrm{G}^{\leq}_{\mathrm{L}}(\varphi_u, \varphi'_u)$ for any $u \in \mathcal{U}$. Then we give a winning strategy $\sigma : \omega^{<\omega} \setminus \{\emptyset\} \to \omega$ for Player **II** in the game $\mathrm{G}^{\leq}_{\mathrm{L}}\left(\bigoplus_{u\in\mathcal{U}} \varphi_u, \bigoplus_{u\in\mathcal{U}} \varphi'_u\right)$ by setting for all $s \in \omega^{<\omega} \setminus \{\emptyset\}$

$$\sigma_u(s) := \begin{cases} s(\mathrm{lh}(s)-1), & \text{if there is } u \in \mathcal{U} \text{s.t. } s \subseteq u, \\ \sigma_u(s \setminus u), & \text{if there is } u \in \mathcal{U} \text{s.t. } u \subsetneq s. \end{cases}$$

This strategy is indeed winning for Player **II**, since for any $v \in \mathcal{U}$ and $x \in \mathbb{R}$ we have that $\overline{\sigma}(v \frown x) = v \frown \overline{\sigma_v}(x)$ and so we have

$$\left(\bigoplus_{u\in\mathcal{U}} \varphi_u\right)(v \frown x) = \varphi_v(x) \leq \varphi'_v(\overline{\sigma_v}(x)) = \left(\bigoplus_{u\in\mathcal{U}} \varphi_u\right)(v \frown \overline{\sigma_v}(x)) = \left(\bigoplus_{u\in\mathcal{U}} \varphi_u\right)(\overline{\sigma}(v \frown x)).$$

□

Thus assuming $\mathbf{AD}$, which we need to choose representatives for countably many degrees simultaneously, we can extend the join operation to Lipschitz and Wadge degrees by setting for a partition $\mathcal{U}$ and a family $\{\varphi_u \mid u \in \mathcal{U}\}$ of regular norms:

$$\bigoplus_{u\in\mathcal{U}} [\varphi_u]_{\mathrm{L}} := \left[\bigoplus_{u\in\mathcal{U}} \varphi_u\right]_{\mathrm{L}},$$

$$\text{and} \quad \bigoplus_{u\in\mathcal{U}} [\varphi_u]_{\mathrm{W}} := \left[\bigoplus_{u\in\mathcal{U}} \varphi_u\right]_{\mathrm{W}}.$$

Next we are going to show that on the level of Wadge ranks and with certain restrictions also of Lipschitz ranks the join indeed acts as a supremum operation.

Lemma 4.2.3. *Let $\mathcal{U}$ be a partition of $\omega^{<\omega}$ and $\{\varphi_u \mid u \in \mathcal{U}\}$ a family of regular norms. Then we have for all $v \in \mathcal{U}$ that $\varphi_v \leq_{\mathrm{L}} \bigoplus_{u\in\mathcal{U}} \varphi_u$. The same is true with $\leq_{\mathrm{L}}$ replaced with $\leq_{\mathrm{W}}$ everywhere.*

Proof. We only show this for $\leq_{\mathrm{L}}$; the argument for $\leq_{\mathrm{W}}$ is completely analogous. We fix $v \in \mathcal{U}$. Then a winning strategy $\sigma : \omega^{<\omega} \setminus \{\emptyset\} \to \omega$ for Player **II** in $\mathrm{G}^{\leq}_{\mathrm{L}}\left(\varphi_v, \bigoplus_{u\in\mathcal{U}} \varphi_u\right)$ is given by setting for all $s \in \omega^{<\omega} \setminus \{\emptyset\}$

$$\sigma(s) := \begin{cases} v(\mathrm{lh}(s)-1), & \text{if } \mathrm{lh}(s) \leq \mathrm{lh}(v), \\ s(\mathrm{lh}(s)-(\mathrm{lh}(v)+1)) & \text{if } \mathrm{lh}(s) > \mathrm{lh}(v). \end{cases}$$

This is indeed winning, since for any $x \in \mathbb{R}$ we have $\overline{\sigma}(x) = v \frown x$ and so

$$\varphi_v(x) = \left(\bigoplus_{u\in\mathcal{U}} \varphi_u\right)(v \frown x) = \left(\bigoplus_{u\in\mathcal{U}} \varphi_u\right)(\overline{\sigma}(x)).$$

□

Lemma 4.2.4. *We assume* $\mathbf{AD}$. *If $\mathcal{U} \subseteq \omega^{<\omega}$ is a partition of $\omega^{<\omega}$, then for any regular norm ψ and any family $\{\varphi_u \mid u \in \mathcal{U}\}$ of regular norms such that $\psi <_{\mathrm{W}} \bigoplus_{u\in\mathcal{U}} \varphi_u$ there is a $u \in \mathcal{U}$ such that $\psi <_{\mathrm{W}} \varphi_u$.*

Proof. By assumption Player **II** has no winning strategy in $\mathrm{G}^{\leq}_{\mathrm{W}}\left(\bigoplus_{u\in\mathcal{U}} \varphi_u, \psi\right)$. Therefore by $\mathbf{AD}$ Player **I** wins $\mathrm{G}^{\leq}_{\mathrm{W}}\left(\bigoplus_{u\in\mathcal{U}} \varphi_u, \psi\right)$ with a winning strategy $\sigma : (\omega \cup \{\mathfrak{p}\})^{<\omega} \to \omega$. Since $\mathcal{U}$ is a partition of $\omega^{<\omega}$ there is a unique $u \in \mathcal{U}$ and a unique $n \in \omega$ such that $u = \overline{\sigma}\left(\mathfrak{p}^{(n)}\right)$. We fix this u and this n and give a winning strategy $\tau : (\omega \cup \{\mathfrak{p}\})^{<\omega} \to \omega$ for Player **I** in $\mathrm{G}^{\leq}_{\mathrm{L}}(\varphi_u, \psi)$ by setting for any $s \in (\omega \cup \{\mathfrak{p}\})^{<\omega}$

$$\tau(s) := \sigma(\mathfrak{p}^{(n)} \frown s).$$

This is a winning strategy for Player **I**, since for any $x \in (\omega \cup \{\mathfrak{p}\})^\omega$ we have that

$$\begin{aligned} \psi(\mathrm{filter}(x)) &= \psi(\mathrm{filter}(\mathfrak{p}^{(n)} \frown x)) \\ &< \left(\bigoplus_{v\in\mathcal{U}} \varphi_v\right)(\overline{\sigma}(\mathfrak{p}^{(n)} \frown x)) = \left(\bigoplus_{v\in\mathcal{U}} \varphi_v\right)(u \frown \overline{\tau}(x)) = \varphi_u(\overline{\tau}(x)). \end{aligned}$$

In total this shows that $\psi <_{\mathrm{W}} \varphi_u$, as claimed. □

This particular result does not hold for the case $\leq_{\mathrm{L}}$, but we still can give a similar result.

Lemma 4.2.5. *We assume* $\mathbf{AD}$*. Let* $\{\varphi_n \mid n \in \omega\}$ *be a family of regular norms. Then for any regular norm* ψ *with* $\psi <_{\mathrm{L}} \bigoplus_{n\in\omega} \varphi_n$ *there is an* $n \in \omega$ *such that* $\psi \leq_{\mathrm{L}} \varphi_n$*.*

Proof. By assumption Player $\mathbf{II}$ has no winning strategy in $\mathrm{G}^{\leq}_{\mathrm{L}}\left(\bigoplus_{u\in\mathcal{U}} \varphi_u, \psi\right)$. Therefore by $\mathbf{AD}$ Player $\mathbf{I}$ wins $\mathrm{G}^{\leq}_{\mathrm{L}}\left(\bigoplus_{n\in\omega} \varphi_n, \psi\right)$ with a winning strategy $\sigma : \omega^{<\omega} \to \omega$. Then we fix $n := \sigma(\emptyset)$ and get a winning strategy $\tau : \omega^{<\omega}\setminus\{\emptyset\} \to \omega$ for Player $\mathbf{II}$ in the game $\mathrm{G}^{\leq}_{\mathrm{L}}(\psi, \varphi_n)$ by setting for all $s \in \omega^{<\omega}\setminus\{\emptyset\}$

$$\tau(s) = \sigma(s).$$

Then for all $x \in \mathbb{R}$ we get that $(\overline{\sigma}(x))(0) = n$ and $\overline{\tau}(x) = \overline{\sigma}(x) \setminus \langle n\rangle$ and so

$$\psi(x) \leq \left(\bigoplus_{m\in\omega} \varphi_m\right)(\overline{\sigma}(x)) = \varphi_n(\overline{\sigma}(x) \setminus \langle n\rangle) = \varphi_n(\overline{\tau}(x)).$$

This shows that $\psi \leq_{\mathrm{L}} \varphi_n$. $\square$

Lemma 4.2.6. *We assume* $\mathbf{AD}$ *and* $\mathbf{DC}$*. If* $\mathcal{U} \subseteq \omega^{<\omega}$ *is a partition of* $\omega^{<\omega}$*, then for any family* $\{\varphi_u \mid u \in \mathcal{U}\}$ *of regular norms we have that*

$$\left|\bigoplus_{u\in\mathcal{U}} \varphi_u\right|_{\mathrm{W}} = \sup\{|\varphi_u|_{\mathrm{W}} \mid u \in \mathcal{U}\}.$$

Furthermore if $\{\varphi_n \mid n \in \omega\}$ *is a family of regular norms such that for all* $n \in \omega$ *there is* $m \in \omega$ *such that* $\varphi_n <_{\mathrm{L}} \varphi_m$*, then we have that*

$$\left|\bigoplus_{n\in\omega} \varphi_n\right|_{\mathrm{L}} = \sup\{|\varphi_n|_{\mathrm{L}} \mid n \in \omega\}$$

Proof. It follows from Lemma 4.2.3 that $\sup\{|\varphi_u|_{\mathrm{W}} \mid u \in \mathcal{U}\} \leq \left|\bigoplus_{u\in\mathcal{U}} \varphi_u\right|_{\mathrm{W}}$. For the other direction we assume towards a contradiction that $\sup\{|\varphi_u|_{\mathrm{W}} \mid u \in \mathcal{U}\} < \left|\bigoplus_{u\in\mathcal{U}} \varphi_u\right|_{\mathrm{W}}$. But then we take a regular norm ψ such that $|\psi|_{\mathrm{W}} = \sup\{|\varphi_u|_{\mathrm{W}} \mid u \in \mathcal{U}\}$ and get that $\psi <_{\mathrm{W}} \bigoplus_{u\in\mathcal{U}} \varphi_u$. But then by Lemma 4.2.4 we get that there is $v \in \mathcal{U}$ such that $\psi < \varphi_v$, contradicting the fact that $|\psi|_{\mathrm{W}} = \sup\{|\varphi_u|_{\mathrm{W}} \mid u \in \mathcal{U}\}$.

The second part of the lemma follows analogously just using in place of Lemma 4.2.4 that by Lemma 4.2.5 it follows that for any $\psi <_{\mathrm{L}} \bigoplus_{n\in\omega} \varphi_n$ we have $n \in \omega$ such that $\psi \leq_{\mathrm{L}} \varphi_n$ and that thus there is $m \in \omega$ such that $\psi \leq_{\mathrm{L}} \varphi_n <_{\mathrm{L}} \varphi_m$. $\square$

Next we are going to show – using the join operation – that under the assumption of $\mathbf{AD}$ and $\mathbf{DC}$ for any regular norm φ we have that φ is self-dual if and only if $|\varphi|_{\mathrm{W}}$ is a limit ordinal and $\mathrm{cf}(|\varphi|_{\mathrm{W}}) = \omega$. The proof that we are going to present is a slight modification of a proof given for a similar fact in the article [Dup03].

Lemma 4.2.7. *We assume* $\mathbf{AD}$*. Let* $\{\varphi_n \mid n \in \omega\}$ *be a family of regular norms such that for all* $n \in \omega$ *there is* $m \in \omega$ *such that* $\varphi_n < \varphi_m$*. Then* $\bigoplus_{n\in\omega} \varphi_n$ *is a self-dual regular norm.*

Proof. We define a function $f : \omega \to \omega$ by setting for any $n \in \omega$:

$$f(n) = \min\{m \in \omega \mid \varphi_n <_{\mathrm{L}} \varphi_m\}.$$

Since for any $n \in \omega$ we have $\varphi_n <_{\mathrm{L}} \varphi_{f(n)}$, we have that Player $\mathbf{II}$ loses $\mathrm{G}^{\leq}_{\mathrm{L}}(\varphi_{f(n)}, \varphi_n)$ and so by $\mathbf{AD}$ Player $\mathbf{I}$ wins $\mathrm{G}^{\leq}_{\mathrm{L}}(\varphi_{f(n)}, \varphi_n)$. Then by $\mathbf{AC}_\omega(\mathbb{R})$ we choose for any $n \in \omega$ a winning strategy $\sigma_n : \omega^{<\omega} \to \omega$ for Player $\mathbf{I}$ in $\mathrm{G}^{\leq}_{\mathrm{L}}(\varphi_{f(n)}, \varphi_n)$. Now we define a winning strategy $\sigma : \omega^{<\omega} \setminus \{\emptyset\} \to \omega$ for Player $\mathbf{II}$ in the game $\mathrm{G}^{<}_{\mathrm{L}}\left(\bigoplus_{n\in\omega} \varphi_n, \bigoplus_{n\in\omega} \varphi_n\right)$ as follows for any $s \in \omega^{<\omega} \setminus \{\emptyset\}$:

$$\sigma(s) := \begin{cases} f(s(0)), & \text{if } \mathrm{lh}(s) = 1 \\ \sigma_{s(0)}(s^{+}\restriction_{\mathrm{lh}(s)-2}), & \text{if } \mathrm{lh}(s) > 1. \end{cases}$$

This is indeed winning for **II**, since for any $m \in \omega$ and $x \in \mathbb{R}$ we then have that $\overline{\sigma}(\langle m \rangle \frown u) = \langle f(m) \rangle \frown \overline{\sigma_m}(x)$ and so

$$\left(\bigoplus_{n\in\omega}\varphi_n\right)(\langle m\rangle^\frown x) = \varphi_m(x) < \varphi_{f(m)}(\overline{\sigma_m}(x)) = \left(\bigoplus_{n\in\omega}\varphi_n\right)(\langle f(m)\rangle^\frown \overline{\sigma_m}(x)) = \left(\bigoplus_{n\in\omega}\varphi_n\right)(\overline{\sigma}(x)).$$

□

Proposition 4.2.8 (Duparc)**.** *We assume* $\mathbf{AD}$ *and* $\mathbf{DC}$*. Let* φ *be a self-dual regular norm. Then there is* $\mathcal{U} \subseteq \omega^{<\omega}$ *such that* $\{u \frown \omega^{<\omega} \mid u \in \mathcal{U}\}$ *is a partition of* $\omega^{<\omega}$ *and for each* $u \in \mathcal{U}$ *the norm* $\varphi_{\lfloor u \rfloor}$ *is self-dual with* $\varphi_{\lfloor u \rfloor} <_{\mathrm{W}} \varphi$ *and* $\varphi = \bigoplus_{u\in\mathcal{U}} \varphi_{\lfloor u \rfloor}$*.*

Proof. We prove this proposition by induction on $|\varphi|_{\mathrm{W}}$ and so assume that the claim was shown for every self-dual regular norm ψ with $\psi <_{\mathrm{W}} \varphi$. Since φ is self-dual, we have that $\mathbf{T}(\varphi)$ is well-founded and so $\partial\mathbf{T}(\varphi)$ is a partition of $\omega^{<\omega}$. Now for any $s \in \partial\mathbf{T}(\varphi)$ we define a set $\mathcal{U}_s \subseteq \omega^{<\omega}$ by distinguishing two cases.

Case 1 is that $\varphi_{\lfloor s \rfloor}$ is non-self-dual. Then we set $\mathcal{U}_s := \{\emptyset\}$.

Case 2 is that $\varphi_{\lfloor s \rfloor}$ is self-dual. Since $s \in \partial\mathbf{T}(\varphi)$ and thus in particular $s \notin \mathbf{T}(\varphi)$ we have that $\varphi_{\lfloor s \rfloor} <_{\mathrm{W}} \varphi$ and so by induction hypothesis we fix a partition $\mathcal{U}_s$ of $\omega^{<\omega}$ such that for all $u \in \mathcal{U}_s$ we have that $\varphi_{\lfloor s^\frown u \rfloor}$ is non-self-dual and $\varphi_{\lfloor s^\frown u \rfloor} <_{\mathrm{W}} \varphi_{\lfloor s \rfloor}$ and furthermore $\varphi_{\lfloor s \rfloor} = \bigoplus_{u\in\mathcal{U}_s} \varphi_{\lfloor s^\frown u \rfloor}$.

Now we set $\mathcal{U} := \{s \frown u \mid s \in \partial\mathbf{T}(\varphi), u \in \mathcal{U}_s\}$ and since both $\partial\mathbf{T}(\varphi)$ and all the $\mathcal{U}_s$ are partitions of $\omega^{<\omega}$ we get that also $\mathcal{U}$ is a partition of $\omega^{<\omega}$ and therefore $\varphi = \bigoplus_{v\in\mathcal{U}} \varphi_{\lfloor v \rfloor}$. By construction for any $v \in \mathcal{U}$ we have that $\varphi_{\lfloor v \rfloor}$ is non-self-dual and $\varphi_{\lfloor v \rfloor} <_{\mathrm{W}} \varphi$. □

Proposition 4.2.9. *We assume* $\mathbf{AD}$ *and* $\mathbf{DC}$*. Then we have for any regular norm* φ *that* φ *is self-dual if and only if* $|\varphi|_{\mathrm{W}}$ *is a limit ordinal and* $\mathrm{cf}(|\varphi|_{\mathrm{W}}) = \omega$*.*

Proof. For the left-to-right direction let φ be self-dual. Then as in Proposition 4.2.8 we find a partition $\mathcal{U}$ of $\omega^{<\omega}$ such that $\varphi = \bigoplus_{u\in\mathcal{U}} \varphi_{\lfloor u \rfloor}$ and for any $u \in \mathcal{U}$, $\varphi_{\lfloor u \rfloor}$ is non-self-dual. But by Lemma 4.2.6 we also have

$$|\varphi|_{\mathrm{W}} = \sup\left\{\left|\varphi_{\lfloor u \rfloor}\right|_{\mathrm{W}} \mid u \in \mathcal{U}\right\}.$$

If now the set $\left\{\left|\varphi_{\lfloor u \rfloor}\right|_{\mathrm{W}} \mid u \in \mathcal{U}\right\}$ has a maximum, then there is $u \in \mathcal{U}$ such that $\varphi \equiv_{\mathrm{W}} \varphi_{\lfloor u \rfloor}$, but this is absurd, since φ is self-dual and $\varphi_{\lfloor u \rfloor}$ is non-self-dual. Thus for all $u \in \mathcal{U}$ we have $\left|\varphi_{\lfloor u \rfloor}\right|_{\mathrm{W}} < |\varphi|_{\mathrm{W}}$, which in total implies that $|\varphi|_{\mathrm{W}}$ is a limit ordinal. Furthermore, since $\mathcal{U}$ is countable, it follows that $\mathrm{cf}\,(|\varphi|_{\mathrm{W}}) = \omega$.

For the right-to-left direction let $\langle \alpha_n \mid n \in \omega \rangle$ be a strictly increasing sequence of ordinals such that $|\varphi|_{\mathrm{W}} = \sup\{\alpha_n \mid n \in \omega\}$. Then – using $\mathbf{AC}_\omega$ – we choose a sequence $\langle \varphi_n \mid n \in \omega \rangle$ of regular norms such that for all $n \in \omega$ we have $|\varphi_n|_{\mathrm{W}} = \alpha_n$. Then we consider the norm $\psi := \bigoplus_{n\in\omega} \varphi_n$. By Lemma 4.2.7 we have that ψ is self-dual and by Lemma 4.2.6 we then have that $|\psi|_{\mathrm{W}} = \sup\{|\varphi_n|_{\mathrm{W}} \mid n \in \omega\}$ and so $\psi \equiv_{\mathrm{W}} \varphi$. It thus follows that φ is self-dual. □

Using this result as a starting point we will show that in the hierarchy of regular norms strictly above every regular norm there is a non-self-dual norm and a self-dual norm and introduce operations on $\mathcal{N}$ pointing to $\leq_{\mathrm{W}}$ minimal such norms. The key step for this endeavor is the definition of an operation that assigns to each regular norm φ a regular norm φ^{succ} such that $|\varphi^{\mathrm{succ}}|_{\mathrm{W}} = |\varphi|_{\mathrm{W}} + 1$. We will give such an operation in the following and note that this operation is considered for the first time in this thesis.

Definition 4.2.10 Let $x, y \in \mathbb{R}$. Then we call y a *stretch of* x iff there is a strictly increasing function $f : \omega \to \omega$ such that for all $n \in \omega$

$$y(n) = \begin{cases} x(f^{-1}(n)) + 1, & \text{if } n \in f[\omega], \\ 0, & \text{otherwise.} \end{cases}$$

We call x the *unstretch of* x iff y is a stretch of x.

We note that for any $y \in \mathbb{R}$ there is at most one unstretch. Furthermore for any $y \in \mathbb{R}$ there is exactly one unstretch if and only if $\exists^\infty n\,(x(n) \neq 0)$. Thus for any $y \in \mathbb{R}$ with $\exists^\infty n\,(x(n) \neq 0)$ we write $\mathrm{unstretch}(y)$ to denote its unique unstretch.

Definition 4.2.11 For any regular norm φ we define a regular norm φ^{succ}, the *successor of* φ, by setting for any $x \in \mathbb{R}$:

$$\varphi^{\mathrm{succ}}(x) = \begin{cases} \varphi(\mathrm{unstretch}(x)) + 1, & \text{if } \exists^\infty n \ (x(n) \neq 0), \\ 0, & \text{if } \forall^\infty n \ (x(n) = 0). \end{cases}$$

We note that for any regular norm φ we have that

$$\mathrm{lh}(\varphi^{\mathrm{succ}}) = \begin{cases} \mathrm{lh}(\varphi), & \text{if } \mathrm{lh}(\varphi) \text{ is a limit ordinal}, \\ \mathrm{lh}(\varphi) + 1, & \text{otherwise}. \end{cases}$$

The intuition behind the definition of φ^{succ} is the following. Given any $x \in \mathbb{R}$ a regular norm of the form φ^{succ} on input x reads x by sequentially reading $x(n)$ for $n < \omega$, ignoring every 0 and assembling for all $n \in \omega$ with $x(n) > 0$ the values $x(n) - 1$, building a new sequence in $\omega^{\leq\omega}$. If this procedure results in a real $y \in \mathbb{R}$, then φ^{succ} outputs $\varphi(y) + 1$. Otherwise φ^{succ} just outputs the default value 0. Thus in the game $\mathrm{G}^{\leq}_{\mathrm{L}}(\varphi^{\mathrm{succ}}, \varphi)$ Player **I** can always just win by playing 0 first and then playing 0, whenever Player **II** passes, and playing $n + 1$, whenever Player **II** plays some $n \in \omega$. The idea is that this somehow gives Player **I** the minimal necessary advantage in the game $\mathrm{G}^{\leq}_{\mathrm{W}}(\varphi^{\mathrm{succ}}, \varphi)$. We will formalize this idea in the proof of the following lemma.

Lemma 4.2.12. *1. If $\varphi \leq_{\mathrm{W}} \psi$ then $\varphi^{\mathrm{succ}} \leq_{\mathrm{W}} \psi^{\mathrm{succ}}$.*
2. For every regular norm φ we have $\varphi <_{\mathrm{W}} \varphi^{\mathrm{succ}}$.
3. We assume $\mathbf{AD}$. For any two regular norms ψ, φ such that $\psi <_{\mathrm{W}} \varphi^{\mathrm{succ}}$ we have $\psi \leq_{\mathrm{W}} \varphi$.
4. We assume $\mathbf{AD}$ and $\mathbf{DC}$. Then for any regular norm φ we have that $|\varphi^{\mathrm{succ}}|_{\mathrm{W}} = |\varphi|_{\mathrm{W}} + 1$.

Proof. 1. Let $\sigma : \omega^{<\omega} \setminus \{\emptyset\} \to \omega \cup \{\mathfrak{p}\}$ be a winning strategy for Player **II** in the game $\mathrm{G}^{\leq}_{\mathrm{W}}(\varphi, \psi)$. Then we can define a winning strategy $\sigma' : \omega^{<\omega} \setminus \{\emptyset\} \to \omega \cup \{\mathfrak{p}\}$ for Player **II** in the game $\mathrm{G}^{\leq}_{\mathrm{W}}(\varphi^{\mathrm{succ}}, \psi^{\mathrm{succ}})$ by setting for all $s \in \omega^{<\omega} \setminus \{\emptyset\}$

$$\sigma'(s) := \begin{cases} \sigma(\mathrm{unstretch}(s)) + 1, & \text{if } s(\mathrm{lh}(s) - 1) > 0, \\ 0, & \text{otherwise}. \end{cases}$$

Then for any x with infinitely many $n \in \omega$ such that $x(n) = 0$ we have that also $\overline{\sigma'}(x)$ has infinitely many $n \in \omega$ such that $(\overline{\sigma'}(x))(n) = 0$ and so

$$\varphi^{\mathrm{succ}}(x) = 0 = \psi^{\mathrm{succ}}(\mathrm{filter}(\overline{\sigma'}(x))).$$

If, however, x has cofinitely many $n \in \omega$ such that $x(n) \neq 0$, then $\mathrm{filter}(\overline{\sigma'}(x))$ has the same property and its unstretch is equal to $\mathrm{filter}(\overline{\sigma}(\mathrm{unstretch}(x)))$. So we get that

$$\varphi^{\mathrm{succ}}(x) = \varphi(\mathrm{unstretch}(x)) \leq \psi(\mathrm{filter}(\overline{\sigma}(\mathrm{unstretch}(x)))) = \psi^{\mathrm{succ}}(\mathrm{filter}(\overline{\sigma'}(x)).$$

Hence in total this shows that σ is winning for Player **II** in $\mathrm{G}^{\leq}_{\mathrm{W}}(\varphi^{\mathrm{succ}}, \psi^{\mathrm{succ}})$ and so $\varphi^{\mathrm{succ}} \leq_{\mathrm{W}} \psi^{\mathrm{succ}}$.

2. We give a winning strategy $\tau : (\omega \cup \{\mathfrak{p}\})^{<\omega} \to \omega$ for Player **I** in the game $\mathrm{G}^{\leq}_{\mathrm{W}}(\varphi^{\mathrm{succ}}, \varphi)$ by setting for all $s \in (\omega \cup \{\mathfrak{p}\})^{<\omega}$

$$\tau(s) := \begin{cases} 0, & \text{if } \mathrm{lh}(s) = 0 \text{ or } \mathrm{lh}(s) > 0 \text{ and } s(\mathrm{lh}(s) - 1) = \mathfrak{p}, \\ s(\mathrm{lh}(s) - 1) + 1, & \text{if } \mathrm{lh}(s) > 0 \text{ and } s(\mathrm{lh}(s) - 1) \in \omega. \end{cases}$$

Now let $x \in (\omega \cup \{\mathfrak{p}\})^\omega$ be such that there are infinitely many $n \in \omega$ such that $x(n) \in \omega$. Then by construction $\mathrm{unstretch}(\overline{\tau}(x)) = \mathrm{filter}(x)$ and so

$$\varphi(\mathrm{filter}(x)) < \varphi(\mathrm{filter}(x)) + 1 = \varphi^{\mathrm{succ}}(\overline{\tau}(x)).$$

If, however, Player **II** plays a sequence $x \in (\omega \cup \{\mathfrak{p}\})^\omega$ such that for cofinitely may $n \in \omega$, $x(n) = \mathfrak{p}$, then Player **II** loses anyway. So in total we have shown that Player **I** wins $\mathrm{G}^{\leq}_{\mathrm{W}}(\varphi^{\mathrm{succ}}, \varphi)$ and so Player **II** has no winning strategy in this game, which implies that $\varphi <_{\mathrm{W}} \varphi^{\mathrm{succ}}$.

3. Since $\psi < \varphi^{\mathrm{succ}}$ we have that $\varphi^{\mathrm{succ}} \not\leq \psi$ and thus by **AD** Player **I** has a winning strategy $\tau : (\omega \cup \{\mathfrak{p}\})^{<\omega} \to \omega$ in the game $G^{\leq}_{W}(\varphi^{\mathrm{succ}}, \psi)$. Hence for any $x \in (\omega \cup \{\mathfrak{p}\})^{\omega}$ with $x(n) \in \omega$ for infinitely many $n \in \omega$, we have that $\psi(x) < \varphi^{\mathrm{succ}}(\overline{\tau}(x))$ and so in particular there are infinitely many $n \in \omega$ such that $(\overline{\tau}(x))(n) > 0$. Now we give a winning strategy $\sigma : \omega^{<\omega} \setminus \{\emptyset\} \to \omega \cup \{\mathfrak{p}\}$ for Player **II** in the game $\mathrm{G}^{\leq}_{\mathrm{W}}(\psi, \varphi)$ by setting for all $s \in \omega^{<\omega} \setminus \{\emptyset\}$

$$\sigma(s) := \begin{cases} \mathfrak{p}, & \text{if } \tau(s{\restriction}_{\mathrm{lh}(s)-1}) = 0, \\ \tau(s) - 1, & \text{if } \tau(s{\restriction}_{\mathrm{lh}(s)-1}) > 0. \end{cases}$$

By construction we then have for any $y \in \mathbb{R}$ that $\mathrm{filter}(\overline{\sigma}(y)) = \mathrm{unstretch}(\overline{\tau}(y))$ and so

$$\psi(y) < \varphi^{\mathrm{succ}}(\overline{\tau}(y)) = \varphi(\mathrm{filter}(\overline{\sigma}(y))) + 1$$

and so $\psi(y) \leq \varphi(\overline{\sigma}(y))$. Hence Player **II** wins $\mathrm{G}^{\leq}_{\mathrm{W}}(\psi, \varphi^{\mathrm{succ}})$ and so $\psi \leq \varphi^{\mathrm{succ}}$.

4. Since $\varphi <_{\mathrm{W}} \varphi^{\mathrm{succ}}$ we have $|\varphi|_{\mathrm{W}} + 1 \leq |\varphi^{\mathrm{succ}}|_{\mathrm{W}}$. But if $|\varphi|_{\mathrm{W}} + 1 < |\varphi^{\mathrm{succ}}|_{\mathrm{W}}$, then there is a norm ψ such that $\varphi <_{\mathrm{W}} \psi$ and $\psi <_{\mathrm{W}} \varphi^{\mathrm{succ}}$, but the latter implies that $\psi \leq_{\mathrm{W}} \varphi$ and so $\psi \leq_{\mathrm{W}} \varphi <_{\mathrm{W}} \psi$, a contradiction. Hence we have $|\varphi^{\mathrm{succ}}|_{\mathrm{W}} = |\varphi|_{\mathrm{W}} + 1$. ☐

Moreover we have that for any regular norm φ the regular norm φ^{succ} is non-self-dual.

Lemma 4.2.13. *For any regular norm φ we have that φ^{succ} is non-self-dual.*

Proof. We give a winning strategy $\tau : \omega^{<\omega} \to \omega$ for Player **I** in the game $\mathrm{G}^{<}_{\mathrm{L}}(\varphi^{\mathrm{succ}}, \varphi^{\mathrm{succ}})$ by setting for all $s \in \omega^{<\omega}$

$$\tau(s) := \begin{cases} 0 & \text{if } \mathrm{lh}(s) = 0, \\ s(\mathrm{lh}(s) - 1) & \text{otherwise.} \end{cases}$$

For any $x \in \mathbb{R}$ we then have that $\overline{\tau}(x) = \langle 0 \rangle ^\frown x$ and so

$$\varphi^{\mathrm{succ}}(x) = \varphi^{\mathrm{succ}}(\langle 0 \rangle ^\frown x) = \varphi(\overline{\tau}(x)),$$

which shows that τ is indeed winning for **I** and hence that Player **II** does not win the game $\mathrm{G}^{<}_{\mathrm{L}}(\varphi^{\mathrm{succ}}, \varphi^{\mathrm{succ}})$. Thus φ^{succ} is non-self-dual. ☐

So we get as an immediate corollary of Lemma 4.2.12 and Lemma 4.2.13 that φ^{succ} is the $\leq_{\mathrm{W}}$-minimal non-self-dual degree strictly above φ.

Proposition 4.2.14. *For any regular norm φ we have that $[\varphi^{\mathrm{succ}}]_{\mathrm{W}}$ is the $\leq_{\mathrm{W}}$-minimal Wadge degree such that $[\varphi]_{\mathrm{W}}$ is non-self-dual and $[\varphi]_{\mathrm{W}} <_{\mathrm{W}} [\varphi^{\mathrm{succ}}]_{\mathrm{W}}$.*

Next for an arbitrary regular norm φ we are going to obtain the $\leq_{\mathrm{W}}$-least self-dual degree strictly above φ by iterating the succ-operation ω-times using the join operation.

Definition 4.2.15 For a regular norm φ we define a regular norm φ^{+0} by setting $\varphi^{+0} := \varphi$.

For $n \in \omega$, $n > 1$ and a regular norm φ we define a regular norm φ^{+n} by setting for all $x \in \mathbb{R}$:

$$\varphi^{+n}(x) := \begin{cases} \varphi(\mathrm{unstretch}(x)) + n + 1, & \text{if } \exists^{\infty} m \ (x(m) \neq 0), \\ n, & \text{otherwise.} \end{cases}$$

For any regular norm φ we furthermore define a regular norm φ^{ω} by setting for all $n \in \omega$ and $x \in \mathbb{R}$:

$$\varphi^{+\omega}(\langle n \rangle ^\frown x) := \begin{cases} \varphi(\mathrm{unstretch}(x)) + n, & \text{if } \exists^{\infty} m \ (x(m) \neq 0), \\ n, & \text{otherwise.} \end{cases}$$

Clearly we have $\varphi^{+1} = \varphi^{\mathrm{succ}} = (\varphi^{+0})^{\mathrm{succ}}$ and $\mathrm{lh}(\varphi^{+n}) = \mathrm{lh}(\varphi^{+\omega}) = \mathrm{lh}(\varphi)$. We are going to show now that for $\alpha \in \omega + 1$ we have that $|\varphi|_{\mathrm{W}} = |\varphi|_{\mathrm{W}} + \alpha$.

Lemma 4.2.16. *For any $n \in \omega$, $n > 1$ and any regular norm φ we have that $\varphi^{+(n+1)} \equiv_{\mathrm{L}} (\varphi^{+n})^{\mathrm{succ}}$.*

Proof. We have that for any $x \in \mathbb{R}$

$$\left(\varphi^{+n}\right)^{\text{succ}}(x) = \begin{cases} \varphi(\text{unstretch}(\text{unstretch}(x))) + n + 2, & \text{if } \exists^\infty m \in \omega\ (x(m) > 1), \\ n+1, & \text{if } \forall^\infty m \in \omega\ (x(m) = 1), \\ 0, & \text{if } \forall^\infty m \in \omega\ (x(m) = 0). \end{cases}$$

Now we give a winning strategy $\tau : \omega^{<\omega} \setminus \{\emptyset\} \to \omega$ for Player $\mathbf{II}$ in the game $\mathrm{G}_{\mathrm{L}}^{\leq}(\varphi^{+(n+1)}, (\varphi^{+n})^{\text{succ}})$ by setting for all $s \in \omega^{<\omega} \setminus \{\emptyset\}$

$$\tau(s) := s(\text{lh}(s) - 1) + 1.$$

Then for any $x \in \mathbb{R}$ with $x(m) > 0$ for infinitely many $m \in \omega$ we have that $\overline{\tau}(x)(m) > 1$ for infinitely many $m \in \omega$ and that $\text{unstretch}(x) = \text{unstretch}(\text{unstretch}(\overline{\tau}(x)))$ and so

$$\varphi^{+(n+1)}(x) = \varphi(\text{unstretch}(x)) + n + 2 = \varphi(\text{unstretch}(\text{unstretch}(\overline{\tau}(x)))) + n + 2 = \left(\varphi^{+n}\right)^{\text{succ}}(\tau(x)).$$

For any $x \in \mathbb{R}$ with $x(m) = 0$ for cofinitely many $m \in \omega$, however, we have that $\overline{\tau}(x)(m) = 1$ for cofinitely many $m \in \omega$ and so that

$$\varphi^{+(n+1)}(x) = n + 1 = \left(\varphi^{+n}\right)^{\text{succ}}(\tau(x)).$$

In total this shows that τ is indeed a winning strategy for Player $\mathbf{II}$ and hence that $\varphi^{+(n+1)} \leq_{\mathrm{L}} (\varphi^{+n})^{\text{succ}}$.

Next we give a winning strategy $\sigma : \omega^{<\omega} \setminus \{\emptyset\} \to \omega$ for Player $\mathbf{II}$ in the game $\mathrm{G}_{\mathrm{L}}^{\leq}((\varphi^{+n})^{\text{succ}}, \varphi^{+(n+1)})$ by setting for all $s \in \omega^{<\omega} \setminus \{\emptyset\}$

$$\sigma(s) := \begin{cases} s(\text{lh}(s) - 1) - 1, & \text{if } s(\text{lh}(s) - 1) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then for any $x \in \mathbb{R}$ with $x(m) > 1$ for infinitely many $m \in \omega$ we have that $\overline{\sigma}(x)$ has the property that for infinitely many $m \in \omega$ we have $\overline{\sigma}(x)(m) > 0$ and furthermore $\text{unstretch}(\text{unstretch}(x)) = \text{unstretch}(\overline{\sigma}(x))$ and so

$$\left(\varphi^{+n}\right)^{\text{succ}}(x) = \varphi(\text{unstretch}(\text{unstretch}(x))) + n + 2 = \varphi(\text{unstretch}(\overline{\sigma}(x))) + n + 2 = \varphi^{+(n+1)}(\overline{\sigma}(x)).$$

For any $x \in \mathbb{R}$ with $x(m) = 1$ for cofinitely many $m \in \omega$ we have that also $\overline{\sigma}(x)(m) = 0$ for cofinitely many $n \in \omega$ and so we have that

$$\left(\varphi^{+n}\right)^{\text{succ}}(x) = n + 1 = \varphi^{+(n+1)}(\overline{\sigma}(x)).$$

Finally for any $x \in \mathbb{R}$ with $x(m) = 0$ for cofinitely many $m \in \omega$ we have that $\overline{\sigma}(x)(m) = 0$ for cofinitely many $n \in \omega$ and so

$$\left(\varphi^{+n}\right)^{\text{succ}}(x) = 0 \leq n + 1 = \varphi^{+(n+1)}(\overline{\sigma}(x)).$$

In total this shows that σ is indeed a winning strategy for Player $\mathbf{II}$ and hence that $(\varphi^{+n})^{\text{succ}} \leq_{\mathrm{L}} \varphi^{+(n+1)}$. □

Corollary 4.2.17. *Assume* $\mathbf{AD}$ *and* $\mathbf{DC}$*. Then for any regular norm* φ *and any* $n \in \omega$ *with* $n \geq 1$ *we have that*

$$\left|\varphi^{+n}\right|_{\mathrm{W}} = |\varphi|_{\mathrm{W}} + n.$$

Proof. We prove this by induction on n. For $n = 1$ this follows from Lemma 4.2.12, since $\varphi^{+1} = \varphi^{\text{succ}}$. Then we assume the claim for some $n \geq 1$. By Lemma 4.2.16 we have that $\varphi^{+(n+1)} \equiv_{\mathrm{W}} (\varphi^{+n})^{\text{succ}}$ and so again by Lemma 4.2.12, using the induction hypothesis, we get

$$\left|\varphi^{+(n+1)}\right|_{\mathrm{W}} = \left|(\varphi^{+n})^{\text{succ}}\right|_{\mathrm{W}} = \left|\varphi^{+n}\right|_{\mathrm{W}} + 1 = |\varphi|_{\mathrm{W}} + n + 1.$$

This concludes the induction and hence the proof. □

Proposition 4.2.18. *Let* φ *be a regular norm. Then we have that*

$$\varphi^{+\omega} \equiv_{\mathrm{L}} \bigoplus_{n \in \omega} \varphi^{+(n+1)}.$$

Proof. First we note that for any $x \in \mathbb{R}$ and $n \in \omega$ we have

$$\bigoplus_{n\in\omega} \varphi^{+(n+1)}(\langle n\rangle ^\frown x) = \begin{cases} \varphi(\mathrm{unstretch}(x)) + n + 1, & \text{if } \exists^\infty m \in \omega\ (x(m) \neq 0),\\ n, & \text{otherwise.}\end{cases}$$

Thus for all $x \in \mathbb{R}$ we have that $\varphi^{+\omega}(x) \leq \left(\bigoplus_{n\in\omega} \varphi^{+(n+1)}\right)(x)$ and so we get that $\varphi^{+\omega} \leq_{\mathrm{L}} \bigoplus_{n\in\omega} \varphi^{+(n+1)}$ as witnessed by the identity function on $\mathbb{R}$. In the other direction we give a winning strategy $\sigma : \omega^{<\omega} \setminus \{\emptyset\} \to \omega$ for Player **II** in the game $\mathrm{G}^{\leq}_{\mathrm{L}}\left(\bigoplus_{n\in\omega} \varphi^{+(n+1)}, \varphi^{+\omega}\right)$ by setting for all $s \in \omega^{<\omega}$

$$\sigma(s) := \begin{cases} s(0) + 1, & \text{if } \mathrm{lh}(s) = 1,\\ s(\mathrm{lh}(s) - 1), & \text{otherwise.}\end{cases}$$

Then for any $m \in \omega$ and $x \in \mathbb{R}$ we have that $\overline{\sigma}(\langle m\rangle ^\frown x) = \langle m+1\rangle ^\frown x$ and so if $x \in \mathbb{R}$ is such that for infinitely many $k \in \omega$ we have that $x(k) > 0$, then

$$\bigoplus_{n\in\omega} \varphi^{+(n+1)}(\langle m\rangle ^\frown x) = \varphi(\mathrm{unstretch}(x)) + m + 1 = \varphi^{+\omega}(\langle m+1\rangle ^\frown x) = \varphi^{+\omega}(\overline{\sigma}(\langle m\rangle ^\frown x)).$$

If, however $x \in \mathbb{R}$ is such that for cofinitely many $m \in \omega$ we have that $x(m) = 0$, then

$$\bigoplus_{n\in\omega} \varphi^{+(n+1)}(\langle m\rangle ^\frown x) = m \leq m + 1 = \varphi^{+\omega}(\langle m+1\rangle ^\frown x) = \varphi^{+\omega}(\overline{\sigma}(\langle m\rangle ^\frown x)).$$

In total this shows that σ indeed is a winning strategy for Player **II** and hence that $\bigoplus_{n\in\omega} \varphi^{+(n+1)} \leq_{\mathrm{L}} \varphi^{+\omega}$. □

Corollary 4.2.19. *Assume* **AD** *and* **DC**. *Let φ be any regular norm. Then $|\varphi^{+\omega}|_{\mathrm{W}} = |\varphi|_{\mathrm{W}} + \omega$. Furthermore $[\varphi^{+\omega}]_{\mathrm{W}}$ is the least self-dual Wadge-degree with $[\varphi]_{\mathrm{W}} <_{\mathrm{W}} [\varphi^{+\omega}]_{\mathrm{W}}$.*

Proof. By Corollary 4.2.17 we have that for any regular norm φ and any $n \in \omega$

$$\left|\varphi^{+n}\right|_{\mathrm{W}} = |\varphi| + n.$$

Then by Proposition 4.2.18 and Lemma 4.2.6 we have that

$$\begin{aligned} \left|\varphi^{+\omega}\right|_{\mathrm{W}} &= \left|\bigoplus_{n\in\omega} \varphi^{+(n+1)}\right|_{\mathrm{W}} \\ &= \sup\left\{\left|\varphi^{+(n+1)}\right|_{\mathrm{W}} \mid n \in \omega\right\} \\ &= \sup\left\{(|\varphi|_{\mathrm{W}} + n + 1) \mid n \in \omega\right\} \\ &= |\varphi|_{\mathrm{W}} + \omega. \end{aligned}$$

Now we let ψ be a regular norm with $\varphi <_{\mathrm{W}} \psi <_{\mathrm{W}} \varphi^{+\omega}$. Then there is $n \in \omega \setminus \{0\}$ such that $|\psi|_{\mathrm{W}} = |\varphi| + n$ and so $\psi \equiv_{\mathrm{W}} \varphi^{+n} \equiv_{\mathrm{W}} (\varphi^{+(n-1)})^{\mathrm{succ}}$. But then ψ is non-self-dual by Lemma 4.2.13. Thus $[\varphi^{+\omega}]_{\mathrm{W}}$ is the minimal Wadge degree strictly $\leq_{\mathrm{W}}$-above $[\varphi]_{\mathrm{W}}$ such that $[\varphi^{+\omega}]_{\mathrm{W}}$ is self-dual. □

Now to conclude this section we will give an alternative, simpler operation yielding for any non-self-dual regular norm φ a $\leq_{\mathrm{W}}$-minimal self-dual regular norm above φ. This alternative operation will come in very handy in Subsection 4.3.3.

Definition 4.2.20 Let φ be a regular norm. We define a regular norm $\varphi^{\mathrm{stretch}}$ by setting for all $x \in \mathbb{R}$

$$\varphi^{\mathrm{stretch}}(x) := \begin{cases} \varphi(\mathrm{unstretch}(x)), & \text{if } \exists^\infty m\ (x(m) \neq 0),\\ 0, & \text{otherwise.}\end{cases}$$

Furthermore for any $n \in \omega$ we define a regular norm $\varphi + n$ by setting for all $x \in \mathbb{R}$

$$(\varphi + n)(x) := \varphi(x) + n.$$

Lemma 4.2.21. *Assume* $\mathbf{AD}$*. If* φ *is non-self-dual, then* $\varphi \equiv_{\mathrm{L}} \varphi^{\mathrm{stretch}}$.

Proof. The map $f : \mathbb{R} \to \mathbb{R}, x \mapsto x + 1$ is Lipschitz and clearly for all $x \in \mathbb{R}$ we have that $\varphi(x) = \varphi^{\mathrm{stretch}}(f(x))$. This shows that $\varphi \leq_{\mathrm{L}} \varphi^{\mathrm{stretch}}$.

For the other direction we note that by assumption φ is non-self-dual and so Player **II** does not have a winning strategy in the game $\mathrm{G}^{<}_{\mathrm{W}}(\varphi, \varphi)$, which by $\mathbf{AD}$ implies that Player **I** has a winning strategy $\sigma : (\omega \cup \{\mathfrak{p}\})^{<\omega} \to \omega$ in the game $\mathrm{G}^{<}_{\mathrm{W}}(\varphi, \varphi)$. Then taking $x \in (\omega \cup \{\mathfrak{p}\})^{\omega}$ such that $x(m) \in \omega$ for infinitely many $m \in \omega$ we have that

$$\varphi(\mathrm{filter}(x)) \leq \varphi(\overline{\sigma}(x)).$$

Now we define a bijection $g : \omega^{\leq\omega} \to (\omega \cup \{\mathfrak{p}\})^{\leq\omega}$ by setting for any $x \in (\omega \cup \{\mathfrak{p}\})^{\omega}$ and $m < \mathrm{lh}(x)$

$$g(x)(m) = \begin{cases} \mathfrak{p}, & \text{if } x(m) = 0, \\ x(m) - 1, & \text{if } x(m) > 0. \end{cases}$$

Then by definition of the stretch we get that for all $x \in \mathbb{R}$ with $x(m) \neq 0$ for infinitely many $m \in \omega$ we have that

$$\mathrm{unstretch}(x) = \mathrm{filter}(g(x)).$$

Now using this we define a winning strategy $\tau : \omega^{<\omega} \setminus \{\emptyset\} \to \omega$ for Player **II** in the game $\mathrm{G}^{\leq}_{\mathrm{L}}(\varphi^{\mathrm{stretch}}, \varphi)$ by setting for all $s \in \omega^{<\omega} \setminus \{\emptyset\}$

$$\tau(s) = \sigma(g(s{\restriction}_{\mathrm{lh}(s)-1})).$$

Then for any $x \in \mathbb{R}$ with infinitely many $m \in \omega$ such that $x(m) \neq 0$ we have that $\overline{\tau}(x) = \overline{\sigma}(g(x))$ and so

$$\varphi^{\mathrm{stretch}}(x) = \varphi(\mathrm{unstretch}(x)) = \varphi(\mathrm{filter}(g(x))) \leq \varphi(\overline{\sigma}(g(x))) = \varphi(\overline{\tau}(x)).$$

If, however $x \in \mathbb{R}$ is such that there are cofinitely many $m \in \omega$ such that $x(m) = 0$, then $\varphi^{\mathrm{stretch}}(x) = 0$ and so vacuously $\varphi^{\mathrm{stretch}}(x) \leq \varphi(\overline{\tau}(x))$. Thus in total we have shown that τ is indeed a winning strategy for Player **II** and hence that $\varphi^{\mathrm{stretch}} \leq_{\mathrm{L}} \varphi$ as claimed. ☐

Lemma 4.2.22. *Let* φ *and* ψ *be regular norms and* $n \in \omega$ *arbitrary. Then* $\varphi \leq_{\mathrm{L}} \psi$ *implies that* $\varphi + n \leq_{\mathrm{L}} \psi + n$ *and the same with* L *replaced by* W.

Proof. We note that if $f : \mathbb{R} \to \mathbb{R}$ is any function such that for all $x \in \mathbb{R}$ we have that $\varphi(x) \leq \varphi(f(x))$, then clearly also $\varphi(x) + n \leq \varphi(f(x)) + n$. Thus taking f to be Lipschitz or continuous shows the claim. ☐

Definition 4.2.23 Let φ be a regular norm. Then we define a regular norm $\varphi^{\vee}$ by setting for all $x \in \mathbb{R}$

$$\varphi^{\vee}(\langle n \rangle ^\frown x) := \varphi(x) + n.$$

Proposition 4.2.24. *Assume* $\mathbf{AD}$*. If* φ *is a non-self-dual regular norm, then* $\varphi^{+\omega} \equiv_{\mathrm{L}} \varphi^{\vee}$.

Proof. We consider the Lipschitz function $f : \mathbb{R} \to \mathbb{R}, x \mapsto x + 1$. Then we have for all $x \in \mathbb{R}$ that $\varphi^{\vee}(x) \leq \varphi^{+\omega}(f(x))$ and so we have that $\varphi^{\vee} \leq_{\mathrm{W}} \varphi^{+\omega}$.

For the other direction we define for any $n \in \omega$ a regular norm φ_n by setting $\varphi_n := \varphi^{\mathrm{stretch}} + n$. Then by definition of $\varphi^{+\omega}$ we have that

$$\varphi^{+\omega} = \bigoplus_{n \in \omega} \varphi_n.$$

Furthermore we define for any $n \in \omega$ a regular norm φ'_n by setting $\varphi'_n = \varphi + n$. Then by definition of $\varphi^{\vee}$ we have that

$$\varphi^{\vee} = \bigoplus_{n \in \omega} \varphi'_n.$$

We begin by showing that for any $n \in \omega$ we have that $\varphi_n \equiv_{\mathrm{W}} \varphi'_n$. For this we first note that since φ is non-self-dual, Lemma 4.2.21 implies that $\varphi \equiv_{\mathrm{W}} \varphi^{\mathrm{stretch}}$ and so by Lemma 4.2.22 we finally get that

$$\varphi_n = \varphi^{\mathrm{stretch}} + n \equiv_{\mathrm{W}} \varphi + n = \varphi'_n.$$

But by Lemma 4.2.2 this already implies that

$$\varphi^{+\omega} = \bigoplus_{n\in\omega} \varphi_n \equiv_{\mathrm{W}} \bigoplus_{n\in\omega} \varphi'_n = \varphi^{\vee},$$

as claimed. □

Now we state and prove another small fact regarding self-duality that will be very useful later.

Lemma 4.2.25. *If φ is self-dual, then we have that $\varphi + n \leq_{\mathrm{L}} \varphi$ for all $n \in \omega$.*

Proof. First we note that if φ is self-dual, then $\varphi + n$ is self-dual for any $n \in \omega$. The reason for this is that any Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$ we have that $\varphi(x) < \varphi(f(x))$ also satisfies that for all $x \in \mathbb{R}$ we have that $\varphi(x) + n < \varphi(f(x)) + n$.

Also if φ is self-dual and again $f : \mathbb{R} \to \mathbb{R}$ is the Lipschitz-function witnessing it, then we have for all $x \in \mathbb{R}$ that $\varphi(x) + 1 \leq \varphi(f(x))$ and so $\varphi + 1 \leq_{\mathrm{L}} \varphi$.

But then inductively we get for self-dual φ, assuming that $\varphi + n \leq_{\mathrm{L}} \varphi$, that $\varphi + n$ is itself self-dual and so $\varphi + n + 1 \leq_{\mathrm{L}} \varphi + n \leq_{\mathrm{L}} \varphi$. This concludes the proof. □

4.3 A New Lower Bound for Σ

In this section we will work towards getting an improved lower bound for Σ by increasing the lower bounds for the Σ_αs for all $\alpha < \Theta$. Our strategy will be to show – using suitable operation on $\mathcal{N}$ – that the ordinals Σ_α are closed under multiplication, which we will then use to show that for all $\alpha < \Theta$ we have that $\Sigma_\alpha \geq \Theta^{(\Theta^\alpha)}$ and therefore $\Sigma \geq \Theta^{(\Theta^\Theta)}$.

We will do this in several steps. In Subsection 4.3.1 we will introduce an operation on $\mathcal{N}$ acting on Wadge ranks as ordinal addition. In Subsection 4.3.2 we will introduce an operation on $\mathcal{N}$ acting on Wadge ranks as multiplication of a rank with ω_1. In Subsection 4.3.3 we will finally introduce an operation on $\mathcal{N}$ acting on Wadge ranks as ordinal multiplication. Then we will wrap up everything and prove the new lower bound for Σ in Subsection 4.3.4.

The operation acting as addition and the operation acting as multiplication with ω_1 on Wadge ranks were – in the context of the original Wadge hierarchy – first considered by William Wadge (see [Wad12]). The operation acting as multiplication on Wadge ranks was developed later. Again in the context of the original Wadge hierarchy a complete treatment of this operation is given in the book draft [And00], which credits the PhD theses [Ste77] and [VW77] as sources for it. The addition and the multiplication operations operations were transfered by Benedikt Löwe to the context of the hierarchy of norms in unpublished notes [Löw10]. He also already showed that the addition indeed acts a addition. For the multiplication, however, he only gave a lower bound for the multiplication. In this thesis, we will give the exact action of all of the noted operations on Wadge ranks.

4.3.1 Closure Under Addition

The goal of this subsection will be to define for any two regular norms φ, ψ a regular norm $\varphi \dotplus \psi$ such that for any self-dual regular norm φ and an arbitrary norm ψ we have that $|\varphi \dotplus \psi|_{\mathrm{W}} = |\varphi|_{\mathrm{W}} + 1 + |\psi|_{\mathrm{W}}$ and $\mathrm{lh}(\varphi \dotplus \psi) = \max\{\mathrm{lh}(\varphi), \mathrm{lh}(\psi)\}$. Towards this end we will first introduce an auxiliary operation $\varphi \mapsto \varphi^{\nabla}$ on $\mathcal{N}$ which assigns to every self-dual regular norm a successor.

Definition 4.3.1 Given a regular norm φ we define a norm φ^{∇} by setting for all $x \in \mathbb{R}$:

$$\varphi^{\nabla}(x) := \begin{cases} \varphi(y), & \text{if there are } n, m \in \omega \text{ such that } x = 0^{(n)} {}^\frown \langle m+1 \rangle {}^\frown y, \\ 0, & \text{if } x = 0^{(\omega)}. \end{cases}$$

Lemma 4.3.2. *Assume* $\mathbf{AD}$. *If* φ *is a self-dual regular norm, then* $\varphi^\nabla \equiv_{\mathrm{W}} \varphi^{\mathrm{succ}}$ *and so* $\varphi <_{\mathrm{W}} \varphi^\nabla$. *If we furthermore assume* $\mathbf{DC}$ *we get as additional consequence that for any self-dual regular norm* φ *we have* $|\varphi^\nabla|_{\mathrm{W}} = |\varphi|_{\mathrm{W}} + 1$.

Proof. First we note that $\varphi^\nabla \leq_{\mathrm{L}} \varphi^{\mathrm{succ}}$ as Player $\mathbf{II}$ wins the game $\mathrm{G}^{\leq}_{\mathrm{L}}(\varphi^\nabla, \varphi^{\mathrm{succ}})$ using the strategy $\sigma : \omega^{<\omega} \setminus \{\emptyset\} \to \omega$ defined by setting for all $s \in \omega^{<\omega} \setminus \{\emptyset\}$

$$\sigma(s) := \begin{cases} 0, & \text{if } \forall n < \mathrm{lh}(s)\ (s(n) = 0), \\ 0, & \text{if } \forall n < \mathrm{lh}(s) - 1\ (s(n) = 0)\ \wedge\ s(\mathrm{lh}(s) - 1) \neq 0, \\ s(\mathrm{lh}(s) - 1) + 1, & \text{otherwise.} \end{cases}$$

This strategy is indeed winning for $\mathbf{II}$; if $x \in \mathbb{R}$ is of the form $0^{(n)} {}^\frown \langle m+1 \rangle {}^\frown y$ for some $n, m \in \omega$ and $y \in \mathbb{R}$, then by construction $\overline{\sigma}(x) = 0^{(n+1)} {}^\frown (y+1)$ and therefore $\mathrm{unstretch}(\overline{\sigma}(x)) = y$. But then we can calculate that

$$\varphi^\nabla(x) = \varphi(y) \leq \varphi(y) + 1 = \varphi^{\mathrm{succ}}(\overline{\sigma}(x)).$$

If, however, we consider the real $0^{(\omega)}$, then we get that $\overline{\sigma}(0^{(\omega)}) = 0^{(\omega)}$ and thus

$$\varphi^\nabla(0^{(\omega)}) = 0 = \varphi^{\mathrm{succ}}(0^{(\omega)}) = \varphi^{\mathrm{succ}}(\overline{\sigma}(0^{(\omega)})).$$

Thus we have shown that indeed $\varphi^\nabla \leq_{\mathrm{L}} \varphi^{\mathrm{succ}}$ and so a fortiori $\varphi^\nabla \leq_{\mathrm{W}} \varphi^{\mathrm{succ}}$.

For the other direction we first show that $\varphi <_{\mathrm{W}} \varphi^\nabla$. To do this we assume towards a contradiction that Player $\mathbf{II}$ has a winning strategy $\tau : \omega^{<\omega} \to \omega \cup \{\mathfrak{p}\}$ in the game $\mathrm{G}^{\leq}_{\mathrm{W}}(\varphi^\nabla, \varphi)$. Since we assumed that φ is self-dual, we have by Corollary 3.5.6 that $\mathbf{T}(\varphi)$ is well-founded and so $[\mathbf{T}(\varphi)] = \emptyset$. Hence in particular $\mathrm{filter}(\overline{\tau}(0^{(\omega)})) \notin [\mathbf{T}(\varphi)]$ and so we can fix the minimal $n \in \omega$ such that $\mathrm{filter}(\overline{\tau}(0^{(n)})) \in \partial\mathbf{T}(\varphi)$ and set $s := \mathrm{filter}(\overline{\tau}(0^{(n)} {}^\frown \langle 1 \rangle))$. Then τ induces a winning strategy τ' for Player $\mathbf{II}$ in the game $\mathrm{G}^{\leq}_{\mathrm{W}}\left((\varphi^\nabla)_{\lfloor 0^{(n)} {}^\frown \langle 1 \rangle \rfloor}, \varphi_{\lfloor s \rfloor} \right)$ by setting $\tau'(t) = \tau(0^{(n)} {}^\frown \langle 1 \rangle {}^\frown t)$ for all $t \in \omega^{<\omega} \setminus \{\emptyset\}$. But clearly we have that $(\varphi^\nabla)_{\lfloor 0^{(n)} {}^\frown \langle 1 \rangle \rfloor} = \varphi$ and that $s \notin \mathbf{T}(\varphi)$, which implies that $\varphi_{\lfloor s \rfloor} <_{\mathrm{W}} \varphi$. But then putting all this together we get that

$$\varphi = (\varphi^\nabla)_{\lfloor 0^{(n)} {}^\frown \langle 1 \rangle \rfloor} \leq_{\mathrm{W}} \varphi_{\lfloor s \rfloor} <_{\mathrm{W}} \varphi,$$

a contradiction. Hence we have shown that $\varphi <_{\mathrm{W}} \varphi^\nabla$. Now by Lemma 4.2.12 and the linearity of $\leq_{\mathrm{W}}$ we can finally conclude that $\varphi^\nabla \leq_{\mathrm{W}} \varphi^{\mathrm{succ}}$, concluding the proof. □

Now we are ready to define the addition operation.

Definition 4.3.3 For any two regular norms φ, ψ we define a regular norm $\varphi \dotplus \psi$ by setting

$$(\varphi \dotplus \psi)(x) := \begin{cases} \varphi(y), & \text{if there is } s \in \omega^{<\omega} \text{ such that } x = (s+1) {}^\frown \langle 0 \rangle {}^\frown y, \\ \psi(y), & \text{if } x = y + 1. \end{cases}$$

Lemma 4.3.4. *For any regular norms* $\varphi_0, \varphi_1, \psi_0, \psi_1$, *if* $\varphi_0 \leq_{\mathrm{L}} \psi_0$ *and* $\varphi_1 \leq_{\mathrm{L}} \psi_1$, *then* $\varphi_0 \dotplus \varphi_1 \leq_{\mathrm{L}} \psi_0 \dotplus \psi_1$. *An analogous statement holds for* $\leq_{\mathrm{W}}$.

Proof. Given winning strategies $\sigma_0, \sigma_1 : \omega^{<\omega} \setminus \{\emptyset\} \to \omega$ for Player $\mathbf{II}$ in the game $\mathrm{G}^{\leq}_{\mathrm{L}}(\varphi_0, \psi_0)$ and $\mathrm{G}^{\leq}_{\mathrm{L}}(\varphi_1, \psi_1)$, respectively, we construct a winning strategy $\tau : \omega^{<\omega} \setminus \{\emptyset\} \to \omega$ for Player $\mathbf{II}$ in the game $\mathrm{G}^{\leq}_{\mathrm{L}}(\varphi_0 \dotplus \varphi_1, \psi_0 \dotplus \psi_1)$ by setting for all $s \in \omega^{<\omega} \setminus \{\emptyset\}$:

$$\tau(s) := \begin{cases} \sigma_1(s-1) + 1, & \text{if } \forall n < \mathrm{lh}(s)\ (s(n) \neq 0), \\ 0, & \text{if } \forall n < \mathrm{lh}(s) - 1\ (s(n) \neq 0)\ \wedge\ s(\mathrm{lh}(s) - 1) = 0, \\ \sigma_0(s \setminus (s{\restriction}_k)), & \text{if } \exists n < \mathrm{lh}(s) - 1\ (s = 0), \\ & \quad \text{where } k = \min\{m < \mathrm{lh}(s) \mid s(m) = 0\} + 1. \end{cases}$$

This is indeed winning for Player $\mathbf{II}$, since for any $x \in \mathbb{R}$ of the form $x = (s+1) {}^\frown \langle 0 \rangle {}^\frown y$ for some $s \in \omega^{<\omega}$ and $y \in \mathbb{R}$ we get that $\overline{\tau}(x) = (\overline{\sigma_1}(s) + 1) {}^\frown \langle 0 \rangle {}^\frown \overline{\sigma_0}(y)$ and so

$$(\varphi_0 \dotplus \varphi_1)(x) = \varphi_0(y) \leq \psi_0(\overline{\sigma_0}(y)) = (\psi_0 \dotplus \psi_1)((\overline{\sigma_1}(s) + 1) {}^\frown \langle 0 \rangle {}^\frown \overline{\sigma_0}(y)) = (\psi_0 \dotplus \psi_1)(\overline{\tau}(x)).$$

and for any $x \in \mathbb{R}$ of the form $x = (y+1)$ we get that $\overline{\tau}(x) = \overline{\sigma_1}(y) + 1$ and so

$$(\varphi_0 + \varphi_1)(x) = \varphi_1(y) \leq \psi_1(\overline{\sigma_1}(y)) = (\psi_0 + \psi_1)(\overline{\sigma_1}(y) + 1) = (\psi_0 + \psi_1)(\overline{\tau}(x)).$$

This establishes that $\varphi_0 \dotplus \varphi_1 \leq_{\mathrm{L}} \psi_0 \dotplus \psi_1$. The argument for $\leq_{\mathrm{W}}$ is completely analogous. $\square$

Lemma 4.3.4 implies that the $\dotplus$-operation acts on Wadge-degrees, i.e., for any two regular norms φ, ψ we can define $[\varphi]_{\mathrm{W}} \dotplus [\psi]_{\mathrm{W}} := [\varphi \dotplus \psi]_{\mathrm{W}}$. The rest of this subsection will be devoted to showing that the $\dotplus$-operation really acts like addition on Wadge-degrees.

Lemma 4.3.5. *1. For any two regular norms φ, ψ we have that $\varphi \leq_{\mathrm{L}} \varphi \dotplus \psi$ and $\psi \leq_{\mathrm{L}} \varphi \dotplus \psi$.*

2. We have $\varphi \dotplus \vec{0} \equiv_{\mathrm{L}} \varphi^{\nabla}$ for any regular norm φ. Hence, assuming $\mathbf{AD}$, if φ is self-dual and ψ is an arbitrary norm, we have that $\varphi <_{\mathrm{W}} \varphi \dotplus \psi$.

3. We have that

$$(\varphi \dotplus \psi)_{\lfloor s+1 \rfloor} = \varphi \dotplus \psi_{\lfloor s \rfloor}$$

and

$$(\varphi \dotplus \psi)_{\lfloor (s+1)^\frown \langle 0 \rangle \rfloor} = \varphi$$

for any $s \in \omega^{<\omega}$.

Proof. 1. We consider the Lipschitz functions $f : \mathbb{R} \to \mathbb{R}, x \mapsto \langle 0 \rangle ^\frown x$ and $g : \mathbb{R} \to \mathbb{R}, x \mapsto x + 1$. Then for any $x \in \mathbb{R}$ we have that

$$\varphi(x) \leq (\varphi \dotplus \psi)(\langle 0 \rangle ^\frown x) = \varphi(f(x))$$

and

$$\psi(x) \leq (\varphi \dotplus \psi)(x+1) = \psi(g(x)).$$

This shows that indeed $\varphi \leq_{\mathrm{L}} \varphi \dotplus \psi$ and $\psi \leq_{\mathrm{L}} \varphi \dotplus \psi$, as claimed.

2. We note that for any $x \in \mathbb{R}$ we have

$$(\varphi \dotplus \vec{0})(x) = \begin{cases} \varphi(y), & \text{if there is } s \in \omega^{<\omega} \text{ such that } x = (s+1) ^\frown \langle 0 \rangle ^\frown y, \\ 0, & \text{otherwise.} \end{cases}$$

Hence we can give a winning strategy $\sigma : \omega^{<\omega} \setminus \{\emptyset\} \to \omega$ for Player **II** in the game $\mathrm{G}^{\leq}_{\mathrm{L}}(\varphi \dotplus \vec{0}, \varphi^{\nabla})$ by setting for all $s \in \omega^{<\omega} \setminus \{\emptyset\}$

$$\sigma(s) := \begin{cases} 0, & \text{if } \forall n < \mathrm{lh}(s)\, (s(n) \neq 0), \\ 1, & \text{if } \forall n < \mathrm{lh}(s) - 1\, (s(n) \neq 0) \ \wedge \ s(\mathrm{lh}(s) - 1) = 0, \\ s(\mathrm{lh}(s) - 1), & \text{otherwise.} \end{cases}$$

This is indeed winning for Player **II**, since for any $x \in \mathbb{R}$ of the form $x = (s+1) ^\frown \langle 0 \rangle ^\frown y$ for some $s \in \omega^{<\omega}$ and $y \in \mathbb{R}$ we then have that $\overline{\sigma}(x) = 0^{(\mathrm{lh}(s))} ^\frown \langle 1 \rangle ^\frown y$ and so

$$(\varphi \dotplus \vec{0})(x) = \varphi(y) = \varphi^{\nabla}(0^{(\mathrm{lh}(s))} ^\frown \langle 1 \rangle ^\frown y) = \varphi^{\nabla}(\overline{\sigma}(x))$$

and for any $x \in \mathbb{R}$ of the form $x = y + 1$ for some $y \in \mathbb{R}$ we have that $\overline{\sigma}(x) = 0^{(\omega)}$ and so

$$(\varphi \dotplus \vec{0})(x) = 0 = \varphi^{\nabla}(0^{(\omega)}) = \varphi^{\nabla}(\overline{\sigma}(x)).$$

We can give a winning strategy $\tau : \omega^{<\omega} \setminus \{\emptyset\} \to \omega$ for Player **II** in the game $\mathrm{G}^{\leq}_{\mathrm{L}}(\varphi^{\nabla}, \varphi \dotplus \vec{0})$ by setting for all $s \in \omega^{<\omega} \setminus \{\emptyset\}$

$$\tau(s) := \begin{cases} 1, & \text{if } \forall n < \mathrm{lh}(s)\, (s(n) = 0), \\ 0, & \text{if } \forall n < \mathrm{lh}(s) - 1\, (s(n) = 0) \ \wedge \ s(\mathrm{lh}(s) - 1) \neq 0, \\ s(\mathrm{lh}(s) - 1), & \text{otherwise.} \end{cases}$$

This strategy is indeed winning for Player **II**, since for any $x \in \mathbb{R}$ of the form $x = 0^{(n)} ^\frown \langle m+1 \rangle ^\frown y$ for some $n, m \in \omega$ and $y \in \mathbb{R}$ we have that $\overline{\tau}(x) = 1^{(n)} ^\frown \langle 0 \rangle ^\frown y$ and so

$$\varphi^{\nabla}(x) = \varphi(y) = (\varphi \dotplus \vec{0})(1^{(n)} ^\frown \langle 0 \rangle ^\frown y) = (\varphi \dotplus \vec{0})(\overline{\tau}(x))$$

and for $0^{(\omega)} \in \mathbb{R}$ we have that $\overline{\tau}(0^{(\omega)}) = 1^{(\omega)}$ and so

$$\varphi^{\nabla}(0^{(\omega)}) = 0 = (\varphi \dotplus \vec{0})(1^{(\omega)}) = (\varphi \dotplus \vec{0})(\overline{\sigma}(x)).$$

This establishes that indeed $\varphi \dotplus \vec{0} \equiv_{\mathrm{L}} \varphi^{\nabla}$.

3. If $x \in \mathbb{R}$ is of the form $x = (t+1) ^\frown \langle 0 \rangle ^\frown y$ for some $t \in \omega^{<\omega}$ and $y \in \mathbb{R}$, then we can calculate that

$$(\varphi \dotplus \psi)_{\lfloor s+1 \rfloor}(x) = (\varphi \dotplus \psi)((s+1) ^\frown x) = (\varphi \dotplus \psi)(((s ^\frown t)+1) ^\frown \langle 0 \rangle ^\frown y) = \varphi(y) = (\varphi \dotplus \psi_{\lfloor s \rfloor})(x)$$

and if $x \in \mathbb{R}$ is of the form $x = (y+1)$ for some $y \in \mathbb{R}$, we can calculate that

$$(\varphi \dotplus \psi)_{\lfloor s+1 \rfloor}(x) = (\varphi \dotplus \psi)((s+1)^\frown x)(\varphi \dotplus \psi)(((s^\frown y)+1)) = \psi(s^\frown y) = \psi_{\lfloor s \rfloor}(y) = \left(\varphi \dotplus \psi_{\lfloor s \rfloor}\right)(x).$$

Furthermore for any $x \in \mathbb{R}$ we can calculate that

$$(\varphi \dotplus \psi)_{\lfloor (s+1)^\frown \langle 0 \rangle \rfloor}(x) = (\varphi \dotplus \psi)((s+1) ^\frown \langle 0 \rangle ^\frown x) = \varphi(x).$$

□

Lemma 4.3.6. *Assume* **AD**. *For regular norms φ, ψ, ψ' and φ self-dual we have that*

$$\varphi \dotplus \psi \leq_{\mathrm{W}} \varphi \dotplus \psi' \quad \Leftrightarrow \quad \psi \leq_{\mathrm{W}} \psi'.$$

Proof. We have already shown the left-to-right direction in Lemma 4.3.4. For the other direction we assume that there is a winning strategy $\sigma : \omega^{<\omega} \setminus \{\emptyset\} \to \omega \cup \{\mathfrak{p}\}$ for Player **II** in the game $\mathrm{G}_{\mathrm{W}}^{\leq}(\varphi \dotplus \psi, \varphi \dotplus \psi')$. Now we claim that according to σ, Player **II** will never be the first player to play 0, i.e.,

$$\forall s \in \omega^{<\omega} \setminus \{\emptyset\} \, (\sigma(s+1) \neq 0).$$

To show this we assume towards a contradiction that there are $s, t \in \omega^{<\omega}$ such that $\mathrm{filter}(\overline{\sigma}(s+1)) = (t+1) ^\frown \langle 0 \rangle$. But then we get that $(\varphi \dotplus \psi)_{\lfloor s+1 \rfloor} \leq_{\mathrm{W}} (\varphi \dotplus \psi')_{\lfloor (t+1)^\frown \langle 0 \rangle \rfloor}$ and thus that

$$\varphi <_{\mathrm{W}} \varphi \dotplus \psi_{\lfloor s \rfloor} = (\varphi \dotplus \psi)_{\lfloor s+1 \rfloor} \leq_{\mathrm{W}} (\varphi \dotplus \psi')_{\lfloor (t+1)^\frown \langle 0 \rangle \rfloor} = \varphi,$$

which is absurd.

Using this we now define a strategy $\sigma' : \omega^{<\omega} \setminus \{\emptyset\} \to \omega \cup \{\mathfrak{p}\}$ for Player **II** in $\mathrm{G}_{\mathrm{W}}^{\leq}(\psi, \psi')$ by setting for every $s \in \omega^{<\omega} \setminus \{\emptyset\}$

$$\sigma'(s) = \begin{cases} \sigma(s+1) - 1, & \text{if } \sigma(s+1) \in \omega, \\ \mathfrak{p}, & \text{if } \sigma(s+1) = \mathfrak{p}. \end{cases}$$

But this strategy is winning for Player **II**, since for any $x \in \mathbb{R}$ we have that $\overline{\sigma'}(x) = \overline{\sigma}(x+1) - 1$ and so

$$\psi(x) = (\varphi \dotplus \psi)(x+1) \leq (\varphi \dotplus \psi')(\mathrm{filter}(\overline{\sigma}(x+1))) = \psi'(\mathrm{filter}(\overline{\sigma}(x+1) - 1)) = \psi'(\mathrm{filter}(\overline{\sigma'}(x)).$$

□

Lemma 4.3.7. *Assume* **AD**. *Suppose φ and ψ are regular norms such that $\varphi <_{\mathrm{W}} \psi$. Then there is a regular norm χ such that*

$$\varphi \dotplus \chi \equiv_{\mathrm{W}} \psi.$$

Proof. Let $T := \{s \in \omega^{<\omega} \mid \varphi <_{\mathrm{W}} \psi_{\lfloor s \rfloor}\}$. By assumption $\varphi <_{\mathrm{W}} \psi$ and so $\emptyset \in T$. Also if $s \in T$ and $t \subseteq s$, then $t \in T$, since $\varphi <_{\mathrm{W}} \psi_{\lfloor s \rfloor} \leq_{\mathrm{W}} \psi_{\lfloor t \rfloor}$. Hence T is a tree. Furthermore T is pruned. To see this we assume to the contrary that there is an $s \in T$ such that for all $n \in \omega$ we have $s ^\frown \langle n \rangle \notin T$. But then for all $n \in \omega$ we have $\varphi_{\lfloor s^\frown \langle n \rangle \rfloor} \leq_{\mathrm{W}} \varphi$ and so by Lemma 4.2.4 we have that $\varphi_{\lfloor s \rfloor} = \bigoplus_{n \in \omega} \varphi_{\lfloor s^\frown \langle n \rangle \rfloor} \leq_{\mathrm{W}} \varphi$ and so $s \notin T$, a contradiction.

Now since T is a pruned tree, its body $[T] \subseteq \mathbb{R}$ is a non-empty closed set and so there is a surjective Lipschitz function $f : \mathbb{R} \to [T]$ such that $f \restriction_{[T]} = \mathrm{id}_{[T]}$. We let $h : \omega^{<\omega} \to T$ be a surjective, monotone and strictly infinitary function inducing f. Next we define a regular norm $\chi := \psi \circ f$ and claim that $\varphi \dotplus \chi \equiv_{\mathrm{W}} \psi$.

To show this claim we first define a winning strategy $\tau : \omega^{<\omega} \setminus \{\emptyset\} \to \omega \cup \{\mathfrak{p}\}$ for Player **II** in the game $\mathrm{G}^{\leq}_{\mathrm{W}}(\psi, \varphi \dotplus \chi)$. For this we first note that the set $\omega^{<\omega}$ is countable and so we choose for any $t \in \omega^{<\omega} \setminus T$ a winning strategy $\sigma_t : \omega^{<\omega} \setminus \{\emptyset\} \to \omega \cup \{\mathfrak{p}\}$ in the game $\mathrm{G}^{\leq}_{\mathrm{W}}(\psi_{\lfloor t \rfloor}, \varphi)$, which exists by definition of T. Now we go on to define τ by setting for all $s \in \omega^{<\omega} \setminus \{\emptyset\}$

$$\tau(s) := \begin{cases} s(\mathrm{lh}(s)-1)+1, & \text{if } s \in T, \\ 0, & \text{if } s \notin T, \text{but } s\restriction_{\mathrm{lh}(s)-1} \in T, \\ \sigma_t(u), & \text{if } s \notin T \text{ and } s\restriction_{\mathrm{lh}(s)-1} \notin T, \\ & \quad \text{where } t \text{ is the unique terminal node in } T \text{ s.t. } t \subseteq s \\ & \quad \text{and } u \in \omega^{<\omega} \text{ is s.t. } s = t ^\frown u. \end{cases}$$

To see that this strategy is indeed winning for **II** in $\mathrm{G}^{\leq}_{\mathrm{W}}(\psi, \varphi \dotplus \chi)$ we note that for any $x \in [T]$ we have $\overline{\tau}(x) = x + 1$ and so

$$\psi(x) = \chi(x) = (\varphi \dotplus \chi)(x+1) = (\varphi \dotplus \chi)(\mathrm{filter}(\overline{\tau}(x))$$

and for $x \in \mathbb{R} \setminus [T]$, considering the unique terminal node in $t \in T$ such that $x = t ^\frown y$ for some $y \in \mathbb{R}$, we get that $\overline{\tau}(x) = (t+1) ^\frown \langle 0 \rangle ^\frown \overline{\sigma_t}(y)$ and so

$$\psi(x) = \psi_{\lfloor t \rfloor}(y) \leq \varphi(\mathrm{filter}(\overline{\sigma_t}(y))) = (\varphi \dotplus \chi)((t+1) ^\frown \langle 0 \rangle ^\frown \mathrm{filter}(\overline{\sigma_t}(y))) = (\varphi \dotplus \chi)(\mathrm{filter}(\overline{\tau}(y))).$$

Thus we have established that $\psi \leq_{\mathrm{W}} \varphi \dotplus \chi$.

Next we define a winning strategy $\tau' : \omega^{<\omega} \setminus \{\emptyset\} \to \omega \cup \{\mathfrak{p}\}$ for Player **II** in the game $\mathrm{G}^{\leq}_{\mathrm{W}}(\varphi \dotplus \chi, \psi)$. For this we choose for any $t \in T$ a winning strategy $\sigma'_t : \omega^{<\omega} \setminus \{\emptyset\} \to \omega \cup \{\mathfrak{p}\}$ in the game $\mathrm{G}^{\leq}_{\mathrm{W}}(\varphi, \psi_{\lfloor s \rfloor})$, which exists by definition of T. Now we go on to define τ' by setting for all $s \in \omega^{<\omega} \setminus \{\emptyset\}$

$$\tau'(s) := \begin{cases} h(s-1)(\mathrm{lh}(s)-1), & \text{if } \forall n < \mathrm{lh}(s)\ (s(n) \neq 0), \\ \min\{n \in \omega \mid h(s\restriction_{\mathrm{lh}(s)-1}) ^\frown \langle n \rangle \in T\}, & \text{if } \forall n < \mathrm{lh}(s)-1\ (s(n) \neq 0)\ \wedge\ s(\mathrm{lh}(s)-1) = 0 \\ \sigma'_t(s \setminus (s\restriction_{m+1})), & \text{if } \exists n < \mathrm{lh}(s)-1\ (s(n) = 0), \\ & \quad \text{where } m \in \omega \text{ is minimal s.t. } s(m) = 0. \end{cases}$$

This strategy is indeed winning for Player **II**; for any $x \in \mathbb{R}$ of the form $x = y + 1$ for some $y \in \mathbb{R}$ we have that $\overline{\tau'}(x) = \lim\{h(y\restriction_m) \mid m \in \omega\} = f(y)$ and therefore

$$(\varphi \dotplus \chi)(x) = \chi(y) = \psi(f(y)) = \psi(\mathrm{filter}(\overline{\tau'}(x))).$$

On the other hand for any $x \in \mathbb{R}$ of the form $x = (s+1) ^\frown \langle 0 \rangle ^\frown y$ for some $s \in \omega^{<\omega}$ and $y \in \mathbb{R}$ we have that there is a $t \in T$ such that $\overline{\tau'}(x) = t ^\frown \overline{\sigma'_t}(y)$ and therefore by choice of σ'_t we get that

$$(\varphi \dotplus \chi)(x) = \varphi(y) \leq \psi_{\lfloor t \rfloor}(\mathrm{filter}(\overline{\sigma'_t}(y))) = \psi(t ^\frown \mathrm{filter}(\overline{\sigma'_t}(y))) = \psi(\mathrm{filter}(t ^\frown \overline{\sigma'_t}(y))) = \psi(\mathrm{filter}(\overline{\tau'}(x)).$$

This establishes that $\varphi \dotplus \chi \leq_{\mathrm{W}} \psi$ and thus in total that $\varphi \dotplus \chi \equiv_{\mathrm{W}} \psi$, as claimed. □

Next we give a purely ordinal arithmetical argument showing that the results we have established up to now suffice to characterize the action of the $\dotplus$-operation on the level of Wadge ranks of self-dual regular norms in terms of ordinal addition.

Lemma 4.3.8. *Let $\mathbf{X}$ be a set of ordinals and δ an ordinal. Let $A : \mathbf{X} \times \delta \to \mathbf{Ord}$ be such that for all $\alpha \in \mathbf{X}$ and $\beta, \beta' \in \delta$ we have*

$$A(\alpha, \beta) \leq A(\alpha, \beta') \quad \Leftrightarrow \quad \beta \leq \beta'$$

and for all $\alpha \in \mathbf{X}$ and $\gamma \in \delta$ such that $\alpha < \gamma$ there is an ordinal $\beta \in \delta$ such that

$$A(\alpha, \beta) = \gamma$$

and for all $\alpha \in \mathbf{X}$ we have

$$A(\alpha, 0) = \alpha + 1.$$

Then for all $\alpha \in \mathbf{X}$ and $\beta \in \delta$ we have that $A(\alpha, \beta) = \alpha + 1 + \beta$.

Proof. We show this by induction on β. Let $\alpha \in \mathbf{X}$ be fixed. The base case is immediate since by assumption $A(\alpha, 0) = \alpha + 1 = \alpha + 1 + 0$.

For the successor step we need to show that $A(\alpha, \beta + 1) = A(\alpha, \beta) + 1$. To see that $A(\alpha, \beta) + 1 \leq A(\alpha, \beta + 1)$ we simply note that by assumption $\beta < \beta + 1$ implies that $A(\alpha, \beta) < A(\alpha, \beta + 1)$. Now we assume towards a contradiction that $A(\alpha, \beta) + 1 < A(\alpha, \beta + 1)$. Then we have that $A(\alpha, \beta) < A(\alpha, \beta) + 1 < A(\alpha, \beta + 1)$ and since there is no ordinal β' such that $\beta < \beta' < \beta + 1$, this implies that for no ordinal γ we have $A(\alpha, \gamma) = A(\alpha, \beta) + 1$, although clearly $\alpha < A(\alpha, 0) < A(\alpha, \beta) + 1$, which is a contradiction. Hence using the induction hypothesis we get that $A(\alpha, \beta + 1) = A(\alpha, \beta) + 1 = \alpha + 1 + (\beta + 1)$.

For β a limit ordinal we have to show that $A(\alpha, \beta) = \sup\{A(\alpha, \beta') \mid \beta' < \beta\}$. To show that $A(\alpha, \beta) \geq \sup\{A(\alpha, \beta') \mid \beta' < \beta\}$ we simply remark that by assumption $A(\alpha, \beta) \geq A(\alpha, \beta')$ for any $\beta' < \beta$. For the converse we assume towards a contradiction that $\sup\{A(\alpha, \beta') \mid \beta' < \beta\} < A(\alpha, \beta)$. Then for all $\beta'' < \beta$ we have $A(\alpha, \beta'') < \sup\{A(\alpha, \beta') \mid \beta' < \beta\} < A(\alpha, \beta)$, again contradicting our second assumption as in the successor case. Hence by induction hypothesis

$$A(\alpha, \beta) = \sup\{A(\alpha, \beta') \mid \beta' < \beta\} = \sup\{(\alpha + 1 + \beta') \mid \beta' < \beta\} = \alpha + 1 + \beta.$$

□

Theorem 4.3.9. *Assume* $\mathbf{AD}$ *and* $\mathbf{DC}$*. If* φ *is a self-dual regular norm and* ψ *an arbitrary regular norm, then*

$$|\varphi \dot{+} \psi|_{\mathrm{W}} = |\varphi|_{\mathrm{W}} + 1 + |\psi|_{\mathrm{W}}.$$

Proof. Let $\mathbf{X} := \{|\varphi|_{\mathrm{W}} \mid \varphi \text{ is self-dual regular norm}\}$. It follows from Lemma 4.3.4 that the map $A : \mathbf{X} \times \Sigma \to \mathbf{Ord}, (|\varphi|_{\mathrm{W}}, |\psi|_{\mathrm{W}}) \mapsto |\varphi \dot{+} \psi|_{\mathrm{W}}$ is well-defined. It follows from Lemma 4.3.6 that for any $\alpha \in \mathbf{X}$ and $\beta, \beta' \in \Sigma$ we have $A(\alpha, \beta) \leq A(\alpha, \beta')$ if and only if $\beta \leq \beta'$. By Lemma 4.3.7 for any $\alpha \in \mathbf{X}$ and any $\gamma \in \mathbf{Ord}$ with $\alpha < \gamma < \Sigma$ there is $\beta < \Sigma$ such that $A(\alpha, \beta) = \gamma$. By Lemma 4.3.5 we have for all $\alpha \in \mathbf{X}$ that $A(\alpha, 0) = \alpha + 1$.

Thus all prerequisites of Lemma 4.3.8 are satisfied and we get that for all $\alpha \in \mathbf{X}$ and $\beta \in Y$ we have $A(\alpha, \beta) = \alpha + 1 + \beta$, i.e., that for any self-dual regular norm φ and any regular norm ψ we have that $|\varphi \dot{+} \psi|_{\mathrm{W}} = |\varphi|_{\mathrm{W}} + 1 + |\psi|_{\mathrm{W}}$. □

Thus we have fully characterized the action of the $\dot{+}$-operation on Wadge degrees if the regular norm on the left is self-dual. Putting together several facts that we have shown until now we can furthermore give an upper bound for the general operation of $\dot{+}$ on Wadge degrees.

Proposition 4.3.10. *Assume* $\mathbf{AD}$ *and* $\mathbf{DC}$*. Then for any two regular norms* φ, ψ *we have that*

$$|\varphi \dot{+} \psi|_{\mathrm{W}} \leq |\varphi|_{\mathrm{W}} + \omega + |\psi|_{\mathrm{W}}.$$

Proof. We have that $\varphi^{+\omega}$ is self-dual and so, since $|\varphi^{+\omega}|_{\mathrm{W}} = |\varphi|_{\mathrm{W}} + \omega$, we get that

$$\left|\varphi^{+\omega} \dot{+} \psi\right|_{\mathrm{W}} = \left|\varphi^{+\omega}\right|_{\mathrm{W}} + 1 + |\psi|_{\mathrm{W}} = |\varphi|_{\mathrm{W}} + \omega + |\psi|_{\mathrm{W}}.$$

But since $\varphi \leq_{\mathrm{L}} \varphi^{+\omega}$ we have that $|\varphi \dot{+} \psi|_{\mathrm{W}} \leq |\varphi^{+\omega} \dot{+} \psi|_{\mathrm{W}}$ and thus the claim follows. □

Next we note that we can iterate addition up to, but not including, ω_1-fold by the following definition.

Definition 4.3.11 Assume $\mathbf{AC}_\omega$. We define for any Wadge degree $[\varphi]_W$ of regular norms Wadge degrees $[\varphi]_{\mathrm{W}} \odot \alpha$ for any ordinal α with $1 \leq \alpha < \omega_1$ by recursively setting:

- We set $[\varphi]_{\mathrm{W}} \odot 1 := [\varphi]_{\mathrm{W}}$.
- For α a successor ordinal, say $\alpha = \gamma + 1$, we define $[\varphi]_{\mathrm{W}} \odot \alpha := ([\varphi]_{\mathrm{W}} \odot \gamma) \dot{+} [\varphi]_{\mathrm{W}}$.
- For α a limit ordinal we take a sequence $\langle \alpha_n \mid n \in \omega \rangle$ of ordinals cofinal in α and choose – using $\mathbf{AC}_\omega$ – a sequence $\langle \psi_n \mid n \in \omega \rangle$ of regular norms such that for any $n \in \omega$ we have that $\psi_n \in [\varphi]_{\mathrm{W}} \cdot \alpha_n$. Then we set

$$[\varphi]_{\mathrm{W}} \odot \alpha = \left[\bigoplus_{n \in \omega} \psi_n\right]_{\mathrm{W}}.$$

That this is already well-defined follows from the next lemma.

Lemma 4.3.12. *Assume* $\mathbf{AC}_\omega(\mathbb{R})$. *If* $\{\varphi_n \mid n \in \omega\}$ *and* $\{\psi_n \mid n \in \omega\}$ *are families of regular norms such that for all* $n \in \omega$ *there is a* $k \in \omega$ *such that* $\varphi_n \leq_{\mathrm{L}} \psi_k$*, then we have that*

$$\bigoplus_{n\in\omega} \varphi_n \leq_{\mathrm{L}} \bigoplus_{k\in\omega} \psi_k.$$

The same is true for $\leq_{\mathrm{L}}$ *replaced with* $\leq_{\mathrm{W}}$ *everywhere.*

Proof. We only show the proposition for the case of $\leq_{\mathrm{L}}$, since the case of the $\leq_{\mathrm{W}}$ is completely analogous. To do this we construct a winning strategy $\sigma : \omega^{<\omega} \setminus \{\emptyset\} \to \omega$ for Player **II** in the game $\mathrm{G}_{\mathrm{L}}^{\leq}\left(\bigoplus_{n\in\omega} \varphi_n, \bigoplus_{k\in\omega} \psi_k\right)$ as follows. First we fix the function

$$f : \omega \to \omega, n \mapsto \min\{k \in \omega \mid \varphi_n \leq_{\mathrm{L}} \varphi_k\}.$$

Then using $\mathbf{AC}_\omega(\mathbb{R})$ we choose for any $n \in \omega$ a winning strategy $\sigma_n : \omega^{<\omega} \setminus \{\emptyset\} \to \omega$ for Player **II** in the game $\mathrm{G}_{\mathrm{L}}^{\leq}(\varphi_n, \varphi_{f(n)})$. Now we set for any $n \in \omega$ and $s \in \omega^{<\omega}$

$$\sigma(\langle n\rangle ^\frown s) := \begin{cases} f(n), & \text{if } s = \emptyset, \\ \sigma_n(s), & \text{otherwise.} \end{cases}$$

Then σ is indeed winning for Player **II**, since for any $m \in \omega$ and $x \in \mathbb{R}$ we have that $\overline{\sigma}(\langle m\rangle ^\frown x) = \langle f(m)\rangle ^\frown \overline{\sigma_m}(x)$ and so

$$\left(\bigoplus_{n\in\omega} \varphi_n\right)(\langle m\rangle^\frown x) = \varphi_m(x) \leq \psi_{f(m)}(\overline{\sigma_m}(x)) = \left(\bigoplus_{k\in\omega} \psi_k\right)(\langle f(m)\rangle^\frown \overline{\sigma_m}(x)) = \left(\bigoplus_{k\in\omega} \psi_k\right)(\overline{\sigma}(x)).$$

□

Now to see that the limit step of Definition 4.3.11 is actually well-defined. For this we first note that given a choice $\langle \alpha_n \mid n \in \omega\rangle$ of a cofinal sequence of ordinals for α, we have that the choice of representatives $\psi_n \in [\varphi]_{\mathrm{W}} \cdot \lambda_n$ does not matter, since the join operation acts on Wadge degrees. Thus we only have to show that the choice of the sequence cofinal in α does not matter. So we let $\langle \alpha_n \mid n \in \omega\rangle$ and $\langle \beta_n \mid n \in \omega\rangle$ be two sequences of ordinals cofinal in α. Then using $\mathbf{AC}_\omega$ we choose sequences $\langle \psi_n \mid n \in \omega\rangle$ and $\langle \psi'_n \mid n \in \omega\rangle$ of regular norms such that for all $n \in \omega$ we have that $\psi_n \in [\varphi]_{\mathrm{W}} \cdot \alpha_n$ and $\psi'_n \in [\varphi]_{\mathrm{W}} \cdot \beta_n$. But then, since $\langle \alpha_n \mid n \in \omega\rangle$ and $\langle \beta_n \mid n \in \omega\rangle$ for any $n \in \omega$ we have a $k \in \omega$ such that $\alpha_n \leq \beta_k$ and so directly by the definition of $[\varphi]_{\mathrm{W}} \cdot \gamma$ for $\gamma < \alpha$, we see that then $\psi_n \leq_{\mathrm{L}} \psi'_k$. Thus by Lemma 4.3.12 we have that $\bigoplus_{n\in\omega} \psi_n \leq_{\mathrm{W}} \bigoplus_{k\in\omega} \psi'_k$. But then by symmetry it follows that $\bigoplus_{n\in\omega} \psi_n \equiv_{\mathrm{W}} \bigoplus_{k\in\omega} \psi'_k$, which shows that $[\varphi]_{\mathrm{W}} \odot \alpha$ is indeed well-defined.

Next we show that this operation acts as multiplication with a given ordinal $\alpha < \omega_1$ on the level of Wadge ranks.

Lemma 4.3.13. *If* φ, ψ *are self-dual regular norms, then the norm* $\varphi \dotplus \psi$ *is also self-dual.*

Proof. Let $\sigma_0, \sigma_1 : \omega^{<\omega} \setminus \{\emptyset\} \to \omega$ be winning strategies for Player **II** in the game $\mathrm{G}_{\mathrm{L}}^{<}(\varphi, \varphi)$ and $\mathrm{G}_{\mathrm{L}}^{<}(\psi, \psi)$, respectively. Then we verbatim repeat the construction in the proof of Lemma 4.3.4 to obtain a strategy $\tau : \omega^{<\omega} \setminus \{\emptyset\} \to \omega$ for Player **II** in the game $\mathrm{G}_{\mathrm{L}}^{<}(\varphi \dotplus \psi, \varphi \dotplus \psi)$. This strategy is winning for Player **II**, since as shown in Lemma 4.3.4 for any $x \in \mathbb{R}$ of the form $x = (s+1) ^\frown \langle 0\rangle ^\frown y$ for some $s \in \omega^{<\omega}$ and $y \in \mathbb{R}$ we get that $\overline{\tau}((s+1) ^\frown \langle 0\rangle ^\frown y) = (\overline{\sigma_1}(s) + 1) ^\frown \langle 0\rangle ^\frown \overline{\sigma_0}(y)$ and so

$$(\varphi \dotplus \psi)(x) = \varphi(y) < \varphi(\overline{\sigma_0}(y)) = (\varphi \dotplus \psi)((\overline{\sigma_1}(s) + 1) ^\frown \langle 0\rangle ^\frown \overline{\sigma_0}(y)) = (\varphi \dotplus \psi)(\overline{\tau}(x)).$$

On the other hand for $x \in \mathbb{R}$ of the form $x = (y + 1)$ we have that $\overline{\tau}(x) = \overline{\sigma_1}(y) + 1$ and so

$$(\varphi \dotplus \psi)(x) = \psi(y) < \psi(\overline{\sigma_1}(y)) = (\varphi \dotplus \psi)(\overline{\sigma_1}(y) + 1) = (\varphi \dotplus \psi)(\overline{\tau}(x)).$$

In total this shows that Player **II** wins the game $\mathrm{G}_{\mathrm{L}}^{<}(\varphi \dotplus \psi, \varphi \dotplus \psi)$ and so that $\varphi \dotplus \psi$ is self-dual. □

Proposition 4.3.14. *Assume* $\mathbf{AD}$ *and* $\mathbf{DC}$. *For all* $\alpha < \omega_1$*, if* $[\varphi]_{\mathrm{W}}$ *is a self-dual Wadge degree, then for any* $\alpha < \omega_1$, $[\varphi]_{\mathrm{W}} \odot \alpha$ *is a self-dual Wadge degree. Furthermore we have for all* $\alpha < \omega_1$ *that the Wadge rank of* $[\varphi]_{\mathrm{W}} \cdot \alpha$ *is* $|\varphi|_{\mathrm{W}} \cdot \alpha$.

Proof. The first part of the proposition follows directly from Lemma 4.3.13 and Lemma 4.2.7 by induction on α.

The second part also follows by induction on α as follows. For $\alpha = 1$ we have that $[\varphi]_{\mathrm{W}} \odot \alpha = [\varphi]_{\mathrm{W}}$ and so $|[\varphi]_{\mathrm{W}} \odot \alpha|_{\mathrm{W}} = |\varphi|_{\mathrm{W}} \cdot \alpha$.

For α a successor ordinal, say $\alpha = \gamma + 1$, we have that $[\varphi]_{\mathrm{W}} \odot \alpha = [\varphi]_{\mathrm{W}} \odot \gamma + [\varphi]_{\mathrm{W}}$ but since $[\varphi]_{\mathrm{W}} \odot \gamma$ is self-dual, we then have that

$$|[\varphi]_{\mathrm{W}} \odot \alpha|_{\mathrm{W}} = |[\varphi]_{\mathrm{W}} \odot \gamma|_{\mathrm{W}} + 1 + |\varphi|_{\mathrm{W}} = |\varphi|_{\mathrm{W}} \cdot \gamma + 1 + |\varphi|_{\mathrm{W}}.$$

But since φ is self-dual we have that $|\varphi|_{\mathrm{W}} \geq \omega$ and so it follows that

$$|[\varphi]_{\mathrm{W}} \odot \alpha|_{\mathrm{W}} = |\varphi|_{\mathrm{W}} \cdot \gamma + 1 + |\varphi|_{\mathrm{W}} = |\varphi|_{\mathrm{W}} \cdot \gamma + |\varphi|_{\mathrm{W}} = |\varphi|_{\mathrm{W}} \cdot \alpha.$$

Finally for α a limit ordinal we have by induction hypothesis that $|[\varphi]_{\mathrm{W}} \cdot \alpha'|_{\mathrm{W}} = |\varphi|_{\mathrm{W}} \cdot \alpha'$ for all $\alpha' < \alpha$. But then we have that for some sequence $\langle \alpha_n \mid n \in \omega \rangle$ of ordinals cofinal in α that

$$|[\varphi]_{\mathrm{W}} \odot \alpha|_{\mathrm{W}} = \left| \bigoplus_{n \in \omega} ([\varphi] \odot \alpha_n) \right|_{\mathrm{W}} = \sup\{ |[\varphi] \odot \alpha_n|_{\mathrm{W}} \mid n \in \omega \} = \sup\{ |\varphi|_{\mathrm{W}} \cdot \alpha_n \mid n \in \omega \} = |\varphi|_{\mathrm{W}} \cdot \alpha.$$

□

This result is the basis for our investigation of an operation acting as multiplication with ω_1 on Wadge degrees in the next subsection.

4.3.2 Multiplication with ω_1

In the last subsection we have constructed an operation on self-dual Wadge degrees for multiplication with an arbitrary ordinal $\alpha < \omega_1$. Now we will define an operation $\varphi \mapsto \varphi^\natural$ on $\mathcal{N}$ such that for any self-dual φ we have that $|\varphi^\natural|_{\mathrm{W}} = |\varphi|_{\mathrm{W}} \cdot \omega_1$.

Definition 4.3.15 For any regular norm φ we define $\varphi^\natural$ by setting for all $x \in \mathbb{R}$

$$\varphi^\natural(x) = \begin{cases} \varphi(y), & \text{if } \exists s \in \omega^{<\omega} \, (x = s \frown \langle 0 \rangle \frown (y+1)), \\ \varphi(y), & \text{if } x = (y+1), \\ 0, & \text{if } \exists^\infty n \in \omega \, (y(n) = 0). \end{cases}$$

The idea behind this definition is that $\varphi^\natural$ on input $x \in \mathbb{R}$ sequentially reads $x(n)$ and assembles a sequence y at the side by at step n appending $x(n) - 1$ to the sequence assembled up to now, if $x(n) > 1$. Otherwise, if $x(n) = 0$, the sequence assembled up to now is erased instead and a new recording of a sequence y potentially begins in the next step. If this procedure results in a sequence $y \in \mathbb{R}$, then $\varphi^\natural$ outputs $\varphi(y)$, otherwise $\varphi^\natural$ outputs the default value 0.

We will formally capture this idea using the following auxiliary definition:

Definition 4.3.16 We fix some $\mathfrak{nd} \notin (\omega \cup \{\mathfrak{p}\})^{\leq\omega} \cup (\omega \cup \{\mathfrak{p}\})$ and use this as a symbol that we read as *"not defined"*. Then we define a function $\mathrm{record} : \omega^{\leq\omega} \to \omega^{\leq\omega} \cup \{\mathfrak{nd}\}$ by setting for all $x \in \omega^{\leq\omega}$

$$\mathrm{record}(x) := \begin{cases} y - 1, & \text{if } \exists m < \mathrm{lh}(x) \, \forall n \geq m \, (n < \mathrm{lh}(x) \Rightarrow x(n) \neq 0), \\ & \quad \text{where } y \text{ is the unique maximal end-segment of } x \text{ s.t. } \forall n < \mathrm{lh}(y) \, (y(n) > 0), \\ \mathfrak{nd}, & \text{otherwise.} \end{cases}$$

Furthermore we define $\mathrm{NRec} : \omega^{\leq\omega} \to \omega + 1$ by setting for all $x \in \omega^{\leq\omega}$

$$\mathrm{NRec}(x) := \begin{cases} \mathrm{card}\{n < \mathrm{lh}(x) \mid x(n) = 0\} + 1, & \text{if } \mathrm{card}\{n < \mathrm{lh}(x) \mid x(n) = 0\} \in \omega, \\ \mathrm{card}\{n < \mathrm{lh}(x) \mid x(n) = 0\}, & \text{if } \mathrm{card}\{n < \mathrm{lh}(x) \mid x(n) = 0\} = \omega. \end{cases}$$

According to this definition we get that for any $x \in \mathbb{R}$ we have that $\operatorname{record}(x) = \mathfrak{nd}$ if and only if $\operatorname{NRec}(x) = \omega$. We can now use these auxiliary function to note that for any regular norm and any $x \in \mathbb{R}$ we have that

$$\varphi^\natural(x) = \begin{cases} \varphi(\operatorname{record}(x)), & \text{if } \operatorname{NRec}(x) \in \omega, \\ 0, & \text{otherwise.} \end{cases}$$

We will not use this presentation of $\varphi^\natural$ in this subsection. However, in the next subsection it will be used to simplify the presentation of norms resulting from successive applications of several operations on $\mathcal{N}$.

In the remainder of this subsection we are going to show that the operation $\varphi \mapsto \varphi^\natural$ really acts on self-dual Wadge ranks as multiplications with ω_1.

Lemma 4.3.17. *For two regular norms φ, ψ with $\varphi \leq_{\mathrm{W}} \psi$ we have that $\varphi^\natural \leq_{\mathrm{W}} \psi^\natural$.*

Proof. We let $\sigma : \omega^{<\omega} \setminus \{\emptyset\} \to \omega \cup \{\mathfrak{p}\}$ be a winning strategy for Player **II** in the game $\mathrm{G}^{\leq}_{\mathrm{W}}(\varphi, \psi)$. Then we define a winning strategy $\tau : \omega^{<\omega} \setminus \{\emptyset\} \to \omega \cup \{\mathfrak{p}\}$ by setting for all $s \in \omega^{<\omega} \setminus \{\emptyset\}$

$$\tau(s) := \begin{cases} 0, & \text{if } s(\operatorname{lh}(s) - 1) = 0, \\ \tau(t-1) + 1, & \text{otherwise, where } t \text{ is the maximal end-segment of } s \text{ s. t. } \forall n < \operatorname{lh}(t)\, (t(n) \neq 0). \end{cases}$$

This is indeed winning for **II**; if $x \in \mathbb{R}$ is such that $x = s \frown \langle 0 \rangle \frown (y+1)$ for some $s \in \omega^{<\omega}$ and $y \in \mathbb{R}$, then by construction there is some $t \in (\omega \cup \{\mathfrak{p}\})^{<\omega}$ such that $\overline{\tau}(x) = t \frown \langle 0 \rangle \frown (\overline{\sigma}(y) + 1)$ and so we have that

$$\varphi^\natural(x) = \varphi(y) \leq \psi(\operatorname{filter}(\overline{\sigma}(y))) = \psi^\natural(\operatorname{filter}(t \frown \langle 0 \rangle \frown (\overline{\sigma}(y) + 1))) = \psi^\natural(\operatorname{filter}(\overline{\tau}(x))).$$

If $x \in \mathbb{R}$ is such that $x = y + 1$ for some $y \in \mathbb{R}$ then we have that $\overline{\tau}(x) = \overline{\sigma}(y) + 1$ and so again

$$\varphi^\natural(x) = \varphi(y) \leq \psi(\operatorname{filter}(\overline{\sigma}(y)) = \psi^\natural(\operatorname{filter}(\overline{\sigma}(y) + 1)) = \psi^\natural(\operatorname{filter}(\overline{\tau}(x))).$$

If, however, $x \in \mathbb{R}$ is such that there are infinitely many $n \in \omega$ such that $x(n) = 0$, then there are also infinitely many $n \in \omega$ such that $(\overline{\tau}(x))(n) = 0$ and thus we have that

$$\varphi^\natural(x) = 0 = \psi^\natural(\operatorname{filter}(\overline{\tau}(x))).$$

This shows that $\varphi^\natural \leq_{\mathrm{W}} \psi^\natural$. □

This shows that the $\natural$-operation acts on Wadge degrees, i.e., for any regular norm φ we can define $[\varphi]^\natural_{\mathrm{W}} := [\varphi^\natural]_{\mathrm{W}}$.

Lemma 4.3.18. *For any self-dual regular norm φ the regular norm $\varphi^\natural$ is non-self-dual.*

Proof. Let $\sigma : \omega^{<\omega} \setminus \{\emptyset\} \to \omega$ be a winning strategy for Player **II** in the game $\mathrm{G}^{<}_{\mathrm{L}}(\varphi, \varphi)$. Then we define a winning strategy $\tau : \omega^{<\omega} \to \omega$ for Player **I** in the game $\mathrm{G}^{<}_{\mathrm{L}}(\varphi^\natural, \varphi^\natural)$ by setting for all $s \in \omega^{<\omega}$

$$\tau(s) := \begin{cases} 0, & \text{if } \operatorname{lh}(s) = 0 \text{ or else } s(\operatorname{lh}(s) - 1) = 0, \\ \sigma(t-1) + 1, & \text{if } t \text{ is the maximal end-segment of } s \text{ s.t. for all } n < \operatorname{lh}(t)\text{: } t(n) \neq 0. \end{cases}$$

This is indeed winning for Player **I**; we note that for any $x \in \mathbb{R}$ of the form $x = y + 1$ for some $y \in \mathbb{R}$ we have that $\overline{\tau}(x) = \langle 0 \rangle \frown (\overline{\sigma}(y) + 1)$ and so

$$\varphi^\natural(\overline{\tau}(x)) = \varphi^\natural(\langle 0 \rangle \frown (\overline{\sigma}(y) + 1)) = \varphi(\overline{\sigma}(y)) > \varphi(y) = \varphi^\natural(x).$$

Furthermore for any $x \in \mathbb{R}$ of the form $x = s \frown \langle 0 \rangle \frown (y+1)$ for some $s \in \omega^{<\omega}$ and $y \in \mathbb{R}$ we have $t \in \omega^{<\omega}$ such that $\overline{\tau}(x) = t \frown \langle 0 \rangle \frown (\overline{\sigma}(y) + 1)$ and so

$$\varphi^\natural(\overline{\tau}(x)) = \varphi^\natural(t \frown \langle 0 \rangle \frown (\overline{\sigma}(y) + 1)) = \varphi(\overline{\sigma}(y)) > \varphi(y) = \varphi^\natural(x).$$

Finally for any $x \in \mathbb{R}$ such that there are infinitely many indices $n \in \omega$ such that $x(n) = 0$ we get that there are infinitely many indices $n \in \omega$ such that $(\overline{\tau}(x))(n) = 0$. Hence we have

$$\varphi^\natural(\overline{\tau}(x)) = 0 = \varphi^\natural(x).$$

In total we thus get that for any $x \in \mathbb{R}$ we have that $\varphi^\natural(\overline{\tau}(x)) \geq \varphi^\natural(x)$, which shows that Player **I** wins $\mathrm{G}^{<}_{\mathrm{L}}(\varphi^\natural, \varphi^\natural)$ and so Player **II** has no winning strategy in $\mathrm{G}^{<}_{\mathrm{L}}(\varphi^\natural, \varphi^\natural)$, which in turn implies that $\varphi^\natural$ is non-self-dual, as claimed. □

Lemma 4.3.19. *Assume* $\mathbf{AD}$*. Let* φ *be a self-dual regular norm. For any two regular norms* $\psi, \chi <_{\mathrm{W}} \varphi^{\natural}$ *we have that* $\psi \dotplus \chi <_{\mathrm{W}} \varphi^{\natural}$.

Proof. We note that – since $\varphi^{\natural}$ is non-self-dual – by Proposition 3.4.7 we have that $\psi, \chi <_{\mathrm{W}} \varphi^{\natural}$ implies that Player $\mathbf{II}$ wins the games $\mathrm{G}^{<}_{\mathrm{W}}(\psi, \varphi^{\natural})$ and $\mathrm{G}^{<}_{\mathrm{W}}(\chi, \varphi^{\natural})$ with winning strategies $\tau_0 : \omega^{<\omega} \setminus \{\emptyset\} \to \omega \cup \{\mathfrak{p}\}$ and $\tau_1 : \omega^{<\omega} \setminus \{\emptyset\} \to \omega \cup \{\mathfrak{p}\}$, respectively. Now we define a winning strategy $\sigma : \omega^{<\omega} \setminus \{\emptyset\} \to \omega \cup \{\mathfrak{p}\}$ in the game $\mathrm{G}^{<}_{\mathrm{W}}(\psi \dotplus \chi, \varphi^{\natural})$ by setting for all $s \in \omega^{<\omega} \setminus \{\emptyset\}$

$$\sigma(s) := \begin{cases} \tau_1(s-1), & \text{if } \forall n < \mathrm{lh}(s)\ (s(n) > 0), \\ 0, & \text{if } s(\mathrm{lh}(s)-1) = 0, \\ \tau_0(s \setminus (s\restriction_{m+1})), & \text{if } \exists n < \mathrm{lh}(s)-1\ (s(n) = 0) \text{ and } s(\mathrm{lh}(s)-1) \neq 0, \\ & \quad \text{where } m < \mathrm{lh}(s)-1 \text{ is minimal s.t. } s(m) = 0. \end{cases}$$

This strategy is indeed winning for $\mathbf{II}$; if $x \in \mathbb{R}$ is of the form $x = y + 1$ for some $y \in \mathbb{R}$, then we get that $\overline{\sigma}(x) = \overline{\tau_1}(y)$ and so by choice of τ_1 we get

$$(\psi \dotplus \chi)(x) = \chi(y) \leq \varphi^{\natural}(\mathrm{filter}(\overline{\tau_1}(y)) = \varphi^{\natural}(\mathrm{filter}(\overline{\sigma}(x)).$$

If $x \in \mathbb{R}$ is of the form $x = (s+1) \frown \langle 0 \rangle \frown y$ for some $s \in \omega^{<\omega}$ and $y \in \mathbb{R}$ we have a $t \in \omega^{<\omega}$ such that $\mathrm{filter}(\overline{\sigma}(x)) = t \frown \langle 0 \rangle \frown (\mathrm{filter}(\overline{\tau_0}(y)))$ and so by choice of τ_0 we get

$$(\psi \dotplus \chi)(x) = \psi(y) \leq \varphi^{\natural}(\mathrm{filter}(\overline{\tau_0}(y)) = \varphi^{\natural}(t \frown \langle 0 \rangle \frown (\mathrm{filter}(\overline{\tau_0}(y)))) = \varphi^{\natural}(\overline{\sigma}(x)).$$

This establishes that Player $\mathbf{II}$ indeed wins $\mathrm{G}^{<}_{\mathrm{W}}(\psi \dotplus \chi, \varphi^{\natural})$ and so again by Proposition 3.4.7 it follows that $\psi \dotplus \chi <_{\mathrm{W}} \varphi^{\natural}$ as claimed. □

Proposition 4.3.20. *Assume* $\mathbf{AD}$ *and* $\mathbf{DC}$*. Let* φ *be a self-dual regular norm. Then for any* $\alpha < \omega_1$ *with* $\alpha > 0$ *we have that* $[\varphi]_{\mathrm{W}} \odot \alpha <_{\mathrm{W}} [\varphi]^{\natural}_{\mathrm{W}}$.

Proof. We show this by induction on α.

For the base case we need to show that $\varphi <_{\mathrm{W}} \varphi^{\natural}$. Since $\varphi^{\natural}$ is non-self-dual, by Proposition 3.4.7 this is equivalent to showing that Player $\mathbf{II}$ wins the game $\mathrm{G}^{<}_{\mathrm{W}}(\varphi, \varphi^{\natural})$. For this we take a winning strategy $\sigma : \omega^{<\omega} \setminus \{\emptyset\} \to \omega \cup \{\mathfrak{p}\}$ for Player $\mathbf{II}$ in the game $\mathrm{G}^{<}_{\mathrm{W}}(\varphi, \varphi)$, which exists, since φ is self-dual, and define a winning strategy $\tau : \omega^{<\omega} \setminus \{\emptyset\} \to \omega \cup \{\mathfrak{p}\}$ by setting for all $s \in \omega^{<\omega} \setminus \{\emptyset\}$

$$\tau(s) := \sigma(s) + 1.$$

Then we have for all $x \in \mathbb{R}$ that $\overline{\tau}(x) = \overline{\sigma}(x) + 1$ and so

$$\varphi(x) < \varphi(\mathrm{filter}(\overline{\sigma}(x))) = \varphi^{\natural}(\mathrm{filter}(\overline{\sigma}(x)) + 1) = \varphi^{\natural}(\mathrm{filter}(\overline{\tau}(x)),$$

which shows that this strategy is indeed winning for Player $\mathbf{II}$ and so that $\varphi <_{\mathrm{W}} \varphi^{\natural}$. But then we have that $[\varphi]_{\mathrm{W}} \odot 1 = [\varphi]_{\mathrm{W}} <_{\mathrm{W}} [\varphi]^{\natural}_{\mathrm{W}}$.

For the successor step we take a successor ordinal α, say $\alpha = \gamma + 1$. Then $[\varphi]_{\mathrm{W}} \odot \alpha = [\varphi]_{\mathrm{W}} \odot \gamma \dotplus [\varphi]_{\mathrm{W}}$. But by induction hypothesis we have $[\varphi]_{\mathrm{W}} \odot \gamma < [\varphi]^{\natural}_{\mathrm{W}}$ and by the base case we have $[\varphi]_{\mathrm{W}} < [\varphi]^{\natural}_{\mathrm{W}}$ and so by Lemma 4.3.19 we have that $[\varphi]_{\mathrm{W}} \odot \gamma \dotplus [\varphi]_{\mathrm{W}} < [\varphi]^{\natural}_{\mathrm{W}}$.

For the limit step we note that for some sequence $\langle \alpha_n \mid n \in \omega \rangle$ of ordinals cofinal in α and a sequence $\langle \chi_n \mid n \in \omega \rangle$ of regular norms with $\chi_n \in [\varphi]_{\mathrm{W}} \odot \alpha_n$ for all $n \in \omega$ we have that

$$[\varphi]_{\mathrm{W}} \odot \alpha = \left[\bigoplus_{n \in \omega} \chi_n \right]_{\mathrm{W}}.$$

But since for all $n \in \omega$ by induction hypothesis we have that $[\chi_n]_{\mathrm{W}} = [\varphi]_{\mathrm{W}} \odot \alpha_n <_{\mathrm{W}} [\varphi]^{\natural}_{\mathrm{W}}$ we get by Lemma 4.2.6 that $[\varphi]_{\mathrm{W}} \odot \alpha = \left[\bigoplus_{n \in \omega} \chi_n\right]_{\mathrm{W}} \leq_{\mathrm{W}} [\varphi]^{\natural}_{\mathrm{W}}$. But by Lemma 4.2.7 the regular norm $\bigoplus_{n \in \omega} \chi_n$ is self-dual and by Lemma 4.3.18 the regular norm $\varphi^{\natural}$ is non-self-dual. Thus we have that $\bigoplus_{n \in \omega} \chi_n \not\equiv_{\mathrm{W}} \varphi^{\natural}$, from which it finally follows that $[\varphi]_{\mathrm{W}} \odot \alpha <_{\mathrm{W}} [\varphi]^{\natural}_{\mathrm{W}}$, concluding the proof. □

Theorem 4.3.21. *Assume* $\mathbf{AD}$ *and* $\mathbf{DC}$*. For any self-dual regular norm* φ *we have that*

$$\left|\varphi^{\natural}\right|_{\mathrm{W}} = |\varphi|_{\mathrm{W}} \cdot \omega_1.$$

Proof. From Proposition 4.3.20 together with Proposition 4.3.14 it follows that for all $\alpha < \omega_1$ we have $\left|\varphi^\natural\right|_{\mathrm{W}} > |\varphi|_{\mathrm{W}} \cdot \alpha$ and thus that $\left|\varphi^\natural\right|_{\mathrm{W}} \geq |\varphi|_{\mathrm{W}} \cdot \omega_1$.

For the other direction we take a regular norm ψ with $\psi <_{\mathrm{W}} \varphi^\natural$ and show that there is $\alpha < \omega_1$ such that $[\psi]_{\mathrm{W}} \leq_{\mathrm{W}} [\varphi] \odot \alpha$. Since $\varphi^\natural$ is non-self-dual we fix by Proposition 3.4.7 a winning strategy $\sigma : \omega^{<\omega} \setminus \{\emptyset\} \to \omega \cup \{\mathfrak{p}\}$ for Player **II** in the game $\mathrm{G}^{<}_{\mathrm{W}}(\psi, \varphi^\natural)$. We note that then for all $x \in \mathbb{R}$ we have that for cofinitely many $n \in \omega$, $(\mathrm{filter}(\overline{\sigma}(x))(n) \neq 0$. To see this we take towards a contradiction an $x \in \mathbb{R}$ denying this. Then by definition of $\varphi^\natural$ we have $\varphi^\natural(\mathrm{filter}(\overline{\sigma}(x))) = 0 \leq \psi(x)$, a contradiction.

Now based on σ we define for any $u \in \omega^{<\omega}$ a winning strategy $\sigma_u : \omega^{<\omega} \setminus \{\emptyset\} \to \omega \cup \{\mathfrak{p}\}$ for Player **II** in the game $\mathrm{G}^{<}_{\mathrm{W}}(\psi_{\lfloor u \rfloor}, \varphi^\natural)$ by setting for any $s \in \omega^{<\omega}$

$$\sigma_u(s) = \sigma(u ^\frown s).$$

This strategy is indeed winning for Player **II**, since for any $x \in \mathbb{R}$ we have that $\overline{\sigma_u}(x) = \overline{\sigma}(u ^\frown x)$ and so

$$\varphi_{\lfloor u \rfloor}(x) = \varphi(u ^\frown x) < \psi(\mathrm{filter}(\overline{\sigma}(u ^\frown x))) = \psi(\mathrm{filter}(\overline{\sigma_u}(x))).$$

Furthermore it is also clear that $\sigma = \sigma_\emptyset$.

Now we define a function $r : \omega^{<\omega} \to \omega$ by setting for any $s \in \omega^{<\omega}$

$$r(s) := \begin{cases} 0, & \text{if } \forall i < \mathrm{lh}(s)\ (s(i) \neq 0), \\ \max\{i < \mathrm{lh}(s) \mid s(i) = 0\} + 1, & \text{otherwise.} \end{cases}$$

Then we let $\preceq$ be the quasi-ordering on $\omega^{<\omega}$ defined by setting for any two $s, t \in \omega^{<\omega}$

$$s \preceq t \quad :\Leftrightarrow \quad t\restriction_{r(t)} \subseteq s\restriction_{r(s)}.$$

Now using the strategies σ_u for $u \in \omega^{<\omega}$ as fixed above, we define for any $u \in \omega^{<\omega}$ a quasi-ordering $\leq_u$ on $\omega^{<\omega}$ by setting for any two $s, t \in \omega^{<\omega}$

$$s \leq_u t \quad :\Leftrightarrow \quad \mathrm{filter}(\overline{\sigma_u}(s)) \preceq \mathrm{filter}(\overline{\sigma_u}(t)).$$

Next we note that for any $u \in \omega^{<\omega}$ the quasi-order $(\omega^{<\omega}, \leq_u)$ is well-founded. To see this, we assume towards a contradiction that $\leq_u$ is ill-founded. Then using $\mathbf{DC}$ we get an infinite strictly $\leq_u$-decreasing sequence $\langle s_i \mid i \in \omega \rangle \in (\omega^{<\omega})^\omega$. But then

$$\mathrm{filter}\left(\overline{\sigma_u}\left(\lim\left\{s_i \mid i \in \omega\right\}\right)\right) = \lim\left\{\mathrm{filter}(\overline{\sigma_u}(s_i)) \mid i \in \omega\right\}$$

and by definition of $\preceq$ clearly there are infinitely many $n \in \omega$ such that

$$\left(\lim\left\{\mathrm{filter}(\overline{\sigma_u}(s_i)) \mid i \in \omega\right\}\right)(n) = 0,$$

which by definition of σ_u implies that there is $x \in \mathbb{R}$ such that for infinitely many $n \in \omega$ we have that $\mathrm{filter}(\overline{\sigma}(x)) = 0$, which contradicts our choice of σ. Hence indeed $(\omega^{<\omega}, \leq_u)$ is well-founded.

Thus we can consider the order types $\|\emptyset\|_{\leq_u}$ for any $u \in \omega^{<\omega}$ and get that $\|\emptyset\|_{\leq_u} < \omega_1$, since $\omega^{<\omega}$ is a countable set. Furthermore we note that for any $s, u \in \omega^{<\omega}$ we have that the two quasi-orders $(\{t \in \omega^{<\omega} \mid s \subseteq t\}, \leq_u)$ and $(\omega^{<\omega}, \leq_{u^\frown s})$ are isomorphic as witnessed by the function

$$F : \{t \in \omega^{<\omega} \mid s \subseteq t\} \to \omega^{<\omega}, t \mapsto t \setminus s.$$

This function is clearly bijective. Also we see that it is order-preserving, since for any $t_1, t_2 \in \{t \in \omega^{<\omega} \mid s \subseteq t\}$ we have by definition of $\leq_u$ and $\leq_{u^\frown s}$ that

$$\begin{aligned} t_1 \leq_u t_2 \quad &\Leftrightarrow \quad \mathrm{filter}(\overline{\sigma_u}(t_1)) \preceq \mathrm{filter}(\overline{\sigma_u}(t_2)) \\ &\Leftrightarrow \quad \mathrm{filter}(\overline{\sigma}(u ^\frown t_1)) \preceq \mathrm{filter}(\overline{\sigma}(u ^\frown t_2)) \\ &\Leftrightarrow \quad \mathrm{filter}(\overline{\sigma}(u ^\frown s ^\frown (t_1 \setminus s))) \preceq \mathrm{filter}(\overline{\sigma}(u ^\frown s ^\frown (t_2 \setminus s))) \\ &\Leftrightarrow \quad \mathrm{filter}(\overline{\sigma_{u^\frown s}}(t_1 \setminus s)) \preceq \mathrm{filter}(\overline{\sigma_{u^\frown s}}(t_2 \setminus s)) \\ &\Leftrightarrow \quad t_1 \setminus s \leq_{u^\frown s} t_2 \setminus s \\ &\Leftrightarrow \quad F(t_1) \leq_{u^\frown s} F(t_2). \end{aligned}$$

Hence F is an isomorphism as claimed. In particular this implies that for any three $s, u, t \in \omega^{<\omega}$ such that $s \subseteq t$ we have that

$$\|t\|_{\leq_u} = \|t \setminus s\|_{\leq_{u^\frown s}}$$

Also we note that it follows from the definition of $\preceq$ that for any $u \in \omega^{<\omega}$ and any $t \in \omega^{<\omega}$ we have that $t \leq_u \emptyset$.

Now we prove simultaneously for all $u \in \omega^{<\omega}$ via induction on the ordinals $\alpha_u := \|\emptyset\|_{\leq_u}$ that

$$\left[\psi_{\lfloor u \rfloor}\right]_{\mathrm{W}} \leq_{\mathrm{W}} [\varphi]_{\mathrm{W}} \odot (\alpha_u + 1).$$

For the base case we assume that $\alpha_u = 0$. Then for all $s \in \omega^{<\omega}$ we have that $\emptyset \leq_u s$ and so by definition of $\preceq$ we have for all $s \in \omega^{<\omega}$ that $r(\mathrm{filter}(\overline{\sigma_u}(s))) = 0$ and so that in particular $\sigma_u(s) \neq 0$. But then we can give a winning strategy $\tau : \omega^{<\omega} \to \omega \cup \{\mathfrak{p}\}$ for Player **II** in the game $\mathrm{G}_{\mathrm{W}}^{\leq}(\psi_{\lfloor u \rfloor}, \varphi)$ by setting for all $s \in \omega^{<\omega}$

$$\tau(s) = \sigma_u(s) - 1.$$

This is indeed winning for Player **II**, since then we have for any $x \in \mathbb{R}$ that

$$\psi_{\lfloor u \rfloor}(x) \leq \varphi^{\natural}(\mathrm{filter}(\overline{\sigma_u}(x)) = \varphi(\mathrm{filter}(\overline{\sigma_u}(x) - 1) = \varphi(\mathrm{filter}(\overline{\tau}(x)).$$

Hence we get that $\left[\psi_{\langle u \rangle}\right]_{\mathrm{W}} \leq_{\mathrm{W}} [\varphi]_{\mathrm{W}} = [\varphi]_{\mathrm{W}} \odot 1$.

Now we assume that $\alpha_u > 0$ and take $\chi \in [\varphi]_{\mathrm{W}} \odot \alpha_u$. Then we have for any $s \in \omega^{<\omega}$ such that $\|s\|_{\leq_u} < \alpha$ that $\|\emptyset\|_{\leq_{u^\frown s}} = \alpha_{u^\frown s} < \alpha$ and so by induction hypothesis we have that $\varphi_{\lfloor u^\frown s \rfloor} \leq_{\mathrm{W}} \chi$. Thus for any such s we choose a winning strategy $\sigma'_{u,s} : \omega^{<\omega} \setminus \{\emptyset\} \to \omega \cup \{\mathfrak{p}\}$ for Player **II** in the game $\mathrm{G}_{\mathrm{W}}^{\leq}(\psi_{\lfloor u^\frown s \rfloor}, \chi)$. Using these strategies we define a winning strategy $\tau' : \omega^{<\omega} \setminus \{\emptyset\} \to \omega \cup \{\mathfrak{p}\}$ for Player **II** in the game $\mathrm{G}_{\mathrm{W}}^{\leq}(\psi_{\lfloor u \rfloor}, \chi \dotplus \psi)$ by setting for any $s \in \omega^{<\omega} \setminus \{\emptyset\}$

$$\tau'(s) := \begin{cases} \sigma_u(s), & \text{if } \|s\|_{\leq_u} = \alpha_u, \\ 0, & \text{if } \left\|s\restriction_{\mathrm{lh}(s)-1}\right\|_{\leq_u} = \alpha_u \text{ and } \|s\|_{\leq_u} < \alpha_u, \\ \sigma'_{u,t}(s \setminus t), & \text{otherwise,} \\ & \quad \text{where } t \text{ is the maximal initial segment of } s \text{ s.t. } \|t\|_{\leq_u} = \alpha_u. \end{cases}$$

This is indeed winning for Player **II**; if $x \in \mathbb{R}$ is such that $\|x\restriction_n\|_{\leq_u} = \alpha_u$ for all $n \in \omega$, then we have that $\overline{\tau'}(x) = \overline{\sigma_u}(x)$ and since furthermore $(\overline{\sigma_u}(x))(n) \neq 0$ for cofinitely many $n \in \omega$ we have by choice of σ_u that

$$\psi_{\lfloor u \rfloor}(x) \leq \varphi^{\natural}(\mathrm{filter}(\overline{\sigma_u}(x))) = \varphi((\mathrm{filter}(\overline{\sigma_u}(x)) - 1) = (\chi \dotplus \varphi)(\mathrm{filter}(\overline{\sigma_u}(x))) = (\chi \dotplus \varphi)(\mathrm{filter}(\overline{\tau'}(x))).$$

If, however, $x \in \mathbb{R}$ is such that there is $n \in \omega$ such that $\|x\restriction_n\|_{\leq_u} < \alpha$, then we let $m \in \omega$ be the least such and get that $\overline{\tau'}(x) = (\overline{\sigma_u}(x\restriction_m)) ^\frown \langle 0 \rangle ^\frown \left(\overline{\sigma'_{u,x\restriction_m}}(x \setminus (x\restriction_m))\right)$ and thus

$$\begin{aligned} \psi_{\lfloor u \rfloor}(x) &= \psi_{\lfloor u^\frown x\restriction_m \rfloor}(x \setminus (x\restriction_m)) \\ &\leq \chi\left(\mathrm{filter}\left(\overline{\sigma'_{u,x\restriction_m}}(x \setminus (x\restriction_m))\right)\right) \\ &= (\chi \dotplus \varphi)\left((\overline{\sigma_u}(x\restriction_m)) ^\frown \langle 0 \rangle ^\frown \left(\overline{\sigma'_{u,x\restriction_m}}(x \setminus (x\restriction_m))\right)\right) \\ &= (\chi \dotplus \varphi)(\mathrm{filter}(\overline{\tau'}(x))). \end{aligned}$$

In total this shows that indeed $\psi_{\lfloor u \rfloor} \leq_{\mathrm{W}} \chi \dotplus \varphi$ and so

$$\left[\psi_{\lfloor u \rfloor}\right]_{\mathrm{W}} \leq_{\mathrm{W}} [\chi]_{\mathrm{W}} \dotplus [\varphi]_{\mathrm{W}} = [\varphi]_{\mathrm{W}} \odot \alpha_u + [\varphi]_{\mathrm{W}} = [\varphi]_{\mathrm{W}} \odot (\alpha_u + 1).$$

This induction in particular has shown that for every regular norm ψ such that $\psi <_{\mathrm{W}} \varphi^{\natural}$ there is an ordinal $\alpha < \omega_1$ such that $[\psi]_{\mathrm{W}} \leq_{\mathrm{W}} [\varphi] \odot \alpha$ and so by Proposition 4.3.14 it follows that $\left|\varphi^{\natural}\right|_{\mathrm{W}} \leq |\varphi| \odot \omega_1$, concluding the proof. □

4.3.3 Closure Under Multiplication

Without further ado we will now define the multiplication operation $(\varphi, \psi) \mapsto \varphi \odot \psi$ on $\mathcal{N}$, for which we are going to show in this subsection that for any self-dual regular norm φ and any regular norm ψ such that $|\psi|_{\mathrm{W}}$ is a limit ordinal of uncountable cofinality we have that

$$|\varphi \odot \psi|_{\mathrm{W}} = |\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot |\psi|_{\mathrm{W}} .$$

Definition 4.3.22 For any two regular norms φ, ψ we define a regular norm $\varphi \odot \psi$ by setting for all $x \in \mathbb{R}$

$$(\varphi \odot \psi)(x) := \begin{cases} \varphi(y), & \text{if } \exists n \in \omega, \langle s_m \mid m < 2n \rangle \in (\omega^{<\omega})^{2n} \\ & \quad \big((s_0{+}1)^\frown\langle 0\rangle^\frown(s_1{+}1)^\frown\langle 0\rangle^\frown\ldots^\frown\langle 0\rangle^\frown(s_{2n}{+}1)^\frown\langle 0\rangle^\frown(y{+}1)\big) \\ \psi(s_0{}^\frown s_2{}^\frown \ldots {}^\frown s_{2n}{}^\frown y) & \text{if } \exists n \in \omega, \langle s_m \mid m < 2n+1 \rangle \in (\omega^{<\omega})^{2n+1} \\ & \quad \big((s_0{+}1)^\frown\langle 0\rangle^\frown(s_1{+}1)^\frown\langle 0\rangle^\frown\ldots^\frown\langle 0\rangle^\frown(s_{2n+1}{+}1)^\frown\langle 0\rangle^\frown(y{+}1)\big) \\ \psi(s_0{}^\frown s_2{}^\frown s_4{}^\frown \ldots), & \text{if } \exists \langle s_m \mid m \in \omega \rangle \in (\omega^{<\omega})^{\omega} \\ & \quad \big((s_0{+}1)^\frown\langle 0\rangle^\frown(s_1{+}1)^\frown\langle 0\rangle^\frown(s_2{+}1)^\frown\langle 0\rangle^\frown\ldots\big) \\ \psi(y), & \text{if } x = y+1, \\ 0, & \text{if } \exists n \in \omega\ (x(n) = x(n+1) = 0). \end{cases}$$

We call $x \in \omega^{\leq\omega}$ *product conform* iff $x = \emptyset$ or

$$x(0) \neq 0 \ \wedge\ (\mathrm{lh}(x) \leq 1 \ \vee\ \forall m < \mathrm{lh}(x) - 1\ (x(m) = 0 \Rightarrow x(m+1) \neq 0)) .$$

Product conform reals are indeed all we need, when working with norms of the form $\varphi \odot \psi$.

Lemma 4.3.23. *Let φ, ψ, χ be regular norms. If Player* **II** *wins the game $\mathrm{G}^{\leq}_{\mathrm{L}}(\varphi, \psi \odot \chi)$, then there is a winning strategy $\sigma : \omega^{<\omega} \setminus \{\emptyset\} \to \omega \cup \{\mathfrak{p}\}$ for Player* **II** *in the game $\mathrm{G}^{\leq}_{\mathrm{L}}(\varphi, \psi \odot \chi)$ such that for all $x \in \mathbb{R}$ we have that $\overline{\sigma}(x)$ is product conform.*

The analogous result holds for games of the form $\mathrm{G}^{<}_{\mathrm{L}}(\varphi, \psi \odot \chi)$, $\mathrm{G}^{\leq}_{\mathrm{W}}(\varphi, \psi \odot \chi)$ and $\mathrm{G}^{<}_{\mathrm{W}}(\varphi, \psi \odot \chi)$.

Proof. We consider a winning strategy $\sigma : \omega^{<\omega} \setminus \{\emptyset\} \to \omega$ for Player **II** in the game $\mathrm{G}^{\leq}_{\mathrm{L}}(\varphi, \psi \odot \chi)$. Now we define another strategy $\sigma' : \omega^{<\omega} \setminus \{\emptyset\} \to \omega$ for Player **II** in the game $\mathrm{G}^{\leq}_{\mathrm{L}}(\varphi, \psi \odot \chi)$ by setting for all $s \in \omega^{<\omega} \setminus \{\emptyset\}$

$$\sigma'(s) := \begin{cases} 1, & \text{if } \sigma(s) = 0 \text{ and either } \mathrm{lh}(s) = 1 \text{ or } \sigma(s\restriction_{\mathrm{lh}(s)-1}) = 0, \\ \sigma(s), & \text{otherwise.} \end{cases}$$

Then by construction for any $x \in \mathbb{R}$ such that $\overline{\sigma}(x)$ is product conform we have that $\overline{\sigma}(x) = \overline{\sigma'}(x)$ and thus

$$\varphi(x) \leq (\psi \odot \chi)(\mathrm{filter}(\overline{\sigma}(x))) = (\psi \odot \chi)(\mathrm{filter}(\overline{\sigma'}(x))).$$

If, however $x \in \mathbb{R}$ is such that $\overline{\sigma}(x)$ is not product conform, then $\overline{\sigma'}(x)$ is still product conform by construction, but we have that

$$\varphi(x) \leq (\psi \odot \chi)(\mathrm{filter}(\overline{\sigma}(x))) = 0$$

and so $\varphi(x) = 0$, which vacuously implies that $\varphi(x) \leq (\psi \odot \chi)(\mathrm{filter}(\overline{\sigma'}(x)))$. But this shows that σ' is a winning strategy for Player **II** in the game $\mathrm{G}^{\leq}_{\mathrm{L}}(\varphi, \psi \odot \chi)$. Analogous constructions work for the other games listed in the claim. □

In light of this lemma, whenever we take an arbitrary winning strategy σ for Player **II** in a game of the form $\mathrm{G}^{\leq}_{\mathrm{W}}(\varphi, \psi \odot \chi)$, $\mathrm{G}^{<}_{\mathrm{W}}(\varphi, \psi \odot \chi)$, $\mathrm{G}^{\leq}_{\mathrm{L}}(\varphi, \psi \odot \chi)$ and $\mathrm{G}^{<}_{\mathrm{L}}(\varphi, \psi \odot \chi)$, we can always without loss of generality assume that for all $s \in \omega^{\leq\omega}$ we have that $\overline{\sigma}(s)$ is product conform.

The idea behind the definition of the multiplication is the following. For regular norms φ, ψ the regular norms $\varphi \odot \psi$ reads a product conform real x by successively going through the natural numbers $x(n)$ for $n \in \omega$ by filling out two rows at the side in the following way. In the beginning, as long a no 0 is encountered row 1 is filled with the read natural numbers minus 1. However, an encountered 0 means a row change and so a change to row 2 begins, where newly encountered natural numbers minus 1 are filled in until another 0 is encountered, which results in a further row change back to row 1 and so on. Thus after reading the whole real x, assuming that it was product conform, we have a distribution of natural numbers in two rows. If there are infinitely many natural numbers filled in in row 1, then we assemble them into a single real z and $\varphi \odot \psi$ outputs $\psi(z)$. If, however, there are only finitely many natural numbers filled in in row 1, then we consider the real y that was assembled in row 2 from the point on, where the last change into row 2 occurred (i.e. we forget everything filled into row 2 beforehand) and $\varphi \odot \psi$ outputs $\varphi(y)$.

Now using this idea we define the following auxiliary functions to simplify the presentation of $\varphi \odot \psi$ considerably.

Definition 4.3.24 We define a function $\mathrm{NCol} : (\omega \cup \{\mathfrak{p}\})^{\leq\omega} \to (\omega+1) \cup \{\mathfrak{nd}\}$ by setting for all $x \in \omega^{\leq\omega}$

$$\mathrm{NCol}(x) := \begin{cases} \mathrm{card}(\{n \in \omega \mid x(n) = 0\}), \\ \quad \text{if } \mathrm{filter}(x) \text{ is product conform and } \mathrm{card}(\{n \in \omega \mid x(n) = 0\}) = \omega, \\ \mathrm{card}(\{n \in \omega \mid x(n) = 0\}) + 1, \\ \quad \text{if } \mathrm{filter}(x) \text{ is product conform and } \mathrm{card}(\{n \in \omega \mid x(n) = 0\}) \in \omega, \\ \mathfrak{nd}, \\ \quad \text{if } \mathrm{filter}(x) \text{ is not product conform.} \end{cases}$$

We define a function $\mathrm{cut} : (\omega \cup \{\mathfrak{p}\})^{\leq\omega} \to ((\omega \cup \{\mathfrak{p}\})^{\leq\omega})^{\leq\omega} \cup \{\mathfrak{nd}\}$ as follows. For any $x \in (\omega \cup \{\mathfrak{p}\})^{\leq\omega}$ such that $\mathrm{filter}(x)$ is not product conform we set $\mathrm{cut}(x) = \mathfrak{nd}$. For any $x \in (\omega \cup \{\mathfrak{p}\})^{\leq\omega}$ such that $\mathrm{filter}(x)$ is product conform, however, we define $\mathrm{cut}(x)_n$ for $n < \mathrm{NCol}(x)$ recursively by the following terms

- $\mathrm{cut}(x)_0$ is the unique maximal $y \in (\omega \cup \{\mathfrak{p}\})^{\leq\omega}$ such that $(y+1) \subseteq x$.
- $\mathrm{cut}(x)_{n+1}$ is the unique maximal $y \in (\omega \cup \{\mathfrak{p}\})^{\leq\omega}$ such that we have

$$(\mathrm{cut}(x)_0 + 1) ^\frown \langle 0 \rangle ^\frown (\mathrm{cut}(x)_1 + 1) ^\frown \langle 0 \rangle ^\frown \ldots ^\frown \langle 0 \rangle ^\frown (\mathrm{cut}(x)_n + 1) ^\frown \langle 0 \rangle ^\frown (y+1) \subseteq x.$$

We define a function $\mathrm{rowchoice} : (\omega \cup \{\mathfrak{p}\})^{\leq\omega} \to \{1, 2, \mathfrak{nd}\}$ by setting for all $x \in (\omega \cup \{\mathfrak{p}\})^{\leq\omega}$

$$\mathrm{rowchoice}(x) := \begin{cases} 1, & \text{if } \mathrm{NCol}(x) = \omega, \\ 1, & \text{if } \mathrm{NCol}(x) \in \omega \text{ is odd}, \\ 2, & \text{if } \mathrm{NCol}(x) \in \omega \text{ is even}, \\ \mathfrak{nd}, & \text{if } \mathrm{NCol}(x) = \mathfrak{nd}. \end{cases}$$

Then we define $\mathrm{Row}_1 : (\omega \cup \{\mathfrak{p}\})^{\leq\omega} \to (\omega \cup \{\mathfrak{p}\})^{\leq\omega} \cup \{\mathfrak{nd}\}$ by setting for all $x \in (\omega \cup \{\mathfrak{p}\})^{\leq\omega}$

$$\mathrm{Row}_1(x) := \begin{cases} \mathrm{cut}(x)_0 ^\frown \mathrm{cut}(x)_2 ^\frown \ldots ^\frown \mathrm{cut}(x)_{\lfloor \frac{\mathrm{NCol}(x)}{2} \rfloor}, & \text{if } \mathrm{NCol}(x) \in \omega, \\ \mathrm{cut}(x)_0 ^\frown \mathrm{cut}(x)_2 ^\frown \ldots, & \text{if } \mathrm{NCol}(x) = \omega, \\ \mathfrak{nd}, & \text{if } \mathrm{NCol}(x) = \mathfrak{nd}. \end{cases}$$

Also we define $\mathrm{Row}_2 : (\omega \cup \{\mathfrak{p}\})^{\leq\omega} \to (\omega \cup \{\mathfrak{p}\})^{\leq\omega} \cup \{\mathfrak{nd}\}$ by setting for all $x \in (\omega \cup \{\mathfrak{p}\})^{\leq\omega}$

$$\mathrm{Row}_2(x) := \begin{cases} \mathrm{cut}(x)_{\mathrm{NCol}-1}, & \text{if } \mathrm{rowchoice}(x) = 2, \\ \mathfrak{nd}, & \text{otherwise.} \end{cases}$$

Now using this we can write down the definition of $\varphi \odot \psi$ for two regular norms φ, ψ slightly more concisely in the following way for any $x \in \mathbb{R}$:

$$(\varphi \odot \psi)(x) := \begin{cases} \psi(\mathrm{Row}_1(x)), & \text{if } \mathrm{rowchoice}(x) = 1, \\ \varphi(\mathrm{Row}_2(x)), & \text{if } \mathrm{rowchoice}(x) = 2, \\ 0, & \text{if } \mathrm{rowchoice}(x) = \mathfrak{nd}. \end{cases}$$

This will be useful to establish results regarding this multiplication operation. We will also use the following facts which are immediate from the above definitions:

- For all $x \in \mathbb{R}$ we have that $\text{rowchoice}(x) = 1$ if and only if there are infinitely many $n \in \omega$ such that $\text{rowchoice}(x\restriction_n) = 1$.
- For all $x \in \mathbb{R}$ we have that $\text{rowchoice}(x) = 2$ if and only if there are cofinitely many $n \in \omega$ such that $\text{rowchoice}(x\restriction_n) = 2$.

Lemma 4.3.25. *If $\varphi_0, \varphi_1, \psi_0, \psi_1$ are regular norms such that $\varphi_0 \leq_{\mathrm{W}} \psi_0$ and $\varphi_1 \leq_{\mathrm{L}} \psi_1$, then $\varphi_0 \odot \varphi_1 \leq_{\mathrm{W}} \psi_0 \odot \psi_1$.*

Proof. We take winning strategies $\sigma_0 : \omega^{<\omega} \setminus \{\emptyset\} \to \omega \cup \{\mathfrak{p}\}$ and $\sigma_1 : \omega^{<\omega} \setminus \{\emptyset\} \to \omega$ for Player **II** in the game $\mathrm{G}^{\leq}_{\mathrm{W}}(\varphi_0, \psi_0)$ and $\mathrm{G}^{\leq}_{\mathrm{L}}(\varphi_1, \psi_1)$, respectively.

Now we define a monotone infinitary function $g : \omega^{<\omega} \to \omega^{<\omega}$ such that for all product conform $s \in \omega^{<\omega}$ the following four properties are satisfied:

1. $\text{NCol}(s) = \text{NCol}(g(s))$;
2. $\text{rowchoice}(s) = \text{rowchoice}(g(s))$;
3. $\text{Row}_1(g(s)) = \overline{\sigma_1}(\text{Row}_1(s))$;
4. If $\text{rowchoice}(s) = 2$, then $\text{Row}_2(g(s)) = \text{filter}(\overline{\sigma_0}(\text{Row}_2(s)))$.

We define $g(s)$ by recursion on the length of $s \in \omega^{<\omega}$. First we set $g(\emptyset) = \emptyset$. Now we assume that $g(s)$ is already defined for some fixed $s \in \omega^{<\omega}$. Then we let $s' = s^\frown\langle n\rangle$ for some $n \in \omega$ and define $g(s')$ by distinguishing several cases. First, if s' is not product conform, then we simply set $g(s') := g(s)^\frown\langle 0\rangle$. Therefore from now on we assume that s' is product conform and distinguish cases as follows.

Case 1 is that $\text{rowchoice}(s) = 1$ and so $\text{rowchoice}(g(s)) = 1$.

Subcase 1.1 is that $n = 0$ and so $\text{rowchoice}(s') = 2$ and $\text{NCol}(s') = \text{NCol}(s) + 1$. Then we set $g(s') := g(s) ^\frown \langle 0\rangle$, which by induction hypothesis directly implies that

$$\text{NCol}(s') = \text{NCol}(s) + 1 = \text{NCol}(g(s)) + 1 = \text{NCol}(s').$$

By assumption we have that $\text{lh}(s) \geq 1$ and $s(\text{lh}(s) - 1) \neq 0$, since otherwise s' were not product conform. Therefore, since by induction hypothesis $\text{Row}_1(g(s)) = \overline{\sigma_1}(\text{Row}_1(s))$ and since $\overline{\sigma_1}\restriction_{\omega^{<\omega}}$ is strictly infinitary, we have that $\text{lh}(g(s)) \geq 1$ and $g(s)(\text{lh}(g(s)) - 1) = \sigma_1(\text{Row}_1(s)) + 1 > 0$. Hence $g(s')$ is product conform and therefore $\text{rowchoice}(g(s)) = 1$ implies that $\text{rowchoice}(g(s')) = 2$ and so $\text{rowchoice}(s') = \text{rowchoice}(g(s'))$. Next we note that $\text{Row}_1(g(s')) = \text{Row}_1(g(s))$ and $\text{Row}_1(s') = \text{Row}_1(s)$ and so by induction hypothesis $\text{Row}_1(g(s)) = \overline{\sigma_1}(\text{Row}_1(s))$. Finally we note that $\text{Row}_2(g(s')) = \emptyset = \text{Row}_2(s')$ and thus vacuously $\text{Row}_2(g(s')) = \text{filter}(\overline{\sigma_0}(\text{Row}_2(s')))$.

Subcase 1.2 is that $n \geq 1$ and so $\text{rowchoice}(s') = \text{rowchoice}(s) = 1$ and $\text{NCol}(s') = \text{NCol}(s)$. Then we set $g(s') := g(s)^\frown\langle\overline{\sigma_1}(\text{Row}_1(s')) + 1\rangle$. We have $\text{NCol}(g(s')) = \text{NCol}(g(s)$ and so by induction hypothesis $\text{NCol}(g(s')) = \text{NCol}(s')$. Also we have that $\text{rowchoice}(g(s')) = \text{rowchoice}(g(s)) = 1$ and so $\text{rowchoice}(s') = \text{rowchoice}(g(s'))$. By induction hypothesis we have that $\text{Row}_1(g(s)) = \overline{\sigma_1}(\text{Row}_1(s))$ and thus

$$\text{Row}_1(g(s')) = \text{Row}_1(g(s)) ^\frown \langle\sigma_1(\text{Row}_1(s'))\rangle = \overline{\sigma_1}(\text{Row}_1(s)) ^\frown \langle\sigma_1(\text{Row}_1(s'))\rangle = \overline{\sigma_1}(\text{Row}_1(s')).$$

Case 2 is that $\text{rowchoice}(s) = 2$ and so $\text{rowchoice}(g(s)) = 2$.

Subcase 2.1 is that $n = 0$ and so $\text{rowchoice}(s') = 1$ and $\text{NCol}(s') = \text{NCol}(s) + 1$. Then we set $g(s') := g(s) ^\frown \langle 1, 0\rangle$, which by induction hypothesis directly implies that

$$\text{NCol}(s') = \text{NCol}(s) + 1 = \text{NCol}(g(s)) + 1 = \text{NCol}(s').$$

Also since $\text{rowchoice}(g(s)) = 2$ and by construction $g(s')(\text{lh}(g(s')) - 2) \neq 0$, we then get that $g(s')$ is product conform and so $\text{rowchoice}(g(s')) = 1$ and thus $\text{rowchoice}(g(s')) = \text{rowchoice}(s')$. Finally we note that $\text{Row}_1(g(s')) = \text{Row}_1(g(s))$ and $\text{Row}_1(s') = \text{Row}_1(s)$ and so by induction hypothesis $\text{Row}_1(g(s)) = \overline{\sigma_1}(\text{Row}_1(s))$.

Subcase 2.2 is that $n \geq 1$ and so $\text{rowchoice}(s') = \text{rowchoice}(s) = 2$ and $\text{NCol}(s') = \text{NCol}(s)$. Then we set $g(s') := \text{filter}(g(s)^\frown\langle\sigma_0(\text{Row}_2(s')) + 1\rangle)$ and get that $\text{rowchoice}(g(s')) = \text{rowchoice}(g(s)) = 2$

and so $\mathrm{rowchoice}(g(s')) = \mathrm{rowchoice}(s')$. Clearly we also have that $\mathrm{NCol}(g(s')) = \mathrm{NCol}(g(s))$ and so by induction hypothesis $\mathrm{NCol}(g(s')) = \mathrm{NCol}(s')$. Also by construction and the induction hypothesis we have that

$$\begin{aligned}\mathrm{Row}_2(g(s')) &= \mathrm{filter}(\mathrm{Row}_2(g(s)) ^\frown \langle \sigma_0(\mathrm{Row}_2(s'))\rangle) \\ &= \mathrm{filter}(\overline{\sigma_0}(\mathrm{Row}_2(s)) ^\frown \langle \sigma_0(\mathrm{Row}_2(s'))\rangle) \\ &= \mathrm{filter}(\overline{\sigma_0}(\mathrm{Row}_2(s'))).\end{aligned}$$

Finally it is immediate that $\mathrm{Row}_1(g(s')) = \mathrm{Row}_1(g(s))$ and $\mathrm{Row}_1(s') = \mathrm{Row}_1(s)$. So by induction hypothesis we have that $\mathrm{Row}_1(g(s')) = \mathrm{Row}_1(s')$.

This finishes the construction of $g : \omega^{<\omega} \to \omega^{<\omega}$. We have verified at every step that the four properties listed above are satisfied. Furthermore by the recursive construction g is monotone. To see that g is infinitary we assume towards a contradiction that g is not infinitary, i.e., that there is $x \in \mathbb{R}$ and $n \in \omega$ such that for all $m \in \omega$ we have that $\mathrm{lh}(g(x\restriction_m)) \leq n$. But considering the above recursive definition of g we note that this can only be the case, if for cofinitely many $m \in \omega$ we have that $\sigma_1(x\restriction_m) = \mathfrak{p}$. But then there are only finitely many $m \in \omega$ such that $(\overline{\sigma_1}(x))(m) \in \omega$, contradicting the fact that σ_1 is a winning strategy for Player **II** in a Wadge game. Thus indeed g is infinitary.

Now since g is monotone and infinitary, g induces a continuous function $G : \mathbb{R} \to \mathbb{R}$. We claim that G witnesses $\varphi_0 \odot \psi_0 \leq_{\mathrm{W}} \varphi_1 \odot \psi_1$, i.e., for all $x \in \mathbb{R}$ we have that $(\varphi_0 \odot \varphi_1)(x) \leq (\psi_0 \odot \psi_1)(G(x))$. To prove this we distinguish different cases for $x \in \mathbb{R}$.

If $x \in \mathbb{R}$ is such that $\mathrm{rowchoice}(x) = 1$, then for infinitely many $m \in \omega$ we have that $\mathrm{rowchoice}(x\restriction_m) = 1$ and so by Property 2 of the function g we have infinitely many $m \in \omega$ such that $\mathrm{rowchoice}(g(x\restriction_m)) = 1$ and so there are infinitely many $m \in \omega$ such that $\mathrm{rowchoice}(G(x)\restriction_m) = 1$, which in turn implies that $\mathrm{rowchoice}(G(x)) = 1$. But now we note that by Property 3 of the function g for all $m \in \omega$ we have that $\mathrm{Row}_1(g(x\restriction_m)) = \overline{\sigma_1}(\mathrm{Row}_1(x\restriction_m))$ and so $\mathrm{Row}_1(G(x)) = \overline{\sigma_1}(\mathrm{Row}_1(x))$. But since σ_1 is winning for Player **II** in the game $\mathrm{G}^{\leq}_{\mathrm{L}}(\varphi_1, \psi_1)$ we finally get that

$$(\varphi_0 \odot \varphi_1)(x) = \varphi_1(\mathrm{Row}_1(x)) \leq \psi_1(\overline{\sigma_1}(\mathrm{Row}_1(x))) = \psi_1(\mathrm{Row}_1(G(x))) = (\psi_0 \odot \psi_1)(G(x)).$$

If, however, $x \in \mathbb{R}$ is such that $\mathrm{rowchoice}(x) = 2$, then for cofinitely many $m \in \omega$ we have that $\mathrm{rowchoice}(x\restriction_m) = 2$ and furthermore $\mathrm{NCol}(x) \in \omega$. By Property 1 of the function g we have for all $m \in \omega$ that $\mathrm{NCol}(x\restriction_m) = \mathrm{NCol}(g(x\restriction_m)$. But then clearly also $\mathrm{NCol}(x) = \mathrm{NCol}(G(x))$. Then since $\mathrm{NCol}(x)$ is odd, since $\mathrm{rowchoice}(x) = 2$, we also have that $\mathrm{NCol}(G(x))$ is odd and so $\mathrm{rowchoice}(G(x)) = 2$. But now we note that by Property 4 of the function g we have for cofinitely many $m \in \omega$ that $\mathrm{Row}_2(g(x\restriction_m)) = \mathrm{filter}(\overline{\sigma_0}(\mathrm{Row}_2(x\restriction_m)))$ and so $\mathrm{Row}_2(G(x)) = \mathrm{filter}(\overline{\sigma_0}(\mathrm{Row}_2(x)))$. But since σ_0 is winning for Player **II** in the game $\mathrm{G}^{\leq}_{\mathrm{W}}(\varphi_0, \psi_0)$ we finally get that

$$(\varphi_0 \odot \varphi_1)(x) = \varphi_0(\mathrm{Row}_2(x)) \leq \psi_0(\mathrm{filter}(\overline{\sigma_0}(\mathrm{Row}_2(x)))) = \psi_0(\mathrm{Row}_2(G(x))) = (\psi_0 \odot \psi_1)(G(x)).$$

Finally if $x \in \mathbb{R}$ is such that $\mathrm{rowchoice}(x) = \mathfrak{nd}$, then x is not product conform and so $(\varphi_0 \odot \varphi_1)(x) = 0$, which implies that vacuously we have

$$(\varphi_0 \odot \varphi_1)(x) \leq (\psi_0 \odot \psi_1)(G(x)).$$

In total this shows that $\varphi_0 \odot \varphi_1 \leq_{\mathrm{W}} \psi_0 \odot \psi_1$ as claimed. □

This result implies that for any regular norms φ and any non-self-dual regular norm ψ we can define the $\odot$-operation on Wadge degrees by setting $[\varphi]_{\mathrm{W}} \odot [\psi]_{\mathrm{W}} = [\varphi \odot \psi]_{\mathrm{W}}$. The reason for this is that for a non-self-dual regular norm ψ we have that $[\psi]_{\mathrm{W}} = [\psi]_{\mathrm{L}}$ by Corollary 3.4.8.

Lemma 4.3.26. *Assume* **AD**. *If φ, ψ and χ are regular norms such that φ is self-dual, then we have that*

$$\varphi \odot \psi \leq_{\mathrm{W}} \varphi \odot \chi \;\Rightarrow\; \psi \leq_{\mathrm{W}} \chi$$

and

$$\varphi \odot \psi \leq_{\mathrm{L}} \varphi \odot \chi \;\Rightarrow\; \psi \leq_{\mathrm{L}} \chi$$

Proof. We only show that from $\varphi \odot \psi \leq_{\mathrm{W}} \varphi \odot \chi$ it follows that $\psi \leq_{\mathrm{W}} \chi$. The argument for $\leq_{\mathrm{L}}$ runs analogously. We take a winning strategy $\tau : \omega^{<\omega} \setminus \{\emptyset\} \to \omega \cup \{\mathfrak{p}\}$ for Player **II** in the game $\mathrm{G}^{\leq}_{\mathrm{W}}(\varphi \odot \psi, \varphi \odot \chi)$. Without loss of generality we assume that for all $x \in \mathbb{R}$ the real $\mathrm{filter}(\overline{\tau}(x))$ is product conform. Furthermore for any $t \in \partial\mathbf{T}(\varphi)$ we choose – using $\mathbf{AD}$ and $\mathbf{AC}_\omega(\mathbb{R})$ – a winning strategy $\sigma_t : (\omega \cup \{\mathfrak{p}\})^{<\omega} \to \omega$ for Player **I** in the game $\mathrm{G}^{\leq}_{\mathrm{W}}(\varphi, \varphi_{\lfloor t \rfloor})$.

Next we recursively define a monotone and infinitary function $f : \omega^{<\omega} \to \omega^{<\omega}$ satisfying the following two properties for any $s \in \omega^{<\omega}$:

1. $\mathrm{rowchoice}(f(s)) = 1$ and $\mathrm{Row}_1(f(s)) = s$;
2. if $\mathrm{rowchoice}(\mathrm{filter}(\overline{\tau}(f(s)))) = 2$, then $\mathrm{Row}_2(\mathrm{filter}(\overline{\tau}(f(s)))) \in \mathbf{T}(\varphi)$.

First we set $f(\emptyset) = \emptyset$. Now we assume that $f(s)$ was already defined for a given $s \in \omega^{<\omega}$, let $n \in \omega$ be arbitrary and set $s' := s ^\frown \langle n \rangle$. To simplify our notation we then furthermore set $f'(s') := f(s) ^\frown \langle n+1 \rangle$. Now to define $f(s')$ we distinguish three cases.

Case 1 is that $\mathrm{rowchoice}(\mathrm{filter}(\overline{\tau}(f'(s')))) = 1$. Then we set $f(s') := f'(s')$. Then, since by induction hypothesis we have $\mathrm{rowchoice}(f(s)) = 1$ and $\mathrm{Row}_1(f(s)) = s$, we clearly have that $\mathrm{rowchoice}(f(s')) = 1$ and $\mathrm{Row}_1(f(s')) = \mathrm{Row}_1(f(s)) ^\frown \langle n \rangle = s'$.

Case 2 is that $\mathrm{rowchoice}(\mathrm{filter}(\overline{\tau}(f'(s')))) = 2$ and $\mathrm{Row}_2(\mathrm{filter}(\overline{\tau}(f'(s')))) \in \mathbf{T}(\varphi)$. Then again we set $f(s') := f'(s')$ and get as in Case 1 that $\mathrm{rowchoice}(f(s')) = 1$ and

$$\mathrm{Row}_1(f(s')) = \mathrm{Row}_1(f(s)) ^\frown \langle n \rangle = s'.$$

But furthermore we get that $\mathrm{rowchoice}(\mathrm{filter}(\overline{\tau}(f(s')))) = 2$ and $\mathrm{Row}_2(\mathrm{filter}(\overline{\tau}(f(s')))) \in \mathbf{T}(\varphi)$.

Case 3 is that $\mathrm{rowchoice}(\mathrm{filter}(\overline{\tau}(f'(s')))) = 2$ and $\mathrm{Row}_2(\mathrm{filter}(\overline{\tau}(f'(s')))) \notin \mathbf{T}(\varphi)$. But then clearly $\mathrm{rowchoice}(\mathrm{filter}(\overline{\tau}(f(s)))) = 2$, since otherwise $\mathrm{Row}_2(\mathrm{filter}(\overline{\tau}(f'(s')))) = \emptyset \notin \mathbf{T}(\varphi)$, a contradiction. Thus by induction hypothesis $\mathrm{Row}_2(\mathrm{filter}(\overline{\tau}(f(s)))) \in \mathbf{T}(\varphi)$ and so $\mathrm{Row}_2(\mathrm{filter}(\overline{\tau}(f'(s')))) \in \partial\mathbf{T}(\varphi)$. To simplify our notation we now abbreviate $\hat{t} := \mathrm{Row}_2(\overline{\tau}(f'(s')))$ and $t := \mathrm{filter}(t)$. But since $t \in \partial\mathbf{T}(\varphi)$ we can take the winning strategy σ_t for Player **I** in the game $\mathrm{G}^{\leq}_{\mathrm{W}}(\varphi, \varphi_{\lfloor t \rfloor})$.

Our idea is now to let Player **I** play according to the strategy σ_t to force Player **II** – when reacting with the strategy τ – to change back to the first row in the game $\mathrm{G}^{\leq}_{\mathrm{W}}(\varphi \odot \psi, \varphi \odot \chi)$. To implement this we recursively define matches p_m in the game $\mathrm{G}^{\leq}_{\mathrm{W}}(\varphi \odot \psi, \varphi \odot \chi)$ for any $m \in \omega$ by setting:

$$\begin{aligned}
(p_0)_{\mathbf{I}} &:= f'(s') ^\frown \langle 0 \rangle, \\
(p_{m+1})_{\mathbf{I}} &:= \begin{cases} (p_m)_{\mathbf{I}} ^\frown \left\langle \sigma_t(\mathrm{Row}_2((p_m)_{\mathbf{II}}) \setminus \hat{t}) + 1 \right\rangle, & \text{if } \forall k < m \ (\mathrm{rowchoice}((p_k)_{\mathbf{II}}) = 2), \\ \emptyset, & \text{otherwise,} \end{cases} \\
(p_m)_{\mathbf{II}} &:= \overline{\tau}((p_m)_{\mathbf{I}}).
\end{aligned}$$

Now we assume towards a contradiction that $\mathrm{rowchoice}((p_m)_{\mathbf{II}}) = 2$ for all $m \in \omega$. Then by construction we have for all $m \in \omega$ that $(p_m)_{\mathbf{I}} \subseteq (p_{m+1})_{\mathbf{I}}$ and $(p_m)_{\mathbf{II}} \subseteq (p_{m+1})_{\mathbf{II}}$ and $\mathrm{rowchoice}((p_m)_{\mathbf{I}}) = 2$ and $(p_m)_{\mathbf{II}} = \overline{\tau}((p_m)_{\mathbf{I}})$ and $\mathrm{Row}_2((p_{m+1})_{\mathbf{I}}) = \overline{\sigma_t}(\mathrm{Row}_2((p_m)_{\mathbf{II}}) \setminus \hat{t})$. Then we define $x \in \mathbb{R}$ and $y \in (\omega \cup \{\mathfrak{p}\})^\omega$ by setting $x := \lim\{(p_m)_{\mathbf{I}} \mid m \in \omega\}$ and $y := \lim\{(p_m)_{\mathbf{II}} \mid m \in \omega\}$ and by the properties just noted we get that $\mathrm{rowchoice}(x) = \mathrm{rowchoice}(y) = 2$, that $y = \overline{\tau}(x)$ and that $\mathrm{Row}_2(x) = \overline{\sigma_t}(\mathrm{Row}_2(y \setminus \hat{t}))$.

Since τ is a winning strategy for Player **II** in the game $\mathrm{G}^{\leq}_{\mathrm{W}}(\varphi \odot \psi, \varphi \odot \chi)$ we furthermore get that $\mathrm{filter}(y) \in \mathbb{R}$ and using this and the fact that σ_t is a winning strategy for Player **I** in the game $\mathrm{G}^{\leq}_{\mathrm{W}}(\varphi, \varphi_{\lfloor t \rfloor})$ we can now calculate that

$$\begin{aligned}
\varphi \odot \chi(\mathrm{filter}(y)) &= \varphi(\mathrm{filter}(\mathrm{Row}_2(y))) \\
&= \varphi(t ^\frown \mathrm{filter}(\mathrm{Row}_2(y) \setminus \hat{t})) \\
&= \varphi_{\lfloor t \rfloor}(\mathrm{filter}(\mathrm{Row}_2(y) \setminus \hat{t})) \\
&< \varphi(\overline{\sigma_t}(\mathrm{Row}_2(y) \setminus \hat{t}) \\
&= \varphi(\mathrm{Row}_2(x)) \\
&= \varphi \odot \psi(x) \\
&\leq \varphi \odot \chi(\mathrm{filter}(\overline{\tau}(x))) \\
&= \varphi \odot \chi(\mathrm{filter}(y))
\end{aligned}$$

and so $\varphi \odot \chi(\text{filter}(y)) < \varphi \odot \chi(\text{filter}(y))$, which is a contradiction. So we let $m > 0$ be the least non-zero natural number such that $\text{rowchoice}((p_m)_{\mathbf{II}}) = 1$. First we note that $(p_m)_{\mathbf{II}}$ is product conform, since $(p_m)_{\mathbf{II}} = \overline{\tau}((p_m)_{\mathbf{II}})$.

Now we set $f(s') := (p_m)_{\mathbf{I}} {}^\frown \langle 0 \rangle$. Then we have that $\text{rowchoice}(f(s')) = 1$ and

$$\text{Row}_1(f(s')) = \text{Row}_1(f'(s')) = s',$$

since by construction there is some $u \in \omega^{<\omega} \setminus \{\emptyset\}$ such that $f(s') = f'(s') {}^\frown \langle 0 \rangle {}^\frown (u+1) {}^\frown \langle 0 \rangle$. Furthermore also directly by construction we have that $\text{rowchoice}(\overline{\tau}(f(s'))) = 1$. This concludes the construction of f.

It is clear from our recursive construction that f is monotone and infinitary. Now we claim that for any $x \in \mathbb{R}$ and any $m \in \omega$ there is $k > m$ such that $\text{rowchoice}(\text{filter}(\overline{\tau}(f(x\restriction_k))) = 1$. To see this we assume towards a contradiction that this is not the case and take $x \in \mathbb{R}$ and $m \in \omega$ such that for all $k > m$ we have that $\text{rowchoice}(\text{filter}(\overline{\tau}(f(x\restriction_k))) = 2$ and so $\text{Row}_2(\text{filter}(\overline{\tau}(f(x\restriction_k))) \in \mathbf{T}(\varphi)$. But then either $\text{filter}(\overline{\tau}(f(x)) \in \omega^{<\omega}$ or $\mathbf{T}(\varphi)$ is ill-founded. The former contradicts the fact that τ is a winning strategy for Player **II** in a Wadge game and the latter contradicts the fact that φ is self-dual and therefore by Corollary 3.5.6 the tree $\mathbf{T}(\varphi)$ is well-founded.

But then considering the continuous function $F : \mathbb{R} \to \mathbb{R}$ induced by f we get for all $x \in \mathbb{R}$ that $\text{rowchoice}(F(x)) = 1$, $\text{Row}_1(F(x)) = x$ and $\text{rowchoice}(\text{filter}(\overline{\tau}(F(x)))) = 1$ and so

$$\psi(x) = \varphi \odot \psi(F(x)) \leq \varphi \odot \chi(\text{filter}(\overline{\tau}(F(x)))) = \chi(\text{Row}_1(\text{filter}(\overline{\tau}(F(x)))).$$

But now we note that the function $H : \mathbb{R} \to \mathbb{R}$ defined as $H := \text{Row}_2 \circ (\text{filter} \circ \overline{\tau}) \circ F$ is continuous and witnesses $\psi \leq_{\text{W}} \chi$, thus concluding the proof. □

Lemma 4.3.27. *Assume* **AD**. *Let φ be a self-dual regular norm and ψ an arbitrary regular norm. Then $\varphi \odot \psi$ is self-dual if and only if ψ is self-dual.*

Proof. For the left-to-right direction we assume that $\varphi \odot \psi$ is self-dual. But then there is a winning strategy $\tau : \omega^{<\omega} \setminus \{\emptyset\} \to \omega \cup \{\mathfrak{p}\}$ for Player **II** in the game $\text{G}^{<}_{\text{W}}(\varphi \odot \psi, \varphi \odot \psi)$. But using the fact that also φ is self-dual we can now from here on verbatim repeat the construction of the continuous function $F : \mathbb{R} \to \mathbb{R}$ in the proof of Lemma 4.3.26 to get that for all $x \in \mathbb{R}$

$$\psi(x) = \varphi \odot \psi(F(x)) < \varphi \odot \psi(\text{filter}(\overline{\tau}(F(x)))) = \psi(\text{Row}_2(\text{filter}(\overline{\tau}(F(x)))).$$

Then setting $H := \text{Row}_2 \circ (\text{filter} \circ \overline{\tau}) \circ F$ as there and noting that $H : \mathbb{R} \to \mathbb{R}$ is also continuous we then have that for all $x \in \mathbb{R}$

$$\psi(x) < \psi(H(x)),$$

which implies that Player **II** has a winning strategy in the game $\text{G}^{<}_{\text{W}}(\psi, \psi)$ and so that ψ is self-dual.

For the right-to-left direction we assume that ψ is self-dual. But then we take winning strategies $\sigma_0 : \omega^{<\omega} \setminus \{\emptyset\} \to \omega \cup \{\mathfrak{p}\}$ and $\sigma_1 : \omega^{<\omega} \setminus \{\emptyset\} \to \omega$ for Player **II** in the games $\text{G}^{<}_{\text{W}}(\varphi, \varphi)$ and $\text{G}^{<}_{\text{L}}(\psi, \psi)$, respectively. Furthermore since φ is self-dual we also note that $\text{lh}(\varphi) \geq \omega$ and so we fix $z \in \mathbb{R}$ such that $\varphi(z) > 0$. Then we define $g : \omega^{<\omega} \to \omega^{<\omega}$ for product conform arguments as in Lemma 4.3.25. For any $s \in \omega^{<\omega}$ that is not product conform, however, we set

$$f(s) := \begin{cases} g(t) {}^\frown \langle 0 \rangle {}^\frown z\restriction_{\text{lh}(s)}, & \text{if for } t \subseteq s \text{maximal s.t. } t \text{ is product conform } \text{rowchoice}(g(t)) = 1, \\ g(t) {}^\frown \langle 0, 1, 0 \rangle {}^\frown z\restriction_{\text{lh}(s)}, & \text{if for } t \subseteq s \text{maximal s.t. } t \text{ is product conform } \text{rowchoice}(g(t)) = 2, \end{cases}$$

But then as in the proof of Lemma 4.3.25 we get that g is monotone and infinitary and we let $G : \mathbb{R} \to \mathbb{R}$ be the continuous function induced by g. Then again as in the proof of Lemme 4.3.25 we get for any product conform $x \in \mathbb{R}$ that

$$(\varphi \odot \psi)(x) < (\varphi \odot \psi)(G(x)).$$

If, however, $x \in \mathbb{R}$ is not product conform, then we get that $\text{rowchoice}(G(x)) = 2$ and $\text{Row}_2(x) = z$ and so

$$(\varphi \odot \psi)(x) = 0 < \varphi(z) = (\varphi \odot \psi)(G(x)).$$

This implies that Player **II** has a winning strategy in the game $\text{G}^{<}_{\text{W}}(\varphi \odot \psi, \varphi \odot \psi)$, which in turn implies that $\varphi \odot \psi$ is self-dual. □

Next we will state three lemmas expressing that the $\odot$-operation behaves well under composition with several other operations on $\mathcal{N}$ that we have already defined before. However, the proofs of these results are very lengthy and give no new insights; so we will skip them here.

Lemma 4.3.28. *Let φ, ψ, χ be regular norms. Then we have that $\varphi \odot (\psi \dotplus \chi) \equiv_{\mathrm{L}} \varphi \odot \psi \dotplus \varphi \odot \chi$.*

Lemma 4.3.29. *Let φ be a regular norm and $\langle \psi_n \mid n \in \omega \rangle$ a sequence of regular norms. Then we have that*

$$\varphi \odot \left(\bigoplus_{n \in \omega} \psi_n \right) \equiv_{\mathrm{W}} \bigoplus_{n \in \omega} (\varphi \odot \psi_n \dotplus \varphi) \,.$$

Lemma 4.3.30. *Let φ and ψ be regular norms. Then we have that $\varphi \odot \psi^{\natural} \equiv_{\mathrm{W}} (\varphi \odot \psi)^{\natural}$.*

Now we are going to show that for φ self-dual and ψ such that $|\psi|_{\mathrm{W}}$ is a limit ordinal of uncountable cofinality we have that $|\varphi \odot \psi|_{\mathrm{W}} = |\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot |\psi|_{\mathrm{W}}$.

To do this we first show that for any self-dual regular norm φ and for any regular norm ψ with $|\psi|_{\mathrm{W}} = \omega_1$ we have that

$$|\varphi \odot \psi|_{\mathrm{W}} = |\varphi|_{\mathrm{W}} \cdot \omega_1^2.$$

Definition 4.3.31 We define a regular norm $\hat{\omega}$ by setting for all $x \in \omega$

$$\hat{\omega}(x) = x(0).$$

We define a regular norm $\hat{\omega_1}$ by setting $\hat{\omega_1} := \hat{\omega}^{\natural}$.

Now to give a simple presentation of $\hat{\omega_1}$ we make the following auxiliary definition.

Definition 4.3.32 We define a function $\mathrm{target} : \omega^{\leq\omega} \to \omega \cup \{\mathfrak{nd}\}$ by setting for all $x \in \omega^{\leq\omega}$

$$\mathrm{target}(x) := \begin{cases} x(0), & \text{if } x \neq \emptyset \text{ and } \forall m < \mathrm{lh}(x)\ (x(m) \neq 0), \\ x(\max\{n < \mathrm{lh}(x) \mid x(n) = 0\} + 1), & \text{if } \exists m < \mathrm{lh}(x)\ (x(m) = 0) \wedge \\ & \quad \exists n < \mathrm{lh}(x) - 1\ \forall m > n \\ & \qquad (m < \mathrm{lh}(x) \Rightarrow x(m) \neq 0), \\ \mathfrak{nd}, & \text{otherwise.} \end{cases}$$

Also we define a function $\mathrm{NTar} : \omega^{\leq\omega} \to \omega + 1$ by setting for all $x \in \omega^{\leq\omega}$

$$\mathrm{NTar}(x) := \begin{cases} \mathrm{card}\{m < \mathrm{lh}(x) \mid x(m) = 0\} + 1, & \text{if}\,\mathrm{card}\{m < \mathrm{lh}(x) \mid x(m) = 0\} \in \omega, \\ \omega, & \text{if}\,\mathrm{card}\{m < \mathrm{lh}(x) \mid x(m) = 0\} = \omega. \end{cases}$$

Then by construction of $\hat{\omega_1}$ we have for any $x \in \mathbb{R}$ that

$$\hat{\omega_1}(x) = \begin{cases} \mathrm{target}(x), & \text{if } \mathrm{target}(x) \in \omega, \\ 0, & \text{otherwise.} \end{cases}$$

Also we note that evidently by the above definitions we have for any $x \in \mathbb{R}$ that $\mathrm{target}(x) \in \omega$ if and only if there is some $m \in \omega$ such that for all $n \geq m$ we have that $\mathrm{NTar}(x\restriction_m) = \mathrm{NTar}(x\restriction_m)$. If these equivalent statements are true, we also have for cofinitely many $m \in \omega$ that $\mathrm{target}(x) = \mathrm{target}(x\restriction_m)$.

Clearly we have that $\hat{\omega} = \bigoplus_{n\in\omega} \vec{n}$ and therefore $\hat{\omega}$ is self-dual and furthermore, assuming $\mathbf{AD}$ and $\mathbf{DC}$ we have that $|\hat{\omega}|_{\mathrm{W}} = \omega$. But this already implies that $|\hat{\omega_1}|_{\mathrm{W}} = \left|\hat{\omega}^{\natural}\right|_{\mathrm{W}} = \omega \cdot \omega_1 = \omega_1$. Thus we have obtained an archetypical regular norm of Wadge rank ω_1. Our goal now will be to show that $|\varphi \odot \omega_1|_{\mathrm{W}} = |\varphi|_{\mathrm{W}} \cdot \omega_1^2$ for any self-dual regular norm φ. The next step will be to look at the other side of this equality and for a given self-dual φ to find a regular norm $\tilde{\varphi}$ such that $|\tilde{\varphi}|_{\mathrm{W}} = |\varphi|_{\mathrm{W}} \cdot \omega_1$.

Definition 4.3.33 Given a regular norm φ we define a regular norm $\tilde{\varphi}$ by setting $\tilde{\varphi} := \left((\varphi^{\natural})^{\vee}\right)^{\natural}$.

Now assuming $\mathbf{AD}$ and $\mathbf{DC}$ again we get for any self-dual φ that $|\varphi^\natural|_{\mathrm{W}} = |\varphi|_{\mathrm{W}} \cdot \omega_1$ and that $\varphi^\natural$ is non-self-dual. This in turn implies that $|(\varphi^\natural)^\vee|_{\mathrm{W}} = |\varphi^\natural|_{\mathrm{W}} + \omega = |\varphi|_{\mathrm{W}} \cdot \omega_1 + \omega$. But then again $(\varphi^\natural)^\vee$ is self-dual and so finally we get that $|\tilde{\varphi}|_{\mathrm{W}} = (|\varphi|_{\mathrm{W}} \cdot \omega_1 + \omega) \cdot \omega_1 = |\varphi|_{\mathrm{W}} \cdot \omega_1^2$.

Thus our actual goal is to show that for all self-dual φ we have that $\varphi \odot \hat{\omega}_1 \equiv_{\mathrm{W}} \tilde{\varphi}$ or equivalently – since $\tilde{\varphi}$ is non-self-dual – that $\varphi \odot \hat{\omega}_1 \equiv_{\mathrm{L}} \tilde{\varphi}$.

To be able to do this we will look at these two norms a bit more carefully. We can calculate that for any $x \in \mathbb{R}$ we have that

$$\tilde{\varphi}(x) = \begin{cases} \varphi(y) + n, & \text{if there are } s,t \in \omega^{<\omega} \text{ such that} \\ & \quad x = s ^\frown \langle 0 \rangle ^\frown \langle n+1 \rangle ^\frown (t+1) ^\frown \langle 1 \rangle ^\frown (y+2) \\ & \quad \text{or } x = s ^\frown \langle 0 \rangle ^\frown \langle n+1 \rangle ^\frown (y+2) \\ & \quad \text{or } x = \langle n+1 \rangle ^\frown (t+1) ^\frown \langle 1 \rangle ^\frown (y+2) \\ & \quad \text{or } x = \langle n+1 \rangle ^\frown (y+2), \\ n, & \text{if there are } s,t \in \omega^{<\omega} \text{ and } z \in \mathbb{R} \text{ with } \forall^\infty m \in \omega\, (z(m) = 1) \text{ s.t.} \\ & \quad x = s ^\frown \langle 0 \rangle ^\frown \langle n+1 \rangle ^\frown (x+1) \\ & \quad \text{or } x = \langle n+1 \rangle ^\frown (x+1), \\ 0 & \text{if } \forall^\infty m \in \omega\ (x(m) = 0). \end{cases}$$

Here in total we distinguish seven cases. We will introduce two natural auxiliary functions to simplify this presentation significantly.

Definition 4.3.34 We define a function $\widetilde{\mathrm{target}} : \omega^{\leq\omega} \to \omega \cup \{\mathfrak{nd}\}$ by setting for all $x \in \omega^{\leq\omega}$

$$\widetilde{\mathrm{target}}(x) := \begin{cases} x(0) - 1, & \text{if } x \neq \emptyset \text{ and } \forall m < \mathrm{lh}(x)\ (x(m) \neq 0), \\ x(\max\{n < \mathrm{lh}(x) \mid x(n) = 0\} + 1) - 1, & \text{if } \exists m < \mathrm{lh}(x)\ (x(m) = 0) \wedge \\ & \quad \exists n < \mathrm{lh}(x) - 1\ \forall m > n \\ & \qquad (m < \mathrm{lh}(x) \Rightarrow x(m) \neq 0), \\ \mathfrak{nd}, & \text{otherwise.} \end{cases}$$

We define a function $\widetilde{\mathrm{record}} : \omega^{\leq\omega} \to \omega^{\leq\omega} \cup \{\mathfrak{nd}\}$ by setting for all $x \in \omega^{\leq\omega}$

$$\widetilde{\mathrm{record}}(x) := \begin{cases} y - 2, & \text{if } \exists n < \mathrm{lh}(x) - 1\ \forall m \geq n\ (m < \mathrm{lh}(x) \Rightarrow x(m) > 1), \\ & \quad \text{where } y \in \omega^{\leq\omega} \text{ is the unique maximal end-segment of } x \text{ s.t.} \\ & \quad \forall m < \mathrm{lh}(y)\ (y(m) > 1), \\ \mathfrak{nd}, & \text{otherwise.} \end{cases}$$

To measure how often $\widetilde{\mathrm{record}}$ and $\widetilde{\mathrm{target}}$ get reset we define functions $\widetilde{\mathrm{NTar}}, \widetilde{\mathrm{NRec}} : \omega^{\leq\omega} \to \omega + 1$ by setting for all $x \in \omega^{<\omega}$:

$$\widetilde{\mathrm{NTar}}(x) = \begin{cases} \mathrm{card}\{m < \mathrm{lh}(x) \mid x(m) = 0\} + 1, & \text{if } \mathrm{card}\{m < \mathrm{lh}(x) \mid x(m) = 0\} \in \omega, \\ \omega, & \text{if } \mathrm{card}\{m < \mathrm{lh}(x) \mid x(m) = 0\} = \omega. \end{cases}$$

and, defining $R_x := \{m < \mathrm{lh}(x) \mid x(m) = 1 \wedge (m > 0 \Rightarrow x(m-1) \neq 0)\}$ as an auxiliary notion

$$\widetilde{\mathrm{NRec}}(x) = \begin{cases} \mathrm{card}(R_x), & \text{if } \widetilde{\mathrm{record}}(x) \in \{\emptyset, \mathfrak{nd}\}, \\ \mathrm{card}(R_x) + 1, & \text{otherwise.} \end{cases}$$

It follows directly from the definition that if $\widetilde{\mathrm{target}}(x) = \mathfrak{nd}$, then also $\widetilde{\mathrm{record}}(x) = \mathfrak{nd}$. Also we have that $\widetilde{\mathrm{record}}(x) \in \mathbb{R} \cup \{\mathfrak{nd}\}$, whenever $x \in \mathbb{R}$. Now using these functions we can simplify our presentation of $\tilde{\varphi}$ by seeing that for all $x \in \mathbb{R}$ we have:

$$\tilde{\varphi}(x) = \begin{cases} \varphi(\widetilde{\mathrm{record}}(x)) + \widetilde{\mathrm{target}}(x), & \text{if } \widetilde{\mathrm{record}}(x) \in \mathbb{R} \text{ and } \widetilde{\mathrm{target}}(x) \in \omega, \\ \widetilde{\mathrm{target}}(x), & \text{if } \widetilde{\mathrm{record}}(x) = \mathfrak{nd}, \text{but } \widetilde{\mathrm{target}}(x) \in \omega, \\ 0, & \text{if } \widetilde{\mathrm{record}}(x) = \mathfrak{nd} \text{ and } \widetilde{\mathrm{target}}(x) = \mathfrak{nd}. \end{cases}$$

We note the following facts for $x \in \mathbb{R}$, which are immediate by the above definitions:

- $\widetilde{\mathrm{record}}(x) \in \mathbb{R}$ if and only if there is $k \in \omega$ such that for cofinitely many $n \in \omega$, $\widetilde{\mathrm{NRec}}(x\restriction_n) = k$. If these two equivalent statements are true, then for such $k \in \omega$ we have that $k = \widetilde{\mathrm{NRec}}(x)$. Also for any $m \in \omega$ such that $\widetilde{\mathrm{NRec}}(x\restriction_m) = \widetilde{\mathrm{NRec}}(x)$ we have that

$$\widetilde{\mathrm{record}}(x) = \lim\{\widetilde{\mathrm{record}}(x\restriction_n) \mid n \geq m\}.$$

- $\widetilde{\mathrm{target}}(x) \in \omega$ if and only if there is $k \in \omega$ such that for cofinitely many $n \in \omega$, $\widetilde{\mathrm{NTar}}(x\restriction_n) = k$. If these two equivalent statements are true, then for such $k \in \omega$ we have that $k = \widetilde{\mathrm{NTar}}(x)$. Also we have for cofinitely many $n \in \omega$ that $\widetilde{\mathrm{target}}(x\restriction_n) = \widetilde{\mathrm{target}}(x)$.

Next we note that directly by its definition the regular norm $\varphi \odot \hat{\omega_1}$ can be characterized as follows for any $x \in \mathbb{R}$:

$$\varphi \odot \hat{\omega_1}(x) = \begin{cases} \mathrm{target}(\mathrm{Row}_1(x)), & \text{if } \mathrm{rowchoice}(x) = 1, \mathrm{target}(\mathrm{Row}_1(x)) \neq \mathfrak{nd}, \\ \varphi(\mathrm{Row}_2(x)), & \text{if } \mathrm{rowchoice}(x) = 2, \\ 0, & \text{otherwise.} \end{cases}$$

In the following two lemmas we are going to calculate that $\tilde{\varphi} \equiv_{\mathrm{L}} \varphi \odot \hat{\omega_1}$.

Lemma 4.3.35. *Assume* **AD**. *Let φ be self-dual. Then we have that $\tilde{\varphi} \leq_{\mathrm{L}} \varphi \odot \hat{\omega_1}$.*

Proof. Since φ is self-dual, we have that **II** wins all games $\mathrm{G}^{\leq}_{\mathrm{L}}(\varphi + n, \varphi)$ for all $n \in \omega$. Now using $\mathbf{AC}_\omega(\mathbb{R})$ we now choose for any $n \in \omega$ a strategy $\sigma_n : \omega^{<\omega} \setminus \{\emptyset\} \to \omega$ for Player **II** in $\mathrm{G}^{\leq}_{\mathrm{L}}(\varphi + n, \varphi)$.

Using this we now define a monotone function $g : \omega^{<\omega} \to \omega^{<\omega}$ such that for all $s \in \omega^{<\omega}$ we have that $\mathrm{lh}(s) \leq \mathrm{lh}(g(s))$ and that the following five properties are satisfied:

1. If $\widetilde{\mathrm{record}}(s) \in \omega^{<\omega} \setminus \{\emptyset\}$ and $\widetilde{\mathrm{target}}(s) \in \omega$, then $\mathrm{rowchoice}(g(s)) = 2$, $\mathrm{Row}_2(g(s)) = \overline{\sigma_{\widetilde{\mathrm{target}}(s)}}(\widetilde{\mathrm{record}}(s))$ and $\mathrm{target}(\mathrm{Row}_1(g(s))) = \widetilde{\mathrm{target}}(s)$.
2. If $\widetilde{\mathrm{record}}(s) = \emptyset$ and $\widetilde{\mathrm{target}}(s) \in \omega$, then $\mathrm{rowchoice}(g(s)) = 1$ and $\mathrm{target}(\mathrm{Row}_1(g(s))) = \widetilde{\mathrm{target}}(s)$.
3. If $\widetilde{\mathrm{record}}(s) = \mathfrak{nd}$ and $\widetilde{\mathrm{target}}(s) = \mathfrak{nd}$, then $\mathrm{rowchoice}(g(s)) = 1$ and $\mathrm{target}(\mathrm{Row}_1(g(s))) = \mathfrak{nd}$.
4. $\mathrm{NCol}(g(s)) = \widetilde{\mathrm{NTar}}(s) + \widetilde{\mathrm{NRec}}(s)$.
5. $\mathrm{NTar}(\mathrm{Row}_1(g(s))) = \widetilde{\mathrm{NTar}}(s)$.

We define $g(s)$ for $s \in \omega^{<\omega}$ by recursion on the length of s. First we set $g(\emptyset) = \emptyset$. Now assume that $g(s)$ is already defined, then we construct $g(s')$, where $s' = s^\frown \langle n \rangle$ for some $n \in \omega$ by the following case distinction:

Case 1 is that $\mathrm{target}(s) \in \omega$ and $\widetilde{\mathrm{record}}(s) \in \omega^{<\omega} \setminus \{\emptyset\}$ and so by induction hypothesis we have that $\mathrm{rowchoice}(g(s)) = 2$, $\mathrm{Row}_2(g(s)) = \overline{\sigma_{\widetilde{\mathrm{target}}(s)}}(\widetilde{\mathrm{record}}(s))$ and $\mathrm{target}(\mathrm{Row}_1(g(s))) = \widetilde{\mathrm{target}}(s)$.

Subcase 1.1 is that $n = 0$ and so $\widetilde{\mathrm{target}}(s') = \widetilde{\mathrm{record}}(s') = \mathfrak{nd}$. Then we set

$$g(s') := g(s)^\frown \langle 0, 1 \rangle .$$

Since $\mathrm{rowchoice}(g(s)) = 2$ we then have that $\mathrm{rowchoice}(g(s')) = 1$. Furthermore we get that $\mathrm{target}(\mathrm{Row}_1(g(s'))) = \mathfrak{nd}$. Hence Property 3 above is satisfied. Furthermore we have that $\mathrm{NTar}(\mathrm{Row}_1(g(s'))) = \mathrm{NTar}(\mathrm{Row}_1(g(s))) + 1$ and $\widetilde{\mathrm{NTar}}(s') = \widetilde{\mathrm{NTar}}(s) + 1$ and so by induction hypothesis $\mathrm{NTar}(\mathrm{Row}_1(g(s'))) = \widetilde{\mathrm{NTar}}(s')$. Finally we have that $\mathrm{NCol}(g(s')) = \mathrm{NCol}(g(s)) + 1$ and that $\widetilde{\mathrm{NRec}}(s') = \widetilde{\mathrm{NRec}}(s)$ and so by induction hypothesis we get that

$$\mathrm{NCol}(g(s')) = \mathrm{NCol}(g(s)) + 1 = (\widetilde{\mathrm{NTar}}(s) + \widetilde{\mathrm{NRec}}(s)) + 1 = \widetilde{\mathrm{NTar}}(s') + \widetilde{\mathrm{NRec}}(s').$$

Subcase 1.2 is that $n = 1$ and so $\widetilde{\text{target}}(s') = \widetilde{\text{target}}(s)$, but $\widetilde{\text{record}}(s') = \emptyset$. Then we set

$$g(s') := g(s) ^\frown \langle 0, 2 \rangle \,.$$

We have that $\text{rowchoice}(g(s)) = 2$ and furthermore, since $\widetilde{\text{record}}(s) \neq \emptyset$ that $\text{Row}_2(g(s)) \neq \emptyset$ and so $\text{rowchoice}(g(s')) = 1$. Furthermore we get that $\widetilde{\text{target}}(s) = \widetilde{\text{target}}(s')$ and $\text{target}(\text{Row}_1(g(s))) = \text{target}(\text{Row}_1(g(s')))$ and so that $\text{target}(\text{Row}_1(g(s'))) = \widetilde{\text{target}}(s')$. Hence Property 2 above is satisfied. Furthermore we have that $\text{NTar}(\text{Row}_1(g(s'))) = \text{NTar}(\text{Row}_1(g(s)))$ and $\widetilde{\text{NTar}}(s') = \widetilde{\text{NTar}}(s)$ and so by induction hypothesis $\text{NTar}(\text{Row}_1(g(s'))) = \widetilde{\text{NTar}}(s')$. Finally we have that $\text{NCol}(g(s')) = \text{NCol}(g(s)) + 1$ and that $\widetilde{\text{NRec}}(s') = \widetilde{\text{NRec}}(s) + 1$ and so by induction hypothesis we get that

$$\text{NCol}(g(s')) = \text{NCol}(g(s)) + 1 = (\widetilde{\text{NTar}}(s) + \widetilde{\text{NRec}}(s)) + 1 = \widetilde{\text{NTar}}(s') + \widetilde{\text{NRec}}(s').$$

Subcase 1.3 is that $n \geq 2$ and so $\widetilde{\text{target}}(s') = \widetilde{\text{target}}(s)$ and $\widetilde{\text{record}}(s') = \widetilde{\text{record}}(s) ^\frown \langle n-2 \rangle$. Then we set

$$g(s') := g(s) ^\frown \left\langle \sigma_{\widetilde{\text{target}}(s)}(\text{record}(s')) + 1 \right\rangle .$$

Then we have $\text{rowchoice}(g(s')) = \text{rowchoice}(g(s)) = 2$ and since nothing is changed in row 1 also $\text{target}(\text{Row}_1(g(s'))) = \text{target}(\text{Row}_1(g(s))) = \widetilde{\text{target}}(s) = \widetilde{\text{target}}(s')$. However, we get that $\widetilde{\text{record}}(s') = \widetilde{\text{record}}(s) ^\frown \langle n-2 \rangle$ and that $\text{Row}_2(g(s')) = \text{Row}_2(g(s)) ^\frown \left\langle \sigma_{\widetilde{\text{target}}(s)}(\text{record}(s')) \right\rangle$. So using all these facts and the induction hypothesis we get that

$$\text{Row}_2(g(s')) = \overline{\sigma_{\widetilde{\text{target}}(s)}}(\widetilde{\text{record}}(s)) ^\frown \left\langle \sigma_{\widetilde{\text{target}}(s)}(\widetilde{\text{record}}(s')) \right\rangle = \overline{\sigma_{\widetilde{\text{target}}(s')}}(\widetilde{\text{record}}(s')).$$

Hence Property 1 above is satisfied. Furthermore we get that $\widetilde{\text{target}}(s) = \widetilde{\text{target}}(s')$ and $\text{target}(\text{Row}_1(g(s))) = \text{target}(\text{Row}_1(g(s')))$ and so that $\text{target}(\text{Row}_1(g(s'))) = \widetilde{\text{target}}(s')$. Hence Property 2 above is satisfied. Furthermore we have that $\text{NTar}(\text{Row}_1(g(s'))) = \text{NTar}(\text{Row}_1(g(s)))$ and $\widetilde{\text{NTar}}(s') = \widetilde{\text{NTar}}(s)$ and so by induction hypothesis $\text{NTar}(\text{Row}_1(g(s'))) = \widetilde{\text{NTar}}(s')$. Finally we have that $\text{NCol}(g(s')) = \text{NCol}(g(s))$ and that $\widetilde{\text{NRec}}(s') = \widetilde{\text{NRec}}(s)$ and so directly by induction hypothesis we get that $\text{NCol}(g(s')) = \widetilde{\text{NTar}}(s') + \widetilde{\text{NRec}}(s')$.

Case 2 is that $\widetilde{\text{target}}(s) \in \omega$ and $\widetilde{\text{record}}(s) = \emptyset$. Then by induction hypothesis we have that $\text{rowchoice}(g(s)) = 1$ and $\text{target}(\text{Row}_1(g(s))) = \widetilde{\text{target}}(s)$.

Subcase 2.1 is that $n = 0$ and so $\widetilde{\text{record}}(s') = \widetilde{\text{target}}(s') = \emptyset$. Then we set

$$g(s') := g(s) ^\frown \langle 1 \rangle$$

and get that $\text{rowchoice}(g(s')) = \text{rowchoice}(g(s)) = 1$. Also we get that $\text{target}(\text{Row}_1(g(s'))) = \mathfrak{nd}$. Hence Property 3 above is satisfied. Furthermore we have that $\text{NTar}(\text{Row}_1(g(s'))) = \text{NTar}(\text{Row}_1(g(s))) + 1$ and $\widetilde{\text{NTar}}(s') = \widetilde{\text{NTar}}(s) + 1$ and so by induction hypothesis $\text{NTar}(\text{Row}_1(g(s'))) = \widetilde{\text{NTar}}(s')$. Finally we have that $\text{NCol}(g(s')) = \text{NCol}(g(s))$ and that $\widetilde{\text{NRec}}(s') = \widetilde{\text{NRec}}(s)$ and so by induction hypothesis we get that

$$\text{NCol}(g(s')) = \text{NCol}(g(s)) + 1 = (\widetilde{\text{NTar}}(s) + \widetilde{\text{NRec}}(s)) + 1 = \widetilde{\text{NTar}}(s') + \widetilde{\text{NRec}}(s').$$

Subcase 2.2 is that $n = 1$ and so $\widetilde{\text{record}}(s') = \emptyset$ and $\widetilde{\text{target}}(s') = \widetilde{\text{target}}(s)$. Then we set

$$g(s') := g(s) ^\frown \langle 2 \rangle$$

and so get that $\text{rowchoice}(g(s')) = \text{rowchoice}(g(s)) = 1$. Furthermore then $\text{target}(\text{Row}_1(g(s'))) = \text{target}(row_1(g(s))) = \widetilde{\text{target}}(s) = \widetilde{\text{target}}(s')$. Hence Property 2 above is satisfied. Furthermore we have that $\text{NTar}(\text{Row}_1(g(s'))) = \text{NTar}(\text{Row}_1(g(s)))$ and $\widetilde{\text{NTar}}(s') = \widetilde{\text{NTar}}(s)$ and so by induction hypothesis $\text{NTar}(\text{Row}_1(g(s'))) = \widetilde{\text{NTar}}(s')$. Finally we have that $\text{NCol}(g(s')) = \text{NCol}(g(s))$ and that $\widetilde{\text{NRec}}(s') = \widetilde{\text{NRec}}(s)$ and so directly by induction hypothesis we get that $\text{NCol}(g(s')) \leq \widetilde{\text{NTar}}(s') + \widetilde{\text{NRec}}(s')$.

Subcase 2.3 is that $n \geq 2$ and so $\widetilde{\mathrm{record}}(s') = n-2$ and $\widetilde{\mathrm{target}}(s') = \widetilde{\mathrm{target}}(s)$. Then we set

$$g(s') := g(s) \frown \left\langle 2, 0, \sigma_{\widetilde{\mathrm{target}}(s')}(\langle n-2 \rangle) + 1 \right\rangle$$

and get that $\mathrm{rowchoice}(g(s')) = 2$, since $\mathrm{rowchoice}(g(s)) = 1$. Also

$$\mathrm{target}(\mathrm{Row}_1(g(s'))) = \mathrm{target}(\mathrm{Row}_1(g(s))) = \widetilde{\mathrm{target}}(s) = \widetilde{\mathrm{target}}(s')$$

and

$$\mathrm{Row}_2(g(s)) = \left\langle \sigma_{\widetilde{\mathrm{target}}(s')}(\langle n-2 \rangle) \right\rangle = \overline{\sigma_{\widetilde{\mathrm{target}}(s')}}(\widetilde{\mathrm{record}}(s')).$$

Thus Property 1 above is satisfied. Furthermore we have that $\mathrm{NTar}(\mathrm{Row}_1(g(s'))) = \mathrm{NTar}(\mathrm{Row}_1(g(s)))$ and $\widetilde{\mathrm{NTar}}(s') = \widetilde{\mathrm{NTar}}(s)$ and so by induction hypothesis $\mathrm{NTar}(\mathrm{Row}_1(g(s'))) = \widetilde{\mathrm{NTar}}(s')$. Finally we have that $\mathrm{NCol}(g(s')) = \mathrm{NCol}(g(s)) + 1$ and that $\widetilde{\mathrm{NRec}}(s') = \widetilde{\mathrm{NRec}}(s) + 1$ and so by induction hypothesis we get that

$$\mathrm{NCol}(g(s')) = \mathrm{NCol}(g(s)) + 1 = (\widetilde{\mathrm{NTar}}(s) + \widetilde{\mathrm{NRec}}(s)) + 1 = \widetilde{\mathrm{NTar}}(s') + \widetilde{\mathrm{NRec}}(s').$$

Case 3 is that $\widetilde{\mathrm{target}}(s) = \mathfrak{nd}$ and $\widetilde{\mathrm{record}}(s) = \mathfrak{nd}$. Then by induction hypothesis we have that $\mathrm{rowchoice}(g(s)) = 1$ and $\mathrm{target}(\mathrm{Row}_1(g(s))) = \mathfrak{nd}$.

Subcase 3.1 is that $n = 0$ and so $\widetilde{\mathrm{record}}(s') = \mathfrak{nd}$ and $\widetilde{\mathrm{target}}(s') = \mathfrak{nd}$. Then we set

$$g(s') := g(s) \frown \langle 1 \rangle$$

and get that $\mathrm{rowchoice}(g(s')) = \mathrm{rowchoice}(g(s)) = 1$ and $\mathrm{target}(\mathrm{Row}_1(g(s'))) = \mathrm{target}(\mathrm{Row}_1(g(s))) = \mathfrak{nd}$. Thus Property 3 above is satisfied. Furthermore we have that $\mathrm{NTar}(\mathrm{Row}_1(g(s'))) = \mathrm{NTar}(\mathrm{Row}_1(g(s))) + 1$ and $\widetilde{\mathrm{NTar}}(s') = \widetilde{\mathrm{NTar}}(s) + 1$ and so by induction hypothesis $\mathrm{NTar}(\mathrm{Row}_1(g(s'))) = \widetilde{\mathrm{NTar}}(s')$. Finally we have that $\mathrm{NCol}(g(s')) = \mathrm{NCol}(g(s))$ and that $\widetilde{\mathrm{NRec}}(s') = \widetilde{\mathrm{NRec}}(s)$ and so directly by induction hypothesis we get that $\mathrm{NCol}(g(s')) = \widetilde{\mathrm{NTar}}(s') + \widetilde{\mathrm{NRec}}(s')$.

Subcase 3.2 is that $n \geq 1$ and so $\widetilde{\mathrm{target}}(s') = n-1$, but $\widetilde{\mathrm{record}}(s') = \emptyset$. But then we set

$$g(s') := g(s) \frown \langle n+1 \rangle$$

and get that $\mathrm{rowchoice}(g(s')) = \mathrm{rowchoice}(g(s)) = 1$ and $\mathrm{target}(\mathrm{Row}_1(g(s'))) = n-1$, since $\mathrm{target}(\mathrm{Row}_1(g(s)) = \mathfrak{nd}$. But the latter implies that $\mathrm{target}(\mathrm{Row}_1(g(s')) = \widetilde{\mathrm{target}}(s')$. Thus Property 2 above is satisfied. Furthermore we have that $\mathrm{NTar}(\mathrm{Row}_1(g(s'))) = \mathrm{NTar}(\mathrm{Row}_1(g(s)))$ and $\widetilde{\mathrm{NTar}}(s') = \widetilde{\mathrm{NTar}}(s)$ and so by induction hypothesis $\mathrm{NTar}(\mathrm{Row}_1(g(s'))) = \widetilde{\mathrm{NTar}}(s')$. Finally we have that $\mathrm{NCol}(g(s')) = \mathrm{NCol}(g(s))$ and that $\widetilde{\mathrm{NRec}}(s') = \widetilde{\mathrm{NRec}}(s)$ and so directly by induction hypothesis we get that $\mathrm{NCol}(g(s')) = \widetilde{\mathrm{NTar}}(s') + \widetilde{\mathrm{NRec}}(s')$.

This concludes our recursive construction of $g : \omega^{<\omega} \to \omega^{<\omega}$. But since evidently for all $s \in \omega$ we have that $\mathrm{lh}(s) \leq \mathrm{lh}(g(s))$and g is monotone we can now define a monotone strictly infinitary function $f : \omega^{<\omega} \to \omega^{<\omega}$ by setting for all $s \in \omega^{<\omega}$

$$f(s) = g(s)\restriction_{\mathrm{lh}(s)}.$$

Now we let $F : \mathbb{R} \to \mathbb{R}$ be the Lipschitz function induced by f. We claim that F is a witnesses that $\tilde{\varphi} \leq_{\mathrm{L}} \varphi \odot \hat{\omega_1}$.

To see this we first consider $x \in \mathbb{R}$ such that $\widetilde{\mathrm{record}}(x) \in \mathbb{R}$. But then $\widetilde{\mathrm{NRec}}(x) \in \omega$ and we can fix some $m \in \omega$ such that for all $n > m$ we have that $\widetilde{\mathrm{NRec}}(x\restriction_n) = \widetilde{\mathrm{NRec}}(x)$. Also we then have that $\widetilde{\mathrm{record}}(x) = \lim\{\widetilde{\mathrm{record}}(x\restriction_n) \mid n > m\}$. Now by construction of g we have for all $n > m$ that $\mathrm{rowchoice}(g(x\restriction_m)) = 2$, which implies that $\mathrm{rowchoice}(F(x)) = 2$. Also we have for all $n > m$ that $\widetilde{\mathrm{target}}(x\restriction_n) = \widetilde{\mathrm{target}}(x)$. Finally we note that for all $n > m$ we have that $\mathrm{Row}_2(g(x\restriction_n)) = \overline{\sigma_{\widetilde{\mathrm{target}}(x\restriction_n)}}(\widetilde{\mathrm{record}}(x\restriction_n))$ and so

$$\mathrm{Row}_2(F(x)) = \overline{\sigma_{\widetilde{\mathrm{target}}(x)}}(\widetilde{\mathrm{record}}(x)).$$

Now we use that $\sigma_{\widetilde{\text{target}}(x)}$ is a winning strategy for Player **II** in the game $\mathrm{G}_{\mathrm{L}}^{\leq}(\varphi + \widetilde{\text{target}}(x), \varphi)$ to conclude that

$$\tilde{\varphi}(x) = \varphi(\widetilde{\text{record}}(x)) + \widetilde{\text{target}}(x) \leq \varphi\left(\overline{\sigma_{\widetilde{\text{target}}}(x)}(\widetilde{\text{record}}(x))\right) = \varphi(\text{Row}_2(F(x))) = (\varphi \odot \hat{\omega_1})(F(x)).$$

Now we consider $x \in \mathbb{R}$ such that $\widetilde{\text{target}}(x) \in \omega$, but $\widetilde{\text{record}}(x) = \mathfrak{nd}$. Then there are infinitely many $n \in \omega$ such that $\widetilde{\text{NTar}}(x\restriction_n) = \widetilde{\text{NTar}}(x)$ and $\widetilde{\text{target}}(x\restriction_n) = \widetilde{\text{target}}(x)$ and furthermore $\widetilde{\text{record}}(x\restriction_n)$. But then by construction of g there are infinitely many $n \in \omega$ such that $\text{rowchoice}(g(x\restriction_n)) = 1$ and $\text{target}(\text{Row}_1(g(x\restriction_n))) = \widetilde{\text{target}}(x\restriction_n) = \widetilde{\text{target}}(x)$ and $\text{NTar}(\text{Row}_1(x\restriction_n)) = \widetilde{\text{NTar}}(x) \in \omega$. But then by construction of F we get that $\text{rowchoice}(F(x)) = 1$ and $\text{target}(\text{Row}_2(F(x))) = \widetilde{\text{target}}(x)$ and so

$$\tilde{\varphi}(x) = \widetilde{\text{target}}(x) = \text{target}(\text{Row}_2(F(x))) = (\varphi \odot \hat{\omega_1})(F(x)).$$

Finally for $x \in \mathbb{R}$ such that $\widetilde{\text{record}}(x) = \mathfrak{nd}$ and $\widetilde{\text{target}}(x) = \mathfrak{nd}$, we have that $\tilde{\varphi}(x) = 0$ and so vacuously $\tilde{\varphi}(x) \leq (\varphi \odot \hat{\omega_1})(F(x))$. Thus in total we have shown for all $x \in \mathbb{R}$ that $\tilde{\varphi}(x) \leq (\varphi \odot \hat{\omega_1})(F(x))$ and since F is Lipschitz thus $\tilde{\varphi} \leq_{\mathrm{L}} \varphi \odot \hat{\omega_1}$, as claimed. ☐

Lemma 4.3.36. *Let φ be an arbitrary regular norm. Then we have that $\varphi \odot \hat{\omega_1} \leq_{\mathrm{L}} \tilde{\varphi}$.*

Proof. We define a monotone and strictly infinitary function $f : \omega^{<\omega} \to \omega^{<\omega}$ such that the following properties are satisfied:

1. $\text{target}(\text{Row}_1(s)) = \widetilde{\text{target}}(f(s))$ and $\widetilde{\text{NTar}}(f(s)) = \text{NTar}(\text{Row}_1(s))$;
2. if $\text{rowchoice}(s) = 1$, then $\text{NCol}(s) = 2 \cdot \widetilde{\text{NRec}}(f(s)) - 1$;
3. if $\text{rowchoice}(s) = 2$, then $\widetilde{\text{record}}(f(s)) = \text{Row}_2(s)$ and $\text{NCol}(s) = 2 \cdot \widetilde{\text{NRec}}(f(s)) - 2$.

We define f recursively. First we set $f(\emptyset) = \emptyset$. Then we assume that $f(s)$ is already defined for some $s \in \omega^{<\omega}$ and take $n \in \omega$ and $s' := s \frown \langle n \rangle$. Then we define $f(s')$ by distinguishing the following cases:

Case 1 is that s' is not product conform. Then we set $f(s') := f(s) \frown \langle 0 \rangle$. Thus from now on we assume that s' is product conform.

Case 2 is that $\text{rowchoice}(s) = 1$ and so $\widetilde{\text{target}}(f(s)) = \text{target}(\text{Row}_1(s))$, $\widetilde{\text{NTar}}(f(s)) = \text{NTar}(\text{Row}_1(s))$ and $\text{NCol}(s) = 2 \cdot \widetilde{\text{NRec}}(f(s)) - 1$.

Subcase 2.1 is that $n = 0$. Then $\text{rowchoice}(s') = 2$ and so $\text{Row}_1(s) = \text{Row}_1(s')$. Then we set $f(s') := f(s) \frown \langle 1 \rangle$ and have that

$$\widetilde{\text{target}}(f(s')) = 1 = \text{target}(\text{Row}_1(s'))$$

and

$$\widetilde{\text{NTar}}(f(s')) = \widetilde{\text{NTar}}(f(s)) = \widetilde{\text{NTar}}(\text{Row}_1(s)) = \widetilde{\text{NTar}}(\text{Row}_1(s')).$$

We also see that $\widetilde{\text{record}}(f(s')) = \emptyset = \text{Row}_2(s')$. To see that $\text{NCol}(s) = 2 \cdot \widetilde{\text{NRec}}(f(s')) - 2$ we note that $\widetilde{\text{NRec}}(f(s')) = \widetilde{\text{NRec}}(f(s)) + 1$ and so by induction hypothesis

$$\text{NCol}(s') = \text{NCol}(s) + 1 = (2 \cdot \widetilde{\text{NRec}}(f(s)) - 1) + 1 = 2 \cdot \widetilde{\text{NRec}}(f(s')) - 2.$$

Subcase 2.2 is that $n = 1$. Then $\text{rowchoice}(s') = 1$ and $\text{target}(\text{Row}_1(s')) = \mathfrak{nd}$ and we set $f(s') := f(s) \frown \langle 0 \rangle$. Then also $\widetilde{\text{target}}(f(s')) = \mathfrak{nd}$ and so $\text{target}(\text{Row}_1(s')) = \widetilde{\text{target}}(f(s')$. Furthermore we have that

$$\widetilde{\text{NTar}}(f(s')) = \widetilde{\text{NTar}}(f(s)) + 1 = \text{NTar}(\text{Row}_2(s)) + 1 = \text{NTar}(\text{Row}_2(s')).$$

On the other hand we have that $\text{NCol}(s') = \text{NCol}(s)$ and $\widetilde{\text{NRec}}(f(s')) = \widetilde{\text{NRec}}(f(s))$ and so we have that

$$\text{NCol}(s') = 2 \cdot \widetilde{\text{NRec}}(f(s')) - 1.$$

Subcase 2.3 is that $n \geq 2$. Then $\text{rowchoice}(s') = 1$ and we set $f(s') := f(s) \frown \langle n \rangle$. If $\text{target}(\text{Row}_1(s)) = \mathfrak{nd}$, then $\text{target}(\text{Row}_1(s')) = n - 2$, but then also $\widetilde{\text{target}}(f(s)) = \mathfrak{nd}$ by induction hypothesis and so $\widetilde{\mathrm{target}}(f(s')) = n - 2 = \mathrm{target}(\mathrm{Row}_1(s'))$. If, however, $\mathrm{target}(\mathrm{Row}_1(s)) \neq \mathfrak{nd}$, then we simply have that

$$\mathrm{target}(\mathrm{Row}_1(s')) = \mathrm{target}(\mathrm{Row}_1(s)) = \widetilde{\mathrm{target}}(f(s)) = \widetilde{\mathrm{target}}(f(s')).$$

In any case we furthermore have that

$$\widetilde{\mathrm{NTar}}(f(s')) = \widetilde{\mathrm{NTar}}(f(s)) = \mathrm{NTar}(\mathrm{Row}_1(s)) = \mathrm{NTar}(\mathrm{Row}_1(s')).$$

Finally we have that

$$\mathrm{NCol}(s') = \mathrm{NCol}(s) = 2 \cdot \widetilde{\mathrm{NRec}}(f(s)) - 1 = 2 \cdot \widetilde{\mathrm{NRec}}(f(s')) - 1.$$

Case 3 is that $\mathrm{rowchoice}(s) = 2$ and so $\mathrm{target}(\mathrm{Row}_1(s)) = \widetilde{\mathrm{target}}(f(s))$, $\widetilde{\mathrm{NTar}}(f(s)) = \mathrm{NTar}(\mathrm{Row}_1(s))$ and $\mathrm{NCol}(s) = 2 \cdot \widetilde{\mathrm{NRec}}(f(s)) - 2$.

Subcase 3.1 is that $n = 0$. Then $\mathrm{rowchoice}(s') = 1$ and we set $f(s') := f(s) ^\frown \langle 2 \rangle$. We then have that $\mathrm{Row}_1(s) = \mathrm{Row}_1(s')$ and $\widetilde{\mathrm{target}}(f(s)) = \widetilde{\mathrm{target}}(f(s'))$ and so by induction hypothesis $\mathrm{target}(\mathrm{Row}_2(s')) = \widetilde{\mathrm{target}}(f(s'))$. Furthermore also

$$\mathrm{NTar}(s') = \mathrm{NTar}(s) = \widetilde{\mathrm{NTar}}(\mathrm{Row}_1(s)) = \widetilde{\mathrm{NTar}}(\mathrm{Row}_1(s')).$$

Finally we have that $\widetilde{\mathrm{NRec}}(f(s')) = \widetilde{\mathrm{NRec}}(f(s))$, but $\mathrm{NCol}(s') = \mathrm{NCol}(s) + 1$ and so by induction hypothesis

$$\mathrm{NCol}(s') = \mathrm{NCol}(s) + 1 = (2 \cdot \widetilde{\mathrm{NRec}}(f(s)) - 2) + 1 = 2 \cdot \widetilde{\mathrm{NRec}}(f(s')) - 1.$$

Subcase 3.2 is that $n > 1$. Then we set $f(s') := f(s)^\frown\langle n+1 \rangle$ and get that $\mathrm{Row}_1(s') = \mathrm{Row}_1(s) = 1$ and $\widetilde{\mathrm{target}}(f(s')) = \widetilde{\mathrm{target}}(f(s)) = 1$ and so $\mathrm{target}(\mathrm{Row}_1(s')) = 1 = \widetilde{\mathrm{target}}(f(s'))$. Furthermore we have that

$$\widetilde{\mathrm{NTar}}(f(s')) = \widetilde{\mathrm{NTar}}(f(s)) = \mathrm{NTar}(\mathrm{Row}_1(s)) = \mathrm{NTar}(\mathrm{Row}_1(s')).$$

Also we have that

$$\widetilde{\mathrm{record}}(f(s')) = \widetilde{\mathrm{record}}(f(s)) ^\frown \langle n-1 \rangle = \mathrm{Row}_2(s) ^\frown \langle n-1 \rangle = \mathrm{Row}_2(s').$$

Finally we clearly have that

$$\mathrm{NCol}(s') = \mathrm{NCol}(s') = 2 \cdot \widetilde{\mathrm{NRec}}(f(s)) - 2 = 2 \cdot \widetilde{\mathrm{NRec}}(f(s')) - 2.$$

This concludes the construction of f. It is immediately clear from the recursive definition of f that f is monotone and strictly infinitary and so induces a Lipschitz function $F : \mathbb{R} \to \mathbb{R}$. We claim that F witnesses that $\varphi \odot \hat{\omega_1} \leq_{\mathrm{L}} \tilde{\varphi}$. To see this we first take $x \in \mathbb{R}$ such that $\mathrm{rowchoice}(x) = 1$ and $\mathrm{target}(\mathrm{Row}_1(x)) \neq \mathfrak{nd}$. Then there are cofinitely many $m \in \omega$ such that $\mathrm{NTar}(\mathrm{Row}_1(x\restriction_m)) = \mathrm{NTar}(\mathrm{Row}_1(x))$ and so by construction of f also $\widetilde{\mathrm{NTar}}(f(x\restriction_m)) = \mathrm{NTar}(\mathrm{Row}_1(x))$. This already shows that $\widetilde{\mathrm{NTar}}(F(x)) = \mathrm{NTar}(\mathrm{Row}_1(x)) \in \omega$. Also by construction of f we then have that $\widetilde{\mathrm{target}}(F(x)) = \mathrm{target}(\mathrm{Row}_1(x))$. So we get that

$$(\varphi \odot \hat{\omega_1})(x) = \mathrm{target}(\mathrm{Row}_1(x)) = \widetilde{\mathrm{target}}(F(x)) \leq \tilde{\varphi}(F(x)).$$

Now we take $x \in \mathbb{R}$ such that $\mathrm{rowchoice}(x) = 2$. Then there are cofinitely many $m \in \omega$ such that $\mathrm{NCol}(x\restriction_m) = \mathrm{NCol}(x)$ and $\mathrm{rowchoice}(x\restriction_m) = 2$. But then by construction of f there are cofinitely many $m \in \omega$ such that

$$\mathrm{NCol}(x) = 2 \cdot \widetilde{\mathrm{NRec}}(f(x\restriction_m)) - 2$$

and so $\mathrm{NCol}(x) = 2 \cdot \widetilde{\mathrm{NRec}}(F(x)) - 2$, which implies that $\widetilde{\mathrm{NRec}}(F(x)) \in \omega$. Then furthermore by construction of f we have that $\widetilde{\mathrm{record}}(F(x)) = \mathrm{Row}_2(x) \in \mathbb{R}$ and $\widetilde{\mathrm{target}}(F(x)) = 1$, which implies that

$$(\varphi \odot \hat{\omega_1})(x) = \varphi(\mathrm{Row}_2(x)) \leq \varphi(\widetilde{\mathrm{record}}(F(x))) + \widetilde{\mathrm{target}}(F(x)) = \tilde{\varphi}(F(x)).$$

Finally for any other $x \in \mathbb{R}$ we have that $(\varphi \odot \hat{\omega_1})(x) = 0$ and so vacuously $(\varphi \odot \hat{\omega_1})(x) \leq \tilde{\varphi}(F(x))$. Thus we have for any $x \in \mathbb{R}$ that $(\varphi \odot \hat{\omega_1})(x) \leq \tilde{\varphi}(F(x))$ and so $\varphi \odot \hat{\omega_1} \leq_{\mathrm{L}} \tilde{\varphi}$, as claimed. $\square$

Now from what we have just shown we immediately get the following result.

Proposition 4.3.37. *Assume* $\mathbf{AD}$ *and* $\mathbf{DC}$. *Then for* φ *a self-dual regular norm and for* ψ *a regular norm such that* $|\psi|_{\mathrm{W}} = \omega_1$ *we have that* $|\varphi \odot \psi|_{\mathrm{W}} = |\varphi| \cdot \omega_1^2$.

Based on this we will now work towards showing that indeed for all self-dual φ and any ψ with $|\psi|_{\mathrm{W}}$ a limit ordinal of uncountable cofinality we have that $|\varphi \odot \psi|_{\mathrm{W}} = |\varphi| \cdot \omega_1 \cdot |\psi|_{\mathrm{W}}$.

Lemma 4.3.38. *Assume* $\mathbf{AD}$ *and* $\mathbf{DC}$. *Let* φ *and* ψ *be regular norms such that* φ *is self-dual and* ψ *is non-self-dual. Then we have that*

$$\left|\varphi \odot \psi^{+\omega}\right|_{\mathrm{W}} \leq |\varphi \odot \psi|_{\mathrm{W}} + \omega + |\varphi|_{\mathrm{W}}.$$

Proof. First we show that for φ self-dual and ψ non-self-dual we have that $\varphi \odot \psi^{\mathrm{succ}} \equiv_{\mathrm{L}} (\varphi \odot \psi)^{\mathrm{succ}}$. To see this we first note that by non-self-duality of ψ it follows from Lemma 4.2.21 and Lemma 4.2.22 that

$$\psi^{\mathrm{succ}} = \psi^{\mathrm{stretch}} + 1 \equiv_{\mathrm{L}} \psi + 1.$$

and so $\varphi \odot \psi^{\mathrm{succ}} \equiv_{\mathrm{L}} \varphi \odot (\psi + 1)$. By Lemma 4.3.27 also $\varphi \odot \psi$ is non-self-dual and so we have that

$$(\varphi \odot \psi)^{\mathrm{succ}} = (\varphi \odot \psi)^{\mathrm{stretch}} + 1 \equiv_{\mathrm{L}} (\varphi \odot \psi) + 1.$$

Thus we only need to show that $\varphi \odot (\psi + 1) \equiv_{\mathrm{L}} (\varphi \odot \psi) + 1$. To see this we first note that for any $x \in \mathbb{R}$ we have that

$$\varphi \odot (\psi + 1)(x) = \begin{cases} \psi(\mathrm{Row}_1(x)) + 1, & \text{if } \mathrm{rowchoice}(x) = 1, \\ \varphi(\mathrm{Row}_2(x)), & \text{if } \mathrm{rowchoice}(x) = 2, \\ 0, & \text{if } \mathrm{rowchoice}(x) = \mathfrak{nd}, \end{cases}$$

and

$$((\varphi \odot \psi) + 1)(x) = \begin{cases} \psi(\mathrm{Row}_1(x)) + 1, & \text{if } \mathrm{rowchoice}(x) = 1, \\ \varphi(\mathrm{Row}_2(x)) + 1, & \text{if } \mathrm{rowchoice}(x) = 2, \\ 1, & \text{if } \mathrm{rowchoice}(x) = \mathfrak{nd}. \end{cases}$$

But then we immediately see that for all $x \in \mathbb{R}$ we have that $\varphi \odot (\psi + 1)(x) \leq ((\varphi \odot \psi) + 1)(x)$, which already gives us that $\varphi \odot (\psi + 1) \leq_{\mathrm{L}} (\varphi \odot \psi) + 1$. To see the other direction we take a winning strategy $\sigma : \omega^{<\omega} \setminus \{\emptyset\} \to \omega$ for Player $\mathbf{II}$ in the game $\mathrm{G}_{\mathrm{L}}^{\leq}(\varphi, \varphi)$, which clearly also is a winning strategy for Player $\mathbf{II}$ in the game $\mathrm{G}_{\mathrm{L}}^{\leq}(\varphi + 1, \varphi)$. Also we note that since φ is self-dual, we have that $\mathrm{lh}(\varphi) > 1$ and so we fix $z \in \mathbb{R}$ such that $\varphi(z) \geq 1$.

Using this we define a winning strategy $\tau : \omega^{<\omega} \setminus \{\emptyset\} \to \omega$ for Player $\mathbf{II}$ in the game $\mathrm{G}_{\mathrm{W}}((\varphi \odot \psi) + 1, \varphi \odot (\psi + 1))$ by setting for any $s \in \omega^{<\omega} \setminus \{\emptyset\}$

$$\tau(s) := \begin{cases} s(\mathrm{lh}(s) - 1), & \text{if } \mathrm{rowchoice}(s) = 1 \\ 0, & \text{if } \mathrm{rowchoice}(s) = 2 \text{ and } s(\mathrm{lh}(s) - 1) = 0, \\ \sigma(\mathrm{Row}_2(s)) + 1, & \text{if } \mathrm{rowchoice}(s) = 2 \text{ and } s(\mathrm{lh}(s) - 1) > 0, \\ (\langle 1, 0 \rangle \frown (z + 1))(\mathrm{lh}(s) - \mathrm{lh}(t) - 1), & \text{if } \mathrm{rowchoice}(s) = \mathfrak{nd} \text{ and } \mathrm{rowchoice}(t) = 1, \\ & \quad \text{where } t \subseteq s \text{ is maximal s.t. } \mathrm{rowchoice}(t) \neq \mathfrak{nd}, \\ (\langle 1, 0, 1, 0 \rangle \frown (z + 1))(\mathrm{lh}(s) - \mathrm{lh}(t) - 1), & \text{if } \mathrm{rowchoice}(s) = \mathfrak{nd} \text{ and } \mathrm{rowchoice}(t) = 2, \\ & \quad \text{where } t \subseteq s \text{ is maximal s.t. } \mathrm{rowchoice}(t) \neq \mathfrak{nd}. \end{cases}$$

This strategy is indeed winning for Player $\mathbf{II}$; let $x \in \mathbb{R}$ be such that $\mathrm{rowchoice}(x) = 1$. Then we have by construction that $\mathrm{rowchoice}(\overline{\tau}(x)) = 1$ and furthermore $\mathrm{Row}_1(x) = \mathrm{Row}_1(\overline{\tau}(x))$ and so we get that

$$((\varphi \odot \psi) + 1)(x) = \psi(\mathrm{Row}_1(x)) + 1 = \psi(\mathrm{Row}_1(\overline{\tau}(x))) + 1 = \varphi \odot (\psi + 1)(\overline{\tau}(x)).$$

Now let $x \in \mathbb{R}$ be such that $\mathrm{rowchoice}(x) = 2$. Then we have that $\mathrm{rowchoice}(\overline{\tau}(x)) = 2$ and furthermore $\mathrm{Row}_2(\overline{\tau}(x)) = \overline{\sigma}(\mathrm{Row}_2(x))$ and so we get that

$$((\varphi \odot \psi) + 1)(x) = \varphi(\mathrm{Row}_2(x)) + 1 \leq \varphi(\overline{\sigma}(\mathrm{Row}_2(x)) = \varphi(\mathrm{Row}_2(\overline{\tau}(x))) = \varphi \odot (\psi + 1)(\overline{\tau}(x)).$$

Finally let $x \in \mathbb{R}$ be such that $\text{rowchoice}(x) = \mathfrak{n}\mathfrak{d}$. Then we have that $\text{rowchoice}(\overline{\tau}(x)) = 2$ and $\text{Row}_2(\overline{\tau}(x)) = z$ and so by choice of z:

$$((\varphi \odot \psi) + 1)(x) = 1 \leq \varphi(z) = \varphi(\text{Row}_2(\overline{\tau}(x))) = \varphi \odot (\psi + 1)(\overline{\tau}(x)).$$

Thus we have shown that $(\varphi \odot \psi) + 1 \leq_{\text{L}} \varphi \odot (\psi + 1)$ and so that $(\varphi \odot \psi) + 1 \equiv_{\text{L}} \varphi \odot (\psi + 1)$, which – by what we have noted above – already shows that $\varphi \odot \psi^{\text{succ}} \equiv_{\text{L}} (\varphi \odot \psi)^{\text{succ}}$. But since we have shown this for arbitrary non-self-dual ψ and for any regular norm χ the regular norm χ^{succ} is non-self-dual, we get by Lemma 4.2.16 that for any self-dual φ, non-self-dual ψ and any $n \in \omega$

$$(\varphi \odot \psi)^{+n} \equiv_{\text{L}} \varphi \odot (\psi^{+n}).$$

Now we note that by Proposition 4.2.18 we have that $\psi^{+\omega} \equiv_{\text{L}} \bigoplus_{n\in\omega} \psi^{+n}$ and so

$$\varphi \odot \psi^{+\omega} \equiv_{\text{W}} \varphi \odot \left(\bigoplus_{n\in\omega} \psi^{+n} \right) \equiv_{\text{W}} \bigoplus_{n\in\omega} (\varphi \odot \psi^{+n} \dotplus \varphi).$$

Hence by Lemma 4.3.29 we get that

$$\left|\varphi \odot \psi^{+\omega}\right|_{\text{W}} = \sup_{n\in\omega} \left|\varphi \odot \psi^{+n} \dotplus \varphi\right|_{\text{W}}.$$

But by what we have just shown we note that $\varphi \odot \psi^{+n} \equiv_{\text{W}} (\varphi \odot \psi)^{+n} \leq_{\text{W}} (\varphi \odot \psi)^{+\omega}$ for all $n \in \omega$ and so

$$\left|\varphi \odot \psi^{+\omega}\right|_{\text{W}} \leq \left|(\varphi \odot \psi)^{+\omega} \dotplus \varphi\right|_{\text{W}}.$$

But then we note that by Corollary 4.2.19 the regular norm $(\varphi \odot \psi)^{+\omega}$ is self-dual and $|(\varphi \odot \psi)^{+\omega}|_{\text{W}} = |\varphi \odot \psi|_{\text{W}} + \omega$ and so by Theorem 4.3.9 we finally get that

$$\left|\varphi \odot \psi^{+\omega}\right|_{\text{W}} \leq |\varphi \odot \psi|_{\text{W}} + \omega + |\varphi|_{\text{W}},$$

which concludes the proof. □

Proposition 4.3.39. *Assume* **AD** *and* **DC**. *Let φ be a self-dual regular norm and ψ a non-self-dual regular norm. Then we have that*

$$|\varphi \odot \psi|_{\text{W}} = \sup\{|\varphi \odot \chi|_{\text{W}} \mid \chi <_{\text{W}} \psi\}.$$

Proof. It already follows from Lemma 4.3.25 that $|\varphi \odot \psi|_{\text{W}} \geq \sup\{|\varphi \odot \chi|_{\text{W}} \mid \chi <_{\text{W}} \psi\}$. To show the other direction we take a regular norm χ with $\chi <_{\text{W}} \varphi \odot \psi$ and show that then there is $\psi^* <_{\text{W}} \psi$ such that $\chi \leq_{\text{W}} \varphi \odot \psi^*$. Since ψ is non-self-dual by assumption we get by Lemma 4.3.27 that also $\varphi \odot \psi$ is non-self-dual. Thus $\chi <_{\text{W}} \varphi \odot \psi$ implies that Player **II** has a winning strategy $\tau : \omega^{<\omega} \setminus \{\emptyset\} \to \omega \cup \{\mathfrak{p}\}$ in the game $\text{G}_{\text{W}}^{\leq}(\chi, \varphi \odot \psi)$. Without loss of generality we assume that for any $x \in \omega^{\leq\omega}$ that $\text{filter}(\overline{\tau}(x))$ is product conform. Now we define a set $Z \subseteq \mathbb{R}$ as follows:

$$Z := \{x \in \mathbb{R} \mid \text{rowchoice}(\overline{\tau}(x)) = 1\}.$$

Using this we define a regular norm ψ^* by setting for any $x \in \mathbb{R}$

$$\psi^*(x) := \begin{cases} \chi(x), & \text{if } x \in Z, \\ 0, & \text{otherwise.} \end{cases}$$

Then we get a winning strategy $\tau' : \omega^{<\omega} \setminus \{\emptyset\} \to \omega \cup \{\mathfrak{p}\}$ for Player **II** in the game $\text{G}_{\text{W}}^{\leq}(\psi^*, \psi^{\text{stretch}})$ by setting for all $s \in \omega^{<\omega}$

$$\tau'(s) := \begin{cases} \tau(s), & \text{if rowchoice}(\overline{\tau}(s)) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

This strategy is indeed winning for Player **II**, since by construction we get for all $x \in Z$ that $\text{rowchoice}(\text{filter}(\overline{\tau}(x)) = 1$ and $\text{unstretch}(\text{filter}(\overline{\tau'}(x))) = \text{Row}_1(\text{filter}(\overline{\tau}(x)))$ and so

$$\psi^*(x) = \chi(x) \leq \psi(\text{Row}_1(\text{filter}(\overline{\tau}(x)))) = \psi(\text{unstretch}(\text{filter}(\overline{\tau'}(x)))) = \psi^{\text{stretch}}(\text{filter}(\overline{\tau'}(x))).$$

If, however $x \notin Z$ then vacuously $\psi^*(x) = 0 \leq \psi^{\text{stretch}}(\text{filter}(\overline{\tau'}(x)))$. Now we note that since ψ is non-self-dual we get by Lemma 4.2.21 that $\psi \equiv_{\mathrm{L}} \psi^{\text{stretch}}$ and so Player **II** wins the game $\mathrm{G}_{\mathrm{W}}^{\leq}(\psi^*, \psi)$, which again by non-self-duality of ψ implies that $\psi^* <_{\mathrm{W}} \psi$.

The next step will be to show that $\chi \leq \varphi \odot \psi^*$, which would conclude the proof. For this we define a monotone infinitary function $g : \omega^{<\omega} \to \omega^{<\omega}$ that satisfies the following two properties for any $s \in \omega^{<\omega}$:

1. $\text{rowchoice}(\text{filter}(\overline{\tau}(s))) = 1$ if and only if $\text{rowchoice}(g(s)) = 1$; and in this case $\text{Row}_1(g(s)) = s$ and $(g(s))(\text{lh}(g(s)) - 1) \neq 0$;

2. $\text{rowchoice}(\text{filter}(\overline{\tau}(s))) = 2$ if and only if $\text{rowchoice}(g(s)) = 2$; and in this case $\text{Row}_2(g(s)) = \text{Row}_2(\text{filter}(\overline{\tau}(s)))$ and $\text{Row}_1(g(s)) = t$, where t is the maximal initial segment of s such that $\text{rowchoice}(\text{filter}(\overline{\tau}(t))) = 1$.

We construct g recursively. We set $g(\emptyset) = \emptyset$. Then we assume that $g(s)$ is already constructed and construct $g(s')$, where $s' = s ^\frown \langle n \rangle$ for some $n \in \omega$, by distinguishing the following cases:

Case 1 is that $\text{rowchoice}(\text{filter}(\overline{\tau}(s))) = 1$ and so $\text{rowchoice}(g(s)) = 1$ and $\text{Row}_1(g(s)) = s$.

Subcase 1.1 is that $\text{rowchoice}(\text{filter}(\overline{\tau}(s'))) = 1$. Then we set $g(s') = g(s) ^\frown \langle n+1 \rangle$ and thus have that $\text{rowchoice}(g(s')) = \text{rowchoice}(g(s)) = 1$ and

$$\text{Row}_1(g(s')) = \text{Row}_1(g(s)) ^\frown \langle n \rangle = s ^\frown \langle n \rangle = s'.$$

Subcase 1.2 is that $\text{rowchoice}(\text{filter}(\overline{\tau}(s'))) = 2$. Then we set $g(s') := g(s) ^\frown \langle 0 \rangle ^\frown (\text{Row}_2(\text{filter}(\overline{\tau}(s'))))$. By construction we ensure that $\text{rowchoice}(g(s')) = 2$, since $(g(s))(\text{lh}(g(s)) - 1) \neq 0$. Also we ensure that $\text{Row}_2(g(s')) = \text{Row}_2(\text{filter}(\overline{\tau}(s')))$. Furthermore clearly $\text{Row}_1(g(s')) = \text{Row}_1(g(s)) = s$ and s is the maximal initial segment of s' such that $\text{rowchoice}(\text{filter}(\overline{\tau}(s))) = 1$.

Case 2 is that $\text{rowchoice}(\text{filter}(\overline{\tau}(s))) = 2$ and so $\text{rowchoice}(g(s)) = 2$ and $\text{Row}_2(g(s)) = \text{Row}_2(\text{filter}(\overline{\tau}(s)))$.

Subcase 2.1 is that $\text{rowchoice}(\text{filter}(\overline{\tau}(s'))) = 1$. Then we let $t \subsetneq s'$ be of maximal length such that $\text{rowchoice}(\text{filter}(\overline{\tau}(t))) = 1$ and we set $g(s') := g(s) ^\frown \langle 0 \rangle ^\frown ((s \setminus t) + 1)$. Since by assumption $\text{filter}(\overline{\tau}(s'))$ is product conform, we have that $\text{Row}_2(\text{filter}(\overline{\tau}(s))) \neq \emptyset$ and so $\text{Row}_2(g(s)) \neq \emptyset$. Thus $(g(s))(\text{lh}(g(s)) - 1) \neq 0$ and so $\text{rowchoice}(g(s')) = 1$. Also we have that $\text{Row}_1(g(s')) = t ^\frown (s \setminus t) = s$.

Subcase 2.2 is that $\text{rowchoice}(\text{filter}(\overline{\tau}(s'))) = 2$. Then we set $g(s') := \text{filter}(g(s) ^\frown \langle \tau(s) + 1 \rangle)$. Evidently $\text{rowchoice}(g(s')) = \text{rowchoice}(g(s)) = 2$ and since $\text{rowchoice}(\overline{\tau}(s')) = 2$, also $\text{Row}_2(g(s')) = \text{Row}_2(\text{filter}(\overline{\tau}(s')))$. Also we have that

$$\begin{aligned}\text{Row}_1(g(s')) &= \text{Row}_1(g(s)) \\ &= \max\{t \subseteq s \mid \text{rowchoice}(\text{filter}(\overline{\tau}(t))) = 1\} \\ &= \max\{t \subseteq s' \mid \text{rowchoice}(\text{filter}(\overline{\tau}(t))) = 1\}.\end{aligned}$$

This concludes the construction of g. It is clear from the recursive construction that g is monotone. To see that g is infinitary, we assume that this is not the case, i.e., there is an $x \in \mathbb{R}$ such that for cofinitely many $m \in \omega$ we have $g(x\restriction_m) = g(x\restriction_{m+1})$. But this can only be the case, when there is an $x \in \mathbb{R}$ such that for cofinitely man $m \in \omega$ we have $\tau(x\restriction_m) = \mathfrak{p}$, contradicting the fact that τ is a winning strategy for Player **II** in a Wadge game. Thus g indeed is infinitary and so induces a continuous function $G : \mathbb{R} \to \mathbb{R}$.

We claim that $G : \mathbb{R} \to \mathbb{R}$ witnesses $\chi \leq_{\mathrm{W}} \varphi \odot \psi^*$, i.e., we have for all $x \in \mathbb{R}$ that $\chi(x) \leq \varphi \odot \psi^*(G(x))$. To see this we first consider $x \in \mathbb{R}$ such that $\text{rowchoice}(G(x)) = 1$. But then we have for infinitely many $m \in \omega$ that $\text{rowchoice}(g(x\restriction_m)) = 1$ and so $\text{rowchoice}(\text{filter}(\overline{\tau}(x\restriction_m))) = 1$, which implies that $\text{rowchoice}(\text{filter}(\overline{\tau}(x))) = 1$ and so $x \in Z$. By construction of g we also have for infinitely many $m \in \omega$ that $\text{Row}_1(g(x\restriction_m)) = x\restriction_m$ and so $\text{Row}_1(G(x)) = x$. But then we get by definition of χ that

$$\chi(x) = \chi(\text{Row}_1(G(x))) = \psi^*(\text{Row}_1(G(x))) = \varphi \odot \psi^*(G(x)).$$

If, however $x \in \mathbb{R}$ is such that $\text{rowchoice}(G(x)) = 2$, then we have for cofinitely many $m \in \omega$ that $\text{rowchoice}(g(x\restriction_m)) = 2$ and so $\text{rowchoice}(\overline{\tau}(x\restriction_m)) = 2$ and $\text{Row}_2(g(x\restriction_m)) = \text{Row}_2(\text{filter}(\overline{\tau}(x\restriction_m)))$. But this implies that $\text{rowchoice}(\overline{\tau}(x)) = 2$ and $\text{Row}_2(G(x)) = \text{Row}_2(\text{filter}(\overline{\tau}(x)))$. But then, since τ is winning for Player **II** in the game $\mathrm{G}_{\mathrm{W}}^{\leq}(\chi, \varphi \odot \psi)$, we get that

$$\chi(x) \leq \varphi \odot \psi(\text{filter}(\overline{\tau}(x))) = \varphi(\text{Row}_2(\text{filter}(\overline{\tau}(x)))) = \varphi(\text{Row}_2(G(x))) \leq \varphi \odot \psi^*(G(x)).$$

To conclude the proof we finally note that by construction of g we have that for all $x \in \mathbb{R}$ the real $G(x)$ is product conform. So the cases considered are the only ones that can occur. $\square$

Theorem 4.3.40. *Assume* $\mathbf{AD}$ *and* $\mathbf{DC}$*. Then for all regular norms* φ, ψ *such that* φ *is self-dual and* $|\psi|_{\mathrm{W}}$ *is a limit ordinal of uncountable cofinality, we have that*

$$|\varphi \odot \psi|_{\mathrm{W}} = |\varphi| \cdot \omega_1 \cdot |\psi|_{\mathrm{W}}.$$

Proof. We prove this by an induction on $\alpha := |\psi|_{\mathrm{W}}$. The case that $|\psi|_{\mathrm{W}} = \omega_1$ is already dealt with in Proposition 4.3.37. Thus we assume that $\alpha > \omega_1$. By Cantor's Normal Form Theorem as stated in Lemma 2.1.3 we then have a unique $k \in \omega$ as well as unique sequences $\langle \xi_m \mid m \leq k \rangle$ and $\langle \eta_m \mid m \leq k \rangle$ of ordinals with $\xi_m > \xi_{m+1}$ for all $m < k$ and $0 < \eta_m < \omega_1$ for all $m \leq k$ such that

$$\alpha = \sum_{m \leq k} \omega_1^{\xi_m} \cdot \eta_m.$$

Now we distinguish two cases.

Case 1 is that $k > 0$. Then we fix a regular norm χ of minimal Lipschitz rank such that $|\chi|_{\mathrm{W}} = \omega_1^{\xi_0} \cdot \eta_0$. Now we distinguish two subcases.

Subcase 1.1 is that $\mathrm{cf}(\omega_1^{\xi_0} \cdot \eta_0) = \omega$, i.e., that χ is self-dual. Then by Lemma 4.3.7 there is a regular norm ψ^* with $\chi \dotplus \psi^* \equiv_{\mathrm{W}} \psi$ and so by Theorem 4.3.9 we have that $|\psi|_{\mathrm{W}} = |\chi|_{\mathrm{W}} + 1 + |\psi^*|_{\mathrm{W}}$. But then clearly $|\psi^*|_{\mathrm{W}}$ is a limit ordinal, which implies that in fact

$$|\psi|_{\mathrm{W}} = |\chi|_{\mathrm{W}} + |\psi^*|_{\mathrm{W}}$$

and so $|\psi^*|_{\mathrm{W}} = \sum_{0 < m \leq k} \omega_1^{\xi_m} \cdot \eta_m$. By elementary ordinal arithmetic we also get that

$$\mathrm{cf}(|\psi^*|_{\mathrm{W}}) = \mathrm{cf}(|\chi|_{\mathrm{W}} + |\psi^*|_{\mathrm{W}}) = \mathrm{cf}(|\psi|_{\mathrm{W}}) > \omega.$$

Hence by induction hypothesis we get that

$$|\varphi \odot \psi^*|_{\mathrm{W}} = |\varphi| \cdot \omega_1 \cdot |\psi^*|_{\mathrm{W}}.$$

Next we note that since $\mathrm{cf}(\omega_1^{\xi_0} \cdot \eta_0) = \omega$ we have that either the ordinal η_0 or the ordinal ξ_0 is a limit ordinal of countable cofinality. In any case we can find a sequence $\langle \nu_n \mid n \in \omega \rangle$ of ordinals cofinal in $\omega_1^{\xi_0} \cdot \eta_0$ such that for all $n \in \omega$ we have that $\mathrm{cf}(\nu_n) > \omega$. Then by $\mathbf{AC}_\omega$ we choose for any $n \in \omega$ a regular norm χ_n with $|\chi_n|_{\mathrm{W}} = \nu_n$. Then by induction hypothesis we have that $|\varphi \odot \chi_n|_{\mathrm{W}} = |\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot |\chi_n|_{\mathrm{W}}$. By Lemma 4.2.6 we also have that $\chi \equiv_{\mathrm{L}} \bigoplus_{n \in \omega} \chi_n$ and so we can calculate that

$$\begin{aligned}
|\varphi \odot \chi|_{\mathrm{W}} &= \left| \varphi \odot \bigoplus_{n \in \omega} \chi_n \right|_{\mathrm{W}} && \text{(by Lemma 4.3.25)} \\
&= \left| \bigoplus_{n \in \omega} (\varphi \odot \chi_n \dotplus \varphi) \right|_{\mathrm{W}} && \text{(by Lemma 4.3.29)} \\
&= \sup\{|\varphi \odot \chi_n \dotplus \varphi|_{\mathrm{W}} \mid n < \omega\}. && \text{(by Lemma 4.2.6)}
\end{aligned}$$

But now for any $n \in \omega$ we can calculate that

$$\begin{aligned}
|\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot |\chi_n|_{\mathrm{W}} &= |\varphi \odot \chi_n|_{\mathrm{W}} \\
&\leq |\varphi \odot \chi_n \dotplus \varphi|_{\mathrm{W}} && \text{(by Lemma 4.3.5)} \\
&\leq |\varphi \odot \chi_n|_{\mathrm{W}} + \omega + |\varphi|_{\mathrm{W}} && \text{(by Proposition 4.3.10)} \\
&\leq |\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot |\chi_n|_{\mathrm{W}} + \omega + |\varphi|_{\mathrm{W}} \\
&\leq |\varphi|_{\mathrm{W}} \cdot (\omega_1 \cdot |\chi_n|_{\mathrm{W}} + 2) && \text{(since } |\varphi| \geq \omega) \\
&< |\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot |\chi_{n+1}|_{\mathrm{W}}.
\end{aligned}$$

Thus we get that

$$\begin{aligned}
\sup\{|\varphi \odot \chi_n \dotplus \varphi|_{\mathrm{W}} \mid n < \omega\} &= \sup\{|\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot |\chi_n| \mid n < \omega\} \\
&= |\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot |\chi|_{\mathrm{W}}.
\end{aligned}$$

Therefore we can finally calculate that

$$\begin{aligned}
|\varphi \odot \psi|_{\mathrm{W}} &= |\varphi \odot (\chi \dotplus \psi^*)|_{\mathrm{W}} \\
&= |\varphi \odot \chi \dotplus \varphi \odot \psi^*|_{\mathrm{W}} && \text{(by Lemma 4.3.28)} \\
&= |\varphi \odot \chi|_{\mathrm{W}} + |\varphi \odot \psi^*|_{\mathrm{W}} && \text{(by Theorem 4.3.9)} \\
&= |\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot |\chi|_{\mathrm{W}} + |\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot |\psi^*|_{\mathrm{W}} \\
&= |\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot (|\chi|_{\mathrm{W}} + |\psi^*|_{\mathrm{W}}) \\
&= |\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot |\psi|_{\mathrm{W}} .
\end{aligned}$$

Subcase 1.2 is that $\mathrm{cf}(\omega_1^{\xi_0} \cdot \eta_0) > \omega$, i.e., that χ is non-self-dual. Then we have to distinguish two subsubcases.

Subsubcase 1.2.1 is that there is $n \in \omega$ such that $|\psi|_{\mathrm{W}} = \omega_1^{\xi_0} \cdot \eta_0 + \omega_1 \cdot (n+1)$. Then we define regular norms χ_ℓ for $\ell < \omega$ recursively by setting $\chi_0 := \chi$ and $\chi_{\ell+1} := \chi_\ell^{+\omega} \dotplus \hat{\omega_1}$ and get for all $\ell \in \omega$ that $|\chi_\ell|_{\mathrm{W}} = |\chi| + \omega_1 \cdot \ell$ as follows. For $\ell = 0$ we vacuously have that $|\chi_0|_{\mathrm{W}} = \chi$. Now we assume that for a given $\ell \in \omega$ we have $|\chi_\ell|_{\mathrm{W}} = |\chi| + \omega_1 \cdot \ell$. Then we use that $\chi_\ell^{+\omega}$ is self-dual and that $\left|\chi_\ell^{+\omega}\right|_{\mathrm{W}} = |\chi_\ell| + \omega$ by Corollary 4.2.19 and that furthermore $|\hat{\omega_1}|_{\mathrm{W}} = \omega_1$ to calculate

$$\begin{aligned}
|\chi_{\ell+1}|_{\mathrm{W}} &= \left|\chi_\ell^{+\omega} \dotplus \hat{\omega_1}\right|_{\mathrm{W}} \\
&= |\chi_\ell|_{\mathrm{W}} + \omega + \omega_1 && \text{(by Theorem 4.3.9)} \\
&= |\chi_\ell|_{\mathrm{W}} + \omega_1 \\
&= |\chi|_{\mathrm{W}} + \omega_1 \cdot \ell + \omega_1 && \text{(by induction hypothesis)} \\
&= |\chi|_{\mathrm{W}} + \omega_1 \cdot (\ell + 1).
\end{aligned}$$

This concludes the induction. In particular we get that for any $\ell \in \omega$ that $|\chi_\ell|_{\mathrm{W}}$ is a limit ordinal of uncountable cofinality and thus by Proposition 4.2.9 that χ_ℓ is non-self-dual.

Now since $|\chi|_{\mathrm{W}} = \omega_1^{\xi_0} \cdot \eta_0$ we get that $|\chi_{n+1}|_{\mathrm{W}} = \omega_1^{\xi_0} \cdot \eta_0 + \omega_1 \cdot (n+1) = |\psi|_{\mathrm{W}}$. Since furthermore ψ is non-self-dual by assumption this also shows that $\chi_{n+1} \equiv_{\mathrm{L}} \psi$. Thus we get by Lemma 4.3.25 and Lemma 4.3.28 that

$$\varphi \odot \psi \equiv_{\mathrm{W}} \varphi \odot \chi_{n+1} = \varphi \odot \left(\chi_n^{+\omega} \dotplus \hat{\omega_1}\right) \equiv_{\mathrm{W}} \varphi \odot \chi_n^{+\omega} \dotplus \varphi \odot \hat{\omega_1}.$$

But since $|\varphi \odot \chi_n|_{\mathrm{W}} \leq |\varphi \odot \chi_n^{+\omega}|_{\mathrm{W}} \leq |\varphi \odot \chi_n|_{\mathrm{W}} + \omega + |\varphi|_{\mathrm{W}}$ by Lemma 4.3.25 and Lemma 4.3.38, we thus get that

$$\begin{aligned}
|\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot |\psi|_{\mathrm{W}} &= |\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot (|\chi_n|_{\mathrm{W}} + \omega_1) \\
&= |\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot |\chi_n|_{\mathrm{W}} + |\varphi|_{\mathrm{W}} \cdot \omega^2 \\
&= |\varphi \odot \chi_n|_{\mathrm{W}} + |\varphi \odot \hat{\omega_1}|_{\mathrm{W}} && \text{(by induction hypothesis)} \\
&\leq \left|\varphi \odot \chi_n^{+\omega}\right|_{\mathrm{W}} + |\varphi \odot \hat{\omega_1}|_{\mathrm{W}} \\
&= \left|\varphi \odot \chi_n^{+\omega} \dotplus \varphi \odot \hat{\omega_1}\right|_{\mathrm{W}} \\
&= |\varphi \odot \chi_{n+1}|_{\mathrm{W}} \\
&= |\varphi \odot \psi|_{\mathrm{W}}
\end{aligned}$$

and that

$$\begin{aligned}
|\varphi \odot \psi|_{\mathrm{W}} &= \left|\varphi \odot \chi_n^{+\omega} \dotplus \varphi \odot \hat{\omega_1}\right|_{\mathrm{W}} \\
&= \left|\varphi \odot \chi_n^{+\omega}\right|_{\mathrm{W}} + |\varphi \odot \hat{\omega_1}|_{\mathrm{W}} \\
&\leq |\varphi \odot \chi_n|_{\mathrm{W}} + \omega + |\varphi|_{\mathrm{W}} + |\varphi|_{\mathrm{W}} \cdot \omega_1^2 \\
&= |\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot |\chi_n| + \omega + |\varphi|_{\mathrm{W}} + |\varphi|_{\mathrm{W}} \cdot \omega_1^2 && \text{(by induction hypothesis)} \\
&= |\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot (|\chi_n|_{\mathrm{W}} + \omega_1) \\
&= |\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot |\psi|_{\mathrm{W}} ,
\end{aligned}$$

which in total shows that $|\varphi \odot \psi|_{\mathrm{W}} = |\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot |\psi|_{\mathrm{W}}$, as claimed.

Subsubcase 1.2.2 is that $|\psi|_{\mathrm{W}} \geq \omega_1^{\xi_0} \cdot \eta_0 + \omega_1 \cdot \omega$. Then we consider $\chi^{+\omega}$ and note that by Lemma 4.3.7 then there is a regular norm ψ^* such that $\psi \equiv_{\mathrm{W}} \chi^{+\omega} \dotplus \psi^*$ and so by non-self-duality of ψ furthermore $\psi \equiv_{\mathrm{L}} \chi^{+\omega} \dotplus \psi^*$. Also ψ^* satisfies $|\psi^*|_{\mathrm{W}} = \sum_{1<m\leq k} \omega_1^{\xi_m} \cdot \eta_m$ and $\mathrm{cf}(|\psi^*|) > \omega$. Then by induction hypothesis we have that $|\varphi \odot \psi^*|_{\mathrm{W}} = |\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot |\psi^*|_{\mathrm{W}}$. Also since $\chi^{+\omega}$ is self-dual and therefore $\varphi \odot \chi^{+\omega}$ is self-dual, we get that

$$|\varphi \odot \psi|_{\mathrm{W}} = \left|\varphi \odot (\chi^{+\omega} \dotplus \psi^*)\right|_{\mathrm{W}} = |\varphi \odot \chi^{\omega} \dotplus \varphi \odot \psi^*|_{\mathrm{W}} = \left|\varphi \odot \chi^{+\omega}\right|_{\mathrm{W}} + |\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot |\psi^*|_{\mathrm{W}}.$$

But since φ is self-dual and $\chi <_{\mathrm{L}} \chi^{+\omega} \leq_{\mathrm{L}} \chi^{+\omega} + \hat{\omega_1}$, we have that

$$\varphi \odot \chi \leq_{\mathrm{W}} \varphi \odot \chi^{+\omega} \leq_{\mathrm{W}} \varphi \odot (\chi^{+\omega} + \hat{\omega_1})$$

and so using the induction hypothesis

$$|\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot |\chi|_{\mathrm{W}} \leq |\varphi \odot \chi|_{\mathrm{W}} \leq |\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot (|\chi|_{\mathrm{W}} + \omega_1).$$

And so finally, since $|\psi^*|_{\mathrm{W}} \geq \omega_1 \cdot \omega$, we get that

$$\begin{aligned}
|\varphi \odot \psi|_{\mathrm{W}} &= \left|\varphi \odot \chi^{+\omega}\right|_{\mathrm{W}} + |\varphi| \cdot \omega_1 \cdot |\psi^*|_{\mathrm{W}} \\
&\leq |\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot (|\chi|_{\mathrm{W}} + \omega_1) + |\varphi| \cdot \omega_1 \cdot |\psi^*|_{\mathrm{W}} \\
&= |\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot (|\chi|_{\mathrm{W}} + \omega_1 + |\psi^*|_{\mathrm{W}}) \\
&= |\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot (|\chi|_{\mathrm{W}} + |\psi^*|_{\mathrm{W}}) \\
&= |\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot |\psi|_{\mathrm{W}}
\end{aligned}$$

and

$$\begin{aligned}
|\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot |\psi|_{\mathrm{W}} &= |\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot |\chi|_{\mathrm{W}} + |\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot |\psi^*|_{\mathrm{W}} \\
&= |\varphi \odot \chi|_{\mathrm{W}} + |\varphi \odot \psi^*|_{\mathrm{W}} \qquad \text{(by induction hypothesis)} \\
&\leq \left|\varphi \odot \chi^{+\omega}\right|_{\mathrm{W}} + |\varphi \odot \psi^*|_{\mathrm{W}} \\
&= \left|\varphi \odot \chi^{+\omega} \dotplus \varphi \odot \psi^*\right|_{\mathrm{W}} \\
&= |\varphi \odot \psi|_{\mathrm{W}},
\end{aligned}$$

which shows that $|\varphi \odot \psi|_{\mathrm{W}} = |\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot |\psi|_{\mathrm{W}}$, as claimed.

Case 2 is that $k = 0$. Then we have that $|\psi|_{\mathrm{W}} = \omega_1^{\xi_0} \cdot \eta_0$. Since $\mathrm{cf}(|\psi|_{\mathrm{W}}) > \omega$ we have that η_0 must be a successor and ξ_0 either a successor or a limit of uncountable cofinality.

Subcase 2.1 is that $\eta_0 > 1$, i.e., there is $\gamma > 0$ such that $\eta_0 = \gamma + 1$. But then $|\psi|_{\mathrm{W}} = \omega_1^{\xi_0} \cdot \gamma + \omega_1^{\xi_0}$ and we take regular norms χ, ψ^* of minimal Lipschitz rank such that $|\chi|_{\mathrm{W}} = \omega_1^{\xi_0} \cdot \gamma$ and $|\psi^*|_{\mathrm{W}} = \omega_1^{\xi_0}$. Then arguing from here exactly as in Subcase 1.2 we can show that

$$|\varphi \odot \psi|_{\mathrm{W}} = |\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot |\psi|_{\mathrm{W}}.$$

Subcase 2.2 is that $\eta_0 = 1$ and ξ_0 is a successor ordinal, i.e., there is an ordinal θ such that $\xi_0 = \theta + 1$. Then we have that $|\psi|_{\mathrm{W}} = \omega_1^{\theta} \cdot \omega_1$ and we fix a regular norm χ of minimal Lipschitz rank such that $|\chi|_{\mathrm{W}} = \omega_1^{\theta}$.

If χ is self-dual, then by Theorem 4.3.21 and the non-self-duality of ψ we have that $\chi^{\natural} \equiv_{\mathrm{L}} \psi$ and so by induction hypothesis and Lemma 4.3.30 we get that

$$|\varphi \odot \psi|_{\mathrm{W}} = \left|\varphi \odot \chi^{\natural}\right|_{\mathrm{W}} = \left|(\varphi \odot \chi)^{\natural}\right|_{\mathrm{W}} = |\varphi \odot \chi|_{\mathrm{W}} \cdot \omega_1.$$

Now since χ is self-dual, we have by Proposition 4.2.9 that $\mathrm{cf}(|\chi|_{\mathrm{W}}) = \mathrm{cf}(\omega_1^{\theta}) = \omega$. But then we take a sequence $\langle \nu_n \mid n \in \omega \rangle$ of ordinals cofinal in $|\chi|_{\mathrm{W}}$ such that $\mathrm{cf}(\nu_n) > \omega$ for all $n \in \omega$. Arguing now exactly as in Subcase 1.1 we get that

$$|\varphi \odot \chi|_{\mathrm{W}} = |\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot |\chi|_{\mathrm{W}}$$

and so in total

$$|\varphi \odot \psi|_{\mathrm{W}} = |\varphi \odot \chi|_{\mathrm{W}} \cdot \omega_1 = |\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot |\chi|_{\mathrm{W}} \cdot \omega_1 = |\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot |\psi|_{\mathrm{W}}.$$

If, however, χ is non-self-dual, then we have that $\psi \equiv_{\mathrm{L}} (\chi^{+\omega})^{\natural}$. Since $\chi^{+\omega}$ is self-dual, we get by the argument just given for the case that χ is self-dual:

$$|\varphi \odot \psi|_{\mathrm{W}} = \left|\varphi \odot \psi^{+\omega}\right|_{\mathrm{W}} \cdot \omega_1.$$

But by Lemma 4.3.25, Lemma 4.3.38 and the induction hypothesis we get that

$$|\varphi|_{\mathrm{W}} \cdot \omega \cdot |\chi|_{\mathrm{W}} = |\varphi \odot \chi|_{\mathrm{W}} \leq \left|\varphi \odot \chi^{+\omega}\right|_{\mathrm{W}} \leq |\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot |\chi|_{\mathrm{W}} + \omega + |\varphi|_{\mathrm{W}}.$$

In total we can now calculate that

$$\begin{aligned} |\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot |\psi|_{\mathrm{W}} &= |\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot |\chi|_{\mathrm{W}} \cdot \omega_1 \\ &= |\varphi \odot \chi|_{\mathrm{W}} \cdot \omega_1 \\ &\leq \left|\varphi \odot \chi^{+\omega}\right|_{\mathrm{W}} \cdot \omega_1 \\ &= |\varphi \odot \psi|_{\mathrm{W}} \end{aligned}$$

and

$$\begin{aligned} |\varphi \odot \psi|_{\mathrm{W}} &= \left|\varphi \odot \chi^{+\omega}\right|_{\mathrm{W}} \cdot \omega_1 \\ &\leq (|\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot |\chi|_{\mathrm{W}} + \omega + |\varphi|_{\mathrm{W}}) \cdot \omega_1 \\ &\leq |\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot (|\chi|_{\mathrm{W}} + 1) \cdot \omega_1 \\ &= |\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot |\chi|_{\mathrm{W}} \cdot \omega_1 \\ &= |\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot |\psi|_{\mathrm{W}}, \end{aligned}$$

which shows that $|\varphi \odot \psi|_{\mathrm{W}} = |\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot |\psi|_{\mathrm{W}}$, as claimed.

Subcase 2.3 is that $\eta_0 = 1$ and ξ_0 is a limit ordinal of uncountable cofinality. Then for any $\alpha < \omega_1^{\xi_0}$ we have that $\beta := \alpha + \omega_1 < \omega_1^{\xi_0}$, where $\alpha < \beta$ and $\mathrm{cf}(\beta) = \omega_1$. Using this fact we get by Proposition 4.3.39 and the induction hypothesis that

$$\begin{aligned} |\varphi \odot \psi|_{\mathrm{W}} &= \sup\{|\varphi \odot \chi|_{\mathrm{W}} \mid \chi <_{\mathrm{W}} \psi\} \\ &= \sup\{|\varphi \odot \chi|_{\mathrm{W}} \mid \chi <_{\mathrm{W}} \psi \wedge \mathrm{cf}(|\chi|_{\mathrm{W}}) > \omega\} \\ &= \sup\{|\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot |\chi|_{\mathrm{W}} \mid \chi <_{\mathrm{W}} \psi \wedge \mathrm{cf}(|\chi|_{\mathrm{W}}) > \omega\} \\ &= \sup\{|\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot |\chi|_{\mathrm{W}} \mid \chi <_{\mathrm{W}} \psi\} \\ &= |\varphi|_{\mathrm{W}} \cdot \omega_1 \cdot |\psi|_{\mathrm{W}}. \end{aligned}$$

This concludes the proof. □

We call an ordinal θ *closed under multiplication* if for any $\beta, \gamma < \theta$ we have that $\beta \cdot \gamma < \theta$. Now it follows as a corollary that for any $\alpha < \omega$ the ordinal Σ_α is closed under multiplication.

Corollary 4.3.41. *Assume* **AD** *and* **DC**. *Then for every $\alpha < \Theta$, Σ_α is closed under multiplication and therefore Σ is closed under multiplication.*

Proof. We fix $\alpha < \Theta$ and let $\beta, \gamma < \Sigma_\alpha$. Then we fix regular norms φ, ψ such that $|\varphi|_{\mathrm{W}} = \beta$ and $|\psi|_{\mathrm{W}} = \gamma$. By definition of Σ_α we then have that $\mathrm{lh}(\varphi), \mathrm{lh}(\psi) \leq \lambda_\alpha$. Also we note that $\mathrm{lh}(\hat{\omega_1}) = \omega$ and since $|\hat{\omega_1}| = \omega = \lambda_0$ so $\omega_1 < \Sigma_0 \leq \Sigma_\alpha$. Then we consider the norms $\varphi' := \varphi^{+\omega}$ and $\psi' := \psi^{+\omega} \dotplus \hat{\omega_1}$. We note that by definition of the operations $(\cdot)^{+\omega}$ and $\dotplus$ and the fact that $\mathrm{lh}(\hat{\omega_1}) = \omega$ we have that

$$\mathrm{lh}(\varphi') = \begin{cases} \mathrm{lh}(\varphi), & \text{if } \mathrm{lh}(\varphi) \text{ is limit ordinal,} \\ \mathrm{lh}(\varphi) + \omega, & \text{if } \mathrm{lh}(\varphi) \text{ otherwise,} \end{cases}$$

and the same with φ' and φ replaced by ψ' and ψ. But since λ_α is a limit ordinal we have for any successor ordinal $\theta \leq \delta_\alpha$ that also $\theta + \omega \leq \delta_\alpha$. Thus clearly $|\varphi'|_{\mathrm{W}}, |\psi'|_{\mathrm{W}} < \Sigma_\alpha$. Now we note that $|\varphi'|_{\mathrm{W}} = |\varphi^{+\omega}|_{\mathrm{W}} = |\varphi|_{\mathrm{W}} + \omega$ by Corollary 4.2.19 and since $\psi^{+\omega}$ is self-dual, we have by Theorem 4.3.9 that $|\psi'|_{\mathrm{W}} = |\psi^{+\omega} \dotplus \chi|_{\mathrm{W}} = |\psi^{+\omega}|_{\mathrm{W}} + 1 + |\chi|_{\mathrm{W}} = |\psi|_{\mathrm{W}} + \omega + \omega_1 = |\psi|_{\mathrm{W}} + \omega_1$.

In particular we have that φ' is self-dual and ψ' is such that $|\psi'|_{\mathrm{W}}$ is a limit ordinal of uncountable cofinality. Now we consider the regular norm $\varphi' \odot \psi'$, for which we note that

$$\mathrm{lh}(\varphi' \odot \psi') = \max\{\mathrm{lh}(\varphi'), \mathrm{lh}(\psi')\} \leq \lambda_\alpha$$

and so $|\varphi' \odot \psi'|_{\mathrm{W}} < \Sigma_\alpha$. By Theorem 4.3.40 we can now calculate that

$$|\varphi' \odot \psi'|_{\mathrm{W}} = |\varphi'|_{\mathrm{W}} \cdot \omega_1 \cdot |\psi'|_{\mathrm{W}} \geq |\varphi'|_{\mathrm{W}} \cdot |\psi'|_{\mathrm{W}} = (|\varphi|_{\mathrm{W}} + \omega) \cdot (|\psi|_{\mathrm{W}} + \omega_1) > |\varphi|_{\mathrm{W}} \cdot |\psi|_{\mathrm{W}} .$$

Thus we get that there is $\delta < \Sigma_\alpha$ such that $\beta \cdot \gamma < \delta$ and so by transitivity $\beta \cdot \gamma < \Sigma_\alpha$. This shows that Σ_α is closed under multiplication.

Now to see that Σ is closed under multiplication we take $\beta, \gamma < \Sigma$. Then there are $\alpha, \alpha' < \Theta$ such that $\beta < \Sigma_\alpha$ and $\gamma < \Sigma_{\alpha'}$ and so $\beta, \gamma < \Sigma_{\max\{\alpha,\alpha'\}}$. So by what we have just shown we get that $\beta \cdot \gamma < \Sigma_{\max\{\alpha,\alpha'\}}$ and so $\beta \cdot \gamma < \Sigma$. □

4.3.4 The Lower Bound

In this section we will deduce from the closure under multiplication of the ordinals Σ_α for $\alpha < \Theta$ that indeed $\Sigma \geq \Theta^{(\Theta^\Theta)}$. This considerably improves the formerly best known lower bound for Σ, which was given in Subsection 4.1.1.

Proposition 4.3.42. *Assume* $\mathbf{AD}$ *and* $\mathbf{DC}$*. For any $\alpha < \Theta$ and any $\beta < \Theta$ we have that if $\gamma < \Sigma_\alpha$, then $\gamma^\beta < \Sigma_\alpha$. As a consequence for every $\gamma < \Sigma_\alpha$ we have that $\gamma^\Theta \leq \Sigma_\alpha$.*

Proof. We take $\gamma < \Sigma_\alpha$ and a regular norm φ such that $|\varphi|_{\mathrm{W}} = \gamma$. Now we fix $\beta < \Theta$. Then also $\beta + 1 < \Theta$ and we can fix a surjection $\pi : \mathbb{R} \to \beta + 1$. Furthermore we note that $\omega_1 < \Sigma_0 \leq \Sigma_\alpha$, since $\mathrm{lh}(\hat{\omega_1}) = \omega$. Now we construct norms φ_δ for $\delta \leq \beta$ s.t. $|\varphi_\delta|_{\mathrm{W}} \geq |\varphi|_{\mathrm{W}}^\delta$ recursively as follows.

We set $\varphi_0 := \varphi$. If δ is a successor ordinal, say $\delta = \theta + 1$, we set $\varphi_\delta := \varphi_\theta^{+\omega} \odot (\varphi^{+\omega} \dotplus \hat{\omega_1})$. If δ is a limit ordinal we finally set for any $x, y \in \mathbb{R}$:

$$\varphi_\delta(x * y) := \begin{cases} \varphi_{\pi(x)}(y), & \text{if } \pi(x) < \delta, \\ \varphi(y), & \text{otherwise.} \end{cases}$$

Clearly by construction we have for any $\delta \leq \beta$ that $\mathrm{lh}(\varphi_\delta) \leq \lambda_\alpha$ and so $|\varphi_\delta|_{\mathrm{W}} \in \Sigma_\alpha$. Now we inductively check that indeed for any $\delta \leq \beta$ we have that $|\varphi_\delta|_{\mathrm{W}} \geq |\varphi|_{\mathrm{W}}^\delta$.

We have $|\varphi_0|_{\mathrm{W}} = |\varphi|_{\mathrm{W}} \geq |\varphi|_{\mathrm{W}}^0$. In case δ is a successor ordinal, say $\delta = \theta + 1$ we get – using the induction hypothesis – that

$$\begin{aligned} |\varphi_\delta|_{\mathrm{W}} &= \left|\varphi_\theta^{+\omega} \odot (\varphi_\theta^{+\omega} \dotplus \chi)\right|_{\mathrm{W}} \\ &= (|\varphi_\theta|_{\mathrm{W}} + \omega) \cdot \omega_1 \cdot (|\varphi|_{\mathrm{W}} + \omega_1) \\ &\geq (|\varphi|_{\mathrm{W}}^\theta + \omega) \cdot (|\varphi|_{\mathrm{W}} + \omega_1) \\ &\geq |\varphi|_{\mathrm{W}}^{\theta+1} = |\varphi|_{\mathrm{W}}^\delta . \end{aligned}$$

In case δ is a limit we note that for any $\delta' < \delta$, the regular norm $\varphi_{\delta'}$ embeds into φ_δ and so $\varphi_{\delta'} \leq_{\mathrm{W}} \varphi_\delta$, which in total implies that $|\varphi_\delta|_{\mathrm{W}} \geq \sup\{|\varphi_{\delta'}|_{\mathrm{W}} \mid \delta' < \delta\}$. Thus by induction hypothesis we get that

$$|\varphi_\delta|_{\mathrm{W}} \geq \sup\{|\varphi|_{\mathrm{W}}^{\delta'} \mid \delta' < \delta\} = |\varphi|_{\mathrm{W}}^\delta .$$

Now in particular we get that $|\varphi_\beta|_{\mathrm{W}} \geq |\varphi|_{\mathrm{W}}^\beta = \gamma^\beta$, but since also $|\varphi_\beta|_{\mathrm{W}} < \Sigma_\alpha$, this implies that by transitivity $\gamma^\beta < \Sigma_\alpha$. Since we have shown this for arbitrary $\beta < \Theta$, we finally get that

$$\gamma^\Theta = \sup\{\gamma^\beta \mid \beta < \Theta\} \leq \Sigma_\alpha,$$

concluding the proof. □

Theorem 4.3.43. *Assume* $\mathbf{AD}$ *and* $\mathbf{DC}$ *. Then we have that $\Sigma \geq \Theta^{(\Theta^\Theta)}$.*

Proof. We show by induction that for any ordinal $\alpha < \Theta$ we have that $\Theta^{(\Theta^\alpha)} \leq \Sigma_\alpha$. As shown in Theorem 4.1.8 we have $\Theta^{(\Theta^0)} = \Theta \leq \Sigma_0$. Let α be a successor ordinal, say $\alpha = \gamma + 1$ and assume that $\Theta^{\Theta^\gamma} \leq \Sigma_\gamma$. Then in particular $\Theta^{\Theta^\gamma} < \Sigma_\alpha$ and so by Proposition 4.3.42 we get that

$$\Theta^{\Theta^\alpha} = \left(\Theta^{\Theta^\gamma}\right)^\Theta \leq \Sigma_\alpha.$$

Finally if α is a limit and assuming that for all $\alpha' < \alpha$ we have that $\Sigma_{\alpha'} \geq \Theta^{(\Theta^\alpha)}$, we get that

$$\Theta^{(\Theta^\alpha)} = \sup\left\{\Theta^{(\Theta^{\alpha'})} \mid \alpha' < \alpha\right\} \leq \sup\left\{\Sigma_{\alpha'} \mid \alpha' < \alpha\right\} = \Sigma_\alpha.$$

Now since for all $\alpha < \Theta$ we have that $\Sigma_\alpha < \Sigma$ we get that $\Theta^{(\Theta^\alpha)} < \Sigma$ and so

$$\Theta^{(\Theta^\Theta)} = \sup\left\{\Theta^{(\Theta^\alpha)} \mid \alpha < \Theta\right\} \leq \sup\left\{\Sigma_\alpha \mid \alpha < \Theta\right\} = \Sigma.$$

□

Chapter 5

Open Questions and Future Work

Regarding the subject matter of this thesis several questions remain open, which could lead to further work in this area. I will note a few of them in the following.

At the conclusion of this thesis we know that $\Theta^{(\Theta^{\Theta})} \leq \Sigma < \Theta^{+}$, but these bounds still lie far apart from each other. To extend the idea of this thesis further one could try to find even stronger closure properties for the ordinals Σ_α for $\alpha < \Theta$. The logical next step would be the following question.

Question 5.1.1. *Assume* $\mathbf{AD}$ *and* $\mathbf{DC}$*. Is there an* $\alpha < \Theta$ *and an operation exp* : $\mathcal{N} \times \mathcal{N} \to \mathcal{N}$ *such that for any* $\varphi, \psi \in \mathcal{N}$ *we have that* $\mathrm{lh}(exp(\varphi, \psi)) = \max\{\mathrm{lh}(\varphi), \mathrm{lh}(\psi)\}$ *and there are upwards* $\leq_{\mathrm{W}}$*-unbounded subsets* $X \subseteq \mathcal{N}$ *and* $Y \subseteq \mathcal{N}$ *such that for all* $\varphi \in X$ *and all* $\psi \in X$*,*

$$|exp(\varphi, \psi)|_{\mathrm{W}} = (|\varphi|_{\mathrm{W}})^{\alpha^{|\varphi|_{\mathrm{W}}}} \text{ ?}$$

This would positively decide the following question, which, however, is also of interest, even if the Question 5.1.1 should be answered negatively.

Question 5.1.2. *Assume* $\mathbf{AD}$ *and* $\mathbf{DC}$*. Is it true that for every* $\alpha < \Theta$ *the ordinal* Σ_α *is closed under ordinal exponentiation ?*

Such a closure under exponentiation would again considerably increase the lower bound for Σ. We will in the following establish the lower bound for Σ that we could show under this assumption.

Definition 5.1.3 For any two ordinals α, β with $\beta > 0$ we define an ordinal $\alpha \uparrow\uparrow \beta$ by recursion on β as follows:

We set $\alpha \uparrow\uparrow 1 := \alpha$. For $\beta > 1$ a successor, say $\beta = \gamma + 1$ we set $\alpha \uparrow\uparrow \beta := (\alpha \uparrow\uparrow \gamma)^{\alpha}$. For β a limit ordinal we set $\alpha \uparrow\uparrow \beta := \bigcup_{\beta' < \beta} (\alpha \uparrow\uparrow \beta')$.

Definition 5.1.4 For every ordinal α we define an ordinal ε_α by recursion on α as follows:

We set $\varepsilon_0 := \omega \uparrow\uparrow \omega$. For α a successor ordinal, say $\alpha = \gamma + 1$, we set $\varepsilon_\alpha := \varepsilon_\gamma \uparrow\uparrow \omega$. Finally for α a limit ordinal we set $\varepsilon_\alpha := \sup\{\varepsilon_{\alpha'} \mid \alpha' < \alpha\}$.

We note that the ordinals ε_α exactly enumerate the ordinals γ with the property that $\gamma = \omega^\gamma$. Also we note that for any uncountable cardinal κ we have that $\varepsilon_\kappa = \kappa$. Now we get the following proposition:

Proposition 5.1.5. *If for every* $\alpha < \Theta$ *the ordinal* Σ_α *is closed under exponentiation, then* $\Sigma \geq \varepsilon_{\Theta+\Theta}$.

Proof. We inductively show that for all $\alpha < \Theta$ we have that $\Sigma_\alpha \geq \varepsilon_{\Theta+\alpha}$. For $\alpha = 0$ we have by Theorem 4.1.8 that $\Sigma_0 \geq \Theta$. But $\Theta = \varepsilon_\Theta$, since Θ is an uncountable cardinal. Thus we have that $\Sigma_0 \geq \varepsilon_\Theta$.

Now let α be a successor ordinal, say $\alpha = \gamma + 1$. Then by induction hypothesis $\Sigma_\gamma \geq \varepsilon_{\Theta+\gamma}$. But then we have that $\varepsilon_{\Theta+\gamma} < \Sigma_\alpha$. But since Σ_α is closed under exponentiation we inductively get that $\varepsilon_{\Theta+\gamma} \uparrow\uparrow n < \Sigma_\alpha$ for all $n \in \omega$. But then we have that $\varepsilon_{\Theta+\alpha} = \varepsilon_{\Theta+\gamma} \uparrow\uparrow \omega \leq \Sigma_\alpha$.

Finally we let α be a limit ordinal. Then by assumption $\Sigma_{\alpha'} \geq \varepsilon_{\Theta+\alpha'}$ for all $\alpha' < \alpha$ and thus

$$\varepsilon_{\Theta+\alpha} = \sup\{\varepsilon_{\Theta+\alpha'} \mid \alpha' < \alpha\} \leq \sup\{\Sigma_{\alpha'} \mid \alpha' < \alpha\} = \Sigma_\alpha.$$

This shows that indeed for all $\alpha < \Theta$ we have that $\Sigma_\alpha \geq \varepsilon_{\Theta+\alpha}$. But then it follows immediately that

$$\Sigma = \sup\{\Sigma_\alpha \mid \alpha < \Theta\} \geq \{\varepsilon_{\Theta+\alpha} \mid \alpha < \Theta\} = \varepsilon_{\Theta+\Theta},$$

as claimed. □

Another interesting open question of course is how to improve the upper bound for Σ. For this it would seem to be a good starting point to try to find an upper bound for Σ_0. We already know that $\Theta \leq \Sigma_0$. So the answer to the following question would be interesting.

Question 5.1.6. *Assume* $\mathbf{AD}$ *and* $\mathbf{DC}$*. Is* $\Sigma_0 = \Theta$*?*

An approach to answer the above question negatively is to try to construct an operation $O : \mathcal{N} \to \mathcal{N}$ such that for all regular norm φ with $\mathrm{lh}(\varphi) \leq \omega$ we have that $\mathrm{lh}(O(\varphi)) \leq \omega$ and there is at least one regular norm φ with $\mathrm{lh}(\varphi) \leq \omega$ such that either $|O(\varphi)|_{\mathrm{W}} = |\varphi|_{\mathrm{W}} + \Theta$ or $|O(\varphi)|_{\mathrm{W}} = |\varphi|_{\mathrm{W}} \cdot \Theta$. If we, however, try to answer above question positively we should try to find a surjection

$$s_\alpha : \mathbb{R} \to \{[\varphi]_{\mathrm{W}} \in \mathcal{N}/_{\equiv_{\mathrm{W}}} \mid |\varphi|_{\mathrm{W}} < \alpha\}$$

for any $\alpha < \Theta$. In any case we should keep in mind that it is conceivable that this question cannot be decided in $\mathbf{ZF} + \mathbf{AD} + \mathbf{DC}$, since it could depend on the cofinality of Θ.

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